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A Discontinuous Galerkin Method for the Coupled Stokes and Darcy Problem

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Abstract
Combining the mixed discontinuous Galerkin method for the Darcy flow and the interior penalty discontinuous Galerkin methods for the Stokes problem, a locally conservative discrete scheme is proposed for numerically solving the coupled Stokes and Darcy problem. We prove the well-posedness of the solution of the proposed numerical scheme by boundedness, K-ellipticity and a discrete inf-sup condition. A priori error estimates, in proper norms are derived, and to verify the theoretical analysis, some numerical experiments are given.

Keywords Stokes and Darcy problem · Discontinuous Galerkin methods · Priori error estimates

1 Introduction

The coupled Stokes and Darcy model describes the interaction between free flow and porous media flow. Such systems arise, for example, in modeling the groundwater (aquifer) contamination through filtration and streams, and numerical modeling of this complicated interaction is a challenging work in both theoretical analysis and practical engineering applications. There are some related works of the coupled system. Based on the Beavers–Joseph–Saffman interface conditions [14] Layton, Schieweck, and Yotov [27] prove the existence and uniqueness of a weak solution of the coupled system and, present and analyze its numerical scheme by adopting continuous finite element methods to discretize the Stokes problem and mixed finite element methods (MFE) to discrete the Darcy problem. Rivière et al. [3,4,13] propose and

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analyze a locally conservative discrete scheme by employing discontinuous Galerkin (DG) methods and mixed finite element methods for the coupled Stokes and Darcy equations, and by utilizing the DG methods for the coupled Navier–Stokes and Darcy problem. In addition, based on DG methods and mixed finite element methods, a strongly conservative numerical scheme is given in [15] by this group. Fu and Lehrenfeld [16] propose a strongly conservative numerical scheme for the coupled system by considering hybrid discontinuous Galerkin methods (HDG) and mixed finite element methods. Based on a continuous trace approximation of velocity and a discontinuous trace approximation of pressure, Cesmelioglu et al. present a embedded-hybridized discontinuous Galerkin (EDG-HDG) finite element method [1] with strong mass conservation for the coupled Stokes–Darcy problem.

Mixed discontinuous Galerkin (MDG) method [12] and discontinuous Galerkin (DG) methods [2,5,6,8,11] are two kinds of locally mass conservative numerical methods. Mixed numerical formulations are popular for porous media problems and DG methods have many attractive properties such as being element-wise conservative, high-order methods, easily implementable on unstructured meshes. Then, we propose a locally conservative discrete scheme to numerically solving the coupled Darcy–Stokes problem, which is constructed by using DG methods to approximate the Stokes problem and MDG method to approximate the Darcy problem. The proposed scheme is different from the above mentioned numerical methods expect for the EDG-HDG finite element method, since we employ the MDG method to approximate the Darcy problem rather than the MFE methods, and the numerical scheme adopts the totally discontinuous polynomial spaces in both Stokes domain and Darcy region. Such choices of discontinuous polynomial spaces avoid the difficulty of the construction of conforming finite element space. It thus is more convenient for us to implement the algorithm in a unified framework of DG methods.

The EDG-HDG finite element method is an efficient and attractive numerical scheme with strong mass conservation, especially when the higher-order polynomial spaces are used. It also utilizes the element discontinuous polynomial spaces in both Stokes domain and Darcy region, even though a continuous approximation of trace of velocity is employed. Comparing to the proposed numerical scheme, from the point of degrees of freedom (DOF), the EDG-HDG finite element method needs fewer DOF in matching triangle (tetrahedra) meshes and quadrilateral (hexahedron) meshes if the lowest order finite element space is used, since a continuous approximation of trace of velocity is applied. However, if the meshes are polygonal and non-matching with hanging nodes, we can't draw this conclusion. Furthermore, our scheme may be superior to this EDG-HDG finite element method if the lowest order finite element space is used and a complete HDG finite element method is utilized, which means the trace of velocity is discontinuous rather than continuous. Thus, our scheme has acceptable DOF for the lowest order finite element space. What's more, our scheme requires less matrix assembly, storage, and is easier to code and implement. In a word, we think our scheme is more suitable for the lowest order polynomial space and the EDG-HDG finite element method is more attractive for higher-order polynomial spaces.

The features of the proposed numerical scheme are that the lowest order finite element space has acceptable DOF. Moreover, it is more convenient for us to implement the algorithm in a unified framework of DG methods and can be generalized to other porous media flow problems such as Stokes-Biot model [17] and Stokes-dual-porosity model [18,21], since it is constructed by a straightforward combination of MDG method and DG methods. In addition, we present the numerical analysis for the proposed scheme in proper norms and show some numerical tests to verify the analysis. The novelty of the analysis mainly includes that we generalize the primal MDG method with Dirichlet boundary condition to an MDG method with Neumann boundary condition, and based on H(div)-like DG norm we also prove the K-ellipticity by using the local lift operator in the kernel space.

The outline of the article is given as follows: In Sect. 2, the coupled Stokes and Darcy equations, notation and numerical scheme are presented. Section 3 recalls some inequalities and approximation operators. In Sect. 4, the boundedness, K-ellipticity and a discrete inf-sup condition are derived. Section 5 proves the priori error estimates. In Sect. 6, some numerical experiments are used to validate the theoretical analysis.

2 Model Equations, Notation, and Scheme

Let Ω be a open bounded domain in $\mathbb{R}^d$, $d = 2, 3$, comprised of two subdomains Ω_1 and Ω_2. Let Γ_{12} be the interface and $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$. Define $\Gamma_i = \partial\Omega_i \setminus \Gamma_{12}$, $i = 1, 2$. Denote by $\mathbf{n}$ the unit outward normal vector to $\partial\Omega$. Let $\mathbf{n}_{12}$ (resp., τ_{12}^j) be the unit normal (resp., tangential) vector to Γ_{12} outward of Ω_1, where $j = 1, \ldots, d-1$. Denote by $\mathbf{u} = (\mathbf{u}_f, \mathbf{u}_s)$ the fluid velocity in (Ω_1, Ω_2) and $p = (p_f, p_s)$ the fluid pressure in (Ω_1, Ω_2). We assume the Stokes equations in Ω_1, and there holds

$$-\nabla \cdot \mathbf{T}(\mathbf{u}_f, p_f) = \mathbf{f} \quad \text{in} \quad \Omega_1, \tag{1}$$

$$\nabla \cdot \mathbf{u}_f = 0 \quad \text{in} \quad \Omega_1, \tag{2}$$

$$\mathbf{u}_f = 0 \quad \text{on} \quad \Gamma_1. \tag{3}$$

Here $\mathbf{T}$ is the stress tensor

$$\mathbf{T}(\mathbf{u}_f, p_f) = -p_f\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}_f),$$

where $\mu > 0$ is the constant viscosity coefficient and the strain tensor is defined by

$$\mathbf{D}(\mathbf{u}_f) = \frac{1}{2}(\nabla\mathbf{u}_f + \nabla\mathbf{u}_f^T).$$

In region Ω_2, the governing equations satisfy the Darcy equations

$$\nabla \cdot \mathbf{u}_s = g \quad \text{in} \quad \Omega_2, \tag{4}$$

$$\mathbf{K}^{-1}\mathbf{u}_s + \nabla p_s = 0 \quad \text{in} \quad \Omega_2, \tag{5}$$

$$\mathbf{u}_s \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_2, \tag{6}$$

where the permeability tensor $\mathbf{K}$ is symmetric and positive definite, and satisfies for some $0 < k_{min} \le k_{max} < \infty$,

$$k_{min}\xi^T\xi \le \xi^T\mathbf{K}(\mathbf{x})\xi \le k_{max}\xi^T\xi \quad \forall\xi \in \mathbb{R}^d.$$

The physical quantities in Ω_1 and Ω_2 are coupled by the following interface conditions on Γ_{12}:

$$\mathbf{u}_f \cdot \mathbf{n}_{12} = \mathbf{u}_s \cdot \mathbf{n}_{12}, \tag{7}$$

$$p_f - 2\mu(\mathbf{D}(\mathbf{u}_f)\mathbf{n}_{12}) \cdot \mathbf{n}_{12} = p_s, \tag{8}$$

$$\mathbf{u}_f \cdot \tau_{12}^j = -2G(\mathbf{D}(\mathbf{u}_f)\mathbf{n}_{12}) \cdot \tau_{12}^j, \quad j = 1, \ldots, d-1. \tag{9}$$

Note that interface condition (7) denotes the mass conservation, interface condition (8) stands for balance of forces, and interface (9) represents the Beaver–Joseph–Saffman law, where $G > 0$ is friction coefficient determined by numerical experiments.

For $i = 1, 2$, let ε_h^i be a non-overlapping and quasi-uniform decomposition [22] of Ω_i, let Γ_h^i be the set of interior facets and let h_i denote the maximum diameter of elements in ε_h^i. For any non-negative integer k and number $r \geq 1$, the classical Sobolev spaces [23] on a domain O is denoted by $W^{k,r}(O) = \{v \in L^r(O) : D^m(v) \in L^r(O), \forall m \geq k\}$, where $D^m(v)$ are the partial derivatives of v of order m. The associated Sobolev norm (respectively, semi-norm) is denoted by $\|\cdot\|_{k,r,O}$ (respectively, $|\cdot|_{k,r,O}$), or by $\|\cdot\|_{k,O}$ (respectively, $|\cdot|_{k,O}$) if $r = 2$. We use the notation $H^k(O)$ for $W^{k,2}(O)$ and $L_0^2(O)$ for the space of square integrable functions with zero average. The L^2 inner-product will be denoted by $(\cdot,\cdot)$. Moreover, let $H(div;\Omega_2) = \{\mathbf{v} \in (L^2(\Omega_2))^d, \nabla\cdot\mathbf{v} \in L^2(\Omega_2)\}$ with norm $\|\mathbf{v}\|^2_{H(div;\Omega_2)} = \|\mathbf{v}\|^2_{0,\Omega_2} + \|\nabla\cdot\mathbf{v}\|^2_{0,\Omega_2}$, and let $H_0(div;\Omega_2) = \{\mathbf{v} \in H(div;\Omega_2) : \mathbf{v}\cdot\mathbf{n}|_{\Gamma_2} = 0\}$. Throughout the paper, c will denote a generic positive constant whose value may vary with different equations but shall be independent of the mesh-sizes h_1 and h_2. Particularly, our scheme requires that the trace of the normal derivatives of $\mathbf{u}_f$ and the trace of $\mathbf{u}_f, p_f, \mathbf{u}_s, p_s$ are well defined, and are square-integrable, therefore, we define the following functional spaces:

$$
\begin{aligned}
\mathbf{X}^f &= \{\mathbf{v}_f \in (L^2(\Omega_1))^d, \quad \forall E \in \varepsilon_h^1, \quad \mathbf{v}_f|_E \in (H^2(E))^d\},\\
\mathbf{X}^s &= \{\mathbf{v}_s \in (L^2(\Omega_2))^d, \quad \forall E \in \varepsilon_h^2, \quad \mathbf{v}_s|_E \in (H^1(E))^d\},\\
M^f &= \{q_f \in L^2(\Omega_1), \quad \forall E \in \varepsilon_h^1, \quad q_f|_E \in H^1(E)\},\\
M^s &= \{q_s \in L^2(\Omega_2), \quad \forall E \in \varepsilon_h^2, \quad q_s|_E \in H^1(E)\}.
\end{aligned}
$$

Let w be any scalar or vector-valued function. Given a fixed unit normal vector $\mathbf{n}_e$ on each interior facet $e \in \partial E_1 \cap \partial E_2$, pointing from E_1 to E_2, the average $\{w\}$ and jump $[w]$ of function w are uniquely defined

$$\{w\} = \frac{1}{2}(w|_{E_1} + w|_{E_2}), \quad [w] = w|_{E_1} - w|_{E_2}.$$

In addition, if $e \in \partial\Omega$ and $e \in E_1$, then the average $\{w\}$ and jump $[w]$ of function w are

$$\{w\} = w|_{E_1}, \quad [w] = w|_{E_1}.$$

Define the general DG norms:

$$
\begin{aligned}
&\|\mathbf{v}_f\|^2_{\mathbf{X}^f} = |||\nabla\mathbf{v}_f|||^2_{0,\Omega_1} + \sum_{e\in\Gamma_h^1\cup\Gamma_1} \frac{\sigma_{1,e}}{|e|}\|[\mathbf{v}_f]\|^2_{0,e} + \frac{\mu}{G}\sum_{j=1}^{d-1}\sum_{e\in\Gamma_{12}}\|\mathbf{v}_f\cdot\tau_{12}^j\|^2_{0,e},\\
&\|q_f\|^2_{M^f} = \|q_f\|^2_{0,\Omega_1}, \qquad \|q_s\|^2_{M^s} = \|q_s\|^2_{0,\Omega_2},
\end{aligned}
$$

and H(div)-like norm:

$$\|\mathbf{v}_s\|^2_{\mathbf{X}^s} = |||\mathbf{v}_s|||^2_{0,\Omega_2} + |||\nabla\cdot\mathbf{v}_s|||^2_{0,\Omega_2} + \sum_{e\in\Gamma_h^2\cup\Gamma_2}\frac{\sigma_{2,e}}{|e|}\|[\mathbf{v}_s\cdot\mathbf{n}_e]\|^2_{0,e},$$

where $|e|$ denotes the measure of facet e, the parameters $\sigma_{1,e}$ and $\sigma_{2,e}$ are positive penalty constants, and the $|||\cdot|||$ norm is the usual "broken" norm with $m = 0$ or $m = 1$

$$|||w|||^2_{m,\Omega_i} = \sum_{E\in\varepsilon_h^i}\|w\|^2_{m,E} \quad \forall i = 1, 2.$$

Now we define

$$\mathbf{X} = \mathbf{X}^f \times \mathbf{X}^s, \quad M = \{q \in L_0^2(\Omega) : q|_{\Omega_1} \in M^f, q|_{\Omega_2} \in M^s\},$$

and the corresponding norms

$$\|\mathbf{v}\|_{\mathbf{X}}^2 = \|\mathbf{v}_f\|_{\mathbf{X}^f}^2 + \|\mathbf{v}_s\|_{\mathbf{X}^s}^2, \quad \|q\|_M^2 = \|q_f\|_{M^f}^2 + \|q_s\|_{M^s}^2. \tag{10}$$

