Journal of Scientific Computing (2020) 85:34
https://doi.org/10.1007/s10915-020-01345-3

A Uniformly Convergent Weak Galerkin Finite Element Method on Shishkin Mesh for 1d Convection–Diffusion Problem

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Received: 1 June 2020 / Revised: 19 September 2020 / Accepted: 12 October 2020 /
Published online: 23 October 2020

Abstract

In this paper, a weak Galerkin finite element method is proposed and analyzed for one-dimensional singularly perturbed convection–diffusion problems. This finite element scheme features piecewise polynomials of degree $k \geq 1$ on interior of each element plus piecewise constant on the node of each element. Our WG scheme is parameter-free and has competitive number of unknowns since the interior unknowns can be eliminated efficiently from the discrete linear system. An ε-uniform error bound of $\mathcal{O}((N^{-1} \ln N)^k)$ in the energy-like norm is established on Shishkin mesh, where N is the number of elements. Finally, the numerical experiments are carried out to confirm the theoretical results. Moreover, the numerical results show that the present method has the optimal convergence rate of $\mathcal{O}(N^{-(k+1)})$ in the L^2-norm and the superconvergence rates of $\mathcal{O}((N^{-1} \ln N)^{2k})$ in the discrete L^∞-norm.

Keywords Singularly perturbed problem · Convection–diffusion equation · Weak Galerkin finite element method · Shishkin mesh

Mathematics Subject Classification 65N15 · 65N30 · 35J50

Supported by Natural Science Foundation of Zhejiang province, China (Grant No.LY19A010008).

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1 Introduction

In this paper, we consider the following one-dimensional singularly perturbed convection–diffusion problem

$$\begin{cases} -\varepsilon u'' + bu' + cu = f & \text{in } \Omega = (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is a small positive parameter, and b, c, f are sufficiently smooth functions with the following properties

$$b(x) \geq b_0 > 0,\ c(x) \geq 0,\ c(x) - \frac{1}{2}b'(x) \geq c_0 > 0, \quad \forall x \in \bar{\Omega}, \tag{1.2}$$

for some constants b_0 and c_0. This assumption guarantees that problem (1.1) has a unique solution in $H^2(\Omega) \cap H_0^1(\Omega)$ for all $f \in L^2(\Omega)$ [16,26].

It is well known that the exact solution of problem (1.1) typically has an exponential boundary layer at $x = 1$, which cause difficulties for classical numerical methods. For example, the standard finite element or finite difference method fails to produce an accurate numerical solution unless the mesh size is comparable or smaller than the parameter ε.

Layer-adapted meshes [9,13], such as Bakhvalov mesh and Shishkin mesh, have been developed to remedy the difficulties caused by the boundary layers. As it is shown in [4], on layer-adapted meshes one can use standard discretization techniques such as conforming finite element method [16,26], but some small oscillations still appear in the discrete solution. Additional stabilization is necessary to improve the situation. Over the past several decades, many stabilized numerical methods such as the up-winding finite difference scheme [8], the streamline-diffusion finite element method [10,11,17], variational multiscale method [19], and the discontinuous Galerkin finite element method [5,6,14,18,23–25,27,30,31], have been developed for the singularly perturbed convection–diffusion problem. Details of these methods can be found in the classical book [15] and the references therein.

Recently, the WG finite element methods have attracted increasing attention. The WG methods, first proposed and analyzed by Wang and Ye [20], provide a general finite element technique for solving partial differential equations. In general, the WG scheme for PDEs by replacing usual derivatives by weakly-defined derivatives in the corresponding weak form with additional parameter-free stabilization term. The WG methods have been successfully applications in the elliptic problems [20,28], the options pricing problem [29], the Stokes equation [21], the Maxwell equations [12], the biharmonic equations [22], and etc.

Most recently, the WG methods demonstrate robust and stable discretizations for singularly perturbed problems (SPP). For example, a WG method with an upwinding-type stabilization was presented and analyzed for the SPP with convection–diffusion type [7]. A P_0-P_0 WG method was investigated in [1] for the SPP with reaction-diffusion type. The WG method was also studied for the fourth order singularly perturbed problems [2]. But the uniform convergence of the WG finite element method on layer-adapted mesh has not been discussed so far. The main concern here is to investigate the uniform convergence of the WG finite element scheme on a Shishkin mesh for one-dimensional singularly perturbed convection–diffusion equations.

The outline of this paper is organized as follows. In Sect. 2, we introduce some preliminaries and notations which will be used later. The formulation of WG finite element method for the singularly perturbed convection–diffusion equation is presented in Sect. 3. The error estimates of the proposed method are discussed in Sect. 4. Some numerical experiments are displayed in Sect. 6. It aims to confirm our theoretical results and investigate some interesting convergence phenomenons.

In the following, C denotes generic positive constants independent of N and ε, and their value will not be the same in different inequalities.

2 Preliminary and Notations

2.1 The Shishkin Mesh

Let N be an even integer. Define the transition parameter

$$\tau = \min\left(\frac{1}{2}, \frac{k+1}{b_0}\varepsilon \ln N\right),$$

where k is the degree of polynomials in the finite element space which will be given later. Then divide each of the subdomains $\Omega_1 = [0, 1-\tau]$ and $\Omega_2 = [1-\tau, 1]$ into $N/2$ equidistant subintervals. Notice that $\varepsilon \ll 1$, here and below we take $\tau = \frac{k+1}{b_0}\varepsilon \ln N$. Now, we have

$$x_0 = 0, x_j = x_{j-1} + h_j, h_j = \begin{cases} h_c, & j = 1, \ldots, N/2, \\ h_f, & j = N/2+1, \ldots, N, \end{cases}$$

where

$$h_c = 2(1-\tau)/N, \qquad h_f = 2\tau/N.$$

It can be easily shown that

$$h_c = \mathcal{O}(N^{-1}), \qquad h_f = \mathcal{O}(\varepsilon N^{-1} \ln N).$$

Denote the mesh by $I_j = [x_{j-1}, x_j]$ for $j = 1, \ldots, N$ and set $\mathcal{T}_N = \{I_j, j = 1, \ldots, N\}$. For each interval $I_j \in \mathcal{T}_N$, we define its outward unit normal $n_{I_j}(x_j) = 1$ and $n_{I_j}(x_{j-1}) = -1$; if there is no confusion, instead of n_{I_j} we simply write n.

2.2 Weak Function and Weak Derivative

On each interval $I_j = [x_{j-1}, x_j]$, a weak function on the interval I_j refers to a function $v = \{v_0, v_b\}$ such that $v_0 \in L^2(I_j)$ and $v_b \in L^\infty(\partial I_j)$, where $\partial I_j = \{x_{j-1}, x_j\}$. That is, for each interval $I_j \in \mathcal{T}_N$, $j = 1, \ldots, N$, we have

$$v = \begin{cases} v_0, & \text{in } I_j, \\ v_b, & \text{on } \partial I_j. \end{cases}$$

Here v_0 can be understood as the value of v in (x_{j-1}, x_j), and v_b represents the values of v on the endpoints of I_j. Denote by $\mathcal{M}(I_j)$ the space of weak functions on I_j, i.e.,

$$\mathcal{M}(I_j) = \{v = \{v_0, v_b\} : v_0 \in L^2(I_j), v_b \in L^\infty(\partial I_j)\}.$$

The local Sobolev space $H^1(I_j)$ can be embedded into the space $\mathcal{M}(I_j)$ by the inclusion map

$$i_{\mathcal{M}}(v) = \{v|_{I_j}, v|_{\partial I_j}\}, \quad \forall v \in H^1(I_j).$$

Let $\mathbb{P}^k(I_j)$ be the set of polynomials defined on I_j with degree no more than k. Denote by $\mathbb{P}^0(\partial I_j)$ is the set of piecewise constants on ∂I_j. For a given integer $k \geq 1$, we define a local WG finite element space $\mathcal{M}_N(I_j)$ on each element $I_j \in \mathcal{T}_N$ as follows

$$\mathcal{M}_N(I_j) = \{v = \{v_0, v_b\} : v_0|_{I_j} \in \mathbb{P}^k(I_j), v_b|_{\partial I_j} \in \mathbb{P}^0(\partial I_j)\}.$$

A global WG finite element space $\mathcal{M}_N$ is then obtained by gluing all the local space $\mathcal{M}_N(I_j)$ with common values on interior nodes. In other words, for any function $v = \{v_0, v_b\} \in \mathcal{M}_N$, it means $v_0|_{I_j}$ belongs to the polynomial space $\mathbb{P}^k(I_j)$ for $j = 1, \ldots, N$, and v_b has a single value on the nodes of the partition $\mathcal{T}_N$.

Let $\mathcal{M}_N^0$ be the subspace of $\mathcal{M}_N$ consisting of discrete weak functions with vanishing boundary values, i.e.,

$$\mathcal{M}_N^0 = \{v = \{v_0, v_b\} : v \in \mathcal{M}_N, v_b(0) = v_b(1) = 0\}. \tag{2.1}$$

The weak derivative of a weak function $v = \{v_0, v_b\} \in \mathcal{M}_N$ is defined as follows.

