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Inertial-Type Algorithm for Solving Split Common Fixed Point Problems in Banach Spaces

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Abstract
In this paper, motivated by the works of Kohsaka and Takahashi (SIAM J Optim 19:824–835, 2008) and Aoyama et al. (J Nonlinear Convex Anal 10:131–147, 2009) on the class of mappings of firmly nonexpansive type, we explore some properties of firmly nonexpansive-like mappings [or mappings of type (P)] in p-uniformly convex and uniformly smooth Banach spaces. We then study the split common fixed point problems for mappings of type (P) and Bregman weak relatively nonexpansive mappings in p-uniformly convex and uniformly smooth Banach spaces. We propose an inertial-type shrinking projection algorithm for solving the two-set split common fixed point problems and prove a strong convergence theorem. Also, we apply our result to the split monotone inclusion problems and illustrate the behaviour of our algorithm with several numerical examples s. The implementation of the algorithm does not require a prior knowledge of the operator norm. Our results complement many recent results in the literature in this direction. To the best of our knowledge, it seems to be the first to use the inertial technique to solve the split common fixed point problems outside Hilbert spaces.

Keywords Split common fixed point problem · Firmly nonexpansive-like mapping · Bregman weak relatively nonexpansive mappings · Inertial-type algorithm · Banach space

Mathematics Subject Classification 47H10 · 47J25 · 47N10 · 65J15 · 90C33

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1 Introduction

Let X be a nonempty set and $T : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of T if $Tx = x$. We shall denote the set of fixed points of T by $F(T)$. The identity mapping on X is denoted by I.

Let E_1 and E_2 be real Banach spaces and $A : E_1 \to E_2$ be a bounded linear operator with the adjoint operator A^*. Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively. The Split Feasibility Problem (in short, SFP) can be formulated as:

$$\text{find } x \in C \text{ such that } Ax \in Q. \tag{1.1}$$

The SFP was first introduced by Censor and Elfving [13] in the framework of Hilbert spaces for modeling inverse problems which arise from phase retrievals and medical image reconstruction. The SFP has applications in signal processing, radiation therapy, data denoising and data compression (see [4,10,12,14,19,22,50] for details).

A generalization of the SFP (1.1) is the Split Common Fixed Point Problem (in short, SCFPP). Let $T_i : E_1 \to E_1, i = 1, 2, ..., n$ and $U_j : E_2 \to E_2, j = 1, 2, ..., m$ be nonlinear mappings such that $F(T_i)$ and $F(U_j)$ are nonempty. The SCFPP is formulated as:

$$\text{find} \quad x \in \cap_{i=1}^{m} F(T_i) \quad \text{such that} \quad Ax \in \cap_{j=1}^{n} F(U_j). \tag{1.2}$$

In particular, for $m = n = 1$, then SCFPP (1.2) becomes the two-set SCFPP, which is formulated as:

$$\text{find } x \in F(T) \text{ such that } Ax \in F(U). \tag{1.3}$$

The two-set SCFPP (1.3) was first studied by Censor and Segal [15] in the framework of Hilbert spaces for the case where T and U are nonexpansive mappings. They proposed the following algorithm:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = T[x_n - \gamma A^*(I - U)Ax_n], \end{cases}$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A, and under some suitable conditions proved a weak convergence theorem. In 2011, Moudafi [32] also studied the SCFPP for quasi-nonexpansive mappings in infinite-dimensional Hilbert spaces. By modifying the Mann's iteration, Moudafi [32] proposed the following algorithm (1.4) for solving the two-set SCFPP and obtained a weak convergence theorem:

$$\begin{cases} x_0 \in C, \\ y_n = x_n - \gamma\beta A^*(I - U)Ax_n, \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n T y_n, \end{cases} \tag{1.4}$$

where $I - T$ and $I - U$ are demiclosed at zero, $\gamma \in (0, \frac{1}{\lambda\beta})$ for $\beta \in (0, 1)$ and λ being the spectral radius of the operator A^*A.

Recently, some authors have studied the SCFPP (1.3) for a pair of mappings of different classes in Banach spaces. In 2015, Tang. et al. [53] studied the SCFPP (1.3) for an asymptotic nonexpansive mapping and a τ-quasi-strict pseudocontractive mapping in the setting of two Banach spaces. They proved weak and strong convergence theorems.

Let E be a smooth, strictly convex and reflexive Banach space. Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let B be a maximal monotone operator of H into 2^H such that $dom(B) \subset C$. Let $A : H \to E$ be a bounded linear operator such that $A \neq 0$, $\{T_i\}_{i=1}^{\infty} : C \to H$ be an infinite family of k_i-demimetric and demiclosed mappings and U be a firmly nonexpansive-like mapping on E. In 2018, Y. Song [43] studied the generalized split feasibility problem of the form:

$$\text{find } x \in \cap_{i=1}^{\infty} F(T_i) \cap B^{-1}(0) \cap A^{-1}F(U). \tag{1.5}$$

It is noted that if $B = 0$, then (1.5) becomes the SCFPP (1.3). They proposed an Halpern type iterative algorithm for solving the problem (1.5) and proved a strong convergence theorem.

Also very recently, Takahashi [52] studied the SCFPP for generalized demimetric mappings in two Banach spaces. They used the hybrid method and shrinking projection method to find a solution to the problem and proved strong convergence theorems.

We note that the algorithms proposed for the SCFPP in [43,52,53] require a prior estimate of the norm of the bounded linear operator. This in practice is not always easy to compute. For more on SCFPP and related optimization problem, see [23–25,27,35,41,47,48,51,54] and the references therein.

In fixed point theory, it is more desirable to work with an algorithm that has a high rate of convergence. A way of achieving this is by incorporating inertial term in the algorithm. This idea was proposed originally by Polyak [37]. It can be seen as a discrete version of a second-order time dynamical system used to speed up convergence rate of the smooth convex minimization problem. The main idea of these methods is to make use of two previous iterates to update the next iterate, which results in speeding up the algorithm's convergence. Recently, authors have shown considerable interest in studying inertial type algorithms, see for example [26,28,42,54] and the references therein.

Motivated by the above works, in this paper we study the two-set SCFPP for mappings of type (P) and Bregman weak relatively nonexpansive mappings in p- uniformly convex and uniformly smooth Banach spaces. We propose an inertial-type shrinking projection algorithm with the step size independent on the prior estimate of the norm of the bounded linear operator and prove strong convergence theorem. Our result seems to be the first to consider an inertial-type algorithm for SCFPP in Banach spaces.

This paper is organized as follows. In Sect. 2, we give some useful definitions, notations and lemmas, which are needed for the analysis of our algorithm. In Sect. 3, the algorithm and its strong convergence theorem are presented. In Sect. 4, we apply our main result to the split monotone inclusion problem. In Sect. 5, we give numerical examples to illustrate the behaviour of our algorithm. We conclude in Sect. 6.

2 Preliminaries

In this section, we give some definitions and results which will be needed in proving our main result in the next section.

Let E be a real Banach space with the norm $\|\cdot\|$, C be a nonempty closed convex subset of E and E^* be the dual with the norm $\|\cdot\|_*$. We shall denote the value of the functional $x^* \in E^*$ at $x \in E$ by $\langle x^*, x\rangle$. For a sequence $\{x_n\}$ of E and $x \in E$, we denote the strong convergence of $\{x_n\}$ and weak convergence of $\{x_n\}$ to x by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|_*^2\}, \tag{2.1}$$

for all $x \in E$. Let $U := \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for all $x, y \in U$. E is said to be *strictly convex* if $\|x + y\| < 2$ whenever $x, y \in E$ and $x \neq y$. Let $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\right\}.$$

E is said to be *uniformly convex* if $\delta_E(\epsilon) > 0$ and p*-uniformly convex* if there exists a constant $C_p > 0$ such that $\delta_E(\epsilon) \geq C_p\epsilon^p$, for any $\epsilon \in (0, 2]$. The L_p space is 2-uniformly convex for $1 < p \leq 2$ and p-uniformly convex for $p \geq 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive.

The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}.$$

E is called *uniformly smooth* if $\lim_{\tau \to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$ and q*-uniformly smooth* if there exists $C_q > 0$ such that $\rho_E(\tau) \leq C_q\tau^q$. Every uniformly smooth Banach space is smooth and reflexive. The *generalized duality mapping* $J_p^E : E \to 2^{E^*}$ is defined by

$$J_p^E(x) = \{u^* \in E^* : \langle u^*, x\rangle = \|x\|^p,\ \|u^*\|_* = \|x\|^{p-1}\}. \tag{2.3}$$

If $p = 2$, (2.3) becomes the normalized duality mapping (2.1). It is known that $J_p^E(x) = \|x\|^{p-2}J(x)$ for all $x \in X, x \neq 0$. It is also known that E is uniformly smooth if and only if J_p^E is norm-to-norm uniformly continuous on bounded subsets of E and E is smooth if and only if J_p^E is single valued. Moreover, E is p-uniformly convex (smooth) if and only if E^* is q-uniformly smooth (convex). If E is p-uniformly convex and uniformly smooth, then the duality mapping J_p^E is norm-to-norm uniformly continuous on bounded subsets of E (see [17,30,56]). Examples of generalized duality mapping are given below:

Example 2.1 [1] Let $E := \ell_p(\mathbb{R})$ and $x = (x_1, x_2, x_3, \ldots) \in \ell_p$ $(1 < p < \infty)$. Then the generalized duality mapping J_p^E is given by

$$J_p^E(x) = (|x_1|^{p-1}sgn(x_1), |x_2|^{p-1}sgn(x_2), \ldots).$$

Example 2.2 [1] Let $E := L_p([\alpha, \beta])$ $(1 < p < \infty)$, where $\alpha, \beta \in \mathbb{R}$ and let $f \in E$. Then the generalized duality mapping J_p^E is given by

$$J_p^E(f)(t) = |f(t)|^{p-1}sgn(f(t)).$$

Xu and Roach [56] proved the following inequality for q-uniformly smooth Banach spaces.

Lemma 2.3 *Let $x, y \in E$. If E is a q-uniformly smooth Banach space, then there exists a $C_q > 0$ such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle J_q^{E^*}(x), y\rangle + C_q\|y\|^q.$$

Definition 2.4 A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be

(1) *proper* if its effective domain $D(f) = \{x \in E : f(x) < +\infty\}$ is nonempty,
(2) *convex* if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for every $\lambda \in (0, 1), x, y \in D(f)$,
(3) *lower semicontinuous* at $x_0 \in D(f)$ if $f(x_0) \leq \liminf_{x \to x_0} f(x)$.

Let $x \in \operatorname{int}\operatorname{dom} f$. For any $y \in E$, the *right-hand derivative* of f at x denoted by $f^0(x, y)$ is defined by

$$f^0(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.4}$$

If the limit as $t \to 0$ in (2.4) exists for any y, then the function f is said to be *Gâteaux differentiable* at x (see, for instance [36], Definition 1.3, p. 3). In this case the gradient of f at x is the function $\nabla f(x)$ which is defined by $\langle \nabla f(x), y\rangle = f^0(x, y)$ for any $y \in E$. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \operatorname{int}\operatorname{dom} f$ (see also, [9]).

Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function and $x \in \operatorname{int}\operatorname{dom} f$. The *subdifferential of* f *at* x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(y) \geq \langle x^*, y - x\rangle + f(x) \quad \forall \quad y \in E\}.$$

If $\partial f(x) \neq \emptyset$, then f is said to be subdifferentiable at x.