Let k_1 and k_2 be positive integers. We consider the finite-dimensional approximation spaces $\mathbf{X}_h^f \subset \mathbf{X}^f$, $\mathbf{X}_h^s \subset \mathbf{X}^s$, $M_h^f \subset M^f$ and $M_h^s \subset M^s$, defined as follows:

$$\begin{aligned}
\mathbf{X}_h^f &= \{\mathbf{v}_f \in \mathbf{X}^f, \quad \forall E \in \varepsilon_h^1, \quad \mathbf{v}_f|_E \in (\mathbb{P}_{k_1}(E))^d\},\\
\mathbf{X}_h^s &= \{\mathbf{v}_s \in \mathbf{X}^s, \quad \forall E \in \varepsilon_h^2, \quad \mathbf{v}_s|_E \in (\mathbb{P}_{k_2}(E))^d\},\\
M_h^f &= \{q_f \in M^f, \quad \forall E \in \varepsilon_h^1, \quad q_f|_E \in \mathbb{P}_{k_1-1}(E)\},\\
M_h^s &= \{q_s \in M^s, \quad \forall E \in \varepsilon_h^2, \quad q_s|_E \in \mathbb{P}_{k_2-1}(E)\},
\end{aligned}$$

where $\mathbb{P}_{k_1}(E)$, $\mathbb{P}_{k_2}(E)$ stand for polynomial spaces of degree less than or equal to k_1, k_2 respectively. Let $\mathbf{X}_h$ and M_h be finite-dimensional subspaces and belong to $\mathbf{X}$ and M, respectively, such that

$$\mathbf{X}_h = \mathbf{X}_h^f \times \mathbf{X}_h^s, \quad M_h = (M_h^f \times M_h^s) \cap L_0^2(\Omega).$$

Before giving the numerical scheme, some bilinear forms shall be introduced. For any $\mathbf{u}_f, \mathbf{v}_f \in \mathbf{X}^f$,

$$\begin{aligned}
a_f(\mathbf{u}_f, \mathbf{v}_f) = 2\mu \sum_{E\in\varepsilon_h^1} \int_E \mathbf{D}(\mathbf{u}_f) : \mathbf{D}(\mathbf{v}_f) d\mathbf{x} &+ \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \frac{\sigma_{1,e}}{|e|}[\mathbf{u}_f]\cdot[\mathbf{v}_f]ds\\
&- 2\mu \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{\mathbf{D}(\mathbf{u}_f)\mathbf{n}_e\}\cdot[\mathbf{v}_f]ds + 2\mu\epsilon \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{\mathbf{D}(\mathbf{v}_f)\mathbf{n}_e\}\cdot[\mathbf{u}_f]ds\\
&+ \sum_{j=1}^{d-1} \sum_{e\in\Gamma_{12}} \int_e \frac{\mu}{G}\mathbf{u}_f\cdot\tau_{12}^j \mathbf{v}_f\cdot\tau_{12}^j ds,
\end{aligned}$$

where $\epsilon = \pm 1$ and $\sigma_{1,e} > 0$ is the penalty constant. For any $\mathbf{u}_s, \mathbf{v}_s \in \mathbf{X}^s$,

$$a_s(\mathbf{u}_s, \mathbf{v}_s) = \sum_{E\in\varepsilon_h^2} \int_E \mathbf{K}^{-1}\mathbf{u}_s\cdot\mathbf{v}_s d\mathbf{x} + \sum_{e\in\Gamma_h^2\cup\Gamma_2} \int_e \frac{\sigma_{2,e}}{|e|}[\mathbf{u}_s\cdot\mathbf{n}_e][\mathbf{v}_s\cdot\mathbf{n}_e]ds,$$

where the stability constant $\sigma_{2,e} > 0$. For any $\mathbf{v}_f \in \mathbf{X}^f$, $p_f \in M^f$ and $\mathbf{u}_s \in \mathbf{X}^s$, $p_s \in M^s$,

$$\begin{aligned}
b_f(\mathbf{v}_f, p_f) &= -\sum_{E\in\varepsilon_h^1} \int_E p_f \nabla\cdot\mathbf{v}_f d\mathbf{x} + \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e [\mathbf{v}_f\cdot\mathbf{n}_e]\{p_f\}ds,\\
b_s(\mathbf{v}_s, p_s) &= -\sum_{E\in\varepsilon_h^2} \int_E p_s \nabla\cdot\mathbf{v}_s d\mathbf{x} + \sum_{e\in\Gamma_h^2\cup\Gamma_2} \int_e [\mathbf{v}_s\cdot\mathbf{n}_e]\{p_s\}ds.
\end{aligned}$$

Define the finite-dimensional space of functions $\Lambda_h = \mathbf{X}_h^s \cdot \mathbf{n}_{12}$ on the interface and let

$$\mathbf{V}_h = \{(\mathbf{v}_f, \mathbf{v}_s) \in \mathbf{X}_h : \sum_{e\in\Gamma_{12}} \int_e \eta(\mathbf{v}_f - \mathbf{v}_s)\cdot\mathbf{n}_{12} ds = 0 \quad \forall \eta \in \Lambda_h\}.$$

Assumption 2.1 We assume $\Lambda_h = \{\eta \in L^2(\Gamma_{12}), \forall e \in \Gamma_{12}, \eta|_e \in \mathbb{P}_{k_2}(e)\}$.

Indeed, the Assumption 2.1 holds true by choosing a proper basis of space $\mathbf{X}_h^s$.

Define $a = a_f + a_s$ and $b = b_f + b_s$, the numerical scheme reads as: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \int_{\Omega_1} \mathbf{f} \cdot \mathbf{v}_h d\mathbf{x} \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{11}$$

$$-b(\mathbf{u}_h, q_h) = \int_{\Omega_2} g q_h d\mathbf{x} \quad \forall q_h \in M_h \qquad . \tag{12}$$

Remark 2.1 We can check that the numerical scheme (11)–(12) is locally mass conservative. Indeed, taking the test function q_h in (12) such that $q_h = 1$ on element E and $q_h = 0$ on the remaining elements E, we obtain

$$\int_E \{\mathbf{u}_h\} \cdot \mathbf{n}_E d\mathbf{x} = \int_E \chi_{\Omega_2} g d\mathbf{x} \quad \forall E \in \varepsilon_h^1 \cup \varepsilon_h^2,$$

where χ_{Ω_2} is the characteristic function taking the value 0 in Ω_1 and 1 in Ω_2.

Remark 2.2 To facilitate the theoretical analysis, we introduce the space $\mathbf{V}_h$ of weakly-continuous-normal velocities on the interface. Clearly, it is difficult to construct this space, thus, an equivalent formulation to (11)–(12) is presented in Sect. 6. It only depends on the space $\mathbf{X}_h^s$, and it is more convenient for implementation. The space Λ_h, as a Lagrange multiplies space, is used to impose the continuity of the normal velocities. The choice $\Lambda_h = \mathbf{X}_h^s \cdot \mathbf{n}_{12}$ is to ensure the well-posedness, stability and accuracy of the discrete scheme (11)–(12).

Next, we show the exact solution of the coupled Stokes and Darcy problem (1)–(9) satisfies the numerical scheme (11)–(12) up to an error term on the interface.

Lemma 2.1 *Let* $(\mathbf{u}, p)$ *satisfy the coupled Stokes–Darcy problem (1)–(9), such that* $\mathbf{u}_f = \mathbf{u}|_{\Omega_f}$, $\mathbf{u}_s = \mathbf{u}|_{\Omega_s}$ *and* $p_f = p|_{\Omega_f}$, $p_s = p|_{\Omega_s}$, *then* $(\mathbf{u}, p)$ *solves the variational problem*

$$a(\mathbf{u}, \mathbf{v}_h) + b(\mathbf{v}_h, p) = \int_{\Omega_1} \mathbf{f} \cdot \mathbf{v}_h d\mathbf{x} - \sum_{e \in \Gamma_{12}} \int_e p_s (\mathbf{v}_{f,h} - \mathbf{v}_{s,h}) \cdot \mathbf{n}_{12} \quad \mathbf{v}_h \in \mathbf{V}_h, \tag{13}$$

$$-b(\mathbf{u}, q_h) = \int_{\Omega_2} g q_h d\mathbf{x} \quad \forall q_h \in M_h. \tag{14}$$

Proof Multiplying the Stokes Eq. (1) by $\mathbf{v}_{f,h} \in \mathbf{X}_h^f$, integrating by parts over element E and summing over all elements E. From the regularity of the exact solution and the boundary condition, we can obtain

$$\begin{aligned}
&\sum_{E \in \varepsilon_h^1} \int_E (2\mu D(\mathbf{u}_f) : D(\mathbf{v}_{f,h}) - p_f \nabla \cdot \mathbf{v}_{f,h}) d\mathbf{x} + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{p_f\}[\mathbf{v}_{f,h} \cdot \mathbf{n}_e] ds \\
&- 2\mu \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{D(\mathbf{u}_f)\mathbf{n}_e\} \cdot [\mathbf{v}_{f,h}] ds + 2\mu\epsilon \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{D(\mathbf{v}_{f,h})\mathbf{n}_e\} \cdot [\mathbf{u}_f] ds \\
&+ \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \frac{\sigma_{1,e}}{|e|} [\mathbf{u}_f] \cdot [\mathbf{v}_{f,h}] ds - \sum_{e \in \Gamma_{12}} \int_e (-p_f \mathbf{I} + 2\mu D(\mathbf{u}_f)) \mathbf{n}_{12} \cdot \mathbf{v}_{f,h} ds \\
&= \int_{\Omega_1} \mathbf{f} \cdot \mathbf{v}_{f,h} d\mathbf{x}.
\end{aligned}$$

The interface term can be rewritten as

$$(-p_f\mathbf{I}+2\mu D(\mathbf{u}_f))\mathbf{n}_{12}=-p_f\mathbf{n}_{12}+(2\mu(D(\mathbf{u}_f)\mathbf{n}_{12})\cdot\mathbf{n}_{12})\mathbf{n}_{12}+\sum_{j=1}^{d-1}(2\mu(D(\mathbf{u}_f)\mathbf{n}_{12})\cdot\tau_{12}^j)\tau_{12}^j,$$

combining the interface conditions (8) and (9), we get

$$\begin{aligned}-\sum_{e\in\Gamma_{12}}\int_e(-p_f\mathbf{I}+2\mu D(\mathbf{u}_f))\mathbf{n}_{12}\cdot\mathbf{v}_{f,h}ds&=\sum_{e\in\Gamma_{12}}\int_e p_s(\mathbf{v}_{f,h}\cdot\mathbf{n}_{12})ds\\&+\frac{\mu}{G}\sum_{j=1}^{d-1}\sum_{e\in\Gamma_{12}}\int_e(\mathbf{u}_f\cdot\tau_{12}^j)(\mathbf{v}_{f,h}\cdot\tau_{12}^j)ds.\end{aligned}$$

Thus, we have

$$a_f(\mathbf{u}_f,\mathbf{v}_{f,h})+b_f(\mathbf{v}_{f,h},p_f)+\sum_{e\in\Gamma_{12}}\int_e p_s(\mathbf{v}_{f,h}\cdot\mathbf{n}_{12})ds=(\mathbf{f},\mathbf{v}_{f,h})_{\Omega_1}\quad\forall\mathbf{v}_{f,h}\in\mathbf{X}_h^f. \tag{15}$$

Similarly, we obtain

$$-b_f(\mathbf{u}_f,q_{f,h})=0\qquad\forall q_{f,h}\in M_h^f, \tag{16}$$

$$a_s(\mathbf{u}_s,\mathbf{v}_{s,h})+b_s(\mathbf{v}_{s,h},p_s)-\sum_{e\in\Gamma_{12}}\int_e p_s(\mathbf{v}_{s,h}\cdot\mathbf{n}_{12})ds=0\qquad\forall\mathbf{v}_{s,h}\in\mathbf{X}_h^s, \tag{17}$$

$$-b_s(\mathbf{u}_s,q_{s,h})=(g,q_{s,h})_{\Omega_2}\qquad\forall q_{s,h}\in M_h^s. \tag{18}$$

Adding (15)–(16) to (17)–(18), we complete the proof. □

Remark 2.3 Note that, if $k_1=k_2$, the exact solution of the coupled system (1)–(9) satisfies the numerical scheme (11)–(12) without the interface error term appearing in (13).

3 Inequalities and Approximation Operators

Recall the standard trace inequalities [2], there holds on a given element E with diameter h_E

$$\forall\phi\in H^1(E),\quad\forall e\subset\partial E,\quad\|\phi\|_{0,e}^2\le c(h_E^{-1}\|\phi\|_{0,E}^2+h_E|\phi|_{1,E}^2), \tag{19}$$

$$\forall\phi\in H^2(E),\quad\forall e\subset\partial E,\quad\|\nabla\phi\cdot\mathbf{n}_e\|_{0,e}^2\le c(h_E^{-1}\|\phi\|_{1,E}^2+h_E|\phi|_{2,E}^2), \tag{20}$$

$$\forall\phi\in\mathbb{P}_k(E),\quad\forall e\subset\partial E,\quad\|\phi\|_{0,e}\le ch_E^{-1/2}\|\phi\|_{0,E}, \tag{21}$$

$$\forall\phi\in\mathbb{P}_k(E),\quad\forall e\subset\partial E,\quad\|\nabla\phi\cdot\mathbf{n}_e\|_{0,e}\le ch_E^{-1/2}|\phi|_{1,E}. \tag{22}$$

Also, recall the discrete Korn's inequality [26]

$$\forall\mathbf{v}_{f,h}\in\mathbf{X}_h^f,\quad|||\nabla\mathbf{v}_{f,h}|||_{0,\Omega_1}^2\le c\left(|||D(\mathbf{v}_{f,h})|||_{0,\Omega_1}^2+\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{1}{|e|}\|[\mathbf{v}_{f,h}]\|_{0,e}^2\right). \tag{23}$$

Let $p \in L^2(\Omega)$, we denote by $\tilde{p}$ the L^2- projection of p in M_h satisfying

$$\forall q_{f,h} \in \mathbb{P}_{k_1-1}(E), \quad \int_E q_{f,h}(p-\tilde{p}) = 0 \quad \forall E \in \varepsilon_h^1, \tag{24}$$

$$\forall q_{s,h} \in \mathbb{P}_{k_2-1}(E), \quad \int_E q_{s,h}(p-\tilde{p}) = 0 \quad \forall E \in \varepsilon_h^2, \tag{25}$$

and, if $p|_{\Omega_1} \in H^{k_1}(\Omega_1)$ and $p|_{\Omega_2} \in H^{k_2}(\Omega_2)$, then the following approximation properties hold

$$\|p-\tilde{p}\|_{m,E} \leq ch_E^{k_1-m}|p|_{k_1,E}, \quad E \in \varepsilon_h^1, \quad m = 0, 1, \tag{26}$$

$$\|p-\tilde{p}\|_{m,E} \leq ch_E^{k_2-m}|p|_{k_2,E}, \quad E \in \varepsilon_h^2, \quad m = 0, 1. \tag{27}$$

