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Numerical Analysis of Two Galerkin Discretizations with Graded Temporal Grids for Fractional Evolution Equations

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Abstract
Two numerical methods with graded temporal grids are analyzed for fractional evolution equations. One is a low-order discontinuous Galerkin (DG) discretization in the case of fractional order $0 < \alpha < 1$, and the other one is a low-order Petrov Galerkin (PG) discretization in the case of fractional order $1 < \alpha < 2$. By a new duality technique, pointwise-in-time error estimates of first-order and $(3-\alpha)$-order temporal accuracies are respectively derived for DG and PG, under reasonable regularity assumptions on the initial value. Numerical experiments are performed to verify the theoretical results.

Keywords Fractional evolution equation · Graded temporal grid · Convergence

1 Introduction

Let X be a separable Hilbert space with inner product $(\cdot,\cdot)_X$. Assume that the linear operator $A : D(A) \subset X \to X$ is densely defined and admits a bounded inverse $A^{-1} : X \to X$, which is compact, symmetric and positive. Consider the following time fractional evolution equation:

$$\left(D_{0+}^{\alpha}(u-u_0)\right)(t) + Au(t) = 0, \quad 0 < t \le T, \tag{1}$$

where $\alpha \in (0,2) \setminus \{1\}$, $0 < T < \infty$, $u_0 \in X$ and D_{0+}^{α} is a Riemann-Liouville fractional derivative operator of order α. Here, we assume that $u(0) = u_0$ for $\alpha \in (0,2) \setminus \{1\}$ and $u'(0) = 0$ for $\alpha \in (1,2)$.

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There are quite a few research works on the numerical treatment of time fractional evolution equations. Let us briefly introduce four types of numerical methods for the discretization of time fractional evolution equations. The first-type method uses the convolution quadrature to approximate the fractional integral (derivative) (cf. [2,6,16,17,36]). The second-type method uses the L1 scheme to approximate the fractional derivative (cf. [3,5,12,15,31,32]). Such methods are popular and easy to implement. The third-type method is the spectral method (cf. [9,14,20,33,34]), which uses nonlocal basis functions to approximate the solution. The accuracy of the spectral method is high, provided that the solution or data is smooth enough. The fourth-type method is the finite element method (cf. [10,13,19,21,24,25]), which uses local basis functions to approximate the solution. It should be mentioned that the finite element method is identical to the L1 scheme in some cases (cf. [7,12]).

Most of the convergence analyses for the numerical methods mentioned above are based on the assumption that the exact solution is smooth enough. However, the solution of fractional equations is generally singular near the origin despite how smooth the data is (cf. [6,8]). In fact, the main difficulty is to derive the error estimates without any regularity restriction on the solution, especially for the case with nonsmooth data. When using the uniform temporal grids, the Laplace transform technique is a powerful tool for error estimation in the case of nonsmooth data (cf. [2,5,12,17,21,32]). We note that the non-uniform temporal grids are also useful to handle the singularity of fractional equations (cf. [15,22,26,30]).

McLean and Mustapha analyzed the DG methods with graded temporal grids for a variant form of (1):

$$\begin{aligned} \partial_t u + D_{0+}^{1-\alpha} A u(t) &= 0, \quad 0 < t \leq T, \\ u(0) &= u_0, \end{aligned} \tag{2}$$

which is obtained by applying $D_{0+}^{1-\alpha}$ to the both sides of (1). For (2) with $0 < \alpha < 1$, they [22] derived first-order temporal accuracy for a piecewise-constant DG under the condition that $u_0 \in D(A^\nu)$ for $\nu > 0$. For the case $1 < \alpha < 2$, they [23] proved optimal error bounds for the piecewise-constant DG and a piecewise-linear DG under the condition that

$$\begin{aligned} t\|A\partial_t u(t)\|_X + t^2\|A\partial_{tt} u(t)\|_X \leq Ct^{M-1}, \quad 0 < t \leq T, \\ \|\partial_t u(t)\|_X + t\|\partial_{tt} u(t)\|_X \leq Ct^{M-1}, \quad 0 < t \leq T, \end{aligned}$$

where $0 < M \leq 1$ is a constant. For a fractional reaction-subdiffusion equation, Mustapha [26] derived second-order temporal accuracy for the L1 approximation with graded temporal grids under the condition that

$$\|u(t)\|_{H^2} \leq C, \quad \|\partial_t u(t)\|_{H^2} + t^{1-\alpha/2}\|\partial_{tt} u(t)\|_{H^1} + t^{2-\alpha/2}\|\partial_{ttt} u(t)\|_{H^1} \leq Ct^{M-1},$$

for all $0 < t \leq T$.

Though being equivalent to (2) in some senses, equation (1) leads to different kinds of numerical methods. For a fractional diffusion equation with nonsmooth data, Li et al. [11] obtained optimal error estimates for a low order DG. It should be noticed that their analysis is optimal in the sense of some space-time Sobolev norms, which is not very sharp when compared with the pointwise-in-time error estimates. For a fractional diffusion equation, Stynes et al. [30] analyzed the L1 scheme with graded temporal grids and derived temporal accuracy $O(N^{\alpha-2})$ (N is the number of nodes in the temporal grids) under the condition that

$$\|\partial_x^{(4)} u(t)\|_{L^\infty} \leq C, \quad \|\partial_{tt} u(t)\|_{L^\infty} \leq Ct^{\alpha-2}, \quad 0 < t \leq T.$$

Liao et al. [15] obtained temporal accuracy $O(N^{\alpha-2})$ for a reaction-subdiffusion equation by assuming that

$$\|\partial_x^{(4)} u(t)\|_{L^2} \leq C, \quad \|\partial_{tt} u(t)\|_{L^2} \leq C t^{M-2}, \quad 0 < t \leq T,$$

where $M \in (0, 2) \setminus \{1\}$. Although the regularity assumptions above are reasonable in some situations, it is worthwhile to carry out error estimation for some numerical methods with weaker regularity assumptions on the data. Moreover, as far as we know, there is no rigorous numerical analysis for (1) with $1 < \alpha < 2$ and graded temporal grids.

In this paper, we consider the DG and PG approximations for time fractional evolution equation (1) with $0 < \alpha < 1$ and $1 < \alpha < 2$ respectively. These methods are identical to the L1 scheme when the temporal grid is uniform. We develop a new duality technique for the pointwise-in-time error estimation, which is inspired by the local error estimation for the standard linear finite element method [1,29]. The key point of the analysis is the estimate of a "regularized Green function" (cf. Lemmas 3.3 and 4.2). For $0 < \alpha < 1$ and $u_0 \in D(A^\nu)$ with $0 < \nu \leq 1$, we obtain first-order temporal accuracy for the DG approximation with graded grids (cf. Theorem 3.1). For $1 < \alpha < 2$ and $u_0 \in D(A^\nu)$ with $1/2 < \nu \leq 1$, we obtain $(3-\alpha)$-order temporal accuracy for the PG approximation with graded grids (cf. Theorem 4.1).

The rest of this paper is organized as follows. Section 2 gives some notations and basic results, including Sobolev spaces, fractional calculus operators, spectral decomposition of A, solution theory and discretization spaces. Sections 3 and 4 establish the error estimates for problem (1) with $0 < \alpha < 1$ and $1 < \alpha < 2$ respectively. Section 5 performs two numerical experiments to verify the theoretical results. The last section is a conclusion.

2 Preliminaries

Throughout this paper, we will use the following conventions: if $\omega \subset \mathbb{R}$ is an interval, then $\langle p, q \rangle_\omega$ denotes the Lebesgue or Bochner integral $\int_\omega pq$ for scalar or vector valued functions p and q whenever the integral makes sense; for a Banach space W, we use $\langle \cdot, \cdot \rangle_W$ to denote a duality paring between W^* (the dual space of W) and W; the notation $C_\times$ denotes a positive constant depending only on its subscript(s), and its value may differ at each occurrence; for any function v defined on $(0, T)$, by $v(t-)$, $0 < t \leq T$ we mean $\lim_{s \to t-} v(s)$ whenever this limit exists; given $0 < a \leq T$, the notation $(a-t)_+$ denotes a function of variable t defined by

$$(a-t)_+ := \begin{cases} a - t & \text{if } 0 \leq t < a, \\ 0 & \text{if } a \leq t \leq T. \end{cases}$$

Sobolev spaces Assume that $-\infty < a < b < \infty$. For any $m \in \mathbb{N}$, define

$${}_0H^m(a, b) := \left\{ v \in H^m(a, b) : \ v^{(k)}(a) = 0 \quad \forall 0 \leq k < m \right\}$$

and endow this space with the norm

$$\|v\|_{{}_0H^m(a,b)} := \|v^{(m)}\|_{L^2(a,b)} \quad \forall v \in {}_0H^m(a, b),$$

where $H^m(a, b)$ is an usual Sobolev space and $v^{(k)}$, $1 \leq k \leq m$, is the k-th order weak derivative of v. For any $m \in \mathbb{N}_{>0}$ and $0 < \theta < 1$, define

$${}_0H^{m-1+\theta}(a, b) := \left({}_0H^{m-1}(a, b), {}_0H^m(a, b) \right)_{\theta,2},$$

where $(\cdot,\cdot)_{\theta,2}$ means the interpolation space defined by the K-method [18]. The space ${}^0H^\gamma(a,b)$, $0 \le \gamma < \infty$, is defined analogously. For each $-\infty < \gamma \le 0$, we use ${}_0H^\gamma(a,b)$ and ${}^0H^\gamma(a,b)$ to denote the dual spaces of ${}^0H^{-\gamma}(a,b)$ and ${}_0H^{-\gamma}(a,b)$, respectively. The embedding $L^2(a,b) \hookrightarrow {}_0H^{-\gamma}(a,b)$, $\gamma > 0$, is understood in the conventional sense that

$$\langle v, w\rangle_{{}^0H^\gamma(a,b)} := \langle v, w\rangle_{(a,b)} \quad \forall w \in {}^0H^\gamma(a,b), \quad \forall v \in L^2(a,b).$$

Fractional calculus operators Assume that $-\infty < a < b < \infty$. For $-\infty < \gamma < 0$, define

$$\left(D_{a+}^\gamma v\right)(t) := \frac{1}{\Gamma(-\gamma)} \int_a^t (t-s)^{-\gamma-1} v(s)\,\mathrm{d}s, \quad a < t < b,$$
$$\left(D_{b-}^\gamma v\right)(t) := \frac{1}{\Gamma(-\gamma)} \int_t^b (s-t)^{-\gamma-1} v(s)\,\mathrm{d}s, \quad a < t < b,$$

for all $v \in L^1(a,b)$, where $\Gamma(\cdot)$ is the gamma function. In addition, let D_{a+}^0 and D_{b-}^0 be the identity operator on $L^1(a,b)$. For $j-1 < \gamma \le j$ with $j \in \mathbb{N}_{>0}$, define

$$D_{a+}^\gamma v := D^j D_{a+}^{\gamma-j} v,$$
$$D_{b-}^\gamma v := (-D)^j D_{b-}^{\gamma-j} v,$$

for all $v \in L^1(a,b)$, where D is the first-order differential operator in the distribution sense. The vector-valued version fractional calculus operators are defined analogously. Assume that $0 < \beta \le \gamma < \beta + 1/2$. For any $v \in {}_0H^\beta(a,b)$, define $D_{a+}^\gamma v \in {}_0H^{\beta-\gamma}(a,b)$ by that

$$\left\langle D_{a+}^\gamma v, w\right\rangle_{{}^0H^{\gamma-\beta}(a,b)} := \left\langle D_{a+}^\beta v, D_{b-}^{\gamma-\beta} w\right\rangle_{(a,b)}$$

for all $w \in {}^0H^{\gamma-\beta}(a,b)$. For any $v \in {}^0H^\beta(a,b)$, define $D_{b-}^\gamma v \in {}^0H^{\beta-\gamma}(a,b)$ by that

$$\left\langle D_{b-}^\gamma v, w\right\rangle_{{}_0H^{\gamma-\beta}(a,b)} := \left\langle D_{b-}^\beta v, D_{a+}^{\gamma-\beta} w\right\rangle_{(a,b)}$$

for all $w \in {}_0H^{\gamma-\beta}(a,b)$. By Lemma A.2 and a standard density argument, it is easy to verify that the above definitions are well-defined and that if

$$\left\langle D_{a+}^\gamma v, w\right\rangle_{{}^0H^{\beta_1}(a,b)} \quad \text{and} \quad \left\langle D_{a+}^\gamma v, w\right\rangle_{{}^0H^{\beta_2}(a,b)}$$

both make sense by the definition, then they are identical.

