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Optimal A Priori Error Estimates for the Finite Element Approximation of Dual-Phase-Lag Bio Heat Model in Heterogeneous Medium

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Abstract

Galerkin finite element method is applied to dual-phase-lag bio heat model in heterogeneous medium. Well-posedness of the model interface problem and a priori estimates of its solutions are established. Optimal a priori error estimates for both semidiscrete and fully discrete schemes are proved in $L^\infty(L^2)$ norm. The fully discrete space-time finite element discretizations is based on second order in time Newmark scheme. Finally, numerical results for two dimensional test problems are presented in support of our theoretical findings. Finite element algorithm presented here can contribute to a variety of engineering and medical applications.

Keywords General hyperbolic equation · Heterogeneous medium · Finite element method · A priori analysis · Optimal error estimates

Mathematics Subject Classification 35L05 · 65M15 · 65M60

1 Introduction

1.1 Modeling Background

It has long been established that body temperature is an indicator of health. Abnormalities in local body surface temperature have been recognized as a sign of disease for centuries. The modeling of heat related phenomena such as bio heat transfer is of great importance for the development of biomedical technologies, such as thermotherapy in treating diseases like tumor and injury involving skin tissue. Heating to the temperatures higher than that required to treat the diseased tissue can result in inadmissible damage to the adjoining healthy regions and insufficient heating can lead to under-treatment. The most commonly used model among many bio heat transfer models is the Pennes [27] bio heat model for simplicity and

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validity. Pennes bio heat transfer equation is based on classical Fourier's law. Pennes model assumes that any thermal disturbance produced at a certain location will be felt throughout the medium at that instant. In fact biological tissue, along with a number of other common materials, exhibits a relatively long thermal lag time (e.g., [7,22,23,33]). Due to implication of such relaxation time, heat conduction in biological media is generally not described by Fourier's law, but rather by the Maxwell–Cattaneo law, known as thermal wave model [10,11]. Although Maxwell–Cattaneo model has taken care of thermal relaxation time, the validity of the thermal wave model becomes debatable in view of the fast-transient response with microstructural interaction effects [32]. In order to consider the effect of micro-structural interaction in the fast transient process of heat transport, a phase lag for temperature gradient, τ_T, which is absent in the Maxwell–Cattaneo model, has been introduced in [20,32,34,38,39]. The corresponding model is called the dual-phase-lag (DPL) model. Mathematically, DPL model is described by a time-dependent equation [38,39]

$$\tau_q \rho c \frac{\partial^2 T}{\partial t^2} = k\nabla^2 T + \tau_T k\nabla^2 \frac{\partial T}{\partial t} - \omega_b \rho_b c_b T - (\tau_q \omega_b \rho_b c_b + \rho c)\frac{\partial T}{\partial t}$$
$$+ \left(\omega_b \rho_b c_b T_a + q_{\text{met}} + q_{\text{ext}} + \tau_q \frac{\partial q_{\text{met}}}{\partial t} + \tau_q \frac{\partial q_{\text{ext}}}{\partial t}\right) \tag{1.1}$$

where ρ, c, k are the density, specific heat and thermal conductivity of skin tissue, respectively; ρ_b, c_b are the density and specific heat of blood, ω_b is the blood perfusion rate; T_a and T are the temperatures of arterial blood and skin tissue respectively; q_{met} is the metabolic heat generation in the skin tissue and q_{ext} is the heat source due to external heating, and τ_q is defined as the thermal relaxation time.
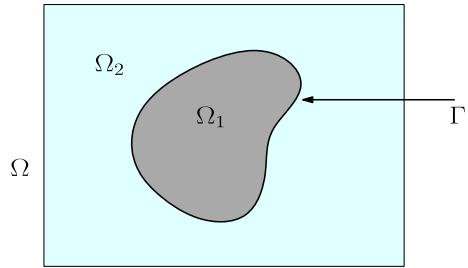
1.2 Basic Notations

Throughout the work, we will follow the usual notation for Sobolev spaces and norms. For any domain $\mathcal{M} \subset \Omega \subset \mathbb{R}^2$, each integer $k \geq 0$ and real p with $1 \leq p \leq \infty$, $W^{k,p}(\mathcal{M})$ denotes the standard Sobolev space of functions with their weak derivatives of order up to k in the Lebesgue space $L^p(\mathcal{M})$. When $p = 2$, we write $H^k(\mathcal{M})$ for $W^{k,2}(\mathcal{M})$. We use $\|\cdot\|_{s,\mathcal{M}}$ and $|\cdot|_{s,\mathcal{M}}$ to denote the norm and seminorm in the Sobolev space $H^s(\mathcal{M})$ for any $s \geq 1$, respectively. The inner product in $H^s(\mathcal{M})$ is denoted by $(\cdot, \cdot)_{s,\mathcal{M}}$. The space $H^0(\mathcal{M})$ coincides with $L^2(\mathcal{M})$, for which the norm and the inner product are denoted by $\|\cdot\|_{\mathcal{M}}$ and $(\cdot, \cdot)_{\mathcal{M}}$, respectively. For simplicity of notation, we skip the subscript $\mathcal{M}$ in the norm and inner product notation when $\mathcal{M} = \Omega$. $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$, which is also closure of $C_0^\infty(\Omega)$ (the set of all C^∞ functions with compact support) with respect to the norm of $H^s(\Omega)$ (cf. [1]).

We also define the standard Bôchner spaces $L^2(J; \mathcal{B})$ and $L^\infty(J; \mathcal{B})$, where $\mathcal{B}$ is a real Banach space with norm $\|.\|_{\mathcal{B}}$ and $J = [0, T]$, consisting of all measurable functions $\phi : J \to \mathcal{B}$ for which

$$\|\phi\|_{L^2(J;\mathcal{B})} := \left(\int_0^T \|\phi(t)\|_{\mathcal{B}}^2 dt\right)^{\frac{1}{2}} < \infty \;\; \text{and}$$
$$\|\phi\|_{L^\infty(J;\mathcal{B})} := \operatorname*{ess\,sup}_{t\in[0,T]} \|\phi(t)\|_{\mathcal{B}} < \infty,$$

Fig. 1 Domain Ω and its subdomains Ω_1, Ω_2 with interface Γ

respectively. We denote by $H^m(J; \mathcal{B})$, $1 \leq m < \infty$, the space of all measurable functions $\phi : J \to \mathcal{B}$ for which

$$\|\phi\|_{H^m(J;\mathcal{B})} = \left(\sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j \phi(t)}{\partial t^j} \right\|_{\mathcal{B}}^2 dt \right)^{\frac{1}{2}} < \infty.$$

For our notational convenience, we will be using $\frac{\partial \phi}{\partial t}$ or ϕ_t or ϕ' interchangeably to denote time differentiation of ϕ. Similar remarks hold for other higher order time derivatives.

When no risk of confusion exists, we shall write $L^2(\mathcal{B})$ for $L^2(J; \mathcal{B})$, $L^\infty(\mathcal{B})$ for $L^\infty(J; \mathcal{B})$ and $H^m(\mathcal{B})$ for $H^m(J; \mathcal{B})$.

1.3 Problem Description

The goal of the present work is to study the following general linear second order hyperbolic equation

$$u'' + \sigma u' + \delta u - \nabla.(\epsilon \nabla u' + \beta \nabla u) = f(x, t) \quad \text{in } \Omega \times (0, T], \ T < \infty \qquad (1.2)$$

with initial and boundary conditions

$$u(x, 0) = u_0, \ u'(x, 0) = v_0 \ \text{in } \Omega \ \& \ u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T] \qquad (1.3)$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. Here, $\sigma = \sigma(x)$, $\delta = \delta(x)$, $\epsilon = \epsilon(x)$, $\beta = \beta(x)$ are non-negative real valued functions defined on Ω and f denotes the source. In this work, it is implicitly assumed that initial data (u_0, v_0) and the source function f are sufficiently smooth so that solution belongs to desired Sobolev spaces.

In realistic applications it is often possible to have heterogeneity of the underlying medium. In particular, media parameters may have jump discontinuities across interfaces in the domain of interest. As a model, we consider DPL bio heat transfer model (cf. [14,20,23,33,34,38,39] and references therein) in multi-layered media. Since the thermal properties of biological media vary between different layers, so, it is natural to have heterogeneity in the underlying media. Of our special interest is the case when the domain Ω consists of two open subdomains Ω_1 and Ω_2 with C^2 smooth interface Γ, and physical coefficients are discontinuous and piecewise constants in Ω (see, Fig. 1). We write

$$(\sigma, \delta, \epsilon, \beta) = \begin{cases} (\sigma_1, \delta_1, \epsilon_1, \beta_1) \ \text{in} \ \ \Omega_1, \\ (\sigma_2, \delta_2, \epsilon_2, \beta_2) \ \text{in} \ \ \Omega_2. \end{cases}$$

The problem (1.2)–(1.3) is completed with the following physical interface conditions (cf. [20])

$$[u] = 0, \quad \left[\epsilon(x) \frac{\partial u'}{\partial \mathbf{n}} + \beta(x) \frac{\partial u}{\partial \mathbf{n}} \right] = 0 \quad \text{along } \Gamma \times [0, T], \tag{1.4}$$

where $[u] = u_1|_\Gamma - u_2|_\Gamma$ and $\left[\epsilon(x) \frac{\partial u'}{\partial \mathbf{n}} + \beta(x) \frac{\partial u}{\partial \mathbf{n}} \right] = \epsilon_1 \frac{\partial u'_1}{\partial \mathbf{n_1}} + \epsilon_2 \frac{\partial u'_2}{\partial \mathbf{n_2}} + \beta_1 \frac{\partial u_1}{\partial \mathbf{n_1}} + \beta_2 \frac{\partial u_2}{\partial \mathbf{n_2}}$ on Γ. Here u_i stands for the restriction of u to Ω_i and $\frac{\partial}{\partial \mathbf{n_i}}$ denotes the outer normal derivative with respect to Ω_i, $i = 1, 2$. The present work regards the temperature and the heat flux at the interface of two regions as continuous. In other words, the heat contact resistance at the interface between the two different media is neglected.

Interface problems are frequently encountered in scientific computing and many applied sciences. Typical examples are the elliptic, parabolic and hyperbolic equations with discontinuous coefficients. Due to the practical relevance of interface problems in many engineering and industrial applications, numerical methods for interface problems have been investigated widely. Finite element method (FEM) is another class of important approaches for interface problems and a wide variety of FEM approaches have been proposed in the literature. Classical finite element methods for interface problems are mainly based on the interface-fitted discretization. The performance of such kind of interface-fitted FEMs depends on the quality of underlying finite element partition and how well the interface is resolved by the finite element mesh (cf. [17]). A fitted finite element method is proposed and analyzed for the interface problem (1.2)–(1.4). The main contribution of the current work is to derive optimal order of convergence of the finite element solution of the BVP (1.2)–(1.4) in the L^∞-in-time/L^2-in-space norm. The fully discrete scheme can be reinterpreted as the Crank–Nicolson discretization of the reformulation of the governing equation in the first-order system, as in Baker [4]. The derivation of the a priori error bound heavily depends on the approximation properties (cf. Lemma 3.6) of a newly introduced non-standard projection operator along with some new analytical tools and techniques, including a λ-strip argument for quantifying the relation of error near the interface in terms of the mismatch parameter λ. There is plenty of literature available on the numerical study of the DPL bio heat model with discontinuous coefficients. One may refer to [20,33,34,39] and references therein. However, to the best of our knowledge, finite element analysis for the general linear second order hyperbolic equation with discontinuous coefficients has not been studied earlier. In this work, we are providing both mathematical and numerical framework for the study of BVP (1.2)–(1.4). Convergence analysis, without the interface, for the general linear second order hyperbolic equation via finite element algorithm has been well studied in literature (cf. [5,13,16,26] just to name a few). More recently, the spatial discretization of Westervelt's quasi-linear strongly damped wave equation by piecewise linear finite elements has been discussed in [25]. A priori error analysis in [25] heavily depends on general linear wave model with time dependent coefficients. Optimal convergence in $L^\infty(L^2)$ norm is obtained for sufficiently smooth solution. Fully discrete error analysis is still open for such problems. Our results are intended to enhance the numerical analysis of strongly damped linear wave equations where physical domain consists of heterogenous media.

The rest of the paper is organized as follows. In Sect. 2, we discuss the existence, uniqueness and regularity for the solution to the interface problem. Finite element discretization and some important theoretical results to be used in this article are discussed in Sect. 3. Section 4 is devoted to the error estimates for the semidiscrete scheme. The error analysis for the fully discrete scheme is presented in Sect. 5. Section 6 focuses on numerical examples. Finally, results are summarized in Sect. 7 with a brief outline on future work.

2 Preliminaries

This section is devoted to existence, uniqueness and regularity for the solutions to the model interface problem (1.2)–(1.4) in a convex polygonal domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary $\partial \Omega$. The solution to the interface problem has a higher regularity in each individual region than in the entire domain. This regularity result is critical for our further numerical analysis.

For the sake of brevity, we write $W = L^2(\Omega)$, $V = H_0^1(\Omega)$ with its dual space $V' = H^{-1}(\Omega)$ and $\mathcal{X} = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2)$ equipped with norm $\|v\|_{\mathcal{X}} := \|v\| + \|v\|_{1,\Omega_1} + \|v\|_{1,\Omega_2}$. We also introduce two bilinear forms $\mathcal{A}_\epsilon(\cdot, \cdot)$ and $\mathcal{A}_\beta(\cdot, \cdot)$ on V as follows

$$\mathcal{A}_\epsilon(w, v) = \int_\Omega \epsilon \nabla w \cdot \nabla v dx = \mathcal{A}_\epsilon^1(w, v) + \mathcal{A}_\epsilon^2(w, v) \quad \forall w, v \in V \tag{2.1}$$

and

$$\mathcal{A}_\beta(w, v) = \int_\Omega \beta \nabla w \cdot \nabla v dx = \mathcal{A}_\beta^1(w, v) + \mathcal{A}_\beta^2(w, v) \quad \forall w, v \in V. \tag{2.2}$$

Here, $\mathcal{A}_\epsilon^l$, $\mathcal{A}_\beta^l : H^1(\Omega_l) \times H^1(\Omega_l) \to \mathbb{R}$, $l = 1, 2$, are the local bilinear forms defined by

$$\mathcal{A}_\epsilon^l(w, v) = \int_{\Omega_l} \epsilon_l \nabla w \cdot \nabla v dx \quad \& \quad \mathcal{A}_\beta^l(w, v) = \int_{\Omega_l} \beta_l \nabla w \cdot \nabla v dx \quad \forall w, v \in H^1(\Omega_l).$$

Further, we define bilinear forms $\mathcal{B}_\sigma(\cdot, \cdot)$, $\mathcal{B}_\delta(\cdot, \cdot) : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$ as

$$\mathcal{B}_\sigma(w, v) = \int_\Omega \sigma w v dx \quad \& \quad \mathcal{B}_\delta(w, v) = \int_\Omega \delta w v dx \quad \forall w, v \in L^2(\Omega).$$

Next, we define the weak form of our model problem (1.2)–(1.4). We adapt following notion of weak solution.

Definition 2.1 A function $u \in H^1(V) \cap H^2(W)$ is called a weak solution of (1.2)–(1.3) if $u(0) = u_0$ and $u'(0) = v_0$, and it satisfies following weak formulation

$$(u'', v) + \mathcal{B}_\sigma(u', v) + \mathcal{B}_\delta(u, v) + \mathcal{A}_\epsilon(u', v) + \mathcal{A}_\beta(u, v) = \langle f(t, \cdot), v \rangle_{V' \times V} \tag{2.3}$$

for all $v \in H_0^1(\Omega)$ and *a.e.* $t \in (0, T]$. Here, $\langle \cdot, \cdot \rangle_{V' \times V}$ denotes the standard duality product.