Given a Gâteaux differentiable function f, the bifunction $\Delta_f : E \times E \to [0, +\infty)$ given as

$$\Delta_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y\rangle, \ \forall\, x, y \in E \tag{2.5}$$

is called the *Bregman distance* with respect to f, (see [49]). In particular, let $f(x) = \frac{1}{p}\|x\|^p$. In this case the duality mapping J_p^E is the derivative of f. The Bregman distance $\Delta_p : E \times E \to [0, +\infty)$ is defined by

$$\begin{aligned} \Delta_p(x, y) &:= \frac{\|x\|^p}{p} - \frac{\|y\|^p}{p} - \langle J_p^E(y), x - y\rangle \\ &= \frac{\|x\|^p}{p} + \frac{\|y\|^p}{q} - \langle J_p^E(y), x\rangle. \end{aligned} \tag{2.6}$$

Note that $\Delta_p(x, y) \geq 0$ and $\Delta_p(x, y) = 0$ if and only if $x = y$ (see e.g. [7]). In general, the Bregman distance is not symmetric and is not a metric. However, it possesses some distance-like properties. From (2.6) one can show that the following equality called three-point identity is satisfied:

$$\Delta_p(x, y) + \Delta_p(y, z) - \Delta_p(x, z) = \langle J_p^E(z) - J_p^E(y), x - y\rangle \ \forall\, x, y, z \in E. \tag{2.7}$$

In particular,

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle J_p^E(x) - J_p^E(y), x - y\rangle \ \forall\, x, y \in E.$$

For p-uniformly convex space, the metric and Bregman distance satisfy the following relation [40]:

$$\tau\|x - y\|^p \leq \Delta_p(x, y) \leq \langle J_p^E(x) - J_p^E(y), x - y\rangle, \tag{2.8}$$

where $\tau > 0$ is some fixed number. If $f(x) = \|x\|^2$, the Bregman distance is the Lyapunov functional $\phi : E \times E \to [0, +\infty)$ defined by

$$\phi(x, y) := \|x\|^2 - 2\langle Jy, x\rangle + \|y\|^2. \tag{2.9}$$

The metric projection

$$P_C x := \arg\min_{y \in C} \|x - y\|, \ x \in E,$$

is the unique minimizer of the norm distance (see [20]). It can be characterized by the following variational inequality:

$$\langle J_p^E(x - P_C x), z - P_C x\rangle \leq 0, \forall z \in C.$$

Moreover, the metric projection is nonexpansive, i.e. $\|P_C x - P_C y\| \leq \|x - y\|, \forall x, y \in C$. Similar to the metric projection, the Bregman projection defined by

$$\Pi_C(x) := argmin_{y\in C}\Delta_p(y, x),\ \ x \in E, \tag{2.10}$$

is the unique minimizer of the Bregman distance (see [39]). It can also be characterized by the variational inequality:

$$\langle J_p^E(x) - J_p^E(\Pi_C x), z - \Pi_C x\rangle \leq 0, \quad \forall z \in C,$$

from which one can derive that

$$\Delta_p(y, \Pi_C x) + \Delta_p(\Pi_C x, x) \leq \Delta_p(y, x), \quad \forall y \in C. \tag{2.11}$$

If E is a real Hilbert space, then $\Pi_C = P_C$, see [2,21] for details. Associated with the Bregman distance is the functional $V_p : E \times E^* \to [0, +\infty)$ defined by

$$V_p(x, \bar{x}) := \frac{1}{p}\|x\|^p - \langle \bar{x}, x\rangle + \frac{1}{q}\|\bar{x}\|^q, \quad x \in E, \bar{x} \in E^*.$$

Clearly, $V_p(x, \bar{x}) \geq 0$ and the following properties are satisfied:

$$V_p(x, \bar{x}) = \Delta_p(x, J_q^{E^*}(\bar{x})),\ \forall x \in E, \bar{x} \in E^*, \tag{2.12}$$

and

$$V_p(x, \bar{x}) + \langle \bar{y}, J_q^{E^*}(\bar{x}) - x\rangle \leq V_p(x, \bar{x} + \bar{y}),\ \forall x \in E, \bar{x}, \bar{y} \in E^*. \tag{2.13}$$

Also, V_p is convex in the second variable. Thus for all $z \in E$,

$$\Delta_p\left(z, J_q^{E^*}\left(\sum_{i=1}^N t_i J_p^E x_i\right)\right) \leq \sum_i^N t_i \Delta_p(z, x_i),$$

where $\{x_i\} \subset E$ and $\{t_i\} \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.
A point $x^* \in C$ is called an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}$ which converges weakly to x^* such that $\lim_{n\to\infty} \|x_n - T x_n\| = 0$. We denote the set of asymptotic fixed points of T by $\hat{F}(T)$. A point $x^* \in C$ is called a *strong asymptotic fixed point* of T if C contains a sequence $\{x_n\}$ which converges strongly to x^* such that $\lim_{n\to\infty} \|x_n - T x_n\| = 0$. We denote the set of strong asymptotic fixed points of T by $\tilde{F}(T)$. It follows from the definitions that $F(T) \subset \tilde{F}(T) \subset \hat{F}(T)$.

Definition 2.5 A mapping T from C to C is said to be

(1) *Bregman quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\Delta_p(x^*, T y) \leq \Delta_p(x^*, y),\ \forall y \in C, x^* \in F(T),$$

(2) *Bregman weak relatively nonexpansive* if $\tilde{F}(T) \neq \emptyset$, $\tilde{F}(T) = F(T)$ and

$$\Delta_p(x^*, T y) \leq \Delta_p(x^*, y),\ \forall y \in C, x^* \in F(T),$$

(3) *Bregman relatively nonexpansive* if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and

$$\Delta_p(x^*, T y) \leq \Delta_p(x^*, y),\ \forall y \in C, x^* \in F(T).$$

It is known that for a Bregman quasi-nonexpansive mapping $T : C \to C$, the fixed point set $F(T)$ is closed and convex (see [38]). From the definitions, it is clearly seen that the class of Bregman quasi-nonexpansive contains the class of Bregman weak relatively nonexpansive and the class of Bregman weak relatively nonexpansive contains the class of Bregman relatively nonexpansive. The next examples illustrate these inclusions.

Example 2.6 (See [16]) Let $E = \ell_2(\mathbb{R})$, where $\ell_2(\mathbb{R}) := \{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots), \sigma_i \in \mathbb{R} : \sum_{i=1}^{\infty} |\sigma_i|^2 < \infty\}$, $\|\sigma\| = (\sum_{i=1}^{\infty} |\sigma_i|)^{\frac{1}{2}}\ \forall\, \sigma \in E$ and let $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$. Let $\{x_n\} \subset E$ be a sequence defined by $x_0 = (1, 0, 0, 0, \ldots), x_1 = (1, 1, 0, 0, \ldots), x_2 = (1, 0, 1, 0, \ldots), \ldots, x_n = (\sigma_{n,1}, \sigma_{n,2}, \sigma_{n,3} \ldots), \ldots,$ where

$$\sigma_{n,k} = \begin{cases} 1 & \text{if} \quad k = 1, n+1, \\ 0 & \text{if} \quad \text{otherwise}, \quad \forall \quad n \geq 0, \end{cases}$$

$n \in \mathbb{N}$. Define the mapping $T : E \to E$ by

$$Tx = \begin{cases} \frac{n}{n+1}x & \text{if} \quad x = x_n, \\ -x & \text{if} \quad x \neq x_n. \end{cases}$$

It can be shown that T is a Bregman quasi-nonexpansive, precisely Bregman weak relatively nonexpansive mapping but not a Bregman relatively nonexpansive mapping (see also [34]).

The next example is a Bregman quasi-nonexpansive mapping which is neither Bregman weak relatively nonexpansive nor Bregman relatively nonexpansive.

Example 2.7 [34,44] Let E be a smooth Banach space, let k be an even number in $\mathbb{N}$ and let $f : E \to \mathbb{R}$ be defined by $f(x) = \frac{1}{k}\|x\|^k$, $x \in E$. Let $x_0 \neq 0$ be an element of E. Define the mapping $T : E \to E$ by

$$Tx = \begin{cases} (\frac{1}{2} + \frac{1}{2^{n+1}})x_0 & \text{if} \quad x = (\frac{1}{2} + \frac{1}{2^n})x_0, \\ -x & \text{if} \quad x \neq (\frac{1}{2} + \frac{1}{2^n})x_0, \end{cases}$$

for all $n \geq 0$. It can be verified that T is a Bregman quasi-nonexpansive mapping which is neither Bregman weak relatively nonexpansive nor Bregman relatively nonexpansive.

One of the most important class of nonlinear mappings in Hilbert space is the class of firmly nonexpansive mappings. It includes all metric projections onto a closed convex set and all resolvents of a monotone operator. Kohsaka and Takahashi [29] proposed the class of firmly nonexpansive type mappings, which contains the firmly nonexpansive mappings in Hilbert spaces and resolvents of maximal monotone operators in Banach spaces. It is classified into three types; namely, type (P), type (Q) and type (R). In this study, we consider the class of firmly nonexpansive-like mappings (or mappings of type (P)).

Definition 2.8 (See [6]) Let E be a smooth Banach space and C a nonempty subset of E. A mapping $U : C \to E$ is said to be a *mapping of type (P)* if

$$\langle J(x - Ux) - J(y - Uy), Ux - Uy \rangle \geq 0, \forall\, x, y \in C. \tag{2.14}$$

From the definition, it is easy to see that if E is a Hilbert space, then U is firmly nonexpansive-like of type (P) if and only if it is firmly nonexpansive, i.e. $\|Ux - Uy\|^2 \leq \langle Ux - Uy, x - y \rangle$, $\forall\, x, y \in C$. We recall the following result.

Lemma 2.9 (See [5]) *Let E be a smooth Banach space, C a nonempty subset of E and $U : C \to E$ a firmly nonexpansive-like mapping (mapping of type (P)). Then the following hold.*

(1) *If C is closed and convex, then so is $F(U)$.*
(2) $\hat{F}(U) = F(U)$.

Henceforth, we refer to a firmly nonexpansive-like mapping as mapping of type (P). In this study, we consider E as p-uniformly convex and uniformly smooth. Consequently, we modify Definition 2.8 to accommodate the generalized duality mapping (2.3). Here and hereafter, E is a p-uniformly convex and uniformly smooth Banach space.

Definition 2.10 Let E be a p-uniformly convex and uniformly smooth Banach space and C a nonempty subset of E. A mapping $U : C \to E$ is said to be of type (P) if

$$\langle J_p^E(x - Ux) - J_p^E(y - Uy), Ux - Uy\rangle \geq 0, \forall\, x, y \in C. \tag{2.15}$$

Example 2.11 Let E be a p-uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E. Then the metric projection P_C is a mapping of type (P).

Example 2.12 Let $E := L_p([\alpha, \beta])$ $(2 \leq p < \infty)$, where $\alpha, \beta \in \mathbb{R}$ and let $f \in E$. The mapping $U : E \to E$ defined by $U(f(x)) = \frac{1}{2} f(x)$ is of type (P). To see this, let $f, g \in E$, we obtain from (2.8) that

$$\begin{aligned}
&\langle J_p^E(f(x) - U(f(x))) - J_p^E(g(x) - U(g(x))), f(x) - g(x)\rangle \\
&= \langle J_p^E(\frac{1}{2} f(x)) - J_p^E(\frac{1}{2} g(x)), f(x) - g(x)\rangle \\
&\geq \frac{\tau}{2^{p-1}} \|f(x) - g(x)\|^p \geq 0.
\end{aligned} \tag{2.16}$$

The following lemmas will be needed in the next section.