Spectral decomposition of A Assume that the separable Hilbert space X is infinite dimensional. It is well known that (cf. [35]) there exists an orthonormal basis, $\{\phi_n : n \in \mathbb{N}\} \subset D(A)$, of X such that

$$A\phi_n = \lambda_n \phi_n,$$

where $\{\lambda_n : n \in \mathbb{N}\}$ is a positive non-decreasing sequence and $\lambda_n \to \infty$ as $n \to \infty$. For any $-\infty < \beta < \infty$, define

$$D\left(A^{\beta/2}\right) := \left\{ \sum_{n=0}^\infty c_n \phi_n : \sum_{n=0}^\infty \lambda_n^\beta c_n^2 < \infty \right\}$$

and equip this space with the norm

$$\left\| \sum_{n=0}^\infty c_n \phi_n \right\|_{D(A^{\beta/2})} := \left(\sum_{n=0}^\infty \lambda_n^\beta c_n^2 \right)^{1/2}.$$

Solution theory Recall that $\alpha \in (0, 2) \setminus \{1\}$. For any $\beta > 0$, define the Mittag-Leffler function $E_{\alpha,\beta}(z)$ by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)} \quad \forall z \in \mathbb{C},$$

which admits the following growth estimate (cf. [27]):

$$|E_{\alpha,\beta}(-t)| \le \frac{C_{\alpha,\beta}}{1+t} \quad \forall t > 0. \tag{3}$$

For any $\lambda > 0$, a straightforward calculation yields

$$D_{0+}^{\alpha}\left(E_{\alpha,1}(-\lambda t^{\alpha}) - 1\right) + \lambda E_{\alpha,1}(-\lambda t^{\alpha}) = 0 \quad \forall t \ge 0. \tag{4}$$

Therefore, the solution to problem (1) is of the form (cf. [28])

$$u(t) = \sum_{n=0}^{\infty} E_{\alpha,1}(-\lambda_n t^{\alpha})(u_0, \phi_n)_X \, \phi_n, \quad 0 \le t \le T. \tag{5}$$

For any $0 < t \le T$, a straightforward calculation gives

$$u'(t) = -\sum_{n=0}^{\infty} \lambda_n t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n t^{\alpha}\right)(u_0, \phi_n)_X \, \phi_n,$$
$$u''(t) = -\sum_{n=0}^{\infty} \lambda_n t^{\alpha-2} E_{\alpha,\alpha-1}\left(-\lambda_n t^{\alpha}\right)(u_0, \phi_n)_X \, \phi_n.$$

Hence, for $1 < \alpha < 2$, by (3) we obtain that

$$t^{-1}\|u'(t)\|_X + \|u''(t)\|_X \le C_\alpha t^{\alpha\nu-2}\|u_0\|_{D(A^\nu)}, \tag{6}$$
$$t^{-1}\|u'(t)\|_{D(A^{1/2})} + \|u''(t)\|_{D(A^{1/2})} \le C_\alpha t^{\alpha(\nu-1/2)-2}\|u_0\|_{D(A^\nu)}, \tag{7}$$

where $0 \le \nu \le 1$.

Discretization spaces Let $t_j := (j/J)^\sigma T$ for each $0 \le j \le J$, where $J \in \mathbb{N}_{>0}$ and $\sigma \ge 1$. Define

$$W_\tau := \left\{v \in L^\infty(0, T; D(A^{1/2})) : v \text{ is constant on } \left(t_{j-1}, t_j\right) \text{ for each } 1 \le j \le J\right\},$$
$$W_\tau^{\mathrm{c}} := \left\{v \in C\left([0, T]; D\left(A^{1/2}\right)\right) : v \text{ is linear on } \left(t_{j-1}, t_j\right) \text{ for each } 1 \le j \le J\right\}.$$

For the particular case $D(A) = \mathbb{R}$, we use $\mathcal{W}_\tau$ and $\mathcal{W}_\tau^{\mathrm{c}}$ to denote W_τ and W_τ^{c}, respectively. Assume that $Y = X$ or $\mathbb{R}$. For any $v \in L^1(0, T; Y)$ and $w \in C([0, T]; Y)$, define $Q_\tau v \in L^\infty(0, T; Y)$ and $\mathcal{I}_\tau w \in C([0, T]; Y)$ respectively by

$$(Q_\tau v)(t) := \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} v \quad \text{and} \quad (\mathcal{I}_\tau w)(t) := \frac{t_j - t}{t_j - t_{j-1}} w\left(t_{j-1}\right) + \frac{t - t_{j-1}}{t_j - t_{j-1}} w\left(t_j\right)$$

for all $t_{j-1} < t < t_j$ and $1 \le j \le J$. In the sequel, we will always assume that $\sigma \ge 1$.

3 Fractional Diffusion Equation ($0 < \alpha < 1$)

This section considers the following discretization (cf. [11]): seek $U \in W_\tau$ such that

$$\int_0^T \left(\left(D_{0+}^\alpha + A\right) U, V\right)_X \, \mathrm{d}t = \int_0^T \left(D_{0+}^\alpha u_0, V\right)_X \, \mathrm{d}t \quad \forall V \in W_\tau. \tag{8}$$

Remark 3.1 By (5), a straightforward calculation yields that

$$\int_0^T \left(\left(D_{0+}^\alpha u + A\right) u, V\right)_X \, \mathrm{d}t = \int_0^T \left(D_{0+}^\alpha u_0, V\right)_X \, \mathrm{d}t \quad \forall V \in W_\tau. \tag{9}$$

Theorem 3.1 *Assume that* $u_0 \in D(A^\nu)$ *with* $0 < \nu \le 1$. *Then*

$$\|u - U\|_{L^\infty(0,T;X)} \le C_{\alpha,\sigma,\nu,T} J^{-\min\{\sigma\nu\alpha,1\}} \|u_0\|_{D(A^\nu)}. \tag{10}$$

The main task of the rest of this section is to prove Theorem 3.1. To this end, we proceed as follows. Assume that $\lambda > 0$. For any $y \in {}_0H^{\alpha/2}(0,T)$, define $\Pi_\tau^\lambda y \in \mathcal{W}_\tau$ by that

$$\left\langle \left(D_{0+}^\alpha + \lambda\right) \left(y - \Pi_\tau^\lambda y\right), w \right\rangle_{{}_0H^{\alpha/2}(0,T)} = 0 \quad \forall w \in \mathcal{W}_\tau. \tag{11}$$

Remark 3.2 Note $\mathcal{W}_\tau$ is the piecewise constant finite element space.

For each $1 \le m \le J$, define $G_\lambda^m \in \mathcal{W}_\tau$ by that $G_\lambda^m|_{(t_m,T)} = 0$ and

$$\left\langle w, \left(D_{t_m-}^\alpha + \lambda\right) G_\lambda^m \right\rangle_{(0,t_m)} = \frac{1}{t_m - t_{m-1}} \int_{t_{m-1}}^{t_m} w \tag{12}$$

for all $w \in \mathcal{W}_\tau$. In addition, let $G_{\lambda,m+1}^m := 0$ and, for each $1 \le j \le m$, let

$$G_{\lambda,j}^m := \lim_{t \to t_j-} G_\lambda^m(t).$$

Remark 3.3 The G_λ^m can be viewed as a regularized Green function with respect to the operator $D_{t_m-}^\alpha + \lambda$.

Lemma 3.1 *For each* $1 \le m \le J$,

$$G_{\lambda,m}^m > G_{\lambda,m-1}^m > \cdots > G_{\lambda,1}^m > 0, \tag{13}$$

$$G_{\lambda,m}^m = \frac{1}{(t_m - t_{m-1})^{1-\alpha} / \Gamma(2-\alpha) + \lambda\,(t_m - t_{m-1})}, \tag{14}$$

$$G_{\lambda,m}^m = \sum_{j=1}^{m-1} \left(G_{\lambda,j+1}^m - G_{\lambda,j}^m\right) \frac{t_j^{1-\alpha} - \left(t_j - t_1\right)^{1-\alpha} + \lambda\Gamma(2-\alpha)t_1}{t_m^{1-\alpha} - (t_m - t_1)^{1-\alpha} + \lambda\Gamma(2-\alpha)t_1}. \tag{15}$$

Proof Let us first prove that

$$G_{\lambda,j+1}^m > G_{\lambda,j}^m \quad \text{for all } 1 \le j < m. \tag{16}$$

For any $1 \le k < m$, by (12) we obtain

$$\sum_{j=k}^m \left(G_{\lambda,j}^m - G_{\lambda,j+1}^m\right) \left(\left(t_j - t_{k-1}\right)^{1-\alpha} - \left(t_j - t_k\right)^{1-\alpha}\right) + \mu\,(t_k - t_{k-1})\, G_{\lambda,k}^m = 0,$$

where $\mu := \lambda\Gamma(2-\alpha)$, so that a simple algebraic computation yields

$$\begin{aligned}&\sum_{j=k}^{m-1}\left(G^m_{\lambda,j+1}-G^m_{\lambda,j}\right)\left(\left(t_j-t_{k-1}\right)^{1-\alpha}-(t_j-t_k)^{1-\alpha}+\mu\,(t_k-t_{k-1})\right)\\&\quad=G^m_{\lambda,m}\left((t_m-t_{k-1})^{1-\alpha}-(t_m-t_k)^{1-\alpha}+\mu\,(t_k-t_{k-1})\right).\end{aligned} \tag{17}$$

Inserting $k=m-1$ into the above equation and noting the fact $G^m_{\lambda,m}>0$ indicate $G^m_{\lambda,m}>G^m_{\lambda,m-1}$. Assume that $G^m_{\lambda,j+1}>G^m_{\lambda,j}$ for all $k\le j<m$, where $2\le k<m$. Multiplying both sides of (17) by

$$\frac{(t_m-t_{k-2})^{1-\alpha}-(t_m-t_{k-1})^{1-\alpha}+\mu\,(t_{k-1}-t_{k-2})}{(t_m-t_{k-1})^{1-\alpha}-(t_m-t_k)^{1-\alpha}+\mu\,(t_k-t_{k-1})},$$

from Lemma B.2 we obtain

$$\begin{aligned}&\sum_{j=k}^{m-1}\left(G^m_{\lambda,j+1}-G^m_{\lambda,j}\right)\left(\left(t_j-t_{k-2}\right)^{1-\alpha}-\left(t_j-t_{k-1}\right)^{1-\alpha}+\mu\,(t_{k-1}-t_{k-2})\right)\\&\quad<G^m_{\lambda,m}\left((t_m-t_{k-2})^{1-\alpha}-(t_m-t_{k-1})^{1-\alpha}+\mu\,(t_{k-1}-t_{k-2})\right).\end{aligned}$$

Similarly to (17), we have

$$\begin{aligned}&\sum_{j=k-1}^{m-1}\left(G^m_{\lambda,j+1}-G^m_{\lambda,j}\right)\left(\left(t_j-t_{k-2}\right)^{1-\alpha}-\left(t_j-t_k-1\right)^{1-\alpha}+\mu\,(t_{k-1}-t_{k-2})\right)\\&\quad=G^m_{\lambda,m}\left((t_m-t_{k-2})^{1-\alpha}-(t_m-t_{k-1})^{1-\alpha}+\mu\,(t_{k-1}-t_{k-2})\right).\end{aligned}$$

Combining the above two equations yields $G^m_{\lambda,k}>G^m_{\lambda,k-1}$. Therefore, (16) is proved by induction.

