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Convergence and Stability in Maximum Norms of Linearized Fourth-Order Conservative Compact Scheme for Benjamin–Bona–Mahony–Burgers' Equation

Qifeng Zhang[1,2] · **Lingling Liu**[1]

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Abstract

In the paper, a newly developed three-point fourth-order compact operator is utilized to construct an efficient compact finite difference scheme for the Benjamin–Bona–Mahony–Burgers' (BBMB) equation. The detailed derivation is carried out based on the reduction order method together with a three-level linearized technique. The conservative invariant, boundedness and unique solvability are studied at length. The convergence is proved by the technical energy argument and induction method with the optimal convergence order $\mathcal{O}(\tau^2 + h^4)$ in the sense of the maximum norm. The stability under mild conditions can be achieved based on the uniform boundedness of the numerical solution. The present scheme is very efficient in practical computation since only a linear system needs to be solved at each time. The extensive numerical examples verify our theoretical results and demonstrate the scheme's superiority when compared with state-of-the-art those in the references.

Keywords BBMB equation · Reduction order method · Linearized compact scheme · Boundedness · Maximum norm

1 Introduction

The classical nonlinear Benjamin–Bona–Mahony (BBM) equation can describe the unidirectional propagation of weakly nonlinear long waves in the presence of dispersion as follows

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✉ Qifeng Zhang
zhangqifeng0504@gmail.com

Lingling Liu
2710751137@qq.com

1 Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China

2 Department of Mathematics, Southeast University, Nanjing 210096, China

$$u_t - \frac{\mu}{6}u_{xxt} + \frac{3\varepsilon}{2}uu_x + u_x = 0, \quad x \in \mathbb{R},\ 0 < t \le T, \tag{1.1}$$

where $\varepsilon > 0$ and μ are the parameters with the same order [8]. Compared with the well-known Korteweg-de Vries (KdV) equation

$$u_t + u_x + \frac{3\varepsilon}{2}uu_x + \frac{\mu}{6}u_{xxx} = 0, \quad x \in \mathbb{R},\ 0 < t \le T, \tag{1.2}$$

(1.1) is proposed as an analytically advantageous alternative. Both (1.1) and (1.2) are derived from the Green–Naghdi equations and they are asymptotically equivalent in the limit $\varepsilon = \mu \to 0$ since $u_{xxx} = u_{xxt} + \mathcal{O}(\mu)$, but enjoying different properties, see [26] for a detailed explanation. In many applications, when the dissipation effect cannot be ignored, $-\nu u_{xx}$ have to be added and (1.1) becomes the known BBMB equation as

$$u_t - \frac{\mu}{6}u_{xxt} + \frac{3\varepsilon}{2}uu_x + u_x - \nu u_{xx} = 0, \quad x \in \mathbb{R},\ 0 < t \le T, \tag{1.3}$$

which describes the propagation of small-amplitude long waves in a nonlinear dispersive media. For the well-posedness, existence, uniqueness, regularity results, long time dynamics and the numerical simulation for (1.3) and its special cases are referred to [22,28,31,32,38, 41,46,47,52].

In this paper, we are aimed to develop and analyze a high-order conservative difference approximation for the BBMB equation as

$$u_t - \mu u_{xxt} + \gamma uu_x + \kappa u_x - \nu u_{xx} = 0, \quad x \in \mathbb{R},\ 0 < t \le T, \tag{1.4}$$

with the periodic boundary condition

$$u(x,t) = u(x+L,t), \quad x \in \mathbb{R},\ 0 < t \le T, \tag{1.5}$$

and the initial-value condition

$$u(x,0) = \varphi(x), \quad x \in \mathbb{R}, \tag{1.6}$$

where μ and ν are positive constants, γ and κ are parameters and L denotes the spatial period.

In order to explore the solutions and their properties of the BBMB equation, researchers racked their wits to develop various analytical methods for seeking the exact solutions of the BBMB equation. For instance, Yin et al. [50] employed the weighted energy method to investigate the time decay rate of traveling waves of Cauchy problem of the BBMB equation. Estévez et al. [16] studied the travelling wave solutions for the generalized BBMB equation systematically by using the factorization technique. Besse et al. [11] developed the exact artificial boundary conditions for the linearized BBM equation. Al-Khaled et al. [3] considered solitary wave solutions of the BBMB equation by using the decomposition method. Fakhari et al. [17] approximated the nonlinear BBMB equation's explicit solutions with high-order nonlinear term via the homotopy analysis method. Tari et al. [45] used He's methods to obtain the explicit solutions of the BBMB equation and compared them with the exact solutions. Ganji et al. [18] solved the special form solutions of the BBMB equation by the Exp-Function method. Based on the well-known tanh-coth method, Cesar et al. [13] obtained new periodic soliton solutions for the generalized BBMB equation. Noor et al. [34] constructed some new solitary solutions of the BBM equation by using the exp-function method. Abbasbandy [1] used the first integral method to find some new exact solutions for the BBMB equation and Bruzón [12] studied some nontrivial conservation laws for the BBMB equation with the help of the multiplier's method.

On the other hand, many attempts have been carried out to approximate the solutions for the BBMB equation and its simplified version numerically. For example, Guo [20] proposed a Laguerre–Galerkin method to solve the BBM equation on a semi-infinite interval. Omrani [35] considered a fully discrete Galerkin method for the BBM equation. Soon afterward, Omrani et al. [2,36] used the Crank–Nicolson-type implicit finite difference method to solve the BBMB equation with the second-order accuracy in maximum norm. They [21] further employed the Galerkin finite element method in space combined with the implicit Euler method in time for solving the generalized BBMB equation. Berikelashvili et al. [9] explored a linearized difference scheme for solving the regularized long-wave equation, which can be viewed as a special case of the BBMB equation with $\nu = 0$. They [10] also analyzed the convergence of a type of difference scheme for the generalized BBMB equation. Based on the meshless method of radial basis functions, Dehghan et al. [14] solved a high dimensional generalized BBMB equation. They [15] further considered the interpolating element-free Galerkin technique for the high dimensional BBMB equation. Zarebnia et al. [40,51] used the collocation method and spectral meshless radial point interpolation, respectively, to solve the BBMB equation. Similar idea applied to BBMB is referred to the work in [4]. Based on the hybridization of Lucas and Fibonacci polynomials, Oruç et al. [37] solved the generalized BBMB equation in one and two dimension, respectively. Kundu et al. [25] proposed a semidiscrete Galerkin method and discussed stabilization results for the semidiscrete scheme with an optimal error estimate. Kundu et al. [24] discussed global stabilization results for the semidiscrete solution based on finite element method. With the help of Legendre wavelets and quasilinearization method, Kumar et al. [23] simulated multidimensional BBMB equation. Zhang et al. [53] established two linearized implicit difference schemes for the BBMB equation, in which the convergence orders both were two.

A review of all the above numerical methods reveals that higher-order algorithms are still scarce, let alone the uniform error estimate of the higher-order algorithms. To the authors' best knowledge, only Mohebbi and Faraz [33] propose a fourth-order algorithm for solving the BBMB equation with five points in space and obtains the infinite error estimate. However, when deal with the points near the boundary, ghost points or fictitious points are requisite. In addition, the stability in [33] is also missing. In order to avoid the difficulty caused by the discretization near the boundary points, we first developed three-point fourth-order compact technique for the Burgers' equation in [48] and further extended it to the BBMB equation in current paper. Moreover, we extensively and deeply studied the convergence and stability of the compact difference scheme for the BBMB equation.

The compact difference scheme as one of the most practical numerical techniques has the significant advantages over standard finite difference methods. Specifically: (1) A smaller matrix stencil generates higher order accuracy; (2) A larger stability domain allows larger spatial and temporal step sizes; (3) It owns a better resolution for high frequency waves; (4) It is more suitable for long time integrations; (5) Fewer boundary point makes the discretization of the boundary easier.

The compact difference scheme in the present paper not only possess all of these advantages, but also does not incur an extra computational cost. Furthermore, our scheme is linearly implicit with the exact well-defined conservative invariant. The main difficulties for the high-order approximation of the strong nonlinear term uu_x involving the optimal convergence and stability are completely overcome based on the newly discovered compact operator, which makes the numerical analysis feasible and toilless.

The main contribution lies in that the maximum error estimate and the optimal convergence order $O(\tau^2 + h^4)$ are obtained. The proof of the convergence in a pointwise sense is novel and technical. Compared our numerical results with those calculated in [33] is carried out, which demonstrates the effectiveness and advantage of the present algorithm. Moreover, the stability under very mild conditions in the maximum norm is also proved in detail.

The rest of the paper is organized as follows. In Sect. 2, some requisite notations and useful lemmas are presented. A three-level linearized compact difference scheme is derived in Sect. 3 based on the reduction order method. Conservative invariant and boundedness are obtained in Sect. 4. The unique solvability is proved strictly in Sect. 5. The uniform convergence and stability are proved at length in Sect. 6, which are the main part of the paper. Several numerical experiments are presented in Sect. 7 followed by a conclusion in Sect. 8.

2 Notations and Lemmas

We firstly introduce some useful notations. Take two positive integers M, N, let $h = L/M$, $\tau = T/N$. Denote $x_i = ih,\ i \in Z,\ t_k = k\tau,\ 0 \le k \le N,\ t_{k+\frac{1}{2}} = (t_k + t_{k+1})/2$; $\Omega_h = \{x_i \,|\, x_i = ih,\ i \in Z\}$, $\Omega_\tau = \{t_k \,|\, t_k = k\tau,\ 0 \le k \le N\}$, $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. For any grid function $u = \{u_i^k \,|\, i \in Z,\ 0 \le k \le N\}$ defined on $\Omega_{h\tau}$, following the notations in [6,7,53], we introduce

$$
\begin{aligned}
\delta_x^+ u_i^k &= \frac{1}{h}(u_{i+1}^k - u_i^k), \quad \delta_x^2 u_i^k = \frac{1}{h}(\delta_x^+ u_i^k - \delta_x^+ u_{i-1}^k), \quad \Delta_x u_i^k = \frac{1}{2h}(u_{i+1}^k - u_{i-1}^k),\\
u_{i+\frac{1}{2}}^k &= \frac{1}{2}(u_i^k + u_{i+1}^k), \quad u_i^{k+\frac{1}{2}} = \frac{1}{2}(u_i^k + u_i^{k+1}), \quad u_i^{\bar{k}} = \frac{1}{2}(u_i^{k-1} + u_i^{k+1}),\\
\delta_t u_i^{k+\frac{1}{2}} &= \frac{1}{\tau}(u_i^{k+1} - u_i^k), \quad \Delta_t u_i^k = \frac{1}{2\tau}(u_i^{k+1} - u_i^{k-1}).
\end{aligned}
$$

Denote

$$
\mathcal{U}_h = \{u \,|\, u = \{u_i\},\ u_{i+M} = u_i\}.
$$

For any grid functions $u, w \in \mathcal{U}_h$, define the discrete inner product $(u, w) = h \sum\limits_{i=1}^{M} u_i w_i$ and the corresponding norms (seminorm)

$$
\|u\| = \sqrt{(u, u)}, \qquad |u|_1 = \sqrt{(\delta_x^+ u, \delta_x^+ u)}, \qquad \|u\|_\infty = \max_{1 \le i \le M} |u_i|.
$$

Moreover, define the function (see [19])

$$
\psi(u, v)_i = \frac{1}{3}[u_i \Delta_x v_i + \Delta_x (uv)_i], \quad 1 \le i \le M.
$$

Lemma 2.1 [44] *For any grid functions $u, w \in \mathcal{U}_h$, we have*

$$
\|u\|_\infty \le \frac{\sqrt{L}}{2}|u|_1, \quad |u|_1 \le \frac{2}{h}\|u\|, \quad \|u\| \le \frac{L}{\sqrt{6}}|u|_1, \quad (\delta_x^2 u, w) = -(\delta_x^+ u, \delta_x^+ w).
$$

Lemma 2.2 *For any grid functions $u, w \in \mathcal{U}_h$, we have*

$$
(\psi(u, w), w) = 0, \quad (\Delta_x u, u) = 0, \quad (\Delta_x u, \delta_x^2 u) = 0.
$$

Proof The first and second equalities come from [42]. We will prove the third equality briefly below. According to the notations defined previously, we have

$$\begin{aligned}\delta_x^+(\Delta_x u)_i =& \frac{1}{h}\Big(\Delta_x u_{i+1} - \Delta_x u_i\Big)\\ &= \frac{1}{h}\Big[\frac{1}{2h}(u_{i+2} - u_i) - \frac{1}{2h}(u_{i+1} - u_{i-1})\Big]\\ &= \frac{1}{2h}\Big[\frac{1}{h}(u_{i+2} - u_{i+1}) - \frac{1}{h}(u_i - u_{i-1})\Big]\\ &= \frac{1}{2h}(\delta_x^+ u_{i+1} - \delta_x^+ u_{i-1}) = \Delta_x(\delta_x^+ u)_i.\end{aligned}$$

From the definition of the discrete inner product and the second equality, we have

$$\begin{aligned}(\Delta_x u, \delta_x^2 u) =& -(\delta_x^+(\Delta_x u), \delta_x^+ u) = -h\sum_{i=1}^{M}\delta_x^+(\Delta_x u)_i\cdot\delta_x^+ u_i\\ &= -h\sum_{i=1}^{M}\Delta_x(\delta_x^+ u)_i\cdot\delta_x^+ u_i = -(\Delta_x(\delta_x^+ u), \delta_x^+ u) = 0.\end{aligned}$$

This completes the proof. □

Lemma 2.3 *Let* $f(x) \in \mathrm{C}^5[x_{i-1}, x_{i+1}]$ *and denote* $F_i = f(x_i)$ *and* $G_i = f''(x_i)$, *then we have*

$$\begin{aligned}f(x_i)f'(x_i) &= \psi(F, F)_i - \frac{h^2}{2}\psi(G, F)_i + \mathcal{O}(h^4), \quad 1 \le i \le M,\\ f'(x_i) &= \Delta_x F_i - \frac{h^2}{6}\Delta_x G_i + \mathcal{O}(h^4), \quad 1 \le i \le M,\\ f''(x_i) &= \delta_x^2 F_i - \frac{h^2}{12}\delta_x^2 G_i + \mathcal{O}(h^4), \quad 1 \le i \le M.\end{aligned}$$

Proof The first and third equalities come from [48] and [43], respectively. The second equality is immediately obtained by Taylor expansion. We omit it here for sake of brevity. □