Definition 2.1 For any weak function $v \in \mathcal{M}_N(I_j)$, the weak derivative of $v = \{v_0, v_b\}$ is defined as the unique polynomial $D_{w,I_j} v \in \mathbb{P}^{k-1}(I_j)$ satisfying

$$(D_{w,I_j} v, q)_{I_j} = -(v_0, q')_{I_j} + \langle v_b, qn\rangle_{\partial I_j}, \quad \forall q \in \mathbb{P}^{k-1}(I_j). \tag{2.2}$$

Here, we have used the notation

$$(\varphi, \psi)_{I_j} := \int_{I_j} \varphi(x)\psi(x)\mathrm{dx}$$

and

$$\langle \varphi, \psi n\rangle_{\partial I_j} := \varphi(x_j)\psi(x_j) - \varphi(x_{j-1})\psi(x_{j-1}).$$

To approximate the convection term bu' in the problem (1.1), we introduce a weak convection derivative as follows.

Definition 2.2 For any weak function $v \in \mathcal{M}_N(I_j)$, the weak convection derivative of $v = \{v_0, v_b\}$ is defined as the unique polynomial $D^b_{w,I_j} v \in \mathbb{P}^k(I_j)$ satisfying

$$(D^b_{w,I_j} v, q)_{I_j} = -(v_0, (bq)')_{I_j} + \langle v_b, bqn\rangle_{\partial I_j}, \quad \forall q \in \mathbb{P}^k(I_j). \tag{2.3}$$

The weak derivatives D_w and D^b_w on the finite element space $\mathcal{M}_N$ can be computed by using the Eqs. (2.2) and (2.3) respectively on each element $I_j \in \mathcal{T}_N$. More precisely, it is given by

$$(D_w v)|_{I_j} = D_{w,I_j}(v|_{I_j}), \quad (D^b_w v)|_{I_j} = D^b_{w,I_j}(v|_{I_j}), \quad \forall v \in \mathcal{M}_N.$$

3 The Weak Galerkin Finite Element Scheme

For simplicity, we adopt the following notations,

$$(\varphi, \psi)_{\mathcal{T}_N} = \sum_{j=1}^{N} (\varphi, \psi)_{I_j}, \quad \langle \varphi, \psi\rangle_{\partial\mathcal{T}_N} = \sum_{j=1}^{N} \langle \varphi, \psi\rangle_{\partial I_j}.$$

To describe our weak Galerkin finite element method, we need to introduce three bilinear forms on $\mathcal{M}_N$ as follows: for any $\varphi = \{\varphi_0, \varphi_b\}$, $\psi = \{\psi_0, \psi_b\} \in \mathcal{M}_N$, we define

$$\mathcal{A}(\varphi, \psi) := \varepsilon(D_w\varphi, D_w\psi)_{\mathcal{T}_N} + (D_w^b\varphi + c\varphi_0, \psi_0)_{\mathcal{T}_N},$$

$$\mathcal{S}_d(\varphi, \psi) := \sum_{j=1}^{N}\langle\sigma_j(\varphi_0 - \varphi_b), \psi_0 - \psi_b\rangle_{\partial I_j},$$

$$\mathcal{S}_c(\varphi, \psi) := \sum_{j=1}^{N}\langle bn_{I_j}(\varphi_0 - \varphi_b), \psi_0 - \psi_b\rangle_{\partial_+ I_j},$$

where $\partial_+ I_j = \{x \in \partial I_j : b(x)n_{I_j}(x) \geq 0\}$, σ_j is a penalty parameter given as follows:

$$\sigma_j = \begin{cases} 1, & \text{if } j = 1, \ldots, N/2, \\ N/\ln N, & \text{if } j = N/2+1, \ldots, N. \end{cases} \tag{3.1}$$

Remark 1 The value of σ_j is chose as $\sigma_j = \varepsilon h_j^{-1}$ in most of existence works of WG finite element method such as [20,28,29]. But ε-uniform error estimates can't be obtained by this choice of σ_j.

With the above notations and definitions, the weak Galerkin finite element approximation of the problem (1.1) is to find an approximate solution $u_N = \{u_0, u_b\} \in \mathcal{M}_N^0$ such that

$$\mathcal{B}(u_N, v_N) = (f, v_0), \quad \forall\, v_N = \{v_0, v_b\} \in \mathcal{M}_N^0, \tag{3.2}$$

where

$$\mathcal{B}(\varphi, \psi) := \mathcal{A}(\varphi, \psi) + \mathcal{S}_d(\varphi, \psi) + \mathcal{S}_c(\varphi, \psi). \tag{3.3}$$

Let $\phi_{0,i}^j$, $i = 1, \ldots, k+1$ be the basis functions of piecewise polynomial space $\mathbb{P}^k(I_j)$. Denote by $\mathcal{E}_N^0 = \{x_j, j = 1, \ldots, N-1\}$ the set of interior nodes of the mesh $\mathcal{T}_N$. And let $\phi_{b,j}$, $j = 0, \ldots, N$ be the nodal basis function of $\mathbb{P}^0(\mathcal{E}_N)$, i.e., $\phi_{b,j}(x_i) = \delta_{ij}$, where $\delta_{ij} = 1$ if $j = i$ else $\delta_{ij} = 0$ if $j \neq i$. Denote $\Phi_{0,m} = \{\phi_{0,i}^j, 0\}$ where $m = i + (j-1)(k+1)$, with $i = 1, \ldots, k+1$, $j = 1, \ldots, N$. Let $\Phi_{b,j} = \{0, \phi_{b,j}\}$ with $j = 1, \ldots, N-1$. Then the WG finite element space $\mathcal{M}_N^0 = span\{\Phi_{0,1}, \ldots, \Phi_{0,(k+1)N}, \Phi_{b,1}, \ldots \Phi_{b,N-1}\}$. Denote by

$$\begin{aligned}
(B_{0,0})_{ij} &= \mathcal{B}_N(\Phi_{0,j}, \Phi_{0,i}), i, j = 1, \ldots, (k+1)N, \\
(B_{0,b})_{ij} &= \mathcal{B}_N(\Phi_{b,j}, \Phi_{0,i}), i = 1, \ldots, (k+1)N, j = 1, \ldots, N-1, \\
(B_{b,0})_{ij} &= \mathcal{B}_N(\Phi_{0,j}, \Phi_{b,i}), j = 1, \ldots, (k+1)N, i = 1, \ldots, N-1, \\
(B_{b,b})_{ij} &= \mathcal{B}_N(\Phi_{b,j}, \Phi_{b,i}), i, j = 1, \ldots, N-1, \\
F_j &= (f, \Phi_{0,j}), j = 1, \ldots, (k+1)N,
\end{aligned}$$

then the matrix form of the WG scheme (3.2) can be written as

$$\begin{pmatrix} B_{0,0} & B_{0,b} \\ B_{b,0} & B_{b,b} \end{pmatrix}\begin{pmatrix} U_0 \\ U_b \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix},$$

where U_0 and U_b represent the vectors of degrees of freedom for u_0 and u_b, respectively. We can write the above system as

$$(B_{b,b} - B_{b,0}B_{0,0}^{-1}B_{0,b})U_b + B_{b,0}B_{0,0}^{-1}F = 0$$

and

$$U_0 = B_{0,0}^{-1}(F - B_{0,b}U_b).$$

We emphasize that the inverse $B_{0,0}^{-1}$ can be computed on each element independently of each other since the matrix $B_{0,0}$ is block-diagonal owing to the discontinuous nature of the approximation space $\mathcal{M}_N^0$.

Remark 2 It can be observed that the interior degrees of freedom U_0 can be locally eliminated in terms of the interface degrees of freedom U_b in practical implementation. This means that, the linear system resulting from WG finite element methods only involves the degrees of freedom on the skeleton of the mesh. Therefore, the degrees of freedom of the WG method is comparable with conforming finite elements, and it is much less than the degrees of freedom of the discontinuous Galerkin method. It is worth to point out that the procedure of elimination of U_0 by U_b is the so-called Schur complement technique in the domain decomposition community, which can be used any dimensional problem.

3.1 Coercivity of the Bilinear form $\mathcal{B}(\cdot, \cdot)$

We introduce an energy norm $|||\cdot|||$ in the finite element space $\mathcal{M}_N$ as follows: for all $v = \{v_0, v_b\} \in \mathcal{M}_N$,

$$|||v|||^2 := |v|_{1,\varepsilon}^2 + \|\sqrt{c_0}v_0\|_{L^2(\mathcal{T}_N)}^2 + |v|_{\mathrm{J}}^2, \tag{3.4}$$

with the seminorm

$$|v|_{1,\varepsilon}^2 := \varepsilon\|D_w v\|_{L^2(\mathcal{T}_N)}^2 + \mathcal{S}_d(v, v),$$

$$|v|_{\mathrm{J}}^2 := \sum_{j=1}^{N} w_j|\sqrt{b}(v_0 - v_b)|^2(x_j^-),$$

where

$$w_j = \begin{cases} \frac{1}{2}, & j = N, \\ 1, & j = 1, \ldots, N-1. \end{cases}$$

In addition, for $v \in \mathcal{M}_N + H_0^1(\Omega)$, define the discrete H^1 energy norm

$$\|v\|_{\mathcal{M}}^2 := |v|_{*,\varepsilon}^2 + \|\sqrt{c_0}v_0\|_{L^2(\mathcal{T}_N)}^2 + |v|_{\mathrm{J}}^2. \tag{3.5}$$

with the seminorm

$$|v|_{*,\varepsilon}^2 := \varepsilon\|v_0'\|_{L^2(\mathcal{T}_N)}^2 + \mathcal{S}_d(v, v).$$

It is worth noting that a function $v \in H_0^1(\Omega)$ can be understood as a weak function $\{v_0, v_b\}$ with $v_0 = v|_{I_j}$ and $v_b = v|_{\partial I_j}$ for any $I_j \in \mathcal{T}_h$.