Next, inserting $k=1$ into (17) yields

$$\begin{aligned}&\sum_{j=1}^{m-1}\left(G^m_{\lambda,j+1}-G^m_{\lambda,j}\right)\left(t_j^{1-\alpha}-\left(t_j-t_1\right)^{1-\alpha}+\mu t_1\right)\\&\quad=G^m_{\lambda,m}\left(t_m^{1-\alpha}-(t_m-t_1)^{1-\alpha}+\mu t_1\right).\end{aligned} \tag{18}$$

Since

$$t_j^{1-\alpha}-\left(t_j-t_1\right)^{1-\alpha}+\mu t_1>t_m^{1-\alpha}-(t_m-t_1)^{1-\alpha}+\mu t_1\quad\forall 1\le j\le m-1,$$

from (16) and (18) it follows that

$$\sum_{j=1}^{m-1}\left(G^m_{\lambda,j+1}-G^m_{\lambda,j}\right)<G^m_{\lambda,m}.$$

This implies $G^m_{\lambda,1}>0$ and hence proves (13) by (16).

Finally, (14) is evident by (12), and dividing both sides of (18) by $t_m^{1-\alpha}-(t_m-t_1)^{1-\alpha}+\mu t_1$ proves (15). This completes the proof. □

Lemma 3.2 *For each* $1 \leq k \leq J$,

$$\sum_{j=1}^{k} j^{(\sigma-1)(\alpha-1)} \|(I - Q_\tau)\,(t_k - t)^{-\alpha}\|_{L^1(t_{j-1},t_j)} \leq C_{\alpha,\sigma,T} J^{\sigma(\alpha-1)}, \tag{19}$$

$$\sum_{j=1}^{k} j^{-\sigma-\alpha+1} \|(I - Q_\tau)(t_k - t)^{-\alpha}\|_{L^1(t_{j-1},t_j)} \leq C_{\alpha,\sigma,T} J^{\sigma(\alpha-1)} k^{-\sigma\alpha}. \tag{20}$$

Proof A straightforward calculation gives

$$\begin{aligned}
&k^{(\sigma-1)(\alpha-1)} \|(I - Q_\tau)\,(t_k - t)^{-\alpha}\|_{L^1(t_{k-1},t_k)} \\
&\quad \leq C_\alpha k^{(\sigma-1)(\alpha-1)}\,(t_k - t_{k-1})^{1-\alpha} \\
&\quad \leq C_{\alpha,\sigma,T} J^{-\sigma(1-\alpha)}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^{k-1} j^{(\sigma-1)(\alpha-1)} \|(I - Q_\tau)\,(t_k - t)^{-\alpha}\|_{L^1(t_{j-1},t_j)} \\
&\quad \leq C_\alpha \sum_{j=1}^{k-1} j^{(\sigma-1)(\alpha-1)} \left(t_j - t_{j-1}\right) \left(\left(t_k - t_j\right)^{-\alpha} - \left(t_k - t_{j-1}\right)^{-\alpha}\right) \\
&\quad \leq C_{\alpha,\sigma,T} J^{-\sigma(1-\alpha)} \sum_{j=1}^{k-1} j^{(\sigma-1)(\alpha-1)} \left(j^\sigma - (j-1)^\sigma\right) \left((k^\sigma - j^\sigma)^{-\alpha} - \left(k^\sigma - (j-1)^\sigma\right)^{-\alpha}\right) \\
&\quad \leq C_{\alpha,\sigma,T} J^{-\sigma(1-\alpha)} \sum_{j=1}^{k-1} j^{(\sigma-1)(\alpha-1)} j^{2(\sigma-1)} \left(k^\sigma - j^\sigma\right)^{-\alpha-1} \\
&\quad = C_{\alpha,\sigma,T} J^{-\sigma(1-\alpha)} \sum_{j=1}^{k-1} j^{(\sigma-1)(\alpha+1)} \left(k^\sigma - j^\sigma\right)^{-\alpha-1} \\
&\quad \leq C_{\alpha,\sigma,T} J^{-\sigma(1-\alpha)} \quad \text{(by Lemma } B.4\text{)}.
\end{aligned}$$

Combining the above two estimates proves (19). Similarly, a simple calculation gives

$$\begin{aligned}
&k^{-\sigma-\alpha+1} \|(I - Q_\tau)\,(t_k - t)^{-\alpha}\|_{L^1(t_{k-1},t_k)} \\
&\quad \leq C_\alpha k^{-\sigma-\alpha+1}\,(t_k - t_{k-1})^{1-\alpha} \\
&\quad \leq C_{\alpha,\sigma,T} J^{-\sigma(1-\alpha)} k^{-\sigma\alpha}
\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{j=1}^{k-1} j^{-\sigma-\alpha+1} \|(I-Q_\tau)\,(t_k-t)^{-\alpha}\|_{L^1(t_{j-1},t_j)} \\
&\leq C_\alpha \sum_{j=1}^{k-1} j^{-\sigma-\alpha+1} \left(t_j - t_{j-1}\right) \left(\left(t_k - t_j\right)^{-\alpha} - \left(t_k - t_{j-1}\right)^{-\alpha} \right) \\
&\leq C_{\alpha,\sigma,T} J^{-\sigma(1-\alpha)} \sum_{j=1}^{k-1} j^{-\sigma-\alpha+1} \left(j^\sigma - (j-1)^\sigma\right) \left((k^\sigma - j^\sigma)^{-\alpha} - \left(k^\sigma - (j-1)^\sigma\right)^{-\alpha} \right) \\
&\leq C_{\alpha,\sigma,T} J^{-\sigma(1-\alpha)} \sum_{j=1}^{k-1} j^{-\sigma-\alpha+1} j^{2(\sigma-1)} \left(k^\sigma - j^\sigma\right)^{-\alpha-1} \\
&= C_{\alpha,\sigma,T} J^{-\sigma(1-\alpha)} \sum_{j=1}^{k-1} j^{\sigma-\alpha-1} \left(k^\sigma - j^\sigma\right)^{-\alpha-1} \\
&\leq C_{\alpha,\sigma,T} J^{-\sigma(1-\alpha)} k^{-\sigma\alpha} \quad \text{(by Lemma } B.4\text{)}.
\end{aligned}
$$

Combining the above two estimates proves (20) and thus concludes the proof. □

Lemma 3.3 *For each* $1 \leq m \leq J$,

$$
\sum_{j=1}^{m} \left(\frac{m}{j}\right)^{(\sigma-1)(1-\alpha)} \|(I-Q_\tau)\, D_{t_m-}^\alpha G_\lambda^m\|_{L^1(t_{j-1},t_j)} \leq C_{\alpha,\sigma,T}. \tag{21}
$$

Proof For each $1 \leq j \leq m$, let

$$
\eta_j^m := \frac{(J/j)^{\sigma\alpha} + \lambda}{(J/m)^{\sigma\alpha} + \lambda} j^{(\sigma-1)(\alpha-1)} J^{\sigma(1-\alpha)}. \tag{22}
$$

Since

$$
\left(D_{t_m-}^\alpha G_\lambda^m\right)(t) = \sum_{j=1}^{m} \left(G_{\lambda,j}^m - G_{\lambda,j+1}^m\right) \frac{\left(t_j - t\right)_+^{-\alpha}}{\Gamma(1-\alpha)},
$$

we have

$$
\begin{aligned}
&\sum_{j=1}^{m} \eta_j^m \|(I - Q_\tau) D_{t_m-}^{\alpha} G_\lambda^m\|_{L^1(t_{j-1},t_j)} \\
&\leq \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{m} \eta_j^m \sum_{k=j}^{m} |G_{\lambda,k}^m - G_{\lambda,k+1}^m| \|(I - Q_\tau)(t_k - t)^{-\alpha}\|_{L^1(t_{j-1},t_j)} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{m} |G_{\lambda,k}^m - G_{\lambda,k+1}^m| \sum_{j=1}^{k} \eta_j^m \|(I - Q_\tau)(t_k - t)^{-\alpha}\|_{L^1(t_{j-1},t_j)} \\
&\leq C_{\alpha,\sigma,T} \sum_{k=1}^{m} |G_{\lambda,k}^m - G_{\lambda,k+1}^m| \frac{J^{\sigma(1-\alpha)}}{(J/m)^{\sigma\alpha} + \lambda} \\
&\quad \times \sum_{j=1}^{k} \left((J/j)^{\sigma\alpha} + \lambda\right) j^{(\sigma-1)(\alpha-1)} \|(I - Q_\tau)(t_k - t)^{-\alpha}\|_{L^1(t_{j-1},t_j)} \\
&\leq C_{\alpha,\sigma,T} \sum_{k=1}^{m} \frac{(J/k)^{\sigma\alpha} + \lambda}{(J/m)^{\sigma\alpha} + \lambda} |G_{\lambda,k}^m - G_{\lambda,k+1}^m| \quad \text{(by (19) and (20))}.
\end{aligned}
$$

Therefore, from Lemma 3.1 and the inequality

$$
\frac{t_k^{1-\alpha} - (t_k - t_1)^{1-\alpha} + \lambda\Gamma(2-\alpha)t_1}{t_m^{1-\alpha} - (t_m - t_1)^{1-\alpha} + \lambda\Gamma(2-\alpha)t_1} \geq C_{\alpha,\sigma,T} \frac{(J/k)^{\sigma\alpha} + \lambda}{(J/m)^{\sigma\alpha} + \lambda},
$$

it follows that

$$
\sum_{j=1}^{m} \eta_j^m \|(I - Q_\tau) D_{t_m-}^{\alpha} G_\lambda^m\|_{L^1(t_{j-1},t_j)} \leq C_{\alpha,\sigma,T} G_{\lambda,m}^m.
$$

In addition, by (14) and (22), it holds

$$
\begin{aligned}
\frac{\eta_j^m}{G_{\lambda,m}^m} &\geq C_{\alpha,\sigma,T} \frac{(J/j)^{\sigma\alpha} + \lambda}{(J/m)^{\sigma\alpha} + \lambda} j^{(\sigma-1)(\alpha-1)} J^{\sigma(1-\alpha)} \left(m^{(\sigma-1)(1-\alpha)} J^{\sigma(\alpha-1)} + \lambda m^{\sigma-1} J^{-\sigma}\right) \\
&\geq C_{\alpha,\sigma,T} \frac{(J/j)^{\sigma\alpha} + \lambda}{(J/m)^{\sigma\alpha} + \lambda} j^{(\sigma-1)(\alpha-1)} m^{(\sigma-1)(1-\alpha)} \\
&\geq C_{\alpha,\sigma,T} (m/j)^{(\sigma-1)(1-\alpha)}.
\end{aligned}
$$

Consequently, combining the above two estimates proves (21) and thus concludes the proof. ◻

Remark 3.4 $D_{t_m-}^{\alpha} G_\lambda^m$ is a non-smooth function in $L^1(0, T)$, but it is smoother away from t_m. This is the starting point of Lemma 3.3.

Lemma 3.4 *If* $y \in {}_0H^{\alpha/2}(0, T) \cap C(0, T]$, *then*

$$
\left(\Pi_\tau^\lambda y - Q_\tau y\right)(t_m-) = \left\langle (I - Q_\tau) y,\ (I - Q_\tau) D_{t_m-}^{\alpha} G_\lambda^m \right\rangle_{(0,t_m)} \tag{23}
$$

for each $1 \leq m \leq J$.