Lemma 2.4 *For any grid functions* $u,\ v,\ S \in \mathcal{U}_h$, *satisfying*

$$v_i = \delta_x^2 u_i - \frac{h^2}{12}\delta_x^2 v_i + S_i, \quad 1 \le i \le M, \tag{2.1}$$

$$u_i = u_{i+M}, \quad 0 \le i \le M, \tag{2.2}$$

$$v_i = v_{i+M}, \quad 0 \le i \le M, \tag{2.3}$$

we have

$$(v, u) = -|u|_1^2 - \frac{h^2}{12}\|v\|^2 + \frac{h^4}{144}|v|_1^2 + \frac{h^2}{12}(v, S) + (S, u), \tag{2.4}$$

$$(v, u) \le -|u|_1^2 - \frac{h^2}{18}\|v\|^2 + \frac{h^2}{12}(v, S) + (S, u), \tag{2.5}$$

$$(\Delta_x v, u) = \frac{h^2}{12}(\Delta_x v, S) + (\Delta_x S, u). \tag{2.6}$$

Proof Taking the inner product of (2.1) with u and noticing (2.2)–(2.3), we have

$$
\begin{aligned}
(v, u) &= \left(\delta_x^2 u - \frac{h^2}{12}\delta_x^2 v + S, u\right) \\
&= (\delta_x^2 u, u) - \frac{h^2}{12}(\delta_x^2 v, u) + (S, u) \\
&= -|u|_1^2 - \frac{h^2}{12}(v, \delta_x^2 u) + (S, u) \\
&= -|u|_1^2 - \frac{h^2}{12}\left(v, v + \frac{h^2}{12}\delta_x^2 v - S\right) + (S, u) \\
&= \ -|u|_1^2 - \frac{h^2}{12}\|v\|^2 + \frac{h^4}{144}|v|_1^2 + \frac{h^2}{12}(v, S) + (S, u).
\end{aligned}
$$

With the help of Lemma 2.1, we have

$$
(v, u) \le -|u|_1^2 - \frac{h^2}{18}\|v\|^2 + \frac{h^2}{12}(v, S) + (S, u).
$$

Combining (2.1) with Lemmas 2.1–2.2, we have

$$
\begin{aligned}
(\Delta_x v, u) &= \ \left(\Delta_x\left(\delta_x^2 u - \frac{h^2}{12}\delta_x^2 v + S\right), u\right) \\
&= \ \left(\Delta_x(\delta_x^2 u), u) - \frac{h^2}{12}(\Delta_x(\delta_x^2 v), u\right) + (\Delta_x S, u) \\
&= \ -(\Delta_x(\delta_x^+ u), \delta_x^+ u) - \frac{h^2}{12}(\Delta_x v, \delta_x^2 u) + (\Delta_x S, u) \\
&= \ -\frac{h^2}{12}(\Delta_x v, \delta_x^2 u) + (\Delta_x S, u) \\
&= \ -\frac{h^2}{12}(\Delta_x v, v + \frac{h^2}{12}\delta_x^2 v - S) + (\Delta_x S, u) \\
&= \ -\frac{h^2}{12}(\Delta_x v, v) + \frac{h^4}{144}(\Delta_x(\delta_x^+ v), \delta_x^+ v) + \frac{h^2}{12}(\Delta_x v, S) + (\Delta_x S, u) \\
&= \ \frac{h^2}{12}(\Delta_x v, S) + (\Delta_x S, u).
\end{aligned}
$$

This completes the proof. □

Lemma 2.5 *For any grid functions u, $v \in \mathcal{U}_h$, we have*

$$
\Delta_x(uv)_i = \frac{1}{2}\left(\delta_x^+ u_i\right) v_{i+1} + \frac{1}{2}\left(\delta_x^+ u_{i-1}\right) v_{i-1} + u_i \Delta_x v_i.
$$

Proof Using the definition of the operator, we directly have

$$
\begin{aligned}
\Delta_x(uv)_i &= \ \frac{1}{2h}(u_{i+1}v_{i+1} - u_{i-1}v_{i-1}) \\
&= \ \frac{1}{2h}[(u_{i+1} - u_i)v_{i+1} + (u_i - u_{i-1})v_{i-1} + u_i(v_{i+1} - v_{i-1})] \\
&= \ \frac{1}{2}\left(\delta_x^+ u_i\right) v_{i+1} + \frac{1}{2}\left(\delta_x^+ u_{i-1}\right) v_{i-1} + u_i \Delta_x v_i.
\end{aligned}
$$

This completes the proof. □

3 Derivation of Compact Difference Scheme

Denote

$$c_0 = \max_{0\le x\le L, 0\le t\le T}\{|u(x,t)|,\ |u_x(x,t)|,\ |u_{xx}(x,t)|,\ |u_{xxx}(x,t)|\}. \tag{3.1}$$

Let $v = u_{xx}$, then the problem (1.4)–(1.6) is equivalent to

$$u_t - \mu v_t + \gamma u u_x + \kappa u_x - \nu v = 0, \quad x \in \mathbb{R},\ 0 < t \le T, \tag{3.2}$$

$$v = u_{xx}, \quad x \in \mathbb{R},\ 0 < t \le T, \tag{3.3}$$

$$u(x,0) = \varphi(x), \quad x \in \mathbb{R}, \tag{3.4}$$

$$u(x,t) = u(x+L,t), \quad v(x,t) = v(x+L,t), \quad x \in \mathbb{R},\ 0 < t \le T. \tag{3.5}$$

Define the grid functions

$$U_i^k = u(x_i,t_k), \quad V_i^k = v(x_i,t_k), \quad 1 \le i \le M,\ 0 \le k \le N.$$

With the help of Lemma 2.3, we have

$$uu_x(x_i,t_k) = \psi(U^k,U^k)_i - \frac{h^2}{2}\psi(V^k,U^k)_i + \mathcal{O}(h^4), \tag{3.6}$$

$$u_x(x_i,t_k) = \Delta_x U_i^k - \frac{h^2}{6}\Delta_x V_i^k + \mathcal{O}(h^4), \tag{3.7}$$

$$u_{xx}(x_i,t_k) = \delta_x^2 U_i^k - \frac{h^2}{12}\delta_x^2 V_i^k + \mathcal{O}(h^4). \tag{3.8}$$

Considering (3.2) at the point $(x_i, t_{\frac{1}{2}})$, with the help of Taylor expansion and (3.6)–(3.8), we have

$$\delta_t U_i^{\frac{1}{2}} - \mu\delta_t V_i^{\frac{1}{2}} + \gamma\Big[\psi(U^0,U^{\frac{1}{2}})_i - \frac{h^2}{2}\psi(V^0,U^{\frac{1}{2}})_i\Big] + \kappa\Big(\Delta_x U_i^{\frac{1}{2}} - \frac{h^2}{6}\Delta_x V_i^{\frac{1}{2}}\Big) - \nu V_i^{\frac{1}{2}} = Q_i^0, \quad 1 \le i \le M, \tag{3.9}$$

where

$$|Q_i^0| \le c_1(\tau + h^4), \quad 1 \le i \le M, \tag{3.10}$$

with c_1 being a positive constant. Analogously, considering (3.2) at the point (x_i, t_k), we have

$$\Delta_t U_i^k - \mu\Delta_t V_i^k + \gamma\Big[\psi(U^k,U^{\bar{k}})_i - \frac{h^2}{2}\psi(V^k,U^{\bar{k}})_i\Big] + \kappa\Big(\Delta_x U_i^{\bar{k}} - \frac{h^2}{6}\Delta_x V_i^{\bar{k}}\Big) - \nu V_i^{\bar{k}} = Q_i^k, \quad 1 \le i \le M,\ 1 \le k \le N-1, \tag{3.11}$$

where

$$|Q_i^k| \le c_2(\tau^2 + h^4), \quad 1 \le i \le M,\ 1 \le k \le N-1, \tag{3.12}$$

with c_2 being a positive constant.

Again considering (3.3) at the point (x_i, t_k), we have

$$V_i^k = \delta_x^2 U_i^k - \frac{h^2}{12}\delta_x^2 V_i^k + R_i^k, \quad 1 \le i \le M,\ 0 \le k \le N, \tag{3.13}$$

where

$$|R_i^k| \le c_3 h^4, \quad |\Delta_t R_i^k| \le c_3(\tau^2 + h^4), \quad 1 \le i \le M,\ 0 \le k \le N, \tag{3.14}$$

with c_3 being a positive constant.

Noticing the initial and boundary conditions (3.4)–(3.5), we have

$$U_i^0 = \varphi(x_i), \quad 1 \le i \le M, \tag{3.15}$$

$$U_i^k = U_{i+M}^k, \quad V_i^k = V_{i+M}^k, \quad 1 \le i \le M,\ 0 \le k \le N. \tag{3.16}$$

Omitting the small terms Q_i^k and R_i^k, replacing the grid functions U_i^k, V_i^k by u_i^k, v_i^k in (3.9), (3.11), (3.13), respectively, and noticing the initial and boundary conditions (3.15)–(3.16), then we construct a finite difference scheme for (3.2)–(3.5) as follows

$$\delta_t u_i^{\frac{1}{2}} - \mu\delta_t v_i^{\frac{1}{2}} + \gamma\Big[\psi(u^0, u^{\frac{1}{2}})_i - \frac{h^2}{2}\psi(v^0, u^{\frac{1}{2}})_i\Big] + \kappa\Big(\Delta_x u_i^{\frac{1}{2}} - \frac{h^2}{6}\Delta_x v_i^{\frac{1}{2}}\Big) - \nu v_i^{\frac{1}{2}} = 0, \quad 1 \le i \le M, \tag{3.17}$$

$$\Delta_t u_i^k - \mu\Delta_t v_i^k + \gamma\Big[\psi(u^k, u^{\bar{k}})_i - \frac{h^2}{2}\psi(v^k, u^{\bar{k}})_i\Big] + \kappa\Big(\Delta_x u_i^{\bar{k}} - \frac{h^2}{6}\Delta_x v_i^{\bar{k}}\Big) - \nu v_i^{\bar{k}} = 0, \quad 1 \le i \le M,\ 1 \le k \le N-1, \tag{3.18}$$

$$v_i^k = \delta_x^2 u_i^k - \frac{h^2}{12}\delta_x^2 v_i^k, \quad 1 \le i \le M,\ 0 \le k \le N, \tag{3.19}$$

$$u_i^0 = \varphi(x_i), \quad 1 \le i \le M, \tag{3.20}$$

$$u_i^k = u_{i+M}^k, \quad v_i^k = v_{i+M}^k, \quad 1 \le i \le M,\ 0 \le k \le N. \tag{3.21}$$

4 Conservative Invariant and Boundedness

Theorem 4.1 *Suppose* $\{u_i^k,\ v_i^k \mid 1 \le i \le M,\ 0 \le k \le N\}$ *is the solution of* (3.17)–(3.21). *Then it holds that*

$$\frac{1}{2}(\|u^1\|^2 + \|u^0\|^2) + \frac{\mu}{2}\Big[(|u^1|_1^2 + |u^0|_1^2) + \frac{h^2}{12}(\|v^1\|^2 + \|v^0\|^2) - \frac{h^4}{144}(|v^1|_1^2 + |v^0|_1^2)\Big] + \nu\tau\Big(|u^{\frac{1}{2}}|_1^2 + \frac{h^2}{12}\|v^{\frac{1}{2}}\|^2 - \frac{h^4}{144}|v^{\frac{1}{2}}|_1^2\Big) = \|u^0\|^2 + \mu|u^0|_1^2 + \frac{\mu h^2}{12}\|v^0\|^2 - \frac{\mu h^4}{144}|v^0|_1^2, \tag{4.1}$$

$$E(u^{k+1}, u^k) = E(u^1, u^0), \quad 1 \le k \le N-1, \tag{4.2}$$

where

$$E(u^{k+1}, u^k) = \frac{1}{2}(\|u^{k+1}\|^2 + \|u^k\|^2) + 2\nu\tau\sum_{l=1}^{k}\Big(|u^{\bar{l}}|_1^2 + \frac{h^2}{12}\|v^{\bar{l}}\|^2 - \frac{h^4}{144}|v^{\bar{l}}|_1^2\Big) + \frac{\mu}{2}\Big[(|u^{k+1}|_1^2 + |u^k|_1^2) + \frac{h^2}{12}(\|v^{k+1}\|^2 + \|v^k\|^2) - \frac{h^4}{144}(|v^{k+1}|_1^2 + |v^k|_1^2)\Big].$$

Proof Taking the inner product of (3.17) with $u^{\frac{1}{2}}$ and applying Lemma 2.2, we have

$$(\delta_t u^{\frac{1}{2}}, u^{\frac{1}{2}}) - \mu(\delta_t v^{\frac{1}{2}}, u^{\frac{1}{2}}) - \frac{\kappa h^2}{6}(\Delta_x v^{\frac{1}{2}}, u^{\frac{1}{2}}) - \nu(v^{\frac{1}{2}}, u^{\frac{1}{2}}) = 0. \tag{4.3}$$

Averaging (3.19) with superscripts $k = 0$ and $k = 1$, it holds

$$v_i^{\frac{1}{2}} = \delta_x^2 u_i^{\frac{1}{2}} - \frac{h^2}{12}\delta_x^2 v_i^{\frac{1}{2}}, \quad 1 \le i \le M. \tag{4.4}$$

With the help of (4.4) and summation by parts, we have

$$\begin{aligned}(\delta_t v^{\frac{1}{2}}, u^{\frac{1}{2}}) &= \left(\delta_t\left(\delta_x^2 u^{\frac{1}{2}} - \frac{h^2}{12}\delta_x^2 v^{\frac{1}{2}}\right), u^{\frac{1}{2}}\right)\\ &= -(\delta_t(\delta_x^+ u^{\frac{1}{2}}), \delta_x^+ u^{\frac{1}{2}}) - \frac{h^2}{12}(\delta_t v^{\frac{1}{2}}, \delta_x^2 u^{\frac{1}{2}})\\ &= -\frac{1}{2\tau}(|u^1|_1^2 - |u^0|_1^2) - \frac{h^2}{12}\left(\delta_t v^{\frac{1}{2}}, v^{\frac{1}{2}} + \frac{h^2}{12}\delta_x^2 v^{\frac{1}{2}}\right)\\ &= -\frac{1}{2\tau}(|u^1|_1^2 - |u^0|_1^2) - \frac{h^2}{12}(\delta_t v^{\frac{1}{2}}, v^{\frac{1}{2}}) - \frac{h^4}{144}(\delta_t v^{\frac{1}{2}}, \delta_x^2 v^{\frac{1}{2}})\\ &= -\frac{1}{2\tau}\left[(|u^1|_1^2 - |u^0|_1^2) + \frac{h^2}{12}(\|v^1\|^2 - \|v^0\|^2) - \frac{h^4}{144}(|v^1|_1^2 - |v^0|_1^2)\right].\end{aligned} \tag{4.5}$$

