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A Fully-Mixed Formulation for the Steady Double-Diffusive Convection System Based upon Brinkman–Forchheimer Equations

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Abstract

We propose and analyze a new mixed finite element method for the problem of steady double-diffusive convection in a fluid-saturated porous medium. More precisely, the model is described by the coupling of the Brinkman–Forchheimer and double-diffusion equations, in which the originally sought variables are the velocity and pressure of the fluid, and the temperature and concentration of a solute. Our approach is based on the introduction of the further unknowns given by the fluid pseudostress tensor, and the pseudoheat and pseudodiffusive vectors, thus yielding a fully-mixed formulation. Furthermore, since the nonlinear term in the Brinkman–Forchheimer equation requires the velocity to live in a smaller space than usual, we partially augment the variational formulation with suitable Galerkin type terms, which forces both the temperature and concentration scalar fields to live in L^4. As a consequence, the aforementioned pseudoheat and pseudodiffusive vectors live in a suitable $H(div)$-type Banach space. The resulting augmented scheme is written equivalently as a fixed point equation, so that the well-known Schauder and Banach theorems, combined with the Lax–Milgram and Banach–Nečas–Babuška theorems, allow to prove the unique solvability of the continuous problem. As for the associated Galerkin scheme we utilize Raviart–Thomas spaces of order $k \geq 0$ for approximating the pseudostress tensor, as well as the pseudoheat and pseudodiffusive vectors, whereas continuous piecewise polynomials of degree $\leq k+1$ are employed for the velocity, and piecewise polynomials of degree $\leq k$ for the temperature and concentration fields. In turn, the existence and uniqueness of the discrete solution is established similarly to its continuous counterpart, applying in this case the Brouwer and Banach fixed-point theorems, respectively. Finally, we derive optimal a priori error estimates and provide several numerical results confirming the theoretical rates of convergence and illustrating the performance and flexibility of the method.

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1 Introduction

The phenomenon of double-diffusive convection, in which two scalar fields, such as heat and concentration of a solute, affect the density distribution in a fluid-saturated porous medium, has a wide range of applications, including processes arising in chemical engineering, energy technology, geophysics, and oceanography. In particular, some applications include groundwater system in karst aquifers, chemical processing, convective flow of carbon nanotubes, propagation of biological fluids, and simulation of bacterial bioconvection and thermohaline circulation problems (see, e.g. [1,3,6,21,35] to name a few). In this regard, we remark that much of the research in porous medium has been focused on the use of Darcy's law. However, this constitutive equation becomes unreliable to model the flow of fluids through highly porous media at higher Reynolds numbers, as in the above applications. To avoid this inconvenient, a first alternative is to employ the Brinkman model [5], which describes Stokes flows through array of obstacles, and therefore can be applied precisely to that kind of media. Another possible option is the Forchheimer law [22], which accounts for faster flows by including a nonlinear inertial term. According to the above, the Brinkman–Forchheimer equation (see, e.g. [13,31]), which combines the advantages of both models, has been used for fast flows in highly porous media. Moreover, this fact has motivated the introduction of the corresponding coupling with a system of advection-diffusion equations (also called double-diffusion equations), through convective terms and the body force.

In this context, and up to the authors' knowledge, one of the first works in analyzing the coupling of the incompressible Brinkman–Forchheimer and double-diffusion equations is [28]. In there, the authors propose a velocity–pressure–temperature-concentration variational formulation and discuss the corresponding analysis of existence, uniqueness, and regularity of solution. To that end, a Galerkin method was employed to prove that the problem has at least one solution and that, under a smallness data assumption, a uniqueness result is established. Later on, the global solvability of a time-dependent double-diffusive convection system coupled with a linearized version of the Brinkman–Forchheimer equations was introduced and analyzed in [30]. In particular, the authors prove that the global solvability in L^2-spaces holds true for the 3-dimensional case. More recently, in [34] a finite volume method was adopted to solve the coupling of the unsteady Brinkman–Forchheimer and double-diffusion equations. The focus of this work was on the validity of the Brinkman–Forchheimer model when various combinations of the thermal Rayleigh number, inclination angle, permeability ratio, thermal conductivity and buoyancy ratio are considered. Meanwhile, a $\mathbf{H}(\mathrm{div})$-conforming method for double-diffusion equations but coupled with the stationary Navier–Stokes–Brinkman model was analyzed in [6]. Here, the solvability analysis results as a combination of compactness arguments and fixed-point theory. The corresponding numerical scheme is based on Brezzi–Douglas–Marini (BDM) elements of order k for the velocity, discontinuous elements of order $k-1$ for the pressure, and Lagrangian finite elements of order k for temperature and the concentration of a solute. We observe that this formulation produces exactly divergence-free velocity approximations.

According to the above bibliographic discussion, the goal of the present paper is to develop and analyze a new fully-mixed formulation for the coupling of the steady Brinkman–Forchheimer and double-diffusion equations and study its numerical approximation by a mixed finite element method. To that end, unlike previous works, we introduce the pseudostress tensor as in [11] and subsequently eliminate the pressure unknown using the incompressibility condition. In turn, and in order to enforce conservation of momentum in a physically compatible way, we proceed similarly to [7,9] and introduce the pseudoheat and pseudodiffusive vectors as additional unknowns. Furthermore, the difficulty given by the fact that the fluid velocity lives in H^1 instead of L^2 as usual, is resolved as in [11,25] by augmenting the variational formulation with residuals arising from the constitutive equation and the Dirichlet boundary condition on the velocity, which forces both the temperature and concentration fields to live in L^4, and consequently the pseudoheat and pseudodiffusive vectors in a suitable H(div)-type Banach space. Then, following [17,25] and [9], we combine classical fixed-point arguments with the Lax–Milgram and Banach–Nečas–Babuška theorems to prove the well-posedness of both the continuous and discrete formulations. In particular, for the continuous formulation, and under a smallness data assumption, we prove existence and uniqueness of solution by means of a fixed-point strategy where the Schauder (for existence) and Banach (for uniqueness) fixed-point theorems are employed. Using similar arguments (but applying Brower's fixed-point theorem instead of Schauder's for the existence result) we prove the well-posedness of the discrete problem for arbitrary conforming discrete spaces. In addition, applying an ad-hoc Strang-type lemma in Banach spaces, we are able to derive the corresponding a priori error estimates. Next, employing Raviart–Thomas spaces of order $k \geq 0$ for approximating the pseudostress tensor, the pseudoheat and pseudodiffusive vectors, continuous piecewise polynomials of degree $k+1$ for velocity, and piecewise polynomials of degree k for the temperature and concentration fields, we prove that the method is convergent with optimal rate.

The rest of this work is organized as follows. The remainder of this section describes standard notation and functional spaces to be employed throughout the paper. In Sect. 2 we introduce the model problem and derive its augmented fully-mixed variational formulation. Next, in Sect. 3 we establish the well-posedness of this continuous scheme by means of a fixed-point strategy and Schauder and Banach fixed-point theorems. The corresponding Galerkin system is introduced and analyzed in Sect. 4, where the discrete analogue of the theory used in the continuous case is employed to prove existence and uniqueness of solution. In Sect. 5, an ad-hoc Strang-type lemma in Banach spaces is utilized to derive the corresponding a priori error estimate and the consequent rates of convergence. Finally, in Sect. 6 we report some numerical experiments illustrating the accuracy and flexibility of our augmented fully-mixed finite element method.

Preliminary Notations

Let $\Omega \subset \mathrm{R}^n, n \in \{2,3\}$, be a bounded domain with polyhedral boundary Γ, and let $\mathbf{n}$ be the outward unit normal vector on Γ. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$, with $s \in \mathrm{R}$ and $p > 1$, whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{s,p;\Omega}$, respectively. In particular, given a non-negative integer m, $\mathrm{W}^{m,2}(\Omega)$ is also denoted by $\mathrm{H}^m(\Omega)$, and the notations of its norm and seminorm are simplified to $\|\cdot\|_{m,\Omega}$ and

$|\cdot|_{m,\Omega}$, respectively. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, and $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space. In turn, for any vector field $\mathbf{v} = (v_i)_{i=1,n}$, we let $\nabla \mathbf{v}$ and $\mathrm{div}(\mathbf{v})$ be its gradient and divergence, respectively. Furthermore, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\, \zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I},$$

where $\mathbb{I}$ is the identity matrix in $\mathrm{R}^{n\times n}$. In what follows, when no confusion arises, $|\cdot|$ will denote the Euclidean norm in R^n or $\mathrm{R}^{n\times n}$. Additionally, we recall that

$$\mathbb{H}(\mathbf{div};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^2(\Omega) \Big\},$$

equipped with the usual norm $\|\boldsymbol{\tau}\|^2_{\mathbf{div};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,\Omega}$, is a standard Hilbert space in the realm of mixed problems. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. Also, by $\langle\cdot,\cdot\rangle_\Gamma$ we will denote the corresponding product of duality between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$ (and also between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$). Finally, throughout the rest of the paper we employ $\mathbf{0}$ to denote a generic null vector (or tensor), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2 The Continuous Formulation

In this section we introduce the model problem and derive the corresponding weak formulation.

2.1 The Model Problem

In what follows we consider the model introduced in [28], which is given by a steady double-diffusive convection system in a fluid saturated porous medium. More precisely, we focus on solving the coupling of the incompressible Brinkman–Forchheimer and double-diffusion equations, which reduces to finding a velocity field $\mathbf{u}$, a pressure field p, a temperature field ϕ_1 and a concentration field ϕ_2, both defining a vector $\underline{\boldsymbol{\phi}} := (\phi_1, \phi_2)$, such that

$$\begin{aligned} -\nu\,\Delta\mathbf{u} + \mathbf{K}^{-1}\mathbf{u} + \mathtt{F}\,|\mathbf{u}|\mathbf{u} + \nabla p &= \mathbf{f}(\underline{\boldsymbol{\phi}}) && \text{in } \Omega, \\ \mathrm{div}(\mathbf{u}) &= 0 && \text{in } \Omega, \\ -\mathrm{div}(\mathbf{Q}_1\,\nabla\phi_1) + \mathtt{R}_1\,\mathbf{u}\cdot\nabla\phi_1 &= 0 && \text{in } \Omega, \\ -\mathrm{div}(\mathbf{Q}_2\,\nabla\phi_2) + \mathtt{R}_2\,\mathbf{u}\cdot\nabla\phi_2 &= 0 && \text{in } \Omega, \end{aligned} \tag{2.1}$$

with parameters $\nu := \mathtt{D}_a\,\widetilde{\mu}/\mu$ and $\mathtt{F} := \vartheta\,\mathtt{D}_a\,\mathtt{R}_1$, where $\mathtt{D}_a$ stands for the Darcy number, $\widetilde{\mu}$ the viscosity, μ the effective viscosity, $\mathtt{R}_1$ the thermal Rayleigh number, $\mathtt{R}_2$ the solute Rayleigh number, and ϑ is a real number that can be calculated experimentally. In addition, the external force $\mathbf{f}$ is defined by

$$\mathbf{f}(\underline{\boldsymbol{\phi}}) := -(\phi_1 - \phi_{1,\mathrm{r}})\,\mathbf{g} + \frac{1}{\varrho}(\phi_2 - \phi_{2,\mathrm{r}})\,\mathbf{g}, \tag{2.2}$$

with $\mathbf{g}$ representing the potential type gravitational acceleration, $\phi_{1,\mathrm{r}}$ the reference temperature, $\phi_{2,\mathrm{r}}$ the reference concentration of a solute, both of them living in $\mathrm{L}^4(\Omega)$, and ϱ is another parameter experimentally valued that can be assumed to be ≥ 1 (see [28, Sect. 2] for details). In turn, the permeability, thermal diffusion and concentration diffusion tensors are denoted, respectively, by $\mathbf{K}$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ living in $\mathbb{L}^\infty(\Omega)$. Moreover, $\mathbf{K}$ and the inverses of $\mathbf{Q}_1$ and $\mathbf{Q}_2$, are uniformly positive definite tensors, which means that there exist positive constants $C_{\mathbf{K}}$, $C_{\mathbf{Q}_1}$, and $C_{\mathbf{Q}_2}$, such that

$$\mathbf{v}\cdot\mathbf{K}(\mathbf{x})\mathbf{v} \geq C_{\mathbf{K}}\,|\mathbf{v}|^2 \quad \text{and} \quad \mathbf{v}\cdot\mathbf{Q}_j^{-1}(\mathbf{x})\mathbf{v} \geq C_{\mathbf{Q}_j}\,|\mathbf{v}|^2 \quad \forall\,\mathbf{v}\in\mathrm{R}^n,\ \forall\,\mathbf{x}\in\Omega, \quad j\in\{1,2\}. \tag{2.3}$$

Equations (2.1) are complemented with Dirichlet boundary conditions for the velocity, the temperature, and the concentration fields, that is

$$\mathbf{u} = \mathbf{u}_{\mathrm{D}}, \quad \phi_1 = \phi_{1,\mathrm{D}}, \quad \text{and} \quad \phi_2 = \phi_{2,\mathrm{D}} \quad \text{on} \quad \Gamma, \tag{2.4}$$

with given data $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma)$, $\phi_{1,\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma)$ and $\phi_{2,\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma)$. Owing to the incompressibility of the fluid and the Dirichlet boundary condition for $\mathbf{u}$, the datum $\mathbf{u}_{\mathrm{D}}$ must satisfy the compatibility condition

$$\int_\Gamma \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0. \tag{2.5}$$

In addition, due to the first equation of (2.1), and in order to guarantee uniqueness of the pressure, this unknown will be sought in the space

$$\mathrm{L}^2_0(\Omega) := \left\{ q\in\mathrm{L}^2(\Omega) : \quad \int_\Omega q = 0 \right\}.$$

Next, in order to derive a fully-mixed formulation for (2.1)–(2.4), in which the Dirichlet boundary conditions become natural ones, we now proceed as in [11] (see similar approaches in [17,18]), and introduce as further unknowns the pseudostress tensor $\boldsymbol{\sigma}$, the pseudoheat vector $\boldsymbol{\rho}_1$, and the pseudodiffusive vector $\boldsymbol{\rho}_2$, which are defined by

$$\boldsymbol{\sigma} := \nu\,\nabla\mathbf{u} - p\,\mathbb{I} \quad \text{and} \quad \boldsymbol{\rho}_j := \mathbf{Q}_j\,\nabla\phi_j - \mathrm{R}_j\,\phi_j\,\mathbf{u}, \quad j\in\{1,2\}, \quad \text{in} \quad \Omega. \tag{2.6}$$

In this way, applying the trace operator to $\boldsymbol{\sigma}$ and utilizing the incompressibility condition $\mathrm{div}(\mathbf{u}) = 0$ in Ω, one arrives at

$$p = -\frac{1}{n}\,\mathrm{tr}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega. \tag{2.7}$$