Existence and uniqueness of a solution to the variational problem (2.3) is proved in [12,31,35, 36], for instance, we refer to ([35], Theorem 3). For suitable initial data (u_0, v_0) and forcing function f, we assume that weak solution $u \in C^1([0, T]; V) \cap C^2([0, T]; W)$.

Remark 2.1 Apart from bio heat modeling, a substantial number of articles deal with model problem (2.3) and it can be applied to any system where elastic bodies interact, provided that the model problem is linear. Numerous examples can be found in [15,18,36], for example, viscous wave equation, networks of linked beams, hybrid chimney etc.

To deal with the strong solution to the interface problem, we introduce a Banach space

$$\mathcal{Y} := H_0^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)$$

equipped with the norm

$$\|v\|_{\mathcal{Y}} := \|v\|_1 + \|v\|_{2,\Omega_1} + \|v\|_{2,\Omega_2}.$$

Definition 2.2 A function $u \in H^1(\mathcal{Y}) \cap H^2(W)$ is called a strong solution of (1.2)–(1.4) if $u(0) = u_0$ and $u'(0) = v_0$ with jump conditions (1.4), and the relation

$$u''(x,t) + \sigma(x)u'(x,t) + \delta(x)u(x,t) - \nabla \cdot (\epsilon(x)\nabla u'(x,t) + \beta(x)\nabla u(x,t))$$
$$= f(x,t) \tag{2.4}$$

holds for a.e. $t \in (0, T]$ and a.e. $x \in \Omega_i$ ($i = 1, 2$).

Before proving the existence of a strong solution to the interface problem, we first establish the following result.

Lemma 2.1 *Let u be the weak solution of (1.2)–(1.3). Assume that $u \in H^1(\mathcal{Y}) \cap H^2(W)$, then u is a strong solution for (1.2)–(1.4).*

Proof For $u \in H^1(\mathcal{Y}) \cap H^2(W)$ and *a.e.* $t \in (0, T]$, upon integration by parts, we obtain

$$\int_{\Omega_i} -\nabla \cdot (\epsilon_i \nabla u' + \beta_i \nabla u)v dx$$

$$= \int_{\Omega_i} \epsilon_i \nabla u' \cdot \nabla v dx + \int_{\Omega_i} \beta_i \nabla u \cdot \nabla v dx$$

$$= (f - u'' - \sigma_i u' - \delta_i u, v)_{\Omega_i} \quad \forall v \in H_0^1(\Omega_i), \tag{2.5}$$

which implies that

$$-\nabla \cdot (\epsilon_i \nabla u'(x,t) + \beta_i \nabla u(x,t)) = f(x,t) - u''(x,t) - \sigma_i u'(x,t) - \delta_i u(x,t)$$

holds for *a.e.* $t \in (0, T]$ and *a.e.* $x \in \Omega_i$ ($i = 1, 2$). It remains to show that the weak solution also satisfies the jump conditions (1.4). Applying integration by parts, for *a.e.* $t \in (0, T]$, we have

$$0 = \sum_{i=1}^{2} \int_{\Omega_i} (u'' + \sigma_i u' + \delta_i u - f)v dx + \sum_{i=1}^{2} \int_{\Omega_i} -\nabla \cdot (\epsilon_i \nabla u' + \beta_i \nabla u)v dx$$

$$= \sum_{i=1}^{2} \int_{\Omega_i} (u'' + \sigma_i u' + \delta_i u - f)v dx + \sum_{i=1}^{2} \int_{\Omega_i} (\epsilon_i \nabla u' \cdot \nabla v + \beta_i \nabla u \cdot \nabla v) dx$$

$$- \int_{\Gamma} \left[\epsilon \frac{\partial u'}{\partial \mathbf{n}} + \beta \frac{\partial u}{\partial \mathbf{n}} \right] v ds$$

$$= \sum_{i=1}^{2} (u'' + \sigma_i u' + \delta_i u - f, v)_{\Omega_i} + \sum_{i=1}^{2} (A_\epsilon^i(u', v) + A_\beta^i(u, v))$$

$$- \int_{\Gamma} \left[\epsilon \frac{\partial u'}{\partial \mathbf{n}} + \beta \frac{\partial u}{\partial \mathbf{n}} \right] v ds \quad \forall v \in V. \tag{2.6}$$

Above relation and the definition of weak solution it follows that

$$\int_{\Gamma} \left[\epsilon \frac{\partial u'}{\partial \mathbf{n}} + \beta \frac{\partial u}{\partial \mathbf{n}} \right] v ds = 0 \quad \forall v \in V.$$

The arbitrariness of v shows that u satisfies the second jump condition in (1.4). The first condition in (1.4) is a direct consequence of the fact that $u \in H^1(V)$. This completes the proof. □

In general, the solution u of the problem (1.2)–(1.4) does not belong to $H^1(H^2(\Omega))$ due to the presence of discontinuous coefficients. We can get better local regularity using local smoothness of the coefficients. From Lemma 2.1, it is clear that the existence of a strong solution depends on higher regularity of the weak solution, which is the main object of the Theorem 2.1. Further, a priori estimates for the solution to the problem (1.2)–(1.4) are also presented in Lemma 2.2 under appropriate regularity conditions on the initial functions u_0, v_0 and source function f.

Theorem 2.1 *Let $u_0 \in \mathcal{Y}$, $v_0 \in \mathcal{X}$ and $f \in H^1(J; W)$, then the interface problem (1.2)–(1.4) admits a unique strong solution.*

Proof Let $u \in C^1(J; V) \cap C^2(J; W)$ be a weak solution to the problem (1.2)–(1.3) satisfying (2.3). We consider following auxiliary problem: Find $w \in H^1(J; \mathcal{Y})$ such that

$$\mathcal{A}_\epsilon(w', v) + \mathcal{A}_\beta(w, v) = (f - u'' - \sigma u' - \delta u, v) \quad \forall v \in V, \tag{2.7}$$

with $[w] = 0$ and $\left[\beta \frac{\partial w}{\partial \mathbf{n}} + \epsilon \frac{\partial w'}{\partial \mathbf{n}} \right] = 0$ across the interface Γ, and $w(x, 0) = u_0$. For the existence and uniqueness of a solution to the problem (2.7), we refer to [2]. Further, $w \in H^1(\mathcal{Y})$ satisfies following a priori estimate

$$\|w\|_{H^1(\mathcal{Y})} \le C(\|f - u'' - \sigma u' - \delta u\|_{L^2(J; W)} + \|u_0\|_{\mathcal{Y}}). \tag{2.8}$$

Subtracting (2.7) from (2.3), we have

$$\mathcal{A}_\epsilon(u' - w', v) + \mathcal{A}_\beta(u - w, v) = 0 \quad \forall v \in V,$$

which implies that $w(x, t) = u(x, t)$ for *a.e.* $t \in (0, T]$ and *a.e.* $x \in \Omega$. Therefore $u \in H^1(J; \mathcal{Y})$ and due to Lemma 2.1 it is a strong solution to the interface problem (1.2)–(1.4). This completes the proof. $\qquad \square$

Remark 2.2 We are well aware of the fact that the rate of convergence of finite element approximations depends on the 'smoothness' of a solution. In Theorem 2.1, we have shown that interface problem (1.2)–(1.4) admits a unique strong solution $u \in H^1(\mathcal{Y}) \cap H^2(W)$ for appropriate initial data and source function. In fact, strong solution $u \in H^1(J; \mathcal{Y}) \cap C^1(J; V) \cap C^2(J; W)$. In articles on the finite element method for general second-order hyperbolic equations without the interface, related to convergence, it is assumed higher order time derivatives of the solutions (cf. [5,13,16,25,26]). Therefore, we will be required additional regularity of u which guarantee the convergence results.

Lemma 2.2 *Let u_0, $v_0 \in H^3(\Omega) \cap H_0^1(\Omega)$ and $f \in H^1(J; H^1(\Omega))$. Then the strong solution u to the interface problem (1.2)–(1.4) satisfies following a priori estimate*

$$\|u\|_{H^2(\mathcal{Y})} \le C(\|u_0\|_3 + \|v_0\|_3 + \|f\|_{H^1(J; H^1(\Omega))}).$$

Proof Suppose $z \in H^1(J; \mathcal{Y}) \cap C^1(J; V) \cap C^2(J; W)$ satisfies following variational formulation

$$(z'', v) + \mathcal{B}_\sigma(z', v) + \mathcal{B}_\delta(z, v) + \mathcal{A}_\epsilon(z', v) + \mathcal{A}_\beta(z, v) = (f', v) \quad \forall v \in V \tag{2.9}$$

with $z(0) = v_0$ and $z'(0) = z_0$. Here, $z_0 \in \mathcal{X}$ is defined as

$$z_0 = -\sigma_l v_0 - \delta_l u_0 + \nabla \cdot (\epsilon_l \nabla v_0 + \beta_l \nabla u_0) + f(0) \text{ in } \Omega_l, \ l = 1, 2.$$

Using the fact that $u \in C^2(J; W)$, it is easy to verify that $(z_0 - u''(0), v) = 0$ for all $v \in V$.

Now, we define $w(t) = u_0 + \int_0^t z \, ds$, $t \in [0, T]$ so that $w(0) = u_0$, $w'(0) = z(0) = v_0$ and $w''(0) = z_0$. Further, for all $v \in V$, we observe that w satisfies following equation

$$(w''', v) + \mathcal{B}_\sigma(w'', v) + \mathcal{B}_\delta(w', v) + \mathcal{A}_\epsilon(w'', v) + \mathcal{A}_\beta(w', v) = (f', v) \quad \forall v \in V \quad (2.10)$$

which can be written as

$$\frac{d}{dt}\Big\{(w'', v) + \mathcal{B}_\sigma(w', v) + \mathcal{B}_\delta(w, v) + \mathcal{A}_\epsilon(w', v) + \mathcal{A}_\beta(w, v) - (f, v)\Big\} = 0. \quad (2.11)$$

Now, we differentiate (2.3) with respect to t to obtain

$$\frac{d}{dt}\Big\{(u'', v) + \mathcal{B}_\sigma(u', v) + \mathcal{B}_\delta(u, v) + \mathcal{A}_\epsilon(u', v) + \mathcal{A}_\beta(u, v) - (f, v)\Big\} = 0, \quad (2.12)$$

for all $v \in V$. For similar type of arguments in the context of wave equations, we refer to ([19], pages 95-98). Then subtracting (2.11) from (2.12) yields

$$\frac{d}{dt}\Big\{(p'', v) + \mathcal{B}_\sigma(p', v) + \mathcal{B}_\delta(p, v) + \mathcal{A}_\epsilon(p', v) + \mathcal{A}_\beta(p, v)\Big\} = 0 \quad \forall v \in V,$$

where $p(t) = u(t) - w(t)$. Integrating the above equation from 0 to t, we derive

$$(p'', v) + \mathcal{B}_\sigma(p', v) + \mathcal{B}_\delta(p, v) + \mathcal{A}_\epsilon(p', v) + \mathcal{A}_\beta(p, v) = 0 \ \forall v \in V. \quad (2.13)$$

Note that $p(0) = 0$ and $p'(0) = 0$, which implies that $u = w$. Then use the fact $u' = w' = z \in H^1(\mathcal{Y})$ to conclude that $u \in H^2(\mathcal{Y})$. Further, for $v = u''$, Eq. (2.3) yields

$$\int_0^t \|u''\|^2 ds \leq C\Big(\sum_{l=1}^2 \big\{\|u_0\|_{1,\Omega_l}^2 + \|v_0\|_{1,\Omega_l}^2\big\} + \|f\|_{L^2(J;W)}^2\Big),$$

which together with (2.8) leads to following a priori estimate

$$\|u\|_{H^1(\mathcal{Y})} \leq C(\|u_0\|_{\mathcal{Y}} + \|v_0\|_{\mathcal{X}} + \|f\|_{L^2(J;W)}). \quad (2.14)$$

Using estimate (2.14) for z satisfying (2.9), we obtain

$$\begin{aligned}
\|z\|_{H^1(\mathcal{Y})} &\leq C(\|v_0\|_{\mathcal{Y}} + \|z'(0)\|_{\mathcal{X}} + \|f'\|_{L^2(J;W)}) \\
&\leq C(\|u_0\|_3 + \|v_0\|_3 + \|f\|_{H^1(J;H^1(\Omega))}). \quad (2.15)
\end{aligned}$$

This together with the fact that $u' = z$ leads to desired estimate. This completes the rest of the proof. □

Remark 2.3 In the previous result, for u_0, $v_0 \in H^3(\Omega) \cap H_0^1(\Omega)$ and $f \in H^1(J; H^1(\Omega))$, we have shown that the strong solution u to the interface problem (1.2)–(1.4) belongs to $H^2(J; \mathcal{Y}) \cap C^2(J; V) \cap C^3(J; W)$. An argument similar to that of the preceding, and after having change the smoothness condition to u_0, $v_0 \in H^4(\Omega) \cap H_0^1(\Omega)$ and $f \in H^2(J; H^1(\Omega))$ leads to an improvement in the regularity of the strong solution u. More precisely, for sufficiently smooth initial data and source function f, we assume $u \in H^3(J; \mathcal{Y})$. Results of this section are also hold true in a bounded and convex domain $\Omega \subset \mathbb{R}^3$ with C^2 smooth interface Γ.

3 Finite Element Discretization

In this section, we describe a finite element discretization, introduce some auxiliary projections and prove their approximation properties.

For the purpose of finite element approximation of the problem (1.2)–(1.4), we now describe the discretization of domain Ω. The following discussion is borrowed from [17]. We assume that the family of triangulation $\{\mathcal{T}_h\}_{h \in (0,h_0)}$, for some fixed $h_0 > 0$, is quasi-uniform. We first approximate the domain Ω_1 by a polyhedral domain $\Omega_{1,h}$ using a quasi-uniform mesh $\mathcal{T}_h^1$ such that all the boundary vertices of $\Omega_{1,h}$ lie on the boundary of Ω_1. Let $\Omega_{2,h}$ be the approximation for the domain Ω_2 due to a quasi-uniform triangulation $\mathcal{T}_h^2$ with simplicial elements of size h. The triangulation $\mathcal{T}_h^2$ is done such that all the vertices of the outer polyhedral boundary $\partial\Omega$ are also the vertices of $\Omega_{2,h}$, while all the vertices on the inner boundary of $\Omega_{2,h}$ match the boundary vertices of $\Omega_{1,h}$. More precisely, the triangulation $\mathcal{T}_h := \mathcal{T}_h^1 \cup \mathcal{T}_h^2$ satisfies the following conditions:

($\mathcal{A}$1) $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$,
($\mathcal{A}$2) if $K_1, K_2 \in \mathcal{T}_h$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is a common vertex, an edge or a face,
($\mathcal{A}$3) for each K, all its vertices are completely contained in either $\overline{\Omega}_1$ or $\overline{\Omega}_2$.