Proof A straightforward calculation gives

$$
\begin{aligned}
&\left(\Pi_\tau^\lambda y - Q_\tau y\right)(t_m-) \\
&= \left\langle \Pi_\tau^\lambda y - Q_\tau y,\ \left(D_{t_m-}^\alpha + \lambda\right) G_\lambda^m \right\rangle_{(0,t_m)} && \text{(by (12))} \\
&= \left\langle \Pi_\tau^\lambda y - Q_\tau y,\ (D_{T-}^\alpha + \lambda) G_\lambda^m \right\rangle_{(0,T)} && \text{(by the fact } G_\lambda^m|_{(t_m,T)} = 0) \\
&= \left\langle \left(D_{0+}^\alpha + \lambda\right)\left(\Pi_\tau^\lambda y - Q_\tau y\right),\ G_\lambda^m \right\rangle_{{}_0H^{\alpha/2}(0,T)} && \text{(by Lemma A.3)} \\
&= \left\langle \left(D_{0+}^\alpha + \lambda\right)(I - Q_\tau)\, y,\ G_\lambda^m \right\rangle_{{}_0H^{\alpha/2}(0,T)} && \text{(by (11))} \\
&= \left\langle (I - Q_\tau)\, y,\ \left(D_{T-}^\alpha + \lambda\right) G_\lambda^m \right\rangle_{(0,T)} && \text{(by Lemma A.3)} \\
&= \left\langle (I - Q_\tau) y,\ (D_{t_m-}^\alpha + \lambda) G_\lambda^m \right\rangle_{(0,t_m)} && \text{(by the fact } G_\lambda^m|_{(t_m,T)} = 0).
\end{aligned}
$$

Hence, (23) follows from the equality

$$
\left\langle (I - Q_\tau)\, y, \left(D_{t_m-}^\alpha + \lambda\right) G_\lambda^m \right\rangle_{(0,t_m)} = \left\langle (I - Q_\tau)\, y, (I - Q_\tau)\, D_{t_m-}^\alpha G_\lambda^m \right\rangle_{(0,t_m)},
$$

which is easily derived by the definition of Q_τ. This completes the proof. □

Lemma 3.5 *Assume that* $y \in {}_0H^{\alpha/2}(0,T) \cap C^1(0,T]$ *satisfies*

$$
|y'(t)| \le t^{-r}, \quad 0 < t \le T, \tag{24}
$$

where $0 < r < 1$. *Then*

$$
\left\| \left(I - \Pi_\tau^\lambda\right) y \right\|_{L^\infty(0,T)} \le C_{\alpha,\sigma,r,T} J^{-\min\{\sigma(1-r),1\}}. \tag{25}
$$

Proof For any $1 \le m \le J$,

$$
\begin{aligned}
&\left| \left(\Pi_\tau^\lambda y - Q_\tau y\right)(t_m-) \right| \\
&\quad = \left| \left\langle (I - Q_\tau) y, (I - Q_\tau) D_{t_m-}^\alpha G_\lambda^m \right\rangle_{(0,t_m)} \right| \quad \text{(by Lemma 3.4)} \\
&\quad \le \sum_{j=1}^m \|(I - Q_\tau)\, y\|_{L^\infty(t_{j-1},t_j)} \|(I - Q_\tau)\, D_{t_m-}^\alpha G_\lambda^m\|_{L^1(t_{j-1},t_j)} \\
&\quad \le \max_{1 \le j \le m} (m/j)^{(\sigma-1)(\alpha-1)} \|(I - Q_\tau)\, y\|_{L^\infty(t_{j-1},t_j)} \\
&\qquad \times \sum_{j=1}^m (m/j)^{(\sigma-1)(1-\alpha)} \|(I - Q_\tau)\, D_{t_m-}^\alpha G_\lambda^m\|_{L^1(t_{j-1},t_j)} \\
&\quad \le C_{\alpha,\sigma,T} \max_{1 \le j \le m} (m/j)^{(\sigma-1)(\alpha-1)} \|(I - Q_\tau) y\|_{L^\infty(t_{j-1},t_j)} \quad \text{(by (21))} \\
&\quad \le C_{\alpha,\sigma,T} \|(I - Q_\tau) y\|_{L^\infty(0,t_m)}.
\end{aligned}
$$

It follows that

$$
\left\| \left(\Pi_\tau^\lambda - Q_\tau\right) y \right\|_{L^\infty(0,T)} = \max_{1 \le m \le J} \left| \left(\Pi_\tau^\lambda y - Q_\tau y\right)(t_m-) \right| \le C_{\alpha,\sigma,T} \|(I - Q_\tau) y\|_{L^\infty(0,T)},
$$

and hence

$$
\begin{aligned}
\left\| \left(I - \Pi_\tau^\lambda\right) y \right\|_{L^\infty(0,T)} &\le \|(I - Q_\tau) y\|_{L^\infty(0,T)} + \left\| \left(\Pi_\tau^\lambda - Q_\tau\right) y \right\|_{L^\infty(0,T)} \\
&\le C_{\alpha,\sigma,T} \|(I - Q_\tau) y\|_{L^\infty(0,T)}.
\end{aligned}
$$

In addition, by (24) we obtain

$$\begin{aligned}\|(I-Q_\tau)\,y\|_{L^\infty(0,T)} &\leq \max_{1\leq j\leq J}\|(I-Q_\tau)\,y\|_{L^\infty(t_{j-1},t_j)}\\ &\leq \max_{1\leq j\leq J}\left(t_j^{1-r}-t_j^{1-r}\right)/(1-r)\\ &\leq C_{\alpha,\sigma,r,T}\max_{1\leq j\leq J} j^{\sigma(1-r)-1}J^{-\sigma(1-r)}\\ &\leq C_{\alpha,\sigma,r,T}\,J^{-\min\{\sigma(1-r),1\}}.\end{aligned}$$

Finally, combining the above two estimates proves (25) and hence this lemma. □

Finally, we are in a position to prove Theorem 3.1 as follows.

Proof of Theorem 3.1. For each $n\in\mathbb{N}$, let

$$u^n(t) := (u(t),\phi_n)_X,\quad 0\leq t\leq T.$$

By (5) we have

$$u^n(t) = E_{\alpha,1}\left(-\lambda_n t^\alpha\right)(u_0,\phi_n)_X\,,\quad 0<t\leq T.$$

A straightforward calculation gives

$$(u^n)'(t) = -\lambda_n t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n t^\alpha\right)(u_0,\phi_n)_X\,\phi_n,\quad 0\leq t\leq T,$$

and hence (3) implies

$$|\left(u^n\right)'(t)| \leq C_\alpha t^{\nu\alpha-1}\lambda_n^\nu|(u_0,\phi_n)_X|,\quad 0<t\leq T. \tag{26}$$

By (8, (9) and (11) we have $U=\sum_{n=0}^\infty(\Pi_\tau^{\lambda_n}u^n)\phi_n$, so that

$$\begin{aligned}\|u-U\|_{L^\infty(0,T;X)} &= \sup_{0<t<T}\left(\sum_{n=0}^\infty\left|\left(u^n-\Pi_\tau^{\lambda_n}u^n\right)(t)\right|^2\right)^{1/2}\\ &\leq \left(\sum_{n=0}^\infty\left\|\left(I-\Pi_\tau^{\lambda_n}\right)u^n\right\|_{L^\infty(0,T)}^2\right)^{1/2}\\ &\leq C_{\alpha,\sigma,T}\,J^{-\min\{\sigma\alpha\nu,1\}}\left(\sum_{n=0}^\infty\lambda_n^{2\nu}(u_0,\phi_n)_X^2\right)^{1/2}\quad\text{(by Lemma 3.5 \textit{and} (26))}\\ &= C_{\alpha,\sigma,T}\,J^{-\min\{\sigma\alpha\nu,1\}}\|u_0\|_{D(A^\nu)}.\end{aligned}$$

This proves (10) and thus concludes the proof. □

4 Fractional Diffusion-Wave Equation ($1<\alpha<2$)

This section considers the following discretization: seek $U\in W_\tau^{\mathrm{c}}$ such that $U(0)=u_0$ and

$$\int_0^T\left(D_{0+}^{\alpha-1}U'+AU,V\right)_X\,\mathrm{d}t = 0\quad\forall V\in W_\tau. \tag{27}$$

Remark 4.1 By (5), a straightforward calculation gives that $u(0) = u_0$, $u'(0) = 0$ and

$$\int_0^T (D_{0+}^{\alpha-1} u' + Au, V)_X \, \mathrm{d}t = 0 \quad \forall V \in W_\tau. \tag{28}$$

Remark 4.2 For the case with uniform temporal grids, the discretization (27) is equivalent to the L1 scheme (cf. [12]),

Theorem 4.1 *Assume that* $u_0 \in D(A^\nu)$ *with* $1/2 < \nu \le 1$. *If*

$$\sigma > \frac{3-\alpha}{\alpha(\nu - 1/2)}, \tag{29}$$

then

$$\max_{1 \le m \le J} \|(u - U)(t_m)\|_X \le C_{\alpha,\sigma,T} J^{\alpha-3} \|u_0\|_{D(A^\nu)}. \tag{30}$$

The main task of the rest of this section is to prove the above theorem. For each $1 \le m \le J$, define $\mathcal{G}^m \in \mathcal{W}_\tau$ by that $\mathcal{G}^m|_{(t_m,T)} = 0$ and

$$\left\langle w, D_{t_m-}^{\alpha-1} \mathcal{G}^m \right\rangle_{(0,t_m)} = \langle 1, w \rangle_{(0,t_m)} \quad \forall w \in \mathcal{W}_\tau. \tag{31}$$

Let $\mathcal{G}_{m+1}^m = 0$ and, for each $1 \le j \le m$, let

$$\mathcal{G}_j^m := \lim_{t \to t_j -} \mathcal{G}^m(t).$$

Since

$$D_{t_m-}^{\alpha-1} \mathcal{G}^m = \sum_{j=1}^m \left(\mathcal{G}_j^m - \mathcal{G}_{j+1}^m \right) \frac{(t_j - t)_+^{1-\alpha}}{\Gamma(2-\alpha)},$$

a straightforward calculation yields, from (31), that

$$\sum_{j=k}^m \left(\mathcal{G}_j^m - \mathcal{G}_{j+1}^m \right) \left((t_j - t_{k-1})^{2-\alpha} - (t_j - t_k)^{2-\alpha} \right) = \Gamma(3-\alpha)(t_k - t_{k-1}) \tag{32}$$

for each $1 \le k \le m$.

Remark 4.3 Although $\mathcal{G}^m$ is not a regularized Green function, it has similar properties.

Lemma 4.1 *For any* $1/2 < \beta < 1$ *and* $1 \le k \le J$,

$$\sum_{j=1}^k (j/J)^{\sigma(1-\alpha)} \left(\left(t_k - t_{j-1}\right)^{1-\beta} - \left(t_k - t_j\right)^{1-\beta} \right) \le C_{\alpha,\sigma,T} (k/J)^{\sigma(2-\alpha-\beta)}. \tag{33}$$

Proof An elementary calculation gives

$$k^{\sigma(1-\alpha)} \left(k^\sigma - (k-1)^\sigma\right)^{1-\beta} \le C_\sigma k^{\sigma(1-\alpha)} k^{(\sigma-1)(1-\beta)} = C_\sigma k^{\sigma(2-\alpha-\beta)+\beta-1}$$

and

$$\begin{aligned}
&\sum_{j=1}^{k-1} j^{\sigma(1-\alpha)} \left(\left(k^\sigma - (j-1)^\sigma\right)^{1-\beta} - \left(k^\sigma - j^\sigma\right)^{1-\beta} \right) \\
&\quad \le C_\sigma (1-\beta) \sum_{j=1}^{k-1} j^{\sigma(1-\alpha)} \left(k^\sigma - j^\sigma\right)^{-\beta} j^{\sigma-1} \\
&\quad = C_\sigma (1-\beta) \sum_{j=1}^{k-1} j^{2\sigma-\sigma\alpha-1} \left(k^\sigma - j^\sigma\right)^{-\beta} \\
&\quad \le C_{\alpha,\sigma} k^{\sigma(2-\alpha-\beta)} \quad \text{(by Lemma } B.5).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{j=1}^{k} (j/J)^{\sigma(1-\alpha)} \left(\left(t_k - t_{j-1}\right)^{1-\beta} - \left(t_k - t_j\right)^{1-\beta} \right) \\
&= J^{-\sigma(2-\alpha-\beta)} T^{1-\beta} \sum_{j=1}^{k} j^{\sigma(1-\alpha)} \left(\left(k^\sigma - (j-1)^\sigma\right)^{1-\beta} - \left(k^\sigma - j^\sigma\right)^{1-\beta} \right) \\
&\le C_{\alpha,\sigma,T} J^{-\sigma(2-\alpha-\beta)} \left(k^{\sigma(2-\alpha-\beta)+\beta-1} + k^{\sigma(2-\alpha-\beta)} \right) \\
&\le C_{\alpha,\sigma,T} (k/J)^{\sigma(2-\alpha-\beta)}.
\end{aligned}$$