The following lemma shows that the $|||\cdot|||$-norm and $\|\cdot\|_{\mathcal{M}}$ are equivalent in the WG finite element space $\mathcal{M}_N^0$.

Lemma 3.1 *For any $v_N = \{v_0, v_b\} \in \mathcal{M}_N^0$, there holds*

$$C_{\mathrm{lb}}\|v_N\|_{\mathcal{M}} \le |||v_N||| \le C_{\mathrm{ub}}\|v_N\|_{\mathcal{M}},$$

where $C_{\mathrm{ub}} := \max\{C_{\mathrm{eq}}, 1\}$ with $C_{\mathrm{eq}} = \max\{\sqrt{2}, \sqrt{1+2C_*}\}$ and $C_{\mathrm{lb}} := 1/C_{\mathrm{ub}}$.

Proof For any $v_N = \{v_0, v_b\} \in \mathcal{M}_N^0$, it follows from the definition of weak derivative (2.1) and integration by parts that

$$(D_w v_N, w)_{I_j} = (v_0', w)_{I_j} - \langle v_0 - v_b, wn\rangle_{\partial I_j}, \quad \forall w \in \mathbb{P}^{k-1}(I_j), \forall I_j \in \mathcal{T}_N. \tag{3.6}$$

Let $w = D_w v_N$ in (3.6), we have

$$(D_w v_N, D_w v_N)_{I_j} = (v_0', D_w v_N)_{I_j} - \langle v_0 - v_b, D_w v_N n\rangle_{\partial I_j}.$$

Using the Cauchy-Schwarz inequality and the trace inequality (4.3), we infer

$$\begin{aligned}\|D_w v_N\|^2_{L^2(I_j)} &\le \|v_0'\|_{L^2(I_j)}\|D_w v_N\|_{L^2(I_j)} + \|v_0 - v_b\|_{L^2(\partial I_j)}\|D_w v_N\|_{L^2(\partial I_j)}\\ &\le (\|v_0'\|_{L^2(I_j)} + C_* h_j^{-1/2}\|v_0 - v_b\|_{L^2(\partial I_j)})\|D_w v_N\|_{L^2(I_j)}.\end{aligned}$$

Thus,

$$\|D_w v_N\|_{L^2(I_j)} \le (\|v_0'\|_{L^2(I_j)} + C_* h_j^{-1/2}\|v_0 - v_b\|_{L^2(\partial I_j)}).$$

Squaring this inequality and summing over $I_j \in \mathcal{T}_N$ yields

$$\varepsilon\|D_w v_N\|^2_{L^2(\mathcal{T}_N)} \le 2(\varepsilon\|v_0'\|^2_{L^2(\mathcal{T}_N)} + C_*^2\sum_{j=1}^N \varepsilon h_j^{-1}\|v_0 - v_b\|^2_{L^2(\partial I_j)}).$$

Recalling (3.1), we have

$$\frac{\varepsilon h_j^{-1}}{\sigma_j} \le C \quad \text{for } j = 1, \dots, N.$$

Then, from the definition of $\mathcal{S}_d(\cdot,\cdot)$, we get

$$\sum_{j=1}^N \varepsilon h_j^{-1}\|v_0 - v_b\|^2_{L^2(\partial I_j)} = \sum_{j=1}^N \frac{\varepsilon h_j^{-1}}{\sigma_j}\cdot\sigma_j\|v_0 - v_b\|^2_{L^2(\partial I_j)} \le C\mathcal{S}_d(v_N, v_N)$$

As a result,

$$\varepsilon\|D_w v_N\|^2_{L^2(\mathcal{T}_N)} \le 2(\varepsilon\|v_0'\|^2_{L^2(\mathcal{T}_N)} + C_*^2\mathcal{S}_d(v_N, v_N)).$$

Moreover,

$$|v_N|^2_{1,\varepsilon} \le 2\varepsilon\|v_0'\|^2_{L^2(\mathcal{T}_N)} + (1 + 2C_*^2)\mathcal{S}_d(v_N, v_N),$$

which yields

$$|v_N|_{1,\varepsilon} \le C_{\text{eq}}|v_N|_{*,\varepsilon} \tag{3.7}$$

with $C_{\text{eq}} = \max\{\sqrt{2}, \sqrt{1 + 2C_*}\}$.

As to the lower bound, we choose $w = v_0'$ in (3.6) to obtain

$$(v_0', v_0')_{I_j} = (D_w v_N, v_0')_{I_j} + \langle v_0 - v_b, v_0' n\rangle_{\partial I_j}.$$

Using the Cauchy-Schwarz inequality and the trace inequality (4.3), we infer

$$\begin{aligned}\|v_0'\|^2_{L^2(I_j)} &\le \|D_w v\|_{L^2(I_j)}\|v_0'\|_{L^2(I_j)} + \|v_0 - v_b\|_{L^2(\partial I_j)}\|v_0'\|_{L^2(\partial I_j)}\\ &\le (\|D_w v_N\|_{L^2(I_j)} + C_* h_j^{-1/2}\|v_0 - v_b\|_{L^2(\partial I_j)})\|v_0'\|_{L^2(I_j)}.\end{aligned}$$

Thus,

$$\|v_0'\|_{L^2(I_j)} \leq (\|D_w v_N\|_{L^2(I_j)} + C_* h_j^{-1/2}\|v_0 - v_b\|_{L^2(\partial I_j)}).$$

As a result,

$$\varepsilon\|v_0'\|^2_{L^2(\mathcal{T}_N)} \leq 2(\varepsilon\|D_w v_N\|^2_{L^2(\mathcal{T}_N)} + C_*^2 \mathcal{S}_d(v_N, v_N)),$$

which yields

$$\|v_N\|^2_{*,\varepsilon} \leq 2\varepsilon\|D_w v_N\|^2_{L^2(\mathcal{T}_N)} + (1 + 2C_*^2)\mathcal{S}_d(v_N, v_N) \leq C_{\text{eq}}^2\|v_N\|^2_{1,\varepsilon}.$$

Then, we arrive at

$$C_{\text{eq}}^{-1}\|v\|_{*,\varepsilon} \leq \|v\|_{1,\varepsilon},$$

which together with (3.7) yields

$$C_{\text{eq}}^{-1}|v_N|_{*,\varepsilon} \leq |v_N|_{1,\varepsilon} \leq C_{\text{eq}}|v_N|_{*,\varepsilon}.$$

From the definition of $|||\cdot|||$-norm and $\|\cdot\|_{\mathcal{M}}$-norm, we observe that

$$C_{\text{lb}}\|v_N\|_{\mathcal{M}} \leq |||v_N||| \leq C_{\text{ub}}\|v_N\|_{\mathcal{M}},$$

with $C_{\text{ub}} := \max\{C_{\text{eq}}, 1\}$ and $C_{\text{lb}} := 1/C_{\text{ub}}$. The proof is completed. □

Now we turn to the coercivity of the WG bilinear form $\mathcal{B}(\cdot,\cdot)$ with respect to the $|||\cdot|||$-norm defined by (3.4).

Lemma 3.2 (Coercivity with respect to the $|||\cdot|||$-norm) *The WG bilinear form defined by* (3.3) *is coercive on $\mathcal{M}_N^0$ with respect to the $|||\cdot|||$-norm, i.e.,*

$$\mathcal{B}(v_N, v_N) \geq |||v_N|||^2, \quad \forall v_N \in \mathcal{M}_N^0. \tag{3.8}$$

Proof Let $v_N = \{v_0, v_b\}, w_N = \{w_0, w_b\} \in \mathcal{M}_N^0$. It follows from (2.2) and integration by parts that

$$\begin{aligned}(D_w^b v_N, w_0)_{\mathcal{T}_N} &= -(v_0, (bw_0)')_{\mathcal{T}_N} + \langle v_b, bnw_0\rangle_{\partial\mathcal{T}_N}\\ &= (bv_0', w_0)_{\mathcal{T}_N} - \langle bn(v_0 - v_b), w_0\rangle_{\partial\mathcal{T}_N}.\end{aligned} \tag{3.9}$$

Since v_b and w_b are single value at the interior nodes of $\mathcal{T}_N$ and vanish at the boundaries nodes of $\mathcal{T}_N$, we have

$$\begin{aligned}\langle bnv_b, w_b\rangle_{\partial\mathcal{T}_N} &= \sum_{j=1}^{N}[(bv_bw_b)(x_j) - (bv_bw_b)(x_{j-1})]\\ &= (bv_bw_b)(1) - (bv_bw_b)(0) = 0,\end{aligned}$$

whence we infer from (2.2) that

$$\begin{aligned}(D_w^b w_N, v_0)_{\mathcal{T}_N} &= -(w_0, (bv_0)')_{\mathcal{T}_N} + \langle w_b, bnv_0\rangle_{\partial\mathcal{T}_N}\\ &= -(w_0, (bv_0)')_{\mathcal{T}_N} + \langle w_b, bn(v_0 - v_b)\rangle_{\partial\mathcal{T}_N}.\end{aligned} \tag{3.10}$$

Summing (3.9) and (3.10), and let $v_N = w_N$, we obtain

$$(D_w^b v_N, v_0)_{\mathcal{T}_N} = -\frac{1}{2}(b'v_0, v_0)_{\mathcal{T}_N} - \frac{1}{2}\langle bn(v_0 - v_b), v_0 - v_b\rangle_{\partial\mathcal{T}_N}. \tag{3.11}$$