Lemma 2.13 [34] *Let E be a smooth and uniformly convex real Banach space. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. Then $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.14 [56] *Let $q \geq 1$ and $r > 0$ be two fixed real numbers. Then, a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$, $g(0) = 0$ such that for all $x, y \in B_r$ and $0 \leq \lambda \leq 1$,*

$$\|\lambda x + (1 - \lambda)y\|^q \leq \lambda \|x\|^q + (1 - \lambda)\|y\|^q - W_q(\lambda) g(\|x - y\|),$$

where $W_q(\lambda) := \lambda^q(1 - \lambda) + \lambda(1 - \lambda)^q$ and $B_r := \{x \in E : \|x\| \leq r\}$.

3 Main Results

In this section, we present our inertial-type algorithm and prove the strong convergence of the sequence generated to a solution of the SCFPP for mapping of type (P) and Bregman weak relatively nonexpansive mapping in p-uniformly convex and uniformly smooth Banach spaces. We assume $1 < q \leq 2 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Let E_1, E_2 be p-uniformly convex and uniformly smooth Banach spaces with duals E_1^*, E_2^*, respectively. Let $C = C_1$ be nonempty closed and convex subset of E_1. Let $T : E_1 \to E_1$ be a Bregman weak relatively nonexpansive mapping and $U : E_2 \to E_2$ be a mapping of type (P). Let $A : E_1 \to E_2$ be a bounded linear operator. We consider the following SCFPP:

$$\text{find } x \in F(T) \text{ such that } Ax \in F(U). \tag{3.1}$$

We shall denote the solution set of the SCFPP (3.1) by Γ and assume that $\Gamma \neq \emptyset$. We first prove the following lemma.

Lemma 3.1 *Let E be a p-uniformly convex and uniformly smooth Banach space, C a nonempty closed convex subset of E, and $U : C \to E$ a mapping of type (P). Then the following hold:*

(1) *$y \in F(U)$ if and only if $\langle J_p^E(x - Ux), Ux - y\rangle \geq 0$, for every $x \in C$;*
(2) *$F(U)$ is closed and convex;*
(3) *$\tilde{F}(U) = F(U)$.*

Proof (1) Let $y \in F(U)$. Then it follows from (2.15) that $\langle J_p^E(x - Ux), Ux - y\rangle \geq 0$, for every $x \in C$. Conversely, suppose $\langle J_p^E(x - Ux), Ux - y\rangle \geq 0$, for every $x \in C$. Then in particular

$$\langle J_p^E(y - Uy), Uy - y\rangle \geq 0.$$

The last inequality implies that $\|y - Uy\|^p \leq 0$. Hence $y = Uy$.
(2) Let $\{x_n\} \subset F(U)$ be a sequence such that $x_n \to x$ as $n \to \infty$. Then

$$\langle J_p^E(x - Ux), Ux - x_n\rangle \to \langle J_p^E(x - Ux), Ux - x\rangle \geq 0 \text{ as } n \to \infty.$$

Therefore, $\|x - Ux\|^p \leq 0$. Hence $x \in F(U)$. Which shows that $F(U)$ is closed.
Let $x^*, y^* \in F(U)$. Then for all $\lambda \in (0, 1)$, $\lambda x^* + (1-\lambda)y^* \in C$. Let $w = \lambda x^* + (1-\lambda)y^*$. We want to show that $w \in F(U)$. Since $x^*, y^* \in F(U)$, we have that

$$\lambda\langle J_p^E(x - Ux), Ux - x^*\rangle \geq 0 \tag{3.2}$$

and

$$(1-\lambda)\langle J_p^E(x - Ux), Ux - y^*\rangle \geq 0. \tag{3.3}$$

From (3.2) and (3.3), we get that $\langle J_p^E(x - Ux), Ux - w\rangle \geq 0$. Hence $w \in F(U)$ and so $F(U)$ is convex.
(3) It is clear that $F(U) \subset \hat{F}(U)$. Let $x \in \hat{F}(U)$. Then there exists a sequence $\{x_n\} \subset C$ which converges weakly to x such that $\|Ux_n - x_n\| \to 0$ as $n \to \infty$. Since U is a mapping of type (P), from (2.15) we obtain

$$\langle J_p^E(x - Ux) - J_p^E(x_n - Ux_n), Ux - Ux_n\rangle \geq 0.$$

Taking limit as $n \to \infty$ in the last inequality gives

$$\langle J_p^E(x - Ux), Ux - x\rangle \geq 0,$$

which implies that $\|x - Ux\|^p \leq 0$. Hence $x \in F(U)$. We then obtain that $F(U) = \hat{F}(U)$. Since $F(U) \subset \tilde{F}(U) \subset \hat{F}(U)$, we conclude that $F(U) = \tilde{F}(U) = \hat{F}(U)$.

In what follows, we present our inertial-type algorithm.

Algorithm 3.2 Let E_1, E_2 be p-uniformly convex and uniformly smooth Banach spaces with duals E_1^*, E_2^*, respectively. Let $C = C_1$ be nonempty closed and convex subset of E_1. Let $T : E_1 \to E_1$ be a Bregman weak relatively nonexpansive mapping and $U : E_2 \to E_2$ be a mapping of type (P). Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$. Select $x_0, x_1 \in E_1$, let $\{\theta_n\}$ be a real sequence such that $-\theta \leq \theta_n \leq \theta$ for some $\theta > 0$ and $\{\alpha_n\} \subset (0, 1)$ be a real sequence satisfying $\liminf_{n\to\infty} \alpha_n > 0$. Assuming that the $(n-1)$th and nth-iterates have been constructed, then we calculate the $(n+1)$th-iterate $x_{n+1} \in E_1$ via the formula

$$\begin{cases} w_n = P_C J_q^{E_1^*}[J_p^{E_1} x_n + \theta_n (J_p^{E_1} x_n - J_p^{E_1} x_{n-1})], \\ v_n = J_q^{E_1^*}[J_p^{E_1} w_n - \mu_n A^* J_p^{E_2}(I-U)Aw_n], \\ y_n = J_q^{E_1^*}[\alpha_n J_p^{E_1} v_n + (1-\alpha_n) J_p^{E_1} T(v_n)], \\ C_{n+1} = \{z \in C_n : \Delta_p(z, y_n) \le \Delta_p(z, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall n \ge 1. \end{cases} \tag{3.4}$$

Assume for small $\epsilon > 0$, the step size μ_n is chosen such that

$$\mu_n^{q-1} \in \left(0, \frac{q\|Aw_n - UAw_n\|^p}{C_q \|A^* J_p^{E_2}(I-U)Aw_n\|_*^q}\right), \quad n \in \Omega, \tag{3.5}$$

where the index set $\Omega := \{n \in \mathbb{N} : Aw_n - UAw_n \ne 0\}$, otherwise $\mu_n = \mu$, where μ is any non-negative real number.

We first prove the following lemmas which will be used to prove the convergence of Algorithm 3.2.

Lemma 3.3 *The sequence $\{\mu_n\}$ defined by (3.5) is well-defined.*

Proof Let $x^* \in \Gamma$, then $x^* = Tx^*$ and $Ax^* = UAx^*$. Thus,

$$\begin{aligned} &\|Aw_n - UAw_n\|^p \\ &= \langle J_p^{E_2}(I-U)Aw_n, Aw_n - UAw_n\rangle \\ &= \langle J_p^{E_2}(I-U)Aw_n, Aw_n - Ax^* + UAx^* - UAw_n\rangle \\ &= \langle J_p^{E_2}(I-U)Aw_n, Aw_n - Ax^*\rangle + \langle J_p^{E_2}(I-U)Aw_n, UAx^* - UAw_n\rangle \\ &= \langle A^* J_p^{E_2}(I-U)Aw_n, w_n - x^*\rangle + \langle J_p^{E_2}(I-U)Aw_n, UAx^* - UAw_n\rangle \\ &\le \|w_n - x^*\|\|A^* J_p^{E_2}(I-U)Aw_n\|_* + \|UAx^* - UAw_n\|\|J_p^{E_2}(I-U)Aw_n\|_* \\ &= \|w_n - x^*\|\|A^* J_p^{E_2}(I-U)Aw_n\|_* + \|UAx^* - UAw_n\|\|(I-U)Aw_n\|^{p-1}. \end{aligned}$$

Consequently, for $n \in \Omega$, that is $\|(I-U)Aw_n\| > 0$, we obtain that $\|w_n - x^*\|\|A^* J_p^{E_2}(I-U)Aw_n\|_* > 0$ and $\|UAx^* - UAw_n\|\|(I-U)Aw_n\|^{p-1} > 0$. Since $\|UAx^* - UAw_n\|\|(I-U)Aw_n\|^{p-1} > 0$, we have that $\|A^* J_p^{E_2}(I-U)Aw_n\|_* \ne 0$. This implies that μ_n is well-defined.

Lemma 3.4 *For every $n \ge 1$, $\Gamma \subset C_n$ and x_{n+1} defined by Algorithm 3.2 is well-defined.*

Proof By our construction, $C_1 = C$ is closed and convex. Suppose C_k is closed and convex for some $k \in \mathbb{N}$. Then

$$\begin{aligned} C_{k+1} &= \{z \in C_k : \Delta_p(z, y_k) \le \Delta_p(z, w_k)\} \\ &= \left\{z \in C_k : \frac{\|z\|^p}{p} + \frac{\|y_k\|^p}{q} - \langle J_p^{E_1} y_k, z\rangle \le \frac{\|z\|^p}{p} + \frac{\|w_k\|^p}{q} - \langle J_p^{E_1} w_k, z\rangle\right\} \\ &= \{z \in C_k : \|y_k\|^p - \|w_k\|^p \le q\langle J_p^{E_1} y_k - J_p^{E_1} w_k, z\rangle\}, \end{aligned}$$

from which it follows that C_{k+1} is closed. Let $z_1, z_2 \in C_{k+1}$ and $\lambda_1, \lambda_2 \in (0, 1)$ such that $\lambda_1 + \lambda_2 = 1$, then we have that

$$\|y_k\|^p - \|w_k\|^p \leq q\langle J_p^{E_1} y_k - J_p^{E_1} w_k, z_1\rangle \tag{3.6}$$

and

$$\|y_k\|^p - \|w_k\|^p \leq q\langle J_p^{E_1} y_k - J_p^{E_1} w_k, z_2\rangle. \tag{3.7}$$

From (3.6) and (3.7) we then have that

$$\|y_k\|^p - \|w_k\|^p \leq q\langle J_p^{E_1} y_k - J_p^{E_1} w_k, \lambda_1 z_1 + \lambda_2 z_2\rangle. \tag{3.8}$$

By convexity, $\lambda_1 z_1 + \lambda_2 z_2 \in C_k$. Therefore from (3.8), we conclude that $\lambda_1 z_1 + \lambda_2 z_2 \in C_{k+1}$ and hence C_{k+1} is convex. Thus, we have that C_n is convex for all $n \in \mathbb{N}$.