This proves (33) and hence this lemma. □

Lemma 4.2 *For any* $1/2 < \beta < 1$ *and* $1 \le m \le J$,

$$\sum_{j=1}^{m} (j/J)^{\sigma(1-\alpha)} \| D_{t_m-}^{\beta} \mathcal{G}^m \|_{L^1(t_{j-1},t_j)} \le C_{\alpha,\sigma,T}. \tag{34}$$

Proof By (32) and Lemma B.3, an inductive argument yields that

$$\mathcal{G}_1^m > \mathcal{G}_2^m > \cdots > \mathcal{G}_m^m = \Gamma(3-\alpha)\,(t_m - t_{m-1})^{\alpha-1}. \tag{35}$$

Plugging $k = 1$ into (32) shows

$$\sum_{j=1}^{m} \left(\mathcal{G}_j^m - \mathcal{G}_{j+1}^m \right) \left(t_j^{2-\alpha} - (t_j - t_1)^{2-\alpha} \right) = \Gamma(3-\alpha) t_1,$$

and hence

$$\sum_{j=1}^{m} \frac{t_j^{2-\alpha} - \left(t_j - t_1\right)^{2-\alpha}}{\Gamma(3-\alpha) t_1} \left(\mathcal{G}_j^m - \mathcal{G}_{j+1}^m \right) = 1.$$

From (35) and the inequality

$$\frac{t_j^{2-\alpha} - \left(t_j - t_1\right)^{2-\alpha}}{\Gamma(3-\alpha) t_1} \ge C_{\alpha,\sigma,T} (j/J)^{\sigma(1-\alpha)},$$

it follows that

$$\sum_{j=1}^{m}(j/J)^{\sigma(1-\alpha)}\left(\mathcal{G}_j^m-\mathcal{G}_{j+1}^m\right)\leq C_{\alpha,\sigma,T}. \tag{36}$$

Since

$$D_{t_m-}^{\beta}\mathcal{G}^m=\sum_{j=1}^{m}\left(\mathcal{G}_j^m-\mathcal{G}_{j+1}^m\right)\frac{(t_j-t)_+^{-\beta}}{\Gamma(1-\beta)},$$

we obtain

$$\begin{aligned}
&\sum_{j=1}^{m}(j/J)^{\sigma(1-\alpha)}\|D_{t_m-}^{\beta}\mathcal{G}^m\|_{L^1(t_{j-1},t_j)}\\
&\leq\sum_{j=1}^{m}(j/J)^{\sigma(1-\alpha)}\sum_{k=j}^{m}(\mathcal{G}_k^m-\mathcal{G}_{k+1}^m)\frac{\left(t_k-t_{j-1}\right)^{1-\beta}-(t_k-t_j)^{1-\beta}}{\Gamma(2-\beta)}\quad(\text{by }(35))\\
&=\sum_{k=1}^{m}\left(\mathcal{G}_k^m-\mathcal{G}_{k+1}^m\right)\sum_{j=1}^{k}(j/J)^{\sigma(1-\alpha)}\frac{\left(t_k-t_{j-1}\right)^{1-\beta}-\left(t_k-t_j\right)^{1-\beta}}{\Gamma(2-\beta)}\\
&\leq C_{\alpha,\sigma,T}\sum_{k=1}^{m}(k/J)^{\sigma(2-\alpha-\beta)}\left(\mathcal{G}_k^m-\mathcal{G}_{k+1}^m\right)\quad(\text{by Lemma 4.1} and (35))\\
&\leq C_{\alpha,\sigma,T}\quad(\text{by}(35)\text{and}(36)).
\end{aligned}$$

This proves (34) and thus completes the proof. □

Remark 4.4 For more details about proving (35), we refer the reader to the proof of (13).

Lemma 4.3 *Assume that* $y\in C^2((0,T];X)$ *satisfies*

$$t^{-1}\|y'(t)\|_X+\|y''(t)\|_X\leq t^{-r},\quad 0<t\leq T, \tag{37}$$

where $0<r<2$. *For each* $1\leq j\leq J$, *the following three estimates hold:*
if $\sigma<2/(3-r)$, *then*

$$\left\|D_{0+}^{\alpha-2}\left(I-Q_\tau\right)y'\right\|_{L^\infty(t_{j-1},t_j;X)}\leq C_{\alpha,\sigma,r,T}J^{-\sigma(3-\alpha-r)}j^{-\sigma\alpha}\left(j^{\sigma(3-r)+\alpha-3}+1\right); \tag{38}$$

if $\sigma=2/(3-r)$, *then*

$$\left\|D_{0+}^{\alpha-2}(I-Q_\tau)y'\right\|_{L^\infty(t_{j-1},t_j;X)}\leq C_{\alpha,r,T}J^{-\sigma(3-\alpha-r)}j^{-\sigma\alpha}\left(j^{\sigma(3-r)+\alpha-3}+\ln j\right); \tag{39}$$

if $\sigma>2/(3-r)$, *then*

$$\left\|D_{0+}^{\alpha-2}\left(I-Q_\tau\right)y'\right\|_{L^\infty(t_{j-1},t_j;X)}\leq C_{\alpha,\sigma,r,T}J^{-\sigma(3-\alpha-r)}j^{\sigma(3-\alpha-r)+\alpha-3}. \tag{40}$$

Proof We only present a proof of (40), the proofs of (38) and (39) being similar. Since the case $r=1$ can be proved analogously, we assume that $r\neq 1$.

Let us first prove that

$$\sup_{t_{j-1}\leq a<t_j}\left\|\left\langle(a-t)^{1-\alpha},\ (I-Q_\tau)\,y'\right\rangle_{(0,t_{j-1})}\right\|_X\leq C_{\alpha,\sigma,r,T}J^{-\sigma(3-\alpha-r)}j^{\sigma(3-\alpha-r)+\alpha-3} \tag{41}$$

for each $2 \leq j \leq J$. Since the case $j = 2$ can be easily verified, we assume that $3 \leq j \leq J$. Let $t_{j-1} \leq a < t_j$. By the definition of Q_τ, we have

$$\left\| \left\langle (a-t)^{1-\alpha}, (I-Q_\tau)y' \right\rangle_{(0,t_{j-1})} \right\|_X \leq \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3, \tag{42}$$

where

$$\begin{aligned}
\mathbb{I}_1 &:= \left\| \left\langle (I-Q_\tau)(a-t)^{1-\alpha}, (I-Q_\tau)y' \right\rangle_{(0,t_1)} \right\|_X, \\
\mathbb{I}_2 &:= \sum_{k=2}^{j-2} \left\| \left\langle (I-Q_\tau)\,(a-t)^{1-\alpha}, (I-Q_\tau)\,y' \right\rangle_{(t_{k-1},t_k)} \right\|_X, \\
\mathbb{I}_3 &:= \left\| \left\langle (I-Q_\tau)(a-t)^{1-\alpha}, (I-Q_\tau)\,y' \right\rangle_{(t_{j-2},t_{j-1})} \right\|_X.
\end{aligned}$$

By (37) and the facts $\sigma > 2/(3-r)$ and $t_{j-1} \leq a$, a routine calculation yields the following three estimates:

$$\begin{aligned}
\mathbb{I}_1 &\leq \|(I-Q_\tau)\,(a-t)^{1-\alpha}\|_{L^\infty(0,t_1)} \|(I-Q_\tau)\,y'\|_{L^1(0,t_1;X)} \\
&\leq C_{\alpha,r} \left((a-t_1)^{1-\alpha} - a^{1-\alpha}\right) t_1^{2-r} \\
&\leq C_{\alpha,r} \left(\left(t_{j-1}-t_1\right)^{1-\alpha} - t_{j-1}^{1-\alpha}\right) t_1^{2-r} \\
&\leq C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)} \left(\left((j-1)^\sigma - 1\right)^{1-\alpha} - (j-1)^{\sigma(1-\alpha)}\right) \\
&\leq C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)}\, j^{-\sigma\alpha} \\
&\leq C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)}\, j^{\sigma(3-\alpha-r)+\alpha-3}, \\
\mathbb{I}_2 &\leq C_\alpha \sum_{k=2}^{j-2} \|(I-Q_\tau)y'\|_{L^\infty(t_{k-1},t_k;X)} (t_k - t_{k-1})\big((a-t_k)^{1-\alpha} - (a-t_{k-1})^{1-\alpha}\big) \\
&\leq C_{\alpha,r} \sum_{k=2}^{j-2} \left| t_k^{1-r} - t_{k-1}^{1-r} \right| (t_k - t_{k-1})((t_{j-1}-t_k)^{1-\alpha} - (t_{j-1}-t_{k-1})^{1-\alpha}) \\
&\leq C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)} \sum_{k=2}^{j-2} k^{\sigma(1-r)-1} k^{2(\sigma-1)} (j^\sigma - k^\sigma)^{-\alpha} \\
&= C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)} \sum_{k=2}^{j-2} k^{3\sigma-\sigma r-3} (j^\sigma - k^\sigma)^{-\alpha} \\
&\leq C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)}\, j^{\sigma(3-\alpha-r)+\alpha-3} \quad \text{(by Lemma } B.4\text{)}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{I}_3 &\leq C_\alpha \|(I-Q_\tau)y'\|_{L^\infty(t_{j-2},t_{j-1};X)} \left(\left(a-t_{j-2}\right)^{2-\alpha} - (a-t_{j-1})^{2-\alpha}\right) \\
&\leq C_{\alpha,r} \left| t_{j-1}^{1-r} - t_{j-2}^{1-r} \right| \left(t_{j-1} - t_{j-2}\right)^{2-\alpha} \\
&\leq C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)}\, j^{\sigma(1-r)-1}\, j^{(\sigma-1)(2-\alpha)} \\
&= C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)}\, j^{\sigma(3-\alpha-r)+\alpha-3}.
\end{aligned}$$

Since a, $t_{j-1} \leq a < t_j$, is arbitrary, combining (42) and the above three estimates proves (41) for $3 \leq j \leq J$.

Next, let us prove that (40) holds for all $2 \leq j \leq J$. For any $t_{j-1} \leq a < t_j$,

$$\left(D_{0+}^{\alpha-2}\left(y' - Q_\tau y'\right)\right)(a) = (\mathbb{I}_4 + \mathbb{I}_5)\,/\,\Gamma(2-\alpha), \tag{43}$$

where

$$\mathbb{I}_4 := \left\langle (a-t)^{1-\alpha},\ (I - Q_\tau)\,y' \right\rangle_{(t_{j-1},a)},$$
$$\mathbb{I}_5 := \left\langle (a-t)^{1-\alpha},\ (I - Q_\tau)\,y' \right\rangle_{(0,t_{j-1})}.$$

We have

$$\begin{aligned}
\|\mathbb{I}_4\|_X &\leq C_\alpha (a - t_{j-1})^{2-\alpha} \|(I - Q_\tau)y'\|_{L^\infty(t_{j-1},t_j;X)} \\
&\leq C_{\alpha,r} (t_j - t_{j-1})^{2-\alpha} \left| t_j^{1-r} - t_{j-1}^{1-r} \right| \quad \text{(by (37))} \\
&\leq C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)}\, j^{(\sigma-1)(2-\alpha)}\, j^{\sigma(1-r)-1} \\
&= C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)}\, j^{\sigma(3-\alpha-r)+\alpha-3}
\end{aligned}$$

and, by (41),

$$\|\mathbb{I}_5\|_X \leq C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)}\, j^{\sigma(3-\alpha-r)+\alpha-3}.$$

Combining the above two estimates and (43) gives

$$\left\| \left(D_{0+}^{\alpha-2}(y' - Q_\tau y')\right)(a) \right\|_X \leq C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)}\, j^{\sigma(3-\alpha-r)+\alpha-3}.$$

Hence, the arbitrariness of $t_{j-1} \leq a < t_j$ proves (40) for $2 \leq j \leq J$.