By a simple manipulation, we have

$$\mathcal{S}_c(v_N, v_N) - \frac{1}{2}\langle bn(v_0 - v_b), v_0 - v_b\rangle_{\partial\mathcal{T}_N} = |v_N|_{\mathrm{J}}^2,$$

which together with (3.11) yields

$$\begin{aligned}(D_w^b v_N + cv_0, v_0)_{\mathcal{T}_N} + \mathcal{S}_c(v_N, v_N) &= ((c - \frac{1}{2}b')v_0, v_0)_{\mathcal{T}_N} + |v_N|_{\mathrm{J}}^2 \\ &\geq \|\sqrt{c_0}v_0\|_{L^2(\mathcal{T}_N)}^2 + |v_N|_{\mathrm{J}}^2 \end{aligned} \tag{3.12}$$

Owing to the definition of $\mathcal{B}(\cdot,\cdot)$ and (3.12), we obtain, for any $v_N \in \mathcal{M}_N^0$,

$$\mathcal{B}(v_N, v_N) \geq \varepsilon(\nabla_w v_N, \nabla_w v_N) + \mathcal{S}_d(v_N, v_N) + \|\sqrt{c_0}v_0\|_{L^2(\mathcal{T}_N)}^2 + |v_N|_{\mathrm{J}}^2 = \|\!|\!| v_N \|\!|\!|^2. \tag{3.13}$$

The proof is completed. □

As a consequent of Lemma 3.1 and Lemma 3.2, the WG bilinear form $\mathcal{B}_h(\cdot,\cdot)$ also has the coercivity with respect to the $\|\cdot\|_{\mathcal{M}}$-norm defined by (3.5).

Lemma 3.3 (Coercivity with respect to the $\|\cdot\|_{\mathcal{M}}$norm) *The WG bilinear form defined by* (3.3) *is coercive on $\mathcal{M}_N^0$ with respect to the $\|\cdot\|_{\mathcal{M}}$-norm, i.e.,*

$$\mathcal{B}(v_N, v_N) \geq C_{\mathrm{lb}}\|v_N\|_{\mathcal{M}}^2, \quad \forall v_N \in \mathcal{M}_N^0.$$

3.2 Interpolation Operator

Usually, the locally defined L^2 projections on each element and its boundaries are used for the error analysis of WG finite element method in all existence references such as [20, 28,29]. Unfortunately, the interpolation error bound of L^2 projection is not ε-uniform on Shishkin mesh because of its anisotropic property. So in our analysis we will adopt a special interpolation introduced in [19].

On each element $I_j \in \mathcal{T}_N$ with $I_j = [x_{j-1}, x_j]$, we define the set of $k+1$ nodal functionals

$$N_0(v) = v(x_{j-1}), \quad N_k(v) = v(x_j), \tag{3.14}$$

$$N_l(v) = h_j^{-l}\int_{I_j}(x - x_{j-1})^{k-1}v(x)\mathrm{d}x, \quad l = 1, \ldots, k-1. \tag{3.15}$$

Now a local interpolation $\mathcal{I} : H^1(I_j) \to \mathbb{P}^k(I_j)$ is defined by

$$N_l(\mathcal{I}v - v) = 0, \quad l = 0, 1, \ldots, k, \tag{3.16}$$

which can be extended to a continuous global interpolation $\mathcal{I}v$.

Obviously, $\mathcal{I}v|_{I_j}$ is continuous on I_j and belongs to $H^1(I_j)$. Then, the weak function $\{(\mathcal{I}v)|_{I_j}, (\mathcal{I}v)|_{\partial I_j}\}$, still denoted by $\mathcal{I}v$ for simplicity, belongs to the local WG finite element space $\mathcal{M}_N(I_j)$.

Lemma 3.4 (Commutativity of $\mathcal{I}$) *Let $\mathcal{I}$ be the interpolation operator defined by* (3.16). *Then, on each element $I_j \in \mathcal{T}_N$, we have*

$$D_w(\mathcal{I}v) = (\mathcal{I}v)', \quad \forall v \in H^1(I_j).$$

Proof It follows from the definition of weak derivative (2.1) that for any $w \in \mathbb{P}^{k-1}(I_j)$

$$(D_w(\mathcal{I}v), w)_{I_j} = -(\mathcal{I}v, w')_{I_j} + \langle \mathcal{I}v, wn\rangle_{\partial I_j}.$$

Applying integration by parts to the first term on the right hand side of the above equation leads to the assertion. □

3.3 Error Equation

The WG finite element scheme (3.2) is not consistent in the sense that for the solution u of problem (1.1), one doesn't have $\mathcal{B}_N(u, v_N) = (f, v_0)$ for some $v_N = \{v_0, v_b\} \in \mathcal{M}_N^0$. As a result of the inconsistency, the usual orthogonality property for the conforming Galerkin finite element methods doesn't hold true for the weak Galerkin method; i.e., $\mathcal{B}_N(u - u_N, v_N) \neq 0$ for some $v_N = \{v_0, v_b\} \in \mathcal{M}_N^0$. In this subsection, we will derive an error equation which will be used in error analysis.

Lemma 3.5 *Let u be the solution of the problem (1.1). Then for $v_N = \{v_0, v_b\} \in \mathcal{M}_N^0$, there holds*

$$-\varepsilon(u'', v_0)_{\mathcal{T}_N} = \varepsilon(D_w(\mathcal{I}u), D_w v_N)_{\mathcal{T}_N} - \mathcal{E}_1(u, v_N). \tag{3.17}$$

where

$$\mathcal{E}_1(u, v_N) = \varepsilon\langle u' - (\mathcal{I}u)', (v_0 - v_b)n\rangle_{\partial\mathcal{T}_N}. \tag{3.18}$$

Proof Let $v_N = \{v_0, v_b\} \in \mathcal{M}_N^0$. We infer from Lemma 3.4 that $D_w(\mathcal{I}u) = (\mathcal{I}u)'$, which yields

$$(D_w(\mathcal{I}u), D_w v_N)_{I_j} = ((\mathcal{I}u)', D_w v_N)_{I_j}, \quad \forall I_j \in \mathcal{T}_N. \tag{3.19}$$

Then, it follows the definition of the weak derivative (2.1) and integration by parts that

$$\begin{aligned}((\mathcal{I}u)', D_w v_N)_{I_j} &= -(v_0, (\mathcal{I}u)'')_{I_j} + \langle v_b n, (\mathcal{I}u)'\rangle_{\partial I_j} \\ &= ((\mathcal{I}u)', v_0')_{I_j} - \langle(\mathcal{I}u)', (v_0 - v_b)n\rangle_{\partial I_j}.\end{aligned} \tag{3.20}$$

The definition of $\mathcal{I}$ and integration by parts implies

$$((u - \mathcal{I}u)', v_0')_{I_j} = -(u - \mathcal{I}u, v_0'')_{I_j} + \langle u - \mathcal{I}u, v_0' n\rangle_{\partial I_j} = 0,$$

thus

$$((\mathcal{I}u)', v_0')_{I_j} = (u', v_0')_{I_j},$$

which together with (3.19) and (3.20), leads to

$$(D_w(\mathcal{I}u), D_w v_N)_{I_j} = (u', v_0')_{I_j} - \langle(\mathcal{I}u)', (v_0 - v_b)n\rangle_{\partial I_j}.$$

Summing the above equation over all element $I_j \in \mathcal{T}_N$, we obtain

$$(D_w(\mathcal{I}u), D_w v_N)_{\mathcal{T}_N} = (u', v_0')_{\mathcal{T}_N} - \langle(\mathcal{I}u)', (v_0 - v_b)n\rangle_{\partial\mathcal{T}_N}. \tag{3.21}$$

Integration by parts shows that

$$-(u'', v_0)_{I_j} = (u', v_0')_{I_j} - \langle u', v_0 n\rangle_{\partial I_j}$$

Summing the above equation over all element $I_j \in \mathcal{T}_N$, and recalling the fact

$$\sum_{j=1}^{N} \langle u', v_b n \rangle_{\partial I_j} = 0,$$

we obtain

$$-(u'', v_0)_{\mathcal{T}_N} = (u', v_0')_{\mathcal{T}_N} - \langle u', (v_0 - v_b)n \rangle_{\partial \mathcal{T}_N},$$

which combining with (3.21) yields the assertion (3.17). □

Lemma 3.6 *Let u be the solution of the problem* (1.1)*. Then for $v_N = \{v_0, v_b\} \in \mathcal{M}_N^0$, there holds*

$$(bu', v_0)_{\mathcal{T}_N} = (D_w^b(\mathcal{I}u), v_0)_{\mathcal{T}_N} - \mathcal{E}_2(u, v_N), \tag{3.22}$$

where

$$\mathcal{E}_2(u, v_N) = (u - \mathcal{I}u, (bv_0)')_{\mathcal{T}_N}. \tag{3.23}$$

Proof It follows from the definition of the weak convection derivative (2.2) that

$$(D_w^b(\mathcal{I}u), v_0)_{\mathcal{T}_N} = -(\mathcal{I}u, (bv_0)')_{\mathcal{T}_N} + \langle \mathcal{I}u, bnv_0 \rangle_{\partial \mathcal{T}_N}. \tag{3.24}$$