Let $\Pi_h^f : (H^1(\Omega_1))^d \to \mathbf{X}_h^f$ be the quasi-local interpolation [24], and the quasi-local interpolation satisfies for any $E \in \varepsilon_h^1$

$$\forall \mathbf{v}_f \in (H^1(\Omega_1))^d, \quad \forall q_f \in \mathbb{P}_{k_1-1}(E), \quad \int_E q_f \nabla \cdot (\Pi_h^f \mathbf{v}_f - \mathbf{v}_f) d\mathbf{x} = 0, \tag{28}$$

$$\forall \mathbf{v}_f \in (H^1(\Omega_1))^d, \quad \forall e \in \Gamma_h^1, \quad \forall \mathbf{q}_f \in (\mathbb{P}_{k_1-1}(e))^d, \quad \int_e \mathbf{q}_f \cdot [\Pi_h^f \mathbf{v}_f] ds = 0, \tag{29}$$

$$\forall \mathbf{v}_f \in (H_0^1(\Omega_1))^d, \quad \forall e \in \Gamma_1, \quad \forall \mathbf{q}_f \in (\mathbb{P}_{k_1-1}(e))^d, \quad \int_e \mathbf{q}_f \cdot \Pi_h^f \mathbf{v}_f ds = 0, \tag{30}$$

$$|||\Pi_h^f \mathbf{v}_f|||_{1,\Omega_1} \leq c\|\mathbf{v}_f\|_{1,\Omega_1}. \tag{31}$$

For any $\mathbf{v}_f \in (H_0^1(\Omega_1))^d$, by (28), (29) and (30) we have

$$b_f(\Pi_h^f \mathbf{v}_f - \mathbf{v}_f, q_f) = 0 \quad \forall q_f \in M_h^f. \tag{32}$$

Moreover, the interpolation operator Π_h^f satisfies the following approximation property

$$|\Pi_h^f \mathbf{v}_f - \mathbf{v}_f|_{m,E} \leq ch_E^{s-m}|\mathbf{v}_f|_{s,\delta(E)} \quad \forall 1 \leq s \leq k_1+1, \forall \mathbf{v}_f \in (H^s(\Omega_1))^d, m = 0, 1, \tag{33}$$

where $\delta(E)$ is a macro-element containing E. Moreover, there exists at least one facet e of every element $E \in \varepsilon_h^1$ such that

$$\int_e (\Pi_h^f \mathbf{v}_f - \mathbf{v}_f) ds = 0 \quad \forall \mathbf{v}_f \in (H^s(\Omega_1))^d. \tag{34}$$

Indeed, if $d = 2$, when $k_1 = 1$ and $k_1 = 2$, (34) holds true for all facets, when $k_1 = 3$, it holds true for all facets of most practical mesh (see [24]), if $d = 3$, when $k_1 = 1$, (34) holds true for all facets. Specially, for the interpolation operator Π_h^f, we have the following bounds.

Lemma 3.1 *Let $1 \leq s \leq k_1 + 1$. For all $\mathbf{v}_f \in (H^s(\Omega_1))^d$ and $\mathbf{v}_f|_{\Gamma_1} = 0$, there holds*

$$\|\Pi_h^f \mathbf{v}_f - \mathbf{v}_f\|_{\mathbf{X}^f} \leq ch_1^{s-1}|\mathbf{v}_f|_{s,\Omega_1}, \tag{35}$$

$$\|\Pi_h^f \mathbf{v}_f\|_{\mathbf{X}^f} \leq c\|\mathbf{v}_f\|_{1,\Omega_1}. \tag{36}$$

Proof From the approximation property (33) and (34) (see Lemma 3.9 of [24]), we have

$$\|\Pi_h^f \mathbf{v}_f - \mathbf{v}_f\|_{\mathbf{X}^f} \le c|||\nabla(\Pi_h^f \mathbf{v}_f - \mathbf{v}_f)|||_{0,\Omega_1} \le ch_1^{s-1}|\mathbf{v}_f|_{s,\Omega_1}.$$

Using the fact that $\|\mathbf{v}_f\|_{\mathbf{X}^f} \le c\|\mathbf{v}_f\|_{1,\Omega_1}$, for any $\mathbf{v}_f \in (H^1(\Omega_1))^d$, the bound (36) follows from (35) with $s = 1$ and triangle inequality. □

Let $\Pi_h^s : (H^\theta(\Omega_2))^d \cap H(div; \Omega_2) \to \tilde{\mathbf{X}}_h^s$ be the MFE interpolant [10] for any $\theta > 0$, where $\tilde{\mathbf{X}}_h^s$ satisfies

$$\tilde{\mathbf{X}}_h^s \equiv H_0(div; \Omega_2) \cap \mathbf{X}_h^s, \tag{37}$$

indeed, the space $\tilde{\mathbf{X}}_h^s$ is BDM_{k_2} [25]. For any $\mathbf{v}_s \in (H^\theta(\Omega_2))^d \cap H_0(div; \Omega_2)$, it holds

$$b_s(\Pi_h^s \mathbf{v}_s - \mathbf{v}_s, q_s) = 0 \quad \forall q_s \in M_h^s, \tag{38}$$

$$\int_e ((\Pi_h^s \mathbf{v}_s - \mathbf{v}_s) \cdot \mathbf{n}_e)\mathbf{w}_s \cdot \mathbf{n}_e = 0 \quad \forall e \in \Gamma_h^2 \cup \Gamma_2 \cup \Gamma_{12}, \forall \mathbf{w}_s \in \mathbf{X}_h^s. \tag{39}$$

For any $E \in \varepsilon_h^2$, Π_h^s satisfies the approximation properties

$$\|\Pi_h^s \mathbf{v}_s - \mathbf{v}_s\|_{m,E} \le ch_E^s |\mathbf{v}_s|_{s-m,E} \quad 1 \le s \le k_2 + 1, m = 0, 1, \tag{40}$$

$$\|\nabla \cdot (\Pi_h^s \mathbf{v}_s - \mathbf{v}_s)\|_{0,E} \le ch_E^s |\nabla \cdot \mathbf{v}_s|_{s,E} \quad 0 \le s \le k_2. \tag{41}$$

In addition, we have the following result [4,25]

$$\|\Pi_h^s \mathbf{v}_s\|_{H(div;\Omega_2)} \le c(\|\mathbf{v}_s\|_{\theta,\Omega_2} + \|\nabla \cdot \mathbf{v}_s\|_{0,\Omega_2}). \tag{42}$$

Remark 3.1 Note that, the interpolation operator Π_h^s holds in any dimension, However, the existence of interpolation operators Π_h^f, in three dimensions, for $k_1 = 1$ is presented in [10]. As for other k_1, we don't know whether the interpolation operators Π_h^f is exist.

4 Well-Posedness

In this section, we prove the boundedness of bilinear operators $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, K-ellipticity of bilinear operator $a(\cdot, \cdot)$ and discrete inf-sup condition of bilinear operator $b(\cdot, \cdot)$. Then, the well-posedness of the numerical scheme (11)–(12) is obtained by using the boundedness, K-ellipticity and discrete inf-sup condition.

The boundedness of bilinear operators $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are proved in the following Lemma.

Lemma 4.1 *There exists a constant c, independent of mesh-sizes h_1 and h_2 such that*

$$a(\mathbf{u}_h, \mathbf{v}_h) \le c\|\mathbf{u}_h\|_{\mathbf{X}}\|\mathbf{v}_h\|_{\mathbf{X}} \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{X}_h,$$
$$b(\mathbf{v}_h, p_h) \le c\|\mathbf{v}_h\|_{\mathbf{X}}\|p_h\|_M \quad \forall \mathbf{v}_h \in \mathbf{X}_h, p_h \in M_h.$$

Proof By Cauchy–Schwarz inequality, trace inequalities (21) and (22), the bilinear operators $a_f(\cdot, \cdot)$, $a_s(\cdot, \cdot)$ and $b_f(\cdot, \cdot)$, $b_s(\cdot, \cdot)$ satisfy

$$a_f(\mathbf{u}_{f,h}, \mathbf{v}_{f,h}) \le c\|\mathbf{u}_{f,h}\|_{\mathbf{X}^f}\|\mathbf{v}_{f,h}\|_{\mathbf{X}^f} \quad \forall \mathbf{u}_{f,h}, \mathbf{v}_{f,h} \in \mathbf{X}_h^f,$$
$$a_s(\mathbf{u}_{s,h}, \mathbf{v}_{s,h}) \le c\|\mathbf{u}_{s,h}\|_{\mathbf{X}^s}\|\mathbf{v}_{s,h}\|_{\mathbf{X}^s} \quad \forall \mathbf{u}_{s,h}, \mathbf{v}_{s,h} \in \mathbf{X}_h^s,$$

$$b_f(\mathbf{v}_{f,h}, p_{f,h}) \le c\|\mathbf{v}_{f,h}\|_{\mathbf{X}^f}\|p_{f,h}\|_{M^f} \quad \forall \mathbf{v}_{f,h} \in \mathbf{X}_h^f,\, p_{f,h} \in M_h^f,$$
$$b_s(\mathbf{v}_{s,h}, p_{s,h}) \le c\|\mathbf{v}_{s,h}\|_{\mathbf{X}^s}\|p_{s,h}\|_{M^s} \quad \forall \mathbf{v}_{s,h} \in \mathbf{X}_h^s,\, p_{s,h} \in M_h^s.$$

From the relations $a(\cdot,\cdot) = a_f(\cdot,\cdot) + a_s(\cdot,\cdot)$ and $b(\cdot,\cdot) = b_f(\cdot,\cdot) + b_s(\cdot,\cdot)$, immediately, we have the boundedness of bilinear operators $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$. □

Next, we present the K-ellipticity of bilinear operator $a(\cdot,\cdot)$. To prove the K-ellipticity, the following conditions shall be given.

1. $\sigma_{1,e} \ge 1$ for all facets in $\Gamma_h^1 \cup \Gamma_1$ if $\epsilon = 1$, e.g., one may choose $\sigma_{1,e} = 2$.
2. $\sigma_{1,e} \ge \sigma_0 > 0$ for σ_0 large enough if $\epsilon = -1$.
3. $\sigma_{2,e} \ge 1$ for all facets in $\Gamma_h^2 \cup \Gamma_2$, e.g., one also may choose $\sigma_{2,e} = 2$.

Specially, the local lifting operator [9] is introduced and used to prove the K-ellipticity.

Lemma 4.2 *The local lifting operator r_e: $L^2(e) \to M_h^s$ is defined by*

$$\int_{\Omega_2} r_e(w) q_{s,h} d\mathbf{x} = -\int_e w\{q_{s,h}\} ds \quad \forall w \in L^2(e), \forall q_{s,h} \in M_h^s. \tag{43}$$

Then, for any $e \in \Gamma_h^2 \cup \Gamma_2$, the following inequality holds

$$\|r_e(w)\|_{0,\Omega_2} \le c h_e^{-1/2} \|w\|_{0,e}. \tag{44}$$

Proof By taking $q_{s,h} = r_e(w)$ in (43) and using the trace inequality (21), we have

$$\forall e \in \Gamma_h^2, \quad \|r_e(w)\|_{0,\Omega_2}^2 \le \frac{1}{2}\|w\|_{0,e}(\|r_e(w)^+\|_{0,e} + \|r_e(w)^-\|_{0,e}) \le c h_e^{-1/2}\|w\|_{0,e}\|r_e(w)\|_{0,\Omega_2},$$

and

$$\forall e \in \Gamma_2, \quad \|r_e(w)\|_{0,\Omega_2}^2 \le \|w\|_{0,e}\|r_e(w)\|_{0,e} \le c h_e^{-1/2}\|w\|_{0,e}\|r_e(w)\|_{0,\Omega_2}.$$

□

Note that $r_e(w)$ vanishes outside the union of the elements containing facet e.

Lemma 4.3 *There exists a constant $C_K > 0$, independent of mesh-sizes h_1 and h_2 such that*

$$a(\mathbf{u}_h, \mathbf{u}_h) > C_K \|\mathbf{u}_h\|_{\mathbf{X}}^2 \quad \forall \mathbf{u}_h \in \mathbf{Z}_h, \tag{45}$$

where $\mathbf{Z}_h$ is the kernel space $\mathbf{Z}_h = \{\mathbf{v}_h \in \mathbf{X}_h;\ \ b(\mathbf{v}_h, q_h) = 0 \quad \forall q_h \in M_h\}$.

Proof Note that, if $\sigma_{1,e}$ is sufficiently large for $\epsilon = -1$ and if $\sigma_{1,e} = 1$ for $\epsilon = 1$, by discrete Korn's inequality (23), we obtain the global coercivity in Stokes domain (see [2])

$$a_f(\mathbf{u}_{f,h}, \mathbf{u}_{f,h}) > C_f \|\mathbf{u}_{f,h}\|_{\mathbf{X}^f}^2 \quad \forall \mathbf{u}_{f,h} \in \mathbf{X}_h^f,$$

where C_f is independent of mesh-size h_1. The proof is a trivial and not presented in this paper. We need to prove the remaining K-ellipticity of $a_s(\cdot,\cdot)$. In light of the definition of the local lifting operator, for any $\mathbf{u}_{s,h} \in \mathbf{Z}_h$ and any $q_{s,h} \in M_h^s$, we have

$$\begin{aligned} b_s(\mathbf{u}_{s,h}, q_{s,h}) &= -\sum_{E \in \varepsilon_h^2} \int_E \nabla \cdot \mathbf{u}_{s,h} q_{s,h} d\mathbf{x} - \int_{\Omega_2} \sum_{e \in \Gamma_h^2 \cup \Gamma_2} r_e([\mathbf{u}_{s,h} \cdot \mathbf{n}_e]) q_{s,h} d\mathbf{x} \\ &= -\sum_{E \in \varepsilon_h^2} \int_E (\nabla \cdot \mathbf{u}_{s,h} + \sum_{e \subset \partial E \setminus \Gamma_{12}} r_e([\mathbf{u}_{s,h} \cdot \mathbf{n}_e])) q_{s,h} d\mathbf{x}. \end{aligned}$$

Due to $\mathbf{u}_{s,h} \in \mathbf{Z}_h$, it satisfies $b_s(\mathbf{u}_{s,h}, q_{s,h}) = 0$. Choosing

$$q_{s,h} = \nabla \cdot \mathbf{u}_{s,h} + \sum_{e \subset \partial E \backslash \Gamma_{12}} r_e([\mathbf{u}_{s,h} \cdot \mathbf{n}_e]) \quad \forall E \in \varepsilon_h^2,$$

yields

$$\nabla \cdot \mathbf{u}_{s,h} = - \sum_{e \subset \partial E \backslash \Gamma_{12}} r_e([\mathbf{u}_{s,h} \cdot \mathbf{n}_e]) \quad \forall E \in \varepsilon_h^2.$$

By (44), we obtain

$$\|\nabla \cdot \mathbf{u}_{s,h}\|_{0,\Omega_2} \leq c \sum_{e \in \Gamma_h^2 \cup \Gamma_2} h_e^{-1/2} \|[\mathbf{u}_{s,h} \cdot \mathbf{n}_e]\|_{0,e}.$$

Note that $\sigma_{2,e} > 1$, therefore

$$a_s(\mathbf{u}_{s,h}, \mathbf{u}_{s,h}) = |||\mathbf{u}_{s,h}|||_{0,\Omega_2}^2 + \sum_{e \in \Gamma_h^2 \cup \Gamma_2} \frac{\sigma_{2,e}}{|e|} \|[\mathbf{u}_{s,h} \cdot \mathbf{n}_e]\|_{0,e}^2 \geq C_s \|\mathbf{u}_{s,h}\|_{\mathbf{X}^s}^2 \quad \forall \mathbf{u}_{s,h} \in \mathbf{Z}_h,$$

where C_s is independent of mesh-size h_2. By combining the global coercivity of $a_f(\cdot, \cdot)$ and the K-ellipticity of $a_s(\cdot, \cdot)$, we finish the proof of the K-ellipticity by taking $C_K = \min(\frac{C_f}{2}, \frac{C_s}{2})$. □

Finally, a discrete inf-sup condition shall be derived.