Let V_h be a finite dimensional subspace of $H_0^1(\Omega)$ defined on $\mathcal{T}_h$ consisting of piecewise linear functions vanishing on the boundary $\partial\Omega$. We now define a tubular neighborhood S_λ of Γ by

$$S_\lambda = \{x \in \Omega : \ \text{dist}(x, \Gamma) < \lambda\}$$

for some $\lambda > 0$ with $\lambda = O(h^2)$ (cf. [17]). Existence of such λ is possible due to the fact that interface Γ is of class C^2. A typical λ-strip is presented in Fig. 2. In fact the mesh $\mathcal{T}_h$ can be decomposed into three disjoint subsets $\mathcal{T}_h = \dot{\mathcal{T}}_h^1 \cup \dot{\mathcal{T}}_h^2 \cup \mathcal{T}_\star$, where

$$\dot{\mathcal{T}}_h^i = \{K \in \mathcal{T}_h : \ K \subset \overline{\Omega}_i \backslash S_\lambda\}, \ i = 1, \ 2,$$

and $\mathcal{T}_\star = \mathcal{T}_h \backslash (\dot{\mathcal{T}}_h^1 \cup \dot{\mathcal{T}}_h^2)$. An element $K \in \mathcal{T}_\ast$ is called an interface element and $K \in \mathcal{T}_h \backslash \mathcal{T}_\ast$ is called a non-interface element. Further, for $i = 1, \ 2$, we define following disjoint collections of interface elements

$$\mathcal{T}_\star^i := \{K \in \mathcal{T}_\star : \ K \subset \bar{\Omega}_i \cup S_\lambda\}.$$

With above notations, we have

$$\Omega_{i,h} = \cup\{\overline{K} : \ K \in \dot{\mathcal{T}}_h^i \cup \mathcal{T}_\ast^i\},$$

so that for any $K \in \mathcal{T}_h$, either $K \subseteq \Omega_{1,h}$ or $K \subseteq \Omega_{2,h}$.
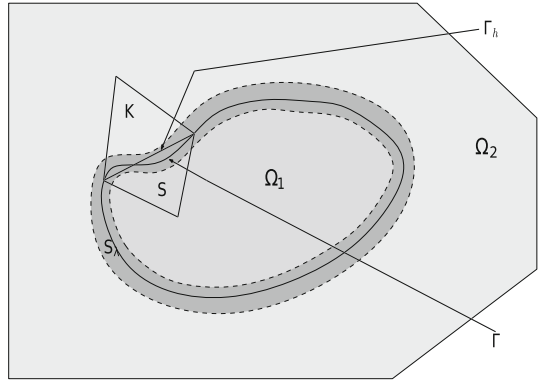
In order to approximate $\mathcal{A}_\epsilon(\cdot, \cdot)$, $\mathcal{A}_\beta(\cdot, \cdot)$, we now introduce approximate bilinear maps $\mathcal{A}_{\epsilon h}, \mathcal{A}_{\beta h} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined as

$$\mathcal{A}_{\epsilon h}(w, v) = \sum_{K \in \mathcal{T}_h} \int_K \epsilon_K(x) \nabla w . \nabla v \, dx \quad \forall w, v \in H^1(\Omega),$$

$$\mathcal{A}_{\beta h}(w, v) = \sum_{K \in \mathcal{T}_h} \int_K \beta_K(x) \nabla w . \nabla v \, dx \quad \forall w, v \in H^1(\Omega),$$

with $\epsilon_K(x) = \epsilon_i$ and $\beta_K(x) = \beta_i$ if $K \subset \Omega_{i,h}$, $i = 1, 2$.

Fig. 2 An illustrative example of interface triangles K and S with λ-strip S_λ

Next, we approximate the bilinear maps $\mathcal{B}_\sigma(\cdot, \cdot)$, $\mathcal{B}_\delta(\cdot, \cdot)$ by $\mathcal{B}_{\sigma h}(\cdot, \cdot)$ and $\mathcal{B}_{\delta h}(\cdot, \cdot)$ respectively, defined as

$$\mathcal{B}_{\sigma h}(w, v) = \sum_{K \in \mathcal{T}_h} \int_K \sigma_K(x) w v dx \quad \forall w, v \in L^2(\Omega),$$

$$\mathcal{B}_{\delta h}(w, v) = \sum_{K \in \mathcal{T}_h} \int_K \delta_K(x) w v dx \quad \forall w, v \in L^2(\Omega),$$

with $\sigma_K(x) = \sigma_i$ and $\delta_K(x) = \delta_i$ if $K \subset \Omega_{i,h}$, $i = 1, 2$.

For the simplicity of the exposition, we write $\mathcal{A}$ for $\mathcal{A}_\epsilon$ (or $\mathcal{A}_\beta$) and $\mathcal{A}_h$ for $\mathcal{A}_{\epsilon h}$ (or $\mathcal{A}_{\beta h}$), respectively. Similarly, we write $\mathcal{B}$ for $\mathcal{B}_\sigma$ (or $\mathcal{B}_\delta$) and $\mathcal{B}_h$ for $\mathcal{B}_{\sigma h}$ (or $\mathcal{B}_{\delta h}$), respectively. For the difference between the bilinear form $\mathcal{A}$ ($\mathcal{B}$) and its approximated bilinear form $\mathcal{A}_h$ ($\mathcal{B}_h$), we have the following results. For a Proof of Lemma 3.1, we refer to [17].

Lemma 3.1 *For $u, v \in H^1(\Omega)$, we define $\mathcal{A}_h^\triangle(u, v) = \mathcal{A}(u, v) - \mathcal{A}_h(u, v)$, then we have*

$$|\mathcal{A}_h^\triangle(u, v)| \le C|u|_{1,S_\lambda}|v|_{1,S_\lambda} \quad \forall u, v \in H^1(\Omega).$$

Lemma 3.2 *For $z \in H^1(\Omega)$, we have*

$$|\mathcal{B}(z, v_h) - \mathcal{B}_h(z, v_h)| \le Ch^2 \|z\|_1 \|v_h\|_1 \quad \forall v_h \in V_h. \tag{3.1}$$

Further, for $z \in \mathcal{Y}$ and $v_h \in V_h$ there holds

$$|\mathcal{B}(z, v_h) - \mathcal{B}_h(z, v_h)| \le C(h^2 + \lambda)\|z\|_{\mathcal{Y}} \|v_h\|. \tag{3.2}$$

Proof We define

$$\tilde{K} = \begin{cases} K \cap \Omega_1 \text{ if } K \in \mathcal{T}_*^2, \\ K \cap \Omega_2 \text{ if } K \in \mathcal{T}_*^1. \end{cases} \tag{3.3}$$

Clearly, $\tilde{K} \subset S_\lambda \cap \Omega_i$, $i = 1, 2$. Then we get

$$|B(z, v_h) - B_h(z, v_h)| \le \sum_{K \in \mathcal{T}_*} (z, v_h)_{\tilde{K}} \le C \sum_{K \in \mathcal{T}_*} \|z\|_{\tilde{K}} \|v_h\|_{\tilde{K}}. \tag{3.4}$$

Here, $\| \cdot \|_{\tilde{K}}$ denotes the L^2 norm over $\tilde{K}$. Now, using Hölder's inequality, we obtain

$$\|z\|_{\tilde{K}} \le Ch^{\frac{3(p-2)}{2p}} \|z\|_{L^p(\tilde{K})} \quad \forall \, p > 2. \tag{3.5}$$

We now recall Sobolev embedding inequality for two dimensions (cf. [29])

$$\|w\|_{L^p(K)} \leq C p^{\frac{1}{2}} \|w\|_{1,K} \quad \forall w \in H^1(K), \quad p > 2. \tag{3.6}$$

Now, setting $p = 6$ in (3.5) and then using the Sobolev embedding inequality (3.6), we obtain

$$\|z\|_{\tilde{K}} \leq Ch \|z\|_{1,K}. \tag{3.7}$$

Proceeding in a similar way, we obtain

$$\|v_h\|_{\tilde{K}} \leq Ch \|v_h\|_{1,K}. \tag{3.8}$$

Using estimates (3.7)–(3.8) in (3.4), we obtain the first inequality.

For the second inequality, let $p \to \infty$ in (3.5) to have

$$\|z\|_{\tilde{K}} \leq Ch^{\frac{3}{2}} \|z\|_{L^\infty(\tilde{K})} \leq Ch^{\frac{3}{2}} \sum_{i=1}^{2} \|z\|_{L^\infty(\Omega_i)} \leq Ch^{\frac{3}{2}} \|z\|_{2,\Omega_i}. \tag{3.9}$$

In the last inequality, we have used standard Sobolev embedding inequality.

From Lemma 2.1 in [17] we then infer

$$\|v_h\|_{S_\lambda \cap \Omega_i}^2 \leq C\lambda \|v_h\|_{\Omega_i} \|v_h\|_{1,\Omega_i}, \quad i = 1, 2. \tag{3.10}$$

Finally, Poincafe inequality, inverse estimate $\|\nabla v_h\| \leq Ch^{-1}\|v_h\|$ together with (3.9)–(3.10) and (3.4) leads to

$$\begin{aligned}
|B(z, v_h) - B_h(z, v_h)| &\leq Ch^{\frac{3}{2}} \|z\|_{\mathcal{Y}} \left(\sum_{K \in \mathcal{T}_*} \|v_h\|_{\tilde{K}}^2 \right)^{\frac{1}{2}} \\
&\leq Ch^{\frac{3}{2}} \|z\|_{\mathcal{Y}} \left(\sum_{i=1}^{2} \|v_h\|_{S_\lambda \cap \Omega_i}^2 \right)^{\frac{1}{2}} \\
&\leq Ch^{\frac{3}{2}} \|z\|_{\mathcal{Y}} \left(\sum_{i=1}^{2} \lambda \|v_h\|_{\Omega_i} \|v_h\|_{1,\Omega_i} \right)^{\frac{1}{2}} \\
&\leq Ch^{\frac{3}{2}} \|z\|_{\mathcal{Y}} \sqrt{\lambda} \|v_h\|^{\frac{1}{2}} \|v_h\|_{1,\Omega}^{\frac{1}{2}} \\
&\leq Ch\sqrt{\lambda} \|z\|_{\mathcal{Y}} \|v_h\| \leq C(h^2 + \lambda) \|z\|_{\mathcal{Y}} \|v_h\|.
\end{aligned}$$

This completes the rest of the proof.   □

We, now, state following approximation result near interface. For a proof, we refer to [17].

Lemma 3.3 *There exists a positive constant μ independent of h such that*

$$\|v_h\|_{H^1(S_\lambda)} \leq C\sqrt{\frac{\lambda}{h}} \|v_h\|_{H^1(S_{\mu h})} \forall v_h \in V_h.$$

Now, we introduce our elliptic projection operators $\mathcal{Q}_{\epsilon h}, \mathcal{Q}_{\beta h} : \mathcal{Y} \to V_h$ defined by

$$\mathcal{A}_{\epsilon h}(\mathcal{Q}_{\epsilon h} v, v_h) = \mathcal{A}_\epsilon^1(v, v_h) + \mathcal{A}_\epsilon^2(v, v_h) \; \forall v_h \in V_h \tag{3.11}$$

and

$$\mathcal{A}_{\beta h}(\mathcal{Q}_{\beta h} v, v_h) = \mathcal{A}_\beta^1(v, v_h) + \mathcal{A}_\beta^2(v, v_h) \; \forall v_h \in V_h, \tag{3.12}$$

respectively. To simplify the notation, we will write $\mathcal{Q}_h$ in place of $\mathcal{Q}_{\epsilon h}$ or $\mathcal{Q}_{\beta h}$ when no risk of confusion arises.

Regarding the approximation properties of $\mathcal{Q}_h$ operator defined by (3.11)–(3.12), we have following result (cf. [17])

Lemma 3.4 *Let $\mathcal{Q}_h$ be defined by (3.11) or (3.12). Then, for any $v \in \mathcal{Y}$, there is a positive constant C independent of the mesh parameter h such that*

$$\|\mathcal{Q}_h v - v\| + h\|\mathcal{Q}_h v - v\|_1 \leq C\left(h + \sqrt{\lambda} + \frac{\lambda}{h}\right)^2 \|v\|_{\mathcal{Y}}.$$

Let $\mathcal{L}_h : L^2(\Omega) \to V_h$ be the standard L^2 projection defined by

$$(\mathcal{L}_h v, \phi) = (v, \phi) \; \forall \phi \in V_h, \; v \in L^2(\Omega). \tag{3.13}$$

Previous result along with definition of L^2 projection leads to the following error estimate.

Lemma 3.5 *Let $\mathcal{L}_h$ be defined by (3.13), then for any $v \in \mathcal{Y}$ there is a positive constant C independent of the mesh parameter h such that*

$$\|\mathcal{L}_h v - v\| + h\|\mathcal{L}_h v - v\|_1 \leq C\left(h + \sqrt{\lambda} + \frac{\lambda}{h}\right)^2 \|v\|_{\mathcal{Y}}.$$

Remark 3.1 Elliptic projection operator $\mathcal{Q}_{\epsilon h}$ defined by (3.11) is also valid in the space $\hat{\mathcal{X}} := \{\xi \in \mathcal{X} : [\xi] = 0 \text{ along } \Gamma \; \& \; \xi = 0 \text{ on } \partial\Omega\}$ and satisfies following stability

$$\|\mathcal{Q}_{\epsilon h} v\|_1 \leq C\|v\|_{\mathcal{X}} \; \forall v \in \hat{\mathcal{X}}. \tag{3.14}$$

Further, following approximation results hold

$$\|v - \mathcal{Q}_{\epsilon h} v\| + h\sum_{l=1}^{2}\|v - \mathcal{Q}_{\epsilon h} v\|_{1,\Omega_l} \leq Ch^2\{\|v\|_{2,\Omega_1} + \|v\|_{2,\Omega_2}\},$$

for all $v \in \hat{\mathcal{X}} \cap H^2(\Omega_1) \cap H^2(\Omega_2)$. Similar remarks hold for the elliptic projection $\mathcal{Q}_{\beta h}$ and L^2 projection $\mathcal{L}_h$. For details, we refer to [8].

For given $v \in \hat{\mathcal{X}}$, there exists $w \in \mathcal{Y}$ (cf. [6]) satisfying

$$\sum_{l=1}^{2} \mathcal{A}_{\epsilon}^{l}(w, \phi) = (v - \mathcal{Q}_{\epsilon h} v, \phi) \; \forall \phi \in \hat{\mathcal{X}}. \tag{3.15}$$

Equation (3.15) together with (3.11), Lemmas 3.1 and 3.4 leads to

$$\|v - \mathcal{Q}_{\epsilon h} v\|^2 = \sum_{l=1}^{2} \mathcal{A}_{\epsilon}^{l}(w - \mathcal{Q}_{\epsilon h} w, v - \mathcal{Q}_{\epsilon h} v) + \sum_{l=1}^{2} \mathcal{A}_{\epsilon}^{l}(\mathcal{Q}_{\epsilon h} w, v - \mathcal{Q}_{\epsilon h} v)$$

$$\leq C\|w - \mathcal{Q}_{\epsilon h} w\|_1 \sum_{l=1}^{2}\|v - \mathcal{Q}_{\epsilon h} v\|_{1,\Omega_l} - \mathcal{A}_{\epsilon h}^{\Delta}(\mathcal{Q}_{\epsilon h} v, \mathcal{Q}_{\epsilon h} w)$$

$$\leq Ch\|w\|_{\mathcal{Y}} \sum_{l=1}^{2}\|v - \mathcal{Q}_{\epsilon h} v\|_{1,\Omega_l} + Ch\|\mathcal{Q}_{\epsilon h} v\|_1 \|\mathcal{Q}_{\epsilon h} w\|_1$$

$$\leq Ch\|v - \mathcal{Q}_{\epsilon h} v\|\|v\|_{\mathcal{X}}. \tag{3.16}$$

In the last inequality, we have used the fact that $\|w\|_{\mathcal{Y}} \le C\|v - \mathcal{Q}_{\epsilon h}v\|$ and stability estimate (3.14). As a consequence of estimate (3.16), we obtain

$$\|v - \mathcal{L}_h v\| \le \|v - \mathcal{Q}_{\epsilon h}v\| \le Ch\|v\|_{\mathcal{X}} \ \forall v \in \hat{\mathcal{X}}. \tag{3.17}$$

Here, we have used the fact that $\mathcal{L}_h v$ is the best approximation of $v \in L^2(\Omega)$ with respect to L^2 norm.