Hence, replacing back (2.7) in the first equation of (2.6), we find that our model problem (2.1)–(2.4) can be rewritten, equivalently, as follows: Find $(\boldsymbol{\sigma},\mathbf{u})$ and $(\boldsymbol{\rho}_j,\phi_j)$, $j\in\{1,2\}$, in suitable spaces to be indicated below such that

$$\begin{aligned} \frac{1}{\nu}\boldsymbol{\sigma}^{\mathrm{d}} &= \nabla\mathbf{u} \quad \text{in } \Omega, \\ -\mathbf{div}(\boldsymbol{\sigma}) + \mathbf{K}^{-1}\mathbf{u} + \mathtt{F}\,|\mathbf{u}|\mathbf{u} &= \mathbf{f}(\underline{\boldsymbol{\phi}}) \quad \text{in } \Omega, \\ \mathbf{Q}_j^{-1}\boldsymbol{\rho}_j + \mathtt{R}_j\,\mathbf{Q}_j^{-1}\phi_j\,\mathbf{u} &= \nabla\phi_j \quad \text{in } \Omega, \\ -\mathrm{div}(\boldsymbol{\rho}_j) &= 0 \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}_{\mathrm{D}} \quad \text{and} \quad \underline{\boldsymbol{\phi}} &= \underline{\boldsymbol{\phi}}_{\mathrm{D}} \quad \text{on } \Gamma, \\ \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) &= 0, \end{aligned} \tag{2.8}$$

where the Dirichlet datum for $\underline{\boldsymbol{\phi}}$ is certainly given by $\underline{\boldsymbol{\phi}}_{\mathrm{D}} := (\phi_{1,\mathrm{D}}, \phi_{2,\mathrm{D}})$. At this point we stress that, as suggested by (2.7), p is eliminated from the present formulation and computed afterwards in terms of $\boldsymbol{\sigma}$ by using that identity. This fact, justifies the last equation in (2.8), which aims to ensure that the resulting p does belong to $\mathrm{L}_0^2(\Omega)$. Notice also that further variables of interest, such as the velocity gradient $\nabla\mathbf{u}$, the heat vector $\widetilde{\boldsymbol{\rho}}_1 := \mathbf{Q}_1\,\nabla\phi_1$ and the diffusive vector $\widetilde{\boldsymbol{\rho}}_2 := \mathbf{Q}_2\,\nabla\phi_2$, can be computed, respectively, as follows

$$\nabla\mathbf{u} = \frac{1}{\nu}\boldsymbol{\sigma}^{\mathrm{d}}, \quad \widetilde{\boldsymbol{\rho}}_1 = \boldsymbol{\rho}_1 + \mathtt{R}_1\,\phi_1\,\mathbf{u}, \quad \text{and} \quad \widetilde{\boldsymbol{\rho}}_2 = \boldsymbol{\rho}_2 + \mathtt{R}_2\,\phi_2\,\mathbf{u}. \tag{2.9}$$

2.2 The Fully-Mixed Variational Formulation

In this section we derive our fully-mixed formulation for the coupled system given by (2.8). To that end, we multiply the first equation of (2.8) by a tensor $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$, integrate the resulting expression by parts, and use the identity $\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau} = \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}$ and the Dirichlet boundary condition $\mathbf{u} = \mathbf{u}_{\mathrm{D}}$ on Γ, to get

$$\frac{1}{\nu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_{\mathrm{D}}\rangle_\Gamma \quad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega). \tag{2.10}$$

In order to have more flexibility for choosing the finite element subspaces, but at the same time avoiding the incorporation of new terms in the resulting variational equation, we now proceed similarly as in [24] (see also [25]), and replace $\mathbf{u}$ in the second term of the left-hand side of (2.10) by the expression arising from the second equation in (2.8), that is

$$\mathbf{u} = \mathbf{K}\Big(\mathbf{div}(\boldsymbol{\sigma}) - \mathtt{F}\,|\mathbf{u}|\mathbf{u} + \mathbf{f}(\underline{\boldsymbol{\phi}})\Big).$$

In this way, we arrive at the variational formulation: Find $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div};\Omega)$ and $\mathbf{u}$ (in a suitable space to be specified below), such that

$$\begin{aligned} &\frac{1}{\nu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{K}\,\mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{div}(\boldsymbol{\tau}) - \mathtt{F}\int_\Omega \mathbf{K}\,|\mathbf{u}|\mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \\ &\quad = \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_{\mathrm{D}}\rangle_\Gamma - \int_\Omega \mathbf{K}\,\mathbf{f}(\underline{\boldsymbol{\phi}})\cdot\mathbf{div}(\boldsymbol{\tau}), \end{aligned} \tag{2.11}$$

for all $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$. Since $\mathbf{K} \in \mathbb{L}^\infty(\Omega)$ and $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^2(\Omega)$, the term $\mathbf{K}|\mathbf{u}|\mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau})$ forces the velocity $\mathbf{u}$, and consequently the test function $\mathbf{v}$, to live in $\mathbf{L}^4(\Omega)$. In order to deal with this fact, we first observe, applying the Cauchy–Schwarz and Hölder inequalities, and then the continuous injection $\mathbf{i}_4$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$ (see, e.g., [32, Theorem 1.3.4]), that

$$\begin{aligned}&\left|\int_\Omega \mathbf{K}\,|\mathbf{w}|\mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau})\right| \\ &\quad\le\ \|\mathbf{K}\|_\infty\,\|\mathbf{w}\|_{0,4;\Omega}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}\ \le\ \|\mathbf{K}\|_\infty\,\|\mathbf{i}_4\|^2\,\|\mathbf{w}\|_{1,\Omega}\,\|\mathbf{u}\|_{1,\Omega}\,\|\boldsymbol{\tau}\|_{\mathbf{div};\Omega},\end{aligned} \tag{2.12}$$

for all $\mathbf{w},\mathbf{u}\in\mathbf{H}^1(\Omega)$ and $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega)$. However, we notice from (2.11) that the lack of a test function in the space where $\mathbf{u}$ lives (now in $\mathbf{H}^1(\Omega)$), makes the well-posedness analysis of (2.11) non-viable. Then, aiming to circumvent this inconvenient, we propose to enrich our formulation with the following residual terms arising from the constitutive equation (first equation of (2.8)) and the Dirichlet boundary condition $\mathbf{u}=\mathbf{u}_D$ on Γ:

$$\begin{aligned}\kappa_1\int_\Omega\left\{\nabla\mathbf{u}-\frac{1}{\nu}\boldsymbol{\sigma}^{\mathrm{d}}\right\}:\nabla\mathbf{v}\ &=\ 0,\\ \kappa_2\int_\Gamma\mathbf{u}\cdot\mathbf{v}\ &=\ \kappa_2\int_\Gamma\mathbf{u}_D\cdot\mathbf{v},\end{aligned} \tag{2.13}$$

for all $\mathbf{v}\in\mathbf{H}^1(\Omega)$, where κ_1,κ_2 are positive parameters to be specified later. We now recall (see, e.g., [4,23,26]) that there holds

$$\mathbb{H}(\mathbf{div};\Omega)\ =\ \mathbb{H}_0(\mathbf{div};\Omega)\oplus\mathrm{R}\,\mathbb{I},$$

where

$$\mathbb{H}_0(\mathbf{div};\Omega)\ :=\ \left\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega):\quad\int_\Omega\mathrm{tr}(\boldsymbol{\tau})=0\right\}.$$

Hence, decomposing $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega)$ as $\boldsymbol{\tau}=\boldsymbol{\tau}_0+c\,\mathbb{I}$, with $\boldsymbol{\tau}_0\in\mathbb{H}_0(\mathbf{div};\Omega)$ and $c\in\mathrm{R}$, noticing that $\boldsymbol{\tau}^{\mathrm{d}}=\boldsymbol{\tau}_0^{\mathrm{d}}$ and $\mathbf{div}(\boldsymbol{\tau})=\mathbf{div}(\boldsymbol{\tau}_0)$, and using the last equation of (2.8) and the compatibility condition (2.5), we deduce that both $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ can be considered hereafter in $\mathbb{H}_0(\mathbf{div};\Omega)$. Therefore, from (2.11) and (2.13), we arrive at the variational problem: Find $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ such that

$$A((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v}))+B_{\mathbf{u}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v}))\ =\ F_D(\boldsymbol{\tau},\mathbf{v})+F_{\underline{\boldsymbol{\phi}}}(\boldsymbol{\tau},\mathbf{v}), \tag{2.14}$$

for all $(\boldsymbol{\tau},\mathbf{v})\in\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$, where given $\mathbf{w}\in\mathbf{H}^1(\Omega)$, A and $B_{\mathbf{w}}$ are the bilinear forms defined, respectively, as

$$\begin{aligned}A((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v}))\ :=\ &\frac{1}{\nu}\int_\Omega\boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}}+\int_\Omega\mathbf{K}\,\mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{div}(\boldsymbol{\tau})\\ &+\kappa_1\int_\Omega\left\{\nabla\mathbf{u}-\frac{1}{\nu}\boldsymbol{\sigma}^{\mathrm{d}}\right\}:\nabla\mathbf{v}+\kappa_2\int_\Gamma\mathbf{u}\cdot\mathbf{v},\end{aligned} \tag{2.15}$$

and

$$B_{\mathbf{w}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v}))\ :=\ -\,\mathrm{F}\int_\Omega\mathbf{K}\,|\mathbf{w}|\mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}), \tag{2.16}$$

whereas, given $\underline{\boldsymbol{\varphi}}:=(\varphi_1,\varphi_2)$ in a suitable space defined next, F_D and $F_{\underline{\boldsymbol{\varphi}}}$ are the bounded linear functionals defined by

$$F_D(\boldsymbol{\tau},\mathbf{v})\ :=\ \langle\boldsymbol{\tau}\mathbf{n},\mathbf{u}_D\rangle_\Gamma+\kappa_2\int_\Gamma\mathbf{u}_D\cdot\mathbf{v},\quad F_{\underline{\boldsymbol{\varphi}}}(\boldsymbol{\tau},\mathbf{v})\ :=\ -\int_\Omega\mathbf{K}\,\mathbf{f}(\underline{\boldsymbol{\varphi}})\cdot\mathbf{div}(\boldsymbol{\tau}). \tag{2.17}$$

On the other hand, for the double-diffusion equations in (2.8) we proceed as in [9] (see also [7,15,16]). In fact, multiplying the third and fourth equations of (2.8) by suitable test functions $\boldsymbol{\eta}_j$ and ψ_j, $j \in \{1,2\}$, respectively, integrating by parts and using the Dirichlet boundary condition $\underline{\boldsymbol{\phi}} = \underline{\boldsymbol{\phi}}_{\mathrm{D}}$ on Γ, we get

$$\begin{aligned} \int_\Omega \mathbf{Q}_j^{-1}\,\boldsymbol{\rho}_j\cdot\boldsymbol{\eta}_j + \int_\Omega \phi_j\,\mathrm{div}(\boldsymbol{\eta}_j) + \mathtt{R}_j \int_\Omega \mathbf{Q}_j^{-1}\,\phi_j\,\mathbf{u}\cdot\boldsymbol{\eta}_j &= \langle \boldsymbol{\eta}_j\cdot\mathbf{n}, \phi_{j,\mathrm{D}}\rangle_\Gamma\,, \\ \int_\Omega \psi_j\,\mathrm{div}(\boldsymbol{\rho}_j) &= 0, \end{aligned} \tag{2.18}$$

for all $(\boldsymbol{\eta}_j, \psi_j)$ in spaces to be derived below. In this regard, we begin by noting that for $\mathbf{Q}_j \in \mathbb{L}^\infty(\Omega)$, $j \in \{1,2\}$, and $\mathbf{u} \in \mathbf{H}^1(\Omega)$, the first and third terms in the first equation of (2.18) are well defined if $\boldsymbol{\rho}_j$, $\boldsymbol{\eta}_j \in \mathbf{L}^2(\Omega)$, and if ϕ_j, and consequently the test function ψ_j, are chosen to live in $\mathrm{L}^4(\Omega)$, respectively. In this way, since the latter forces both $\mathrm{div}(\boldsymbol{\rho}_j)$ and $\mathrm{div}(\boldsymbol{\eta}_j)$ to live in $\mathrm{L}^{4/3}(\Omega)$, we now introduce the Banach space

$$\mathbf{H}(\mathrm{div}_{4/3};\Omega) := \Big\{ \boldsymbol{\eta} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\eta}) \in \mathrm{L}^{4/3}(\Omega) \Big\},$$

equipped with the norm

$$\|\boldsymbol{\eta}\|_{\mathrm{div}_{4/3};\Omega} := \|\boldsymbol{\eta}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\eta})\|_{0,4/3;\Omega}.$$

Notice that $\mathbf{H}(\mathrm{div};\Omega) \subset \mathbf{H}(\mathrm{div}_{4/3};\Omega)$. Moreover, as remarked in [9, Eq. (2.5)] (see also [15, Eq. (3.2)]), the right-hand side of (2.18) is well defined in the sense that $\boldsymbol{\eta}_j \cdot \mathbf{n} \in \mathrm{H}^{-1/2}(\Gamma)$, $j \in \{1,2\}$, for all $\boldsymbol{\eta}_j \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)$. Thus, the weak formulation for the double-diffusion equations in (2.8) reads: Find $(\boldsymbol{\rho}_j, \phi_j) \in \mathbf{H}(\mathrm{div}_{4/3};\Omega) \times \mathrm{L}^4(\Omega)$, $j \in \{1,2\}$, such that

$$\begin{aligned} a_j(\boldsymbol{\rho}_j,\boldsymbol{\eta}_j) + b(\boldsymbol{\eta}_j,\phi_j) + c_j(\mathbf{u};\phi_j,\boldsymbol{\eta}_j) &= G_j(\boldsymbol{\eta}_j) \quad \forall\, \boldsymbol{\eta}_j \in \mathbf{H}(\mathrm{div}_{4/3};\Omega), \\ b(\boldsymbol{\rho}_j,\psi_j) &= 0 \qquad\quad\ \forall\, \psi_j \in \mathrm{L}^4(\Omega), \end{aligned} \tag{2.19}$$

where, given $\mathbf{w} \in \mathbf{H}^1(\Omega)$, a_j, b, and $c_j(\mathbf{w};\cdot,\cdot)$ are the forms defined, respectively, as

$$\begin{gathered} a_j(\boldsymbol{\rho}_j,\boldsymbol{\eta}_j) := \int_\Omega \mathbf{Q}_j^{-1}\,\boldsymbol{\rho}_j\cdot\boldsymbol{\eta}_j, \quad b(\boldsymbol{\eta}_j,\psi_j) := \int_\Omega \psi_j\,\mathrm{div}(\boldsymbol{\eta}_j), \\ c_j(\mathbf{w};\psi_j,\boldsymbol{\eta}_j) := \mathtt{R}_j \int_\Omega \mathbf{Q}_j^{-1}\,\psi_j\,\mathbf{w}\cdot\boldsymbol{\eta}_j, \end{gathered} \tag{2.20}$$

for all $\boldsymbol{\rho}_j, \boldsymbol{\eta}_j \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)$ and $\psi_j \in \mathrm{L}^4(\Omega)$. In turn, G_j is the bounded linear functional defined by

$$G_j(\boldsymbol{\eta}_j) := \langle \boldsymbol{\eta}_j\cdot\mathbf{n}, \phi_{j,\mathrm{D}}\rangle_\Gamma \qquad \forall\, \boldsymbol{\eta}_j \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\,. \tag{2.21}$$

Then, the augmented fully-mixed formulation for the coupled problem (2.8) reduces to (2.14) and (2.19), that is: Find $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ and $(\boldsymbol{\rho}_j,\phi_j) \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\times\mathrm{L}^4(\Omega)$, $j \in \{1,2\}$, such that

$$\begin{aligned} A((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) + B_{\mathbf{u}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) &= F_{\mathrm{D}}(\boldsymbol{\tau},\mathbf{v}) + F_{\underline{\boldsymbol{\phi}}}(\boldsymbol{\tau},\mathbf{v}), \\ a_j(\boldsymbol{\rho}_j,\boldsymbol{\eta}_j) + b(\boldsymbol{\eta}_j,\phi_j) + c_j(\mathbf{u};\phi_j,\boldsymbol{\eta}_j) &= G_j(\boldsymbol{\eta}_j), \\ b(\boldsymbol{\rho}_j,\psi_j) &= 0, \end{aligned} \tag{2.22}$$

for all $(\boldsymbol{\tau},\mathbf{v}) \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ and for all $(\boldsymbol{\eta}_j,\psi_j) \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\times\mathrm{L}^4(\Omega)$.