Applying (2.4) and (2.6) in Lemma 2.4, we have

$$\begin{aligned}&(\varDelta_x v^{\frac{1}{2}}, u^{\frac{1}{2}}) = 0,\\ &(v^{\frac{1}{2}}, u^{\frac{1}{2}}) = -|u^{\frac{1}{2}}|_1^2 - \frac{h^2}{12}\|v^{\frac{1}{2}}\|^2 + \frac{h^4}{144}|v^{\frac{1}{2}}|_1^2.\end{aligned} \tag{4.6}$$

Substituting (4.5)–(4.6) into (4.3), we have

$$\begin{aligned}&\frac{1}{2\tau}(\|u^1\|^2 - \|u^0\|^2) + \frac{\mu}{2\tau}\left[(|u^1|_1^2 - |u^0|_1^2) + \frac{h^2}{12}(\|v^1\|^2 - \|v^0\|^2) - \frac{h^4}{144}(|v^1|_1^2 - |v^0|_1^2)\right]\\ &\quad + \nu\left(|u^{\frac{1}{2}}|_1^2 + \frac{h^2}{12}\|v^{\frac{1}{2}}\|^2 - \frac{h^4}{144}|v^{\frac{1}{2}}|_1^2\right) = 0.\end{aligned}$$

Rearranging the above formula, we have

$$\begin{aligned}&\frac{1}{2}(\|u^1\|^2 + \|u^0\|^2) + \frac{\mu}{2}\left[(|u^1|_1^2 + |u^0|_1^2) + \frac{h^2}{12}(\|v^1\|^2 + \|v^0\|^2) - \frac{h^4}{144}(|v^1|_1^2 + |v^0|_1^2)\right]\\ &\quad + \nu\tau\left(|u^{\frac{1}{2}}|_1^2 + \frac{h^2}{12}\|v^{\frac{1}{2}}\|^2 - \frac{h^4}{144}|v^{\frac{1}{2}}|_1^2\right) = \|u^0\|^2 + \mu|u^0|_1^2 + \frac{\mu h^2}{12}\|v^0\|^2 - \frac{\mu h^4}{144}|v^0|_1^2.\end{aligned} \tag{4.7}$$

Taking the inner product of (3.18) with $u^{\bar{k}}$ and applying Lemma 2.2, we have

$$(\varDelta_t u^k, u^{\bar{k}}) - \mu(\varDelta_t v^k, u^{\bar{k}}) - \frac{\kappa h^2}{6}(\varDelta_x v^{\bar{k}}, u^{\bar{k}}) - \nu(v^{\bar{k}}, u^{\bar{k}}) = 0, \quad 1 \le k \le N-1. \tag{4.8}$$

Averaging (3.19) with superscripts $k-1$ and $k+1$, it holds

$$v_i^{\bar{k}} = \delta_x^2 u_i^{\bar{k}} - \frac{h^2}{12}\delta_x^2 v_i^{\bar{k}}, \quad 1 \le i \le M, \ 1 \le k \le N-1. \tag{4.9}$$

With the help of (3.19), (4.9), summation by parts and similar to the derivation of (4.5), we have

$$(\Delta_t v^k, u^{\bar{k}}) = -\frac{1}{4\tau}\Big[(|u^{k+1}|_1^2 - |u^{k-1}|_1^2) + \frac{h^2}{12}(\|v^{k+1}\|^2 - \|v^{k-1}\|^2) - \frac{h^4}{144}(|v^{k+1}|_1^2 - |v^{k-1}|_1^2)\Big], \quad 1 \le k \le N-1. \tag{4.10}$$

Applying (2.4) and (2.6) in Lemma 2.4, we have

$$(\Delta_x v^{\bar{k}}, u^{\bar{k}}) = 0, \quad 1 \le k \le N-1,$$
$$(v^{\bar{k}}, u^{\bar{k}}) = -|u^{\bar{k}}|_1^2 - \frac{h^2}{12}\|v^{\bar{k}}\|^2 + \frac{h^4}{144}|v^{\bar{k}}|_1^2, \quad 1 \le k \le N-1. \tag{4.11}$$

Substituting (4.10)–(4.11) into (4.8), we have for $1 \le k \le N-1$

$$\frac{1}{4\tau}(\|u^{k+1}\|^2 - \|u^{k-1}\|^2) + \nu\Big[|u^{\bar{k}}|_1^2 + \frac{h^2}{12}\|v^{\bar{k}}\|^2 - \frac{h^4}{144}|v^{\bar{k}}|_1^2\Big]$$
$$+ \frac{\mu}{4\tau}\Big[(|u^{k+1}|_1^2 - |u^{k-1}|_1^2) + \frac{h^2}{12}(\|v^{k+1}\|^2 - \|v^{k-1}\|^2) - \frac{h^4}{144}(|v^{k+1}|_1^2 - |v^{k-1}|_1^2)\Big] = 0.$$

Consequently

$$E(u^{k+1}, u^k) = E(u^k, u^{k-1}), \quad 1 \le k \le N-1.$$

By the recursion, we have

$$E(u^{k+1}, u^k) = E(u^1, u^0), \quad 1 \le k \le N-1.$$

□

Remark 4.1 (4.1) and (4.2) can be rewritten as

$$\frac{1}{2}(\|u^{k+1}\|^2 + \|u^k\|^2)$$
$$+ \frac{\mu}{2}\Big[(|u^{k+1}|_1^2 + |u^k|_1^2) + \frac{h^2}{12}(\|v^{k+1}\|^2 + \|v^k\|^2) - \frac{h^4}{144}(|v^{k+1}|_1^2 + |v^k|_1^2)\Big]$$
$$+ \nu\tau\Big(|u^{\frac{1}{2}}|_1^2 + \frac{h^2}{12}\|v^{\frac{1}{2}}\|^2 - \frac{h^4}{144}|v^{\frac{1}{2}}|_1^2\Big) + 2\nu\tau\sum_{l=1}^{k}\Big(|u^{\bar{l}}|_1^2 + \frac{h^2}{12}\|v^{\bar{l}}\|^2 - \frac{h^4}{144}|v^{\bar{l}}|_1^2\Big)$$
$$= \|u^0\|^2 + \mu|u^0|_1^2 + \frac{\mu h^2}{12}\|v^0\|^2 - \frac{\mu h^4}{144}|v^0|_1^2, \quad 0 \le k \le N-1. \tag{4.12}$$

Remark 4.2 Combining (4.7) with (4.12), we have

$$\|u^k\| \le 2\Big(\|u^0\| + \mu|u^0|_1^2 + \frac{\mu h^2}{12}\|v^0\|^2 - \frac{\mu h^4}{144}|v^0|_1^2\Big), \quad 1 \le k \le N.$$

5 Uniqueness

Theorem 5.1 *The finite difference scheme* (3.17)–(3.21) *is uniquely solvable.*

Proof From (3.19)–(3.21), it is easy to know that u^0 and v^0 have been determined. From (3.17) and (3.19), a linear system of equations about u^1 and v^1 can be obtained with respect to the first level. Now we consider its homogenous linear system of equations

$$\begin{aligned} &\frac{1}{\tau}u_i^1 - \frac{\mu}{\tau}v_i^1 + \frac{\gamma}{2}\psi(u^0, u^1)_i - \frac{\gamma h^2}{4}\psi(v^0, u^1)_i \\ &\quad + \frac{\kappa}{2}\Delta_x u_i^1 - \frac{\kappa h^2}{12}\Delta_x v_i^1 - \frac{\nu}{2}v_i^1 = 0, \quad 1 \le i \le M, \end{aligned} \tag{5.1}$$

$$v_i^1 = \delta_x^2 u_i^1 - \frac{h^2}{12}\delta_x^2 v_i^1, \quad 1 \le i \le M. \tag{5.2}$$

Taking the inner product of (5.1) with u^1, and combining Lemma 2.2 with (5.2), we have

$$\frac{1}{\tau}\|u^1\|^2 - \frac{\mu}{\tau}(v^1, u^1) - \frac{\kappa h^2}{12}(\Delta_x v^1, u^1) - \frac{\nu}{2}(v^1, u^1) = 0. \tag{5.3}$$

Applying (2.5) in Lemma 2.4, we have

$$(v^1, u^1) \le -|u^1|_1^2 - \frac{h^2}{18}\|v^1\|^2, \tag{5.4}$$

$$(\Delta_x v^1, u^1) = 0. \tag{5.5}$$

Substituting (5.4)–(5.5) into (5.3) and a calculation shows that

$$\frac{1}{\tau}\|u^1\|^2 + \left(\frac{\mu}{\tau} + \frac{\nu}{2}\right)\cdot\left(|u^1|_1^2 + \frac{h^2}{18}\|v^1\|^2\right) \le 0.$$

Thus, it holds that

$$\|u^1\| = 0, \quad \|v^1\| = 0.$$

Therefore, (5.1) and (5.2) only allow zero solutions, which implies that (3.17) and (3.19) determine u^1, v^1 uniquely.

Now we suppose that u^{k-1}, u^k, v^{k-1}, v^k have been determined. From (3.18)–(3.19), a linear system of equations with respect to u^{k+1} and v^{k+1} is obtained. Now we consider the homogenous system of equations as follows

$$\begin{aligned} &\frac{1}{2\tau}u_i^{k+1} - \frac{\mu}{2\tau}v_i^{k+1} + \frac{\gamma}{2}\psi(u^k, u^{k+1})_i - \frac{\gamma h^2}{4}\psi(v^k, u^{k+1})_i + \frac{\kappa}{2}\Delta_x u_i^{k+1} \\ &\quad - \frac{\kappa h^2}{12}\Delta_x v_i^{k+1} - \frac{\nu}{2}v_i^{k+1} = 0, \quad 1 \le i \le M, \end{aligned} \tag{5.6}$$

$$v_i^{k+1} = \delta_x^2 u_i^{k+1} - \frac{h^2}{12}\delta_x^2 v_i^{k+1}, \quad 1 \le i \le M. \tag{5.7}$$

Taking the inner product of (5.6) with u^{k+1} and applying Lemma 2.2 and (5.7), we have

$$\frac{1}{2\tau}\|u^{k+1}\|^2 - \left(\frac{\mu}{2\tau} + \frac{\nu}{2}\right)(v^{k+1}, u^{k+1}) - \frac{\kappa h^2}{12}(\Delta_x v^{k+1}, u^{k+1}) = 0. \tag{5.8}$$

Combining (2.5) in Lemma 2.4 and noticing $S^k = 0$, we have

$$(v^{k+1}, u^{k+1}) \le -|u^{k+1}|_1^2 - \frac{h^2}{18}\|v^{k+1}\|^2, \tag{5.9}$$

$$(\Delta_x v^{k+1}, u^{k+1}) = 0. \tag{5.10}$$

Substituting (5.9)–(5.10) into (5.8), we have

$$\frac{1}{2\tau}\|u^{k+1}\|^2 + \Big(\frac{\mu}{2\tau} + \frac{\nu}{2}\Big)\cdot\Big(|u^{k+1}|_1^2 + \frac{h^2}{18}\|v^{k+1}\|^2\Big) \le 0.$$

Then it holds that

$$\|u^{k+1}\| = 0, \quad \|v^{k+1}\| = 0.$$

Therefore, (5.6) and (5.7) only allow zero solutions, which implies that (3.18)–(3.19) determine u^{k+1} and v^{k+1} uniquely. By the mathematical induction, this completes the proof. □

6 Convergence and Stability

6.1 Convergence

Let h_0 and τ_0 be two positive constants. Denote

$$\begin{aligned}
c_4 =& \Big[\frac{27}{4}\Big(\frac{\nu\tau_0}{2} - \mu\Big)^2 + \frac{3}{16}\kappa^2 h_0^2\tau_0^2 + \frac{3}{4}\gamma^2 c_0^2\tau_0^2 h_0^2(h_0+1)^2 + \frac{\mu h_0^2}{16} + \frac{\nu h_0^2\tau_0}{32}\\
&+ \frac{3}{2}\Big(\mu + \frac{\nu\tau_0}{2}\Big)^2 + \frac{1}{576\mu}\kappa^2 h_0^4\tau_0^2 + \frac{1}{24}\kappa^2 h_0^2\tau_0^2\Big]Lc_3^2 + \frac{3Lc_1^2}{2},\\
c_5 =& \frac{5}{8}\gamma^2(Lc_0 + \sqrt{L})^2 + \frac{5\gamma^2}{32}\Big(Lc_0 h_0^2\\
&+ 3\sqrt{L}\sqrt{8 + 2Lc_3^2 h_0^{10}}\Big)^2 + \frac{5\kappa^2}{2} + \frac{10\kappa^2}{9} + \frac{5}{18}\gamma^2(Lc_0 + c_0)^2,\\
c_6 =& 5\gamma^2(h_0c_0 + c_0)^2 + \frac{3\kappa^2 h_0^2}{16\mu} + \frac{\kappa^2 h_0^2}{48\mu} + \frac{\kappa^2 h_0^4}{72} + \frac{\nu^2 h_0^2}{8},\\
c_7 =& \Big(\frac{3\mu h_0^2}{32} + \frac{5\mu^2}{2} + \frac{1}{2} + \frac{5\kappa^2 h_0^2}{18} + \frac{5\nu^2}{2}\Big)Lc_3^2 + \frac{5Lc_2^2}{2},\\
c_8 =& \max\{c_5, c_6, c_7\}, \quad c_9 = \exp\Big(\frac{6Tc_8}{\nu}\Big)\cdot\Big(\frac{5c_4}{4\mu} + \frac{3}{8}h_0^2 Lc_3^2 + \frac{1}{2}\Big),\\
c_{10} =& \max\Big\{\sqrt{\frac{c_4}{\mu}}, \sqrt{2c_9}\Big\}.
\end{aligned}$$

We have the following convergence result.