Now, inverse inequality and estimates (3.16)–(3.17) lead to following stability for L^2 projection

$$\begin{aligned}
\|\mathcal{L}_h v\|_1 &\le \|\mathcal{L}_h v - \mathcal{Q}_{\epsilon h}v\|_1 + \|\mathcal{Q}_{\epsilon h}v\|_1 \\
&\le Ch^{-1}\|\mathcal{L}_h v - \mathcal{Q}_h v\| + C\|v\|_{\mathcal{X}} \\
&\le C\|v\|_{\mathcal{X}} \ \forall v \in \hat{\mathcal{X}}.
\end{aligned} \tag{3.18}$$

Remark 3.2 In Lemma 2.2, we have proved that the solution to the interface problem is sufficiently smooth in each individual subdomain Ω_1 and Ω_2 for smooth given data. Assuming $u \in C^2(J; \mathcal{X})$ with $[u] = 0$ along Γ and $u = 0$ on $\partial\Omega$, we obtain

$$[u''(t)] = 0 \text{ on } \Gamma \ \& \ u''(t) = 0 \text{ on } \partial\Omega \ \text{ for } t \in [0, T].$$

This together with (3.18) yields

$$\|\mathcal{L}_h u''(0)\|_1 \le C\|u''(0)\|_{\mathcal{X}} \le C(\|u_0\|_3 + \|v_0\|_3 + \|f\|_{H^1(H^1)}). \tag{3.19}$$

Now, we are in a position to define our new non-standard projection operator which is crucial for our error analysis. For $v \in H^1(J; \mathcal{Y})$, find $\xi_v \in H^1(J; V_h)$ such that for *a.e.* $t \in [0, T]$

$$\mathcal{A}_{\epsilon h}(\xi'_v(t), v_h) + \mathcal{A}_{\beta h}(\xi_v(t), v_h) = \mathcal{A}_\epsilon(v'(t), v_h) + \mathcal{A}_\beta(v(t), v_h) \ \forall v_h \in V_h, \tag{3.20}$$

with $\xi_v(0) = \mathcal{Q}_{\beta h}v(0) \in V_h$.

One can follow the Proof of Theorem 3.1 and Theorem 3.2 in [9] to derive the following optimal point-wise-in time error estimates for the newly introduced projection operator.

Lemma 3.6 *For any $v \in H^1(J; \mathcal{Y})$ and a.e. $t \in J$, there is a positive constant C independent of the mesh parameter h such that*

$$\|v(t) - \xi_v(t)\| + h\|v(t) - \xi_v(t)\|_1 \le C(h^2 + \lambda)\Big(\|v\|_{H^1(\mathcal{Y})} + \|v(0)\|_{\mathcal{Y}}\Big).$$

4 Semidiscrete Finite Element Approximation

In this section, we discuss the semidiscrete finite element method for the problem (1.2)–(1.4) and derive optimal order error estimate in L^2 norm.

The continuous time Galerkin finite element approximation to (2.3) is stated as follows: Find $u_h \in C^2(J; V_h)$ such that

$$\begin{aligned}
(u''_h, v_h) &+ \mathcal{B}_{\sigma h}(u'_h, v_h) + \mathcal{B}_{\delta h}(u_h, v_h) + \mathcal{A}_{\epsilon h}(u'_h, v_h) + \mathcal{A}_{\beta h}(u_h, v_h) \\
&= (f, v_h) \ \forall v_h \in V_h, \ t \in (0, T],
\end{aligned} \tag{4.1}$$

with $u_h(0) = \mathcal{Q}_h u_0$ and $u'_h(0) = \mathcal{Q}_h v_0$.

Following result deals with the existence and regularity of u_h. The basic technique is borrowed from [25].

Theorem 4.1 *For each $h \in (0, h_0)$, there exists a unique function $u_h \in C^2(J; V_h)$ satisfying (4.1).*

Proof Let $V_h \subset H_0^1(\Omega)$ be the finite element space defined on $\mathcal{T}_h$ with basis functions $\{\phi_i\}_{i=1}^{N_h}$. We consider Galerkin approximations in space

$$u_h(x, t) = \sum_{i=1}^{N_h} c_i(t)\phi_i(x)$$

where $c_i : (0, T] \to \mathbb{R}$ are coefficient functions for $i \in [1, N_h]$.

We denote by $\mathbf{c}_{h,0} = [c_{1,0}, ..., c_{N_h,0}]^T$ and $\mathbf{c}_{h,1} = [c_{1,1}, \ldots, c_{N_h,1}]^T$ the components of the given initial approximations $u_h(0)$ and $u_h'(0)$, respectively. Then our semidiscrete problem is to find $\mathbf{c}_h(t) = [c_1(t), \ldots, c_{N_h}(t)]^T$, for $t \in (0, T]$, such that

$$\begin{cases} M_h \mathbf{c}_h''(t) + K_h \mathbf{c}_h'(t) + L_h \mathbf{c}_h(t) + C_h \mathbf{c}_h'(t) + D_h \mathbf{c}_h(t) = F_h(t), \\ \mathbf{c}_h(0) = \mathbf{c}_{h,0} \text{ and } \mathbf{c}_h'(0) = \mathbf{c}_{h,1}. \end{cases} \tag{4.2}$$

Coefficient matrices are given by

$$\begin{aligned} M_h &= [M_{i,j}], & M_{i,j} &= (\phi_i, \phi_j), \\ K_h &= [K_{i,j}], & K_{i,j} &= \mathcal{B}_\sigma(\phi_i, \phi_j), \\ L_h &= [L_{i,j}], & L_{i,j} &= \mathcal{B}_\delta(\phi_i, \phi_j), \\ C_h &= [C_{i,j}], & C_{i,j} &= \mathcal{A}_\epsilon(\phi_i, \phi_j), \\ D_h &= [D_{i,j}], & D_{i,j} &= \mathcal{A}_\beta(\phi_i, \phi_j) \end{aligned}$$

and the source term is given by $F_h = [F_1, ..., F_{N_h}]^T$, $F_j = (f, \phi_j)$, with $1 \le i, j \le N_h$. Note that the matrices and the right-hand-side vectors are all well-defined since

$$\begin{aligned} |(\phi_i, \phi_j)| &\le \|\phi_i\|\|\phi_j\|, \\ |\mathcal{B}(\phi_i, \phi_j)| &\le C_1\|\phi_i\|\|\phi_j\|, \\ |\mathcal{A}(\phi_i, \phi_j)| &\le C_2\|\phi_i\|_1\|\phi_j\|_1, \\ |(f, \phi_j)| &\le \|f\|\|\phi_j\| \le \|f\|_{L^\infty(L^2)}\|\phi_j\|, \end{aligned}$$

for all $t \in J$. Furthermore, for any $z \in \mathbb{R}^{N_h} \setminus 0$, we have

$$z^T M_h z = \int_\Omega \Big| \sum_{i=1}^{N_h} z_i \varphi_i \Big|^2 dx \ge \Big| \sum_{i=1}^{N_h} z_i \varphi_i \Big|_{L^2}^2 > 0$$

for all $t \in J$. Hence, the matrix M_h is invertible for all $t \in J$ and the matrix equation in (4.2) can be rewritten as

$$\mathbf{c}_h'' + M_h^{-1} K_h \mathbf{c}_h' + M_h^{-1} L_h \mathbf{c}_h + M_h^{-1} C_h \mathbf{c}_h' + M_h^{-1} D_h \mathbf{c}_h = M_h^{-1} F_h. \tag{4.3}$$

Now the existence of a solution $u_h \in C^2(J; V_h)$ follows from the standard ODE theory. This completes the rest of the proof.

Remark 4.1 Assuming $f \in C^1(J; W)$ and setting

$$\mathbf{c}_h''(0) = M_h^{-1} F_h(0) - M_h^{-1} K_h \mathbf{c}_{h,1} - M_h^{-1} L_h \mathbf{c}_{h,0} - M_h^{-1} C_h \mathbf{c}_{h,1} - M_h^{-1} D_h \mathbf{c}_{h,0},$$

we further observe that $u_h \in C^3(J; V_h)$. Next Lemma assumes $f \in H^3(J; H^2(\Omega))$ and which guarantee the existence of $u_h \in C^4(J; V_h)$ satisfying (4.1).

Regarding the stability of u_h at the initial stage, we have the following result. For a proof, we refer to Appendix.

Lemma 4.1 *Let u_h satisfy (4.1). Then, for $i = 2, 3, 4$, we have*

$$\|D_t^i u_h(0)\| \leq C \big(\|u_0\|_{2i-2} + \|v_0\|_{2i-2} + \|f\|_{H^{i-1}(H^2)} \big),$$

$$\|D_t^{i-1} u_h(0)\|_1 \leq C \big(\|u_0\|_{2i-3} + \|v_0\|_{2i-3} + \|f\|_{H^{i-2}(H^1)} \big),$$

where $D_t^i = \frac{\partial^i}{\partial t^i}$.

Differentiating (4.1) twice with respect to t and substitute $v_h = u_h'''$ to have

$$\frac{1}{2}\frac{d}{dt}\Big\{ \|u_h'''\|^2 + \mathcal{B}_{\delta h}(u_h'', u_h'') + \mathcal{A}_{\beta h}(u_h'', u_h'') \Big\} + \mathcal{B}_{\sigma h}(u_h''', u_h''') + \mathcal{A}_{\epsilon h}(u_h''', u_h''')$$
$$= (f'', u_h'''). \tag{4.4}$$

Integration from 0 to t and using standard arguments lead to

$$\|u_h''(t)\|^2 + \|u_h'''(t)\|^2 + \|u_h''(t)\|_1^2 + \int_0^t \|u_h'''\|^2 dt + \int_0^t \|u_h'''\|_1^2 dt$$
$$\leq C \Big(\|u_h''(0)\|^2 + \|u_h'''(0)\|^2 + \|u_h''(0)\|_1^2 + \int_0^t \|f''\|^2 dt \Big).$$

Using Lemma 4.1 in the above equation, we get

$$\|u_h'''\|^2 + \|u_h''\|_1^2 + \int_0^t \|u_h'''\|_1^2 dt \leq C \Big(\|u_0\|_4^2 + \|v_0\|_4^2 + \|f\|_{H^2(H^2)}^2 \Big). \tag{4.5}$$

Similarly, we obtain

$$\|u_h''''\|^2 + \|u_h'''\|_1^2 + \int_0^t \|u_h''''\|_1^2 dt \leq C \Big(\|u_0\|_6^2 + \|v_0\|_6^2 + \|f\|_{H^3(H^2)}^2 \Big). \tag{4.6}$$

Now, we prove the convergence result for the semidiscrete scheme in $L^\infty(L^2)$ norm.

Theorem 4.2 *Let u and u_h be the solutions of problems (1.2)–(1.4) and (4.1), respectively. Then, for u_0, $v_0 \in \mathcal{Y}$ and $f \in L^2(J; W)$, we have*

$$\|u - u_h\|_{L^\infty(J;L^2(\Omega))} \leq C(u) \Big(h + \sqrt{\lambda} + \frac{\lambda}{h} \Big)^2, \tag{4.7}$$

where $C(u) := C \Big\{ \|u_0\|_{\mathcal{Y}}^2 + \|v_0\|_{\mathcal{Y}}^2 + \|u\|_{H^1(\mathcal{Y})} \Big\}^{\frac{1}{2}}$.

Proof Define the error $e(t)$ as $e(t) := u(t) - u_h(t)$ and then subtracting (2.3) from (4.1) with some natural rearrangements, we obtain

$$(e'', v_h) + \mathcal{B}_{\sigma h}(e', v_h) + \mathcal{B}_{\delta h}(e, v_h) + \mathcal{A}_{\epsilon h}(e', v_h) + \mathcal{A}_{\beta h}(e, v_h)$$
$$= -\mathcal{B}_{\sigma h}^\Delta(u', v_h) - \mathcal{B}_{\delta h}^\Delta(u, v_h) - \mathcal{A}_{\epsilon h}^\Delta(u', v_h) - \mathcal{A}_{\beta h}^\Delta(u, v_h) \ \forall v_h \in V_h. \tag{4.8}$$

Now, we split $e(t)$ into standard ρ and θ as

$$e = \rho + \theta, \ \rho := u - \xi_u, \ \theta := \xi_u - u_h, \tag{4.9}$$

where ξ_u is the projection operator defined as in (3.20).

Then Eq. (4.8) reduces to

$$
\begin{aligned}
&(\theta'', v_h) + \mathcal{B}_{\sigma h}(\theta', v_h) + \mathcal{B}_{\delta h}(\theta, v_h) + \mathcal{A}_{\epsilon h}(\theta', v_h) + \mathcal{A}_{\beta h}(\theta, v_h) \\
&= -(\rho'', v_h) - \mathcal{B}_{\sigma h}(\rho', v_h) - \mathcal{B}_{\delta h}(\rho, v_h) - \mathcal{A}_{\epsilon h}(\rho', v_h) - \mathcal{A}_{\beta h}(\rho, v_h) \\
&\quad -\mathcal{B}_{\sigma h}^{\Delta}(u', v_h) - \mathcal{B}_{\delta h}^{\Delta}(u, v_h) - \mathcal{A}_{\epsilon h}^{\Delta}(u', v_h) - \mathcal{A}_{\beta h}^{\Delta}(u, v_h) \; \forall v_h \in V_h.
\end{aligned} \tag{4.10}
$$

Using the definition of ξ_u, we observe that

$$
\begin{aligned}
&\mathcal{A}_{\epsilon h}(\rho', v_h) + \mathcal{A}_{\beta h}(\rho, v_h) \\
&= \mathcal{A}_{\epsilon h}(u', v_h) + \mathcal{A}_{\beta h}(u, v_h) - \left\{ \mathcal{A}_{\epsilon h}(\xi_u', v_h) + \mathcal{A}_{\beta h}(\xi_u, v_h) \right\} \\
&= \mathcal{A}_{\epsilon h}(u', v_h) + \mathcal{A}_{\beta h}(u, v_h) - \left\{ \mathcal{A}_{\epsilon}(u', v_h) + \mathcal{A}_{\beta}(u, v_h) \right\}.
\end{aligned}
$$