Lemma 4.4 *There exists a positive constant β, independent of mesh-sizes h_1 and h_2 such that*

$$\inf_{q_h \in M_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{X}} \|q_h\|_M} \geq \beta. \tag{46}$$

Proof To this end, we consider the space $\tilde{\mathbf{X}}_h = \mathbf{X}_h^f \times \tilde{\mathbf{X}}_h^s$ and $\tilde{\mathbf{X}}_h \subset \mathbf{X}_h$, where the space $\tilde{\mathbf{X}}_h^s$ is introduced in (37). Define

$$\tilde{\mathbf{V}}_h = \left\{ (\mathbf{v}_f, \mathbf{v}_s) \in \tilde{\mathbf{X}}_h : \sum_{e \in \Gamma_{12}} \int_e \eta(\mathbf{v}_f - \mathbf{v}_s) \cdot \mathbf{n}_{12} = 0 \quad \forall \eta \in \Lambda_h \right\}.$$

If the following inf-sup condition holds

$$\inf_{q_h \in M_h} \sup_{\mathbf{v}_h \in \tilde{\mathbf{V}}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{X}} \|q_h\|_M} \geq \beta, \tag{47}$$

immediately, we finish the proof of this Lemma. Let $q_h \in M_h$ be fixed, then there exists a $\mathbf{v} \in (H^1(\Omega))^d$ such that

$$\nabla \cdot \mathbf{v} = -q_h \quad \text{in} \quad \Omega, \qquad \mathbf{v} = 0 \quad \text{on} \quad \partial\Omega,$$

satisfying

$$\|\mathbf{v}\|_{1,\Omega} \leq c \|q_h\|_{0,\Omega}.$$

Note that, by $\mathbf{v} \in (H^1(\Omega))^d$,

$$b(\mathbf{v}, q_h) = -\int_\Omega (\nabla \cdot \mathbf{v}) q_h d\mathbf{x} = \|q_h\|_{0,\Omega}^2,$$

which, combining with the given priori bound, yields

$$b(\mathbf{v}, q_h) \geq \frac{1}{c}\|q_h\|_{0,\Omega}\|\mathbf{v}\|_{1,\Omega}.$$

The idea of the proof of the inf-sup condition is that we construct a $\pi_h\mathbf{v} \in \tilde{\mathbf{V}}_h$ such that the inf-sup condition (47) holds. To this end, let $\pi_h : \mathbf{X}^f \times (\mathbf{X}^s \cap (H^1(\Omega_2))^2) \to \tilde{\mathbf{V}}_h$ satisfying

$$b(\pi_h\mathbf{v} - \mathbf{v}, q_h) = 0 \quad \forall q_h \in M_h, \quad \text{and} \quad \|\pi_h\mathbf{v}\|_{\mathbf{X}} \leq c\|\mathbf{v}\|_{1,\Omega}. \tag{48}$$

Let $\pi_h\mathbf{v} = (\pi_h^f\mathbf{v}, \pi_h^s\mathbf{v}) \in \mathbf{X}_h^f \times \tilde{\mathbf{X}}_h^s$. We take $\pi_h^f\mathbf{v} = \Pi_h^f\mathbf{v}$ where $\pi_h^f : \mathbf{X}^f \to \mathbf{X}_h^f$ is the quasi-local interpolation defined in (28)–(30). Clearly, by (30) and (36) we have

$$b_f(\pi_h^f\mathbf{v} - \mathbf{v}, q_h) = 0 \quad \forall q_h \in M_h, \quad \text{and} \quad \|\pi_h^1\mathbf{v}\|_{\mathbf{X}^f} \leq c\|\mathbf{v}\|_{1,\Omega_1}. \tag{49}$$

To define π_h^s, we consider the auxiliary problem

$$\begin{aligned} \nabla\cdot\nabla\phi = 0 \ \text{ in } \ \Omega_2, \\ \nabla\phi\cdot\mathbf{n} = 0 \ \text{ on } \ \Gamma_2, \\ \nabla\phi\cdot\mathbf{n}_{12} = (\pi_h^f\mathbf{v} - \mathbf{v})\cdot\mathbf{n}_{12} \ \text{ on } \ \Gamma_{12}. \end{aligned}$$

The auxiliary problem is well-defined, since

$$\int_{\Gamma_{12}} (\pi_h^f\mathbf{v} - \mathbf{v})\cdot\mathbf{n}_{12}ds = 0,$$

due to (34). Let $\mathbf{z} = \nabla\phi$. Note that, the piecewise smooth function $\pi_h^f\mathbf{v}\cdot\mathbf{n}_{12} \in H^\theta(\Gamma_{12})$ for any $0 < \theta < 1/2$. By elliptic regularity [20], we can obtain

$$\|\mathbf{z}\|_{\theta,\Omega_2} \leq c\|(\pi_h^f\mathbf{v} - \mathbf{v})\cdot\mathbf{n}_{12}\|_{\theta-1/2,\Gamma_{12}} \quad 0 \leq \theta \leq 1/2. \tag{50}$$

Let $\mathbf{w} = \mathbf{v} + \mathbf{z}$, the auxiliary problem implies $\nabla\cdot\mathbf{w} = \nabla\cdot\mathbf{v}$ in Ω_2 and $\mathbf{w}\cdot\mathbf{n}_{12} = \pi_h^f\mathbf{v}\cdot\mathbf{n}_{12}$ on Γ_{12}. We now define $\pi_h^s\mathbf{v} = \Pi_h^s\mathbf{w}$, where $\Pi_h^s : (H^\theta(\Omega_2))^d \cap H_0(div;\Omega_2) \to \tilde{\mathbf{X}}_h^s$ is the MFE interpolation defined in (38). Employing (38), it holds

$$\begin{aligned} b_s(\pi_h^s\mathbf{v}, q_{s,h}) &= b_s(\Pi_h^s\mathbf{w}, q_{s,h}) = b_s(\mathbf{w}, q_{s,h}) \\ &= -\int_{\Omega_2} \nabla\cdot\mathbf{w}q_{s,h}d\mathbf{x} = -\int_{\Omega_2} \nabla\cdot\mathbf{v}q_{s,h}d\mathbf{x} = b_s(\mathbf{v}, q_{s,h}) \quad \forall q_{s,h} \in M_h^s, \end{aligned}$$

due to the regularity $\mathbf{w} \in H_0(div;\Omega_2)$. Thus, $\pi_h\mathbf{v} = (\pi_h^f\mathbf{v}, \pi_h^s\mathbf{v})$ satisfies

$$b(\pi_h\mathbf{v}, q_h) = 0 \quad \forall q_h \in M_h.$$

We can check that $\pi_h\mathbf{v} \in \tilde{\mathbf{V}}_h$. Indeed, for every $e \in \Gamma_{12}$ and $\eta \in \Lambda_h$, by (39) and the fact that $\Lambda_h = \mathbf{X}_h^s\cdot\mathbf{n}_{12}$,

$$\int_e \pi_h^s\mathbf{v}\cdot\mathbf{n}_{12}\eta ds = \int_e \Pi_h^s\mathbf{w}\cdot\mathbf{n}_{12}\eta ds = \int_e \mathbf{w}\cdot\mathbf{n}_{12}\eta ds = \int_e \pi_h^f\mathbf{v}\cdot\mathbf{n}_{12}\eta ds.$$

Using (40), (42) and (50), we have

$$\begin{aligned} \|\pi_h^s\mathbf{v}\|_{\mathbf{X}^s} &= \|\Pi_h^s\mathbf{w}\|_{\mathbf{X}^s} \\ &\leq \|\Pi_h^s\mathbf{v}\|_{\mathbf{X}^s} + \|\Pi_h^s\mathbf{z}\|_{\mathbf{X}^s} \end{aligned}$$

$$\le c(\|\mathbf{v}\|_{1,\Omega} + \|\mathbf{z}\|_{\theta,\Omega_2})$$
$$\le c(\|\mathbf{v}\|_{1,\Omega} + \|(\pi_h^f \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{12}\|_{0,\Gamma_{12}}).$$

It remains to bound the last term. For every $e \in \Gamma_{12}$, and facet of $E \in \varepsilon_h^1$, using (19) and (33)

$$\|(\pi_h^f \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_{12}\|_{0,e} \le ch_E^{-1/2}(\|\pi_h^f \mathbf{v} - \mathbf{v}\|_{0,E} + h_K\|\pi_h^f \mathbf{v} - \mathbf{v}\|_{1,E}) \le ch_K^{1/2}|\mathbf{v}|_{1,\delta(E)}. \tag{51}$$

Therefore

$$\|\pi_h^s \mathbf{v}\|_{\mathbf{X}^s} \le c\|\mathbf{v}\|_{1,\Omega},$$

combining with (49), which proves (48). Now using (48) we have

$$\frac{1}{c}\|q_h\|_M \le \frac{b(\mathbf{v}, q_h)}{\|\mathbf{v}\|_{1,\Omega}} = \frac{b(\pi_h \mathbf{v}, q_h)}{\|\mathbf{v}\|_{1,\Omega}} \le \frac{b(\pi_h \mathbf{v}, q_h)}{\frac{1}{c}\|\pi_h \mathbf{v}\|_{\mathbf{X}}} \quad \forall q_h \in M_h,$$

and finish the proof of inf-sup condition (47). □

Now, in light of boundedness, K-ellipticity and the discrete inf-sup condition, we analyze the existence and uniqueness, and stability of solution of discrete scheme (11)–(12). However, the stability is a direct result of saddle point problem [10]. Thus, we only present the existence and uniqueness of solution.

Theorem 4.1 *The numerical scheme (11)–(12) has a unique solution.*

Proof Since the scheme (11)–(12) is square and finite-dimensional system, it is equivalent to the uniqueness of homogeneous system. The homogeneous system is obtain by setting $\mathbf{f} = 0$ and $g = 0$. Thus, we have

$$b(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in M_h,$$

which implies $\mathbf{u}_h \in \mathbf{Z}_h$. Taking $\mathbf{v}_h = \mathbf{u}_h$ and $q_h = p_h$ in (11) and (12), respectively, we can obtain $a(\mathbf{u}_h, \mathbf{u}_h) = 0$. The K-ellipticity (45), immediately, yields $\mathbf{u}_h = 0$. In light of the discrete inf-sup condition (47), we have $p_h = 0$ and finish the proof. □

5 A Priori Error Estimates

In this section, a priori error estimates under proper norms are obtained for both velocity field and pressure field. Before giving the error estimates, an approximation conclusion is obtained in the space $\mathbf{V}_h$.

Lemma 5.1 *Let $\mathbf{v} \in (H^1(\Omega))^d$ such that $\mathbf{v}|_{\Omega_1} \in (H^{k_1+1}(\Omega_1))^d$ and $\mathbf{v}|_{\Omega_2} \in (H^{k_2+1}(\Omega_2))^d$, there exists $\tilde{\mathbf{v}}_h \in \tilde{\mathbf{V}}_h \subset \mathbf{V}_h$ such that*

$$b(\mathbf{v} - \tilde{\mathbf{v}}_h, q_h) = 0 \quad \forall q_h \in M_h, \tag{52}$$

$$\forall e \in \Gamma_h^1 \cup \Gamma_1, \quad \int_e [\tilde{\mathbf{v}}_h] \cdot \mathbf{q}_h ds = 0 \quad \forall \mathbf{q}_h \in (\mathbb{P}_{k_1}(e))^d, \tag{53}$$

$$\|\mathbf{v} - \tilde{\mathbf{v}}_h\|_{\mathbf{X}} \le c(h_1^{k_1}|\mathbf{v}|_{k_1+1,\Omega_1} + h_2^{k_2+1}|\mathbf{v}|_{k_2+1,\Omega_2} + h_2^{k_2}|\nabla \cdot \mathbf{v}|_{k_2,\Omega_2}). \tag{54}$$

Proof Let $\tilde{\mathbf{v}}_h = \pi_h \mathbf{v}$, by the construction of $\pi_h \mathbf{v}$ in Lemma 4.4, we can easily obtain (52) and (53). To show approximation (54), we first note that (35) implies that

$$\|\mathbf{v} - \pi_h^f \mathbf{v}\|_{\mathbf{X}^f} \leq ch_1^{k_1} |\mathbf{v}|_{k_1+1,\Omega_1}. \tag{55}$$

Next,

$$\|\mathbf{v} - \pi_h^s \mathbf{v}\|_{\mathbf{X}^s} = \|\mathbf{v} - \Pi_h^s \mathbf{w}\|_{\mathbf{X}^s} \leq \|\mathbf{v} - \Pi_h^s \mathbf{v}\|_{\mathbf{X}^s} + \|\Pi_h^s(\mathbf{w} - \mathbf{v})\|_{\mathbf{X}^s}. \tag{56}$$