Furthermore, since $\Gamma \neq \emptyset$ by assumption, it implies that $C_{n+1} \neq \emptyset$. To show that $\Gamma \subset C_n$, $\forall n \geq 1$. Let $x^* \in \Gamma$. Then $x^* \in F(T)$ and $Ax^* \in F(U)$, and therefore by construction i.e. (3.4), $\Gamma \subset C_1$. Suppose $x^* \in \Gamma \subset C_n$, then

$$\begin{aligned}
\Delta_p(x^*, y_n) &= \Delta_p(x^*, (1-\alpha_n)J_p^{E_1} v_n + \alpha_n J_p^{E_1} T(v_n)) \\
&\leq (1-\alpha_n)\Delta_p(x^*, v_n) + \alpha_n \Delta_p(x^*, T(v_n)) \\
&\leq \Delta_p(x^*, v_n).
\end{aligned} \tag{3.9}$$

Also using (2.10), Lemma 2.3 and definition of Bregman distance, we get

$$\begin{aligned}
\Delta_p(x^*, v_n) &= \Delta_p(x^*, J_q^{E_1^*}(J_p^{E_1} w_n - \mu_n A^* J_p^{E_2}(I-U)Aw_n)) \\
&= \frac{\|x^*\|^p}{p} - \langle J_p^{E_1} w_n - \mu_n A^* J_p^{E_2}(I-U)AW_n, x^*\rangle \\
&\quad + \frac{\|J_p^{E_1} w_n - \mu_n A^* J_p^{E_2}(I-U)Aw_n\|_*^q}{q} \\
&\leq \frac{\|x^*\|^p}{p} - \langle J_p^{E_1} w_n - \mu_n A^* J_p^{E_2}(I-U)AW_n, x^*\rangle + \frac{\|J_p^{E_1} w_n\|_*^q}{q} \\
&\quad - \mu_n\langle J_p^{E_2}(I-U)Aw_n, Aw_n\rangle + \frac{C_q}{q}\mu_n^q \|A^* J_p^{E_2}(I-U)Aw_n\|_*^q \\
&= \frac{\|x^*\|^p}{p} - \langle J_p^{E_1} w_n, x^*\rangle + \frac{\|J_p^{E_1} w_n\|_*^q}{q} - \mu_n\langle J_p^{E_2}(I-U)Aw_n, Aw_n - Ax^*\rangle \\
&\quad + \frac{C_q}{q}\mu_n^q \|A^* J_p^{E_2}(I-U)Aw_n\|_*^q \\
&= V_p(x^*, J_p^{E_1} w_n) - \mu_n\langle J_p^{E_2}(I-U)Aw_n, Aw_n - Ax^*\rangle \\
&\quad + \frac{C_q}{q}\mu_n^q \|A^* J_p^{E_2}(I-U)Ax_n\|_*^q \\
&= \Delta_p(x^*, w_n) - \mu_n\langle J_p^{E_2}(I-U)Aw_n, Aw_n \\
&\quad - Ax^*\rangle + \frac{C_q}{q}\mu_n^q \|A^* J_p^{E_2}(I-U)Aw_n\|_*^q.
\end{aligned} \tag{3.10}$$

We know from Lemma 3.1 that $\langle J_p^{E_2}(I-U)Aw_n, UAw_n - Ax^*\rangle \geq 0$. Therefore,

$$\begin{aligned}\langle J_p^{E_2}(I-U)Aw_n, Aw_n - Ax^*\rangle &= \langle J_p^{E_2}(I-U)Aw_n, Aw_n - UAw_n + UAw_n - Ax^*\rangle \\ &= \|Aw_n - UAw_n\|^p + \langle J_p^{E_2}(I-U)Aw_n, UAw_n - Ax^*\rangle \\ &\geq \|Aw_n - UAw_n\|^p. \end{aligned} \tag{3.11}$$

Substituting (3.11) in (3.10) will yield

$$\begin{aligned}\Delta_p(x^*, v_n) &\leq \Delta_p(x^*, w_n) - \mu_n\|Aw_n - UAw_n\|^p + \frac{C_q}{q}\mu_n^q\|A^*J_p^{E_2}(I-U)Aw_n\|_*^q \\ &\leq \Delta_p(x^*, w_n) - \mu_n\Big(\|Aw_n - UAw_n\|^p - \frac{C_q}{q}\mu_n^{q-1}\|A^*J_p^{E_2}(I-U)Aw_n\|_*^q\Big) \quad (3.12) \\ &\leq \Delta_p(x^*, w_n), \quad (3.13)\end{aligned}$$

where (3.13) follows from the condition on the step size (3.5). Hence from (3.9) and (3.13), we obtain that $\Delta_p(x^*, y_n) \leq \Delta_p(x^*, w_n)$, which shows that $\Gamma \subset C_{n+1}$, $\forall\, n \in \mathbb{N}$.

Lemma 3.5 *The sequences* $\{x_n\}$, $\{y_n\}$, $\{v_n\}$ *and* $\{w_n\}$ *are bounded.*

Proof We know from Algorithm 3.2 that $x_n = \Pi_{C_n}x_0$ and $C_{n+1} \subseteq C_n$, $\forall\, n \geq 1$. Then from (2.10), we have that $\Delta_p(x_n, x_0) \leq \Delta_p(x_{n+1}, x_0)$. This shows that $\{\Delta_p(x_n, x_0)\}$ is nondecreasing. Also, since $\Gamma \subset C_{n+1}$ it implies that $\Delta_p(x_n, x_0) \leq \Delta_p(x_{n+1}, x_0) \leq \Delta_p(x^*, x_0)$, $\forall\, x^* \in \Gamma$. Therefore from (2.8), we conclude that $\{x_n\}$ is bounded. Since $\{x_n\}$ is bounded, it follows from the construction that $\{y_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded.

Lemma 3.6 *Let the sequences* $\{x_n\}$, $\{y_n\}$, $\{v_n\}$ *and* $\{w_n\}$ *be as defined in Algorithm 3.2. Assuming that for small* $\epsilon > 0$,

$$\mu_n \in \left(\epsilon, \left(\frac{q\|Aw_n - UAw_n\|^p}{C_q\|A^*J_p^{E_2}(I-U)Aw_n\|_*^q} - \epsilon\right)^{\frac{1}{q-1}}\right), \quad n \in \Omega. \tag{3.14}$$

Then

(i) $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$;
(ii) $\lim_{n\to\infty}\|w_n - x_n\| = 0$;
(iii) $\lim_{n\to\infty}\|v_n - Tv_n\| = 0$;
(iv) $\lim_{n\to\infty}\|x_n - v_n\| = 0$;
(v) $\lim_{n\to\infty}\|A^*J_p^{E_2}(I-U)Aw_n\|_* = 0$ *and* $\lim_{n\to\infty}\|(I-U)Aw_n\| = 0$.

Proof From the proof of Lemma 3.5 we have that $\{\Delta_p(x_n, x_0)\}$ is a nondecreasing bounded sequence in $\mathbb{R}$. Hence $\lim_{n\to\infty}\Delta_p(x_n, x_0)$ exists. Using (2.11),

$$\Delta_p(x_{n+1}, \Pi_{C_n}x_0) + \Delta_p(\Pi_{C_n}x_0, x_0) \leq \Delta_p(x_{n+1}, x_0). \tag{3.15}$$

Therefore

$$\Delta_p(x_{n+1}, x_n) \leq \Delta_p(x_{n+1}, x_0) - \Delta_p(x_n, x_0) \to 0. \tag{3.16}$$

Applying Lemma 2.13, we obtain that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.17}$$

This establishes (i).

Let $t_n = J_q^{E_1^*}[J_p^{E_1}x_n + \theta_n(J_p^{E_1}x_n - J_p^{E_1}x_{n-1})]$. It then follows that

$$J_p^{E_1}t_n - J_p^{E_1}x_n = \theta_n(J_p^{E_1}x_n - J_p^{E_1}x_{n-1}).$$

Then by the uniform continuity of $J_p^{E_1}$ on bounded subsets of E_1, we obtain that

$$\begin{aligned}\|J_p^{E_1}t_n - J_p^{E_1}x_n\|_* &= \|\theta_n(J_p^{E_1}x_n - J_p^{E_1}x_{n-1})\|_* \\ &\leq \theta\|J_p^{E_1}x_n - J_p^{E_1}x_{n-1}\|_* \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.18}$$

By the uniform continuity of $J_q^{E_1^*}$ on bounded subsets of E_1^* and (3.17), we obtain that $\lim_{n\to\infty}\|t_n - x_n\| = 0$. Therefore,

$$\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\|P_C t_n - x_n\| \leq \lim_{n\to\infty}\|t_n - x_n\| = 0. \tag{3.19}$$

This establishes (ii). Combining (i) and (ii) will give $\lim_{n\to\infty}\|x_{n+1} - w_n\| = 0$.

Furthermore, since $x_{n+1} \in C_{n+1}$, it follows from our construction that

$$\Delta_p(x_{n+1}, y_n) \leq \Delta_p(x_{n+1}, w_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore by Lemma 2.13, we get that $\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$. This together with (3.17) yields

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.20}$$

From (3.19) and (3.20), we obtain that

$$\lim_{n\to\infty}\|y_n - w_n\| = 0. \tag{3.21}$$

Let $x^* \in F(T)$. Then

$$\begin{aligned}\Delta_p(x^*, y_n) &= \Delta_p(x^*, J_q^{E_1^*}[(1-\alpha_n)J_p^{E_1}v_n + \alpha_n J_p^{E_1}Tv_n]) \\ &= V_p(x^*, (1-\alpha_n)J_p^{E_1}v_n + \alpha_n J_p^{E_1}Tv_n) \\ &= \frac{\|x^*\|^p}{p} - \langle(1-\alpha_n)J_p^{E_1}v_n + \alpha_n J_p^{E_1}Tv_n, x^*\rangle \\ &\quad + \frac{1}{q}\|(1-\alpha_n)J_p^{E_1}v_n + \alpha_n J_p^{E_1}Tv_n\|_*^q \\ &\leq \frac{\|x^*\|^p}{p} - \langle(1-\alpha_n)J_p^{E_1}v_n, x^*\rangle - \alpha_n\langle J_p^{E_1}Tv_n, x^*\rangle + \frac{(1-\alpha_n)}{q}\|v_n\|^p \\ &\quad + \frac{\alpha_n}{q}\|Tv_n\|^p - \frac{W_q(\alpha_n)}{q}g(\|J_p^{E_1}v_n - J_p^{E_1}Tv_n\|_*) \qquad (3.22) \\ &= (1-\alpha_n)\Delta_p(x^*, v_n) + \alpha_n\Delta_p(x^*, Tv_n) - \frac{W_q(\alpha_n)}{q}g(\|J_p^{E_1}v_n - J_p^{E_1}Tv_n\|_*) \\ &= \Delta_p(x^*, v_n) - \frac{W_q(\alpha_n)}{q}g(\|J_p^{E_1}v_n - J_p^{E_1}Tv_n\|_*) \\ &\leq \Delta_p(x^*, w_n) - \frac{W_q(\alpha_n)}{q}g(\|J_p^{E_1}v_n - J_p^{E_1}Tv_n\|_*), \qquad (3.23)\end{aligned}$$

where (3.22) and (3.23) follow from Lemma 2.14 and (3.13), respectively. Then from (3.23), we have that

$$\begin{aligned}\frac{W_q(\alpha_n)}{q} g(\|J_p^{E_1} v_n - J_p^{E_1} T v_n\|_*) &\leq \Delta_p(x^*, w_n) - \Delta_p(x^*, y_n)\\ &= \langle J_p^{E_1} y_n - J_p^{E_1} w_n, x^* - w_n\rangle - \Delta_p(y_n, w_n)\\ &\leq \langle J_p^{E_1} y_n - J_p^{E_1} w_n, x^* - w_n\rangle\\ &= \|x^* - w_n\| \|J_p^{E_1} y_n - J_p^{E_1} w_n\|_*. \end{aligned} \tag{3.24}$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, taking the limit of (3.24) as $n \to \infty$ gives $\frac{W_q(\alpha_n)}{q} g(\|J_p^{E_1} v_n - J_p^{E_1} T v_n\|_*) \to 0$. Thus we obtain that

$$g(\|J_p^{E_1} v_n - J_p^{E_1} T v_n\|_*) \to 0 \quad \text{as} \quad n \to \infty.$$

By the continuity of g, the last limit implies that

$$\|J_p^{E_1} v_n - J_p^{E_1} T v_n\|_* \to 0 \quad \text{as} \quad n \to \infty. \tag{3.25}$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, (3.25) implies that

$$\lim_{n\to\infty} \|v_n - T v_n\| = 0. \tag{3.26}$$

This establishes (iii).