Above equation together with (4.10) leads to

$$
\begin{aligned}
&(\theta'', v_h) + \mathcal{B}_{\delta h}(\theta, v_h) + \mathcal{A}_{\epsilon h}(\theta', v_h) + \mathcal{A}_{\beta h}(\theta, v_h) \\
&= -(\rho'', v_h) - \mathcal{B}_{\sigma h}(e', v_h) - \mathcal{B}_{\delta h}(\rho, v_h) \\
&\quad -\mathcal{B}_{\sigma h}^{\Delta}(u', v_h) - \mathcal{B}_{\delta h}^{\Delta}(u, v_h) \; \forall v_h \in V_h,
\end{aligned} \tag{4.11}
$$

which can be rewritten as

$$
\begin{aligned}
&\frac{d}{dt}(\theta', v_h) - (\theta', v_h') + \mathcal{B}_{\delta h}(\theta, v_h) + \frac{d}{dt}\mathcal{A}_{\epsilon h}(\theta, v_h) - \mathcal{A}_{\epsilon h}(\theta, v_h') + \mathcal{A}_{\beta h}(\theta, v_h) \\
&= -\frac{d}{dt}(\rho', v_h) + (\rho', v_h') - \frac{d}{dt}\mathcal{B}_{\sigma h}(e, v_h) + \mathcal{B}_{\sigma h}(e, v_h') - \mathcal{B}_{\delta h}(\rho, v_h) \\
&\quad -\mathcal{B}_{\sigma h}^{\Delta}(u', v_h) - \mathcal{B}_{\delta h}^{\Delta}(u, v_h) \; \forall v_h \in V_h.
\end{aligned} \tag{4.12}
$$

Following Baker [4], we define $\hat{v} : [0, T] \times \Omega \to \mathbb{R}$ as

$$
\hat{v}(., t) = \int_t^{\zeta} \theta(., s)ds, \quad 0 \le t \le T,
$$

for some fixed $\zeta \in [0, T]$. Then, clearly $\hat{v} \in V_h$ as $\theta = \xi_u - u_h \in V_h$. Also, observe that

$$
\hat{v}(., \zeta) = 0 \quad \text{and} \quad \frac{d}{dt}\hat{v}(., t) = -\theta(., t), \quad 0 \le t \le T. \tag{4.13}
$$

Setting $v_h = \hat{v}$ in (4.12) and making some rearrangements, we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\theta', \hat{v}) + \frac{1}{2}\frac{d}{dt}(\theta, \theta) + \mathcal{B}_{\sigma h}(\theta, \theta) - \frac{1}{2}\frac{d}{dt}\mathcal{B}_{\delta h}(\hat{v}, \hat{v}) \\
&\quad +\frac{d}{dt}\mathcal{A}_{\epsilon h}(\theta, \hat{v}) + \mathcal{A}_{\epsilon h}(\theta, \theta) - \frac{1}{2}\frac{d}{dt}\mathcal{A}_{\beta h}(\hat{v}, \hat{v}) \\
&= -\frac{d}{dt}(\rho', \hat{v}) - (\rho', \theta) - \frac{d}{dt}\mathcal{B}_{\sigma h}(e, \hat{v}) - \mathcal{B}_{\sigma h}(\rho, \theta) - \mathcal{B}_{\delta h}(\rho, \hat{v}) \\
&\quad -\mathcal{B}_{\sigma h}^{\Delta}(u', \hat{v}) - \mathcal{B}_{\delta h}^{\Delta}(u, \hat{v}).
\end{aligned}
$$

Integrating from 0 to ζ and using $\hat{v}(\zeta) = 0$, we get

$$
\begin{aligned}
&-(\theta'(0), \hat{v}(0)) + \frac{1}{2}\|\theta(\zeta)\|^2 - \frac{1}{2}\|\theta(0)\|^2 + \int_0^{\zeta} \mathcal{B}_{\sigma h}(\theta, \theta)ds + \frac{1}{2}\mathcal{B}_{\delta h}(\hat{v}(0), \hat{v}(0)) \\
&\quad -\mathcal{A}_{\epsilon h}(\theta(0), \hat{v}(0)) + \int_0^{\zeta} \mathcal{A}_{\epsilon h}(\theta, \theta)ds + \frac{1}{2}\mathcal{A}_{\beta h}(\hat{v}(0), \hat{v}(0))
\end{aligned}
$$

$$= (\rho'(0), \hat{v}(0)) - \int_0^\zeta (\rho', \theta)ds + \mathcal{B}_{\sigma h}(e(0), \hat{v}(0)) - \int_0^\zeta \mathcal{B}_{\sigma h}(\rho, \theta)ds$$

$$- \int_0^\zeta \mathcal{B}_{\delta h}(\rho, \hat{v})ds - \int_0^\zeta \mathcal{B}_{\sigma h}^\Delta(u', \hat{v})ds - \int_0^\zeta \mathcal{B}_{\delta h}^\Delta(u, \hat{v})ds. \qquad (4.14)$$

Observe that $\theta(0) = \xi_u(0) - u_h(0) = \mathcal{Q}_h u(0) - \mathcal{Q}_h u_0 = 0$, hence (4.14) becomes

$$\frac{1}{2}\|\theta(\zeta)\|^2 + \int_0^\zeta \|\theta\|^2 ds + \int_0^\zeta \|\theta\|_1^2 ds + \frac{1}{2}\|\hat{v}(0)\|_1^2$$

$$\leq (e'(0), \hat{v}(0)) - \int_0^\zeta (\rho', \theta)ds + \mathcal{B}_{\sigma h}(e(0), \hat{v}(0)) - \int_0^\zeta \mathcal{B}_{\sigma h}(\rho, \theta)ds$$

$$- \int_0^\zeta \mathcal{B}_{\delta h}(\rho, \hat{v})ds - \int_0^\zeta \mathcal{B}_{\sigma h}^\Delta(u', \hat{v})ds - \int_0^\zeta \mathcal{B}_{\delta h}^\Delta(u, \hat{v})ds. \qquad (4.15)$$

Then Cauchy-Schwartz inequality, Lemma 3.2 and continuity of $\mathcal{B}_h$ operator leads to

$$\frac{1}{2}\|\theta(\zeta)\|^2 + \int_0^\zeta \|\theta\|^2 ds + \int_0^\zeta \|\theta\|_1^2 ds + \frac{1}{2}\|\hat{v}(0)\|_1^2$$

$$\leq C\Bigg(\|e'(0)\|\|\hat{v}(0)\| + \int_0^\zeta \|\rho'\|\|\theta\|ds + \|e(0)\|\|\hat{v}(0)\| + \int_0^\zeta \|\rho\|\|\theta\|ds$$

$$+ \int_0^\zeta \|\rho\|\|\hat{v}\|ds + \left(h + \sqrt{\lambda} + \frac{\lambda}{h}\right)^2 \int_0^\zeta \|u'\|_\mathcal{Y}\|\hat{v}\|ds$$

$$+ \left(h + \sqrt{\lambda} + \frac{\lambda}{h}\right)^2 \int_0^\zeta \|u\|_\mathcal{Y}\|\hat{v}\|ds \Bigg). \qquad (4.16)$$

Since θ is continuous in the time variable, we select ζ such that $\|\theta(\zeta)\| = \max_{0 \leq t \leq T}\|\theta(t)\|$. Then we observe that $\|\hat{v}(t)\| \leq C(T)\|\theta(\zeta)\|$, $t \in [0, T]$, which in combination with (4.16) leads to

$$\|\theta(\zeta)\| \leq C\Bigg(\|e'(0)\| + \|e(0)\| + \int_0^\zeta (\|\rho'\| + \|\rho\|)ds$$

$$+ \left(h + \sqrt{\lambda} + \frac{\lambda}{h}\right)^2 \int_0^\zeta (\|u'\|_\mathcal{Y} + \|u\|_\mathcal{Y})ds \Bigg).$$

This together with Lemma 3.6 leads to Theorem 4.2.   □

Remark 4.2 Theorem 4.2 is an extension of Theorem 5.6 in [25] to general linear hyperbolic equation with interface. It is worth to note that Theorem 5.6 in [25] is concerned on the convergence of finite element solution to the exact solution of linearized Westervelt equation with variable coefficients without interface.

5 Fully Discrete Scheme

This section is dedicated to the derivation of the L^2 norm error estimate. The basic technique used here is borrowed from Baker [4] with a modification to include the damping term.

First we divide the time interval $I = [0, T]$ into N equally spaced subintervals $I_n = (t_{n-1}, t_n]$, $n = 1, 2, \ldots, N$ with $t_0 = 0$, and $t_N = T$ and $\tau = t_n - t_{n-1}$, the time step. For a

sequence $\{p^n\}_{n=0}^N \subset L^2(\Omega)$, we define

$$\partial_\tau p^n = \frac{p^{n+1} - p^n}{\tau} \quad \text{and} \quad p^{n+\frac{1}{2}} = \frac{1}{2}(p^{n+1} + p^n), \quad n = 0, 1, \ldots, N - 1.$$

Also, for a continuous mapping $\phi : [0, T] \to L^2(\Omega)$, we define $\phi^n = \phi(., t_n), 0 \le n \le N$. Then the fully discrete finite element approximation to the problem (1.2)–(1.4) is defined as follows: Find $U^n \in V_h$ such that

$$\partial_\tau U^n = p^{n+\frac{1}{2}} \text{ for } n = 0, 1, \ldots, N - 1 \tag{5.1}$$

and

$$(\partial_\tau p^n, \psi) + \mathcal{B}_{\sigma h}(p^{n+\frac{1}{2}}, \psi) + \mathcal{B}_{\delta h}(U^{n+\frac{1}{2}}, \psi) + \mathcal{A}_{\epsilon h}(p^{n+\frac{1}{2}}, \psi) + \mathcal{A}_{\beta h}(U^{n+\frac{1}{2}}, \psi)$$
$$= (f^{n+\frac{1}{2}}, \psi) \ \forall \psi \in V_h, \tag{5.2}$$

with $U^0 = \mathcal{Q}_h u_0$ and $p^0 = \mathcal{Q}_h v_0$.

The following Lemma gives the existence and uniqueness of the fully discrete solution U^n of u^n in terms of the auxiliary variable p^n and in fact gives a computational algorithm to find U^n.

Lemma 5.1 *There exists a unique sequence $\{U^n\}_{n=0}^N \subset V_h$ and a corresponding unique sequence $\{p^n\}_{n=0}^N \subset V_h$ satisfying (5.1)–(5.2).*

Proof From (5.1), we have

$$U^{n+1} = \frac{\tau}{2}(p^{n+1} + p^n) + U^n. \tag{5.3}$$

Using (5.3) in (5.2), we get

$$\mathcal{A}_\tau(p^{n+1}, \psi) = \mathcal{F}^n(\psi) \ \forall \psi \in V_h, \tag{5.4}$$

where $\mathcal{A}_\tau$ is the bilinear form given by

$$\mathcal{A}_\tau(w, v) = (w, v) + \frac{\tau}{2}\mathcal{B}_{\sigma h}(w, v) + \frac{\tau^2}{4}\mathcal{B}_{\delta h}(w, v)$$
$$+ \frac{\tau}{2}\mathcal{A}_{\epsilon h}(w, v) + \frac{\tau^2}{4}\mathcal{A}_{\beta h}(w, v) \ \forall w, v \in V$$

and $\mathcal{F}^n$ is the linear functional given by

$$\mathcal{F}^n(\psi) = (p^n, \psi) - \frac{\tau}{2}\mathcal{B}_{\sigma h}(p^n, \psi) - \tau \mathcal{B}_{\delta h}(U^n, \psi) - \frac{\tau^2}{4}\mathcal{B}_{\delta h}(p^n, \psi)$$
$$- \frac{\tau}{2}\mathcal{A}_{\epsilon h}(p^n, \psi) - \frac{\tau^2}{4}\mathcal{A}_{\beta h}(p^n, \psi) - \tau \mathcal{A}_{\beta h}(U^n, \psi) + \tau(f^{n+\frac{1}{2}}, \psi) \ \forall \psi \in V.$$

Due to the positivity of bilinear forms $\mathcal{B}_h$ and $\mathcal{A}_h$, there exists uniquely defined $p^{n+1} \in V_h$ satisfying equation (5.4) and subsequently U^{n+1} exists uniquely for $n = 0, 1, \ldots, N - 1$. $\square$

Later on, we will need the following results. The proofs involve the use of Taylor's series and standard arguments, and therefore, details are omitted.

Lemma 5.2 *For any $v \in H^3(J; L^2(\Omega))$, we have*

$$\|\partial_\tau v^n - v_t^{n+\frac{1}{2}}\|^2 \le C\tau^3 \int_{t_n}^{t_{n+1}} \|v'''\|^2 dt.$$

In order to compute the error between U^n and u^n, it suffices to establish the error $\omega^n := u_h^n - U^n$, for $1 \leq n \leq N$ and $u_h^n = u_h(\cdot, t_n)$. Once we have estimate for ω^n, we can easily get the error estimate for $e^n := U^n - u^n$ by using the triangle inequality, Theorem 4.2 and the Lemma 5.3 given below.

Lemma 5.3 *Let u and U^n be the solutions of the interface problem* (1.2)–(1.4) *and the finite element approximation* (5.1)–(5.2), *respectively. Then, we have*

$$\max_{1 \leq n \leq N} \|\omega^n\|^2 \leq C\tau^4 \Big(\int_0^T \|u_h''''\|^2 dt + \int_0^T \|u_h'''\|_1^2 dt \Big).$$

Proof Substitute $t = t_n$ and $t = t_{n+1}$ in (4.1) and then add to have

$$(\partial_\tau u_{ht}^n, \psi) + \mathcal{B}_{\sigma h}(u_{ht}^{n+\frac{1}{2}}, \psi) + \mathcal{B}_{\delta h}(u_h^{n+\frac{1}{2}}, \psi) + \mathcal{A}_{\epsilon h}(u_{ht}^{n+\frac{1}{2}}, \psi) + \mathcal{A}_{\beta h}(u_h^{n+\frac{1}{2}}, \psi)$$
$$= (f^{n+\frac{1}{2}}, \psi) + (\rho^n, \psi) \ \forall \psi \in V_h, \tag{5.5}$$

where $\rho^n := \partial_\tau u_{ht}^n - u_{htt}^{n+\frac{1}{2}}$.

Now, subtracting (5.2) from (5.5), we have

$$(\partial_\tau q^n, \psi) + \mathcal{B}_{\sigma h}(q^{n+\frac{1}{2}}, \psi) + \mathcal{B}_{\delta h}(\omega^{n+\frac{1}{2}}, \psi) + \mathcal{A}_{\epsilon h}(q^{n+\frac{1}{2}}, \psi) + \mathcal{A}_{\beta h}(\omega^{n+\frac{1}{2}}, \psi)$$
$$= (\rho^n, \psi) \ \forall \psi \in V_h, \tag{5.6}$$

with $q^n := u_{ht}^n - p^n$.

From (5.1), it is easy to observe that

$$\partial_\tau \omega^n = q^{n+\frac{1}{2}} + \partial_\tau u_h^n - u_{ht}^{n+\frac{1}{2}} = q^{n+\frac{1}{2}} + \alpha^n, \ \alpha^n := \partial_\tau u_h^n - u_{ht}^{n+\frac{1}{2}}, \tag{5.7}$$

so that

$$\omega^n = \tau \sum_{k=0}^{n-1} \partial_\tau \omega^k = \tau \sum_{k=0}^{n-1} q^{k+\frac{1}{2}} + \tau \sum_{k=0}^{n-1} \alpha^k \ \& \ q^n = \tau \sum_{k=0}^{n-1} \partial_\tau q^k.$$

Here, we have used the fact that $\omega^0 = u_h^0 - U^0 = \mathcal{Q}_h u_0 - \mathcal{Q}_h u_0 = 0$ and $q^0 = u_{ht}^0 - p^0 = \mathcal{Q}_h v_0 - \mathcal{Q}_h v_0 = 0$.