Using (40) and (41), there holds

$$\|\mathbf{v} - \Pi_h^s \mathbf{v}\|_{\mathbf{X}^s} \leq ch_2^{k_2+1} |\mathbf{v}|_{k_2+1,\Omega_2} + h_2^{k_2} |\nabla \cdot \mathbf{v}|_{k_2,\Omega_2}. \tag{57}$$

The last term in (56) can be bounded by using (42), (50), (19) and (33)

$$\begin{aligned} \|\Pi_h^s(\mathbf{w} - \mathbf{v})\|_{\mathbf{X}^s} &= \|\Pi_h^s \mathbf{z}\|_{\mathbf{X}^s} \leq \|\mathbf{z}\|_{\theta,\Omega_2} \\ &\leq c\|(\mathbf{v} - \pi_h^f \mathbf{v}) \cdot \mathbf{n}_{12}\|_{0,\Gamma_{12}} \leq ch_1^{k_1+1/2} |\mathbf{v}|_{k_1+1,\Omega_1}. \end{aligned} \tag{58}$$

Combing (55)–(58), we finish the proof. □

Theorem 5.1 *Let* $(\mathbf{u}, p)$ *be the solution of the coupled Stokes and Darcy problem (1)–(9). Assume that* $\mathbf{u}|_{\Omega_i} \in (H^{k_i+1}(\Omega_i))^d$, $p|_{\Omega_i} \in H^{k_i}(\Omega_i)$ *for* $i = 1, 2$. *Let* $(\mathbf{u}_h, p_h)$ *be the numerical solution of discrete scheme (11)–(12). Then, we have the following estimate*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{X}} \leq & ch_1^{k_1}(|\mathbf{u}|_{k_1+1,\Omega_1} + |p|_{k_1,\Omega_1}) \\ & + ch_2^{k_2}(|\mathbf{u}|_{k_2+1,\Omega_2} + |p|_{k_2,\Omega_2}) + ch_2^{k_2-1/2} h_1^{1/2} |p|_{k_2,\Omega_2}. \end{aligned}$$

Proof Subtracting (13)–(14) from (11)–(12), the error equations are

$$a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) + b(\mathbf{v}_h, p_h - p) - \sum_{e \in \Gamma_{12}} \int_e p_s(\mathbf{v}_{f,h} - \mathbf{v}_{s,h}) \cdot \mathbf{n}_{12} ds = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{59}$$

$$b(\mathbf{u}_h - \mathbf{u}, q_h) = 0 \quad \forall q_h \in M_h. \tag{60}$$

Let $\tilde{\mathbf{u}}_h$ be the interpolation of $\mathbf{u}$ defined in Lemma 5.1 and let $\tilde{p}_h$ be the L^2- projection of p, satisfying (24) and (25), we then introduce the following notions

$$\begin{aligned} \chi &= \mathbf{u}_h - \tilde{\mathbf{u}}_h, & \theta &= \mathbf{u} - \tilde{\mathbf{u}}_h, \\ \xi &= p_h - \tilde{p}_h, & \zeta &= p - \tilde{p}_h. \end{aligned}$$

Based on the above notions, the error Eqs. (59)–(60) can be rewritten as

$$a(\chi, \mathbf{v}_h) + b(\mathbf{v}_h, \xi) = a(\theta, \mathbf{v}_h) + b(\mathbf{v}_h, \zeta) + \sum_{e \in \Gamma_{12}} \int_e p_s(\mathbf{v}_{f,h} - \mathbf{v}_{s,h}) \cdot \mathbf{n}_{12} ds \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{61}$$

$$b(\chi, q_h) = b(\theta, q_h) \quad \forall q_h \in M_h. \tag{62}$$

By (52) in Lemma 5.1, we have $b(\theta, q_h) = 0$, thus

$$b(\chi, q_h) = 0 \quad \forall q_h \in M_h,$$

which implies $\chi \in \mathbf{Z}_h$. Choosing $\mathbf{v}_h = \chi$ and $q_h = \xi$ in (61) and (62) yields

$$a(\chi, \chi) + b(\chi, \xi) = a(\theta, \chi) + b(\chi, \zeta) + \sum_{e \in \Gamma_{12}} \int_e p_s(\chi_f - \chi_s) \cdot \mathbf{n}_{12} ds,$$

$$b(\chi, \xi) = 0.$$

equivalently,

$$a(\chi, \chi) = a(\theta, \chi) + b(\chi, \zeta) + \sum_{e \in \Gamma_{12}} \int_e p_s(\chi_f - \chi_s) \cdot \mathbf{n}_{12} ds. \tag{63}$$

By $\chi \in \mathbf{Z}_h$, the K-ellipticity (45) yields $a(\chi, \chi) \geq C_K \|\chi\|_{\mathbf{X}}^2$. We only to bound the right hand sides of (63). The first term can be bounded as follows:

$$\begin{aligned} a_f(\theta, \chi) &= 2\mu \sum_{E \in \varepsilon_h^1} \int_E \mathbf{D}(\theta) : \mathbf{D}(\chi) d\mathbf{x} \\ &\quad + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \frac{\sigma_{1,e}}{|e|} [\theta] \cdot [\chi] ds - 2\mu \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\mathbf{D}(\theta) \mathbf{n}_e\} \cdot [\chi] ds \\ &\quad + 2\mu\epsilon \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\mathbf{D}(\chi) \mathbf{n}_e\} \cdot [\theta] ds + \sum_{j=1}^{d-1} \sum_{e \in \Gamma_{12}} \int_e \frac{\mu}{G} \theta \cdot \tau_{12}^j \chi \cdot \tau_{12}^j ds \\ &= T_1 + T_2 + T_3 + T_4 + T_5. \end{aligned}$$

Using Cauchy–Schwarz inequality, Young inequality and the approximation property (33),

$$\begin{aligned} T_1 &\leq 2\mu \sum_{E \in \varepsilon_h^1} \|\nabla \theta\|_{0,E} \|\nabla \chi\|_{0,E} \leq \frac{C_K}{8} |||\nabla \chi|||_{0,\Omega_1}^2 + c|||\nabla \theta|||_{0,\Omega_1}^2 \\ &\leq \frac{C_K}{8} |||\nabla \chi|||_{0,\Omega_1}^2 + c h_1^{2k_1} |\mathbf{u}|_{k_1+1,\Omega_1}^2. \end{aligned}$$

By Cauchy–Schwarz inequality, Young inequality, trace inequality (19) and the approximation property (33)

$$\begin{aligned} T_2 &\leq \frac{C_K}{8} \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_{1,e}}{|e|} \|[\chi]\|_{0,e}^2 + c \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_{1,e}}{|e|} \|[\theta]\|_{0,e}^2 \\ &\leq \frac{C_K}{8} \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_{1,e}}{|e|} \|[\chi]\|_{0,e}^2 + c h_1^{2k_1} |\mathbf{u}|_{k_1+1,\Omega_1}^2. \end{aligned}$$

Let $L_h(\mathbf{u})$, defined in Ω_1, stand for the classic Lagrange interpolation of degree k_1, and note that $L_h(\mathbf{u})$ satisfies the optimal approximation, for any $E \in \varepsilon_h^1$

$$|L_h(\mathbf{u}) - \mathbf{u}|_{m,E} \leq c h_1^{s-m} |\mathbf{u}|_{s,E} \quad \forall 2 \leq s \leq k_1 + 1, \quad m = 0, 1, 2. \tag{64}$$

For a fixed $e \in \Gamma_h^1 \cup \Gamma_1$, using the Lagrange interpolation in T_3, we have

$$\begin{aligned} \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\mathbf{D}(\theta) \mathbf{n}_e\} \cdot [\chi] ds &= \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\mathbf{D}(\mathbf{u} - L_h(\mathbf{u})) \mathbf{n}_e\} \cdot [\chi] ds \\ &\quad + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\mathbf{D}(L_h(\mathbf{u}) - \tilde{\mathbf{u}}_h) \mathbf{n}_e\} \cdot [\chi] ds. \end{aligned}$$

The first part can be bounded by using trace inequality (20) and the approximation property of the Lagrange interpolation (64)

$$\begin{aligned}
&\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e \{\mathbf{D}(\mathbf{u}-L_h(\mathbf{u}))\mathbf{n}_e\}\cdot[\chi]ds\\
&\quad\le \sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_{1,e}^{1/2}}{|e|^{1/2}}\|[\chi]\|_{0,e}\frac{|e|^{1/2}}{\sigma_{1,e}^{1/2}}\|\{\mathbf{D}(\mathbf{u}-L_h(\mathbf{u}))\mathbf{n}_e\}\|_{0,e}\\
&\quad\le \frac{C_K}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_{1,e}}{|e|}\|[\chi]\|_{0,e}^2+c\frac{|e|}{\sigma_{1,e}}(h_e^{-1}|L_h(\mathbf{u})-\tilde{\mathbf{u}}_h|_{1,E_e^{12}}^2+h_e|L_h(\mathbf{u})-\tilde{\mathbf{u}}_h|_{2,E_e^{12}}^2)\\
&\quad\le \frac{C_K}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_{1,e}}{|e|}\|[\chi]\|_{0,e}^2+ch_1^{2k_1}|\mathbf{u}|_{k_1+1,\Omega_1}^2,
\end{aligned}$$

where E_e^{12} represents the union of E_e^1 and E_e^2 ($e=E_e^1\cap E_e^2$). Similarly, by the trace inequality (21), triangle inequality, and the approximation (33), we have

$$\begin{aligned}
&\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e \{\mathbf{D}(L_h(\mathbf{u})-\tilde{\mathbf{u}}_h)\mathbf{n}_e\}\cdot[\chi]ds\\
&\quad\le \frac{C_K}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_{1,e}}{|e|}\|[\chi]\|_{0,e}^2+c\sum_{e\in\Gamma_h^1\cup\Gamma_1}|L_h(\mathbf{u})-\tilde{\mathbf{u}}_h|_{1,E_e^{12}}^2\\
&\quad\le \frac{C_K}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_{1,e}}{|e|}\|[\chi]\|_{0,e}^2+ch_1^{2k_1}|\mathbf{u}|_{k_1+1,\Omega_1}^2.
\end{aligned}$$

Therefore,

$$T_3\le \frac{C_K}{4}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_{1,e}}{|e|}\|[\chi]\|_{0,e}^2+ch_1^{2k_1}|\mathbf{u}|_{k_1+1,\Omega_1}^2.$$

The fourth term vanishes due to the continuity of $\mathbf{u}$ and the property (53) of $\tilde{\mathbf{u}}_h$,

$$T_4=0.$$

The last term can be estimated by using the trace inequality (19),

$$\begin{aligned}
T_5&\le \frac{\mu}{G}\sum_{j=1}^{d-1}\sum_{e\in\Gamma_{12}}\|\theta\|_{0,e}\|\chi\cdot\tau_{12}^j\|_{0,e}\\
&\le \frac{\mu}{2G}\sum_{j=1}^{d-1}\sum_{e\in\Gamma_{12}}\|\chi\cdot\tau_{12}^j\|_{0,e}^2+C\sum_{e\in\Gamma_{12}}(h_e^{-1}\|\theta\|_{0,E}+h_e|\theta|_{1,E})\\
&\le \frac{\mu}{2G}\sum_{j=1}^{d-1}\sum_{e\in\Gamma_{12}}\|\chi\cdot\tau_{12}^j\|_{0,e}^2+ch_1^{2k_1+1}|\mathbf{u}|_{k_1+1,\Omega_1}^2.
\end{aligned}$$

Let us now estimate $a_s(\theta,\chi)$,

$$a_s(\theta,\chi)=\sum_{E\in\varepsilon_h^2}\int_E \mathbf{K}^{-1}\theta\cdot\chi d\mathbf{x}+\sum_{e\in\Gamma_h^2\cup\Gamma_2}\int_e\frac{\sigma_{2,e}}{|e|}[\theta\cdot\mathbf{n}_e][\chi\cdot\mathbf{n}_e]ds.$$

Using the Cauchy–Schwarz inequality, Young inequality and the approximation property (40) to the first part, we have

$$\sum_{E\in\varepsilon_h^2}\int_E \mathbf{K}^{-1}\theta\cdot\chi d\mathbf{x} \le \frac{C_K}{8}\|\chi\|_{0,\Omega_2}^2 + ch_2^{2k_2+2}|\mathbf{u}|_{k_2+1,\Omega_2}^2.$$

The second part is bounded by using trace inequality (19) and the approximation (40)

$$\begin{aligned}\sum_{e\in\Gamma_h^2\cup\Gamma_2}\int_e \frac{\sigma_{2,e}}{|e|}[\theta\cdot\mathbf{n}_e][\chi\cdot\mathbf{n}_e]ds &\le \frac{C_K}{8}\sum_{e\in\Gamma_h^2\cup\Gamma_2}\frac{\sigma_{2,e}}{|e|}\|[\chi\cdot\mathbf{n}_e]\|_{0,e}^2 + c\sum_{e\in\Gamma_h^2\cup\Gamma_2}\frac{\sigma_{2,e}}{|e|}\|[\theta\cdot\mathbf{n}_e]\|_{0,e}^2\\ &\le \frac{C_K}{8}\sum_{e\in\Gamma_h^2\cup\Gamma_2}\frac{\sigma_{2,e}}{|e|}\|[\chi\cdot\mathbf{n}_e]\|_{0,e}^2 + ch_2^{2k_2}|\mathbf{u}|_{k_2+1,\Omega_2}^2.\end{aligned}$$

Next, we estimate $b_f(\chi,\zeta)$, by the trace inequality (19), and properties (24) and (26),

$$\begin{aligned}b_f(\chi,\zeta) &= -\sum_{E\in\varepsilon_h^1}\int_E \zeta\nabla\cdot\chi d\mathbf{x} + \sum_{e\in\Gamma_1\cup\Gamma_h^1}\int_e [\chi\cdot\mathbf{n}_e]\{\zeta\}ds\\ &= \sum_{e\in\Gamma_1\cup\Gamma_h^1}\int_e [\chi\cdot\mathbf{n}_e]\{\zeta\}ds\\ &\le \frac{C_K}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_{1,e}}{|e|}\|[\chi]\|_{0,e}^2 + ch_1^{2k_1}|p|_{k_1,\Omega_1}^2.\end{aligned}$$