3 Analysis of the Coupled Problem

In this section we combine the Lax–Milgram, Banach–Nečas–Babuška, and Babuška–Brezzi theories, with the classical Schauder and Banach fixed-point theorems, to prove the well-posedness of (2.22) under suitable smallness assumptions on the data.

3.1 Preliminaries

We begin by recalling the Banach–Nečas–Babuška theorem, which is the Banach version of the generalized Lax–Milgram lemma in Hilbert spaces (see for instance [23, Theorem 1.1]). More precisely, we have the following result [20, Theorem 2.6].

Theorem 3.1 *Let* H *be a reflexive Banach space, and let* $a : \mathrm{H} \times \mathrm{H} \to \mathrm{R}$ *be a bounded bilinear form. In addition, assume that*

(i) *there exists* $\alpha > 0$ *such that*

$$\sup_{0 \neq v \in \mathrm{H}} \frac{a(u,v)}{\|v\|_{\mathrm{H}}} \geq \alpha\, \|u\|_{\mathrm{H}} \quad \forall\, u \in \mathrm{H}, \tag{3.1}$$

(ii) *there holds*

$$\sup_{u \in \mathrm{H}} a(u,v) > 0 \quad \forall\, v \in \mathrm{H},\ v \neq 0. \tag{3.2}$$

Then, for each $F \in \mathrm{H}'$ *there exists a unique* $u \in \mathrm{H}$ *such that* $a(u,v) = F(v) \quad \forall\, v \in \mathrm{H}$*, and the following a priori estimate holds:*

$$\|u\|_{\mathrm{H}} \leq \frac{1}{\alpha}\, \|F\|_{\mathrm{H}'}.$$

Let us now discuss the stability properties of the forms involved in (2.22). In fact, using (2.12) and performing simple computations, we deduce from (2.15), (2.16), and (2.20) that the forms A, $B_{\mathbf{w}}$, a_j, b and $c_j(\mathbf{w};\cdot,\cdot)$, $j \in \{1,2\}$, are bounded as indicated in what follows

$$\big|A((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v}))\big| \leq C_A\, \|(\boldsymbol{\sigma},\mathbf{u})\|\, \|(\boldsymbol{\tau},\mathbf{v})\|\,, \tag{3.3}$$

$$\begin{aligned} \big|B_{\mathbf{w}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v}))\big| &\leq \mathrm{F}\, \|\mathbf{K}\|_\infty\, \|\mathbf{w}\|_{0,4;\Omega}\, \|\mathbf{u}\|_{0,4;\Omega}\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega} \\ &\leq \mathrm{F}\, \|\mathbf{K}\|_\infty\, \|\mathbf{w}\|_{0,4;\Omega}\, \|\mathbf{i}_4\|\, \|\mathbf{u}\|_{1,\Omega}\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega} \\ &\leq \mathrm{F}\, \|\mathbf{K}\|_\infty\, \|\mathbf{i}_4\|^2\, \|\mathbf{w}\|_{1,\Omega}\, \|\mathbf{u}\|_{1,\Omega}\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}\,, \end{aligned} \tag{3.4}$$

$$\big|a_j(\boldsymbol{\rho}_j,\boldsymbol{\eta}_j)\big| \leq \|\mathbf{Q}_j^{-1}\|_\infty\, \|\boldsymbol{\rho}_j\|_{\mathrm{div}_{4/3};\Omega}\, \|\boldsymbol{\eta}_j\|_{\mathrm{div}_{4/3};\Omega}\,, \tag{3.5}$$

$$\big|b(\boldsymbol{\eta}_j,\psi_j)\big| \leq \|\boldsymbol{\eta}_j\|_{\mathrm{div}_{4/3};\Omega}\, \|\psi_j\|_{0,4;\Omega}\,, \tag{3.6}$$

and

$$\begin{aligned} \big|c_j(\mathbf{w};\phi_j,\boldsymbol{\eta}_j)\big| &\leq \mathrm{R}_j\, \|\mathbf{Q}_j^{-1}\|_\infty\, \|\mathbf{w}\|_{0,4;\Omega}\, \|\phi_j\|_{0,4;\Omega}\, \|\boldsymbol{\eta}_j\|_{\mathrm{div}_{4/3};\Omega} \\ &\leq \mathrm{R}_j\, \|\mathbf{Q}_j^{-1}\|_\infty\, \|\mathbf{i}_4\|\, \|\mathbf{w}\|_{1,\Omega}\, \|\phi_j\|_{0,4;\Omega}\, \|\boldsymbol{\eta}_j\|_{\mathrm{div}_{4/3};\Omega}\,, \end{aligned} \tag{3.7}$$

where C_A is a positive constant depending on ν, $\|\mathbf{K}\|_\infty$, κ_1, and κ_2. In addition, employing the Cauchy–Schwarz and Young inequalities, and recalling the definition of $\mathbf{f}$ (cf. (2.2)), it is readily seen that F_{D}, $F_{\underline{\boldsymbol{\varphi}}}$ and G_j (cf. (2.21)) are bounded, which means that there exist constants $C_{\mathrm{D}}, C_F, C_G > 0$, such that

$$\big|F_{\mathrm{D}}(\boldsymbol{\tau},\mathbf{v})\big| \leq C_{\mathrm{D}} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} \Big\}\, \|(\boldsymbol{\tau},\mathbf{v})\|, \tag{3.8}$$

$$\big|F_{\underline{\boldsymbol{\varphi}}}(\boldsymbol{\tau},\mathbf{v})\big| \leq C_F\, \|\mathbf{g}\|_{0,4;\Omega} \Big(\|\underline{\boldsymbol{\varphi}}\|_{0,4;\Omega} + \|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega} \Big)\, \|(\boldsymbol{\tau},\mathbf{v})\|, \tag{3.9}$$

and

$$\left|G_j(\boldsymbol{\eta}_j)\right| \,\le\, C_{G_j}\,\|\phi_{j,\mathrm{D}}\|_{1/2,\Gamma}\,\|\boldsymbol{\eta}_j\|_{\mathrm{div}_{4/3};\Omega}, \tag{3.10}$$

where $C_{\mathrm{D}} := \max\left\{1, \kappa_2\,\|\gamma_0\|\right\}$, $C_F := \|\mathbf{K}\|_\infty$, $\underline{\boldsymbol{\phi}}_{\mathrm{r}} := (\phi_{1,\mathrm{r}}, \phi_{2,\mathrm{r}}) \in \mathbf{L}^4(\Omega)$, and C_{G_j} is a positive constant depending on $\|\mathbf{i}_4\|$ (cf. [7, Lemma 3.5]). Next, we let $\mathbf{V}$ be the kernel of the operator induced by the bilinear form b, that is

$$\mathbf{V} \,:=\, \Big\{\boldsymbol{\eta} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega): \quad \mathrm{div}(\boldsymbol{\eta}) = 0 \quad \text{in} \quad \Omega\Big\},$$

and observe, thanks to the definition of a_j (cf. (2.20)) and the fact that the inverses of $\mathbf{Q}_j$ are uniformly positive definite tensors (cf. (2.3)), that a_j is elliptic on $\mathbf{V}$, that is

$$a_j(\boldsymbol{\eta},\boldsymbol{\eta}) \,\ge\, \alpha_j\,\|\boldsymbol{\eta}\|^2_{\mathrm{div}_{4/3};\Omega} \quad \forall\,\boldsymbol{\eta} \in \mathbf{V}, \tag{3.11}$$

with $\alpha_j = C_{\mathbf{Q}_j}$. In turn, according to [9, Lemma 2.1] with $p = 4/3$, we know that there exists a constant $\beta > 0$ such that b verify the following inf-sup condition

$$\sup_{\mathbf{0}\neq\boldsymbol{\eta}\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)} \frac{b(\boldsymbol{\eta},\psi)}{\|\boldsymbol{\eta}\|_{\mathrm{div}_{4/3};\Omega}} \,\ge\, \beta\,\|\psi\|_{0,4;\Omega} \quad \forall\,\psi \in \mathrm{L}^4(\Omega)\,. \tag{3.12}$$

We end this section by recalling, for later use, that there exist positive constants $c_1(\Omega)$ and $c_2(\Omega)$ such that (see [23, Lemma 2.3] and [29, Theorem 5.11.2], respectively, for details)

$$\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,\Omega} \,\ge\, c_1(\Omega)\,\|\boldsymbol{\tau}\|^2_{0,\Omega} \quad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega) \tag{3.13}$$

and

$$\|\nabla\mathbf{v}\|^2_{0,\Omega} + \|\mathbf{v}\|^2_{0,\Gamma} \,\ge\, c_2(\Omega)\,\|\mathbf{v}\|^2_{1,\Omega} \quad \forall\,\mathbf{v} \in \mathbf{H}^1(\Omega). \tag{3.14}$$

3.2 A Fixed-Point Approach

We now rewrite (2.22) as an equivalent fixed-point equation. To this end, we first let $\mathbf{S}: \mathbf{H}^1(\Omega)\times\mathbf{L}^4(\Omega) \to \mathbf{H}^1(\Omega)$ be the operator defined as

$$\mathbf{S}(\mathbf{w},\underline{\boldsymbol{\varphi}}) \,:=\, \mathbf{u} \quad \forall\,(\mathbf{w},\underline{\boldsymbol{\varphi}}) \in \mathbf{H}^1(\Omega)\times\mathbf{L}^4(\Omega), \tag{3.15}$$

where $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ is the unique solution (to be confirmed below) of the problem:

$$A((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) + B_{\mathbf{w}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) \,=\, F_{\mathrm{D}}(\boldsymbol{\tau},\mathbf{v}) + F_{\underline{\boldsymbol{\varphi}}}(\boldsymbol{\tau},\mathbf{v}), \tag{3.16}$$

for all $(\boldsymbol{\tau},\mathbf{v}) \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$. In turn, for each $j \in \{1,2\}$ we let $\widetilde{\mathbf{S}}_j: \mathbf{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$ be the operator given by

$$\widetilde{\mathbf{S}}_j(\mathbf{w}) \,:=\, \phi_j \quad \forall\,\mathbf{w} \in \mathbf{H}^1(\Omega), \tag{3.17}$$

where $(\boldsymbol{\rho}_j,\phi_j) \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\times\mathrm{L}^4(\Omega)$ is the unique solution (to be confirmed below) of the problem:

$$\begin{aligned} a_j(\boldsymbol{\rho}_j,\boldsymbol{\eta}_j) + b(\boldsymbol{\eta}_j,\phi_j) + c_j(\mathbf{w};\phi_j,\boldsymbol{\eta}_j) &= G_j(\boldsymbol{\eta}_j) \;\forall\,\boldsymbol{\eta}_j \in \mathbf{H}(\mathrm{div}_{4/3};\Omega),\\ b(\boldsymbol{\rho}_j,\psi_j) &= 0 \qquad \forall\,\psi_j \in \mathrm{L}^4(\Omega), \end{aligned} \tag{3.18}$$

so that we can introduce $\widetilde{\mathbf{S}}(\mathbf{w}) := (\widetilde{\mathbf{S}}_1(\mathbf{w}), \widetilde{\mathbf{S}}_2(\mathbf{w})) \in \mathbf{L}^4(\Omega)$ for all $\mathbf{w} \in \mathbf{H}^1(\Omega)$. Consequently, we can define the operator $\mathbf{T} : \mathbf{H}^1(\Omega) \to \mathbf{H}^1(\Omega)$ as

$$\mathbf{T}(\mathbf{w}) := \mathbf{S}(\mathbf{w}, \widetilde{\mathbf{S}}(\mathbf{w})) \quad \forall\, \mathbf{w} \in \mathbf{H}^1(\Omega), \tag{3.19}$$

and realize that solving (2.22) is equivalent to finding $\mathbf{u} \in \mathbf{H}^1(\Omega)$ such that

$$\mathbf{T}(\mathbf{u}) = \mathbf{u}. \tag{3.20}$$

3.3 Well-Definedness of T

We begin by establishing a result that provides sufficient conditions under which the operator $\mathbf{S}$ (cf. (3.15)) is well-defined, or equivalently, the problem (3.16) is well-posed.