Finally, for any $0 < a \leq t_1$,

$$\begin{aligned}
&\left\| \left(D_{0+}^{\alpha-2}(I - Q_\tau)y'\right)(a) \right\|_X \\
&\leq C_{\alpha,r} \int_0^a (a-t)^{1-\alpha} (t^{1-r} + t_1^{1-r})\,\mathrm{d}t \quad \text{(by (37))} \\
&\leq C_{\alpha,r} \left(a^{3-\alpha-r} \int_0^1 (1-s)^{1-\alpha} s^{1-r}\,\mathrm{d}s + a^{2-\alpha} t_1^{1-r} \right) \\
&\leq C_{\alpha,r} t_1^{3-\alpha-r} \leq C_{\alpha,\sigma,r,T}\, J^{-\sigma(3-\alpha-r)}.
\end{aligned}$$

This proves (40) for $j = 1$ and thus concludes the proof. □

For any $y \in H^{(\alpha+1)/2}(0,T)$, define $\mathcal{P}_\tau y \in \mathcal{W}_\tau^{\mathrm{c}}$ by

$$\begin{cases} (y - \mathcal{P}_\tau y)(0) = 0, \\ \left\langle D_{0+}^{\alpha-1}\left(y - \mathcal{P}_\tau y\right)', w \right\rangle_{{}^0H^{(\alpha-1)/2}(0,T)} = 0 \quad \forall w \in \mathcal{W}_\tau, \end{cases} \tag{44}$$

and define $\Xi_\tau^\lambda y \in \mathcal{W}_\tau^{\mathrm{c}}$ by

$$\begin{cases} \left(y - \Xi_\tau^\lambda y\right)(0) = 0, \\ \left\langle D_{0+}^{\alpha-1}\left(y - \Xi_\tau^\lambda y\right)' + \lambda\left(y - \Xi_\tau^\lambda y\right), w \right\rangle_{{}^0H^{(\alpha-1)/2}(0,T)} = 0 \quad \forall w \in \mathcal{W}_\tau. \end{cases} \tag{45}$$

Lemma 4.4 *If $\alpha - 1 < \beta < 1$ and $y \in H^{(\alpha+1)/2}(0,T)$, then*

$$(y - \mathcal{P}_\tau y)(t_m) = \left\langle D_{0+}^{\alpha-1-\beta}(Q_\tau - I)y',\, D_{t_m-}^{\beta}\mathcal{G}^m \right\rangle_{(0,t_m)} \tag{46}$$

for each $1 \leq m \leq J$.

Proof A straightforward calculation gives

$$
\begin{aligned}
&(y-\mathcal{P}_\tau y)(t_m) = (\mathcal{I}_\tau y-\mathcal{P}_\tau y)(t_m) = \left\langle (\mathcal{I}_\tau y-\mathcal{P}_\tau y)', 1\right\rangle_{(0,t_m)} \\
&= \left\langle (\mathcal{I}_\tau y-\mathcal{P}_\tau y)', D_{t_m-}^{\alpha-1}\mathcal{G}^m\right\rangle_{(0,t_m)} \qquad \text{(by (31))} \\
&= \left\langle (\mathcal{I}_\tau y-\mathcal{P}_\tau y)', D_{t_m-}^{\alpha-1}\mathcal{G}^m\right\rangle_{(0,T)} \qquad \text{(by the fact } \mathcal{G}^m|_{(t_m,T)}=0) \\
&= \left\langle D_{0+}^{\alpha-1}(\mathcal{I}_\tau y-\mathcal{P}_\tau y)', \mathcal{G}^m\right\rangle_{{}^0H^{(\alpha-1)/2}(0,T)} \qquad \text{(by Lemma A.3)} \\
&= \left\langle D_{0+}^{\alpha-1}(\mathcal{I}_\tau y-y)', \mathcal{G}^m\right\rangle_{{}^0H^{(\alpha-1)/2}(0,T)} \qquad \text{(by (44))}.
\end{aligned}
$$

For any $\alpha-1<\beta<1$,

$$
\begin{aligned}
&\left\langle D_{0+}^{\alpha-1}(\mathcal{I}_\tau y-y)', \mathcal{G}^m\right\rangle_{{}^0H^{(\alpha-1)/2}(0,T)} \\
&= \left\langle D_{0+}^{\beta}D_{0+}^{\alpha-1-\beta}(\mathcal{I}_\tau y-y)', \mathcal{G}^m\right\rangle_{{}^0H^{(\alpha-1)/2}(0,T)} \\
&= \left\langle D_{0+}^{\alpha-1-\beta}(\mathcal{I}_\tau y-y)', D_{T-}^{\beta}\mathcal{G}^m\right\rangle_{(0,T)} \qquad \text{(by Lemma A.3)} \\
&= \left\langle D_{0+}^{\alpha-1-\beta}(\mathcal{I}_\tau y-y)', D_{t_m-}^{\beta}\mathcal{G}^m\right\rangle_{(0,t_m)} \qquad \text{(by the fact } \mathcal{G}^m|_{(t_m,T)}=0) \\
&= \left\langle D_{0+}^{\alpha-1-\beta}(Q_\tau-I)y', D_{t_m-}^{\beta}\mathcal{G}^m\right\rangle_{(0,t_m)}.
\end{aligned}
$$

Combining the above two equations proves (46) and hence this lemma. □

For any

$$
y\in H^{(\alpha+1)/2}(0,T;X) := \left\{\sum_{n=0}^{\infty} c_n\phi_n : \sum_{n=0}^{\infty}\|c_n\|^2_{H^{(\alpha+1)/2}(0,T)}<\infty\right\},
$$

define

$$
\mathcal{P}_\tau^X y := \sum_{n=0}^{\infty}\left(\mathcal{P}_\tau (y,\phi_n)_X\right)\phi_n. \tag{47}
$$

Remark 4.5 By (44), (47), Lemmas A.1 and A.2, we obtain

$$
\|\mathcal{P}_\tau^X y\|_{H^{(\alpha+1)/2}(0,T;X)} \le C_{\alpha,T}\|y\|_{H^{(\alpha+1)/2}(0,T;X)} \tag{48}
$$

for all $y\in H^{(\alpha+1)/2}(0,T;X)$.

Lemma 4.5 *Assume that $y\in H^{(\alpha+1)/2}(0,T;X)\cap C^2((0,T];X)$ satisfies*

$$
t^{-1}\|y'(t)\|_X + \|y''(t)\|_X \le t^{-r}, \quad 0<t\le T,
$$

where $0<r<2$. Then

$$
\left\|\left(y-\mathcal{P}_\tau^X y\right)(t_m)\right\|_X \le C_{\alpha,\sigma,r,T}J^{-\sigma(2-r)}\max_{1\le j\le m} j^{2\sigma-\sigma r+\alpha-3} \tag{49}
$$

for each $1\le m\le J$.

Proof For each $n \in \mathbb{N}$, let

$$y^n(t) := (y(t), \phi_n)_X, \quad 0 \le t \le T.$$

A straightforward calculation gives

$$\begin{aligned}
\left\|(y - \mathcal{P}_\tau^X y)(t_m)\right\|_X &= \left(\sum_{n=0}^{\infty} \left|(y^n - \mathcal{P}_\tau y^n)(t_m)\right|^2\right)^{1/2} \quad \text{(by (47))} \\
&= \left(\sum_{n=0}^{\infty} \left|\left\langle D_{0+}^{\alpha-1-\beta}(I - \mathcal{Q}_\tau)\left(y^n\right)', D_{t_m-}^{\beta}\mathcal{G}^m\right\rangle_{(0,t_m)}\right|^2\right)^{1/2} \quad \text{(by Lemma 4.4)} \\
&\le \int_0^{t_m} \left(\sum_{n=0}^{\infty} \left|D_{0+}^{\alpha-1-\beta}(I - \mathcal{Q}_\tau)(y^n)'\right|^2 |D_{t_m-}^{\beta}\mathcal{G}^m|^2\right)^{1/2} \mathrm{d}t \text{ (by the Minkowski inequality)} \\
&= \int_0^{t_m} \left\|D_{0+}^{\alpha-1-\beta}(I - \mathcal{Q}_\tau)\, y'\right\|_X |D_{t_m-}^{\beta}\mathcal{G}^m| \,\mathrm{d}t,
\end{aligned}$$

for any $\alpha - 1 < \beta < 1$. From Lemma 4.2 it follows that

$$\begin{aligned}
\left\|\left(y - \mathcal{P}_\tau^X y\right)(t_m)\right\|_X &\le \sum_{j=1}^{m} (J/j)^{\sigma(\alpha-1)} \|D_{t_m-}^{\beta}\mathcal{G}^m\|_{L^1(t_{j-1},t_j)} \\
&\qquad \times \max_{1\le j\le m} (J/j)^{\sigma(1-\alpha)} \|D_{0+}^{\alpha-1-\beta}(I - \mathcal{Q}_\tau)y'\|_{L^\infty(t_{j-1},t_j;X)} \\
&\le C_{\alpha,\sigma,T} \max_{1\le j\le m} (J/j)^{\sigma(1-\alpha)} \|D_{0+}^{\alpha-1-\beta}(I - \mathcal{Q}_\tau)y'\|_{L^\infty(t_{j-1},t_j;X)}.
\end{aligned}$$

Passing to the limit $\beta \to 1-$ then yields

$$\left\|\left(y - \mathcal{P}_\tau^X y\right)(t_m)\right\|_X \le C_{\alpha,\sigma,T} \max_{1\le j\le m} \left(J/j\right)^{\sigma(1-\alpha)} \|D_{0+}^{\alpha-2}(I - \mathcal{Q}_\tau)y'\|_{L^\infty(t_{j-1},t_j;X)},$$

so that a straightforward calculation proves (49) by Lemma 4.3. This completes the proof. □

Lemma 4.6 *Assume that* $y \in H^{(\alpha+1)/2}(0, T; X) \cap C^2((0, T]; D(A^{1/2}))$ *satisfies*

$$t^{-1}\|y'(t)\|_{D(A^{1/2})} + \|y''(t)\|_{D(A^{1/2})} \le t^{-r}, \quad 0 < t \le T,$$

where $0 < r < 2$. *If* $\sigma > (3 - \alpha)/(2 - r)$, *then*

$$\|(I - \mathcal{P}_\tau)\, y\|_{L^{2/\alpha}(0,T;D(A^{1/2}))} \le C_{\alpha,\sigma,r,T}\, J^{\alpha-3}. \tag{50}$$

Proof A simple modification of the proof of (49) yields

$$\max_{1\le m\le J} \|(y - \mathcal{P}_\tau y)(t_m)\|_{D(A^{1/2})} \le C_{\alpha,\sigma,r,T}\, J^{\alpha-3}, \tag{51}$$

which implies

$$\|(\mathcal{I}_\tau - \mathcal{P}_\tau)\, y\|_{L^\infty(0,T;D(A^{1/2}))} \le C_{\alpha,\sigma,r,T}\, J^{\alpha-3}.$$

It follows that

$$\|(\mathcal{I}_\tau - \mathcal{P}_\tau)\, y\|_{L^{2/\alpha}(0,T;D(A^{1/2}))} \le C_{\alpha,\sigma,r,T}\, J^{\alpha-3}.$$

In addition, a routine calculation gives

$$\|(I - \mathcal{I}_\tau)\, y\|_{L^{2/\alpha}(0,T;D(A^{1/2}))} \le C_{\alpha,\sigma,r,T}\, J^{-2}.$$

Combining the above two estimates proves (50) and hence this lemma. □

Lemma 4.7 *If* $y \in H^{(\alpha+1)/2}(0,T)$*, then*

$$\left|\left(y - \Xi_\tau^\lambda y\right)(t_m)\right| \le C_{\alpha,T}\left(|(y - \mathcal{P}_\tau y)(t_m)| + \lambda^{1/2}\|(I - \mathcal{P}_\tau)\, y\|_{L^{2/\alpha}(0,t_m)}\right) \tag{52}$$

for each $1 \le m \le J$.