Similarly, by the trace inequality (19), and properties (25) and (27),

$$\begin{aligned}b_s(\chi,\zeta) &= -\sum_{E\in\varepsilon_h^2}\int_E \zeta\nabla\cdot\chi d\mathbf{x} + \sum_{e\in\Gamma_h^2\cup\Gamma_2}\int_e [\chi\cdot\mathbf{n}_e]\{\zeta\}ds\\ &= \sum_{e\in\Gamma_h^2\cup\Gamma_2}\int_e [\chi\cdot\mathbf{n}_e]\{\zeta\}ds\\ &\le \frac{C_K}{8}\sum_{e\in\Gamma_h^2\cup\Gamma_2}\frac{\sigma_{2,e}}{|e|}\|[\chi\cdot\mathbf{n}_e]\|_{0,e}^2 + ch_2^{2k_2}|p|_{k_2,\Omega_2}^2.\end{aligned}$$

It remains to estimate the last term in (63). Since χ belongs to $\mathbf{V}_h$, we obtain

$$\sum_{e\in\Gamma_{12}}\int_e p_s(\chi_f-\chi_s)\cdot\mathbf{n}_{12}ds = \sum_{e\in\Gamma_{12}}\int_e (p_s-\tilde{p}_h^s)(\chi_f-\chi_s)\cdot\mathbf{n}_{12}ds,$$

where $\tilde{p}_h^s\in\Lambda_h$ is the L^2- projection of p_s with respect to L^2 inner product on the interface. Thus, from the definition of the Lagrange multiplier space Λ_h, we have

$$\sum_{e\in\Gamma_{12}}\int_e (p_s-\tilde{p}_h^s)\chi_s\cdot\mathbf{n}_{12}ds = 0.$$

For any interface facet e and any piecewise vector-valued constant $\mathbf{c}_e$, there holds

$$\sum_{e\in\Gamma_{12}}\int_e p_s(\chi_f-\chi_s)\cdot\mathbf{n}_{12}ds = \sum_{e\in\Gamma_{12}}\int_e (p_s-\tilde{p}_h^s)\chi_f\cdot\mathbf{n}_{12}ds$$

$$= \sum_{e\in\Gamma_{12}} \int_e (p_s - \tilde{p}_h^s)(\chi_f - \mathbf{c}_e)\cdot \mathbf{n}_{12} ds = \sum_{e\in\Gamma_{12}} \int_e (p_s - \tilde{p}_h^s)(\chi_f - \mathbf{c}_e)\cdot \mathbf{n}_{12} ds.$$

Assume that each interface facet e is shared by the element $E_e^2 \in \varepsilon_h^2$ and parts of the elements $E_{e,i}^1 \in \varepsilon_h^1$, $i = 1, k_e$. Then, by the trace inequality (19) and approximation property of L^2-projection, we have (see [4])

$$\int_e (p_s - \tilde{p}_h^s)(\chi_f - \mathbf{c}_e)\cdot \mathbf{n}_{12} ds \le ch_2^{k_2-1/2}|p|_{k_2,E_e^2} \sum_{i=1}^{k_e}(h_1^{-1/2}\|\chi_f - \mathbf{c}_e\|_{0,E_{e,i}^1} + h_1^{1/2}\|\nabla\chi_f\|_{0,E_{e,i}^1}),$$

therefore

$$\begin{aligned}\sum_{e\in\Gamma_{12}} \int_e (p_s - \tilde{p}_h^s)(\chi_f - \mathbf{c}_e)\cdot \mathbf{n}_{12} ds &\le c\sum_{e\in\Gamma_{12}}\left(h_2^{k_2-1/2}|p|_{k_2,E_e^2}\sum_{i=1}^{k_e} h_1^{1/2}\|\nabla\chi_f\|_{0,E_{e,i}^1}\right)\\ &\le \frac{C_K}{8}|||\nabla\chi|||_{0,\Omega_1}^2 + ch_2^{2k_2-1}h_1|p|_{k_2,\Omega_2}^2.\end{aligned}$$

Indeed, we can estimate the interface term by using the discrete PoincarK inequality [2] and not introduce the piecewise vector-valued constant $\mathbf{c}_e$ if p has sufficient smoothness ($p|_{\Omega_2} \in H^{k_2+1}(\Omega_2)$). Then, based on the above estimates we obtain

$$\begin{aligned} a(\chi,\chi) \le &\frac{C_K}{4}|||\nabla\chi|||_{0,\Omega_1}^2 + \frac{C_K}{2}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_{1,e}}{|e|}\|[\chi]\|_{0,e}^2 + \frac{\mu}{2G}\sum_{j=1}^{d-1}\sum_{e\in\Gamma_{12}}\|\chi\cdot\tau_{12}^j\|_{0,e}^2\\ &+ \frac{C_K}{4}|||\chi|||_{0,\Omega_2}^2 + \frac{C_K}{4}\sum_{e\in\Gamma_h^2\cup\Gamma_2}\frac{\sigma_{2,e}}{|e|}\|[\chi\cdot\mathbf{n}_e]\|_{0,e}^2 + ch_1^{2k_1}|\mathbf{u}|_{k_1+1,\Omega_1}^2\\ &+ ch_2^{2k_2}|\mathbf{u}|_{k_2+1,\Omega_2}^2 + ch_1^{2k_1}|p|_{k_1,\Omega_1}^2 + ch_2^{2k_2}|p|_{k_2,\Omega_2}^2 + ch_2^{2k_2-1}h_1|p|_{k_2,\Omega_2}^2.\end{aligned}$$

Combing the K-ellipticity, we have

$$\begin{aligned}\|\chi\|_{\mathbf{X}}^2 \le\ & ch_1^{2k_1}|\mathbf{u}|_{k_1+1,\Omega_1}^2 + ch_2^{2k_2}|\mathbf{u}|_{k_2+1,\Omega_2}^2\\ &+ ch_1^{2k_1}|p|_{k_1,\Omega_1}^2 + ch_2^{2k_2}|p|_{k_2,\Omega_2}^2 + ch_2^{2k_2-1}h_1|p|_{k_2,\Omega_2}^2,\end{aligned}$$

which complete the proof by using (54) and the triangle inequality. □

Theorem 5.2 *Under the same assumptions and notions of Theorem 5.1, we obtain*

$$\begin{aligned}\|p_h - p\|_{0,\Omega} \le\ & ch_1^{k_1}(|\mathbf{u}|_{k_1+1,\Omega_1} + |p|_{k_1,\Omega_1})\\ &+ ch_2^{k_2}(|\mathbf{u}|_{k_2+1,\Omega_2} + |p|_{k_2,\Omega_2}) + ch_2^{k_2-1/2}h_1^{1/2}|p|_{k_2,\Omega_2}.\end{aligned}$$

Proof The error equation (59) can be written as

$$a(\mathbf{u}_h - u, \mathbf{v}_h) + b(\mathbf{v}_h, p_h - \tilde{p}_h) = b(\mathbf{v}_h, p - \tilde{p}_h) + \sum_{e\in\Gamma_{12}}\int_e p_s(\mathbf{v}_{f,h} - \mathbf{v}_{s,h})\cdot\mathbf{n}_{12} ds \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{65}$$

From the discrete inf-sup condition (46),

$$\|p_h - \tilde{p}_h\|_{0,\Omega} \le \frac{1}{\beta}\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{b(\mathbf{v}_h, p_h - \tilde{p}_h)}{\|\mathbf{v}_h\|_{\mathbf{X}}}. \tag{66}$$

For any $\mathbf{v}_h \in \mathbf{V}_h$, we assume that $p_h - \tilde{p}_h$ and $\mathbf{v}_h$ satisfy (66). From (65), it holds

$$b(\mathbf{v}_h, p_h - \tilde{p}_h) = -a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) + b(\mathbf{v}_h, p - \tilde{p}_h) + \sum_{e\in\Gamma_{12}} \int_e p_s(\mathbf{v}_{f,h} - \mathbf{v}_{s,h}) \cdot \mathbf{n}_{12} ds. \tag{67}$$

To bound the term $a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h)$ in (67),

$$\begin{aligned}
a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) &= 2\mu \sum_{E\in\varepsilon_h^1} \int_E \mathbf{D}(\mathbf{u}_h - \mathbf{u}) : \mathbf{D}(\mathbf{v}_h) d\mathbf{x} + \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \frac{\sigma_{1,e}}{|e|} [\mathbf{u}_h - \mathbf{u}] \cdot [\mathbf{v}_h] ds \\
&\quad - 2\mu \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{\mathbf{D}(\mathbf{u}_h - \mathbf{u})\mathbf{n}_e\} \cdot [\mathbf{v}_h] ds + 2\mu\epsilon \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{\mathbf{D}(\mathbf{v}_h)\mathbf{n}_e\} \cdot [\mathbf{u}_h - \mathbf{u}] ds \\
&\quad + \sum_{j=1}^{d-1} \sum_{e\in\Gamma_{12}} \int_e \frac{\mu}{G} (\mathbf{u}_h - \mathbf{u}) \cdot \tau_{12}^j \mathbf{v}_h \cdot \tau_{12}^j ds + \sum_{E\in\varepsilon_h^2} \int_E (\mathbf{u}_h - \mathbf{u}) \cdot \mathbf{v}_h d\mathbf{x} \\
&\quad + \sum_{e\in\Gamma_h^2\cup\Gamma_2} \int_e \frac{\sigma_{2,e}}{|e|} [(\mathbf{u}_h - \mathbf{u}) \cdot \mathbf{n}_e][\mathbf{v}_h \cdot \mathbf{n}_e] ds \\
&= Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 + Q_7.
\end{aligned}$$

We now estimate each Q_i terms for $i = 1, 7$. The terms Q_1, Q_2, Q_5, Q_6 and Q_7 are bounded by Cauchy-Schwarz inequality,

$$Q_1 + Q_2 + Q_5 + Q_6 + Q_7 \le c\|\mathbf{v}_h\|_{\mathbf{X}} \|\mathbf{u}_h - \mathbf{u}\|_{\mathbf{X}}.$$

Q_3 is estimated by utilizing the Lagrange interpolation

$$\begin{aligned}
Q_3 &\le c \sum_{e\in\Gamma_h^1\cup\Gamma_1} \left(\frac{|e|}{\sigma_{1,e}}\right)^{1/2} \|\nabla(\mathbf{u}_h - \mathbf{u})\|_{0,e} \left(\frac{\sigma_{1,e}}{|e|}\right)^{1/2} \|[\mathbf{v}_h]\|_{0,e} \\
&\le c\|\mathbf{v}_h\|_{\mathbf{X}} \left(\sum_{e\in\Gamma_h^1\cup\Gamma_1} (h_1 \|\nabla(\mathbf{u}_h - \tilde{\mathbf{u}}_h)\|_{0,e}^2 + h_1 \|\nabla(\mathbf{u} - \tilde{\mathbf{u}}_h)\|_{0,e}^2) \right)^{1/2} \\
&\le c\|\mathbf{v}_h\|_{\mathbf{X}} (\|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{\mathbf{X}}^2 + ch_1^{2k_1} |\mathbf{u}|_{k_1+1,\Omega_1}^2)^{1/2}.
\end{aligned}$$

By using the trace inequality (21), there holds

$$\begin{aligned}
Q_4 &\le c \sum_{e\in\Gamma_h^1\cup\Gamma_1} \|\{D(\mathbf{v}_h)\mathbf{n}_e\}\|_{0,e} \|[\mathbf{u}_h - \mathbf{u}]\|_{0,e} \\
&\le c \sum_{e\in\Gamma_h^1\cup\Gamma_1} h_1^{-1/2} \|\nabla \mathbf{v}_h\|_{0,E_e^{12}} \left(\frac{\sigma_{1,e}}{|e|}\right)^{1/2-1/2} \|[\mathbf{u}_h - \mathbf{u}]\|_{0,e} \\
&\le c\|\mathbf{v}_h\|_{\mathbf{X}} \|\mathbf{u}_h - \mathbf{u}\|_{\mathbf{X}}.
\end{aligned}$$

For the term $b(\mathbf{v}_h, p-\tilde{p}_h)$ in (67), by the properties (24) and (26), we have

$$
\begin{aligned}
b_f(\mathbf{v}_h, p-\tilde{p}_h) &= \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e \{p-\tilde{p}_h\}[\mathbf{v}_h\cdot\mathbf{n}_e]ds\\
&\le \sum_{e\in\Gamma_h^1\cup\Gamma_1}\left(\frac{\sigma_{1,e}}{|e|}\right)^{1/2}\|[\mathbf{v}_h]\|_{0,e}\left(\frac{|e|}{\sigma_{1,e}}\right)^{1/2}\|\{p-\tilde{p}_h\}\|_{0,e}\\
&\le ch_1^{k_1}|p|_{k_1,\Omega_1}\|\mathbf{v}_h\|_{\mathbf{X}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
b_s(\mathbf{v}_h, p-\tilde{p}_h) &= \sum_{e\in\Gamma_h^2\cup\Gamma_2}\int_e \{p-\tilde{p}_h\}[\mathbf{v}_h\cdot\mathbf{n}_e]ds\\
&\le \sum_{e\in\Gamma_h^2\cup\Gamma_2}\left(\frac{\sigma_{2,e}}{|e|}\right)^{1/2}\|[\mathbf{v}_h\cdot\mathbf{n}_e]\|_{0,e}\left(\frac{|e|}{\sigma_{2,e}}\right)^{1/2}\|\{p-\tilde{p}_h\}\|_{0,e}\\
&\le ch_2^{k_2}|p|_{k_2,\Omega_2}\|\mathbf{v}_h\|_{\mathbf{X}}.
\end{aligned}
$$

Thus,

$$
b(\mathbf{v}_h, p-\tilde{p}_h) \le c\|\mathbf{v}_h\|_{\mathbf{X}}(h_1^{k_1}|p|_{k_1,\Omega_1} + h_2^{k_2}|p|_{k_2,\Omega_2}).
$$

Similar to the proof in Theorem 5.1, the last interface integral term in (67) is bounded by

$$
\begin{aligned}
\sum_{e\in\Gamma_{12}}\int_e p_s(\mathbf{v}_{f,h}-\mathbf{v}_{s,h})\cdot\mathbf{n}_{12}ds &= \sum_{e\in\Gamma_{12}}\int_e (p_s-\tilde{p}_h^s)\mathbf{v}_{f,h}\cdot\mathbf{n}_{12}ds\\
&\le ch_2^{k_2-1/2}h_1^{1/2}|p|_{k_2,\Omega_2}\|\mathbf{v}_h\|_{\mathbf{X}}.
\end{aligned}
$$

Combing the above bounds and the discrete inf-sup condition (66), we have

$$
\begin{aligned}
\|p_h-\tilde{p}\|_{0,\Omega} \le c(&\|\mathbf{u}_h-\mathbf{u}\|_{\mathbf{X}} + \|\mathbf{u}_h-\tilde{\mathbf{u}}_h\|_{\mathbf{X}} + h_1^{k_1}|\mathbf{u}|_{k_1+1,\Omega_1}\\
&+ h_1^{k_1}|p|_{k_1,\Omega_1} + h_2^{k_2}|p|_{k_2,\Omega_2} + h_2^{k_2-1/2}h_1^{1/2}|p|_{k_2,\Omega_2}).
\end{aligned}
$$