Theorem 6.1 (Convergence) *Suppose* $\{U_i^k, V_i^k \,|\, 1 \le i \le M,\ 0 \le k \le N\}$ *is the solution of* (3.9), (3.11), (3.13), (3.15), (3.16) $\{u_i^k, v_i^k \,|\, 1 \le i \le M,\ 0 \le k \le N\}$ *is the solution of* (3.17)–(3.21). *Denote* $e_i^k = U_i^k - u_i^k,\ f_i^k = V_i^k - v_i^k,\ 1 \le i \le M,\ 0 \le k \le N$, *when* $h \le h_0$, $\tau \le \tau_0$ *and* $\tau^2 + h^4 \le 1/c_{10}$, *we have the error estimate*

$$|e^k|_1 \le c_{10}(\tau^2 + h^4), \quad 0 \le k \le N.$$

Proof Subtracting (3.9), (3.11), (3.13), (3.15), (3.16) from (3.17)–(3.21), the error system is written as

$$\delta_t e_i^{\frac{1}{2}} - \mu\delta_t f_i^{\frac{1}{2}} + \gamma\psi(u^0, e^{\frac{1}{2}})_i - \frac{\gamma h^2}{2}[\psi(V^0, U^{\frac{1}{2}})_i - \psi(v^0, u^{\frac{1}{2}})_i] + \kappa\Delta_x e_i^{\frac{1}{2}} - \frac{\kappa h^2}{6}\Delta_x f_i^{\frac{1}{2}} - \nu f_i^{\frac{1}{2}} = Q_i^0, \quad 1 \le i \le M, \tag{6.1}$$

$$\Delta_t e_i^k - \mu\Delta_t f_i^k + \gamma[\psi(U^k, U^{\bar{k}})_i - \psi(u^k, u^{\bar{k}})_i] - \frac{\gamma h^2}{2}[\psi(V^k, U^{\bar{k}})_i - \psi(v^k, u^{\bar{k}})_i] + \kappa\Delta_x e_i^{\bar{k}} - \frac{\kappa h^2}{6}\Delta_x f_i^{\bar{k}} - \nu f_i^{\bar{k}} = Q_i^k, \quad 1 \le i \le M,\ 1 \le k \le N-1, \tag{6.2}$$

$$f_i^k = \delta_x^2 e_i^k - \frac{h^2}{12}\delta_x^2 f_i^k + R_i^k, \quad 1 \le i \le M,\ 0 \le k \le N, \tag{6.3}$$

$$e_i^0 = 0, \quad 1 \le i \le M, \tag{6.4}$$

$$e_i^k = e_{i+M}^k, \quad f_i^k = f_{i+M}^k, \quad 1 \le i \le M,\ 0 \le k \le N. \tag{6.5}$$

Denote

$$F^k = \frac{1}{2}\Big[(|e^k|_1^2 + |e^{k-1}|_1^2) + \frac{h^2}{12}(\|f^k\|^2 + \|f^{k-1}\|^2) - \frac{h^4}{144}(|f^k|_1^2 + |f^{k-1}|_1^2)\Big], \quad 1 \le k \le N. \tag{6.6}$$

From (3.1), we have

$$|U^k|_1 \le \sqrt{L}c_0, \quad \|U^k\|_\infty \le c_0, \quad 0 \le k \le N, \tag{6.7}$$

$$\|V^k\| \le \sqrt{L}c_0, \quad \|V^k\|_\infty \le c_0, \quad |V^k|_1 \le \sqrt{L}c_0, \quad 0 \le k \le N. \tag{6.8}$$

Taking the inner product of (6.3) with f^k, we have

$$\begin{aligned}
\|f^k\|^2 =\ & (\delta_x^2 e^k, f^k) - \frac{h^2}{12}(\delta_x^2 f^k, f^k) + (R^k, f^k) \\
\le\ & \|\delta_x^2 e^k\| \cdot \|f^k\| + \frac{h^2}{12}|f^k|_1^2 + \|R^k\| \cdot \|f^k\| \\
\le\ & \frac{1}{6}\|f^k\|^2 + \frac{3}{2}\|\delta_x^2 e^k\|^2 + \frac{1}{3}\|f^k\|^2 + \frac{1}{6}\|f^k\|^2 + \frac{3}{2}\|R^k\|^2 \\
\le\ & \frac{2}{3}\|f^k\|^2 + \frac{6}{h^2}|e^k|_1^2 + \frac{3}{2}\|R^k\|^2, \quad 0 \le k \le N.
\end{aligned}$$

Thus, we have

$$\|f^k\|^2 \le \frac{18}{h^2}|e^k|_1^2 + \frac{9}{2}\|R^k\|^2, \quad 0 \le k \le N. \tag{6.9}$$

Then the proof is divided into three main steps.

Step 1: *Establish the error estimate in the first level.* Taking the inner product of (6.1) with $\delta_t e^{\frac{1}{2}}$, we have

$$\|\delta_t e^{\frac{1}{2}}\|^2 - \mu(\delta_t f^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) + \gamma(\psi(u^0, e^{\frac{1}{2}}), \delta_t e^{\frac{1}{2}}) - \frac{\gamma h^2}{2}(\psi(V^0, U^{\frac{1}{2}}) - \psi(v^0, u^{\frac{1}{2}}), \delta_t e^{\frac{1}{2}}) + \kappa(\Delta_x e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) - \frac{\kappa h^2}{6}(\Delta_x f^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) - \nu(f^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) = (Q^0, \delta_t e^{\frac{1}{2}}). \tag{6.10}$$

From (6.4), we have

$$\|\delta_t e^{\frac{1}{2}}\|^2 = \frac{1}{\tau^2}\|e^1\|^2, \tag{6.11}$$

$$(Q^0, \delta_t e^{\frac{1}{2}}) = \frac{1}{\tau}(Q^0, e^1). \tag{6.12}$$

Applying (2.4) in Lemma 2.4, we have

$$\begin{aligned}(\delta_t f^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) &= \frac{1}{\tau^2}(f^1, e^1) - \frac{1}{\tau^2}(f^0, e^1)\\ &= \frac{1}{\tau^2}\Big[-|e^1|_1^2 - \frac{h^2}{12}\|f^1\|^2 + \frac{h^4}{144}|f^1|_1^2\\ &\quad + \frac{h^2}{12}(f^1, R^1) + (R^1, e^1)\Big] - \frac{1}{\tau^2}(f^0, e^1).\end{aligned} \tag{6.13}$$

According to the definition of $\psi(u, v)_i$ and applying Lemma 2.2, we have

$$(\psi(u^0, e^{\frac{1}{2}}), \delta_t e^{\frac{1}{2}}) = \frac{1}{2\tau}(\psi(u^0, e^1), e^1) = 0. \tag{6.14}$$

Moreover, combining (6.7) and Lemma 2.5, we have

$$\begin{aligned}&(\psi(V^0, U^{\frac{1}{2}}) - \psi(v^0, u^{\frac{1}{2}}), \delta_t e^{\frac{1}{2}})\\ &= (\psi(V^0, e^{\frac{1}{2}}) + \psi(f^0, U^{\frac{1}{2}}) - \psi(f^0, e^{\frac{1}{2}}), \delta_t e^{\frac{1}{2}})\\ &= \frac{1}{2\tau}(\psi(V^0, e^1), e^1) + \frac{1}{\tau}(\psi(f^0, U^{\frac{1}{2}}), e^1) - \frac{1}{2\tau}(\psi(f^0, e^1), e^1)\\ &= \frac{1}{\tau}(\psi(f^0, U^{\frac{1}{2}}), e^1) = \frac{h}{3\tau}\sum_{i=1}^{M}\Big[f_i^0 \Delta_x U_i^{\frac{1}{2}} + \Delta_x(f^0 U^{\frac{1}{2}})_i\Big]\cdot e_i^1\\ &= \frac{h}{3\tau}\sum_{i=1}^{M}\Big(2f_i^0\cdot\Delta_x U_i^{\frac{1}{2}} + \frac{1}{2}U_{i-1}^{\frac{1}{2}}\delta_x^+ f_{i-1}^0 + \frac{1}{2}U_{i+1}^{\frac{1}{2}}\delta_x^+ f_i^0\Big)\cdot e_i^1\\ &\le \frac{c_0}{3\tau}\Big(2 + \frac{2}{h}\Big)\|f^0\|\cdot\|e^1\|.\end{aligned} \tag{6.15}$$

Noticing (6.4) and applying Lemma 2.2, we have

$$(\Delta_x e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) = \frac{1}{2\tau}(\Delta_x e^1, e^1) = 0. \tag{6.16}$$

Applying (2.4) and (2.6) in Lemma 2.4, we have

$$\begin{aligned}(\Delta_x f^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) &= \frac{1}{2\tau}(\Delta_x f^1, e^1) + \frac{1}{2\tau}(\Delta_x f^0, e^1)\\ &= \frac{1}{2\tau}\Big[\frac{h^2}{12}(\Delta_x f^1, R^1) + (\Delta_x R^1, e^1)\Big] + \frac{1}{2\tau}(\Delta_x f^0, e^1)\end{aligned} \tag{6.17}$$

and

$$\begin{aligned}&(f^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) = \frac{1}{2\tau}(f^1, e^1) + \frac{1}{2\tau}(f^0, e^1)\\ &= \frac{1}{2\tau}\Big[-|e^1|_1^2 - \frac{h^2}{12}\|f^1\|^2 + \frac{h^4}{144}|f^1|_1^2 + \frac{h^2}{12}(f^1, R^1) + (R^1, e^1)\Big] + \frac{1}{2\tau}(f^0, e^1).\end{aligned} \tag{6.18}$$

Substituting (6.11)–(6.18) into (6.10), we have

$$
\begin{aligned}
\|e^1\|^2 \leq\ & \mu\Big[-|e^1|_1^2 - \frac{h^2}{12}\|f^1\|^2 + \frac{h^4}{144}|f^1|_1^2 \\
& + \frac{h^2}{12}(f^1, R^1) + (R^1, e^1) - (f^0, e^1)\Big] + \tau(Q^0, e^1) \\
& + \frac{\gamma h^2}{2}\cdot\frac{c_0\tau}{3}\Big(2+\frac{2}{h}\Big)\|f^0\|\cdot\|e^1\| + \frac{\kappa h^2\tau}{6}\Big[\frac{h^2}{24}(\Delta_x f^1, R^1) \\
& + \frac{1}{2}(\Delta_x R^1, e^1) + \frac{1}{2}(\Delta_x f^0, e^1)\Big] \\
& + \frac{\nu\tau}{2}\Big[-|e^1|_1^2 - \frac{h^2}{12}\|f^1\|^2 + \frac{h^4}{144}|f^1|_1^2 + \frac{h^2}{12}(f^1, R^1) + (R^1, e^1) + (f^0, e^1)\Big] \\
=\ & -\Big(\mu+\frac{\nu\tau}{2}\Big)|e^1|_1^2 - \Big(\frac{\mu h^2}{12}+\frac{\nu h^2\tau}{24}\Big)\|f^1\|^2 \\
& + \Big(\frac{\mu h^4}{144}+\frac{\nu h^4\tau}{288}\Big)|f^1|_1^2 + \tau(Q^0, e^1) \\
& + \Big(\frac{\mu h^2}{12}+\frac{\nu h^2\tau}{24}\Big)\cdot(f^1, R^1) + \Big(\mu+\frac{\nu\tau}{2}\Big)\cdot(R^1, e^1) + \Big(\frac{\nu\tau}{2}-\mu\Big)\cdot(f^0, e^1) \\
& + \frac{\kappa h^4\tau}{144}\cdot(\Delta_x f^1, R^1) \\
& + \frac{\kappa h^2\tau}{12}\cdot(\Delta_x R^1, e^1) + \frac{\kappa h^2\tau}{12}\cdot(\Delta_x f^0, e^1) + \frac{\gamma c_0\tau h(h+1)}{3}\cdot\|f^0\|\cdot\|e^1\| \\
\leq\ & -\Big(\mu+\frac{\nu\tau}{2}\Big)|e^1|_1^2 - \Big(\frac{\mu h^2}{12}+\frac{\nu h^2\tau}{24}\Big)\|f^1\|^2 + \Big(\frac{\mu h^2}{36}+\frac{\nu h^2\tau}{72}\Big)\|f^1\|^2 + \frac{1}{6}\|e^1\|^2 \\
& + \frac{3\tau^2}{2}\|Q^0\|^2 + \Big(\frac{\mu h^2}{36}+\frac{\nu h^2\tau}{72}\Big)\|f^1\|^2 + \frac{3}{4}\Big(\frac{\mu h^2}{12}+\frac{\nu h^2\tau}{24}\Big)\|R^1\|^2 + \frac{1}{6}\|e^1\|^2 \\
& + \frac{3}{2}\Big(\mu+\frac{\nu\tau}{2}\Big)^2\|R^1\|^2 + \frac{1}{6}\|e^1\|^2 \\
& + \frac{3}{2}\Big(\frac{\nu\tau}{2}-\mu\Big)^2\|f^0\|^2 + \frac{h^2}{4}\Big(\frac{\mu h^2}{36}+\frac{\nu h^2\tau}{72}\Big)|f^1|_1^2 \\
& + \Big(\frac{\kappa h^4\tau}{144}\Big)^2\cdot\frac{1}{h^2\Big(\frac{\mu h^2}{36}+\frac{\nu h^2\tau}{72}\Big)}\|R^1\|^2 + \frac{1}{6}\|e^1\|^2 + \frac{\kappa^2 h^4\tau^2}{96}\|\Delta_x R^1\|^2 + \frac{1}{6}\|e^1\|^2 \\
& + \frac{\kappa^2 h^4\tau^2}{96}\|\Delta_x f^0\|^2 + \frac{1}{6}\|e^1\|^2 + \frac{\gamma^2 c_0^2\tau^2 h^2(h+1)^2}{6}\|f^0\|^2 \\
\leq\ & -\Big(\mu+\frac{\nu\tau}{2}\Big)|e^1|_1^2 + \|e^1\|^2 + \Big[\frac{3}{2}\Big(\frac{\nu\tau}{2}-\mu\Big)^2 + \frac{\kappa^2 h^2\tau^2}{24} + \frac{\gamma^2 c_0^2\tau^2 h^2(h+1)^2}{6}\Big]\|f^0\|^2 \\
& + \frac{3\tau^2}{2}\|Q^0\|^2 + \Big[\frac{3}{4}\Big(\frac{\mu h^2}{12}+\frac{\nu h^2\tau}{24}\Big) + \frac{3}{2}\Big(\mu+\frac{\nu\tau}{2}\Big)^2 \\
& + \frac{\kappa^2 h^4\tau^2}{288(2\mu+\nu\tau)} + \frac{\kappa^2 h^2\tau^2}{24}\Big]\|R^1\|^2.
\end{aligned}
$$

When $h \le h_0$, $\tau \le \tau_0$, we have

$$\begin{aligned}\|e^1\|^2 \le & -\Big(\mu + \frac{\nu\tau}{2}\Big)|e^1|_1^2 + \|e^1\|^2 \\ & + \Big[\frac{3}{2}\Big(\frac{\nu\tau_0}{2} - \mu\Big)^2 + \frac{\kappa^2 h_0^2 \tau_0^2}{24} + \frac{\gamma^2 c_0^2 \tau_0^2 h_0^2 (h_0+1)^2}{6}\Big]\|f^0\|^2 \\ & + \frac{3\tau^2}{2}\|Q^0\|^2 + \Big[\frac{3}{4}\Big(\frac{\mu h_0^2}{12} + \frac{\nu h_0^2 \tau_0}{24}\Big) + \frac{3}{2}\Big(\mu + \frac{\nu\tau_0}{2}\Big)^2 \\ & + \frac{\kappa^2 h_0^4 \tau_0^2}{576\mu} + \frac{\kappa^2 h_0^2 \tau_0^2}{24}\Big]\|R^1\|^2. \end{aligned} \tag{6.19}$$