Proof Letting $\theta := (\Xi_\tau^\lambda - \mathcal{P}_\tau)y$, by (44), (45) and Lemma A.3 we obtain

$$\left\langle D_{0+}^{\alpha-1}\theta', \theta'\right\rangle_{(0,t_m)} + \lambda\left\langle \theta, \theta'\right\rangle_{(0,t_m)} = \lambda\left\langle y - \mathcal{P}_\tau y, \theta'\right\rangle_{(0,t_m)},$$

so that using Lemmas A.1 and A.2 and integration by parts yields

$$\|\theta'\|^2_{{}_0H^{(\alpha-1)/2}(0,t_m)} + \lambda|\theta\,(t_m)|^2 \le C_\alpha\lambda\|(I - \mathcal{P}_\tau)\,y\|_{L^{2/\alpha}(0,t_m)}\|\theta'\|_{L^{2/(2-\alpha)}(0,t_m)}.$$

Since

$$\|\theta'\|_{L^{2/(2-\alpha)}(0,t_m)} \le C_{\alpha,T}\|\theta'\|_{{}_0H^{(\alpha-1)/2}(0,t_m)},$$

it follows that

$$|\theta(t_m)| \le C_{\alpha,T}\lambda^{1/2}\|(I - \mathcal{P}_\tau)\,y\|_{L^{2/\alpha}(0,t_m)}. \tag{53}$$

Hence, (52) follows from the triangle inequality

$$\left|\left(y - \Xi_\tau^\lambda y\right)(t_m)\right| \le |\theta\,(t_m)| + |(y - \mathcal{P}_\tau y)\,(t_m)|\,.$$

This completes the proof. □

Proof of Theorem 4.1. For each $n \in \mathbb{N}$, let

$$u^n(t) := (u(t), \phi_n)_X, \quad 0 < t \le T.$$

By (27), (28), (45) and Lemma A.3, we have

$$U = \sum_{n=0}^{\infty}(\Xi_\tau^{\lambda_n}u^n)\phi_n,$$

so that

$$\begin{aligned}\|(u - U)\,(t_m)\|_X &= \left(\sum_{n=0}^{\infty}\left|\left(u^n - \Xi_\tau^{\lambda_n}u^n\right)(t_m)\right|^2\right)^{1/2}\\ &\le C_{\alpha,T}\left(\|(u - \mathcal{P}_\tau^X u)(t_m)\|_X + \left(\sum_{n=0}^{\infty}\lambda_n\|(I - \mathcal{P}_\tau)\,u^n)\|^2_{L^{2/\alpha}(0,t_m)}\right)^{1/2}\right) \quad \text{(by (52))}.\end{aligned}$$

Applying the Minkowski inequality gives

$$\left(\sum_{n=0}^{\infty}\lambda_n\|(I - \mathcal{P}_\tau)u^n)\|^2_{L^{2/\alpha}(0,t_m)}\right)^{1/2} \le \|(I - \mathcal{P}_\tau^X)u\|_{L^{2/\alpha}(0,t_m;D(A^{1/2}))}.$$

The above two estimates yield

$$\|(u - U)\,(t_m)\|_X \le C_{\alpha,T}\left(\|\left(u - \mathcal{P}_\tau^X u\right)(t_m)\|_X + \|(I - \mathcal{P}_\tau^X)u\|_{L^{2/\alpha}(0,t_m;D(A^{1/2}))}\right).$$

In addition, using (6), (29) and Lemma 4.5 gives

$$\|\left(u - \mathcal{P}_\tau^X u\right)(t_m)\|_X \le C_{\alpha,\sigma,\nu,T}\,J^{\alpha-3}\|u_0\|_{D(A^\nu)},$$

and using (7), (29) and Lemma 4.6 shows

$$\|\left(I-\mathcal{P}_\tau^X\right)u\|_{L^{2/\alpha}(0,T;D(A^{1/2}))} \leq C_{\alpha,\sigma,\nu,T}\,J^{\alpha-3}\|u_0\|_{D(A^\nu)}.$$

Finally, combining the above three estimates proves (30) and thus concludes the proof. □

Remark 4.6 Assume that $u_0 = 0$ and $u'(0) = u_1 \in X$. Similar to (6) and (7), we have

$$t^{-1}\|u'(t)\|_X + \|u''(t)\|_X \leq C_\alpha t^{\alpha\nu-1}\|u_1\|_{D(A^\nu)}, \tag{54}$$

$$t^{-1}\|u'(t)\|_{D(A^{1/2})} + \|u''(t)\|_{D(A^{1/2})} \leq C_\alpha t^{\alpha(\nu-1/2)-1}\|u_1\|_{D(A^\nu)}, \tag{55}$$

for all $0 < t \leq T$ and $0 \leq \nu \leq 1$, provided that $u_1 \in D(A^\nu)$. Discretization (27) will be modified as follows: seek $U \in W_\tau^c$ such that $U(0) = 0$ and

$$\int_0^T (D_{0+}^{\alpha-1}(U' - u_1) + AU, V)_X \,\mathrm{d}t = 0 \quad \forall V \in W_\tau.$$

Following the proof of Theorem 4.1, we have

$$\max_{1\leq m\leq J}\|(u-U)(t_m)\|_X \leq C_{\alpha,T}\Big(\max_{1\leq m\leq J}\|u-\mathcal{P}_\tau^X u)(t_m)\|_X + \|(I-\mathcal{P}_\tau^X)u\|_{L^{2/\alpha}(0,T;D(A^{1/2}))}\Big).$$

By (54), (55) and Lemmas 4.5 and 4.6 we can estimate of the right hand side of the above inequality in terms of $\|u_1\|_{D(A^\nu)}$ and J, and thus obtain the convergence of $\max_{1\leq m\leq J}\|(u-U)(t_m)\|_X$. We leave the details to the interested readers.

5 Numerical Experiments

This section performs two numerical experiments to verify Theorems 3.1 and 4.1, respectively, in the following settings:

$$\begin{cases} T = 1; \\ X := \left\{w \in H_0^1(0,1) : \ w \text{ is linear on } \big((m-1)/2^{11}, m/2^{11}\big) \ \text{ for all } 1 \leq m \leq 2^{11}\right\}; \\ A : X \to X \text{ is defined by } \int_0^1 (Av)w \,\mathrm{d}x = -\int_0^1 v'w' \quad \forall v, w \in X. \end{cases}$$

Experiment 1. The purpose of this experiment is to verify Theorem 3.1. Let u_0 be the L^2-orthogonal projection of $x^{0.51}(1-x)$, $0 < x < 1$, onto X. Define

$$\mathcal{E}_1 := \|U^* - U\|_{L^\infty(0,T;L^2(0,1))},$$

where U^* is the numerical solution of discretization (8) with $J = 2^{15}$ and $\sigma = 2/\alpha$. Clearly, regarding ν as 0.5 is reasonable. The numerical results in Tables 1, 2 and 3 illustrate that $\mathcal{E}_1$ is close to $O(J^{-\min\{\sigma\alpha/2,1\}})$, which agrees well with estimate (10) in Theorem 3.1.
Experiment 2. The purpose of this experiment is to verify Theorem 4.1. Let u_0 be the L^2-orthogonal projection of $x^{1.51}(1-x)^2$, $0 < x < 1$, onto X. Let

$$\mathcal{E}_2 := \max_{1\leq j\leq j}\|\left(U^* - U\right)\left(t_j\right)\|_{L^2(\Omega)},$$

where U^* is the numerical solution of discretization (27) with $J = 2^{15}$ and $\sigma = 2(3-\alpha)/\alpha$. Evidently, regarding $u_0 \in D(A)$ is reasonable. The numerical results in Table 4 clearly demonstrate that $\mathcal{E}_2$ is close to $O(J^{\alpha-3})$, which agrees well with Theorem 4.1.

Table 1 $\alpha = 0.2$

J	$\sigma = 1$ $\mathcal{E}_1$	Order	$\sigma = 5$ $\mathcal{E}_1$	Order	$\sigma = 10$ $\mathcal{E}_1$	Order
2^9	2.12e−1	–	1.60e−2	–	6.58e−4	–
2^{10}	2.05e−1	0.05	1.09e−2	0.55	3.23e−4	1.03
2^{11}	1.97e−1	0.06	7.54e−3	0.54	1.58e−4	1.03
2^{12}	1.89e−1	0.06	5.23e−3	0.53	7.64e−5	1.05

Table 2 $\alpha = 0.5$

J	$\sigma = 1$ $\mathcal{E}_1$	Order	$\sigma = 2$ $\mathcal{E}_1$	Order	$\sigma = 4$ $\mathcal{E}_1$	Order
2^7	1.36e−1	–	3.42e−2	–	3.02e−3	–
2^8	1.14e−1	0.25	2.29e−2	0.58	1.45e−3	1.06
2^9	9.47e−2	0.27	1.55e−2	0.56	7.04e−4	1.04
2^{10}	7.76e−2	0.29	1.06e−2	0.55	3.43e−4	1.04

Table 3 $\alpha = 0.8$

J	$\sigma = 1$ $\mathcal{E}_1$	Order	$\sigma = 2$ $\mathcal{E}_1$	Order	$\sigma = 2.5$ $\mathcal{E}_1$	Order
2^7	6.20e−2	–	6.95e−3	–	3.77e−3	–
2^8	4.46e−2	0.48	3.89e−3	0.84	1.82e−3	1.05
2^9	3.22e−2	0.47	2.19e−3	0.83	8.81e−4	1.05
2^{10}	2.34e−2	0.46	1.24e−3	0.82	4.26e−4	1.05

Table 4 $\sigma = 2(3 - \alpha)/\alpha$

J	$\alpha = 1.2$ $\mathcal{E}_2$	Order	$\alpha = 1.5$ $\mathcal{E}_2$	Order	$\alpha = 1.8$ $\mathcal{E}_2$	Order
2^6	2.44e−5	–	1.27e−4	–	7.97e−4	–
2^7	6.81e−6	1.84	4.58e−5	1.47	3.57e−4	1.16
2^8	1.90e−6	1.84	1.65e−5	1.47	1.57e−4	1.18
2^9	5.35e−7	1.83	5.97e−6	1.47	6.87e−5	1.20

6 Conclusions

For the fractional evolution equation, we have analyzed a low-order discontinuous Galerkin (DG) discretization with fractional order $0 < \alpha < 1$ and a low-order Petrov Galerkin (PG) discretization with fractional order $1 < \alpha < 2$. When using uniform temporal grids, the two discretizations are equivalent to the L1 scheme with $0 < \alpha < 1$ and $1 < \alpha < 2$, respectively. For the DG discretization with graded temporal grids, sharp error estimates are rigorously established for smooth and nonsmooth initial data. For the PG discretization, the optimal $(3 - \alpha)$-order temporal accuracy is derived on appropriately graded temporal grids. The theoretical results have been verified by numerical results.

However, our analysis of the PG discretization requires $u_0 \in D(A^\nu)$ with $1/2 < \nu \leq 1$. Hence, how to analyze the case $0 < \nu \leq 1/2$ remains an open problem. It appears that the results and techniques developed in this paper can be used to analyze the semilinear fractional diffusion-wave equations with graded temporal grids, and this is our ongoing work.

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A Properties of Fractional Calculus Operators

Lemma A.1 *For any $v \in {}_0H^\gamma(a,b)$ with $0 < \gamma < 1/2$,*

$$\cos(\gamma\pi)\,\|D_{a+}^\gamma v\|_{L^2(a,b)}^2 \leq \left\langle D_{a+}^\gamma v, D_{b-}^\gamma v\right\rangle_{(a,b)} \leq \sec(\gamma\pi)\|D_{a+}^\gamma v\|_{L^2(a,b)}^2,$$
$$\cos(\gamma\pi)\|D_{b-}^\gamma v\|_{L^2(a,b)}^2 \leq \left\langle D_{a+}^\gamma v, D_{b-}^\gamma v\right\rangle_{(a,b)} \leq \sec(\gamma\pi)\|D_{b-}^\gamma v\|_{L^2(a,b)}^2.$$

Lemma A.2 *For any $v \in {}_0H^\gamma(a,b)$ and $w \in {}^0H^\gamma(a,b)$ with $0 < \gamma < \infty$,*

$$C_1\|v\|_{{}_0H^\gamma(a,b)} \leq \|D_{a+}^\gamma v\|_{L^2(a,b)} \leq C_2\|v\|_{{}_0H^\gamma(a,b)},$$
$$C_1\|w\|_{{}^0H^\gamma(a,b)} \leq \|D_{b-}^\gamma w\|_{L^2(a,b)} \leq C_2\|w\|_{{}^0H^\gamma(a,b)},$$

where C_1 and C_2 are two positive constants depending only on γ.