Also, using Lemma (2.13), (3.26) implies that $\lim_{n\to\infty} \Delta_p(v_n, T v_n) = 0$. Therefore,

$$\begin{aligned}\Delta_p(v_n, y_n) &= \Delta_p(v_n, J_q^{E_1^*}[(1-\alpha_n) J_p^{E_1} v_n + \alpha_n J_p^{E_1} T v_n])\\ &\leq (1-\alpha_n)\Delta_p(v_n, v_n) + \alpha_n \Delta_p(v_n, T v_n)\\ &= \alpha_n \Delta_p(v_n, T v_n) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.27}$$

Furthermore by Lemma 2.13, we obtain that

$$\lim_{n\to\infty} \|v_n - y_n\| = 0. \tag{3.28}$$

Consequently, from (3.20) and (3.28), we obtain that

$$\lim_{n\to\infty} \|x_n - v_n\| = 0, \tag{3.29}$$

which establishes (iv), and from (3.21) and (3.28), we obtain that

$$\lim_{n\to\infty} \|v_n - w_n\| = 0. \tag{3.30}$$

Also from (3.12), we have that

$$\begin{aligned}\mu_n\Big(\|A w_n - U A w_n\|^p - \frac{C_q}{q}\mu_n^{q-1} \left\|A^* J_p^{E_2}(I-U) A w_n\right\|_*^q \Big)\\ \leq \Delta_p(x^*, w_n) - \Delta_p(x^*, v_n)\\ = \langle J_p^{E_1} v_n - J_p^{E_1} w_n, x^* - w_n\rangle - \Delta_p(v_n, w_n)\\ \leq \langle J_p^{E_1} v_n - J_p^{E_1} w_n, x^* - w_n\rangle\\ = \|x^* - w_n\| \|J_p^{E_1} v_n - J_p^{E_1} w_n\|_*. \end{aligned} \tag{3.31}$$

Passing to the limit as $n \to \infty$ in (3.31) and using (3.30), we obtain that

$$\lim_{n\to\infty} (\|Aw_n - UAw_n\|^p - \frac{C_q}{q}\mu_n^{q-1}\|A^* J_p^{E_2}(I-U)Aw_n\|_*^q) = 0. \tag{3.32}$$

Note that by the choice of our step size, it holds that

$$\mu_n^{q-1} < \frac{q\|Aw_n - UAw_n\|^p}{C_q\|A^* J_p^{E_2}(I-U)Aw_n\|_*^q} - \epsilon. \tag{3.33}$$

Simplifying (3.33) further gives

$$\frac{\epsilon C_q}{q}\|A^* J_p^{E_2}(I-U)Aw_n\|_*^q < (\|Aw_n - UAw_n\|^p - \frac{C_q}{q}\mu_n^{q-1}\|A^* J_p^{E_2}(I-U)Aw_n\|_*^q). \tag{3.34}$$

By passing to the limit as $n \to \infty$ in (3.34) and using (3.32), we obtain that

$$\lim_{n\to\infty} \|A^* J_p^{E_2}(I-U)Aw_n\|_*^q = 0, \tag{3.35}$$

and consequently,

$$\lim_{n\to\infty} \|A^* J_p^{E_2}(I-U)Aw_n\|_* = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(I-U)Aw_n\| = 0. \tag{3.36}$$

This establishes (v).

Theorem 3.7 *The sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to $u \in \Gamma$, where $u = \Pi_\Gamma x_0$.*

Proof Since $\{\Delta_p(x_n, x_0)\}$ is nondecreasing and bounded in $\mathbb{R}$, it implies that there exists $l \in \mathbb{R}$ such that $\Delta_p(x_n, x_0) \to l$ as $n \to \infty$. Using (2.11), we get that for every $m, n \in \mathbb{N}$,

$$\begin{aligned}\Delta_p(x_m, x_n) &= \Delta_p(x_m, \Pi_{C_n} x_0) \\ &\leq \Delta_p(x_m, x_0) - \Delta_p(x_n, x_0) \to 0.\end{aligned}$$

Therefore from Lemma 2.13, we have that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed convex subset of a Banach space, it implies that there exists $u \in C$ such that $x_n \to u$ as $n \to \infty$. It then follows from Lemma 3.6 that $w_n \to u$ and $v_n \to u$ as $n \to \infty$. By the linearity of A, we have that $Aw_n \to Au$ as $n \to \infty$. We have shown in Lemma 3.6 that $\|v_n - Tv_n\| \to 0$ as $n \to \infty$, this together with the fact that T is Bregman weak relatively nonexpansive implies that $u \in F(T)$. We have also shown in Lemma 3.6 that $\|(I-U)Aw_n\| \to 0$ as $n \to \infty$, which implies that $Au \in \tilde{F}(U)$. Then from Lemma 3.1(iii), we obtain that $Au \in F(U)$. It then implies that $u \in \Gamma$.

Lastly, we prove that $u = \Pi_\Gamma x_0$. Suppose there exists $v \in \Gamma$ such that $v = \Pi_\Gamma x_0$. Then

$$\Delta_p(v, x_0) \leq \Delta_p(u, x_0). \tag{3.37}$$

Since $\Gamma \in C_n$ for all $n \geq 1$, we have that $\Delta_p(x_n, x_0) \leq \Delta_p(v, x_0)$. Now by the lower semicontinuity of the norm, we have that

$$\begin{aligned}\Delta_p(u, x_0) &= \frac{\|u\|^p}{p} + \frac{\|x_0\|^p}{q} - \langle J_p^{E_1} x_0, u\rangle \\ &\leq \liminf_{n\to\infty} \left(\frac{\|x_n\|^p}{p} + \frac{\|x_0\|^p}{q} - \langle J_p^{E_1} x_0, x_n\rangle\right) \\ &= \liminf_{n\to\infty} \Delta_p(x_n, x_0) \\ &\leq \limsup_{n\to\infty} \Delta_p(x_n, x_0) \\ &\leq \Delta_p(v, x_0). \end{aligned} \tag{3.38}$$

Then from (3.37) and (3.38), we have that

$$\Delta_p(v, x_0) \leq \Delta_p(u, x_0) \leq \Delta_p(v, x_0). \tag{3.39}$$

(3.39) implies that $u = v$. Hence $u = \Pi_\Gamma x_0$.

We next present some consequences of our main results. Firstly, if $\theta_n = 0$, we obtain the following non-inertial shrinking projection algorithm.

Corollary 3.8 *Let E_1, E_2 be p-uniformly convex and uniformly smooth Banach spaces with duals E_1^*, E_2^*, respectively. Let $C = C_1$ be nonempty closed and convex subset of E_1. Let $T : E_1 \to E_1$ be a Bregman weak relatively nonexpansive mapping and $U : E_2 \to E_2$ be a mapping of type (P). Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint $A^* : E_2^* \to E_1^*$. Select $x_0 \in E_1$ and let $\{\alpha_n\} \subset (0, 1)$ be a real sequence satisfying $\liminf_{n\to\infty} \alpha_n > 0$. Assuming that the nth-iterate $x_n \in E_1$ has been constructed, then we calculate the $(n+1)th$-iterate $x_{n+1} \in E_1$ via the formula*

$$\begin{cases} w_n = P_C x_n, \\ v_n = J_q^{E_1^*}[J_p^{E_1} w_n - \mu_n A^* J_p^{E_2}(I - U)Aw_n], \\ y_n = J_q^{E_1^*}[\alpha_n J_p^{E_1} v_n + (1-\alpha_n) J_p^{E_1} T(v_n)], \\ C_{n+1} = \{z \in C_n : \Delta_p(z, y_n) \leq \Delta_p(z, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall n \geq 1. \end{cases} \tag{3.40}$$

Assume for small $\epsilon > 0$, the step size μ_n is chosen such that

$$\mu_n \in \left(\epsilon, \left(\frac{q\|Aw_n - UAw_n\|^p}{C_q \|A^* J_p^{E_2}(I - U)Aw_n\|_*^q} - \epsilon\right)^{\frac{1}{q-1}}\right), \quad n \in \Omega, \tag{3.41}$$

where the index set $\Omega := \{n \in \mathbb{N} : Aw_n - UAw_n \neq 0\}$, otherwise $\mu_n = \mu$, where μ is any non-negative real number. Then $\{x_n\}$ converges strongly to $u \in \Gamma$, where $u = \Pi_\Gamma x_0$.

Also, by letting U be the metric projection mapping onto a closed convex subset Q of E_2 in Algorithm 3.2, i.e. $U = P_Q$, we obtain the following result as a solution to split feasibility and fixed point problems.

Corollary 3.9 *With reference to the data in Algorithm 3.2, let Q be a nonempty closed convex subset of E_2 and $U = P_Q$. Assuming $\Theta := \{x \in C : x \in F(T), Ax \in Q\} \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to $u \in \Theta$, where $u = \Pi_\Theta x_0$.*

4 Application

4.1 Split Monotone Inclusion Problem

Let E be a smooth, strictly convex and reflexive Banach space with dual E^*. Let $B : E \to 2^{E^*}$ be a multivalued mapping. The graph of B denoted by $gr(B)$ is defined by $gr(B) = \{(x, u) \in E \times E^* : u \in Bx\}$. B is called a non-trivial operator if $gr(B) \neq \emptyset$. B is called a monotone mapping if $\forall\, (x, u), (y, v) \in gr(B)$, $\langle x - y, u - v\rangle \geq 0$. B is said to be a maximal monotone operator if the graph of B is not a proper subset of the graph of any other monotone mapping. Let $T_1 : E_1 \to 2^{E_1^*}$ and $T_2 : E_2 \to 2^{E_2^*}$ be maximal monotone mappings and $A : E_1 \to E_2$ be bounded linear operator. The Split Monotone Inclusion Problem (in short, SMIP) is to find

$$x \in E_1 \text{ such that } x \in T_1^{-1}(0) \cap A^{-1}(T_2^{-1}(0)). \tag{4.1}$$

Many authors have studied the SMIP (see [11,31,33,46,55]) and applied it to solve some real-life problems which include modeling intensity-modulated radiation therapy treatment planning, sensor networks in computerized tomography and data compression, see [10,12,13]. Very recently, Bello and Sheu [8] studied the problem in p-uniformly convex and uniformly smooth Banach spaces. They proposed an algorithm and proved a strong convergence theorem with the step size not depending on the prior knowledge of the norm of the bounded linear operator. Our purpose here is to apply our algorithm to solve the SMIP (4.1).