Taking $k = 0$ in (6.9), substituting the result into (6.19) and using (3.10), (3.14), we have

$$\begin{aligned}\Big(\mu + \frac{\nu\tau}{2}\Big)|e^1|_1^2 \le & \frac{9}{2}\Big[\frac{3}{2}\Big(\frac{\nu\tau_0}{2} - \mu\Big)^2 \\ & + \frac{\kappa^2 h_0^2 \tau_0^2}{24} + \frac{\gamma^2 c_0^2 \tau_0^2 h_0^2 (h_0+1)^2}{6}\Big]\|R^0\|^2 + \frac{3\tau^2}{2}\|Q^0\|^2 \\ & + \Big[\frac{3}{4}\Big(\frac{\mu h_0^2}{12} + \frac{\nu h_0^2 \tau_0}{24}\Big) + \frac{3}{2}\Big(\mu + \frac{\nu\tau_0}{2}\Big)^2 \\ & + \frac{\kappa^2 h_0^4 \tau_0^2}{576\mu} + \frac{\kappa^2 h_0^2 \tau_0^2}{24}\Big]\|R^1\|^2 \\ \le & \; c_4(\tau^2 + h^4)^2. \end{aligned}$$

Rearranging the above term, we have

$$|e^1|_1^2 \le \frac{c_4}{\mu + \frac{\nu\tau}{2}}(\tau^2 + h^4)^2 \le \frac{c_4}{\mu}(\tau^2 + h^4)^2. \tag{6.20}$$

Consequently

$$|e^1|_1 \le c_{10}(\tau^2 + h^4). \tag{6.21}$$

From (6.6), (6.9) and (6.20), we have

$$\begin{aligned} F^1 = & \frac{1}{2}\Big[|e^1|_1^2 + \frac{h^2}{12}(\|f^1\|^2 + \|f^0\|^2) - \frac{h^4}{144}(|f^1|_1^2 + |f^0|_1^2)\Big] \\ \le & \frac{1}{2}|e^1|_1^2 + \frac{h^2}{24}(\|f^1\|^2 + \|f^0\|^2) \\ \le & \frac{1}{2}|e^1|_1^2 + \frac{h^2}{24}\Big(\frac{18}{h^2}|e^1|_1^2 + \frac{9}{2}\|R^1\|^2 + \frac{9}{2}\|R^0\|^2\Big) \\ \le & \Big(\frac{5c_4}{4\mu} + \frac{3h^2 L c_3^2}{8}\Big)(\tau^2 + h^4)^2. \end{aligned} \tag{6.22}$$

Step 2: *Establish the error estimate for* $2 \le k \le N$ *by induction method.* Taking the inner product of (6.2) with $\Delta_t e^k$, we have

$$\begin{aligned} & \|\Delta_t e^k\|^2 - \mu(\Delta_t f^k, \Delta_t e^k) + \gamma(\psi(U^k, U^{\bar{k}}) - \psi(u^k, u^{\bar{k}}), \Delta_t e^k) - \frac{\gamma h^2}{2}(\psi(V^k, U^{\bar{k}}) \\ & - \psi(v^k, u^{\bar{k}}), \Delta_t e^k) + \kappa(\Delta_x e^{\bar{k}}, \Delta_t e^k) - \frac{\kappa h^2}{6}(\Delta_x f^{\bar{k}}, \Delta_t e^k) \\ & - \nu(f^{\bar{k}}, \Delta_t e^k) = (Q^k, \Delta_t e^k), \quad 1 \le k \le N-1. \end{aligned} \tag{6.23}$$

Now we suppose that $|e^k|_1 \le c_{10}(\tau^2 + h^4)$ holds for $k = 1, 2, \cdots, l$ with $1 \le l \le N-1$. When $(\tau^2 + h^4) \le 1/c_{10}$, using (6.7)–(6.9), we have

$$\|f^k\| \le 3\sqrt{\frac{2}{h^2} + \frac{Lc_3^2 h^8}{2}}, \quad 1 \le k \le l, \tag{6.24}$$

$$|u^k|_1 \le |U^k|_1 + |e^k|_1 \le \sqrt{L}c_0 + 1, \quad 1 \le k \le l, \tag{6.25}$$

$$\|u^k\|_\infty \le \frac{\sqrt{L}}{2}|u^k|_1 \le \frac{\sqrt{L}}{2}\left(\sqrt{L}c_0 + 1\right), \quad 1 \le k \le l, \tag{6.26}$$

$$|v^k|_1 \le |V^k|_1 + |f^k|_1 \le \sqrt{L}c_0 + \frac{2}{h}\|f^k\| \le \sqrt{L}c_0 + 3\sqrt{\frac{8}{h^4} + 2Lc_3^2 h^6}, \quad 1 \le k \le l, \tag{6.27}$$

$$\|v^k\|_\infty \le \frac{\sqrt{L}}{2}|v^k|_1 \le \frac{Lc_0}{2} + \frac{3\sqrt{L}}{2}\sqrt{\frac{8}{h^4} + 2Lc_3^2 h^6}, \quad 1 \le k \le l. \tag{6.28}$$

Using (6.3) and applying (2.4) in Lemma 2.4, we have $(1 \le k \le l)$

$$\begin{aligned}
&(\Delta_t f^k, \Delta_t e^k) \\
&= -|\Delta_t e^k|_1^2 - \frac{h^2}{12}\|\Delta_t f^k\|^2 + \frac{h^4}{144}|\Delta_t f^k|_1^2 + \frac{h^2}{12}(\Delta_t f^k, \Delta_t R^k) + (\Delta_t R^k, \Delta_t e^k) \\
&\le -|\Delta_t e^k|_1^2 - \frac{h^2}{12}\|\Delta_t f^k\|^2 + \frac{h^4}{144}|\Delta_t f^k|_1^2 + \frac{h^2}{12}\|\Delta_t f^k\| \cdot \|\Delta_t R^k\| + \|\Delta_t R^k\| \cdot \|\Delta_t e^k\|.
\end{aligned} \tag{6.29}$$

Noticing that

$$\begin{aligned}
&\psi(U^k, U^{\bar{k}})_i - \psi(u^k, u^{\bar{k}})_i \\
&= \psi(U^k, U^{\bar{k}})_i - \psi(U^k - e^k, U^{\bar{k}} - e^{\bar{k}})_i = \psi(u^k, e^{\bar{k}})_i + \psi(e^k, U^{\bar{k}})_i \\
&= \frac{1}{3}\left[u_i^k \Delta_x e_i^{\bar{k}} + \Delta_x(u^k e^{\bar{k}})_i\right] + \frac{1}{3}\left[e_i^k \Delta_x U_i^{\bar{k}} + \Delta_x(e^k U^{\bar{k}})_i\right],
\end{aligned}$$

and applying Lemma 2.5, we have

$$\begin{aligned}
&\psi(U^k, U^{\bar{k}})_i - \psi(u^k, u^{\bar{k}})_i \\
&= \frac{1}{3}\Big[u_i^k \Delta_x e_i^{\bar{k}} + \frac{1}{2}\left(\delta_x^+ e_i^{\bar{k}}\right)u_{i+1}^k + \frac{1}{2}\left(\delta_x^+ e_{i-1}^{\bar{k}}\right)u_{i-1}^k + e_i^{\bar{k}} \Delta_x u_i^k\Big] \\
&\quad + \frac{1}{3}\Big[e_i^k \Delta_x U_i^{\bar{k}} + \frac{1}{2}\left(\delta_x^+ e_i^k\right)U_{i+1}^{\bar{k}} + \frac{1}{2}\left(\delta_x^+ e_{i-1}^k\right)U_{i-1}^{\bar{k}} + e_i^k \Delta_x U_i^{\bar{k}}\Big].
\end{aligned} \tag{6.30}$$

Combining (6.7), (6.25), (6.26) with (6.30), we have

$$
\begin{aligned}
&-\left(\psi(U^k, U^{\bar{k}}) - \psi(u^k, u^{\bar{k}}), \Delta_t e^k\right)\\
&= -\frac{h}{3}\sum_{i=1}^{M}\Big[u_i^k \Delta_x e_i^{\bar{k}} + \frac{1}{2}\Big(\delta_x^+ e_i^{\bar{k}}\Big)u_{i+1}^k + \frac{1}{2}\Big(\delta_x^+ e_{i-1}^{\bar{k}}\Big)u_{i-1}^k + e_i^{\bar{k}}\Delta_x u_i^k\Big]\cdot \Delta_t e_i^k\\
&\quad - \frac{h}{3}\sum_{i=1}^{M}\Big[e_i^k \Delta_x U_i^{\bar{k}} + \frac{1}{2}\Big(\delta_x^+ e_i^k\Big)U_{i+1}^{\bar{k}} + \frac{1}{2}\Big(\delta_x^+ e_{i-1}^k\Big)U_{i-1}^{\bar{k}} + e_i^k\Delta_x U_i^{\bar{k}}\Big]\cdot \Delta_t e_i^k\\
&\le \frac{1}{3}\Big(\|u^k\|_\infty \cdot |e^{\bar{k}}|_1 + \frac{1}{2}|e^{\bar{k}}|_1 \cdot \|u^k\|_\infty + \frac{1}{2}|e^{\bar{k}}|_1 \cdot \|u^k\|_\infty + \|e^{\bar{k}}\|_\infty \cdot |u^k|_1\Big)\cdot \|\Delta_t e^k\|\\
&\quad + \frac{1}{3}\Big(\|e^k\|_\infty \cdot |U^{\bar{k}}|_1 + \frac{1}{2}|e^k|_1 \cdot \|U^{\bar{k}}\|_\infty\\
&\quad + \frac{1}{2}|e^k|_1 \cdot \|U^{\bar{k}}\|_\infty + \|e^k\|_\infty \cdot |U^{\bar{k}}|_1\Big)\cdot \|\Delta_t e^k\|\\
&\le \frac{1}{3}\Big[\frac{\sqrt{L}}{2}\Big(\sqrt{L}c_0 + 1\Big)\cdot |e^{\bar{k}}|_1\\
&\quad + \frac{\sqrt{L}}{2}\Big(\sqrt{L}c_0 + 1\Big)\cdot |e^{\bar{k}}|_1 + \Big(\sqrt{L}c_0 + 1\Big)\cdot \frac{\sqrt{L}}{2}|e^{\bar{k}}|_1\Big]\cdot \|\Delta_t e^k\|\\
&\quad + \frac{1}{3}\Big[\sqrt{L}c_0 \cdot \frac{\sqrt{L}}{2}|e^k|_1 + c_0 \cdot |e^k|_1 + \sqrt{L}c_0 \cdot \frac{\sqrt{L}}{2}|e^k|_1\Big]\cdot \|\Delta_t e^k\|\\
&= \frac{Lc_0 + \sqrt{L}}{2}\cdot |e^{\bar{k}}|_1 \cdot \|\Delta_t e^k\| + \frac{Lc_0 + c_0}{3}\cdot |e^k|_1 \cdot \|\Delta_t e^k\|, \quad 1 \le k \le l.
\end{aligned}
\tag{6.31}
$$

Similarly, it is concluded that

$$
\begin{aligned}
&\psi(V^k, U^{\bar{k}})_i - \psi(v^k, u^{\bar{k}})_i = \psi(v^k, e^{\bar{k}})_i + \psi(f^k, U^{\bar{k}})_i\\
&\quad = \frac{1}{3}\Big[v_i^k \Delta_x e_i^{\bar{k}} + \frac{1}{2}\Big(\delta_x^+ e_i^{\bar{k}}\Big)v_{i+1}^k + \frac{1}{2}\Big(\delta_x^+ e_{i-1}^{\bar{k}}\Big)v_{i-1}^k + e_i^{\bar{k}}\Delta_x v_i^k\Big]\\
&\qquad + \frac{1}{3}\Big[f_i^k \Delta_x U_i^{\bar{k}} + \frac{1}{2}\Big(\delta_x^+ f_i^k\Big)U_{i+1}^{\bar{k}} + \frac{1}{2}\Big(\delta_x^+ f_{i-1}^k\Big)U_{i-1}^{\bar{k}} + f_i^k\Delta_x U_i^{\bar{k}}\Big].
\end{aligned}
\tag{6.32}
$$

Combining (6.7), (6.27) and (6.28), we have for $1 \le k \le l$

$$
\begin{aligned}
&\left(\psi(V^k, U^{\bar{k}}) - \psi(v^k, u^{\bar{k}}), \Delta_t e^k\right)\\
&= \frac{h}{3}\sum_{i=1}^{M-1}\Big[v_i^k \Delta_x e_i^{\bar{k}} + \frac{1}{2}\Big(\delta_x^+ e_i^{\bar{k}}\Big)v_{i+1}^k + \frac{1}{2}\Big(\delta_x^+ e_{i-1}^{\bar{k}}\Big)v_{i-1}^k + e_i^{\bar{k}}\Delta_x v_i^k\Big]\cdot \Delta_t e_i^k\\
&\quad + \frac{h}{3}\sum_{i=1}^{M-1}\Big[f_i^k \Delta_x U_i^{\bar{k}} + \frac{1}{2}\Big(\delta_x^+ f_i^k\Big)U_{i+1}^{\bar{k}} + \frac{1}{2}\Big(\delta_x^+ f_{i-1}^k\Big)U_{i-1}^{\bar{k}} + f_i^k\Delta_x U_i^{\bar{k}}\Big]\cdot \Delta_t e_i^k\\
&\le \frac{1}{3}\Big(\|v^k\|_\infty \cdot |e^{\bar{k}}|_1 + \frac{1}{2}|e^{\bar{k}}|_1 \cdot \|v^k\|_\infty + \frac{1}{2}|e^{\bar{k}}|_1 \cdot \|v^k\|_\infty + \|e^{\bar{k}}\|_\infty \cdot |v^k|_1\Big)\cdot \|\Delta_t e^k\|\\
&\quad + \frac{1}{3}\Big(\|f^k\| \cdot \|\Delta_x U^{\bar{k}}\|_\infty + \frac{1}{2}|f^k|_1 \cdot \|U^{\bar{k}}\|_\infty\\
&\quad + \frac{1}{2}|f^k|_1 \cdot \|U^{\bar{k}}\|_\infty + \|f^k\| \cdot \|\Delta_x U^{\bar{k}}\|_\infty\Big)\cdot \|\Delta_t e^k\|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{3}\Big[2\Big(\frac{Lc_0}{2}+\frac{3\sqrt{L}}{2}\sqrt{\frac{8}{h^4}+2Lc_3^2h^6}\Big)\\
&\quad+\frac{\sqrt{L}}{2}\Big(\sqrt{L}c_0+3\sqrt{\frac{8}{h^4}+2Lc_3^2h^6}\Big)\Big]\cdot|e^{\bar{k}}|_1\cdot\|\Delta_t e^k\|\\
&\quad+\frac{1}{3}\Big(2c_0+\frac{2}{h}\cdot c_0\Big)\cdot\|f^k\|\cdot\|\Delta_t e^k\|\\
&=\Big(\frac{Lc_0}{2}+\frac{3\sqrt{L}}{2}\sqrt{\frac{8}{h^4}+2Lc_3^2h^6}\Big)\cdot|e^{\bar{k}}|_1\cdot\|\Delta_t e^k\|+\frac{2}{3}\Big(c_0+\frac{c_0}{h}\Big)\cdot\|f^k\|\cdot\|\Delta_t e^k\|.
\end{aligned}
\tag{6.33}
$$