Integration by parts shows that

$$(bu', v_0)_{\mathcal{T}_N} = -(u, (bv_0)')_{\mathcal{T}_N} + \langle u, bnv_0 \rangle_{\partial \mathcal{T}_N},$$

which together with (3.24) and recalling the fact $\mathcal{I}u = u$ on ∂I_j yields the assertion (3.22). □

Lemma 3.7 (Error equation) *Let u and $u_N \in \mathcal{M}_N^0$ be the solutions of problem* (1.1) *and* (3.2)*, respectively. Then, for any $v_N \in \mathcal{M}_N^0$, there holds*

$$\mathcal{B}(\mathcal{I}u - u_N, v_N) = \mathcal{E}(u, v_N), \tag{3.25}$$

where

$$\mathcal{E}(u, v_N) := \mathcal{E}_1(u, v_N) + \mathcal{E}_2(u, v_N) + \mathcal{E}_3(u, v_N). \tag{3.26}$$

Here $\mathcal{E}_1(u, v_N)$ and $\mathcal{E}_2(u, v_N)$ are defined by (3.18) *and* (3.23) *respectively, and $\mathcal{E}_3(u, v_N)$ is given as*

$$\mathcal{E}_3(u, v_N) = (c(\mathcal{I}u - u), v_0)_{\mathcal{T}_N}. \tag{3.27}$$

Proof Testing (1.1) by $v_N = \{v_0, v_b\} \in \mathcal{M}_N^0$, we arrive at

$$-\varepsilon(u'', v_0)_{\mathcal{T}_N} + (bu', v_0)_{\mathcal{T}_N} + (cu, v_0)_{\mathcal{T}_N} = (f, v_0)_{\mathcal{T}_N}.$$

Plugging (3.17) and (3.22) into the above equation yields

$$\mathcal{A}(\mathcal{I}u, v_N) = (f, v_0)_{\mathcal{T}_N} + \mathcal{E}(u, v_N).$$

Since $\mathcal{I}u$ is continuous in Ω, with the aid of the definitions of $\mathcal{S}_c(\cdot, \cdot)$ and $\mathcal{S}_d(\cdot, \cdot)$, we conclude

$$\mathcal{S}_c(\mathcal{I}u, v_N) = 0, \quad \mathcal{S}_d(\mathcal{I}u, v_N) = 0.$$

Thus,

$$\mathcal{B}(\mathcal{I}u, v_N) = (f, v_0)_{\mathcal{T}_N} + \mathcal{E}(u, v_N). \tag{3.28}$$

Subtracting (3.2) from (3.28) yields the error equation (3.25). The proof is completed. □

4 Error Analysis on a Shishkin Mesh

In this section, we will provide a ε-uniform error estimate for the error $u - u_N$ in the $\|\cdot\|_{\mathcal{M}}$-norm defined by (3.5). The error analysis relies on a layer-adapted mesh — the Shishkin mesh, S-decomposition and a priori estimate of the exact solution of (1.1) and a special interpolation introduced in [19]. In the following analysis, we will assume $\varepsilon \leq N^{-1}$ which is realistic for singularly perturbed problem.

The following trace inequality and inverse inequality from [3] will be used frequently in our analysis:

$$\|v\|^2_{L^2(\partial I_j)} \leq C_{\text{tr}}(h_j^{-1}\|v\|^2_{L^2(I_j)} + \|v\|_{L^2(I_j)}\|v'\|_{L^2(I_j)}), \quad \forall v \in H^1(I_j), \tag{4.1}$$

$$\|v'_N\|_{L^2(\partial I_j)} \leq C_{\text{inv}} h_j^{-1}\|v_N\|_{L^2(I_j)}, \quad \forall v_N \in \mathbb{P}^k(I_j), \tag{4.2}$$

$$\|v_N\|_{L^p(\partial I_j)} \leq C_* h_j^{-1/p}\|v_N\|_{L^p(I_j)}, \quad \forall 1 \leq p \leq \infty, \forall v_N \in \mathbb{P}^k(I_j), \tag{4.3}$$

where C_{tr}, C_{inv} and C_* are positive constants, and independent of both I_j and h_j.

The following statements present a decomposition of the exact solution u of problem (1.1) into a sum of a smooth part and a layer part, which is necessary to the ε-uniform error estimates of numerical methods for singularly perturbed problems [8].

Lemma 4.1 (S-decomposition) [15] *Let q be some positive integer. Consider the problem* (1.1) *with the assumption of* (1.2)*. The exact solution u can be composed as $u = u_S + u_E$, where the smooth part u_S and the layer part u_E satisfies*

$$-\varepsilon u''_S + bu'_S + cu_S = f,$$
$$-\varepsilon u''_E + bu'_E + cu_E = 0,$$

and

$$|u_S^{(l)}(x)| \leq C, \quad |u_E^{(l)}(x)| \leq C\varepsilon^{-l}\exp(-b_0(1-x)/\varepsilon) \quad \textit{for } 0 \leq l \leq q. \tag{4.4}$$

The following lemma shows the approximation properties of the interpolation operator $\mathcal{I}$ defined by (3.16).

Lemma 4.2 [19] *For any element $I_j \in \mathcal{T}_N$ with $I_j = [x_{j-1}, x_j]$ and $v \in H^{k+1}(I_j)$, the interpolation $\mathcal{I}v$ defined by* (3.16) *has the following approximation properties:*

$$|v - \mathcal{I}v|_{H^l(I_j)} \leq Ch_j^{k+1-l}|v|_{H^{k+1}(I_j)}, \quad l = 0, 1, \ldots, k+1, \tag{4.5}$$

$$\|v - \mathcal{I}v\|_{L^\infty(I_j)} \leq Ch_j^{k+1}|v|_{W^{k+1,\infty}(I_j)}, \tag{4.6}$$

where C is independent of h_j and ε.

From Lemmas 4.1 and 4.2, we have the following interpolation error estimates on the Shishkin mesh $\mathcal{T}_N$.

Lemma 4.3 [19,31] *Let the exact solution $u = u_S + u_E$ of the problem* (1.1) *can be decomposed into a smooth and layer part, respectively. Denote $\mathcal{I}u_S$ and $\mathcal{I}u_E$ by the interpolations u_S and u_E on a Shishkin mesh, respectively. Assume $\varepsilon \ln N \leq b_0/2(k+1)$. Then, we have*

$\mathcal{I}u = \mathcal{I}u_S + \mathcal{I}u_E$ *and the estimates*

$$\|u - \mathcal{I}u\|_{L^\infty(\Omega_1)} \leq CN^{-(k+1)}, \tag{4.7a}$$

$$\|u - \mathcal{I}u\|_{L^\infty(\Omega_2)} \leq C(N^{-1}\ln N)^{k+1}, \tag{4.7b}$$

$$\|(u_S - \mathcal{I}u_S)^{(l)}\|_{L^2(\Omega)} \leq CN^{l-(k+1)},\ l = 0, \dots, k, \tag{4.7c}$$

$$\|u_E - \mathcal{I}u_E\|_{L^2(\Omega_2)} \leq C\varepsilon^{1/2}(N^{-1}\ln N)^{k+1}, \tag{4.7d}$$

$$N^{-1}\|(\mathcal{I}u_E)'\|_{L^2(\Omega_1)} + \|\mathcal{I}u_E\|_{L^2(\Omega_1)} \leq C(\varepsilon^{1/2} + N^{-1/2})N^{-(k+1)}, \tag{4.7e}$$

$$\|u_E\|_{L^\infty(\Omega_1)} + \varepsilon^{-1/2}\|u_E\|_{L^2(\Omega_1)} \leq CN^{-(k+1)}, \tag{4.7f}$$

$$\|u_E'\|_{L^2(\Omega_1)} \leq C\varepsilon^{-1/2}N^{-(k+1)}. \tag{4.7g}$$

Lemma 4.4 *Assume $u \in H^{k+1}(\Omega)$. Under the conditions of Lemma 4.3, there holds*

$$\begin{aligned} \|(u_E - \mathcal{I}u_E)^{(l)}\|_{L^2(\Omega_1)} &\leq C\varepsilon^{1/2-l}N^{-(k+1)}, \\ \|(u_E - \mathcal{I}u_E)^{(l)}\|_{L^2(\Omega_2)} &\leq C\varepsilon^{1/2-l}(N^{-1}\ln N)^{k+1-l} \end{aligned}$$

with $l = 1, 2$.