For all $r > 0$, the mapping $K_r := (I + rJ_q^{E_2^*} T_2)^{-1}$ is called the metric resolvent of T_2. It is easy to see that $F(K_r) = T_2{}^{-1}(0)$. Also note that if $x \in ran(I + rJ_q^{E_2^*} T_2)$, then $J_p^{E_2}(x - K_r x) \in r^{p-1} T_2 K_r x$. Therefore for every $x, y \in ran(I + rJ_q^{E_2^*} T_2)$, we have that

$$\langle J_p^{E_2}(x - K_r x) - J_p^{E_2}(y - K_r y), K_r x - K_r y\rangle \geq 0, \tag{4.2}$$

by the monotonicity of T_2. This implies that K_r is a mapping of type (P). Similarly, let $T_1 : E_1 \to 2^{E_1{}^*}$ be a maximal monotone operator. For every $r > 0$, the Bregman resolvent associated with T_1 is denoted by Res_{rT_1} and is defined by

$$Res_{rT_1} := (J_p^{E_1} + rT_1)^{-1} \circ J_p^{E_1} : E_1 \to 2^{E_1}.$$

It is known that Res_{rT_1} is Bregman weak relatively nonexpansive and $F(Res_{rT_1}) = T_1{}^{-1}(0)$ for each $r > 0$.

It then implies that our algorithm can be used to solve the SMIP (4.1). We shall denote the solution set of (4.1) by $SMIP(T_1, T_2)$. An application of our main result is the following.

Theorem 4.1 *Let $U = K_r$ and $T = Res_{rT_1}$ in Algorithm 3.2. Assuming $SMIP(T_1, T_2) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to $u \in SMIP(T_1, T_2)$, where $u = \Pi_{SMIP(T_1,T_2)} x_0$.*

5 Numerical Examples

We next give some numerical examples to validate our results and to illustrate the performance of our Algorithm 3.2.

Example 5.1 Let $E_1 = E_2 = \mathbb{R}$ and $C = C_1 = [0, 3]$. Let $T : E_1 \to E_1$ be defined by

$$Tx = \begin{cases} 0 \text{ if } x \neq 3, \\ 2 \text{ if } x = 3, \end{cases} \tag{5.1}$$

Table 1 Numerical results for Example 5.1

	Algorithm 3.2	Algorithm (3.40)
Case Ia		
CPU time (s)	0.0112	0.0144
No of iter.	38	51
Case Ib		
CPU time (s)	0.0130	0.0164
No. of iter.	38	52
Case Ic		
CPU time (s)	0.0089	0.0113
No of iter.	39	52
Case Id		
CPU time (s)	0.0099	0.0138
No of iter.	37	52

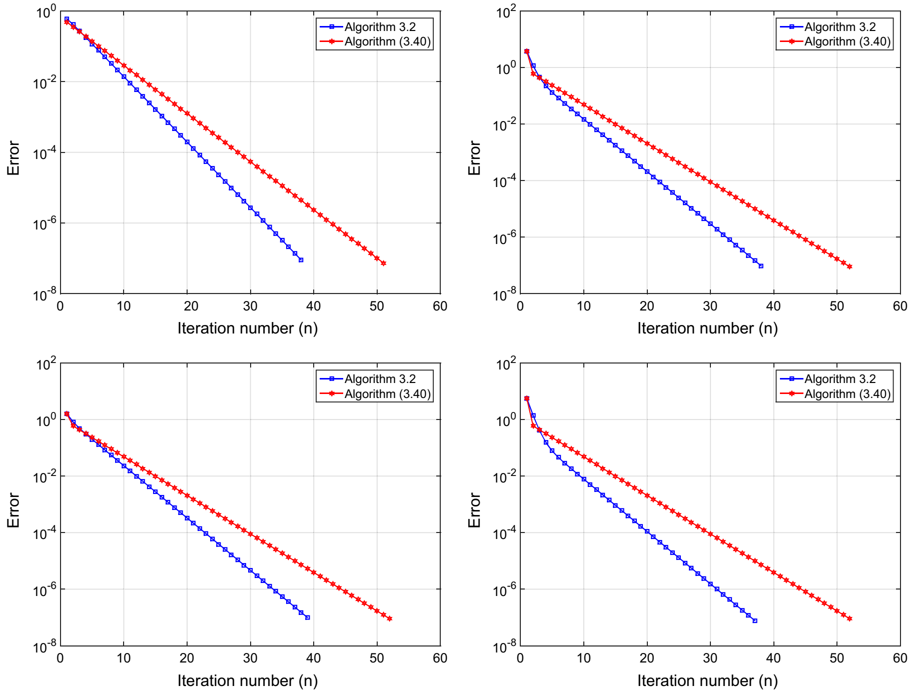


Fig. 1 Example 5.1: Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id

$\forall x \in E_1$ and $U : E_2 \to E_2$ be defined by $Ux = \frac{1}{2}x$, $\forall x \in E_2$. Then T is weak relatively nonexpansive and U is firmly nonexpansive. Let $A : E_1 \to E_2$ be a mapping defined by

$Ax = \frac{2}{3}x, \forall x \in E_1$. We choose $\theta_n = \frac{(-1)^n+3}{10n}$ and $\alpha_n = \frac{n+1}{4n}$. Then Algorithm 3.2 gives

$$\begin{cases} w_n = P_C[x_n + \frac{(-1)^n+3}{10n}(x_n - x_{n-1})], \\ v_n = [w_n - \mu_n(\frac{2}{9}w_n)], \\ y_n = [\frac{n+1}{4n}v_n + \frac{3n-1}{4n}T(v_n)], \\ C_{n+1} = \{z \in C_n : |z - y_n| \leq |z - w_n|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \forall\, n \geq 1, \end{cases} \tag{5.2}$$

where the step size μ_n is chosen such that

$$\mu_n \in \left(0, \frac{2|Aw_n - UAw_n|^2}{|A^*(I-U)Aw_n|^2}\right).$$

Using MATLAB R2015(a), we compute and compare the numerical outputs of Algorithms 3.2 and (3.40). We choose different values of x_0 and x_1 and plot the graphs of errors = $|x_{n+1} - x_n|$ against number of iterations n. The stopping criterion used for the computation is $|x_{n+1} - x_n| < 10^{-7}$ and the initial values are given below:

Case Ia: $x_0 = 2.6;\ \ x_1 = 1.8;$
Case Ib: $x_0 = 9.4;\ \ x_1 = 6.0;$
Case Ic: $x_0 = 2.6;\ \ x_1 = 3.8;$
Case Id: $x_0 = -9.4;\ \ x_1 = 7.8.$

The computational results are shown in Table 1 and Fig. 1.

Example 5.2 Let $E_1 = E_2 = \ell_2(\mathbb{R})$, where $\ell_2(\mathbb{R}) := \{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_i, \ldots), \sigma_i \in \mathbb{R} : \sum_{i=1}^{\infty} |\sigma_i|^2 < \infty\}$, $\|\sigma\|_{\ell_2} = (\sum_{i=1}^{\infty} |\sigma_i|^2)^{\frac{1}{2}}, \forall\, \sigma \in E_1$. Let $C = C_1 := \{x \in E_1 : ||x||_{\ell_2} \leq 1\}$. Let $T : E_1 \to E_1$ be as defined in Example 2.6 and define $U : E_2 \to E_2$ by $Ux = \frac{1}{2}x$, $\forall x \in E_2$. Then U is a mapping of type (P). Let $A : E_1 \to E_2$ be a mapping defined by $Ax = \frac{2}{3}x$. We choose $\theta_n = \frac{2n+1}{10n}$ and $\alpha_n = \frac{n+1}{4n}$. Then Algorithm 3.2 becomes

$$\begin{cases} w_n = P_C J_q^{E_1}[J_p^{E_1}x_n + \frac{1}{2n}(J_p^{E_1}x_n - J_p^{E_1}x_{n-1})], \\ v_n = J_q^{E_1}[J_p^{E_1}w_n - \mu_n\frac{2}{3}J_p^{E_2}(\frac{1}{3}w_n)], \\ y_n = J_q^{E_1}[\frac{n+1}{4n}J_p^{E_1}v_n + \frac{3n-1}{4n}J_p^{E_1}T(v_n)], \\ C_{n+1} = \{z \in C_n : \Delta_p(z, y_n) \leq \Delta_p(z, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \forall\, n \geq 1, \end{cases} \tag{5.3}$$

where the stepsize μ_n is chosen as defined in (3.5). Using MATLAB R2015(a) and $\|x_{n+1} - x_n\|_{\ell_2} < 10^{-7}$ as stopping criterion, we compute and compare the numerical outputs of Algorithms 3.2 and (3.40) using four different starting values as follows:

Case IIa: $x_0 = (4, 2, 1, \ldots);\ \ x_1 = (25, 5, 1, \ldots);$
Case IIb: $x_0 = (-9, 3, -1, \ldots);\ \ x_1 = (10, -1, 0.1, \ldots);$
Case IIc: $x_0 = (1, \frac{-1}{4}, \frac{1}{16}, \ldots);\ \ x_1 = (\frac{1}{\sqrt{3}}, \frac{1}{3}, \frac{1}{\sqrt{27}}, \ldots);$
Case IId: $x_0 = (3, \frac{3}{2}, \frac{3}{4}, \ldots);\ \ x_1 = (-5, 1, -0.2, \ldots).$

We thus plot the graphs of errors against number of iterations in each case. The computational result can be found in Table 2 and Fig. 2.

The next example is to illustrate the application given in Sect. 4.

Table 2 Numerical results for Example 5.2

	Algorithm 3.2	Algorithm (3.40)
Case IIa		
CPU time (s)	0.0587	0.0695
No of iter.	19	24
Case IIb		
CPU time (s)	0.0617	0.0637
No. of iter.	19	24
Case IIc		
CPU time (s)	0.0188	0.0271
No of iter.	17	23
Case IId		
CPU time (s)	0.0263	0.0367
No of iter.	19	24

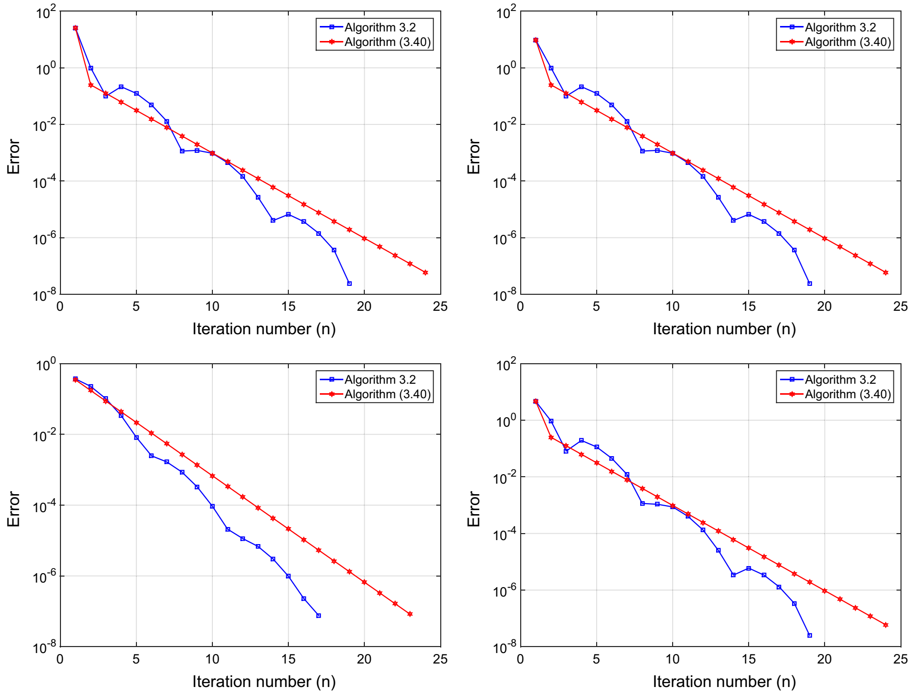


Fig. 2 Example 5.2: Top left: Case IIa; Top right: Case IIb; Bottom left: Case IIc; Bottom right: Case IId

Example 5.3 Let $E_1 = E_2 = \ell_2(\mathbb{R})$, where $\ell_2(\mathbb{R}) := \{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_i, \ldots), \sigma_i \in \mathbb{R} : \sum_{i=1}^{\infty} |\sigma_i|^2 < \infty\}$, $\|\sigma\|_{\ell_2} = (\sum_{i=1}^{\infty} |\sigma_i|^2)^{\frac{1}{2}}, \forall\, \sigma \in E_1$. Let $C = C_1 := \{x \in E_1 : ||x||_{\ell_2} \leq 1\}$. Let $T_1 : E_1 \to E_1$ be defined by $T_1 x = 3x, \forall x \in E_1$ and define $T_2 : E_2 \to E_2$ by $T_2 y = 7y, \forall y \in E_2$. Let $A : E_1 \to E_2$ be a mapping defined by $Ax = \frac{1}{4}x$. Then T_1 and T_2 are maximal monotone operators. One can easily verify that $Res_{rT_1} x = \frac{x}{1+3r}, \forall x \in E_1$ and $K_r y = \frac{y}{1+7r}, \forall y \in E_2, r > 0$. Note that in this case, $SMIP(T_1, T_2) = \{\mathbf{0} = (0, 0, \ldots)\}$.