Lemma 3.2 *Assume that for $\delta \in (0, 2\nu)$ we choose $\kappa_1 \in (0, 2\delta)$ and $\kappa_2 > 0$. Then, there exists $r_1 > 0$ such that for each $r \in (0, r_1)$, and for each $(\mathbf{w}, \underline{\boldsymbol{\varphi}}) \in \mathbf{H}^1(\Omega) \times \mathbf{L}^4(\Omega)$ satisfying $\|\mathbf{w}\|_{1,\Omega} \le r$, the problem (3.16) has a unique solution $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega)$. Moreover, there exists a constant $c_{\mathbf{S}} > 0$, independent of $(\mathbf{w}, \underline{\boldsymbol{\varphi}})$, such that there holds*

$$\begin{aligned}\|\mathbf{S}(\mathbf{w}, \underline{\boldsymbol{\varphi}})\|_{1,\Omega} &= \|\mathbf{u}\|_{1,\Omega} \le \|(\boldsymbol{\sigma}, \mathbf{u})\| \\ &\le c_{\mathbf{S}} \left\{ \|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega} \left(\|\underline{\boldsymbol{\varphi}}\|_{0,4;\Omega} + \|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega} \right) \right\}.\end{aligned} \tag{3.21}$$

Proof We proceed as in [11, Lemma 3.2]. In fact, given $(\mathbf{w}, \underline{\boldsymbol{\varphi}}) \in \mathbf{H}^1(\Omega) \times \mathbf{L}^4(\Omega)$, we observe from (2.15) and (2.16) that $A + B_{\mathbf{w}}$ is clearly a bilinear form. Then, thanks to (3.3) and (3.4), we find that there exists a positive constant, which we denote by $\|A + B_{\mathbf{w}}\|$, only depending on ν, $\|\mathbf{K}\|_\infty$, $\|\mathbf{i}_4\|$, κ_1, κ_2, $\mathtt{F}$, and $\|\mathbf{w}\|_{1,\Omega}$, such that

$$\left| (A + B_{\mathbf{w}})((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \mathbf{v})) \right| \le \|A + B_{\mathbf{w}}\| \, \|(\boldsymbol{\sigma}, \mathbf{u})\| \, \|(\boldsymbol{\tau}, \mathbf{v})\|,$$

for all $(\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}_0(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega)$. In turn, from the definition of A (cf. (2.15)), we have

$$\begin{aligned} A((\boldsymbol{\tau}, \mathbf{v}), (\boldsymbol{\tau}, \mathbf{v})) &= \frac{1}{\nu} \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \int_\Omega \mathbf{K}\, \mathbf{div}(\boldsymbol{\tau}) \cdot \mathbf{div}(\boldsymbol{\tau}) - \frac{\kappa_1}{\nu} \int_\Omega \boldsymbol{\tau}^{\mathrm{d}} : \nabla \mathbf{v} \\ &\quad + \kappa_1 \|\nabla \mathbf{v}\|_{0,\Omega}^2 + \kappa_2 \|\mathbf{v}\|_{0,\Gamma}^2, \end{aligned}$$

and hence, using (2.3), and Cauchy–Schwarz and Young's inequalities, we obtain that for any $\delta > 0$ and for all $(\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}_0(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega)$, there holds

$$\begin{aligned} A((\boldsymbol{\tau}, \mathbf{v}), (\boldsymbol{\tau}, \mathbf{v})) &\ge \frac{1}{\nu} \left(1 - \frac{\kappa_1}{2\delta} \right) \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + C_{\mathbf{K}} \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \\ &\quad + \kappa_1 \left(1 - \frac{\delta}{2\nu} \right) \|\nabla \mathbf{v}\|_{0,\Omega}^2 + \kappa_2 \|\mathbf{v}\|_{0,\Gamma}^2. \end{aligned}$$

In this way, applying the inequalities (3.13) and (3.14), we can define the constants

$$\begin{aligned} \alpha_0(\Omega) &:= \min\left\{ \frac{1}{\nu} \left(1 - \frac{\kappa_1}{2\delta} \right), \frac{C_{\mathbf{K}}}{2} \right\}, \quad \alpha_1(\Omega) := \min\left\{ c_1(\Omega)\, \alpha_0(\Omega), \frac{C_{\mathbf{K}}}{2} \right\}, \\ \alpha_2(\Omega) &:= c_2(\Omega) \min\left\{ \kappa_1 \left(1 - \frac{\delta}{2\nu} \right), \kappa_2 \right\}, \end{aligned} \tag{3.22}$$

which are positive thanks to the hypotheses on δ, κ_1, and κ_2. In this way, it follows that

$$A((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\tau},\mathbf{v})) \,\ge\, \alpha_A\,\|(\boldsymbol{\tau},\mathbf{v})\|^2 \quad \forall\,(\boldsymbol{\tau},\mathbf{v})\in\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega), \tag{3.23}$$

with $\alpha_A := \min\Big\{\alpha_1(\Omega),\alpha_2(\Omega)\Big\}$, which shows that A is elliptic on $\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$. Therefore, combining now (3.4) and (3.23), and using the injection $\mathbf{i}_4$, we deduce that for all $(\boldsymbol{\tau},\mathbf{v})\in\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ there holds

$$(A+B_{\mathbf{w}})((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\tau},\mathbf{v})) \,\ge\, \Big\{\alpha_A - \mathrm{F}\,\|\mathbf{K}\|_\infty\,\|\mathbf{i}_4\|^2\,\|\mathbf{w}\|_{1,\Omega}\Big\}\,\|(\boldsymbol{\tau},\mathbf{v})\|^2.$$

Consequently, requiring $\|\mathbf{w}\|_{1,\Omega}\le r_1$, with

$$r_1 \,=\, \frac{\alpha_A}{2\,\mathrm{F}\,\|\mathbf{K}\|_\infty\,\|\mathbf{i}_4\|^2}, \tag{3.24}$$

we arrive at

$$(A+B_{\mathbf{w}})((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\tau},\mathbf{v})) \,\ge\, \frac{\alpha_A}{2}\,\|(\boldsymbol{\tau},\mathbf{v})\|^2 \quad \forall\,(\boldsymbol{\tau},\mathbf{v})\in\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega). \tag{3.25}$$

Summing up, and owing to the hypotheses on κ_1 and κ_2, we have proved that for any $(\mathbf{w},\underline{\boldsymbol{\varphi}})\in\mathbf{H}^1(\Omega)\times\mathbf{L}^4(\Omega)$ such that $\|\mathbf{w}\|_{1,\Omega}\le r_1$, the bilinear form $A+B_{\mathbf{w}}$ and the functional $F_{\mathrm{D}}+F_{\underline{\boldsymbol{\varphi}}}$ satisfy the hypotheses of the Lax–Milgram theorem (see, e.g., [23, Theorem 1.1]), which guarantees the well-posedness of (3.16). Finally, using (3.25) with $(\boldsymbol{\tau},\mathbf{v})=(\boldsymbol{\sigma},\mathbf{u})$, (3.16), and the bounds of F_{D} and $F_{\underline{\boldsymbol{\varphi}}}$ (cf. (3.8) and (3.9)), we readily obtain that

$$\frac{\alpha_A}{2}\,\|(\boldsymbol{\sigma},\mathbf{u})\| \,\le\, C_{\mathrm{D}}\big(\|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma}+\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\big)+C_F\,\|\mathbf{g}\|_{0,4;\Omega}\,\big(\|\underline{\boldsymbol{\varphi}}\|_{0,4;\Omega}+\|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega}\big),$$

which implies (3.21) with $c_{\mathbf{S}} := 2\,\max\big\{C_{\mathrm{D}},C_F\big\}/\alpha_A$, thus completing the proof. □

Now, we establish the well-posedness of problem (3.18), or equivalently, that the operator $\widetilde{\mathbf{S}}$ (cf. (3.17)) is well-defined. To that end, let us consider the space $\mathbf{H} := \mathbf{H}(\mathrm{div}_{4/3};\Omega)\times\mathrm{L}^4(\Omega)$ and the bilinear form $\mathcal{A}_j:\mathbf{H}\times\mathbf{H}\to\mathrm{R}$, $j\in\{1,2\}$, defined by

$$\mathcal{A}_j((\boldsymbol{\rho}_j,\phi_j),(\boldsymbol{\eta}_j,\psi_j)) \,:=\, a_j(\boldsymbol{\rho}_j,\boldsymbol{\eta}_j)+b(\boldsymbol{\eta}_j,\phi_j)+b(\boldsymbol{\rho}_j,\psi_j), \tag{3.26}$$

for all $(\boldsymbol{\rho}_j,\phi_j),(\boldsymbol{\eta}_j,\psi_j)\in\mathbf{H}$. Then, owing to (3.5), (3.6), (3.11), (3.12), and a direct application of [20, Proposition 2.36], we deduce, equivalently, that $\mathcal{A}_j$ satisfies the following inf-sup condition:

$$\sup_{\mathbf{0}\neq(\boldsymbol{\eta}_j,\psi_j)\in\mathbf{H}} \frac{\mathcal{A}_j((\boldsymbol{\chi}_j,\varphi_j),(\boldsymbol{\eta}_j,\psi_j))}{\|(\boldsymbol{\eta}_j,\psi_j)\|} \,\ge\, \gamma_j\,\|(\boldsymbol{\chi}_j,\varphi_j)\| \quad \forall\,(\boldsymbol{\chi}_j,\varphi_j)\in\mathbf{H}, \tag{3.27}$$

where $\gamma_j>0$ is the constant defined by

$$\gamma_j \,:=\, \frac{\alpha_j\,\beta^2}{\beta^2+\big(2\,\beta+\|\mathbf{Q}_j^{-1}\|_\infty\big)\big(\alpha_j+\|\mathbf{Q}_j^{-1}\|_\infty\big)}. \tag{3.28}$$

In this way, bearing in mind (3.27), we are able to establish the following result.

Lemma 3.3 *There exists $r_2>0$ such that for each $r\in(0,r_2)$, for each $\mathbf{w}\in\mathbf{H}^1(\Omega)$ satisfying $\|\mathbf{w}\|_{1,\Omega}\le r$, and for each $j\in\{1,2\}$, the problem (3.18) has a unique solution $(\boldsymbol{\rho}_j,\phi_j)\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)\times\mathrm{L}^4(\Omega)$, $j\in\{1,2\}$. Moreover, there exists a constant $c_{\widetilde{\mathbf{S}}}>0$, independent of $\mathbf{w}$, such that there holds*

$$\|\widetilde{\mathbf{S}}(\mathbf{w})\|_{0,4;\Omega} \,=\, \|(\phi_1,\phi_2)\|_{0,4;\Omega} \,\le\, \sum_{j=1}^{2}\|(\boldsymbol{\rho}_j,\phi_j)\| \,\le\, c_{\widetilde{\mathbf{S}}}\,\|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma}. \tag{3.29}$$

Proof We proceed as in [7, Theorem 3.6]. In fact, given $\mathbf{w} \in \mathbf{H}^1(\Omega)$ and $j \in \{1, 2\}$, we begin by defining the bilinear form

$$\mathcal{A}_{j,\mathbf{w}}((\boldsymbol{\rho}_j, \phi_j), (\boldsymbol{\eta}_j, \psi_j)) := \mathcal{A}_j((\boldsymbol{\rho}_j, \phi_j), (\boldsymbol{\eta}_j, \psi_j)) + c_j(\mathbf{w}; \phi_j, \boldsymbol{\eta}_j), \tag{3.30}$$

where $\mathcal{A}_j$ and $c_j(\mathbf{w}; \cdot, \cdot)$ are the forms defined in (3.26) and (2.20), respectively. Then, the problem (3.18) can be rewritten, equivalently, as: Find $(\boldsymbol{\rho}_j, \phi_j) \in \mathbf{H} := \mathbf{H}(\mathrm{div}_{4/3}; \Omega) \times \mathrm{L}^4(\Omega)$ such that

$$\mathcal{A}_{j,\mathbf{w}}((\boldsymbol{\rho}_j, \phi_j), (\boldsymbol{\eta}_j, \psi_j)) = G_j(\boldsymbol{\eta}_j) \quad \forall\, (\boldsymbol{\eta}_j, \psi_j) \in \mathbf{H}. \tag{3.31}$$

Therefore, in order to conclude the well-definedness of $\widetilde{\mathbf{S}}$, in the sequel we use the Banach–Nečas–Babuška theorem (cf. Theoremm 3.1) to prove that problem (3.31) is well-posed. Indeed, given $(\boldsymbol{\chi}_j, \varphi_j), (\widetilde{\boldsymbol{\eta}}_j, \widetilde{\psi}_j) \in \mathbf{H}$ with $(\widetilde{\boldsymbol{\eta}}_j, \widetilde{\psi}_j) \neq \mathbf{0}$, we first deduce from (3.30) and (3.7) that

$$\begin{aligned}
&\sup_{\mathbf{0}\neq(\boldsymbol{\eta}_j,\psi_j)\in\mathbf{H}} \frac{\mathcal{A}_{j,\mathbf{w}}((\boldsymbol{\chi}_j, \varphi_j), (\boldsymbol{\eta}_j, \psi_j))}{\|(\boldsymbol{\eta}_j, \psi_j)\|} \ge \frac{|\mathcal{A}_j((\boldsymbol{\chi}_j, \varphi_j), (\widetilde{\boldsymbol{\eta}}_j, \widetilde{\psi}_j))|}{\|(\widetilde{\boldsymbol{\eta}}_j, \widetilde{\psi}_j)\|} - \frac{|c_j(\mathbf{w}; \varphi_j, \widetilde{\boldsymbol{\eta}}_j)|}{\|(\widetilde{\boldsymbol{\eta}}_j, \widetilde{\psi}_j)\|}\\
&\ge \frac{|\mathcal{A}_j((\boldsymbol{\chi}_j, \varphi_j), (\widetilde{\boldsymbol{\eta}}_j, \widetilde{\psi}_j))|}{\|(\widetilde{\boldsymbol{\eta}}_j, \widetilde{\psi}_j)\|} - \mathrm{R}_j\, \|\mathbf{Q}_j^{-1}\|_\infty\, \|\mathbf{i}_4\|\, \|\mathbf{w}\|_{1,\Omega}\, \|(\boldsymbol{\chi}_j, \varphi_j)\|,
\end{aligned}$$

which, together with (3.27) and the fact that $(\widetilde{\boldsymbol{\eta}}_j, \widetilde{\psi}_j) \in \mathbf{H}$ is arbitrary, implies

$$\begin{aligned}
&\sup_{\mathbf{0}\neq(\boldsymbol{\eta}_j,\psi_j)\in\mathbf{H}} \frac{\mathcal{A}_{j,\mathbf{w}}((\boldsymbol{\chi}_j, \varphi_j), (\boldsymbol{\eta}_j, \psi_j))}{\|(\boldsymbol{\eta}_j, \psi_j)\|}\\
&\quad \ge \Big(\gamma_j - \mathrm{R}_j\, \|\mathbf{Q}_j^{-1}\|_\infty\, \|\mathbf{i}_4\|\, \|\mathbf{w}\|_{1,\Omega}\Big)\, \|(\boldsymbol{\chi}_j, \varphi_j)\| \quad \forall\, (\boldsymbol{\chi}_j, \varphi_j) \in \mathbf{H}.
\end{aligned}$$

Consequently, requiring now $\|\mathbf{w}\|_{1,\Omega} \le r_2$, with

$$r_2 := \min\left\{r_2^1, r_2^2\right\} \quad \text{and} \quad r_2^j := \frac{\gamma_j}{2\,\mathrm{R}_j\, \|\mathbf{Q}_j^{-1}\|_\infty\, \|\mathbf{i}_4\|}, \tag{3.32}$$

we find that

$$\sup_{\mathbf{0}\neq(\boldsymbol{\eta}_j,\psi_j)\in\mathbf{H}} \frac{\mathcal{A}_{j,\mathbf{w}}((\boldsymbol{\chi}_j, \varphi_j), (\boldsymbol{\eta}_j, \psi_j))}{\|(\boldsymbol{\eta}_j, \psi_j)\|} \ge \frac{\gamma_j}{2}\, \|(\boldsymbol{\chi}_j, \varphi_j)\| \quad \forall\, (\boldsymbol{\chi}_j, \varphi_j) \in \mathbf{H}. \tag{3.33}$$