In addition, applying Cauchy-Schwarz inequality, we have

$$
-(\Delta_x e^{\bar{k}},\Delta_t e^k)\le\|\Delta_x e^{\bar{k}}\|\cdot\|\Delta_t e^k\|,\quad 1\le k\le l.
$$

Moreover, it holds for $1\le k\le l$

$$
\begin{aligned}
(\Delta_x f^{\bar{k}},\Delta_t e^k)&=\Big(\Delta_x\Big(\delta_x^2 e^{\bar{k}}-\frac{h^2}{12}\delta_x^2 f^{\bar{k}}+R^{\bar{k}}\Big),\Delta_t e^k\Big)\\
&=(\Delta_x(\delta_x^2 e^{\bar{k}}),\Delta_t e^k)-\frac{h^2}{12}(\Delta_x(\delta_x^2 f^{\bar{k}}),\Delta_t e^k)+(\Delta_x R^{\bar{k}},\Delta_t e^k)\\
&=-(\Delta_x(\delta_x^+ e^{\bar{k}}),\Delta_t(\delta_x^+ e^k))-\frac{h^2}{12}(\Delta_x f^{\bar{k}},\Delta_t(\delta_x^2 e^k))+(\Delta_x R^{\bar{k}},\Delta_t e^k)\\
&\le|\Delta_x e^{\bar{k}}|_1\cdot|\Delta_t e^k|_1-\frac{h^2}{12}\Big(\Delta_x f^{\bar{k}},\Delta_t\Big(f^k+\frac{h^2}{12}\delta_x^2 f^k-R^k\Big)\Big)+|R^{\bar{k}}|_1\cdot\|\Delta_t e^k\|\\
&=|\Delta_x e^{\bar{k}}|_1\cdot|\Delta_t e^k|_1-\frac{h^2}{12}(\Delta_x f^{\bar{k}},\Delta_t f^k)+\frac{h^4}{144}(\delta_x^+(\Delta_x f^{\bar{k}}),\delta_x^+(\Delta_t f^k))\\
&\quad+\frac{h^2}{12}(\Delta_x f^{\bar{k}},\Delta_t R^k)+|R^{\bar{k}}|_1\cdot\|\Delta_t e^k\|\\
&\le|\Delta_x e^{\bar{k}}|_1\cdot|\Delta_t e^k|_1+\frac{h^2}{12}|f^{\bar{k}}|_1\cdot\|\Delta_t f^k\|+\frac{h^4}{144}|\Delta_x f^{\bar{k}}|_1\cdot|\Delta_t f^k|_1\\
&\quad+\frac{h^2}{12}|f^{\bar{k}}|_1\cdot\|\Delta_t R^k\|+|R^{\bar{k}}|_1\cdot\|\Delta_t e^k\|
\end{aligned}
\tag{6.34}
$$

and

$$
\begin{aligned}
(f^{\bar{k}},\Delta_t e^k)&=\Big(\delta_x^2 e^{\bar{k}}-\frac{h^2}{12}\delta_x^2 f^{\bar{k}}+R^{\bar{k}},\Delta_t e^k\Big)\\
&=(\delta_x^2 e^{\bar{k}},\Delta_t e^k)-\frac{h^2}{12}(\delta_x^2 f^{\bar{k}},\Delta_t e^k)+(R^{\bar{k}},\Delta_t e^k)\\
&=-(\delta_x^+ e^{\bar{k}},\Delta_t(\delta_x^+ e^k))-\frac{h^2}{12}(f^{\bar{k}},\Delta_t(\delta_x^2 e^k))+(R^{\bar{k}},\Delta_t e^k)\\
&=-\frac{1}{4\tau}(|e^{k+1}|_1^2-|e^{k-1}|_1^2)-\frac{h^2}{12}\Big(f^{\bar{k}},\Delta_t\Big(f^k+\frac{h^2}{12}\delta_x^2 f^k-R^k\Big)\Big)+(R^{\bar{k}},\Delta_t e^k)\\
&=-\frac{1}{4\tau}(|e^{k+1}|_1^2-|e^{k-1}|_1^2)-\frac{h^2}{12}(f^{\bar{k}},\Delta_t f^k)-\frac{h^4}{144}(f^{\bar{k}},\Delta_t(\delta_x^2 f^k))\\
&\quad+\frac{h^2}{12}(f^{\bar{k}},\Delta_t R^k)+(R^{\bar{k}},\Delta_t e^k)\\
&\le-\frac{1}{4\tau}(|e^{k+1}|_1^2-|e^{k-1}|_1^2)-\frac{h^2}{12}\cdot\frac{1}{4\tau}(\|f^{k+1}\|^2-\|f^{k-1}\|^2)
\end{aligned}
$$

$$+\frac{h^4}{144}\cdot\frac{1}{4\tau}(|f^{k+1}|_1^2-|f^{k-1}|_1^2)+\frac{h^2}{12}\|f^{\bar{k}}\|\cdot\|\Delta_t R^k\|+\|R^{\bar{k}}\|\cdot\|\Delta_t e^k\|. \qquad (6.35)$$

Substituting (6.29), (6.31)–(6.35) into (6.23), we have

$$\begin{aligned}
\|\Delta_t e^k\|^2 \le\ & \mu\Big[-|\Delta_t e^k|_1^2-\frac{h^2}{12}\|\Delta_t f^k\|^2\\
&+\frac{h^4}{144}|\Delta_t f^k|_1^2+\frac{h^2}{12}\|\Delta_t f^k\|\cdot\|\Delta_t R^k\|+\|\Delta_t R^k\|\cdot\|\Delta_t e^k\|\Big]\\
&+\gamma\Big[\frac{Lc_0+\sqrt{L}}{2}\cdot|e^{\bar{k}}|_1\cdot\|\Delta_t e^k\|+\frac{Lc_0+c_0}{3}\cdot|e^k|_1\cdot\|\Delta_t e^k\|\Big]\\
&+\kappa\|\Delta_x e^{\bar{k}}\|\cdot\|\Delta_t e^k\|+\frac{\gamma h^2}{2}\Big[\Big(\frac{Lc_0}{2}+\frac{3\sqrt{L}}{2}\sqrt{\frac{8}{h^4}+2Lc_3^2h^6}\Big)\cdot|e^{\bar{k}}|_1\cdot\|\Delta_t e^k\|\\
&+\frac{2}{3}\Big(c_0+\frac{c_0}{h}\Big)\cdot\|f^k\|\cdot\|\Delta_t e^k\|\Big]+\frac{\kappa h^2}{6}\Big[|\Delta_x e^{\bar{k}}|_1\cdot|\Delta_t e^k|_1\\
&+\frac{h^2}{12}|f^{\bar{k}}|_1\cdot\|\Delta_t f^k\|+\frac{h^4}{144}|\Delta_x f^{\bar{k}}|_1\cdot|\Delta_t f^k|_1+\frac{h^2}{12}|f^{\bar{k}}|_1\cdot\|\Delta_t R^k\|\\
&+|R^{\bar{k}}|_1\cdot\|\Delta_t e^k\|\Big]+\nu\Big[-\frac{1}{4\tau}(|e^{k+1}|_1^2-|e^{k-1}|_1^2)\\
&-\frac{h^2}{12}\cdot\frac{1}{4\tau}(\|f^{k+1}\|^2-\|f^{k-1}\|^2)+\frac{h^4}{144}\cdot\frac{1}{4\tau}(|f^{k+1}|_1^2-|f^{k-1}|_1^2)\\
&+\frac{h^2}{12}\|f^{\bar{k}}\|\cdot\|\Delta_t R^k\|+\|R^{\bar{k}}\|\cdot\|\Delta_t e^k\|\Big]+\|Q^k\|\cdot\|\Delta_t e^k\|\\
\le\ & -\frac{\mu h^2}{12}\|\Delta_t f^k\|^2+\frac{\mu h^2}{36}\|\Delta_t f^k\|^2+\frac{\mu h^2}{54}\|\Delta_t f^k\|^2\\
&+\frac{3\mu h^2}{32}\|\Delta_t R^k\|^2+\frac{1}{10}\|\Delta_t e^k\|^2+\frac{5\mu^2}{2}\|\Delta_t R^k\|^2\\
&+\frac{1}{10}\|\Delta_t e^k\|^2+\frac{5\gamma^2(Lc_0+\sqrt{L})^2}{8}|e^{\bar{k}}|_1^2\\
&+\frac{1}{10}\|\Delta_t e^k\|^2+\frac{5\gamma^2(Lc_0+c_0)^2}{18}|e^k|_1^2+\frac{1}{10}\|\Delta_t e^k\|^2\\
&+\frac{5\kappa^2}{2}\|\Delta_x e^{\bar{k}}\|^2+\frac{1}{10}\|\Delta_t e^k\|^2+\frac{5\gamma^2h^4}{8}\Big(\frac{Lc_0}{2}+\frac{3\sqrt{L}}{2}\sqrt{\frac{8}{h^4}+2Lc_3^2h^6}\Big)^2|e^{\bar{k}}|_1^2\\
&+\frac{1}{10}\|\Delta_t e^k\|^2+\frac{5\gamma^2h^2(hc_0+c_0)^2}{18}\|f^k\|^2+\frac{1}{10}\cdot\frac{h^2}{4}|\Delta_t e^k|_1^2\\
&+\frac{5\kappa^2h^2}{18}|\Delta_x e^{\bar{k}}|^2+\frac{\mu h^2}{54}\|\Delta_t f^k\|^2+\frac{\kappa^2h^6}{384\mu}|f^{\bar{k}}|_1^2\\
&+\frac{\mu h^2}{54}\cdot\frac{h^2}{4}|\Delta_t f^k|_1^2+\Big(\frac{\kappa h^6}{864}\Big)^2\cdot\frac{54}{\mu h^4}|\Delta_x f^{\bar{k}}|_1^2+\Big(\frac{\kappa h^4}{72}\Big)^2|f^{\bar{k}}|^2+\frac{1}{4}\|\Delta_t R^k\|^2\\
&+\frac{1}{10}\|\Delta_t e^k\|^2+\frac{5\kappa^2h^4}{72}|R^{\bar{k}}|_1^2-\frac{\nu}{4\tau}\Big[(|e^{k+1}|_1^2-|e^{k-1}|_1^2)\\
&+\frac{h^2}{12}(\|f^{k+1}\|^2-\|f^{k-1}\|^2)-\frac{h^4}{144}(|f^{k+1}|_1^2-|f^{k-1}|_1^2)\Big]\\
&+\frac{\nu^2h^4}{144}\|f^{\bar{k}}\|^2+\frac{1}{4}\|\Delta_t R^k\|^2+\frac{1}{10}\|\Delta_t e^k\|^2+\frac{5\nu^2}{2}\|R^{\bar{k}}\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{10}\|\Delta_t e^k\|^2+\frac{5}{2}\|Q^k\|^2\\
&\le \|\Delta_t e^k\|^2-\frac{\nu}{4\tau}\Big[(|e^{k+1}|_1^2-|e^{k-1}|_1^2)+\frac{h^2}{12}(\|f^{k+1}\|^2-\|f^{k-1}\|^2)\\
&-\frac{h^4}{144}(|f^{k+1}|_1^2-|f^{k-1}|_1^2)\Big]+\Big[\frac{5\gamma^2(Lc_0+\sqrt{L})^2}{8}\\
&+\frac{5\gamma^2}{8}\Big(\frac{Lc_0h^2}{2}+\frac{3\sqrt{L}}{2}\sqrt{8+2Lc_3^2h^{10}}\Big)^2+\frac{5\kappa^2}{2}+\frac{10\kappa^2}{9}\Big]\cdot|e^{\bar{k}}|_1^2\\
&+\frac{5\gamma^2(Lc_0+c_0)^2}{18}|e^k|_1^2+\frac{5\gamma^2h^2(hc_0+c_0)^2}{18}\|f^k\|^2+\Big(\frac{\kappa^2h^4}{96\mu}\\
&+\frac{\kappa^2h^4}{864\mu}+\frac{\kappa^2h^6}{1296}+\frac{\nu^2h^4}{144}\Big)\|f^{\bar{k}}\|^2\\
&+\Big(\frac{3\mu h^2}{32}+\frac{5\mu^2}{2}+\frac{1}{2}\Big)\|\Delta_t R^k\|^2+\Big(\frac{5\kappa^2h^2}{18}+\frac{5\nu^2}{2}\Big)\|R^{\bar{k}}\|^2+\frac{5}{2}\|Q^k\|^2. \qquad (6.36)
\end{aligned}
$$