Table 3 Numerical results for Example 5.3

	Algorithm 3.2	Algorithm (3.40)
Case IIIa		
CPU time (s)	0.0198	0.0244
No of iter.	34	49
Case IIIb		
CPU time (s)	0.0187	0.0409
No. of iter.	36	49
Case IIIc		
CPU time (s)	0.0338	0.0437
No of iter.	36	49
Case IIId		
CPU time (s)	0.0307	0.0347
No of iter.	36	49

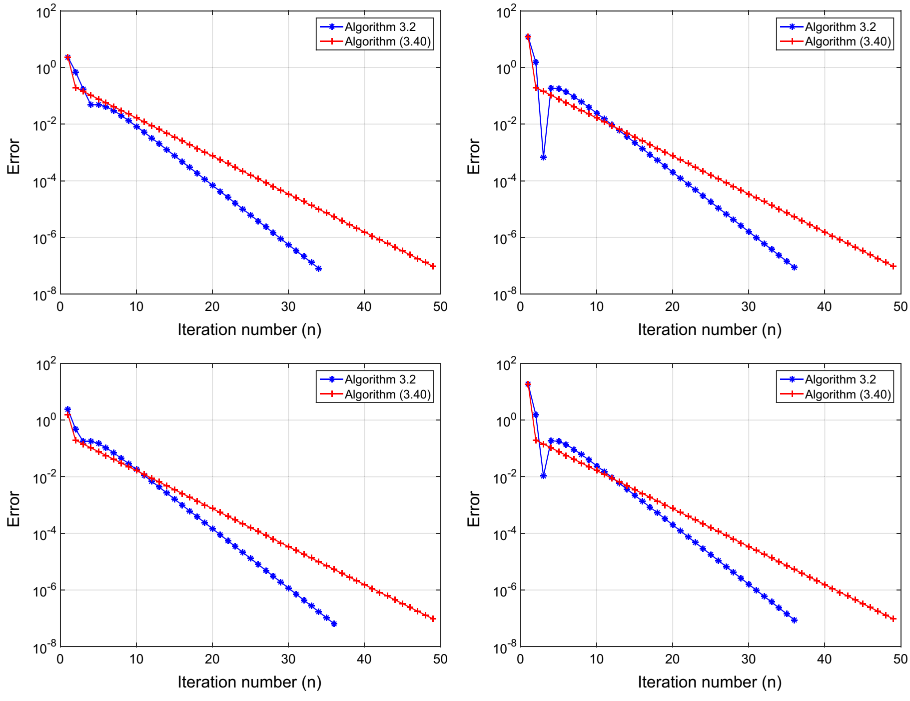


Fig. 3 Example 5.3: Top left: Case IIIa; Top right: Case IIIb; Bottom left: Case IIIc; Bottom right: Case IIId

Therefore by Theorem 4.1, the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to $\mathbf{0}$. We choose $r = 3$, $\theta_n = \frac{2n+1}{10n}$, $\alpha_n = \frac{n+1}{4n}$ and $\mu = 0.35$. Using MATLAB R2015(a), we test Algorithms 3.2 and (3.40) for the following initial values:

Case IIIa: $x_0 = (4, 2, 1, \dots)$; $x_1 = (3, -\frac{1}{3}\frac{1}{27}, \dots)$;
Case IIIb: $x_0 = (-2, \frac{1}{2}, -\frac{1}{8}, \dots)$; $x_1 = (-12, 4, -\frac{4}{3}, \dots)$;
Case IIIc: $x_0 = (10, 2, 0.4, \dots)$; $x_1 = (2, 1, \frac{1}{2}, \dots)$;

Table 4 Numerical results for Example 5.4

	Algorithm 3.2	Algorithm (3.40)
Case IVa		
CPU time (s)	0.0154	0.0184
No of iter.	13	17
Case IVb		
CPU time (s)	0.0667	0.0754
No. of iter.	55	74
Case IVc		
CPU time (s)	0.0681	0.0902
No of iter.	79	106
Case IVd		
CPU time (s)	0.0358	0.0411
No of iter.	30	41

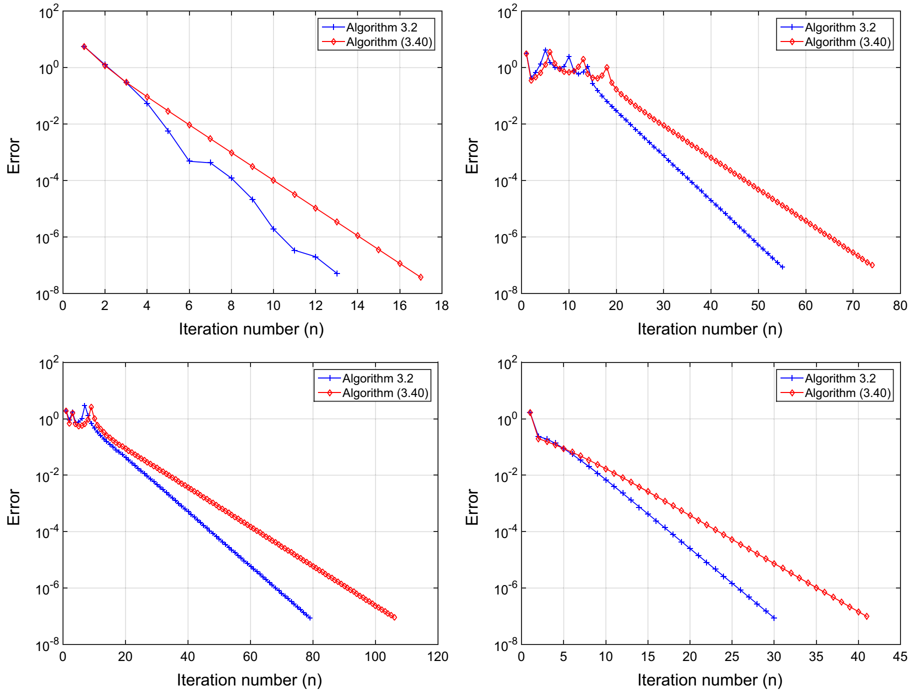


Fig. 4 Example 5.4: Top left: Case IVa; Top right: Case IVb; Bottom left: Case IVc; Bottom right: Case IVd

Case IIId: $x_0 = (7, \sqrt{7}, 1, \ldots)$; $x_1 = (18, 6, 2, \ldots)$.

We thus plot the graphs of errors against number of iterations in each case. The computational result can be found in Table 3 and Fig. 3.

Example 5.4 Let $E_1 = E_2 = \ell_3(\mathbb{R})$, where $\ell_3(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_i, \ldots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^3 < \infty\}$ with the norm $\|x\|_{\ell_3} = (\sum_{i=1}^{\infty} |x_i|^3)^{\frac{1}{3}}$, $\forall x \in E_1$. Let $C = C_1 := \{x \in$

$E_1 : ||x||_{\ell_3} \leq 1\}$. For all $x \in E_1$, we define $A : E_1 \to E_2$ by

$$Ax = \left(x_1, \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{3}}, \dots\right).$$

Let (e_n) be a sequence in $\ell_3(\mathbb{R})$ defined by $e_n = (\delta_{n,1}, \delta_{n,2}, \dots,)$ for each $n \in \mathbb{N}$, where

$$\delta_{n,i} = \begin{cases} 1 \text{ if } n = i, \\ 0 \text{ if } n \neq i. \end{cases} \tag{5.4}$$

Let $f(x) = \frac{1}{3}\|x\|^3_{\ell_3}$ for all $x \in \ell_3(\mathbb{R})$. We define $T : E_1 \to E_1$ by

$$Tx = \begin{cases} \frac{x}{n+1} \text{ if } x = e_n, \\ \frac{x}{2} \text{ if } x \neq e_n. \end{cases} \tag{5.5}$$

Note that $\tilde{F}(T) = \{\mathbf{0} = (0, 0, \dots)\} = F(T)$. Let $x \in E_1$. If $x = e_n$, for some $n \in \mathbb{N}$, then

$$\begin{aligned} \Delta_3(\mathbf{0}, Tx) &= f(\mathbf{0}) - f(Tx) - \langle J_3^{E_1}(Tx), \mathbf{0} - Tx\rangle \\ &= f(\mathbf{0}) - \frac{1}{(n+1)^3} f(x) - \frac{1}{(n+1)^3}\langle J_3^{E_1}x, \mathbf{0} - x\rangle \\ &= \frac{1}{(n+1)^3}\big(f(\mathbf{0}) - f(x) - \langle J_3^{E_1}x, \mathbf{0} - x\rangle\big) \\ &\leq \Delta_3(\mathbf{0}, x). \end{aligned}$$

If $x \neq e_n$, then

$$\begin{aligned} \Delta_3(\mathbf{0}, Tx) &= f(\mathbf{0}) - f(Tx) - \langle J_3^{E_1}(Tx), \mathbf{0} - Tx\rangle \\ &= f(\mathbf{0}) - \frac{1}{8} f(x) - \frac{1}{8}\langle J_3^{E_1}x, \mathbf{0} - x\rangle \\ &= \frac{1}{8}\big(f(\mathbf{0}) - f(x) - \langle J_3^{E_1}x, \mathbf{0} - x\rangle\big) \\ &\leq \Delta_3(\mathbf{0}, x). \end{aligned}$$

It therefore follows that T is Bregman weak relatively nonexpansive. We define $U : E_2 \to E_2$ by $Ux = \frac{3x}{4}$, $x \in E_2$. Furthermore, we choose $\mu_n = 0.0001$, $\theta_n = \frac{2n}{10n+3}$ and $\alpha_n = \frac{2n+1}{13n}$. We make different choices of initial values x_0 and x_1 as follows:

Case IVa: $x_0 = (8, 4, 2, \dots)$, $x_1 = (6, -2, \frac{2}{3}, \dots)$;
Case IVb: $x_0 = (3, -\frac{3}{2}, \frac{3}{4}, \dots)$, $x_1 = (-1, \frac{1}{\sqrt[3]{2}}, -\frac{1}{\sqrt[3]{4}}, \dots)$;
Case IVc: $x_0 = (-2, \sqrt[3]{4}, -\sqrt[3]{2}, \dots)$, $x_1 = (3, \sqrt[3]{3}, 1, \dots)$;
Case IVd: $x_0 = (4, 0.4, 0.04 \dots)$, $x_1 = (5, \frac{5}{4}, \frac{5}{16}, \dots)$.