On the other hand, for a given $(\boldsymbol{\chi}_j, \varphi_j) \in \mathbf{H}$, we observe that

$$\begin{aligned}
&\sup_{(\boldsymbol{\eta}_j,\psi_j)\in\mathbf{H}} \mathcal{A}_{j,\mathbf{w}}((\boldsymbol{\eta}_j, \psi_j), (\boldsymbol{\chi}_j, \varphi_j)) \ge \sup_{\mathbf{0}\neq(\boldsymbol{\eta}_j,\psi_j)\in\mathbf{H}} \frac{\mathcal{A}_{j,\mathbf{w}}((\boldsymbol{\eta}_j, \psi_j), (\boldsymbol{\chi}_j, \varphi_j))}{\|(\boldsymbol{\eta}_j, \psi_j)\|}\\
&= \sup_{\mathbf{0}\neq(\boldsymbol{\eta}_j,\psi_j)\in\mathbf{H}} \frac{\mathcal{A}_j((\boldsymbol{\eta}_j, \psi_j), (\boldsymbol{\chi}_j, \varphi_j)) + c_j(\mathbf{w}; \psi_j, \boldsymbol{\chi}_j)}{\|(\boldsymbol{\eta}_j, \psi_j)\|}
\end{aligned}$$

which, employing again (3.7), yields

$$\begin{aligned}
&\sup_{(\boldsymbol{\eta}_j,\psi_j)\in\mathbf{H}} \mathcal{A}_{j,\mathbf{w}}((\boldsymbol{\eta}_j, \psi_j), (\boldsymbol{\chi}_j, \varphi_j))\\
&\ge \sup_{\mathbf{0}\neq(\boldsymbol{\eta}_j,\psi_j)\in\mathbf{H}} \frac{\mathcal{A}_j((\boldsymbol{\eta}_j, \psi_j), (\boldsymbol{\chi}_j, \varphi_j))}{\|(\boldsymbol{\eta}_j, \psi_j)\|} - \mathrm{R}_j\, \|\mathbf{Q}_j^{-1}\|_\infty\, \|\mathbf{i}_4\|\, \|\mathbf{w}\|_{1,\Omega}\, \|(\boldsymbol{\chi}_j, \varphi_j)\|.
\end{aligned}$$

Therefore, using the symmetry of $\mathcal{A}_j$ and the inf-sup condition (3.27), and considering $\|\mathbf{w}\|_{1,\Omega} \le r_2$ (cf. (3.32)), we obtain

$$\sup_{(\boldsymbol{\eta}_j,\psi_j)\in\mathbf{H}} \mathcal{A}_{j,\mathbf{w}}((\boldsymbol{\eta}_j,\psi_j),(\boldsymbol{\chi}_j,\varphi_j)) \ge \frac{\gamma_j}{2}\,\|(\boldsymbol{\chi}_j,\varphi_j)\| \,>\, 0 \quad \forall\,(\boldsymbol{\chi}_j,\varphi_j)\in\mathbf{H},\ (\boldsymbol{\chi}_j,\varphi_j)\neq\mathbf{0}. \tag{3.34}$$

In this way, it is clear from (3.33) and (3.34) that $\mathcal{A}_{j,\mathbf{w}}$ satisfies the hypotheses of the Banach–Nečas–Babuška theorem (cf. Theorem 3.1), which guarantees the well-posedness of (3.18). Finally, using (3.33) with $(\boldsymbol{\chi}_j,\varphi_j) = (\boldsymbol{\rho}_j,\phi_j)$, (3.31), and the continuity bound of G_j (cf. (3.10)), we get

$$\frac{\gamma_j}{2}\,\|(\boldsymbol{\rho}_j,\phi_j)\| \,\le\, C_{G_j}\,\|\phi_{j,\mathrm{D}}\|_{1/2,\Gamma}, \tag{3.35}$$

which gives (3.29) with $c_{\widetilde{\mathbf{S}}} := \max\left\{c_{\widetilde{\mathbf{S}}_1}, c_{\widetilde{\mathbf{S}}_2}\right\}$ and $c_{\widetilde{\mathbf{S}}_j} := 2\,C_{G_j}/\gamma_j$, $j\in\{1,2\}$, thus ending proof. □

Now, concerning the practical choice of the stabilization parameters κ_1 and κ_2, and particularly for sake of the computational implementation of the Galerkin method to be introduced and analyzed later on, we first select the midpoints of the corresponding feasible intervals for δ and κ_1, namely $\delta = \nu$ and $\kappa_1 = \delta$, respectively. Then, in order to define κ_2, we aim to maximize the constant $\alpha_2(\Omega)$ in (3.22), which is attained by taking $\kappa_2 = \kappa_1\left(1-\frac{\delta}{2\nu}\right)$. In this way, we propose to consider:

$$\kappa_1 = \nu \quad \text{and} \quad \kappa_2 = \frac{\nu}{2}. \tag{3.36}$$

3.4 Solvability Analysis of the Fixed-Point Equation

Having proved the well-posedness of the uncoupled problems (3.16) and (3.18), which ensures that the operators $\mathbf{S}, \widetilde{\mathbf{S}}$ and $\mathbf{T}$ are well defined, we now aim to establish the existence of a unique fixed point of the operator $\mathbf{T}$. For this purpose, in what follows we verify the hypothesis of the Schauder and Banach fixed-point theorems. We begin the analysis with the following straightforward consequence of Lemmas 3.2 and 3.3.

Lemma 3.4 *Given $r\in(0,r_0)$, with $r_0 := \min\{r_1,r_2\}$, r_1 as in (3.24) and r_2 as in (3.32), we let $\mathbf{W}_r$ be the closed and convex subset of $\mathbf{H}^1(\Omega)$ defined by*

$$\mathbf{W}_r := \left\{\mathbf{w}\in\mathbf{H}^1(\Omega): \quad \|\mathbf{w}\|_{1,\Omega} \le r\right\}. \tag{3.37}$$

In addition, we take the stabilization parameters κ_1 and κ_2 as in Lemma 3.2 (particularly, as suggested in (3.36)), and assume that the data satisfy

$$c_{\mathbf{T}}\left\{\|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega}\left(\|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\underline{\boldsymbol{\phi}}_{r}\|_{0,4;\Omega}\right)\right\} \le r. \tag{3.38}$$

Then $\mathbf{T}(\mathbf{W}_r) \subseteq \mathbf{W}_r$.

Proof Given $\mathbf{w}\in\mathbf{W}_r$, we let $\underline{\boldsymbol{\phi}} := \widetilde{\mathbf{S}}(\mathbf{w})$ and observe that certainly $(\mathbf{w},\underline{\boldsymbol{\phi}})$ satisfies the hypotheses of Lemma 3.2. Hence, employing the corresponding estimate (3.21) in combination with (3.29), we get

$$\begin{aligned}\|\mathbf{T}(\mathbf{w})\|_{1,\Omega} &= \|\mathbf{S}(\mathbf{w},\widetilde{\mathbf{S}}(\mathbf{w}))\|_{1,\Omega} = \|\mathbf{S}(\mathbf{w},\underline{\boldsymbol{\phi}})\|_{1,\Omega}\\ &\le c_{\mathbf{S}}\left\{\|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma}+\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}+\|\mathbf{g}\|_{0,4;\Omega}\left(c_{\widetilde{\mathbf{S}}}\,\|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma}+\|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega}\right)\right\}\\ &\le c_{\mathbf{T}}\left\{\|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma}+\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}+\|\mathbf{g}\|_{0,4;\Omega}\left(\|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma}+\|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega}\right)\right\},\end{aligned} \tag{3.39}$$

with $c_{\mathbf{T}} := c_{\mathbf{S}} \max\left\{1, c_{\widetilde{\mathbf{S}}}\right\}$. In this way, (3.39) and assumption (3.38) imply that $\mathbf{T}(\mathbf{w}) \in \mathbf{W}_r$, which proves that $\mathbf{T}(\mathbf{W}_r) \subseteq \mathbf{W}_r$. □

Next, we aim to prove the continuity and compactness properties of $\mathbf{T}$, which basically will be direct consequences of the following two auxiliary lemmas for the operators $\mathbf{S}$ and $\widetilde{\mathbf{S}}$, respectively.

Lemma 3.5 *Let $r \in (0, r_1)$ with r_1 given as in (3.24). Then there exists a positive constant $C_{\mathbf{S}}$ independent of r, such that*

$$\begin{aligned}&\|\mathbf{S}(\mathbf{w},\underline{\boldsymbol{\varphi}})-\mathbf{S}(\widetilde{\mathbf{w}},\underline{\widetilde{\boldsymbol{\varphi}}})\|_{1,\Omega}\\ &\quad\le \frac{1}{\|\mathbf{i}_4\|\,r_1}\,\|\mathbf{S}(\mathbf{w},\underline{\boldsymbol{\varphi}})\|_{1,\Omega}\,\|\mathbf{w}-\widetilde{\mathbf{w}}\|_{0,4;\Omega}+C_{\mathbf{S}}\,\|\mathbf{g}\|_{0,4;\Omega}\,\|\underline{\boldsymbol{\varphi}}-\underline{\widetilde{\boldsymbol{\varphi}}}\|_{0,4;\Omega}\end{aligned} \tag{3.40}$$

for all $(\mathbf{w},\underline{\boldsymbol{\varphi}}),\ (\widetilde{\mathbf{w}},\underline{\widetilde{\boldsymbol{\varphi}}}) \in \mathbf{H}^1(\Omega)\times\mathbf{L}^4(\Omega)$ *such that* $\|\mathbf{w}\|_{1,\Omega},\ \|\widetilde{\mathbf{w}}\|_{1,\Omega} \le r$.

Proof Given $(\mathbf{w},\underline{\boldsymbol{\varphi}}), (\widetilde{\mathbf{w}},\underline{\widetilde{\boldsymbol{\varphi}}}) \in \mathbf{H}^1(\Omega)\times\mathbf{L}^4(\Omega)$, such that $\|\mathbf{w}\|_{1,\Omega}, \|\widetilde{\mathbf{w}}\|_{1,\Omega} \le r$, we let $(\boldsymbol{\sigma},\mathbf{u}), (\widetilde{\boldsymbol{\sigma}},\widetilde{\mathbf{u}}) \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ be the corresponding solutions of (3.16), so that $\mathbf{u} := \mathbf{S}(\mathbf{w},\underline{\boldsymbol{\varphi}})$ and $\widetilde{\mathbf{u}} := \mathbf{S}(\widetilde{\mathbf{w}},\underline{\widetilde{\boldsymbol{\varphi}}})$. Then, using the bilinearity of A and $B_{\mathbf{w}}$ for any $\mathbf{w}$, it follows easily from (3.16) that

$$(A+B_{\widetilde{\mathbf{w}}})((\boldsymbol{\sigma},\mathbf{u})-(\widetilde{\boldsymbol{\sigma}},\widetilde{\mathbf{u}}),(\boldsymbol{\tau},\mathbf{v})) = \big(B_{\widetilde{\mathbf{w}}}-B_{\mathbf{w}}\big)((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) + F_{\underline{\boldsymbol{\varphi}}-\underline{\widetilde{\boldsymbol{\varphi}}}}(\boldsymbol{\tau},\mathbf{v}),$$

for all $(\boldsymbol{\tau},\mathbf{v}) \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$. Hence, taking $(\boldsymbol{\tau},\mathbf{v}) = (\boldsymbol{\sigma},\mathbf{u})-(\widetilde{\boldsymbol{\sigma}},\widetilde{\mathbf{u}})$ in the foregoing identity, and then employing the ellipticity of $A+B_{\mathbf{w}}$ (cf. (3.25)), the fact that $|\,|\widetilde{\mathbf{w}}|-|\mathbf{w}|\,| \le |\widetilde{\mathbf{w}}-\mathbf{w}|$, the boundedness property of $B_{\mathbf{w}}$ (cf. (3.4)), and the definition and continuity of $F_{\underline{\boldsymbol{\varphi}}}$ (cf. (2.17), (3.9)) in combination with Cauchy–Schwarz and Hölder's inequalities, we readily get

$$\begin{aligned}\frac{\alpha_A}{2}\,\|(\boldsymbol{\sigma},\mathbf{u})-(\widetilde{\boldsymbol{\sigma}},\widetilde{\mathbf{u}})\|^2 &\le \big(B_{\widetilde{\mathbf{w}}}-B_{\mathbf{w}}\big)((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\sigma},\mathbf{u})-(\widetilde{\boldsymbol{\sigma}},\widetilde{\mathbf{u}})) + F_{\underline{\boldsymbol{\varphi}}-\underline{\widetilde{\boldsymbol{\varphi}}}}((\boldsymbol{\sigma},\mathbf{u})-(\widetilde{\boldsymbol{\sigma}},\widetilde{\mathbf{u}}))\\ &\le \left\{\mathrm{F}\,\|\mathbf{K}\|_\infty\,\|\mathbf{i}_4\|\,\|\mathbf{u}\|_{1,\Omega}\|\mathbf{w}-\widetilde{\mathbf{w}}\|_{0,4;\Omega}+C_F\,\|\mathbf{g}\|_{0,4;\Omega}\|\underline{\boldsymbol{\varphi}}-\underline{\widetilde{\boldsymbol{\varphi}}}\|_{0,4;\Omega}\right\}\|(\boldsymbol{\sigma},\mathbf{u})-(\widetilde{\boldsymbol{\sigma}},\widetilde{\mathbf{u}})\|,\end{aligned}$$

which, together with the definition of r_1 (cf. (3.24)) and the fact that $\mathbf{u} = \mathbf{S}(\mathbf{w},\underline{\boldsymbol{\varphi}})$, implies (3.40) with constant $C_{\mathbf{S}} := 2\,C_F/\alpha_A$, thus ending the proof. □

Lemma 3.6 *Let $r \in (0, r_2)$ with r_2 given as in (3.32). Then there holds*

$$\|\widetilde{\mathbf{S}}(\mathbf{w})-\widetilde{\mathbf{S}}(\widetilde{\mathbf{w}})\|_{0,4;\Omega} \le \frac{1}{\|\mathbf{i}_4\|\,r_2}\,\|\widetilde{\mathbf{S}}(\mathbf{w})\|_{0,4;\Omega}\,\|\mathbf{w}-\widetilde{\mathbf{w}}\|_{0,4;\Omega}, \tag{3.41}$$

for all $\mathbf{w},\ \widetilde{\mathbf{w}} \in \mathbf{H}^1(\Omega)$ *such that* $\|\mathbf{w}\|_{1,\Omega},\ \|\widetilde{\mathbf{w}}\|_{1,\Omega} \le r$.