Simplifying and rearranging (6.36), then using (3.12) and (3.14), we have

$$
\begin{aligned}
&\frac{\nu}{2\tau}\Big[\frac{1}{2}(|e^{k+1}|_1^2+|e^k|_1^2)+\frac{h^2}{24}(\|f^{k+1}\|^2+\|f^k\|^2)-\frac{h^4}{288}(|f^{k+1}|_1^2+|f^k|_1^2)\Big]\\
&\quad-\frac{\nu}{2\tau}\Big[\frac{1}{2}(|e^k|_1^2+|e^{k-1}|_1^2)+\frac{h^2}{24}(\|f^k\|^2+\|f^{k-1}\|^2)-\frac{h^4}{288}(|f^k|_1^2+|f^{k-1}|_1^2)\Big]\\
&\le \Big[\frac{5\gamma^2(Lc_0+\sqrt{L})^2}{8}+\frac{5\gamma^2}{8}\Big(\frac{Lc_0h^2}{2}+\frac{3\sqrt{L}}{2}\sqrt{8+2Lc_3^2h^{10}}\Big)^2+\frac{5\kappa^2}{2}+\frac{10\kappa^2}{9}\Big]|e^{\bar{k}}|_1^2\\
&\quad+\frac{5\gamma^2(Lc_0+c_0)^2}{18}|e^k|_1^2+\frac{5\gamma^2h^2(hc_0+c_0)^2}{18}\|f^k\|^2\\
&\quad+\Big(\frac{\kappa^2h^4}{96\mu}+\frac{\kappa^2h^4}{864\mu}+\frac{\kappa^2h^6}{1296}+\frac{\nu^2h^4}{144}\Big)\|f^{\bar{k}}\|^2\\
&\quad+\Big(\frac{3\mu h^2}{32}+\frac{5\mu^2}{2}+\frac{1}{2}\Big)\|\Delta_t R^k\|^2+\Big(\frac{5\kappa^2h^2}{18}+\frac{5\nu^2}{2}\Big)\|R^{\bar{k}}\|^2+\frac{5}{2}\|Q^k\|^2\\
&\le c_5\Big(\frac{|e^{k+1}|_1^2+|e^k|_1^2}{2}+\frac{|e^k|_1^2+|e^{k-1}|_1^2}{2}\Big)\\
&\quad+c_6\Big[\frac{h^2}{36}(\|f^{k+1}\|^2+\|f^k\|^2)+\frac{h^2}{36}(\|f^k\|^2+\|f^{k-1}\|^2)\Big]+c_7(\tau^2+h^4)^2\\
&\le c_8\Big[\frac{|e^{k+1}|_1^2+|e^k|_1^2}{2}+\frac{h^2}{36}(\|f^{k+1}\|^2+\|f^k\|^2)\Big]\\
&\quad+c_8\Big[\frac{|e^k|_1^2+|e^{k-1}|_1^2}{2}+\frac{h^2}{36}(\|f^k\|^2+\|f^{k-1}\|^2)\Big]+c_8(\tau^2+h^4)^2, \qquad (6.37)
\end{aligned}
$$

when $h\le h_0$, $\tau\le\tau_0$ for $1\le k\le l$. Thanks to

$$
\begin{aligned}
&\frac{h^2}{24}(\|f^{k+1}\|^2+\|f^k\|^2)-\frac{h^4}{288}(|f^{k+1}|_1^2+|f^k|_1^2)\\
&\quad\ge\Big(\frac{h^2}{24}-\frac{4}{h^2}\cdot\frac{h^4}{288}\Big)(\|f^{k+1}\|^2+\|f^k\|^2)=\frac{h^2}{36}(\|f^{k+1}\|^2+\|f^k\|^2),
\end{aligned}
$$

we have

$$
\frac{1}{2}(|e^k|_1^2+|e^{k-1}|_1^2)+\frac{h^2}{36}(\|f^k\|^2+\|f^{k-1}\|^2)\le F^k,\quad 1\le k\le N. \qquad (6.38)
$$

Combining (6.37) with (6.38), we have

$$\frac{\nu}{2\tau}(F^{k+1}-F^k)\le c_8(F^k+F^{k+1})+c_8(\tau^2+h^4)^2,\quad 1\le k\le l.$$

According to the Gronwall inequality, when $2c_8\tau/\nu\le 1/3$, we have

$$F^{k+1}\le \exp\left(\frac{6Tc_8}{\nu}\right)\cdot\left[F^1+\frac{1}{2}(\tau^2+h^4)^2\right],\quad 1\le k\le l.$$

From (6.22), when $h\le h_0$, we have

$$F^{k+1}\le c_9(\tau^2+h^4)^2,\quad 1\le k\le l. \tag{6.39}$$

A combination of (6.6), (6.38) and (6.39), we have

$$|e^{k+1}|_1\le\sqrt{2F^{k+1}}\le\sqrt{2c_9}(\tau^2+h^4)\le c_{10}(\tau^2+h^4),\quad 1\le k\le l.$$

By the mathematical induction, we have

$$|e^{k+1}|_1\le c_{10}(\tau^2+h^4),\quad 1\le k\le N-1. \tag{6.40}$$

Step 3: **Step 1** *adding* **Step 2** *yields the final result.* Combining (6.4), (6.21) and (6.40), we have

$$|e^k|_1\le c_{10}(\tau^2+h^4),\quad 0\le k\le N.$$

This completes the proof. □

Remark 6.1

$$\|e^k\|_\infty\le\frac{\sqrt{L}}{2}|e^k|_1\le\frac{c_{10}\sqrt{L}}{2}(\tau^2+h^4),\quad 0\le k\le N.$$

6.2 Stability

In the below, we will discuss the stability of the difference scheme (3.17)–(3.21). Here we only consider the stability with respect to the initial value. Suppose $\{\hat{u}_i^k,\ \hat{v}_i^k\mid 1\le i\le M,\ 0\le k\le N\}$ is the solution of

$$\delta_t\hat{u}_i^{\frac12}-\mu\delta_t\hat{v}_i^{\frac12}+\gamma\left[\psi(\hat{u}^0,\hat{u}^{\frac12})_i-\frac{h^2}{2}\psi(\hat{v}^0,\hat{u}^{\frac12})_i\right]+\kappa\left(\Delta_x\hat{u}_i^{\frac12}-\frac{h^2}{6}\Delta_x\hat{v}_i^{\frac12}\right)-\nu\hat{v}_i^{\frac12}=0,\quad 1\le i\le M, \tag{6.41}$$

$$\Delta_t\hat{u}_i^k-\mu\Delta_t\hat{v}_i^k+\gamma\left[\psi(\hat{u}^k,\hat{u}^{\bar{k}})_i-\frac{h^2}{2}\psi(\hat{v}^k,\hat{u}^{\bar{k}})_i\right]+\kappa\left(\Delta_x\hat{u}_i^{\bar{k}}-\frac{h^2}{6}\Delta_x\hat{v}_i^{\bar{k}}\right)-\nu\hat{v}_i^{\bar{k}}=0,\quad 1\le i\le M,\ 1\le k\le N-1, \tag{6.42}$$

$$\hat{v}_i^k=\delta_x^2\hat{u}_i^k-\frac{h^2}{12}\delta_x^2\hat{v}_i^k,\quad 1\le i\le M,\ 0\le k\le N, \tag{6.43}$$

$$\hat{u}_i^0=\varphi(x_i)+\phi^0(x_i),\quad 1\le i\le M, \tag{6.44}$$

$$\hat{u}_i^k=\hat{u}_{i+M}^k,\quad \hat{v}_i^k=\hat{v}_{i+M}^k,\quad 1\le i\le M,\ 0\le k\le N. \tag{6.45}$$

Denote

$$\eta_i^k=\hat{u}_i^k-u_i^k,\quad \xi_i^k=\hat{v}_i^k-v_i^k,\quad 1\le i\le M,\ 0\le k\le N.$$

Subtracting (3.17)–(3.21) from (6.41)–(6.45), we have

$$\delta_t\eta_i^{\frac{1}{2}} - \mu\delta_t\xi_i^{\frac{1}{2}} + \gamma\Big[\psi(\hat{u}^0, \hat{u}^{\frac{1}{2}})_i - \frac{h^2}{2}\psi(\hat{v}^0, \hat{u}^{\frac{1}{2}})_i\Big] - \gamma\Big[\psi(u^0, u^{\frac{1}{2}})_i - \frac{h^2}{2}\psi(v^0, u^{\frac{1}{2}})_i\Big]$$
$$+ \kappa\Big(\Delta_x\eta_i^{\frac{1}{2}} - \frac{h^2}{6}\Delta_x\xi_i^{\frac{1}{2}}\Big) - \nu\xi_i^{\frac{1}{2}} = 0, \quad 1 \le i \le M, \tag{6.46}$$

$$\Delta_t\eta_i^k - \mu\Delta_t\xi_i^k + \gamma\Big[\psi(\hat{u}^k, \hat{u}^{\bar{k}})_i - \frac{h^2}{2}\psi(\hat{v}^k, \hat{u}^{\bar{k}})_i\Big] - \gamma\Big[\psi(u^k, u^{\bar{k}})_i - \frac{h^2}{2}\psi(v^k, u^{\bar{k}})_i\Big]$$
$$+ \kappa\Big(\Delta_x\eta_i^{\bar{k}} - \frac{h^2}{6}\Delta_x\xi_i^{\bar{k}}\Big) - \nu\xi_i^{\bar{k}} = 0, \quad 1 \le i \le M, \ 1 \le k \le N-1, \tag{6.47}$$

$$\xi_i^k = \delta_x^2\eta_i^k - \frac{h^2}{12}\delta_x^2\xi_i^k, \quad 1 \le i \le M, \ 0 \le k \le N, \tag{6.48}$$

$$\eta_i^0 = \phi^0(x_i), \quad 1 \le i \le M, \tag{6.49}$$

$$\eta_i^k = \eta_{i+M}^k, \quad \xi_i^k = \xi_{i+M}^k, \quad 1 \le i \le M, \ 0 \le k \le N. \tag{6.50}$$

Similar to the proof of Theorem 6.1, we can obtain the stability with respect to the initial value.

Theorem 6.2 (Stability) *Suppose* $\{\eta_i^k, \ \xi_i^k \mid 1 \le i \le M, \ 0 \le k \le N\}$ *is the solution of* (6.46)–(6.50)*, then there are positive constants* h_0 *and* τ_0*, such that when* $h \le h_0$*,* $\tau \le \tau_0$*, we have*

$$|\eta^k|_1 \le c_{11}|\phi^0|_1, \quad 0 \le k \le N,$$

where c_{11} *only depends on the coefficients of* (1.4)–(1.6)*, the final time* T *and the spatial period* L*, however independent of the temporal step size* h *and spatial step size* τ*.*

7 Numerical Experiments

In the section, we will implement several numerical examples to verify the effectiveness of our scheme and the correctness of theoretical results.

When the exact solution is known, we define the discrete error in the L^∞-norm as follows

$$\mathrm{E}_\infty(h, \tau) = \max_{1 \le i \le M, \ 0 \le k \le N} |U_i^k - u_i^k|,$$

where U_i^k and u_i^k represent the exact solution and the numerical solution, respectively. Furthermore, denote the spatial and temporal convergence orders, respectively, as

$$\mathrm{Order}_\infty^h = \log_2 \frac{\mathrm{E}_\infty(2h, \tau)}{\mathrm{E}_\infty(h, \tau)}, \quad \mathrm{Order}_\infty^\tau = \log_2 \frac{\mathrm{E}_\infty(h, 2\tau)}{\mathrm{E}_\infty(h, \tau)}.$$

When the exact solution is unknown, we use the posterior error estimation to testify the convergence orders in temporal direction and spatial direction, respectively. For sufficient small h, we denote

$$\mathrm{F}_\infty(h, \tau) = \max_{1 \le i \le M, \ 0 \le k \le N} |u_i^k(h, \tau) - u_{2i}^k(h/2, \tau)|, \quad \mathrm{Order}_\infty^h = \log_2 \left(\frac{\mathrm{F}_\infty(2h, \tau)}{\mathrm{F}_\infty(h, \tau)}\right),$$

and for sufficient small τ, we denote

$$\mathrm{G}_\infty(h, \tau) = \max_{0 \le i \le M, \ 0 \le k \le N} |u_i^k(h, \tau) - u_i^{2k}(h, \tau/2)|, \quad \mathrm{Order}_\infty^\tau = \log_2 \left(\frac{\mathrm{G}_\infty(h, 2\tau)}{\mathrm{G}_\infty(h, \tau)}\right).$$

Table 1 Maximum norm errors behavior versus h-grid size reduction with the fixed temporal step-size $\tau = 1/5000$ in Example 1

h	Difference scheme (3.17)–(3.21)		Difference scheme in [33]	
	$E_\infty(h,\tau)$	Order_∞^h	$E_\infty(h,\tau)$	Order_∞^h
1/4	9.0677e−3	*	1.8968e−2	*
1/8	5.9120e−4	3.9390	1.3213e−3	3.8436
1/16	3.7491e−5	3.9790	8.7856e−5	3.9107
1/32	2.3538e−6	3.9935	5.5096e−6	3.9951
1/64	1.2326e−7	4.2552	3.1792e−7	4.1152

Example 1 We first consider the following BBMB equation (see [37])

$$u_t - u_{xxt} + uu_x + u_x - u_{xx} = f(x,t), \quad 0 < x < 2,\ 0 < t \le 1,$$

where

$$f(x,t) = (1 + 2\pi^2)e^t \sin \pi x + \frac{\pi}{2} e^{2t} \sin 2\pi x + \pi e^t \cos \pi x.$$

The initial condition is determined by the exact solution $u(x,t) = e^t \sin \pi x$ with the period $L = 2$.

The numerical results are reported in Tables 1–2 and Figs. 1–2.

In Table 1, we fix the temporal step-size $\tau = 1/5000$, meanwhile, reduce the spatial step-size h half by half ($h = 1/4, 1/8, 1/16, 1/32, 1/64$). As we can see, the spatial convergence order approaches to four order approximately, which is consistent with our convergence results.

In Table 2, we fix the spatial step-size $h = 1/50$, meanwhile, reduce the temporal step-size τ half by half ($\tau = 1/20, 1/40, 1/80, 1/160, 1/320$). We observe that the temporal convergence order approaches to two order in maximum norm.

Compared our numerical results with those in [33] from Tables 1 and 2, we find our scheme is more efficient and accurate.

Moreover, in order to verify the stability of the difference scheme (3.17)–(3.21), we have drawn the stable error curves in Fig. 1. For each curve, we fixed different temporal step-size ($\tau = 1/8, 1/16, 1/32, 1/64, 1/128$) by reducing the spatial step-size h half by half ($h = 1/2, 1/4, 1/8, 1/16, 1/32, 1/64$). We observe that the spatial error in maximum norm approaches to a fixed value since the numerical errors mainly come from the discretization in time, which verifies the difference scheme (3.17)–(3.21) is almost unconditional stable. In Fig. 2, the numerical panorama for $u(x,t)$ and numerical profiles are displayed, which further demonstrate the high accuracy of our scheme in practical simulation.

Example 2 Then, we consider the following BBMB equation

$$u_t - \mu u_{xxt} + uu_x + u_x - \nu u_{xx} = 0, \quad -25 < x < 25,\ 0 < t \le 1,$$

with the initial condition

$$u(x,0) = \frac{1}{2}\text{sech}^2\left(\frac{x}{4}\right), \quad -25 \le x \le 25,$$

where the exact solution is unknown and the period $L = 50$.