Hence, using the above relations it follows that

$$\partial_\tau \omega^n = \frac{\tau}{2} \Big(\sum_{k=0}^{n} \partial_\tau q^k + \sum_{k=0}^{n-1} \partial_\tau q^k \Big) + \alpha^n, \tag{5.8}$$

$$\omega^{n+\frac{1}{2}} = \frac{\tau}{2} \Big(\sum_{k=0}^{n} q^{k+\frac{1}{2}} + \sum_{k=0}^{n-1} q^{k+\frac{1}{2}} \Big) + \frac{\tau}{2} \Big(\sum_{k=0}^{n} \alpha^k + \sum_{k=0}^{n-1} \alpha^k \Big). \tag{5.9}$$

Now, we define a sequence $\{s^n\}_{n=0}^N$ such that $s^0 = 0$ and

$$s^n = \tau \sum_{k=0}^{n-1} \omega^{k+\frac{1}{2}}, \ n = 1, \ldots, N - 1,$$

so that

$$s^{n+\frac{1}{2}} = \frac{\tau}{2} \Big(\sum_{k=0}^{n} \omega^{k+\frac{1}{2}} + \sum_{k=0}^{n-1} \omega^{k+\frac{1}{2}} \Big). \tag{5.10}$$

Hence, for any $\psi \in V_h$, using the identities (5.8)–(5.10) we obtain

$$
(\partial_\tau \omega^n, \psi) + \mathcal{B}_{\sigma h}(\omega^{n+\frac{1}{2}}, \psi) + \mathcal{B}_{\delta h}(s^{n+\frac{1}{2}}, \psi) + \mathcal{A}_{\epsilon h}(\omega^{n+\frac{1}{2}}, \psi) + \mathcal{A}_{\beta h}(s^{n+\frac{1}{2}}, \psi)
$$

$$
= \frac{\tau}{2} \sum_{k=0}^{n} \left\{ (\partial_\tau q^k, \psi) + \mathcal{B}_{\sigma h}(q^{k+\frac{1}{2}}, \psi) + \mathcal{B}_{\delta h}(\omega^{k+\frac{1}{2}}, \psi) + \mathcal{A}_{\epsilon h}(q^{k+\frac{1}{2}}, \psi) \right.
$$

$$
\left. + \mathcal{A}_{\beta h}(\omega^{k+\frac{1}{2}}, \psi) \right\} + \frac{\tau}{2} \sum_{k=0}^{n-1} \left\{ (\partial_\tau q^k, \psi) + \mathcal{B}_{\sigma h}(q^{k+\frac{1}{2}}, \psi) + \mathcal{B}_{\delta h}(\omega^{k+\frac{1}{2}}, \psi) \right.
$$

$$
\left. + \mathcal{A}_{\epsilon h}(q^{k+\frac{1}{2}}, \psi) + \mathcal{A}_{\beta h}(\omega^{k+\frac{1}{2}}, \psi) \right\} + (\alpha^n, \psi) + \frac{\tau}{2} \mathcal{B}_{\sigma h}\left(\sum_{k=0}^{n} \alpha^k + \sum_{k=0}^{n-1} \alpha^k, \psi \right)
$$

$$
+ \frac{\tau}{2} \mathcal{A}_{\epsilon h}\left(\sum_{k=0}^{n} \alpha^k + \sum_{k=0}^{n-1} \alpha^k, \psi \right).
$$

Using (5.6), for $1 \le n \le N-1$, we derive

$$
(\partial_\tau \omega^n, \psi) + \mathcal{B}_{\sigma h}(\omega^{n+\frac{1}{2}}, \psi) + \mathcal{B}_{\delta h}(s^{n+\frac{1}{2}}, \psi) + \mathcal{A}_{\epsilon h}(\omega^{n+\frac{1}{2}}, \psi) + \mathcal{A}_{\beta h}(s^{n+\frac{1}{2}}, \psi)
$$
$$
= (T_1^n, \psi) + \mathcal{B}_{\sigma h}(T_2^n, \psi) + \mathcal{A}_{\epsilon h}(T_2^n, \psi) \; \forall \psi \in V_h, \tag{5.11}
$$

where

$$
T_1^n := \frac{\tau}{2} \rho^n + \tau \sum_{k=0}^{n-1} \rho^k + \alpha^n \quad \& \quad T_2^n := \frac{\tau}{2} \alpha^n + \tau \sum_{k=0}^{n-1} \alpha^k.
$$

Substituting $\psi = \omega^{n+\frac{1}{2}} = \partial_\tau s^n$ in (5.11) and making some rearrangements, we arrive at

$$
(\omega^{n+1}, \omega^{n+1}) + 2\tau \mathcal{B}_{\sigma h}(\omega^{n+\frac{1}{2}}, \omega^{n+\frac{1}{2}}) + \mathcal{B}_{\delta h}(s^{n+1}, s^{n+1}) + 2\tau \mathcal{A}_{\epsilon h}(\omega^{n+\frac{1}{2}}, \omega^{n+\frac{1}{2}})
$$
$$
+ \mathcal{A}_{\beta h}(s^{n+1}, s^{n+1}) = (\omega^n, \omega^n) + \mathcal{B}_{\delta h}(s^n . s^n) + \mathcal{A}_{\beta h}(s^n, s^n) + 2\tau (T_1^n, \omega^{n+\frac{1}{2}})
$$
$$
+ 2\tau \mathcal{B}_{\sigma h}(T_2^n, \omega^{n+\frac{1}{2}}) + 2\tau \mathcal{A}_{\epsilon h}(T_2^n, \omega^{n+\frac{1}{2}}).
$$

Next, using Cauchy-Schwartz inequality, coercivity and continuity of the bilinear maps $\mathcal{B}$ and $\mathcal{A}$, we obtain

$$
(\omega^{n+1}, \omega^{n+1}) + 2\tau \|\omega^{n+\frac{1}{2}}\|^2 + \mathcal{B}_{\delta h}(s^{n+1}, s^{n+1}) + 2\tau \|\omega^{n+\frac{1}{2}}\|_1^2 + \mathcal{A}_{\beta h}(s^{n+1}, s^{n+1})
$$
$$
\le (\omega^n, \omega^n) + \mathcal{B}_{\delta h}(S^n, S^n) + \mathcal{A}_{\beta h}(s^n, s^n) + 2\tau \|T_1^n\| \|\omega^{n+\frac{1}{2}}\| + 2\tau \|T_2^n\| \|\omega^{n+\frac{1}{2}}\|
$$
$$
+ 2\tau \|T_2^n\|_1 \|\omega^{n+\frac{1}{2}}\|_1.
$$

Finally, applying the Young's inequality $ab \le \kappa a^2 + \frac{1}{\kappa} b^2$ for $a, b > 0$ and choosing $\kappa > 0$ appropriately, above relation leads us to

$$
(\omega^{n+1}, \omega^{n+1}) + \mathcal{B}_{\delta h}(s^{n+1}, s^{n+1}) + \mathcal{A}_{\beta h}(s^{n+1}, s^{n+1})
$$
$$
\le (\omega^n, \omega^n) + \mathcal{B}_{\delta h}(s^n, s^n) + \mathcal{A}_{\beta h}(s^n, s^n)
$$
$$
+ 2\tau \left(\|T_1^n\|^2 + \|T_2^n\|^2 + \|T_2^n\|_1^2 \right). \tag{5.12}
$$

Summing (5.12) from $n = 1$ to $n = l - 1$ with $2 \le l \le N$, we obtain

$$
\max_{2 \le n \le N} \|\omega^n\|^2 \le \|\omega^1\|^2 + \|s^1\|_1^2 + 2\tau \sum_{n=0}^{l-1} \left(\|T_1^n\|^2 + \|T_2^n\|_1^2 \right). \tag{5.13}
$$

For estimation of the terms ω^1 and s^1, we note that

$$s^1 = \tau \omega^{\frac{1}{2}} = \frac{\tau}{2}\omega^1 \quad \& \quad q^{\frac{1}{2}} = \frac{q^1}{2} = \frac{\omega^1}{\tau} - \alpha^0.$$

Now, putting $n = 0$ in the error Eq. (5.6) and using the above identities, we have

$$\frac{2}{\tau^2}(\omega^1, \psi) + \frac{1}{\tau}\mathcal{B}_{\sigma h}(\omega^1, \psi) + \frac{1}{2}\mathcal{B}_{\delta h}(\omega^1, \psi) + \frac{1}{\tau}\mathcal{A}_{\epsilon h}(\omega^1, \psi) + \frac{1}{\tau}\mathcal{A}_{\beta h}(s^1, \psi)$$

$$= (\rho^0, \psi) + \frac{2}{\tau}(\alpha^0, \psi) + \mathcal{B}_{\sigma h}(\alpha^0, \psi) + \mathcal{A}_{\epsilon h}(\alpha^0, \psi) \quad \forall \psi \in V_h. \tag{5.14}$$

Substituting $\psi = \omega^1 = \frac{2}{\tau}$ in (5.14) and using coercivity of the operators $\mathcal{B}$ and $\mathcal{A}$, we obtain

$$\|\omega^1\|^2 + \|s^1\|_1^2 \le \frac{\tau^2}{2}(\rho^0, \omega^1) + \tau(\alpha^0, \omega^1) + \frac{\tau^2}{2}\mathcal{B}_{\sigma h}(\alpha^0, \omega^1) + \tau\mathcal{A}_{\epsilon h}(\alpha^0, s^1).$$

Next, use Cauchy–Schwartz and Young's inequality to have

$$\|\omega^1\|^2 + \|s^1\|_1^2 \le \frac{\tau^4}{4}\|\rho^0\|^2 + \kappa_1\|\omega^1\|^2 + \left(\tau^2 + \frac{\tau^4}{4}\right)\|\alpha^0\|^2$$

$$+ \kappa_2\|\omega^1\|^2 + \tau^2\|\alpha^0\|_1^2 + \kappa_3\|s^1\|_1^2.$$

Finally, choosing $\kappa_i > 0$ appropriately leads us to

$$\|\omega^1\|^2 + \|s^1\|_1^2 \le C\left(\tau^4\|\rho^0\|^2 + \tau^2\|\alpha^0\|_1^2\right). \tag{5.15}$$

Combining (5.13) and (5.15), we have

$$\max_{1 \le n \le N} \|\omega^n\|^2 \le C\left(\tau^4\|\rho^0\|^2 + \tau^2\|\alpha^0\|_1^2 + 2\tau\sum_{n=0}^{l-1}\left(\|T_1^n\|^2 + \|T_2^n\|_1^2\right)\right). \tag{5.16}$$

Now, we shall estimate both terms T_1^n and T_2^n. For the estimation of T_1^n, use triangle inequality and Cauchy-Schwartz inequality to have

$$\|T_1^n\|^2 \le C\left(\frac{\tau^2}{4}\|\rho^n\|^2 + \tau^2\|\sum_{k=0}^{n-1}\rho^k\|^2 + \|\alpha^n\|^2\right)$$

$$\le C\left(\frac{\tau^2}{4}\|\rho^n\|^2 + \tau^2 N\sum_{k=0}^{n-1}\|\rho^k\|^2 + \|\alpha^n\|^2\right)$$

$$\le C\left(\frac{\tau^2}{4}\|\rho^n\|^2 + \tau\sum_{k=0}^{n-1}\|\rho^k\|^2 + \|\alpha^n\|^2\right).$$

Then, using Lemma 5.2, we obtain

$$\|T_1^n\|^2 \le C\left(\tau^5\int_{t_n}^{t_{n+1}}\|u_h''''\|^2 dt + \tau^4\int_0^T\|u_h'''\|^2 dt\right.$$

$$\left. + \tau^3\int_{t_n}^{t_{n+1}}\|u_h'''\|^2 dt\right). \tag{5.17}$$

The following estimate for T_2^n is achieved using the same technique as used for deriving T_1^n

$$\|T_2^n\|_1^2 \le C\left(\tau^5\int_{t_n}^{t_{n+1}}\|u_h'''\|_1^2 dt + \tau^4\int_0^T\|u_h'''\|_1^2 dt\right). \tag{5.18}$$

Finally, using (5.17)–(5.18) in (5.16), we obtain

$$\max_{1 \leq n \leq N} \|\omega^n\|^2 \leq C\tau^4 \left(\int_0^T \|u_h''''\|^2 dt + \int_0^T \|u_h'''\|_1^2 dt \right).$$

$\square$

Now, we are in a position to state the main result of this section.

Theorem 5.1 *Let u and U^n be the solutions of the interface problem* (1.2)–(1.4) *and the finite element approximation* (5.1)-(5.2), *respectively. Assume that $u_0 \in H^6(\Omega) \cap H_0^1(\Omega)$, $v_0 \in H^6(\Omega) \cap H_0^1(\Omega)$ and $f \in H^3(J; H^2(\Omega))$, then we have*

$$\max_{0 \leq n \leq N} \|u^n - U^n\| \leq C(u) \left(\left(h + \sqrt{\lambda} + \frac{\lambda}{h} \right)^2 + \tau^2 \right),$$

where $C(u) = C \left\{ \|u_0\|_{H^6(\Omega)}^2 + \|v_0\|_{H^6(\Omega)}^2 + \|u\|_{H^2(\mathcal{Y})}^2 \right\}^{\frac{1}{2}}$.

Proof Applying the triangle inequality to

$$u^n - U^n = u^n - u_h^n + u_h^n - U^n,$$

followed by estimates (4.5)–(4.6), Theorem 4.2 and Lemma 5.3 leads to desire result. $\square$

Remark 5.1 (a) The proposed fully discrete finite element scheme can be easily extended for the numerical approximation of the solutions to the following IBVP

$$u'' + \sigma u' + \delta u - \nabla \cdot (\epsilon \nabla u' + \beta \nabla u) = f \quad \text{in } \Omega \times (0, T], \tag{5.19}$$

coupled with the following jump conditions

$$[u] = 0, \quad \left[\epsilon(x) \frac{\partial u'}{\partial \mathbf{n}} + \beta(x) \frac{\partial u}{\partial \mathbf{n}} \right] = g \quad \text{along } \Gamma \times [0, T]. \tag{5.20}$$

For numerical validation, we refer to numerical examples 6.1.-6.2.

(b) In developing numerical methods for interface problems, higher order of convergence is always one of the major research goals, because high order methods are more accurate and cost-efficient. Present analysis provides a scope for the generalization of these works to higher order finite element methods by combining the theory in this work with the analysis in [17]. A higher order finite element approximation and its convergence is illustrated in Example 6.3.

6 Numerical Results

In this section, we present some numerical experiments to validate the theoretical findings presented in the previous section. To illustrate the flexibility of the method, different forms of interfaces along with a large scale of variation in the physical coefficients are considered. The nodes of the triangulations of Ω_1 and Ω_2 coincide on the interface Γ as stated in Sect. 2. All the numerical computations are done in the time interval $J = (0, 1]$.