Proof Owing to the triangle inequality and (4.7e) and (4.7g) of Lemma 4.3,

$$\|(u_E - \mathcal{I}u_E)'\|_{L^2(\Omega_1)} \leq \|u_E'\|_{L^2(\Omega_1)} + \|(\mathcal{I}u_E)'\|_{L^2(\Omega_1)} \leq C\varepsilon^{-1/2}N^{-(k+1)}.$$

As the same procedure, and using the inverse inequality, we get

$$\begin{aligned} \|(u_E - \mathcal{I}u_E)''\|_{L^2(\Omega_1)} &\leq \|u_E''\|_{L^2(\Omega_1)} + CN\|(\mathcal{I}u_E)'\|_{L^2(\Omega_1)} \\ &\leq C\varepsilon^{-3/2}[1 + (\varepsilon N)^{3/2} + (\varepsilon N)^2]N^{-(k+1)} \\ &\leq C\varepsilon^{-3/2}N^{-(k+1)}. \end{aligned}$$

Due to (4.5) of Lemma 4.2 and (4.4), we obtain, for $l = 1, 2$,

$$\begin{aligned} \|(u_E - \mathcal{I}u_E)^{(l)}\|^2_{L^2(\Omega_2)} &= \sum_{I_j \subset \Omega_2} \|(u_E - \mathcal{I}u_E)^{(l)}\|^2_{L^2(I_j)} \\ &\leq \sum_{I_j \subset \Omega_2} Ch_j^{2(k+1-l)}\|u_E^{(k+1)}\|^2_{L^2(I_j)} \\ &\leq Ch_f^{2(k+1-l)} \cdot \int_{1-\tau}^1 \varepsilon^{-2(k+1)}\exp(-2b_0(1-x)/\varepsilon)\mathrm{d}x \\ &\leq C\varepsilon^{1-2l}(N^{-1}\ln N)^{2(k+1-l)}. \end{aligned}$$

The proof is completed. □

Lemma 4.5 *Assume $u \in H^{k+1}(\Omega)$. Let σ_j is given by (3.1). Under the conditions of Lemma 4.3, there holds*

$$\left\{\sum_{j=1}^N \frac{\varepsilon^2}{\sigma_j}\|(u - \mathcal{I}u)'\|^2_{L^2(\partial I_j)}\right\}^{1/2} \leq C(N^{-1}\ln N)^k.$$

Proof To simplify notation in the proof, let $\eta_S := u_S - \mathcal{I}u_S$ and $\eta_E := u_E - \mathcal{I}u_E$ denote the interpolation errors of u_S and u_E, respectively. Then, the total interpolation error $\eta := u - \mathcal{I}u$ can be written as $\eta = \eta_S + \eta_E$.

By the triangle inequality, we have

$$\sum_{j=1}^{N} \frac{\varepsilon^2}{\sigma_j} \|\eta'\|^2_{L^2(\partial I_j)} \le 2 \sum_{j=1}^{N} \frac{\varepsilon^2}{\sigma_j} (\|\eta'_S\|^2_{L^2(\partial I_j)} + \|\eta'_E\|^2_{L^2(\partial I_j)}). \tag{4.8}$$

Owing to the trace inequality (4.1),

$$\|\eta'_S\|^2_{L^2(\partial I_j)} \le C_{\mathrm{tr}} (h_j^{-1} \|\eta'_S\|^2_{L^2(I_j)} + \|\eta'_S\|_{L^2(I_j)} \|\eta''_S\|_{L^2(I_j)}),$$

then, by (4.5) of Lemma 4.2, we arrive at

$$\begin{aligned} \sum_{j=1}^{N} \frac{\varepsilon^2}{\sigma_j} \|\eta'_S\|^2_{L^2(\partial I_j)} &\le C_{\mathrm{tr}} \sum_{j=1}^{N} \frac{\varepsilon^2}{\sigma_j} (h_j^{-1} \|\eta'_S\|^2_{L^2(I_j)} + \|\eta'_S\|_{L^2(I_j)} \|\eta''_S\|_{L^2(I_j)}) \\ &\le C(\varepsilon^2 N \|\eta'_S\|^2_{L^2(\Omega_1)} + \varepsilon \|\eta'_S\|^2_{L^2(\Omega_2)} \\ &\quad + \varepsilon^2 \|\eta'_S\|_{L^2(\Omega_1)} \|\eta''_S\|_{L^2(\Omega_1)} + \varepsilon^2 N^{-1} \ln N \|\eta'_S\|_{L^2(\Omega_1)} \|\eta''_S\|_{L^2(\Omega_1)}) \\ &\le C \varepsilon N^{-2k}, \end{aligned} \tag{4.9}$$

where $\varepsilon N < 1$ and $\varepsilon \ln N < 1$ are used.

Using the trace inequality (4.1) again, we have

$$\|\eta'_E\|^2_{L^2(\partial I_j)} \le C_{\mathrm{tr}} (h_j^{-1} \|\eta'_E\|^2_{L^2(I_j)} + \|\eta'_E\|_{L^2(I_j)} \|\eta''_E\|_{L^2(I_j)}).$$

As a result,

$$\begin{aligned} \sum_{j=1}^{N} \frac{\varepsilon^2}{\sigma_j} \|\eta'_E\|^2_{L^2(\partial I_j)} &\le C_{\mathrm{tr}} \sum_{j=1}^{N} \frac{\varepsilon^2}{\sigma_j} (h_j^{-1} \|\eta'_E\|^2_{L^2(I_j)} + \|\eta'_E\|_{L^2(I_j)} \|\eta''_E\|_{L^2(I_j)}) \\ &\le C(\varepsilon^2 N \|\eta'_E\|^2_{L^2(\Omega_1)} + \varepsilon \|\eta'_E\|^2_{L^2(\Omega_2)}) \\ &\quad + C\varepsilon^2 (\|\eta'_E\|_{L^2(\Omega_1)} \|\eta''_E\|_{L^2(\Omega_1)} + N^{-1} \ln N \|\eta'_E\|_{L^2(\Omega_2)} \|\eta''_E\|_{L^2(\Omega_2)}). \end{aligned}$$

Then, it follows from Lemma 4.4 that

$$\sum_{j=1}^{N} \frac{\varepsilon^2}{\sigma_j} \|\eta'_E\|^2_{L^2(\partial I_j)} \le C[(\varepsilon + N^{-1}) N^{-(2k+1)} + (N^{-1} \ln N)^{2k}],$$

which combining with (4.8) and (4.9) yields

$$\sum_{j=1}^{N} \frac{\varepsilon^2}{\sigma_j} \|\eta'\|^2_{L^2(\partial I_j)} \le C[(\varepsilon + N^{-2}) N^{-2k} + (N^{-1} \ln N)^{2k}].$$

Thus,

$$\left\{ \sum_{j=1}^{N} \frac{\varepsilon^2}{\sigma_j} \|\eta'\|^2_{L^2(\partial I_j)} \right\}^{1/2} \le C(N^{-1} \ln N)^k.$$

The proof is completed. □

Lemma 4.6 *Let $u \in H^{k+1}(\Omega)$ solve the problem (1.1) and σ_j is given by (3.1). Then, for $v_N \in \mathcal{M}_N^0$, there holds*

$$|\mathcal{E}(u, v_N)| \le C(N^{-1} \ln N)^k \|v_N\|_{\mathcal{M}}, \tag{4.10}$$

where C is independent of N and ε.

Proof It follows from the Cauchy-Schwarz inequality and Lemma 4.5 that

$$
\begin{aligned}
|\mathcal{E}_1(u, v_h)| &\le \sum_{j=1}^{N} \varepsilon |\langle u' - (\mathcal{I}u)', (v_0 - v_b)n\rangle_{\partial I_j}| \\
&\le \sum_{j=1}^{N} \varepsilon \|(u - \mathcal{I}u)'\|_{L^2(\partial I_j)} \|v_0 - v_b\|_{L^2(\partial I_j)} \\
&\le \left\{ \sum_{j=1}^{N} \frac{\varepsilon^2}{\sigma_j} \|(u - \mathcal{I}u)'\|^2_{L^2(\partial I_j)} \right\}^{1/2} \left\{ \sum_{j=1}^{N} \sigma_j \|v_0 - v_b\|^2_{L^2(\partial I_j)} \right\}^{1/2} \\
&\le C(N^{-1} \ln N)^k S_d^{1/2}(v_N, v_N).
\end{aligned} \tag{4.11}
$$

From (3.23) and (3.27), we observe that

$$
\begin{aligned}
\mathcal{E}_2(u, v_N) + \mathcal{E}_3(u, v_N) &= (u - \mathcal{I}u, bv_0') + (u - \mathcal{I}u, (b' - c)v_0) \\
&= T_1 + T_2.
\end{aligned}
$$

With the aid of the Cauchy-Schwarz inequality and the estimates (4.7a), (4.7b) of Lemma 4.3, we have

$$
\begin{aligned}
|T_1| &\le C[\|u - \mathcal{I}u\|_{L^\infty(\Omega_1)} \|v_0'\|_{L^1(\Omega_1)} + \|u - \mathcal{I}u\|_{L^\infty(\Omega_2)} \|v_0'\|_{L^1(\Omega_2)}] \\
&\le C[N^{-(k+1)} \|v_0'\|_{L^1(\Omega_1)} + (N^{-1} \ln N)^{k+1} \|v_0'\|_{L^1(\Omega_2)}]
\end{aligned}
$$

On Ω_1, the inverse inequality implies

$$
\|v_0'\|_{L^1(\Omega_1)} \le CN\|v_0\|_{L^1(\Omega_1)} \le CN|\Omega_1|^{1/2} \|v_0\|_{L^2(\Omega_1)} \le CN\|v_N\|_{\mathcal{M}},
$$

while on Ω_2 the Cauchy-Schwarz inequality gives

$$
\|v_0'\|_{L^1(\Omega_2)} \le \sqrt{\tau} \|v_0'\|_{L^2(\Omega_2)} \le C(\ln N)^{1/2} \|v_N\|_{\mathcal{M}}.
$$

As a result,

$$
\begin{aligned}
|T_1| &\le C[N^{-k} + N^{-1}(\ln N)^{3/2} \cdot (N^{-1} \ln N)^k] \|v_N\|_{\mathcal{M}} \\
&\le C(N^{-1} \ln N)^k \|v_N\|_{\mathcal{M}},
\end{aligned} \tag{4.12}
$$

where we use the fact $N^{-1}(\ln N)^{3/2} < 1$.