Proof We prooced as in [7, Theorem 3.7]. In fact, given $\mathbf{w}, \widetilde{\mathbf{w}} \in \mathbf{H}^1(\Omega)$ such that $\|\mathbf{w}\|_{1,\Omega},\ \|\widetilde{\mathbf{w}}\|_{1,\Omega} \le r$, for each $j \in \{1,2\}$ we let $(\boldsymbol{\rho}_j, \phi_j),\ (\widetilde{\boldsymbol{\rho}}_j, \widetilde{\phi}_j) \in \mathbf{H} := \mathbf{H}(\mathrm{div}_{4/3};\Omega) \times \mathrm{L}^4(\Omega)$ be the solution of (3.18) (equivalently of (3.31)), so that $(\phi_1,\phi_2) = (\widetilde{\mathbf{S}}_1(\mathbf{w}), \widetilde{\mathbf{S}}_2(\mathbf{w})) = \widetilde{\mathbf{S}}(\mathbf{w})$ and $(\widetilde{\phi}_1, \widetilde{\phi}_2) = (\widetilde{\mathbf{S}}_1(\widetilde{\mathbf{w}}), \widetilde{\mathbf{S}}_2(\widetilde{\mathbf{w}})) = \widetilde{\mathbf{S}}(\widetilde{\mathbf{w}})$. Then, using the linearity of the form $\mathcal{A}_{j,\mathbf{w}}$ (cf. (3.30)), we deduce after simple computations that

$$\mathcal{A}_{j,\mathbf{w}}((\boldsymbol{\rho}_j - \widetilde{\boldsymbol{\rho}}_j, \phi_j - \widetilde{\phi}_j), (\boldsymbol{\eta}_j, \psi_j)) = -c_j(\mathbf{w} - \widetilde{\mathbf{w}}; \phi_j, \boldsymbol{\eta}_j) \quad \forall\, (\boldsymbol{\eta}_j, \psi_j) \in \mathbf{H}.$$

Then, employing (3.33) with $(\boldsymbol{\chi}_j, \varphi_j) = (\boldsymbol{\rho}_j - \widetilde{\boldsymbol{\rho}}_j, \phi_j - \widetilde{\phi}_j) \in \mathbf{H}$ and the continuity bound of $c_j(\mathbf{w};\cdot,\cdot)$ (cf. (3.7)), we obtain

$$\begin{aligned}\frac{\gamma_j}{2}\,\|(\boldsymbol{\rho}_j,\phi_j) - (\widetilde{\boldsymbol{\rho}}_j, \widetilde{\phi}_j)\| &\le \sup_{\mathbf{0}\neq(\boldsymbol{\eta}_j,\psi_j)\in\mathbf{H}} \frac{-c_j(\mathbf{w}-\widetilde{\mathbf{w}};\phi_j,\boldsymbol{\eta}_j)}{\|(\boldsymbol{\eta}_j,\psi_j)\|}\\ &\le \mathrm{R}_j\,\|\mathbf{Q}_j^{-1}\|_\infty\,\|\phi_j\|_{0,4;\Omega}\,\|\mathbf{w}-\widetilde{\mathbf{w}}\|_{0,4;\Omega}\,,\end{aligned}$$

which, together with the definition of r_2 (cf. (3.32)) and the fact that $\phi_j = \widetilde{\mathbf{S}}_j(\mathbf{w})$, implies (3.41) and conclude the proof. □

As a consequence of Lemmas 3.5 and 3.6 we establish the following result providing an estimate needed to derive next the required continuity and compactness properties of the operator $\mathbf{T}$.

Lemma 3.7 *Let $r \in (0, r_0)$, with $r_0 := \min\{r_1, r_2\}$, r_1 as in (3.24) and r_2 as in (3.32). Then, for all $\mathbf{w}, \widetilde{\mathbf{w}} \in \mathbf{W}_r$ (cf. (3.37)), there holds*

$$\begin{aligned}\|\mathbf{T}(\mathbf{w}) - \mathbf{T}(\widetilde{\mathbf{w}})\|_{1,\Omega} \le \tfrac{c_{\mathbf{T}}}{\|\mathbf{i}_4\|\,r_0}\Big\{&\|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}\\ &+\|\mathbf{g}\|_{0,4;\Omega}\Big(2\,\|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\underline{\boldsymbol{\phi}}_{r}\|_{0,4;\Omega}\Big)\Big\}\,\|\mathbf{w}-\widetilde{\mathbf{w}}\|_{0,4;\Omega}.\end{aligned} \tag{3.42}$$

Proof Let $\mathbf{w}, \widetilde{\mathbf{w}} \in \mathbf{W}_r$, such that $\mathbf{u} = \mathbf{T}(\mathbf{w})$ and $\widetilde{\mathbf{u}} = \mathbf{T}(\widetilde{\mathbf{w}})$. Then, from the definition of $\mathbf{T}$ (cf. (3.19)), and Lemmas 3.5 and 3.6 (cf. (3.40) and (3.41)), we deduce that

$$\begin{aligned}\|\mathbf{T}(\mathbf{w}) - \mathbf{T}(\widetilde{\mathbf{w}})\|_{1,\Omega} &= \|\mathbf{S}(\mathbf{w}, \widetilde{\mathbf{S}}(\mathbf{w})) - \mathbf{S}(\widetilde{\mathbf{w}}, \widetilde{\mathbf{S}}(\widetilde{\mathbf{w}}))\|_{1,\Omega}\\ &\le \frac{1}{\|\mathbf{i}_4\|\,r_1}\,\|\mathbf{T}(\mathbf{w})\|_{1,\Omega}\,\|\mathbf{w}-\widetilde{\mathbf{w}}\|_{0,4;\Omega} + C_{\mathbf{S}}\,\|\mathbf{g}\|_{0,4;\Omega}\,\|\widetilde{\mathbf{S}}(\mathbf{w}) - \widetilde{\mathbf{S}}(\widetilde{\mathbf{w}})\|_{0,4;\Omega}\\ &\le \left(\frac{1}{\|\mathbf{i}_4\|\,r_1}\,\|\mathbf{T}(\mathbf{w})\|_{1,\Omega} + \frac{C_{\mathbf{S}}}{\|\mathbf{i}_4\|\,r_2}\,\|\mathbf{g}\|_{0,4;\Omega}\,\|\widetilde{\mathbf{S}}(\mathbf{w})\|_{0,4;\Omega}\right)\,\|\mathbf{w}-\widetilde{\mathbf{w}}\|_{0,4;\Omega}.\end{aligned}$$

Hence, using (3.29) and the fact that $C_{\mathbf{S}}\,c_{\widetilde{\mathbf{S}}}$ is bounded by $c_{\mathbf{T}}$, and then bounding $\|\mathbf{T}(\mathbf{w})\|_{1,\Omega}$ by (3.39) instead of by r, we conclude from the foregoing inequality that

$$\begin{aligned}\|\mathbf{T}(\mathbf{w}) - \mathbf{T}(\widetilde{\mathbf{w}})\|_{1,\Omega} \le \frac{c_{\mathbf{T}}}{\|\mathbf{i}_4\|}\Big\{&\frac{1}{r_1}\Big(\|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega}\,\|\underline{\boldsymbol{\phi}}_{r}\|_{0,4;\Omega}\Big)\\ &+\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\|\mathbf{g}\|_{0,4;\Omega}\,\|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}\,\|\mathbf{w}-\widetilde{\mathbf{w}}\|_{0,4;\Omega}\,.\end{aligned} \tag{3.43}$$

Finally, (3.42) follows from (3.43) by noting that both $1/r_1$ and $1/r_2$ are bounded by $1/r_0$. □

Owing to the above analysis, we establish now the announced properties of the operator $\mathbf{T}$.

Lemma 3.8 *Let $r \in (0, r_0)$, with $r_0 := \min\{r_1, r_2\}$, r_1 as in (3.24) and r_2 as in (3.32). Assume that the stabilization parameters κ_1 and κ_2 are taken as in Lemma 3.2, and that the data satisfy (3.38). Then $\mathbf{T} : \mathbf{W}_r \to \mathbf{W}_r$ is continuous and $\overline{\mathbf{T}(\mathbf{W}_r)}$ is compact.*

Proof The required result follows straightforwardly from estimate (3.42), the compactness of the injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$ (see, e.g., [32, Theorem 1.3.5]), and the well-known fact that every bounded sequence in a Hilbert space has a weakly convergent subsequence. We omit further details and refer to [8, Lemma 3.8]. □

Finally, the main result of this section is stated as follows.

Theorem 3.9 *Assume the same hypothesis of Lemma 3.8. Then the operator $\mathbf{T}$ has a fixed point $\mathbf{u} \in \mathbf{W}_r$ (cf. (3.37)). Equivalently, the coupled problem (2.22) has a solution $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega)$ and $(\boldsymbol{\rho}_j, \phi_j) \in \mathbf{H}(\mathrm{div}_{4/3}; \Omega) \times \mathrm{L}^4(\Omega)$, $j \in \{1, 2\}$, with $\mathbf{u} \in \mathbf{W}_r$. Moreover, there holds*

$$\|(\boldsymbol{\sigma}, \mathbf{u})\| \le c_{\mathbf{T}} \left\{ \|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega} \left(\|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega} \right) \right\} \tag{3.44}$$

and

$$\sum_{j=1}^{2} \|(\boldsymbol{\rho}_j, \phi_j)\| \le c_{\widetilde{\mathbf{S}}} \|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma}. \tag{3.45}$$

In addition, if the data satisfy

$$\frac{c_{\mathbf{T}}}{r_0} \left\{ \|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega} \left(2\, \|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega} \right) \right\} < 1\,, \tag{3.46}$$

then the aforementioned fixed point (equivalently, the solution of (2.22)) is unique.

Proof The equivalence between (2.22) and the fixed point equation (3.20), together with Lemmas 3.4 and 3.8, confirm the existence of solution of (2.22) as a direct application of the Schauder fixed-point theorem [14, Theorem 9.12-1(b)]. In addition, it is clear that the estimate (3.45) follows from (3.29), whereas combining (3.21) with (3.29) we obtain (3.44) (cf. (3.39)). On the other hand, using the estimate (3.42) and the continuous injection $\mathbf{i}_4$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, we easily obtain

$$\begin{aligned} &\|\mathbf{T}(\mathbf{w}) - \mathbf{T}(\widetilde{\mathbf{w}})\|_{1,\Omega} \\ &\quad \le \frac{c_{\mathbf{T}}}{r_0} \left\{ \|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega} \Big(2\, \|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega} \Big) \right\} \|\mathbf{w} - \widetilde{\mathbf{w}}\|_{1,\Omega} \end{aligned}$$

which, thanks to (3.46) and the Banach fixed-point theorem, yields the uniqueness. □

We end this section by remarking that we are able to avoid the augmentation procedure developed for the Brinkman–Forchheimer equations by proceeding as in [15], that is, introducing the gradient of the velocity as a new unknown of the system (besides the pseudostress tensor and the velocity of the fluid), which yields a new three-field Banach mixed formulation. We are addressing this issue in an ongoing work [10].

4 The Galerkin Scheme

In this section, we introduce and analyze the corresponding Galerkin scheme for the fully-mixed formulation (2.22). The solvability of this scheme is addressed following analogous tools to those employed throughout Sect. 3.

4.1 Preliminaries

We begin by considering arbitrary finite dimensional subspaces

$$\mathbb{H}_h^{\boldsymbol{\sigma}} \subseteq \mathbb{H}_0(\mathbf{div};\Omega), \quad \mathbf{H}_h^{\mathbf{u}} \subseteq \mathbf{H}^1(\Omega), \quad \mathbf{H}_h^{\boldsymbol{\rho}} \subseteq \mathbf{H}(\mathrm{div}_{4/3};\Omega), \quad \mathrm{H}_h^{\phi} \subseteq \mathrm{L}^4(\Omega), \tag{4.1}$$

whose specific choices are postponed to Sect. 4.3 below. Hereafter, $h := \max\{h_T : T \in \mathcal{T}_h\}$ stands for the size of a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ made up of triangles T (when $n = 2$) or tetrahedra T (when $n = 3$) of diameter h_T. In what follows, we set $\underline{\boldsymbol{\phi}}_h := (\phi_{1,h}, \phi_{2,h})$, $\underline{\boldsymbol{\varphi}}_h := (\varphi_{1,h}, \varphi_{2,h}) \in \mathbf{H}_h^{\phi} := \mathrm{H}_h^{\phi} \times \mathrm{H}_h^{\phi}$. Then the Galerkin scheme associated with (2.22) reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ and $(\boldsymbol{\rho}_{j,h}, \phi_{j,h}) \in \mathbf{H}_h^{\boldsymbol{\rho}} \times \mathrm{H}_h^{\phi}$, $j \in \{1,2\}$, such that

$$\begin{aligned} A((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)) + B_{\mathbf{u}_h}((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)) &= F_{\mathrm{D}}(\boldsymbol{\tau}_h, \mathbf{v}_h) + F_{\underline{\boldsymbol{\phi}}_h}(\boldsymbol{\tau}_h, \mathbf{v}_h), \\ a_j(\boldsymbol{\rho}_{j,h}, \boldsymbol{\eta}_{j,h}) + b(\boldsymbol{\eta}_{j,h}, \phi_{j,h}) + c_j(\mathbf{u}_h; \phi_{j,h}, \boldsymbol{\eta}_{j,h}) &= G_j(\boldsymbol{\eta}_{j,h}), \\ b(\boldsymbol{\rho}_{j,h}, \psi_{j,h}) &= 0, \end{aligned} \tag{4.2}$$

for all $(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ and $(\boldsymbol{\eta}_{j,h}, \psi_{j,h}) \in \mathbf{H}_h^{\boldsymbol{\rho}} \times \mathrm{H}_h^{\phi}$. We now develop the discrete analogue of the fixed-point approach utilized in Sect. 3.2. To this end, we first consider the operator $\mathbf{S}_h : \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\phi} \to \mathbf{H}_h^{\mathbf{u}}$ defined by

$$\mathbf{S}_h(\mathbf{w}_h, \underline{\boldsymbol{\varphi}}_h) := \mathbf{u}_h \quad \forall\, (\mathbf{w}_h, \underline{\boldsymbol{\varphi}}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\phi}, \tag{4.3}$$

where $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ is the unique solution (to be confirmed below) of the problem:

$$A((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)) + B_{\mathbf{w}_h}((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)) = F_{\mathrm{D}}(\boldsymbol{\tau}_h, \mathbf{v}_h) + F_{\underline{\boldsymbol{\varphi}}_h}(\boldsymbol{\tau}_h, \mathbf{v}_h), \tag{4.4}$$

for all $(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$. In turn, for each $j \in \{1,2\}$ we let $\widetilde{\mathbf{S}}_{j,h} : \mathbf{H}_h^{\mathbf{u}} \to \mathrm{H}_h^{\phi}$ be the operator given by

$$\widetilde{\mathbf{S}}_{j,h}(\mathbf{w}_h) := \phi_{j,h} \quad \forall\, \mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}, \tag{4.5}$$

where $(\boldsymbol{\rho}_{j,h}, \phi_{j,h}) \in \mathbf{H}_h^{\boldsymbol{\rho}} \times \mathrm{H}_h^{\phi}$ is the unique solution (to be confirmed below) of the problem:

$$\begin{aligned} a_j(\boldsymbol{\rho}_{j,h}, \boldsymbol{\eta}_{j,h}) + b(\boldsymbol{\eta}_{j,h}, \phi_{j,h}) + c_j(\mathbf{w}_h; \phi_{j,h}, \boldsymbol{\eta}_{j,h}) &= G_j(\boldsymbol{\eta}_{j,h}) \ \forall\, \boldsymbol{\eta}_{j,h} \in \mathbf{H}_h^{\boldsymbol{\rho}}, \\ b(\boldsymbol{\rho}_{j,h}, \psi_{j,h}) &= 0 \qquad\quad \forall\, \psi_{j,h} \in \mathrm{H}_h^{\phi}, \end{aligned} \tag{4.6}$$

and then we set $\widetilde{\mathbf{S}}_h(\mathbf{w}_h) := (\widetilde{\mathbf{S}}_{1,h}(\mathbf{w}_h), \widetilde{\mathbf{S}}_{2,h}(\mathbf{w}_h)) \in \mathbf{H}_h^{\phi}$ for all $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$. Hence, introducing the operator $\mathbf{T}_h : \mathbf{H}_h^{\mathbf{u}} \to \mathbf{H}_h^{\mathbf{u}}$ as

$$\mathbf{T}_h(\mathbf{w}_h) := \mathbf{S}_h(\mathbf{w}_h, \widetilde{\mathbf{S}}_h(\mathbf{w}_h)) \quad \forall\, \mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}, \tag{4.7}$$

we realize that solving (4.2) is equivalent to seeking a fixed point of $\mathbf{T}_h$, that is: Find $\mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that

$$\mathbf{T}_h(\mathbf{u}_h) = \mathbf{u}_h. \tag{4.8}$$

4.2 Solvability Analysis

We begin by remarking that the same tools employed in the proof of Lemma 3.2 can be used now to prove the unique solvability of (4.4). In fact, it is straightforward to see that for each $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$, the bilinear form $A + B_{\mathbf{w}_h}$ is bounded with a constant depending on ν, $\mathtt{F}$, $\|\mathbf{K}\|_\infty$, $\|\mathbf{i}_4\|$, κ_1, κ_2, and $\|\mathbf{w}_h\|_{1,\Omega}$. In addition, under the same assumptions from Lemma 3.2 on the stabilization parameters, we find that for each $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$, $A + B_{\mathbf{w}_h}$ is elliptic on $\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ with the same constant obtained in (3.25). In turn, it is clear that for each $\underline{\boldsymbol{\varphi}}_h \in \mathbf{H}_h^{\phi}$, the functional $F_{\underline{\boldsymbol{\varphi}}_h}$ is linear and bounded as in (3.9). The foregoing discussion and the Lax–Milgram theorem allow us to conclude the following result.