Table 2 Maximum norm errors behavior versus τ-grid size reduction with the fixed spatial step-size $h = 1/50$ in Example 1

τ	Difference scheme (3.17)–(3.21)		Difference scheme in [33]	
	$E_\infty(h, \tau)$	$Order_\infty^\tau$	$E_\infty(h, \tau)$	$Order_\infty^\tau$
1/20	1.9486e−3	*	2.3772e−3	*
1/40	4.8670e−4	2.0013	6.0794e−4	1.9673
1/80	1.2144e−4	2.0028	1.5342e−4	1.9865
1/160	3.0162e−5	2.0094	3.8242e−5	2.0042
1/320	7.3615e−6	2.0346	9.2928e−6	2.0410

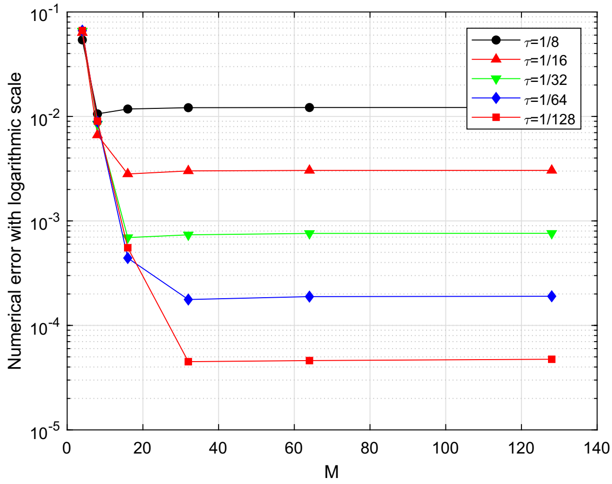


Fig. 1 Numerical stability test chart

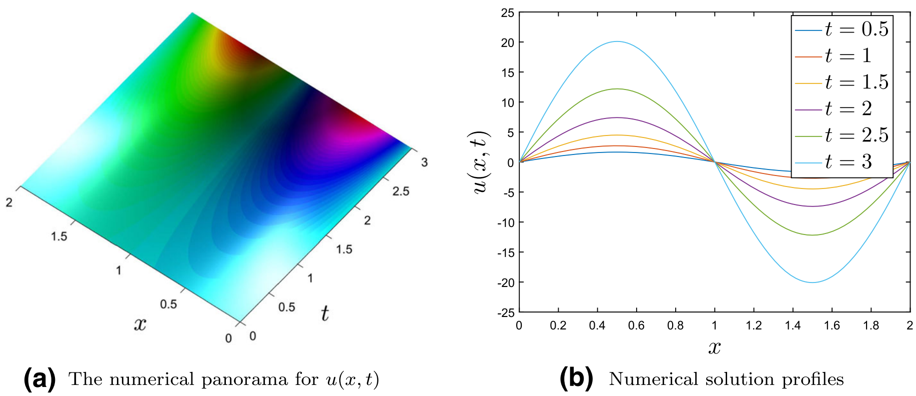


Fig. 2 **a** The numerical solution, **b** the solution profiles for $u(x, t)$ with $t = 0.5, 1, 1.5, 2, 2.5, 3$

Table 3 Maximum norm errors behavior versus h-grid size reduction with the fixed temporal step-size $\tau = 1/2000$ in Example 2

h	Difference scheme (3.17)–(3.21)		Difference scheme in [33]	
	$F_\infty(h,\tau)$	Order^h_∞	$F_\infty(h,\tau)$	Order^h_∞
5/4	4.2025e−4	*	8.1169e−3	*
5/8	3.2284e−5	3.7024	7.6856e−4	3.4007
5/16	2.0457e−6	3.9801	5.7840e−5	3.7320
5/32	1.2833e−7	3.9946	3.8500e−6	3.9091
5/64	7.9715e−9	4.0089	2.4304e−7	3.9856

Table 4 Maximum norm errors behavior versus τ-grid size reduction with the fixed spatial step-size $h = 1/2$ ($M = 100$) in Example 2

τ	Difference scheme (3.17)–(3.21)		Difference scheme in [33]	
	$G_\infty(h,\tau)$	Order^τ_∞	$G_\infty(h,\tau)$	Order^τ_∞
1/20	2.1054e−5	*	3.8007e−4	*
1/40	5.4491e−6	1.9500	9.9663e−5	1.9311
1/80	1.3852e−6	1.9759	3.0017e−5	1.7313
1/160	3.4915e−7	1.9882	1.3680e−5	1.1337
1/320	8.7641e−8	1.9942	1.0438e−5	0.3903

The numerical results are showed in Tables 3, 4, 5 and 6 and Fig. 3 with $\mu = 1$ and $\nu = 1$.

Firstly, we fix the temporal step-size $\tau = 1/2000$, in the meantime, decrease the spatial step-size h half by half ($M = 20, 40, 80, 160, 320, 640$). As we can see from Table 3, the spatial convergence orders approach to fourth order for both schemes. However, our scheme is more accurate than that in the reference [33].

Next, we fix the spatial step-size $h = 1/2$, and then reduce the temporal step-size τ half by half. The maximum norm error and the temporal convergence orders are listed in Table 4. The temporal convergence order approaches to $\mathcal{O}(\tau^2)$ approximately. However, the difference scheme in [33] is less than two and the accuracy is far from enough. We further refine the spatial grid (fixed step size $h = 1/100$) and decrease the temporal step-size τ half by half again in Table 5, though both schemes can achieve orders two, our scheme (3.17)–(3.21) is still better than that in [33] with respect to the accuracy. Combining Tables 4 and 5, we conclude that our scheme is more robust and stable that the scheme in [33], which illustrates the superiority of our scheme.

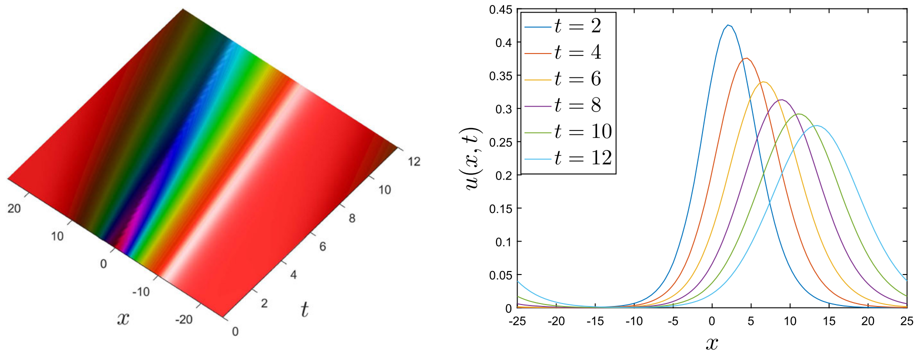
To further verify the performance of the numerical scheme (3.17)–(3.21) more rigorously, we test the energy conservation invariants (4.12) with different μ and ν. The conservation invariants of E^n at different time are demonstrated in Table 6. It is easy to see from Table 6 that the three-point four-order compact difference scheme can keep the conservative invariant even for the very small parameters, which demonstrate that our numerical scheme is stable and robust. The result is also consistent with Theorem 4.1.

Table 5 Maximum norm errors behavior versus τ-grid size reduction with the refined spatial step-size $h = 1/100$ ($M = 5000$) in Example 2

τ	Difference scheme (3.17)–(3.21)		Difference scheme in [33]	
	$G_\infty(h,\tau)$	Order^τ_∞	$G_\infty(h,\tau)$	Order^τ_∞
1/10	3.0686e−4	*	1.4755e−3	*
1/20	7.8967e−5	1.9583	3.7422e−4	1.9793
1/40	2.1227e−5	1.8954	9.4215e−5	1.9898
1/80	5.4878e−6	1.9516	2.3636e−5	1.9950
1/160	1.3945e−6	1.9765	5.9192e−6	1.9975

Table 6 Numerical invariants of E^n at time t with $h = 1/5$ and $\tau = 1/256$ in Example 2

t	$(\mu,\nu) = (100, 1)$	$(\mu,\nu) = (1, 1)$	$(\mu,\nu) = (0.01, 0.01)$	$(\mu,\nu) = (0.0001, 0.0001)$
0	7.999997216956726	1.399999972059210	1.333999999610235	1.333339999885745
2	7.999997216861070	1.399999972053378	1.333999999610103	1.333339999885731
4	7.999997216774900	1.399999972048419	1.333999999609973	1.333339999885733
6	7.999997216690209	1.399999972044242	1.333999999609848	1.333339999885742
8	7.999997216644139	1.399999972040209	1.333999999609709	1.333339999885743



(a) The numerical panorama for $u(x,t)$ **(b)** Numerical solution profiles

Fig. 3 **a** The numerical solution $t = 12$, **b** the solution profiles for $u(x,t)$ with $t = 2, 4, 6, 8, 10, 12$

Example 3 Finally, we consider a nonlinear BBMB equation

$$\begin{aligned} &u_t - \mu u_{xxt} + \gamma u u_x + \kappa u_x - \nu u_{xx} + F'(u) = 0, \quad x_l < x < x_r,\ 0 < t \le T, \\ &u(x,0) = \phi(x), \quad x_l \le x \le x_r, \end{aligned}$$

where $F(u) = 1/4 \cdot (1-u^2)^2$, $x_l = -50$, $x_r = 50$, $\mu = \gamma = \kappa = \nu = 1$. The initial condition is $\phi(x) = \frac{\sqrt{6}}{3}\text{sech}^2\left(\frac{x}{3}\right)$.

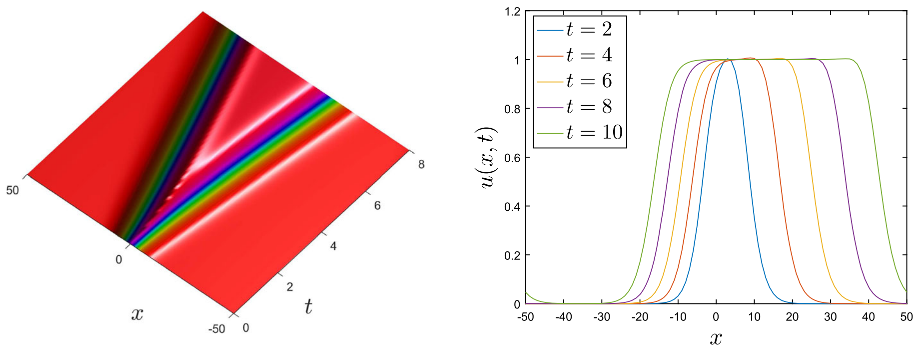
Since the above problem is nonlinear, we use Newton linearized technique (see [53]) for practical implementation. In order to demonstrate the superiority of the present scheme, we compare it with the numerical result in [53] with the period boundary condition. The corresponding convergence orders in spatial direction and temporal direction are reported in

Table 7 Maximum norm errors behavior versus h-grid size reduction with the fixed temporal step-size $\tau = 1/100$ in Example 3

h	Difference scheme (3.17)–(3.21)		Difference scheme in [53]	
	$F_\infty(h,\tau)$	Order_∞^h	$F_\infty(h,\tau)$	Order_∞^h
1/10	1.7767e−5	*	5.2399e−4	*
1/20	1.1166e−6	3.9920	1.3105e−4	1.9994
1/40	6.9998e−8	3.9957	3.2769e−5	1.9997
1/80	4.4432e−9	3.9776	8.1942e−6	1.9996
1/160	3.0875e−10	3.8471	2.0495e−6	1.9993

Table 8 Maximum norm errors behavior versus τ-grid size reduction with the fixed spatial step-size $h = 1/100$ ($M = 10000$) in Example 3

τ	Difference scheme (3.17)–(3.21)		Difference scheme in [53]	
	$G_\infty(h,\tau)$	Order_∞^τ	$G_\infty(h,\tau)$	Order_∞^τ
1/10	9.7617e−3	*	2.4014e−3	*
1/20	2.9675e−3	1.7179	6.7036e−4	1.8409
1/40	7.5462e−4	1.9754	1.7703e−4	1.9209
1/80	1.9088e−4	1.9831	4.5492e−5	1.9603
1/160	4.8043e−5	1.9903	1.1531e−5	1.9801

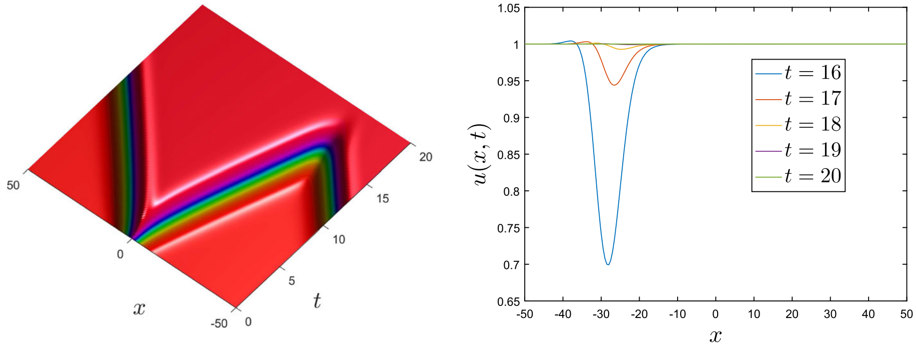


(a) The numerical panorama for $u(x,t)$ **(b)** Numerical solution profiles

Fig. 4 **a** The numerical solution $t = 8$, **b** the solution profiles for $u(x,t)$ with $t = 2, 4, 6, 8, 10$

Tables 7 and 8, respectively. We see from Table 7 that the numerical errors are better than those in [53] along with the spatial direction. According to the results in Tables 7 and 8, we know that the convergence orders are two in time and four in space for difference scheme (3.17)–(3.21), which are consistent with our theoretical results.

The numerical surfaces and the numerical curves are simulated by difference scheme (3.17)–(3.21) in Figs. 4, 5. Our scheme is much more accurate than that in [53] and clearly depicts the evolutionary process of the solution.



(a) The numerical panorama for $u(x,t)$ **(b)** Numerical solution profiles

Fig. 5 **a** The numerical solution $t = 20$, **b** the solution profiles for $u(x, t)$ with $t = 16, 17, 18, 19, 20$

8 Conclusions

In the work, incorporating the reduction order method, a three-point four-order compact difference scheme and a three-level linearized technique, we propose and analyze a linearized implicit, fourth-order compact scheme for the BBMB equation. We have obtained the unique solvability, conservative invariant, and boundedness. Moreover, we have rigorously proved the maximum error estimation and stability. Compared presented scheme with those in the references, the novel fourth-order compact scheme reliably improve the computational accuracy. Moreover, presented scheme can be extended to the BBMB equation with homogeneous boundary conditions without any difficulty. In the future, extended our technique and idea to other nonlocal or nonlinear evolution equations [5,6,27,29,30,39,49] will be our on-going project.

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Data Availability All data or codes generated or used during the study are available from the corresponding author by request.

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