Using MATLAB R2015(a), we compare the performance of Algorithms 3.2 and (3.40). The stopping criterion used for our computation is $\|x_{n+1} - x_n\|_{\ell_3} < 10^{-7}$. The duality mapping is computed using the formula in Example 2.1 and the Bregman projection is calculated using Proposition 5.1 in Alber and Butnariu [3] for a fixed constant $k > 0$. We plot the graphs of errors against the number of iterations in each case. The numerical results and figures are shown in Table 4 and Fig. 4, respectively.

Next, we apply our main result to an inverse problem stemming from image restoration problem. For most of the contents, we follow the recent works of Cholamjiak et al. [18] and Suparatulatorn et al. [45].

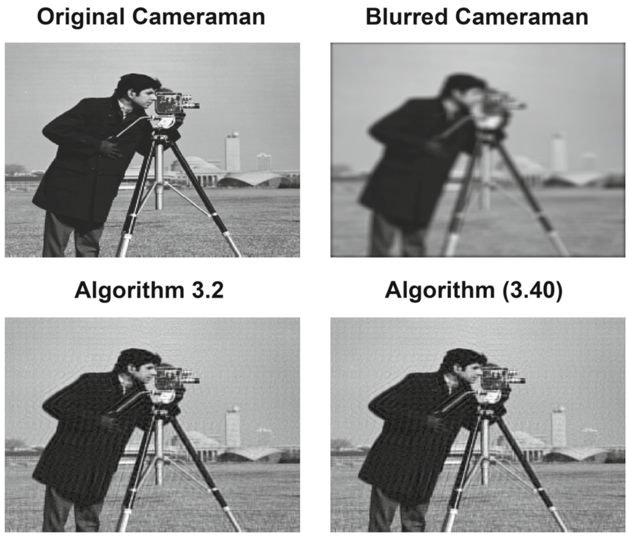


Fig. 5 Example 5.5: Top left: Original image; Top right: Blurred Image; Bottom left: Restored image by Algorithm 3.2 with SNR = 35.0321; Bottom right: Restored image by Algorithm (3.40) with SNR = 34.9926

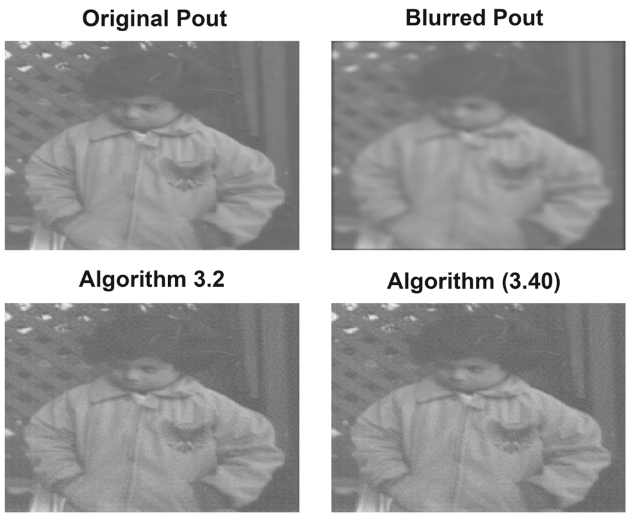


Fig. 6 Example 5.5: Top left: Original image; Top right: Blurred Image; Bottom left: Restored image by Algorithm 3.2 with SNR = 42.1303; Bottom right: Restored image by Algorithm (3.40) with SNR = 42.0827

Example 5.5 (Image Deblurring) We recall the following linear model used in image restoration problem:

$$y = A\bar{x} + \xi,$$

where $\bar{x}$ is the original image, y is the degraded image, A is a blurring matrix and ξ is the noise. For a grayscale image of M pixels wide and N pixels height, each pixel value is known to be in the range $[0, 255]$. Let $D := M \times N$. Then the underlying real Hilbert space is $\mathbb{R}^D$ equipped with the standard Euclidean norm $\|\cdot\|_2$, and $C = [0, 255]^D$. Our aim here is to

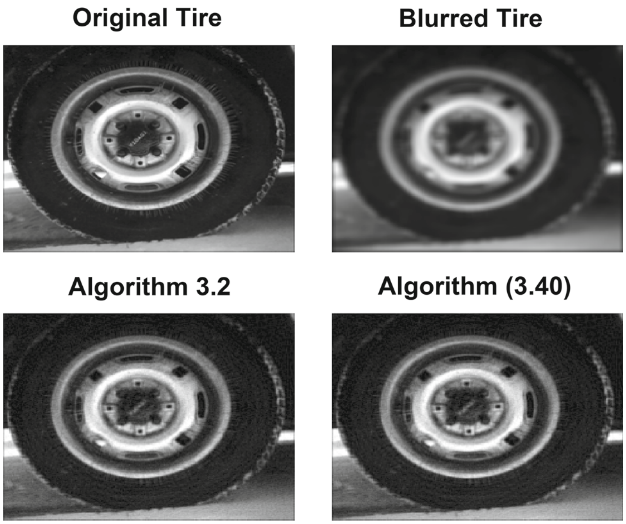


Fig. 7 Example 5.5: Top left: Original image; Top right: Blurred Image; Bottom left: Restored image by Algorithm 3.2 with SNR = 32.8327; Bottom right: Restored image by Algorithm (3.40) with SNR = 32.8150

Table 5 Numerical comparison of SNR (dB) values of Algorithms 3.2 and (3.40)

Images	n	Algorithm 3.2	Algorithm (3.40)
Cameraman.tif	100	29.0561	28.8228
(256 × 256)	500	33.8089	33.7228
	1000	35.0321	34.9926
Pout.tif	100	33.8904	33.5328
(291 × 240)	500	40.5099	40.3971
	1000	42.1303	42.0827
Tire.tif	100	27.7237	27.3897
(205 × 232)	500	32.2460	32.2039
	1000	32.8327	32.8150

recover the original image $\bar{x}$ given the data of the blurred image y and A. An approach to estimate an approximation of $\bar{x}$ is to recast the deblurring problem as the following convex minimization problem:

$$\min_{x} \|Ax - y\|_2. \tag{5.6}$$

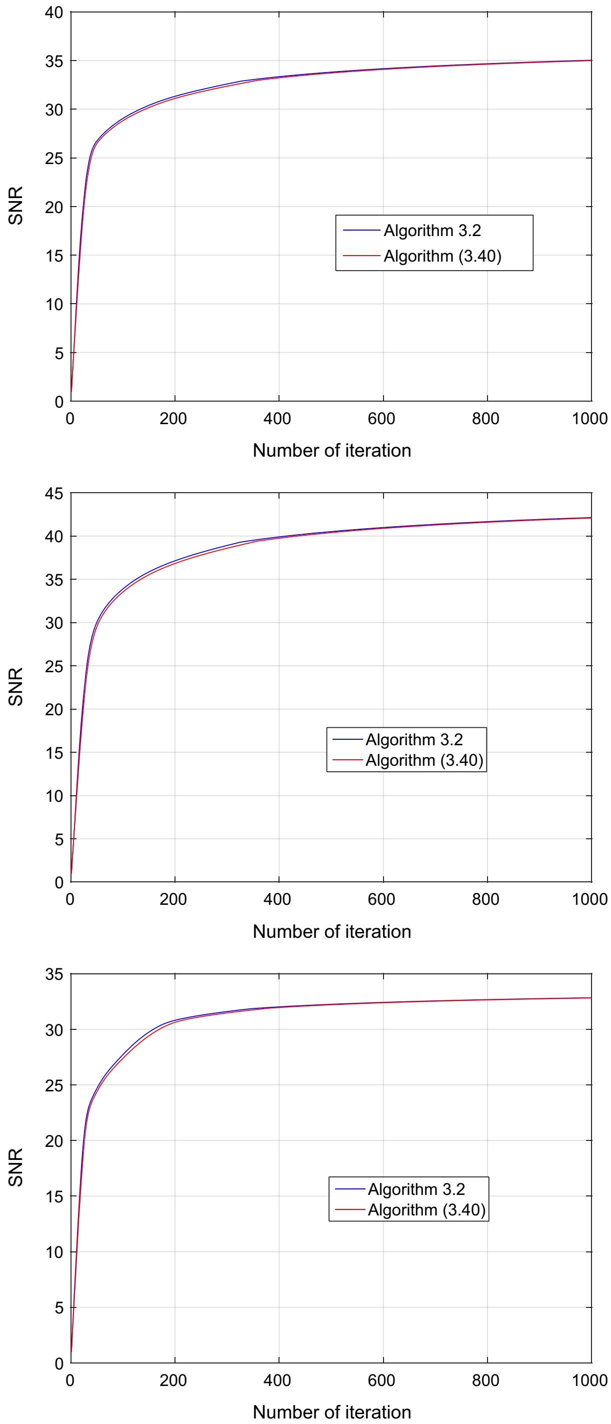
Setting $Q = \{y\}$, $T = P_C$ and $U = P_Q$, then (5.6) is equivalent to the following SCFPP:

$$\text{find } x \in F(T) \text{ such that } Ax \in F(Q).$$

It then follows that our Algorithm 3.2 can be used to solve the problem. Using MATLAB R2015(a), we apply Algorithm 3.2 to recover the original image $\bar{x}$ from the burred image y. The quality of the restored image is measured by the signal-to-noise ratio (SNR) in decibel (dB) as follows:

$$SNR := 20 \log_{10} \frac{\|\bar{x}\|_2}{\|x - \bar{x}\|_2},$$

Fig. 8 Example 5.5: Top: Cameraman.tif (256 × 256); Middle: Pout.tif (291 × 240); Bottom: Tire.tif (205 × 232)



where $\bar{x}$ is the original image and x is the restored image. The larger the SNR, the better the quality of the restored image. The initial values for our experiments are $x_0 = 0 \in \mathbb{R}^D$ and $x_1 = 1 \in \mathbb{R}^D$. The gray test images for our experiments are Cameraman, Pout and Tire. Each test image is degraded by Gaussian 7×7 blur kernel with standard deviation 4. We choose $\theta_n = \frac{n}{10n+5}$ and $\alpha_n = \frac{n}{3n+1}$. We test our Algorithms 3.2 and (3.40). The original, blurred and restored images by each of the algorithms are shown in Figs. 5, 6 and 7.
The computational results are shown in Table 5 and Fig. 8.

Remark 5.6 From the computational results, we see that Algorithm 3.2 performs better than Algorithm (3.40) in both CPU time taken and number of iteration. This illustrates the efficiency of the inertial extrapolation term.

6 Conclusion

We study the Split Common Fixed Point Problem (SCFPP) for a new mapping of type (P) in p-uniformly convex and uniformly smooth Banach spaces. We then propose an inertial shrinking projection algorithm and proved a strong convergence theorem for solving the SCFPP for mapping of type (P) and Bregman weak relatively nonexpansive mapping in p-uniformly convex and uniformly smooth Banach space. In addition, the implementation of our algorithm does not require an a priori estimate of the norm of the bounded linear operator. Lastly, we give numerical examples to demonstrate the performance of our algorithm and also apply our results to image deblurring problem.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interests.

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