In light of Theorem 5.1 and triangle inequality, we complete the proof. □

6 Implementation and Numerical Experiments

6.1 Implementation

In this section, an equivalent discrete scheme (see [19]) is given because it is hard to directly construct the space of function $\mathbf{V}_h$. Defining the following bilinear forms

$$
\Lambda_f(\eta, \mathbf{v}_{f,h}) = \sum_{e\in\Gamma_{12}}\int_e \eta\mathbf{v}_{s,h}\cdot\mathbf{n}_{12}ds \quad \forall\eta\in\Lambda_h, \forall\mathbf{v}_{f,h}\in\mathbf{X}_h^f,
$$

$$
\Lambda_s(\eta, \mathbf{v}_{s,h}) = \sum_{e\in\Gamma_{12}}\int_e \eta\mathbf{v}_{s,h}\cdot\mathbf{n}_{12}ds \quad \forall\eta\in\Lambda_h, \forall\mathbf{v}_{s,h}\in\mathbf{X}_h^s.
$$

The numerical scheme (11)-(12) can be rewritten as: Find $(\mathbf{u}_h, p_h, \lambda_h) \in \mathbf{X}_h \times M_h \times \Lambda_h$ such that $\mathbf{u}_{f,h} = \mathbf{u}_h|_{\Omega_1}$, $\mathbf{u}_{s,h} = \mathbf{u}_h|_{\Omega_2}$ and $p_{f,h} = p_h|_{\Omega_1}$, $p_{s,h} = p_h|_{\Omega_2}$ satisfy

$$a_f(\mathbf{u}_{f,h}, \mathbf{v}_{f,h}) + b_f(\mathbf{v}_{f,h}, p_{f,h}) + \Lambda_f(\lambda_h, \mathbf{v}_{f,h}) = (\mathbf{f}, \mathbf{v}_{f,h})_{\Omega_1} \qquad \forall \mathbf{v}_{f,h} \in \mathbf{X}_h^f, \tag{68}$$

$$-b_f(\mathbf{u}_{f,h}, q_{f,h}) = 0 \qquad \forall q_{f,h} \in M_h^f, \tag{69}$$

$$a_s(\mathbf{u}_{s,h}, \mathbf{v}_{s,h}) + b_f(\mathbf{v}_{s,h}, p_{s,h}) - \Lambda_s(\lambda_h, \mathbf{v}_{s,h}) = 0 \qquad \forall \mathbf{v}_{s,h} \in \mathbf{X}_h^s, \tag{70}$$

$$-b_s(\mathbf{u}_{s,h}, q_{s,h}) = (g, q_{s,h})_{\Omega_2} \qquad \forall q_{s,h} \in M_h^s, \tag{71}$$

$$\Lambda_f(\eta_h, \mathbf{u}_{f,h}) - \Lambda_s(\eta_h, \mathbf{u}_{s,h}) = 0 \qquad \forall \eta_h \in \Lambda_h, \tag{72}$$

$$\int_{\Omega_1} p_{f,h} d\mathbf{x} + \int_{\Omega_2} p_{s,h} d\mathbf{x} = 0. \tag{73}$$

We can easily verify that the numerical schemes (11)-(12) and (68)-(73) are equivalent. In the following numerical examples, the discrete scheme (68)-(73) is applied.

For simplicity, we show how to choose a suitable basis for $\mathbf{X}_h^s$ such that the Assumption 2.1 always holds true for $d = 2$. In general, for any element E and for $i + j \leq k_2$, we use the following basis

$$\begin{pmatrix} span\{1 \ x \ y \ xy \ x^2 \ y^2 \ \ldots \ x^i y^j\} \\ span\{1 \ x \ y \ xy \ x^2 \ y^2 \ \ldots \ x^i y^j\} \end{pmatrix}.$$

However, $\mathbf{X}_h^s \cdot \mathbf{n}_{12}$ doesn't contain constant if the interface edge $e \subset \{(x, y) : y = x + \text{constant}\}$, thus, it doesn't belong to discontinuous piecewise polynomials of degree k_2. To avoid this problem, let constants $a > 0$ and $b > 0$, and $a \neq b$, then the Assumption 2.1 always holds true by taking the following basis

$$\begin{pmatrix} span\{a \ x \ y \ xy \ x^2 \ y^2 \ \ldots \ x^i y^j\} \\ span\{b \ x \ y \ xy \ x^2 \ y^2 \ \ldots \ x^i y^j\} \end{pmatrix}.$$

This conclusion is trivial, thus we don't present the concrete proof here. Similarly, let $i + j + m \leq k_2$ and let constants $a > 0$, $b > 0$, $c > 0$ and $a \neq b$, $a \neq c$, $b \neq c$, then the Assumption 2.1 always holds true for $d = 3$ by taking the following basis

$$\begin{pmatrix} span\{a \ x \ y \ z \ xy \ xz \ yz \ \ldots \ x^i y^j z^m\} \\ span\{b \ x \ y \ z \ xy \ xz \ yz \ \ldots \ x^i y^j z^m\} \\ span\{c \ x \ y \ z \ xy \ xz \ yz \ \ldots \ x^i y^j z^m\} \end{pmatrix}.$$

6.2 Numerical Experiments

In this section, under uniformly matching mesh, the convergence analysis of the coupled system shall be reported by some numerical tests. In these numerical examples, the domain $\Omega = [0, 1] \times [0, 1]$, Stokes domain $\Omega_1 = [0, 1] \times [0.5, 1]$, Darcy domain $\Omega_2 = [0, 1] \times [0, 0.5]$, the interface $\Gamma_{12} = [0, 1] \times \{0.5\}$. In addition, we consider the stability constants $\sigma_{1,e} = 30\mu$, $\sigma_{2,e} = 1$ and $\epsilon = \pm 1$.

6.2.1 Rates of Convergence

In this part, some tests are given to verify the rates of convergence. Let the permeability tensor $\mathbf{K} = \tilde{k}\mathbf{I}$, we consider the coupled system with the following exact solution [1]

$$u_{1,f} = -sin(\pi x)exp(y/2)/(2\pi^2), \quad u_{2,f} = cos(\pi x)exp(y/2)/\pi,$$

Table 1 The convergence rates under $k_1 = k_2 = 1$, $\mu = \tilde{k} = 1$ and $\epsilon = -1$

$h_1 = h_2$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^2(\Omega_1)}$	1.6245e-03	4.1008e-04	1.0351e-04	2.5944e-05	6.4860e-06
Rate	–	1.986	1.986	1.996	2.000
$\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{\mathbf{X}^f}$	8.4094e-02	4.1871e-02	2.0896e-02	1.0442e-02	5.2204e-03
Rate	–	1.006	1.003	1.001	1.000
$\|p_f - p_{f,h}\|_{M^f}$	1.2953e-01	7.0904e-02	3.7000e-02	1.8874e-02	9.5285e-03
Rate	–	0.869	0.938	0.971	0.986
$\|\mathbf{u}_s - \mathbf{u}_{s,h}\|_{L^2(\Omega_2)}$	1.1541e-02	2.9438e-03	7.4116e-04	1.8583e-04	4.6517e-05
Rate	–	1.971	1.990	1.996	1.998
$\|\mathbf{u}_s - \mathbf{u}_{s,h}\|_{\mathbf{X}^s}$	3.2648e-01	1.6340e-01	8.1717e-02	4.0861e-02	2.0431e-02
Rate	–	0.998	0.999	0.999	1.000
$\|p_s - p_{s,h}\|_{M^s}$	3.3935e-02	1.6985e-02	8.4948e-03	4.2477e-03	2.1239e-03
Rate	–	0.998	0.999	0.999	1.000

Table 2 The convergence rates under $k_1 = k_2 = 1$, $\mu = \tilde{k} = 1$ and $\epsilon = 1$

$h_1 = h_2$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^2(\Omega_1)}$	1.6239e-03	4.0591e-04	1.0183e-04	2.5475e-05	6.3664e-06
Rate	–	2.000	1.995	1.999	2.001
$\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{\mathbf{X}^f}$	8.4114e-02	4.1873e-02	2.0897e-02	1.0442e-02	5.2205e-03
Rate	–	1.006	1.003	1.001	1.000
$\|p_f - p_{f,h}\|_{M^f}$	1.3053e-01	7.1144e-02	3.7049e-02	1.8885e-02	9.5308e-03
Rate	–	0.875	0.941	0.972	0.986
$\|\mathbf{u}_s - \mathbf{u}_{s,h}\|_{L^2(\Omega_2)}$	1.1533e-02	2.9412e-03	7.4036e-04	1.8560e-04	4.6456e-05
Rate	–	1.971	1.990	1.996	1.998
$\|\mathbf{u}_s - \mathbf{u}_{s,h}\|_{\mathbf{X}^s}$	3.2648e-01	1.6340e-01	8.1717e-02	4.0861e-02	2.0431e-02
Rate	–	0.998	0.999	0.999	1.000
$\|p_s - p_{s,h}\|_{M^s}$	3.3935e-02	1.6985e-02	8.4948e-03	4.2477e-03	2.1239e-03
Rate	–	0.998	0.999	0.999	1.000

$$
\begin{aligned}
p_f &= \frac{\tilde{k}\mu - 2}{\tilde{k}\pi} cos(\pi x) exp(y/2),\\
u_{1,s} &= -2sin(\pi x)exp(y/2), \qquad u_{2,s} = cos(\pi x)exp(y/2)/\pi,\\
p_s &= -\frac{2}{\tilde{k}\pi} cos(\pi x)exp(y/2),
\end{aligned}
$$

with $G = 2/(1+4\pi^2)$. Then, using the exact solution, the source terms $\mathbf{f}$ and g are determined by the coupled Stokes-Darcy system (1) and (4), respectively, and the boundary conditions are obtained by restricting the corresponding true solution to boundary $\partial\Omega$. To fully verify our analysis, we consider the coupled system with different μ and $\tilde{k}$ in the following tests.

Let $\mu = 1$ and $\tilde{k} = 1$, we report the numerical results given in Tables 1, 2, 3, 4 for different ϵ and finite element spaces, which are optimal and predicted by the analysis. To adequately

Table 3 The convergence rates under $k_1 = k_2 = 2$, $\mu = \tilde{k} = 1$ and $\epsilon = -1$

$h_1 = h_2$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^2(\Omega_1)}$	4.9203e-04	6.3048e-05	8.0006e-06	1.0088e-06	1.2668e-07
Rate	–	2.964	2.978	2.987	2.993
$\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{\mathbf{X}^f}$	1.6032e-02	4.0749e-03	1.0265e-03	2.5756e-04	6.4507e-05
Rate	–	1.976	1.989	1.995	1.997
$\|p_f - p_{f,h}\|_{M^f}$	9.7501e-03	2.2329e-03	5.1498e-04	1.2210e-04	2.9611e-05
Rate	–	2.126	2.116	2.076	2.044
$\|\mathbf{u}_s - \mathbf{u}_{s,h}\|_{L(\Omega_2)}$	3.6081e-03	4.8556e-04	6.2704e-05	7.9554e-06	1.1052e-06
Rate	–	2.894	2.953	2.979	2.848
$\|\mathbf{u}_s - \mathbf{u}_{s,h}\|_{\mathbf{X}^s}$	6.5739e-02	1.6737e-02	4.2161e-03	1.0576e-03	2.6480e-04
Rate	–	1.974	1.989	1.995	1.998
$\|p_s - p_{s,h}\|_{M^s}$	6.5463e-03	1.6453e-03	4.1188e-04	1.0301e-04	2.5754e-05
Rate	–	1.992	1.998	1.999	2.000

Table 4 The convergence rates under $k_1 = k_2 = 2$, $\mu = \tilde{k} = 1$ and $\epsilon = 1$

$h_1 = h_2$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^2(\Omega_1)}$	4.7673e-04	6.2237e-05	7.9996e-06	1.0463e-06	1.4900e-07
Rate	–	2.937	2.960	2.935	2.812
$\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{\mathbf{X}^f}$	1.5778e-02	4.0073e-03	1.0089e-03	2.5307e-04	6.3373e-05
Rate	–	1.977	1.990	1.995	1.998
$\|p_f - p_{f,h}\|_{M^f}$	8.9027e-03	2.0009e-03	4.5569e-04	1.0734e-04	2.5950e-05
Rate	–	2.154	2.135	2.086	2.048
$\|\mathbf{u}_s - \mathbf{u}_{s,h}\|_{L(\Omega_2)}$	3.6132e-03	4.8574e-04	6.2714e-05	7.9575e-06	1.1116e-06
Rate	–	2.895	2.953	2.978	2.840
$\|\mathbf{u}_s - \mathbf{u}_{s,h}\|_{\mathbf{X}^s}$	6.5740e-02	1.6737e-02	4.2161e-03	1.0576e-03	2.6480e-04
Rate	–	1.974	1.989	1.995	1.998
$\|p_s - p_{s,h}\|_{M^s}$	6.5463e-03	1.6453e-03	4.1188e-04	1.0301e-04	2.5754e-05
Rate	–	1.992	1.998	1.999	2.000

Table 5 The convergence rates under $k_1 = k_2 = 1$, $\mu = 10^{-3}$, $\tilde{k} = 1$ and $\epsilon = -1$

$h_1 = h_2$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^2(\Omega_1)}$	1.1216e-01	2.8021e-02	7.0724e-03	1.7824e-03	4.4776e-04
Rate	–	2.001	1.986	1.988	1.993
$\|p_f - p_{f,h}\|_{M^f}$	4.4457e-02	2.2164e-02	1.1062e-02	5.5249e-03	2.7609e-03
Rate	–	1.004	1.003	1.002	1.001
$\|\mathbf{u}_s - \mathbf{u}_{s,h}\|_{L^2(\Omega_2)}$	2.1723e-02	4.6953e-03	1.0542e-03	2.4882e-04	6.0536e-05
Rate	–	2.210	2.155	2.083	2.039
$\|p_s - p_{s,h}\|_{M^s}$	3.3944e-02	1.6986e-02	8.4949e-03	4.2477e-03	2.1239e-03
Rate	–	0.998	0.997	0.999	1.000

Table 6 The convergence rates under $k_1 = k_2 = 1$, $\mu = 1$, $\tilde{k} = 10^3$ and $\epsilon = -1$