Our main emphasis here is to understand the behavior of the true errors obtained in Theorem 5.1 on uniform meshes with uniform time steps. For each quantities of interest we observe its experimental order of convergence (EOC). For a given finite sequence of successive runs (indexed by i), let
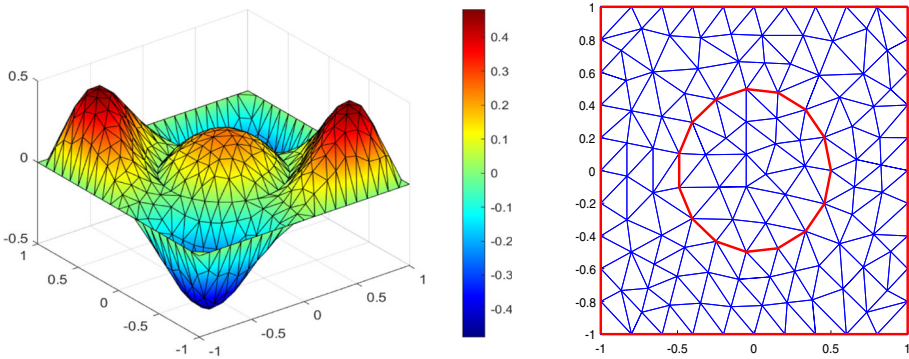
Fig. 3 Exact solution (left) and triangulation(right) of Ω with $h = 0.305091$ (Test Example 6.1)

Table 1 Parameters used in computation (see, Xu et al. [39])

	Parameters	Domain Ω_1	Domain Ω_2
$k(W/mK)$	Thermal conductivity	0.235	0.445
$\rho(kg/m^3)$	Skin density	1500	1116
$c(J/kgK)$	Specific heat	3600	3300
$w_b(kg/m^3 s)$	Blood perfusion rate	0.5	0.5
$c_b(J/kgK)$	Specific heat of blood	3770	43770
$\rho_b(kg/m^3)$	Density of blood	1060	1060

- $e(i) =$ the error corresponding to the L^2-norm and H^1-norm on the ith iteration and
- $h(i) =$ the mesh size of the run i .

Then the experimental order of convergence (EOC) is computed by

$$EOC(i+1) = \frac{log(e(i+1)/e(i))}{log(h(i+1)/h(i))}.$$

Example 6.1 For our first numerical experiment, we consider a square domain $\Omega = (-1, 1) \times (-1, 1)$, where interface Γ is a circle centered at $(0, 0)$ with radius 0.5. We select the data in (5.19)-(5.20) such that the exact solution u is given by

$$u(x, y, t) = \begin{cases} (r_0^2 - r^2)t^2 & \text{if } r \leq r_0, \\ (r_0^2 - r^2)t \sin(\pi x) \sin(\pi y) & \text{if } r > r_0, \end{cases}$$

where $r^2 = x^2 + y^2$ and $r_0 = 0.5$.

In Fig. 3 we show the exact solution and triangulation of the domain Ω with mesh size $h = 0.305091$. In our numerical convergence test, we choose two different sets of physical coefficients borrowed from Xu et al. [39] that corresponds to two different forms of bio heat transfer model. Following [39], physical parameters employed in the computation are as in Table 1. Dual-phase-lag (DPL) bio heat transfer is characterized by thermal relaxation time τ_q and phase lag for temperature gradient τ_T. Vedavarz et al. [37] found that τ_q for some biological tissues lies in the range of $1s - 100s$ at room temperature. Following the paper by Mitra et al. [22], we take the thermal lag time (τ_q) and phase lag time (τ_T) as 16 s and

Table 2 Example 6.1. EOC for $\tau_T \neq 0$ at $t = 1$ and $\tau = 10^{-3}$

h	$\|u - u_h\|_{L^2(\Omega)}$	EOC	$\|u - u_h\|_{H^1(\Omega)}$	EOC
0.3050910	5.03668e-002	–	9.99850e-001	–
0.1673780	1.11142e-002	2.2414	4.64219e-001	1.1025
0.0828717	2.77700e-003	2.1093	2.30304e-001	0.9971
0.0420952	7.65119e-004	2.0718	1.20741e-001	1.0877

Table 3 Example 6.1. EOC for $\tau_T = 0$ at $t = 1$ and $\tau = 10^{-3}$

h	$\|u - u_h\|_{L^2(\Omega)}$	EOC	$\|u - u_h\|_{H^1(\Omega)}$	EOC
0.3050910	1.17510e-002	–	7.11408e-001	–
0.1673780	3.24310e-003	2.1444	3.54219e-001	1.1615
0.0828717	7.31190e-004	2.1191	1.70304e-001	1.0418
0.0420952	1.85119e-004	2.0280	8.47410e-002	1.0305

0.043 s, respectively. Then using Table 1, we have the first set of physical coefficients for the DPL bio heat model:

$$(\sigma, \delta, \epsilon, \beta) = \left(\frac{\tau_q w_b \rho_b c_b + \rho c}{\tau_q \rho c}, \frac{w_b \rho_b c_b}{\tau_q \rho c}, \frac{\tau_T \kappa}{\tau_q \rho c}, \frac{\kappa}{\tau_q \rho c} \right)$$

$$= \begin{cases} \left(0.4325, 0.0231, 1.1696 \times 10^{-10}, 2.7199 \times 10^{-9} \right) & \text{if } r \leq r_0, \\ \left(0.6050, 0.0339, 1.2083 \times 10^{-7}, 7.5520 \times 10^{-9} \right) & \text{if } r > r_0. \end{cases}$$

In the absence of phase lag time (τ_T), Eq. (1.1) reduces to the thermal wave model of bio heat transfer [10,11]. The second set of physical coefficients that corresponds to the thermal wave model of bio heat transfer is given by

$$(\sigma, \delta, \epsilon, \beta) = \begin{cases} \left(0.4325, 0.0231, 0, 2.7199 \times 10^{-9} \right) & \text{if } r \leq r_0, \\ \left(0.6050, 0.0339, 0, 7.5520 \times 10^{-9} \right) & \text{if } r > r_0. \end{cases}$$

Tables 2 and 3 represent the numerical solution errors and convergence rates in both L^2 and H^1 norms for $\tau_T \neq 0$ (DPL bio heat transfer) and $\tau_T = 0$ (thermal wave bio heat transfer), respectively. In both cases, we choose the uniform time step size $\tau = 10^{-3}$. The errors at time $t = 1$ are listed in the Tables 2 and 3. Figure 4 clearly demonstrates the second order of convergence in L^2 norm and first order of convergence in H^1 norm. Note that the second set of physical coefficients are chosen to emphasize the fact that our numerical scheme is consistent for the thermal wave model of bio heat transfer and is clearly depicted in Table 3.

Example 6.2 For our second numerical example, we consider the interface to be a curve given by $y = x^2$ in the computational domain $\Omega = (-1, 1) \times (-1, 1)$. We select the data appearing in (5.19)–(5.20) setting the exact solution as

$$u(x, y, t) = \begin{cases} 0.25 \exp(t)(y - x^2) \sin(\pi x) \sin(\pi y) & \text{if } y \leq x^2, \\ -5t^2(y - x^2)(y - 1) & \text{if } y > x^2. \end{cases}$$

With the development of high-power short impulse lasers, use of dual-phase-lag (DPL) model has become common in the study of heat transport in metallic films during ultrafast laser
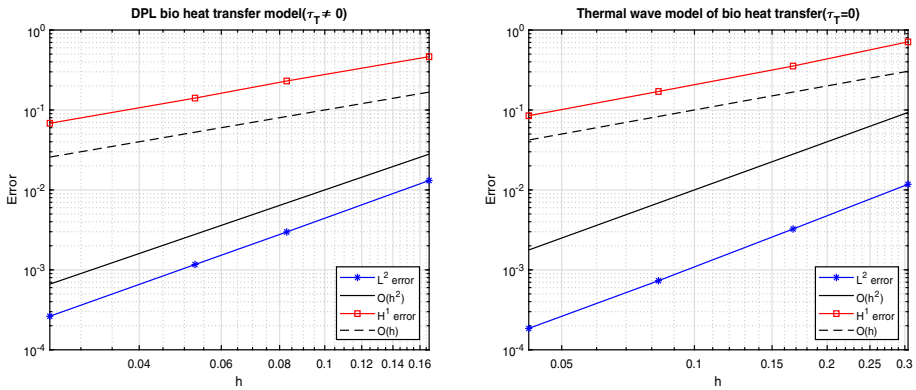
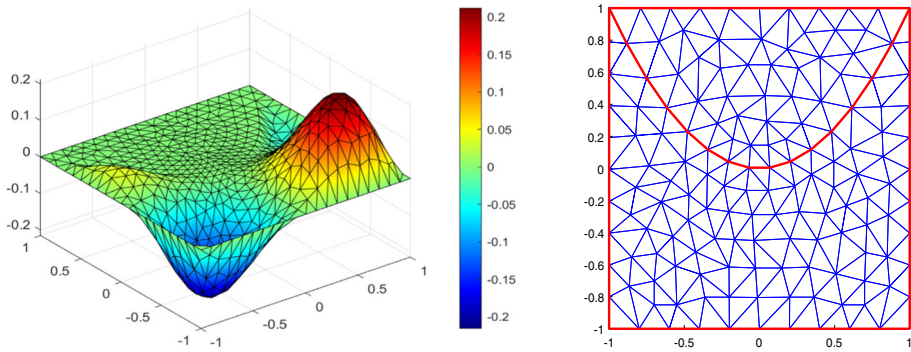**Fig. 4** Log-log plot of the L^2 norm and H^1 norm versus the mesh size at time $t = 1$ in Example 6.1

Fig. 5 Exact solution (left) and triangulation(right) of Ω with $h = 0.286172$ (Test Example 6.2)

Table 4 Example 6.2. EOC at $t = 10^{-12}$ and $\tau = 10^{-14}$

h	$\|u - u_h\|_{L^2(\Omega)}$	EOC	$\|u - u_h\|_{H^1(\Omega)}$	EOC
0.2861720	1.14180e-002	–	2.16910e-001	–
0.1656240	3.05842e-003	2.4088	1.09996e-001	1.2417
0.0888431	7.67597e-004	2.2195	5.48601e-002	1.1169
0.0478408	1.94671e-004	2.2164	2.74689e-002	1.1175

heating [28,34]. The phase-lag time varies for different materials and it may take values in the range of $10^{-3}s - 10^3 s$ for heterogeneous materials (cf. [21]). To mark the significance of our model problem, we choose the physical coefficients from the paper by Tzou et al. [32]

$$(\sigma, \delta, \epsilon, \beta) = \left(\frac{C_E^2}{\alpha_E}, 0, \alpha_e, C_E^2\right)$$

$$= \begin{cases} (1.2 \times 10^{12}, 0, 1.2 \times 10^{-4}, 1.44 \times 10^8) & \text{if } y \leq x^2, \\ (1.2 \times 10^{12}, 0, 1.6 \times 10^{-4}, 1.96 \times 10^8) & \text{if } y > x^2. \end{cases}$$

Fig. 6 Log-log plot of the L^2 norm and H^1 norm versus the mesh size at time $t = 10^{-12}$ in Example 6.2
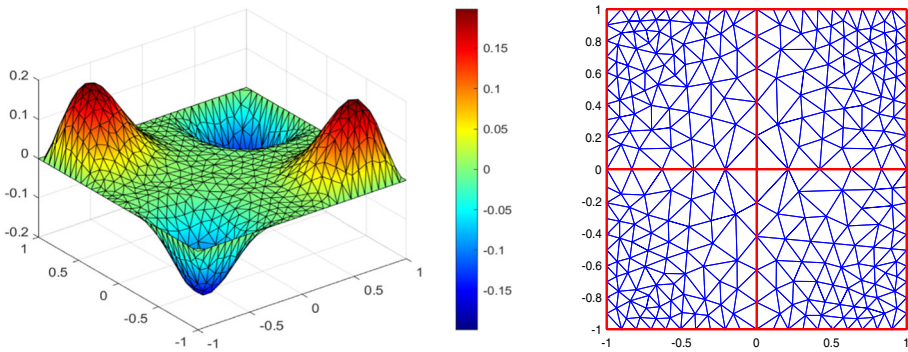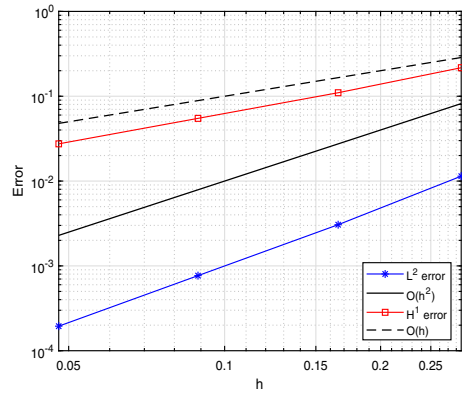

Fig. 7 Exact solution (left) and triangulation(right) of Ω with $h = 0.2969850$ (Test Example 6.3)

Here C_E represents the equivalent thermal wave speed, α_E denotes the equivalent thermal diffusivity and α_e is the electron thermal diffusivity of the material. In Fig. 5, we show the exact solution and the triangulation of the domain Ω with mesh size $h = 0.285956$. The numerical solution errors and convergence rates in both L^2 and H^1 norms at final time $t = 10^{-12}$ are listed in Table 4. The final time step is taken in pico-second (ps) as the thermal lagging model describes the pico-second (ps) heat transport in metal films (cf. [32,34]). It is clear from Fig. 6 that we have achieved optimal order of convergence in both L^2 and H^1 norms, which confirm the theoretical prediction as proved in Theorem 5.1

Example 6.3 For our final numerical example, the computational domain $\Omega = (-1, 1) \times (-1, 1)$ is divided into four subdomains Ω_i, $i = 1, 2, 3, 4$ using the interface $\Gamma := \{(x, y) \in \Omega : xy = 0\}$. We select the data appearing in (5.19)–(5.20) setting the exact solution as

$$u(x, y, t) = \begin{cases} -0.5x^2 \sin(\pi x) \sin(\pi y^2) & \text{if } (x, y) \in \Omega_1, \\ 0.5ty^2 \sin(\pi x^2) \sin(\pi y) & \text{if } (x, y) \in \Omega_2, \\ 0.5 \sin(t)x^2 \sin(\pi x) \sin(\pi y^2) & \text{if } (x, y) \in \Omega_3, \\ -0.5y^2 \sin(\pi x^2) \sin(\pi y) & \text{if } (x, y) \in \Omega_2. \end{cases}$$

In Fig. 7, we show the exact solution and triangulation of the domain Ω with mesh size $h = 0.2969850$. Equation (1.2) also represents the linearized Westervelt's equation for classical model for nonlinear ultrasound propagation through thermoviscous fluids [25].

Table 5 Parameters used in computation for Example 6.3 (cf. [3,25])

Parameters	Domain Ω_1	Domain Ω_2	Domain Ω_3	Domain Ω_4
σ	5	4	4	7
δ	0	0	0	0
ϵ	6×10^{-9}	4×10^{-9}	4×10^{-2}	4×10^{-9}
β	1500^2	1000^2	1000^2	3000^2

Table 6 Example 6.3 EOC at $t = 1$ and $\tau = 10^{-2}$ for $\mathbb{P}_1$ elements

h	$\|u - u_h\|_{L^2(\Omega)}$	EOC	$\|u - u_h\|_{H^1(\Omega)}$	EOC
0.2969850	9.90197e-003	–	2.39631e-002	–
0.1542410	2.70301e-003	1.9817	1.20437e-002	1.0501
0.0765776	6.69095e-004	1.9940	6.01226e-003	0.9922
0.0410173	1.80135e-004	2.1019	2.92771e-003	1.1526

Table 7 Example 6.3 EOC at $t = 1$ and $\tau = 10^{-2}$ for $\mathbb{P}_2$ elements

h	$\|u - u_h\|_{L^2(\Omega)}$	EOC	$\|u - u_h\|_{H^1(\Omega)}$	EOC
0.2969850	4.75802e-004	–	2.77468e-002	–
0.1542410	6.63167e-005	3.0077	7.11948e-003	2.0762
0.0781625	8.56805e-006	3.0106	1.82065e-003	2.0062
0.0435878	1.47599e-006	3.0114	5.47462e-004	2.0576

Table 8 Example 6.3 EOC at $t = 1$ and $\tau = 10^{-2}$ for $\mathbb{P}_3$ elements

h	$\|u - u_h\|_{L^2(\Omega)}$	EOC	$\|u - u_h\|_{H^1(\Omega)}$	EOC
0.2969850	4.32487e-005	–	1.9120e-003	–
0.1542410	3.02790e-006	4.0587	2.60082e-004	3.0449
0.0781625	1.94952e-007	4.0353	2.97168e-005	3.1914
0.0435878	1.85633e-008	4.0266	4.63975e-006	3.1798

Following [3,25], in each subdomain we use different material parameters for the physical coefficients, given in Table 5.