From (4.15) we observe that

$$
\|u - \mathcal{I}u\|_{L^2(\Omega)} \le CN^{-(k+1)}.
$$

Hence, T_2 can be bounded by

$$
|T_2| \le C\|u - \mathcal{I}u\|_{L^2(\Omega)} \|v_0\|_{L^2(\Omega)} \le CN^{-(k+1)} \|v_N\|_{\mathcal{M}},
$$

which together with (4.11) and (4.12) completed the proof. □

Theorem 4.1 *Let u solve the problem (1.1) and $u_N \in \mathcal{M}_N^0$ be the WG finite element solution of (3.2) calculated on Shishkin mesh $\mathcal{T}_N$. Then, there holds*

$$
\|\mathcal{I}u - u_N\|_{\mathcal{M}} \le C(N^{-1} \ln N)^k,
$$

where C is independent of N and ε.

Proof Let $\xi := \mathcal{I}u - u_N$. Owing to Lemma 3.3,

$$C_{\mathrm{lb}}\|\xi\|_{\mathcal{M}}^2 \le \mathcal{B}(\xi,\xi) \tag{4.13}$$

Taking $v_N = \xi$ in the error equation (3.25) leads to

$$\mathcal{B}(\xi,\xi) = \mathcal{E}(u,\xi).$$

It follows from Lemma 4.6 that

$$\mathcal{B}(\xi,\xi) \le C(N^{-1}\ln N)^k\|\xi\|_{\mathcal{M}},$$

which together with (4.13) complete the proof. □

Theorem 4.2 *Assume $u \in H^{k+1}(\Omega)$ and $\sqrt{\varepsilon}(\ln N)^{k+1} < C$. Under the conditions of Lemma 4.3, there holds*

$$\|u - \mathcal{I}u\|_{\mathcal{M}} \le C(N^{-1}\ln N)^k,$$

where C is independent of N and ε.

Proof Let $\eta = u - \mathcal{I}u$. Since η is continuous in Ω, we have $|\eta|_{\mathrm{J}} = 0$ and $\mathcal{S}_d(\eta,\eta) = 0$. Then,

$$\|\eta\|_{\mathcal{M}}^2 = \varepsilon\|\eta'\|_{L^2(\Omega)}^2 + c_0\|\eta\|_{L^2(\Omega)}^2 \tag{4.14}$$

Applying the estimates (4.7c)–(4.7f) of Lemma 4.3 and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\|u - \mathcal{I}u\|_{L^2(\Omega)} &\le \|u_S - \mathcal{I}u_S\|_{L^2(\Omega)} + \|u_E - \mathcal{I}u_E\|_{L^2(\Omega_2)}\\ &\quad + \|u_E\|_{L^2(\Omega_1)} + \|\mathcal{I}u_E\|_{L^2(\Omega_1)}\\ &\le CN^{-(k+1)}[1 + \varepsilon^{1/2}(\ln N)^{k+1} + \varepsilon^{1/2} + N^{-1/2}]\\ &\le CN^{-(k+1)}[1 + \varepsilon^{1/2}(\ln N)^{k+1}]\\ &\le CN^{-(k+1)}.\end{aligned} \tag{4.15}$$

Due to Lemma 4.4, we obtain

$$\begin{aligned}\|(u_E - \mathcal{I}u_E)'\|_{L^2(\Omega_2)}^2 &\le C\varepsilon^{-1}(N^{-1}\ln N)^{2k},\\ \|(u_E - \mathcal{I}u_E)'\|_{L^2(\Omega_1)}^2 &\le C\varepsilon^{-1}N^{-2(k+1)},\end{aligned}$$

which together with (4.7c) of Lemma 4.3 yields

$$\begin{aligned}\varepsilon\|(u - \mathcal{I}u)'\|_{L^2(\Omega)}^2 &\le \varepsilon\|(u_S - \mathcal{I}u_S)'\|_{L^2(\Omega)}^2 + \varepsilon\|(u_E - \mathcal{I}u_E)'\|_{L^2(\Omega_1)}^2\\ &\quad + \varepsilon\|(u_E - \mathcal{I}u_E)'\|_{L^2(\Omega_2)}^2\\ &\le C[\varepsilon N^{-2k} + (N^{-1}\ln N)^{2k} + N^{-2(k+1)}]\\ &\le C(\varepsilon N^{-2k} + (N^{-1}\ln N)^{2k} + N^{-2(k+1)})\end{aligned} \tag{4.16}$$

Combining (4.14), (4.15), and (4.16) leads to

$$\|u - \mathcal{I}u\|_{\mathcal{M}} \le C(N^{-1}\ln N)^k,$$

which completes the proof. □

Using the triangle inequality and the results of Theorems 4.2 and 4.1, we arrive at the following statements.

Theorem 4.3 *Let $u \in H^{k+1}(\Omega)$ and $u_N \in \mathcal{M}_N^0$ solve the problem (1.1) and (3.2), respectively. Then, there holds*

$$\|u - u_N\|_{\mathcal{M}} \leq C(N^{-1} \ln N)^k,$$

where C is independent of N and ε.

5 Numerical Experiments

In this section, we carried out some numerical experiments to verify our theoretical findings in previous section. The Shishkin mesh with N elements is called mesh N. Let e_N denote the error of the approximate solution computed on the mesh N. Then the approximate order of convergence, i.e., $order(2N)$, is computed by

$$order(2N) := \frac{\ln(e_N/e_{2N})}{\ln(2\ln(N)/\ln(2N))}.$$

Firstly, we confirm the convergence rate of the errors between the exact solution u and the WG finite element solution $u_N = \{u_0, u_b\}$ computed by (3.2) measured in the $\|\cdot\|_{\mathcal{M}}$-norm defined by (3.5). Furthermore, we investigate the convergence properties of the error $u - u_N$ measured in the L^2-norm defined by

$$\|u - u_0\|_{L^2(\mathcal{T}_N)} := \left\{ \sum_{j=1}^{N} \|u - u_0\|^2_{L^2(I_j)} \right\}^{1/2},$$

and the discrete L^∞-norm given by

$$\|u - u_b\|_{L^\infty(\mathcal{T}_N)} := \max_{0 \leq j \leq N} |u(x_j) - u_b(x_j)|.$$

Example 1 Consider the following convection–diffusion problem

$$\begin{cases} -\varepsilon u'' + (2-x)u' + u = f & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

with the right-hand side f chosen such that

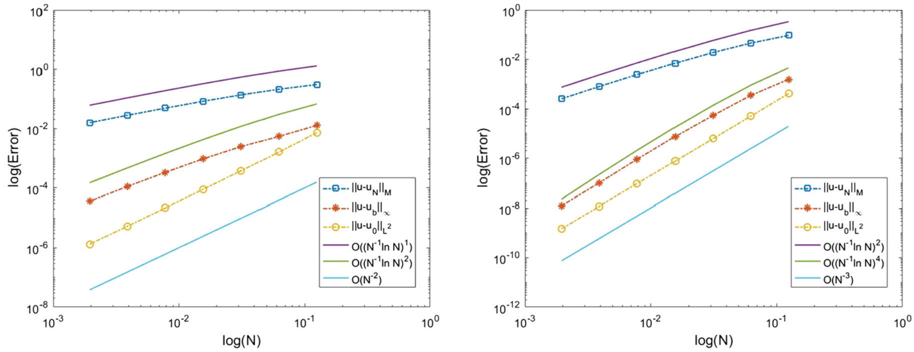
$$u(x) = \sin\left(\frac{1}{2}\pi x\right) - \frac{e^{-(1-x)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}$$

is the exact solution, which has a boundary layer with the width $\mathcal{O}(\varepsilon \ln \frac{1}{\varepsilon})$ at the outflow boundary $x = 1$.

Table 1 displays the history of convergence of the WG finite element method for Example 1. They are clear illustrations of the k-th order convergence rate in the energy-like norm (3.5), which is agree with the theoretical result of Theorem 4.3. The errors $\|u - u_N\|_{\mathcal{M}}$, $\|u-u_0\|_{L^2(\mathcal{T}_N)}$ and $\|u-u_b\|_{L^\infty(\mathcal{T}_N)}$ for Example 1 with $\varepsilon = 10^{-9}$ are plotted on log-log scales in Fig. 1. It is observed that the rate of convergence in the $\|\cdot\|_{\mathcal{M}}$-norm is $\mathcal{O}((N^{-1}\ln N)^k)$, which verifies the theoretical findings in Theorem 4.3. Fig. 1 indicates that our WG finite element scheme (3.2) has the optimal convergence rates of $\mathcal{O}(N^{-(k+1)})$ in the L^2-norm and the super-convergence rate of $\mathcal{O}((N^{-1}\ln N)^{2k})$ in the discrete L^∞-norm.