Lemma A.3 *Assume that $v \in {}_0H^{\gamma/2}(a,b)$ and $w \in {}^0H^{\gamma/2}(a,b)$ with $0 < \gamma < 1$. Then*

$$\left\langle D_{a+}^\gamma v, w\right\rangle_{{}^0H^{\gamma/2}(a,b)} = \left\langle D_{b-}^\gamma w, v\right\rangle_{{}_0H^{\gamma/2}(a,b)}. \tag{56}$$

If $D_{a+}^\gamma v \in L^{2/(1+\gamma)}(a,b)$, then

$$\left\langle D_{a+}^\gamma v, w\right\rangle_{{}^0H^{\gamma/2}(a,b)} = \left\langle D_{a+}^\gamma v, w\right\rangle_{(a,b)}. \tag{57}$$

If $D_{b-}^\gamma w \in L^{2/(1+\gamma)}(a,b)$, then

$$\left\langle D_{b-}^\gamma w, v\right\rangle_{{}_0H^{\gamma/2}(a,b)} = \left\langle D_{b-}^\gamma w, v\right\rangle_{(a,b)}. \tag{58}$$

For the proof of Lemma A.1, we refer the reader to [4]. For the proof of Lemma A.2, we refer the reader to [19]. Since the proof of Lemma A.3 is a standard density argument by Lemmas A.1 and A.2, it is omitted here.

B Some Inequalities

Lemma B.1 *For any $0 < \beta < 1$ and $0 \leq t < a < b < c < d$,*

$$\frac{(d-t)^{1-\beta} - (d-a)^{1-\beta}}{(d-a)^{1-\beta} - (d-b)^{1-\beta}} > \frac{(c-t)^{1-\beta} - (c-a)^{1-\beta}}{(c-a)^{1-\beta} - (c-b)^{1-\beta}}. \tag{59}$$

Proof Let

$$w(y) := \begin{cases} \beta/(1-\beta) & \text{if } y = 1, \\ \frac{1-y^{-\beta}}{y^{1-\beta}-1} & \text{if } y \in (0,\infty)\setminus\{1\}. \end{cases}$$

A routine argument proves that w is strictly decreasing on $(0, \infty)$, so that

$$w\left((d-t-x)/(d-a-x)\right) < w\left((d-b-x)/(d-a-x)\right) \quad \forall 0 \le x \le d-c.$$

It follows that, for any $0 \le x \le d-c$,

$$\frac{(d-a-x)^{-\beta} - (d-t-x)^{-\beta}}{(d-t-x)^{1-\beta} - (d-a-x)^{1-\beta}} < \frac{(d-b-x)^{-\beta} - (d-a-x)^{-\beta}}{(d-a-x)^{1-\beta} - (d-b-x)^{1-\beta}},$$

which implies

$$\begin{aligned}&\left((d-a-x)^{-\beta} - (d-t-x)^{-\beta}\right)\left((d-a-x)^{1-\beta} - (d-b-x)^{1-\beta}\right)\\ &- \left((d-b-x)^{-\beta} - (d-a-x)^{-\beta}\right)\left((d-t-x)^{1-\beta} - (d-a-x)^{1-\beta}\right) < 0\end{aligned}$$

for all $0 \le x \le d-c$. A simple calculation then yields $g'(x) < 0$ for all $0 \le x \le d-c$, where

$$g(x) := \frac{(d-t-x)^{1-\beta} - (d-a-x)^{1-\beta}}{(d-a-x)^{1-\beta} - (d-b-x)^{1-\beta}}, \quad 0 \le x \le d-c.$$

This proves $g(d-c) < g(0)$, namely (59), and thus concludes the proof. □

Lemma B.2 *For any* $0 < \beta < 1$, $\mu \ge 0$ *and* $0 \le t < a < b < c < d$,

$$\frac{(d-t)^{1-\beta} - (d-a)^{1-\beta} + \mu(a-t)}{(d-a)^{1-\beta} - (d-b)^{1-\beta} + \mu(b-a)} > \frac{(c-t)^{1-\beta} - (c-a)^{1-\beta} + \mu(a-t)}{(c-a)^{1-\beta} - (c-b)^{1-\beta} + \mu(b-a)}. \tag{60}$$

Proof Define

$$g(s) := (d-b+s(b-a))^{1-\beta} - (c-b+s(b-a))^{1-\beta} \quad \forall 0 \le s \le a.$$

By the mean value theorem, there exists $\theta \in (0, 1)$ such that

$$\begin{aligned}g(1) - g(0) &= g'(\theta)\\ &= (1-\beta)\left((d-b+\theta(b-a))^{-\beta} - (c-b+\theta(b-a))^{-\beta}\right)(b-a).\end{aligned}$$

Since

$$(d-b+\theta(b-a))^{-\beta} - (c-b+\theta(b-a))^{-\beta} < (d-a)^{-\beta} - (c-a)^{-\beta},$$

it follows that

$$g(1) - g(0) < (1-\beta)\left((d-a)^{-\beta} - (c-a)^{-\beta}\right)(b-a),$$

which implies

$$\frac{1}{b-a} > \frac{(1-\beta)\left((d-a)^{-\beta} - (c-a)^{-\beta}\right)}{(d-a)^{1-\beta} - (d-b)^{1-\beta} - (c-a)^{1-\beta} + (c-b)^{1-\beta}}. \tag{61}$$

Hence, by the estimate

$$(d-s)^{-\beta} - (c-s)^{-\beta} < (d-a)^{-\beta} - (c-a)^{-\beta} \quad \forall 0 \le s < a,$$

we obtain

$$\frac{1}{b-a} > \frac{(1-\beta)\left((d-s)^{-\beta} - (c-s)^{-\beta}\right)}{(d-a)^{1-\beta} - (d-b)^{1-\beta} - (c-a)^{1-\beta} + (c-b)^{1-\beta}} \quad \forall 0 \le s < a. \tag{62}$$

Integrating both sides of the above equation with respect to s from t to a yields

$$\frac{a-t}{b-a} > \frac{(c-t)^{1-\beta} - (c-a)^{1-\beta} - (d-t)^{1-\beta} + (d-a)^{1-\beta}}{(c-a)^{1-\beta} - (c-b)^{1-\beta} - (d-a)^{1-\beta} + (d-b)^{1-\beta}}. \tag{63}$$

Let

$$\begin{aligned}
\mathcal{A} :=&(d-t)^{1-\beta} - (d-a)^{1-\beta}, \quad \mathcal{B} := (d-a)^{1-\beta} - (d-b)^{1-\beta},\\
\mathcal{C} :=&(c-t)^{1-\beta} - (c-a)^{1-\beta}, \quad \mathcal{D} := (c-a)^{1-\beta} - (c-b)^{1-\beta},\\
\mathcal{M} :=&\mu(a-t), \qquad \mathcal{N} := \mu(b-a).
\end{aligned}$$

Since Lemma B.1 implies $\mathcal{AD} > \mathcal{BC}$ and (63) implies $\mathcal{M}(\mathcal{D}-\mathcal{B}) \geq \mathcal{N}(\mathcal{C}-\mathcal{A})$, we obtain

$$(\mathcal{A}+\mathcal{M})\,(\mathcal{D}+\mathcal{N}) > (\mathcal{B}+\mathcal{N})\,(\mathcal{C}+\mathcal{M})\,,$$

which proves (60). This completes the proof. □

Lemma B.3 *For any* $1 < \beta < 2$ *and* $0 \leq t < a < b \leq c$,

$$\frac{(c-t)^{2-\beta} - (c-a)^{2-\beta}}{(c-a)^{2-\beta} - (c-b)^{2-\beta}} < \frac{a-t}{b-a}. \tag{64}$$

Proof By the mean value theorem, there exists $0 < \theta < 1$ such that

$$(c-a)^{2-\beta} - (c-b)^{2-\beta} = (2-\beta)\,(c-b+\theta(b-a))^{1-\beta}\ (b-a),$$

and so

$$\frac{(2-\beta)(c-a)^{1-\beta}}{(c-a)^{2-\beta} - (c-b)^{2-\beta}} = \left(\frac{c-a}{c-b+\theta(b-a)}\right)^{1-\beta} \frac{1}{b-a} < \frac{1}{b-a}.$$

Since

$$(c-a)^{1-\beta} \geq (c-s)^{1-\beta} \quad \text{for all } 0 \leq s \leq a,$$

it follows that

$$\frac{(2-\beta)(c-s)^{1-\beta}}{(c-a)^{2-\beta} - (c-b)^{2-\beta}} < \frac{1}{b-a} \quad \text{for all } 0 \leq s \leq a.$$

Hence, for any $0 \leq t < a$,

$$\int_t^a \frac{(2-\beta)(c-s)^{1-\beta}}{(c-a)^{2-\beta} - (c-b)^{2-\beta}}\,\mathrm{d}t < \int_t^a \frac{1}{b-a}\,\mathrm{d}t,$$

which implies (64). This completes the proof. □

Lemma B.4 *If* $\beta > -1$ *and* $\gamma > 1$, *then*

$$\sum_{j=1}^{k-1} j^{\beta}(k^{\sigma} - j^{\sigma})^{-\gamma} \leq C_{\beta,\gamma,\sigma} k^{\beta-(\sigma-1)\gamma} \tag{65}$$

for all $k \geq 2$.

Proof A routine calculation gives

$$C_0 \le \frac{j^\beta \, (k^\sigma - j^\sigma)^{-\gamma}}{(j-x)^\beta \, (k^\sigma - (j-x)^\sigma)^{-\gamma}} \le C_1,$$

for all $2 \le j \le k-1$ and $0 < x \le 1$, where C_0 and C_1 are two positive constants depending only on β, γ and σ. Hence,

$$\begin{aligned}
&\sum_{j=1}^{k-1} j^\beta (k^\sigma - j^\sigma)^{-\gamma} \\
&\le C_{\beta,\gamma,\sigma} \int_1^{k-1} x^\beta \left(k^\sigma - x^\sigma\right)^{-\gamma} \, \mathrm{d}x \\
&\le C_{\beta,\gamma,\sigma} k^{-\sigma\gamma+\beta+1} \int_{k^{-\sigma}}^{((k-1)/k)^\sigma} s^{(1+\beta)/\sigma-1} (1-s)^{-\gamma} \, \mathrm{d}s \\
&\le C_{\beta,\gamma,\sigma} k^{-\sigma\gamma+\beta+1} \int_0^{((k-1)/k)^\sigma} s^{(1+\beta)/\sigma-1} (1-s)^{-\gamma} \, \mathrm{d}s \\
&\le C_{\beta,\gamma,\sigma} k^{-\sigma\gamma+\beta+1} \left(1 - ((k-1)/k)^\sigma\right)^{1-\gamma} \\
&\le C_{\beta,\gamma,\sigma} k^{-\sigma\gamma+\beta+1+\gamma-1} \\
&= C_{\beta,\gamma,\sigma} k^{\beta-(\sigma-1)\gamma}.
\end{aligned}$$

This proves the lemma. □

A trivial modification of the proof of Lemma B.4 yields the following estimate.

Lemma B.5 *If* $\beta > -1$ *and* $1/2 \le \gamma < 1$, *then*

$$\sum_{j=1}^{k-1} j^\beta \left(k^\sigma - j^\sigma\right)^{-\gamma} \le C_{\beta,\sigma} (1-\gamma)^{-1} k^{\beta-\sigma\gamma+1} \tag{66}$$

for all $k \ge 2$.

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