$h_1 = h_2$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\Vert\mathbf{u}_f - \mathbf{u}_{f,h}\Vert_{L^2(\Omega_1)}$	1.6367e-03	4.1382e-04	1.0463e-04	2.6244e-05	6.5623e-06
Rate	–	1.984	1.984	1.995	2.000
$\Vert p_f - p_{f,h}\Vert_{M^f}$	1.3493e-01	7.3963e-02	3.8597e-02	1.9688e-02	9.9388e-03
Rate	–	0.867	0.938	0.971	0.986
$\Vert\mathbf{u}_s - \mathbf{u}_{s,h}\Vert_{L^2(\Omega_2)}$	8.1755e-03	2.0626e-03	5.1700e-04	1.2937e-04	3.3390e-05
Rate	–	1.987	1.996	1.999	1.954
$\Vert p_s - p_{s,h}\Vert_{M^s}$	3.3924e-05	1.6984e-05	8.4946e-06	4.2476e-06	2.1239e-06
Rate	–	0.998	0.995	0.999	1.000

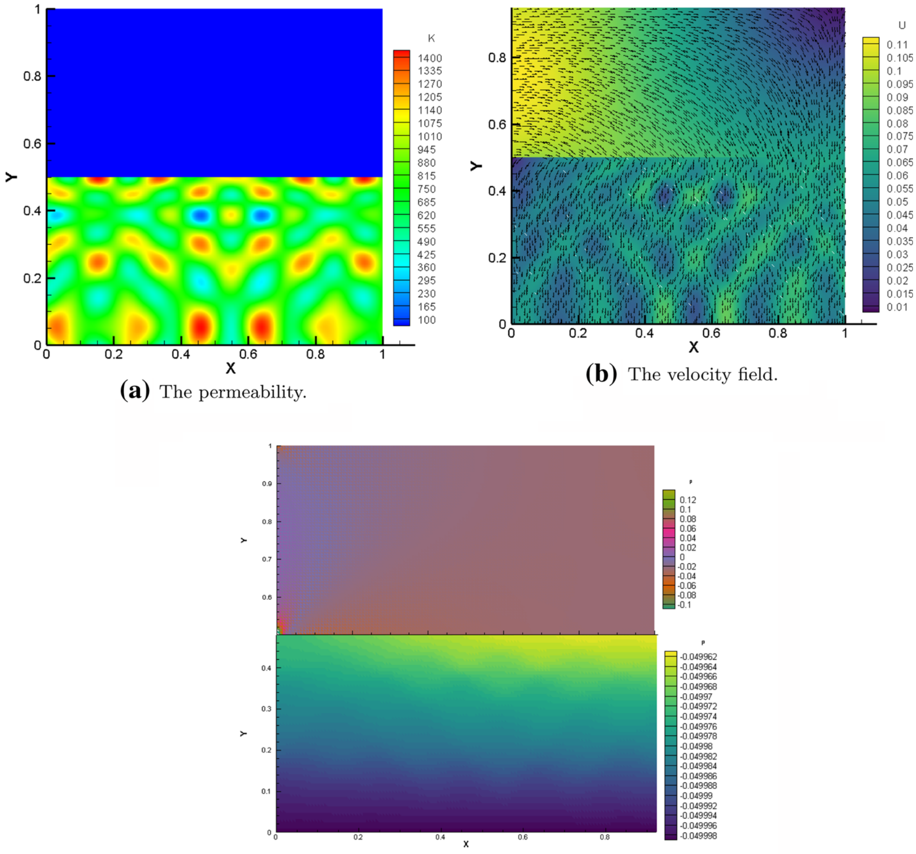
Table 7 The convergence rates under $k_1 = k_2 = 1$, $\mu = 10^{-6}$, $\tilde{k} = 10^3$ and $\epsilon = -1$

$h_1 = h_2$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\Vert\mathbf{u}_f - \mathbf{u}_{f,h}\Vert_{L^2(\Omega_1)}$	1.3558e-01	3.0981e-02	7.4113e-03	1.8150e-03	4.4933e-04
Rate	–	2.130	2.064	2.030	2.014
$\Vert p_f - p_{f,h}\Vert_{M^f}$	4.4156e-05	2.2086e-05	1.1041e-05	5.5196e-06	2.7596e-06
Rate	–	0.995	1.000	1.000	1.000
$\Vert\mathbf{u}_s - \mathbf{u}_{s,h}\Vert_{L^2(\Omega_2)}$	1.9495e-02	4.1853e-03	9.3404e-04	2.1850e-04	5.3370e-05
Rate	–	2.220	2.164	2.096	2.034
$\Vert p_s - p_{s,h}\Vert_{M^s}$	3.3939e-05	1.6986e-05	8.4948e-06	4.2477e-06	2.1239e-06
Rate	–	0.998	0.999	0.999	1.000

Table 8 The convergence rates under $k_1 = k_2 = 1$, $\mu = 10^{-3}$, $\tilde{k} = 10^{-3}$ and $\epsilon = -1$

$h_1 = h_2$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\Vert\mathbf{u}_f - \mathbf{u}_{f,h}\Vert_{L^2(\Omega_1)}$	1.3837e+02	3.2181e+01	7.7913e+00	1.9209e+00	4.7704e-01
Rate	–	2.104	2.046	2.020	2.010
$\Vert p_f - p_{f,h}\Vert_{M^f}$	7.3498e+01	4.0756e+01	1.8081e+01	6.9980e+00	2.9806e+00
Rate	–	0.850	1.172	1.369	1.231
$\Vert\mathbf{u}_s - \mathbf{u}_{s,h}\Vert_{L^2(\Omega_2)}$	8.9480e-01	5.5280e-01	2.3239e-01	7.5284e-02	2.0777e-02
Rate	–	0.694	1.250	1.626	1.857
$\Vert p_s - p_{s,h}\Vert_{M^s}$	4.3590e+01	2.4334e+01	1.1530e+01	4.9014e+00	2.2213e+00
Rate	–	0.841	1.087	1.234	1.142

verify our analysis, let $k_1 = k_2 = 1$ and $\epsilon = -1$, some numerical results with different μ and $\tilde{k}$ are presented in Tables 5, 6, 7, 8. Note that the exact solution pressure changes with different viscosity μ and permeability $\tilde{k}$, the errors of the pressure and velocity will also change since the pressure depends on the permeability $\tilde{k}$ and the velocity error is related to the pressure (see Theorem 5.1). Particularly, it is obvious when $\tilde{k}$ is relatively small and pressure is relatively large (see Table 8). However, these numerical results with different μ and $\tilde{k}$ are consistent with our convergence analysis. From Tables 1, 2, 3, 4,5, 6, 7, 8, we can



(a) The permeability.

(b) The velocity field.

(c) The pressure field.

Fig. 1 The numerical result of Sect. 6.2.2 including the permeability, velocity field and pressure field

conclude that the numerical results support the theoretical analysis derived in Theorems 5.1 and 5.2. In addition, the L^2 error of velocity in both Stokes and Darcy regions are optimal, even though the L^2 optimal convergence can not be proved in this paper.

6.2.2 Coupled Surface and Subsurface Flow

In this part, we consider an example proposed in [1, Example 6.2]. This example is representative of surface flow coupled to subsurface flow. Let the boundary of Darcy region be partitioned as $\Gamma_2 = \Gamma_2^a \cup \Gamma_2^b$, where $\Gamma_2^a = \{x = 0 \; or \; x = 1\}$ and $\Gamma_2^b = \{y = 0\}$. Similarly, let the boundary of Stokes region $\Gamma_1 = \Gamma_1^a \cup \Gamma_1^b \cup \Gamma_1^c$, where $\Gamma_1^a = \{x = 0\}$, $\Gamma_1^b = \{x = 1\}$ and $\Gamma_1^c = \{y = 1\}$. Then, we consider the following boundary conditions:

$$
\begin{aligned}
\mathbf{u}_f &= (y(1.5 - y)/5, 0), && \text{on } \Gamma_1^a, \\
\mathbf{T}(\mathbf{u}_f, p_f) &= 0, && \text{on } \Gamma_1^b, \\
\mathbf{u}_f \cdot \mathbf{n} &= 0, \quad \text{and } \mathbf{T}(\mathbf{u}_f, p_f)^t = 0, && \text{on } \Gamma_1^c, \\
\mathbf{u}_s \cdot \mathbf{n} &= 0, && \text{on } \Gamma_2^a,
\end{aligned}
$$

$$p_s = -0.05, \qquad \text{on } \Gamma_2^b,$$

where $\mathbf{T}(\mathbf{u}_f, p_f)^t$ stands for tangential stress (see [1, Example 6.2] and [7, Example 7.2]). Let $\epsilon = -1$, $k_1 = k_2 = 1$, $\mu = 0.1$, $5G = \mathbf{K}^{-1/2}$, $\mathbf{f} = 0$, $g = 0$ and the permeability

$$\tilde{k} = 700(1 + 0.5(\sin(10\pi x)\cos(20\pi y^2) + \cos^2(6.4\pi x)\sin(9.2\pi y))) + 100.$$

Based on these choices we numerically solve this coupled system on a mesh with $h_1 = h_2 = 1/128$.

The numerical results are given in Fig. 1, which shows the permeability field, velocity field and pressure field. As shown in Fig. 1b, the fluid flow from inlet into interface and then Darcy region, which is similar with the one presented in [1, Example 6.2] and [7, Example 7.2]. The tangential velocity of flow is discontinuous along the interface and the flow field has relatively small velocity at low permeability in the Darcy region Ω_2. The pressure field given in Fig. 1c is highest at the entrance (around the inlet Γ_1^a) and discontinuous across the interface. In summary, the proposed scheme can deal with the physical problem and capture the discontinuity of velocity field and pressure field on the interface.

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Data Availability Statement The data used to support the findings of this study are available from the corresponding author upon request.

References

1. Cesmelioglu, A., Rhebergen, S., Wells, G.N.: An embedded-hybridized discontinuous Galerkin method for the coupled Stokes–Darcy system. J. Comput. Appl. Math. (2019). https://doi.org/10.1016/j.cam.2019.112476
2. Rivière, B.: Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation. SIAM, Philadelphia (2008)
3. Rivière, B.: Analysis of a discontinuous finite element method for the coupled Stokes and Darcy problems. J. Sci. Comput. **22–23**(1–3), 479–500 (2005)
4. Rivière, B., Yotov, I.: Locally conservative coupling of Stokes and Darcy flow. SIAM. J. Numer. Anal. **42**(5), 1959–1977 (2005)
5. Cockburn, B., Shu, C.-W.: The local discontinuous Galerkin method for time-dependent convection-diffusion systems. SIAM. J. Numer. Anal. **35**, 2440–2463 (1998)
6. Cockburn, B., Shu, C.-W.: Runge-Kutta discontinuous Galerkin methods for convection-dominated problems. J. Sci. Comput. **16**, 173–261 (2001)
7. Vassilev, D., Yotov, I.: Coupling Stokes–Darcy flow with transport. SIAM J. Sci. Comput. **31**, 3661–3684 (2009)
8. Cockburn, B., Kanschat, G., Schötzau, D., Schwab, C.: Local discontinuous Galerkin methods for the stokes systems. SIAM J. Numer. Anal. **1**, 319–343 (2002)
9. Arnold, D.N., Brezzi, F., Cockburn, B., Marini, L.D.: Unified analysis of discontinuous Galerkin methods for elliptic problems. SIAM J. Numer. Anal. **39**, 1749–1779 (2002)
10. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer-Verlag, New York (1991)
11. Jing, F.F., Han, W., Yan, W.J., Wang, F.: Discontinuous Galerkin methods for a stationary Navier–Stokes problem with a nonlinear slip boundary condition of friction type. J. Sci. Comput. **76**, 888–912 (2018)
12. Wang, F., Wu, S., Xu, J.: A mixed discontinuous Galerkin method for linear elasticity with strongly imposed symmetry. J. Sci. Comput. (2020). https://doi.org/10.1007/s10915-020-01191-3
13. Vivette, G., Rivière, B.: DG approximation of coupled Navier–Stokes and Darcy equations by Beaver–Joseph–Safeman interface condition. SIAM. J. Numer. Anal. **47**(3), 2052–2089 (2009)
14. Beavers, G.S., Joseph, D.D.: Boundary conditions at a naturally impermeable wall. J. Fluid Mech. **30**, 197–207 (1967)
15. Kanschat, G., Rivière, B.: A strongly conservative finite element method for the coupling Stokes and Darcy flow. J. Comput. Phys. **229**, 5933–5943 (2018)

16. Fu, G., Lehrenfeld, C.: A strongly conservative Hybrid DG\Mixed FEM for the coupling Stokes and Darcy flow. J. Sci. Comput. **77**, 1605–1620 (2010)
17. Ambartsumyan, I., Khattatov, E., Yotov, I., Zunino, P.: A Lagrange multiplier method for a Stokes-Biot fluid-poroelastic structure interaction model. Numer. Math. **1**, 1–41 (2017)
18. Hou, J.Y., Qiu, M.L., He, X.M., et al.: A dual-porosity-Stokes model and finite element method for coupling dual-porosity flow and free flow. SIAM J. Sci. Comput. **38**, B710–B739 (2016)
19. Galvis, J., Sarkis, M.: Non-matching mortar discretization analysis for the coupling Stokes–Darcy equations. Electron. T. Numer. Ana. **26**(29), 350–384 (2007)
20. Lions, J.L., Magenes, E.: Non-Homogeneous Boundary Value Problems and Applications. Springer-Verlag, New York (1972)
21. Shan, L., Hou, J.Y., Yan, W.J., Chen, J.: Partitioned time stepping method for a dual-porosity-Stokes Model. J. Sci. Comput. **79**(1), 389–413 (2019)
22. Ciarlet, P.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
23. Adams, R.A.: Sobolev Spaces. Academic Press, New York (2003)
24. Girult, R., Rivière, B., Wheeler, M.: A discontinuous Galerkin method with non-overlapping domain decompsition for the Stokes and Navier-Stokes problems. Math. Comp. **74**, 53–84 (2004)
25. Raviart, R.A., Thomas, J.M.: A Mixed Finite Element Method for Second Order Elliptic Problems. In Mathematical Aspects of Finite Element Methods, Lecture Notes in Math, p. 606, Springer-Verlag, New York, (1977)
26. Brenner, S.: Korn's inequalities for piecewise H^1 vector fields. Math. Comp. **73**, 1067–1087 (2004)
27. Layton, W.J., Schieweck, F., Yotov, I.: Coupling fluid flow with porous media flow. SIAM. J. Numer. Anal. **40**, 2195–2218 (2003)