Table 1 History of convergence of the WG method, under the norm $\|\cdot\|_{\mathcal{M}}$

k	N	$\varepsilon=1.0\text{E}{-03}$		$\varepsilon=1.0\text{E}{-05}$		$\varepsilon=1.0\text{E}{-07}$	
		$\|u-u_N\|_{\mathcal{M}}$	Order	$\|u-u_N\|_{\mathcal{M}}$	Order	$\|u-u_N\|_{\mathcal{M}}$	Order
1	8	3.0536E−01	–	3.0522E−01	–	3.0522E−01	–
	16	2.1181E−01	0.90	2.1179E−01	0.90	2.1179E−01	0.90
	32	1.3588E−01	0.94	1.3587E−01	0.94	1.3587E−01	0.94
	64	8.2684E−02	0.97	8.2683E−02	0.97	8.2683E−02	0.97
	128	4.8578E−02	0.99	4.8577E−02	0.99	4.8577E−02	0.99
	256	2.7858E−02	0.99	2.7858E−02	0.99	2.7858E−02	0.99
	512	1.5698E−02	1.00	1.5698E−02	1.00	1.5698E−02	1.00
2	8	9.3756E−02	–	9.3752E−02	–	9.3752E−02	–
	16	4.5808E−02	1.77	4.5809E−02	1.77	4.5809E−02	1.77
	32	1.8928E−02	1.88	1.8928E−02	1.88	1.8928E−02	1.88
	64	7.0095E−03	1.94	7.0096E−03	1.94	7.0096E−03	1.94
	128	2.4172E−03	1.98	2.4172E−03	1.98	2.4172E−03	1.98
	256	7.9420E−04	1.99	7.9421E−04	1.99	7.9421E−04	1.99
	512	2.5204E−04	1.99	2.5204E−04	1.99	2.5204E−04	1.99
3	8	2.8937E−02	–	2.8939E−02	–	2.8939E−02	–
	16	1.0029E−02	2.61	1.0030E−03	2.61	1.0030E−02	2.61
	32	2.6862E−03	2.80	2.6863E−03	2.80	2.6863E−03	2.80
	64	6.0721E−04	2.91	6.0721E−04	2.91	6.0721E−04	2.91
	128	1.2303E−04	2.96	1.2303E−04	2.96	1.2303E−04	2.96
	256	2.3168E−05	2.98	2.3168E−05	2.98	2.3253E−05	2.98
	512	4.1406E−06	2.99	4.1411E−06	2.99	4.6166E−06	2.81



(a) $\mathbb{P}^1$ element **(b)** $\mathbb{P}^2$ element

Fig. 1 Example 1. Convergence curve of error with $\varepsilon=10^{-9}$. **a** For $\mathbb{P}^1$ element, and **b** for $\mathbb{P}^2$ element

Example 2 Consider the following convection–diffusion problem

$$\begin{cases} -\varepsilon u''+(1+x)u'+(2+x)u=4\sin(\pi x) & \text{in } (0,1),\\ u(0)=u(1)=0. \end{cases}$$

Table 2 History of convergence of the WG method, under the norm $\|\cdot\|_{\mathcal{M}}$

		$\varepsilon = 1.0E{-}03$		$\varepsilon = 1.0E{-}05$		$\varepsilon = 1.0E{-}07$	
k	N	$\|u_N - u_{2N}\|_{\mathcal{M}}$	Order	$\|u_N - u_{2N}\|_{\mathcal{M}}$	Order	$\|u_N - u_{2N}\|_{\mathcal{M}}$	Order
1	8	3.3932E−01	–	3.3950E−01	–	3.3950E−01	–
	16	2.6021E−01	0.65	2.6029E−01	0.66	2.6029E−01	0.66
	32	1.7385E−01	0.86	1.7388E−01	0.86	1.7388E−01	0.86
	64	1.0600E−01	0.97	1.0602E−01	0.97	1.0602E−01	0.97
	128	6.1573E−02	1.01	6.1581E−02	1.01	6.1582E−02	1.01
	256	3.4891E−02	1.01	3.4895E−02	1.02	3.4895E−02	1.02
	512	1.9488E−02	1.01	1.9491E−02	1.01	1.9491E−02	1.01
2	8	2.0186E−01	–	2.0186E−01	–	2.0186E−01	–
	16	1.1525E−01	1.38	1.1522E−01	1.38	1.1522E−01	1.38
	32	5.1769E−02	1.70	5.1742E−02	1.70	5.1742E−02	1.70
	64	1.9708E−02	1.89	1.9696E−02	1.89	1.9696E−02	1.89
	128	6.8107E−03	1.97	6.8061E−03	1.97	6.8061E−03	1.97
	256	2.2270E−03	2.00	2.2255E−03	2.00	2.2254E−03	2.00
	512	7.0333E−04	2.00	7.0284E−04	2.00	7.0283E−04	2.00
3	8	1.1006E−01	–	1.1001E−01	–	1.1001E−01	–
	16	4.8045E−02	2.04	4.7999E−02	2.05	4.7998E−02	2.05
	32	1.5015E−02	2.47	1.4994E−02	2.48	1.4994E−02	2.48
	64	3.6572E−03	2.76	3.6512E−03	2.77	3.6512E−03	2.77
	128	7.6146E−04	2.91	7.6014E−04	2.91	7.6005E−04	2.91
	256	1.4446E−04	2.97	1.4421E−04	2.97	1.4412E−04	2.97
	512	2.5850E−05	2.99	2.5809E−05	2.99	3.4582E−05	2.48

The exact solution of this test problem is unknown. Therefore, we use the following variant of the double mesh principle to estimate the errors. Compute

$$e_N = \|u_N - u_{2N}\|_{\mathcal{T}_N},$$

where $\|\cdot\|_{\mathcal{T}_N}$ refers one of the three norm $\|\cdot\|_{\mathcal{M}}$, $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^\infty}$, and u_{2N} is the WG solution obtained on a mesh containing the mesh points of the original Shishkin mesh $\mathcal{T}_N$ and its midpoints $x_j = (x_j + x_{j+1})/2$, $j = 0, 1, \ldots, N-1$.

For different $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$, the numerical solutions of Example 2 computed by the WG scheme (3.2) with $\mathbb{P}^1$ element on Shishkin meshes of $N = 32$ elements are displayed in Fig. 2. It can be observed that there is a boundary layer near $x = 1$ for small ε.

We show the history of convergence of the WG finite element method for Example 2 in Table 2. The errors $\|u - u_N\|_{\mathcal{M}}$, $\|u - u_0\|_{L^2(\mathcal{T}_N)}$ and $\|u - u_b\|_{L^\infty(\mathcal{T}_N)}$ for Example 2 with $\varepsilon = 10^{-9}$ are plotted on log-log scales in Fig. 3. From Table 2 and Fig. 3, we observe the same convergence behavior as in Example 1.

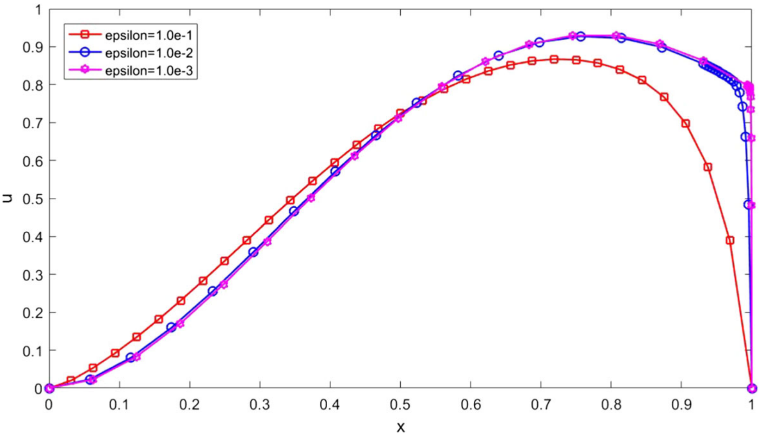
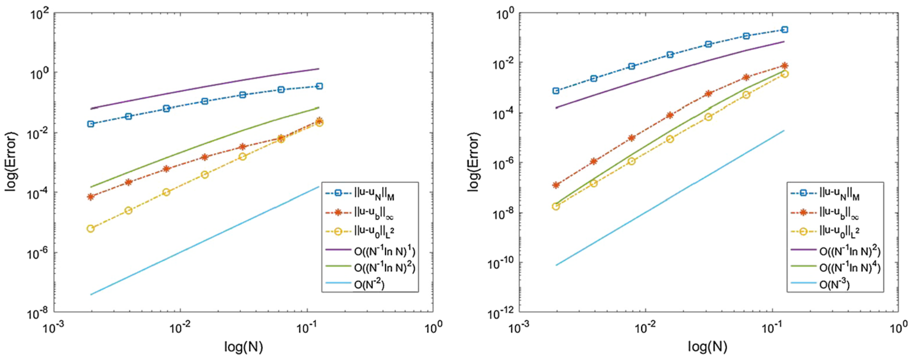


Fig. 2 Example 2. The WG solution computed by $\mathbb{P}^1$ element with $N = 32$ and different $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$



(a) $\mathbb{P}^1$ element **(b)** $\mathbb{P}^2$ element

Fig. 3 Example 2. Convergence curve of error with $\varepsilon = 10^{-9}$. **a** For $\mathbb{P}^1$ element, and **b** for $\mathbb{P}^2$ element

6 Conclusion

In this article, a WG finite element method is presented and analyzed for the one-dimensional singularly perturbed problem of convection–diffusion type. To obtain ε-independent error estimate, a special stabilization term is proposed for the discretization of the diffusion term. Optimal and uniformly convergent error estimates in the energy-like norm of the present method is proved on the Shishkin mesh for any high order element. In the view of implementation, the presented WG finite element method and the technique of elimination of interior unknowns can be extended to two-dimensional singularly perturbed problem of convection–diffusion type. Using our error analysis approach, it is not hard to prove optimal and uniformly convergent error estimates in the energy-like norm of our presented method with linear element on Shishkin meshes. As for the uniform convergence of high order element case, the main difficulty is to construct a special type of interpolation satisfying two following conditions: (1) its interpolation error is uniformly convergent on Shishkin meshes; (2) it is suitable for the analysis of the WG finite element method. We will investigate this topic in future work.

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