Lemma 4.1 *Let $\widetilde{r} \in (0, r_1)$, with r_1 given as in (3.24), and assume that κ_1 and κ_2 are taken as in Lemma 3.2. Then, for each $(\mathbf{w}_h, \underline{\boldsymbol{\varphi}}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\phi}$ satisfying $\|\mathbf{w}_h\|_{1,\Omega} \le \widetilde{r}$, the problem (4.4) has a unique solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$. Moreover, with the same constant $c_{\mathbf{S}} > 0$ from (3.21), which is independent of $(\mathbf{w}_h, \underline{\boldsymbol{\varphi}}_h)$, there holds*

$$\begin{aligned} \|\mathbf{S}_h(\mathbf{w}_h, \underline{\boldsymbol{\varphi}}_h)\|_{1,\Omega} &= \|\mathbf{u}_h\|_{1,\Omega} \le \|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \\ &\le c_{\mathbf{S}} \left\{ \|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega} \left(\|\underline{\boldsymbol{\varphi}}_h\|_{0,4;\Omega} + \|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega} \right) \right\}. \end{aligned} \tag{4.9}$$

We emphasize here that there is no restriction on $\mathbb{H}_h^{\boldsymbol{\sigma}}$ and $\mathbf{H}_h^{\mathbf{u}}$, and hence they can be chosen as any finite element subspaces of $\mathbb{H}_0(\mathbf{div}; \Omega)$ and $\mathbf{H}^1(\Omega)$, respectively.

On the other hand, in order to analyze problem (4.6), we need to incorporate further hypotheses on the discrete spaces $\mathbf{H}_h^{\boldsymbol{\rho}}$ and H_h^{ϕ}. For this purpose, we now let $\mathbf{V}_h$ be the discrete kernel of b, i.e.,

$$\mathbf{V}_h = \left\{ \boldsymbol{\eta}_h \in \mathbf{H}_h^{\boldsymbol{\rho}} : \quad b(\boldsymbol{\eta}_h, \psi_h) = 0 \quad \forall\, \psi_h \in \mathrm{H}_h^{\phi} \right\}. \tag{4.10}$$

Then, we assume that the following discrete inf-sup conditions hold:
(**H.1**) for each $j \in \{1, 2\}$ there exists a constant $\widetilde{\alpha}_j > 0$, independent of h, such that

$$\sup_{\mathbf{0} \neq \boldsymbol{\eta}_h \in \mathbf{V}_h} \frac{a_j(\boldsymbol{\eta}_h, \boldsymbol{\chi}_h)}{\|\boldsymbol{\eta}_h\|_{\mathrm{div}_{4/3};\Omega}} \ge \widetilde{\alpha}_j \, \|\boldsymbol{\chi}_h\|_{\mathrm{div}_{4/3};\Omega} \quad \forall\, \boldsymbol{\chi}_h \in \mathbf{V}_h\,, \tag{4.11}$$

(**H.2**) there exists a constant $\widetilde{\beta} > 0$, independent of h, such that

$$\sup_{\mathbf{0} \neq \boldsymbol{\eta}_h \in \mathbf{H}_h^{\boldsymbol{\rho}}} \frac{b(\boldsymbol{\eta}_h, \psi_h)}{\|\boldsymbol{\eta}_h\|_{\mathrm{div}_{4/3};\Omega}} \ge \widetilde{\beta} \, \|\psi_h\|_{0,4;\Omega} \quad \forall\, \psi_h \in \mathrm{H}_h^{\phi}\,. \tag{4.12}$$

Specific examples of spaces verifying (**H.1**) and (**H.2**) are described later on in Sect. 4.3. Notice that (4.11) is more general, and hence less restrictive, than assuming that the bilinear forms a_j are elliptic in $\mathbf{V}_h$. In other words, the latter is not necessary but only a sufficient condition for (4.11), which is precisely what we apply below in Sect. 4.3 for a particular choice of subspaces. In turn, unless $\mathbf{V}_h$ is contained in $\mathbf{V}$, which occurs in many cases but not always, the $\mathbf{V}_h$-ellipticity of a_j does not follow from its eventual $\mathbf{V}$-ellipticity.

Next, we consider the bilinear form $\mathcal{A}_j$ (cf. (3.26)) restricted to the discrete space $\mathbf{H}_h := \mathbf{H}_h^{\boldsymbol{\rho}} \times \mathrm{H}_h^{\phi}$. Thus, employing (3.5), (4.11), and (4.12), and applying again [20, Proposition 2.36], we are able to show that $\mathcal{A}_j$ verifies the following discrete inf-sup condition

$$\begin{aligned} \sup_{\mathbf{0} \neq (\boldsymbol{\eta}_{j,h}, \psi_{j,h}) \in \mathbf{H}_h} & \frac{\mathcal{A}_j((\boldsymbol{\chi}_{j,h}, \varphi_{j,h}), (\boldsymbol{\eta}_{j,h}, \psi_{j,h}))}{\|(\boldsymbol{\eta}_{j,h}, \psi_{j,h})\|} \\ &\ge \widetilde{\gamma}_j \, \|(\boldsymbol{\chi}_{j,h}, \varphi_{j,h})\| \quad \forall\, (\boldsymbol{\chi}_{j,h}, \varphi_{j,h}) \in \mathbf{H}_h, \end{aligned} \tag{4.13}$$

where $\widetilde{\gamma}_j$ (cf. (3.28)), a positive constant independent of h, is defined by

$$\widetilde{\gamma}_j \,:=\, \frac{\widetilde{\alpha}_j\,\widetilde{\beta}^2}{\widetilde{\beta}^2 + \big(2\,\widetilde{\beta} + \|\mathbf{Q}_j^{-1}\|_\infty\big)\big(\widetilde{\alpha}_j + \|\mathbf{Q}_j^{-1}\|_\infty\big)}\,. \tag{4.14}$$

We are now in a position to establish the discrete version of Lemma 3.3.

Lemma 4.2 *There exists $\widetilde{r}_2 > 0$ such that for each $\widetilde{r} \in (0, \widetilde{r}_2)$, and for each $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$ satisfying $\|\mathbf{w}_h\|_{1,\Omega} \le \widetilde{r}$, the problem (4.6) has a unique solution $(\boldsymbol{\rho}_{j,h}, \phi_{j,h}) \in \mathbf{H}_h^{\boldsymbol{\rho}} \times \mathrm{H}_h^{\phi}$, for each $j \in \{1,2\}$. Moreover, there exists a constant $c_{\widetilde{\mathbf{S}}_h} > 0$, independent of $\mathbf{w}_h$, such that there holds*

$$\|\widetilde{\mathbf{S}}_h(\mathbf{w}_h)\|_{0,4;\Omega} \,=\, \|(\phi_{1,h},\phi_{2,h})\|_{0,4;\Omega} \,\le\, \sum_{j=1}^{2} \|(\boldsymbol{\rho}_{j,h},\phi_{j,h})\| \,\le\, c_{\widetilde{\mathbf{S}}_h}\,\|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma}\,. \tag{4.15}$$

Proof Given $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$, we proceed analogously to the proof of Lemma 3.3 and utilize the continuity bound of c_j (cf. (3.7)), and the discrete inf-sup condition of $\mathcal{A}_j$ (4.13), to deduce that for each $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$ satisfying $\|\mathbf{w}_h\|_{1,\Omega} \le \widetilde{r}_2$, with

$$\widetilde{r}_2 \,:=\, \min\big\{\widetilde{r}_2^1, \widetilde{r}_2^2\big\} \quad \text{and} \quad \widetilde{r}_2^j \,:=\, \frac{\widetilde{\gamma}_j}{2\,\mathrm{R}_j\,\|\mathbf{Q}_j^{-1}\|_\infty\,\|\mathbf{i}_4\|}\,, \qquad j \in \{1,2\}\,, \tag{4.16}$$

$\mathcal{A}_{j,\mathbf{w}_h}$ (cf. (3.30)) satisfies the discrete inf-sup condition

$$\begin{aligned} \sup_{\mathbf{0}\neq(\boldsymbol{\eta}_{j,h},\psi_{j,h})\in\mathbf{H}_h} & \frac{\mathcal{A}_{j,\mathbf{w}_h}((\boldsymbol{\chi}_{j,h},\varphi_{j,h}),(\boldsymbol{\eta}_{j,h},\psi_{j,h}))}{\|(\boldsymbol{\eta}_{j,h},\psi_{j,h})\|} \\ & \ge\, \frac{\widetilde{\gamma}_j}{2}\,\|(\boldsymbol{\chi}_{j,h},\varphi_{j,h})\| \quad \forall\,(\boldsymbol{\chi}_{j,h},\varphi_{j,h}) \in \mathbf{H}_h\,. \end{aligned} \tag{4.17}$$

Therefore, owing to the fact that for finite dimensional linear problems, surjectivity and injectivity are equivalent, we conclude from (4.17) and Theorem 3.1 that for each $j \in \{1,2\}$ there exists a unique $(\boldsymbol{\rho}_{j,h},\phi_{j,h}) \in \mathbf{H}_h^{\boldsymbol{\rho}} \times \mathrm{H}_h^{\phi}$ satisfying

$$\mathcal{A}_{j,\mathbf{w}_h}((\boldsymbol{\rho}_{j,h},\phi_{j,h}),(\boldsymbol{\eta}_{j,h},\psi_{j,h})) \,=\, G_j(\boldsymbol{\eta}_{j,h}) \quad \forall\,(\boldsymbol{\eta}_{j,h},\psi_{j,h}) \in \mathbf{H}_h\,, \tag{4.18}$$

which means that (4.6) is well-posed. In addition, proceeding similarly to (3.29) we obtain (4.15), with $c_{\widetilde{\mathbf{S}}_h} := \max\big\{c_{\widetilde{\mathbf{S}}_{1,h}}, c_{\widetilde{\mathbf{S}}_{2,h}}\big\}$ and $c_{\widetilde{\mathbf{S}}_{j,h}} := 2\,C_{G_j}/\widetilde{\gamma}_j$, $j \in \{1,2\}$, which ends the proof. □

We now proceed to analyze the fixed-point equation (4.8). We begin with the discrete version of Lemma 3.4, whose proof, being a simple translation of the arguments proving that lemma, is omitted.

Lemma 4.3 *Let $\widetilde{r} \in (0, \widetilde{r}_0)$, with $\widetilde{r}_0 := \min\{r_1, \widetilde{r}_2\}$, r_1 as in (3.24) and $\widetilde{r}_2$ as in (4.16), and let $\mathbf{W}_{\widetilde{r}}$ be the bounded subset of $\mathbf{H}_h^{\mathbf{u}}$ defined by*

$$\mathbf{W}_{\widetilde{r}} \,:=\, \Big\{\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}} : \quad \|\mathbf{w}_h\|_{1,\Omega} \,\le\, \widetilde{r}\Big\}. \tag{4.19}$$

Assume that κ_1 and κ_2 are taken as in Lemma 3.2, and that the data satisfy

$$c_{\mathbf{T}_h}\Big\{\|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega}\big(\|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega}\big)\Big\} \,\le\, \widetilde{r}, \tag{4.20}$$

where $c_{\mathbf{T}_h} := c_{\mathbf{S}}\,\max\big\{1, c_{\widetilde{\mathbf{S}}_h}\big\}$. Then $\mathbf{T}_h(\mathbf{W}_{\widetilde{r}}) \subseteq \mathbf{W}_{\widetilde{r}}$.

Next, we address the discrete counterparts of the auxiliary Lemmas 3.5 and 3.6, whose proofs, being almost verbatim of the continuous ones, are omitted. We just remark that Lemma 4.5 below is derived using the discrete inf-sup condition (4.13). Thus, we simply state the corresponding results as follows.

Lemma 4.4 *Let $\widetilde{r} \in (0, r_1)$ with r_1 given as in (3.24). Then, with the same constant $C_{\mathbf{S}} > 0$ from (3.40), which is independent of $\widetilde{r}$, there holds*

$$\|\mathbf{S}_h(\mathbf{w}_h, \underline{\boldsymbol{\varphi}}_h) - \mathbf{S}_h(\widetilde{\mathbf{w}}_h, \widetilde{\underline{\boldsymbol{\varphi}}}_h)\|_{1,\Omega} \leq \frac{1}{\|\mathbf{i}_4\|\, r_1} \|\mathbf{S}_h(\mathbf{w}_h, \underline{\boldsymbol{\varphi}}_h)\|_{1,\Omega} \|\mathbf{w}_h - \widetilde{\mathbf{w}}_h\|_{0,4;\Omega} + C_{\mathbf{S}} \|\mathbf{g}\|_{0,4;\Omega} \|\underline{\boldsymbol{\varphi}}_h - \widetilde{\underline{\boldsymbol{\varphi}}}_h\|_{0,4;\Omega}$$

for all $(\mathbf{w}_h, \underline{\boldsymbol{\varphi}}_h),\ (\widetilde{\mathbf{w}}_h, \widetilde{\underline{\boldsymbol{\varphi}}}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\phi}$ such that $\|\mathbf{w}_h\|_{1,\Omega},\ \|\widetilde{\mathbf{w}}_h\|_{1,\Omega} \leq \widetilde{r}$.