Tables 6, 7 and 8 represent the numerical solution errors and convergence rates in both L^2 and H^1 norms for $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ finite elements, respectively. In all cases, we choose the uniform time step size $\tau = 10^{-2}$. The errors at time $t = 1$ are listed in the Tables 6, 7 and 8. Note that the finite element spaces $\mathbb{P}_2$ and $\mathbb{P}_3$ are chosen to emphasize the fact that our numerical scheme is consistent for the higher order finite element spaces under the assumption that $\lambda = O(h^3)$ and $\lambda = O(h^4)$, respectively. It is clear from Fig. 8 that we have achieved optimal order of convergence in both L^2 and H^1 norms which consolidates our theoretical findings.
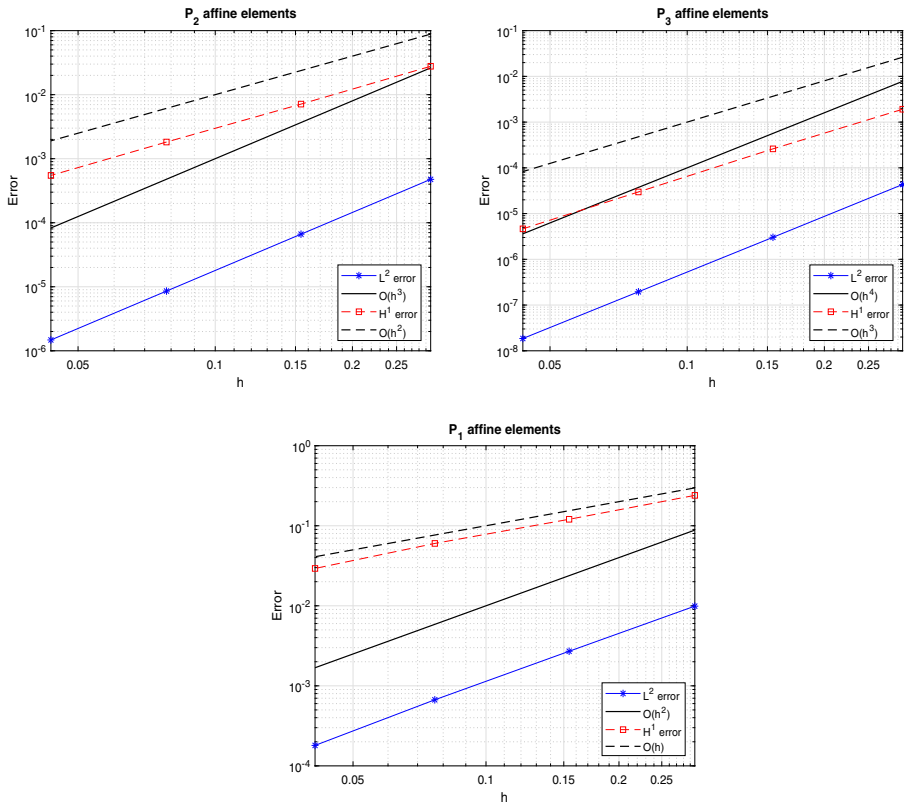
Fig. 8 Log-log plot of the L^2 norm and H^1 norm versus the mesh size at time $t = 1$ in Example 6.3

7 Conclusion

Time-dependent interface problems are frequently encountered in scientific computing and many applied sciences. The typical mathematical models are the heat or wave type equations with discontinuous coefficients, which arise when the physical processes involve two or more materials or media with non-identical properties. In this article, we have presented finite element analysis for linear general hyperbolic equations with interfaces. The discretization with respect to space is by the piecewise linear finite elements and in time we have applied the Crank-Nicolsion scheme by setting the governing equation as a first-order system in time. We have established second order convergence in time and optimal order convergence in space with respect to $L^\infty(L^2)$-norm. Present analysis provides a scope for the generalization of these works to higher order finite element methods under the assumption that $\lambda = O(h^{2p})$ and solution belongs to $L^\infty(L^2(\Omega) \cap H^{p+1}(\Omega_1 \cup \Omega_2))$, p is the order of approximating polynomial spaces (see, [17] for elliptic type problems with interfaces). Further, we believe that present work can be easily extended to following linearized Westervelt equation with variable coefficients (cf. [25])

$$\gamma(x,t)u'' + \sigma(x,t)u' - \nabla \cdot (\epsilon \nabla u' + \beta \nabla u) = f(x,t) \text{ in } \Omega \times (0,T], \qquad (7.1)$$

where $\Omega \subset \mathbb{R}^3$ is a bounded and convex domain with C^2 smooth interface Γ. It is worth to note that only semidiscrete error analysis has been discussed in Nikolić et al. [25] for non-interface problems.

Future work will be focussed on the extension of this theory to the Westervelt's quasi-linear acoustic wave equation

$$c^{-2}u'' - \nabla \cdot \big(\nabla u(x,t) + \beta(x)\nabla u'\big) = \gamma(u^2)'' \quad \text{in } (0,T] \times \Omega, \tag{7.2}$$

with interfaces. Equation (7.2) with interfaces are motivated by lithotripsy where a silicone acoustic lens focuses the ultrasound traveling through a nonlinearly acoustic fluid to a kidney stone (cf. [24]). In [24], the authors have investigated interface coupling problems involving Westervelt equation for different types of boundary conditions. Currently, we are working on the extension of the work of Nikolić et al. [25] for interface problems.

Appendix

Proof of Lemma 4.1: Taking $t \rightarrow 0^+$ in (4.1) and then using definition of $\mathcal{Q}_h$ operator, we obtain

$$
\begin{aligned}
(u_h''(0), v_h) &= -\mathcal{B}_{\sigma h}(u_h'(0), v_h) - \mathcal{B}_{\delta h}(u_h(0), v_h) \\
&\quad -\mathcal{A}_{\epsilon h}(u_h'(0), v_h) - \mathcal{A}_{\beta h}(u_h(0), v_h) + (f(0), v_h) \\
&= -\mathcal{B}_{\sigma h}(\mathcal{Q}_{\epsilon h}v_0, v_h) - \mathcal{B}_{\beta h}(\mathcal{Q}_{\beta h}u_0, v_h) \\
&\quad -\mathcal{A}_{\epsilon h}(\mathcal{Q}_{\epsilon h}v_0, v_h) - \mathcal{A}_{\beta h}(\mathcal{Q}_{\beta h}u_0, v_h) + (f(0), v_h) \\
&= -\mathcal{B}_{\sigma h}(\mathcal{Q}_{\epsilon h}v_0, v_h) - \mathcal{B}_{\beta h}(\mathcal{Q}_{\beta h}u_0, v_h) \\
&\quad -\mathcal{A}_{\epsilon}(v_0, v_h) - \mathcal{A}_{\beta}(u_0, v_h) + (f(0), v_h).
\end{aligned}
\tag{7.3}
$$

Here, we have used the fact that $u_h \in C^2(J; V_h)$. For the third term and fourth term in (7.3), we use Green's formula and boundary condition to derive

$$
\begin{aligned}
\mathcal{A}_{\epsilon}(v_0, v_h) &= -(\nabla \cdot (\epsilon \nabla v_0), v_h) \leq C\|v_0\|_2 \|v_h\|, \\
\mathcal{A}_{\beta}(u_0, v_h) &= -(\nabla \cdot (\beta \nabla u_0), v_h) \leq C\|u_0\|_2 \|v_h\|.
\end{aligned}
$$

Hence, (7.3) yields

$$\|u_h''(0)\| \leq C\big(\|u_0\|_2 + \|v_0\|_2 + \|f\|_{H^1(L^2)}\big). \tag{7.4}$$

In the previous estimate, we have used the fact that

$$\sup_{0 \leq t \leq T} \|f(t)\| \leq C(T)\|f\|_{H^1(J;W)}.$$

In fact, for any Banach space $\mathcal{B}$, we know that (cf. [30], Proposition 7.1)

$$\sup_{0 \leq t \leq T} \|v(t)\|_{\mathcal{B}} \leq C(T)\|v\|_{H^1(J;\mathcal{B})} \quad \forall v \in H^1(J;\mathcal{B}). \tag{7.5}$$

Also, from the definition of $\mathcal{Q}_h$ operator, we can easily derive

$$\|u_h'(0)\|_1 = \|\mathcal{Q}_{\epsilon h}v_0\|_1 \leq C\|v_0\|_1. \tag{7.6}$$

For $i = 3$, taking $t \to 0^+$ in (2.3) and using (7.3), we have

$$
\begin{aligned}
(u_h''(0) - u''(0), v_h) &= \mathcal{B}_\sigma(v_0, v_h) - \mathcal{B}_{\sigma h}(\mathcal{Q}_{\epsilon h} v_0, v_h) + \mathcal{B}_\delta(u_0, v_h) \\
&\quad - \mathcal{B}_{\delta h}(\mathcal{Q}_{\beta h} u_0, v_h) \\
&= \mathcal{B}_\sigma^\Delta(v_0, v_h) + \mathcal{B}_{\sigma h}(v_0 - \mathcal{Q}_{\epsilon h} v_0, v_h) + \mathcal{B}_\delta^\Delta(u_0, v_h) \\
&\quad + \mathcal{B}_{\delta h}(u_0 - \mathcal{Q}_{\beta h} u_0, v_h) \\
&\leq C(h^2 + \lambda)(\|u_0\|_2 + \|v_0\|_2)\|v_h\|.
\end{aligned}
\tag{7.7}
$$

In the last inequality, we have used Lemmas 3.2 and 3.4, and the fact that $u \in C^2(J; W)$. Then use definition of L^2 projection and (7.7) to obtain

$$
\begin{aligned}
(u_h''(0) - \mathcal{L}_h u''(0), v_h) &= (u_h''(0) - u''(0), v_h) \\
&\leq C(h^2 + \lambda)(\|u_0\|_2 + \|v_0\|_2)\|v_h\|,
\end{aligned}
\tag{7.8}
$$

which imply

$$
\|u_h''(0) - \mathcal{L}_h u''(0)\| \leq Ch(\|u_0\|_2 + \|v_0\|_2).
\tag{7.9}
$$

Estimate (7.9) together with inverse inequality and (3.19) yields

$$
\begin{aligned}
\|u_h''(0)\|_1 &\leq Ch^{-1}\|u_h''(0) - \mathcal{L}_h u''(0)\| + \|\mathcal{L}_h u''(0)\|_1 \\
&\leq C(\|u_0\|_3 + \|v_0\|_3 + \|f\|_{H^1(H^1)}).
\end{aligned}
\tag{7.10}
$$

Next, for $u_h \in C^3(J; V_h)$, we differentiate (4.1) with respect to t and then take $t \to 0^+$ to have

$$
\begin{aligned}
(u_h'''(0), v_h) &= -\mathcal{B}_{\sigma h}(u_h''(0), v_h) - \mathcal{B}_{\delta h}(u_h'(0), v_h) \\
&\quad - \mathcal{A}_{\epsilon h}(u_h''(0), v_h) - \mathcal{A}_{\beta h}(u_h'(0), v_h) + (f'(0), v_h) \\
&= -\mathcal{B}_{\sigma h}(u_h''(0), v_h) - \mathcal{B}_{\delta h}(\mathcal{Q}_{\epsilon h} v_0, v_h) \\
&\quad - \mathcal{A}_{\epsilon h}(u_h''(0) - \mathcal{Q}_{\epsilon h} u''(0), v_h) - \mathcal{A}_{\beta h}(\mathcal{Q}_{\epsilon h} v_0 - \mathcal{Q}_{\beta h} u'(0), v_h) \\
&\quad - \sum_{l=1}^{2} \left\{ \mathcal{A}_\epsilon^l(u''(0), v_h) + \mathcal{A}_\beta^l(u'(0), v_h) \right\} + (f'(0), v_h).
\end{aligned}
$$

Now, for $u \in H^3(J; \mathcal{Y})$ or equivalently $u \in C^2(J; \mathcal{Y})$, use the fact that

$$
\left[\epsilon(x) \frac{\partial u''(t)}{\partial \mathbf{n}} + \beta(x) \frac{\partial u'(t)}{\partial \mathbf{n}} \right] = 0 \quad \text{along } \Gamma \times [0, T]
$$

in the Eq. (7.11) to obtain

$$
\begin{aligned}
(u_h'''(0), v_h) &= -\mathcal{B}_{\sigma h}(u_h''(0), v_h) - \mathcal{B}_{\delta h}(\mathcal{Q}_{\epsilon h} v_0, v_h) \\
&\quad - \mathcal{A}_{\epsilon h}(u_h''(0) - \mathcal{Q}_{\epsilon h} u''(0), v_h) - \mathcal{A}_{\beta h}(\mathcal{Q}_{\epsilon h} v_0 - \mathcal{Q}_{\beta h} v_0, v_h) \\
&\quad + \sum_{l=1}^{2} \left\{ (\nabla \cdot \epsilon_l \nabla u''(0), v_h)_{\Omega_l} + (\nabla \cdot \beta \nabla u'(0), v_h)_{\Omega_l} \right\} + (f'(0), v_h) \\
&\leq C \Bigg\{ \|u_h''(0)\| + h^{-1}(\|u_h''(0) - \mathcal{Q}_{\epsilon h} u''(0)\|_1 + \|\mathcal{Q}_{\epsilon h} v_0 - \mathcal{Q}_{\beta h} v_0\|_1) \\
&\quad + \sum_{l=1}^{2} \|u''(0)\|_{2, \Omega_l} + \|v_0\|_2 + \|f\|_{H^2(L^2)} \Bigg\} \|v_h\|.
\end{aligned}
\tag{7.11}
$$

From (7.8) and Remark 3.1, we have

$$
\begin{aligned}
&\|u_h''(0) - \mathcal{Q}_{\epsilon h} u''(0)\|_1 \\
&\quad \le Ch^{-1}\|u_h''(0) - \mathcal{L}_h u''(0)\| + \|\mathcal{L}_h u''(0) - \mathcal{Q}_{\epsilon h} u''(0)\|_1 \\
&\quad \le C(\|u_0\|_2 + \|v_0\|_2) + Ch \sum_{l=1}^{2} \|u''(0)\|_{2,\Omega_l} \\
&\quad \le Ch(\|u_0\|_4 + \|v_0\|_4 + \|f\|_{H^1(H^2)}).
\end{aligned}
\tag{7.12}
$$

In the last inequality, we have used the fact that $\|u_0\|_K \le Ch\|u_0\|_{2,K}$ for all $K \in \mathcal{T}_h$. Using (7.12) in (7.11), we obtain

$$
\|u_h'''(0)\| \le C(\|u_0\|_4 + \|v_0\|_4 + \|f\|_{H^2(H^2)}).
\tag{7.13}
$$

The case $i = 4$ can be done in a similar way and hence details are omitted. This completes the rest of the proof. □

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