Lemma 4.5 *Let $\widetilde{r} \in (0, \widetilde{r}_2)$ with $\widetilde{r}_2$ given as in (4.16). Then there holds*

$$\|\widetilde{\mathbf{S}}_h(\mathbf{w}_h) - \widetilde{\mathbf{S}}_h(\widetilde{\mathbf{w}})\|_{0,4;\Omega} \leq \frac{1}{\|\mathbf{i}_4\|\, \widetilde{r}_2} \|\widetilde{\mathbf{S}}_h(\mathbf{w}_h)\|_{0,4;\Omega} \|\mathbf{w}_h - \widetilde{\mathbf{w}}_h\|_{0,4;\Omega}$$

for all $\mathbf{w}_h,\ \widetilde{\mathbf{w}}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that $\|\mathbf{w}_h\|_{1,\Omega},\ \|\widetilde{\mathbf{w}}_h\|_{1,\Omega} \leq \widetilde{r}$.

As a straightforward consequence of Lemmas 4.4 and 4.5, and the continuous injection $\mathbf{i}_4$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, we can prove the Lipschitz-continuity of the operator $\mathbf{T}_h$ (cf. Lemma 3.7).

Lemma 4.6 *Let $\widetilde{r} \in (0, \widetilde{r}_0)$, with $\widetilde{r}_0 := \min\{r_1, \widetilde{r}_2\}$, r_1 as in (3.24) and $\widetilde{r}_2$ as in (4.16). Then, with the same constant $c_{\mathbf{T}_h} > 0$ from (4.20), for all $\mathbf{w}_h,\ \widetilde{\mathbf{w}}_h \in \mathbf{W}_{\widetilde{r}}$ (cf. (4.19)) there holds*

$$\|\mathbf{T}_h(\mathbf{w}_h) - \mathbf{T}_h(\widetilde{\mathbf{w}}_h)\|_{1,\Omega} \leq \frac{c_{\mathbf{T}_h}}{\widetilde{r}_0} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega} \big(2\, \|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega} \big) \Big\} \|\mathbf{w}_h - \widetilde{\mathbf{w}}_h\|_{1,\Omega}. \tag{4.21}$$

We are now in position of establishing the well-posedness of (4.2).

Theorem 4.7 *Assume the same hypothesis of Lemma 4.3 and that the data satisfy*

$$\frac{c_{\mathbf{T}_h}}{\widetilde{r}_0} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega} \big(2\, \|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega} \big) \Big\} < 1. \tag{4.22}$$

Then, the operator $\mathbf{T}_h$ has a unique fixed point $\mathbf{u}_h \in \mathbf{W}_{\widetilde{r}}$ (cf. (4.19)). Equivalently, the coupled problem (4.2) has a unique solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ and $(\boldsymbol{\rho}_{j,h}, \phi_{j,h}) \in \mathbf{H}_h^{\boldsymbol{\rho}} \times \mathrm{H}_h^{\phi}$, $j \in \{1, 2\}$, with $\mathbf{u}_h \in \mathbf{W}_{\widetilde{r}}$. Moreover, there holds

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \leq c_{\mathbf{T}_h} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma} + \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega} \big(\|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma} + \|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega} \big) \Big\} \tag{4.23}$$

and

$$\sum_{j=1}^{2} \|(\boldsymbol{\rho}_{j,h}, \phi_{j,h})\| \leq c_{\widetilde{\mathbf{S}}_h} \|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma}. \tag{4.24}$$

Proof It follows similarly to the proof of Theorem 3.9. Indeed, we first notice from Lemma 4.3 that $\mathbf{T}_h$ maps the ball $\mathbf{W}_{\widetilde{r}}$ into itself. In turn, it is easy to see from (4.21) (cf. Lemma 4.6) that $\mathbf{T}_h : \mathbf{W}_{\widetilde{r}} \to \mathbf{W}_{\widetilde{r}}$ is continuous, and hence the existence result follows from the Brouwer fixed-point theorem [14, Theorem 9.9-2]. In addition, it is clear that the estimate (4.24) follows from (4.15), whereas combining (4.9) with (4.15), we obtain (4.23) (cf. (4.20)). On the other hand, the estimate (4.21) and the assumption (4.22) show that $\mathbf{T}_h$ is a contraction mapping, which, thanks to the Banach fixed-point theorem, implies the uniqueness result and concludes the proof. □

4.3 Specific Finite Element Subspaces

In this section, we introduce specific finite element subspaces satisfying (4.1) and the crucial discrete inf-sup conditions given by hypotheses (**H.1**) and (**H.2**). These discrete spaces arise naturally as consequence of the same analysis developed in [11,25] and [9, Sect. 3] (see also [15, Sect. 5]). Then, with the same notations from Sect. 4.1, for each $T \in \mathcal{T}_h$ we let $\mathbf{RT}_k(T)$ be the local Raviart–Thomas element of order $k \geq 0$, i.e., $\mathbf{RT}_k(T) := [\mathrm{P}_k(T)]^n \oplus \mathrm{P}_k(T)\,\mathbf{x}$, where $\mathbf{x} := (x_1, \ldots, x_n)^{\mathrm{t}}$ is a generic vector of R^n and $\mathrm{P}_k(T)$ is the space of polynomials defined on T of degree $\leq k$. Then, the finite element subspaces on Ω are defined as

$$\begin{aligned} \mathbb{H}_h^{\boldsymbol{\sigma}} &:= \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div};\Omega) : \quad \mathbf{c}^{\mathrm{t}}\boldsymbol{\tau}_h|_T \in \mathbf{RT}_k(T) \quad \forall\, \mathbf{c} \in \mathrm{R}^n, \quad \forall\, T \in \mathcal{T}_h \Big\}, \\ \mathbf{H}_h^{\mathbf{u}} &:= \Big\{ \mathbf{v}_h \in \mathbf{C}(\overline{\Omega}) : \quad \mathbf{v}_h|_T \in [\mathrm{P}_{k+1}(T)]^n \quad \forall\, T \in \mathcal{T}_h \Big\}, \\ \mathbf{H}_h^{\boldsymbol{\rho}} &:= \Big\{ \boldsymbol{\eta}_h \in \mathbf{H}(\mathrm{div}_{4/3};\Omega) : \quad \boldsymbol{\eta}_h|_T \in \mathbf{RT}_k(T) \quad \forall\, T \in \mathcal{T}_h \Big\}, \\ \mathrm{H}_h^{\phi} &:= \Big\{ \psi_h \in \mathrm{L}^4(\Omega) : \quad \psi_h|_T \in \mathrm{P}_k(T) \quad \forall\, T \in \mathcal{T}_h \Big\}. \end{aligned} \tag{4.25}$$

It is clear from (4.25) that $\mathrm{div}(\mathbf{H}_h^{\boldsymbol{\rho}}) \subseteq \mathrm{H}_h^{\phi}$, and hence (4.10) becomes

$$\mathbf{V}_h = \Big\{ \boldsymbol{\eta}_h \in \mathbf{H}_h^{\boldsymbol{\rho}} : \quad \mathrm{div}(\boldsymbol{\eta}_h) = 0 \quad \text{in} \quad \Omega \Big\}.$$

This fact together with the uniform positiveness of tensors $\mathbf{Q}_j^{-1}$ (cf. (2.3)), imply that the bilinear forms a_j (cf. (2.20)) are $\mathbf{V}_h$-elliptic with the same constants α_j defined in (3.11), and thus the assumption (**H.1**) is trivially satisfied. In turn, we know from [15, Lemma 5.5] (see also [7, Lemma 4.4] or [9, Lemma 3.3] with $p = 4/3$) that there holds (**H.2**).

We end this section by collecting next the approximation properties of the finite element subspaces $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, $\mathbf{H}_h^{\boldsymbol{\rho}}$, and H_h^{ϕ} (cf. (4.25)), whose derivations can be found in [23], [26], and [9, Sect. 3.1] (see also [15, Sect. 5.5]):

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) there exists $C > 0$, independent of h, such that for each $l \in (0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div};\Omega)$ with $\mathbf{div}\,(\boldsymbol{\tau}) \in \mathbf{H}^l(\Omega)$, there holds

$$\mathrm{dist}\,(\boldsymbol{\tau}, \mathbb{H}_h^{\boldsymbol{\sigma}}) := \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div};\Omega} \leq C\, h^l \Big\{ \|\boldsymbol{\tau}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{l,\Omega} \Big\}.$$

($\mathbf{AP}_h^{\mathbf{u}}$) there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{u} \in \mathbf{H}^{l+1}(\Omega)$, there holds

$$\mathrm{dist}\,(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) := \inf_{\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}} \|\mathbf{v} - \mathbf{v}_h\|_{1,\Omega} \leq C\, h^l\, \|\mathbf{v}\|_{l+1,\Omega}.$$

($\mathbf{AP}_h^{\boldsymbol{\rho}}$) there exists $C > 0$, independent of h, such that for each $l \in (0, k+1]$, and for each $\boldsymbol{\eta} \in \mathbf{H}^l(\Omega)$ with $\mathrm{div}(\boldsymbol{\eta}) \in \mathrm{W}^{l,4/3}(\Omega)$, there holds

$$\mathrm{dist}\,(\boldsymbol{\eta}, \mathbf{H}_h^{\boldsymbol{\rho}}) := \inf_{\boldsymbol{\eta}_h \in \mathbf{H}_h^{\boldsymbol{\rho}}} \|\boldsymbol{\eta} - \boldsymbol{\eta}_h\|_{\mathrm{div}_{4/3};\Omega} \leq C\, h^l \Big\{ \|\boldsymbol{\eta}\|_{l,\Omega} + \|\mathrm{div}(\boldsymbol{\eta})\|_{l,4/3;\Omega} \Big\}.$$

$(\mathbf{AP}_h^{\phi})$ there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\psi \in \mathrm{W}^{l,4}(\Omega)$, there holds

$$\operatorname{dist}(\psi, \mathrm{H}_h^{\phi}) := \inf_{\psi_h \in \mathrm{H}_h^{\phi}} \|\psi - \psi_h\|_{0,4;\Omega} \leq C\, h^l\, \|\phi\|_{l,4;\Omega}.$$

5 A Priori Error Analysis

In this section, we first derive a Céa estimate for our Galerkin scheme with arbitrary finite element subspaces satisfying the hypothesis stated in Sect. 4.2. Next, using the specific discrete spaces stated in Sect. 4.3, we establish the corresponding rates of convergence. In fact, let $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega)$ and $(\boldsymbol{\rho}_j, \phi_j) \in \mathbf{H}(\mathrm{div}_{4/3}; \Omega) \times \mathrm{L}^4(\Omega)$, $j \in \{1, 2\}$, with $\mathbf{u} \in \mathbf{W}_r$ (cf. (3.37)), be the unique solution of the coupled problem (2.22), and let $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ and $(\boldsymbol{\rho}_{j,h}, \phi_{j,h}) \in \mathbf{H}_h^{\boldsymbol{\rho}} \times \mathrm{H}_h^{\phi}$, $j \in \{1, 2\}$, with $\mathbf{u}_h \in \mathbf{W}_{\widetilde{r}}$ (cf. (4.19)), be the unique solution of the discrete coupled problem (4.2). Then, we are interested in obtaining an a priori estimate for the error

$$\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\| + \sum_{j=1}^{2} \|(\boldsymbol{\rho}_j, \phi_j) - (\boldsymbol{\rho}_{j,h}, \phi_{j,h})\|.$$

To this end, we establish next an ad-hoc Strang-type estimate. In what follows, given a subspace X_h of a generic Banach space $(X, \|\cdot\|_X)$, we set for each $x \in X$

$$\operatorname{dist}(x, X_h) := \inf_{x_h \in X_h} \|x - x_h\|_X.$$

Lemma 5.1 *Let* H *be a reflexive Banach space, and let* $a : \mathrm{H} \times \mathrm{H} \to \mathrm{R}$ *be a bounded bilinear form with induced operator* $\mathcal{A} \in \mathcal{L}(\mathrm{H}, \mathrm{H}')$*, such that* a *satisfies the hypotheses of Theorem 3.1. Furthermore, let* $\{\mathrm{H}_h\}_{h>0}$ *be a sequence of finite dimensional subspaces of* H*, and for each* $h > 0$*, consider a bounded bilinear form* $a_h : \mathrm{H}_h \times \mathrm{H}_h \to \mathrm{R}$ *with induced operator* $\mathcal{A}_h \in \mathcal{L}(\mathrm{H}_h, \mathrm{H}_h')$*, such that* $a_h|_{\mathrm{H}_h \times \mathrm{H}_h}$ *satisfies the hypotheses of Theorem 3.1 as well, with constant* $\widetilde{\alpha}$ *independent of* h*. In turn, given* $F \in \mathrm{H}'$*, and a sequence of functionals* $\{F_h\}_{h>0}$*, with* $F_h \in \mathrm{H}_h'$ *for each* $h > 0$*, we let* $u \in \mathrm{H}$ *and* $u_h \in \mathrm{H}_h$ *be the unique solutions, respectively, to the problems*

$$a(u, v) = F(v) \quad \forall\, v \in \mathrm{H}, \tag{5.1}$$

and

$$a_h(u_h, v_h) = F_h(v_h) \quad \forall\, v_h \in \mathrm{H}_h. \tag{5.2}$$

Then, there holds

$$\|u - u_h\|_{\mathrm{H}} \leq C_{S,1} \operatorname{dist}(u, \mathrm{H}_h) + C_{S,2} \left\{ \|F - F_h\|_{\mathrm{H}_h'} + \|a(u, \cdot) - a_h(u, \cdot)\|_{\mathrm{H}_h'} \right\}, \tag{5.3}$$

where $C_{S,1}$ *and* $C_{S,2}$ *are the positive constants given by*

$$C_{S,1} := \left(1 + \frac{2\, \|\mathcal{A}\|}{\widetilde{\alpha}} + \frac{\|\mathcal{A}_h\|}{\widetilde{\alpha}}\right) \quad \text{and} \quad C_{S,2} := \frac{1}{\widetilde{\alpha}}. \tag{5.4}$$

Proof We proceed similarly to the proof of [33, Theorem 11.1]. Indeed, employing the inf-sup condition of the bilinear form a_h (cf. (3.1)), the identities (5.1) and (5.2), and the continuity of $\mathcal{A}$, $\mathcal{A}_h$, F, and F_h, we obtain

$$\|u - u_h\|_{\mathrm{H}} \le \left(1 + \frac{\|\mathcal{A}\|}{\widetilde{\alpha}}\right) \operatorname{dist}(u, \mathrm{H}_h) + \frac{1}{\widetilde{\alpha}} \left\{ \|F - F_h\|_{\mathrm{H}_h'} + \inf_{w_h \in \mathrm{H}_h} \|a(w_h, \cdot) - a_h(w_h, \cdot)\|_{\mathrm{H}_h'} \right\}, \tag{5.5}$$

where

$$\|a(w_h, \cdot) - a_h(w_h, \cdot)\|_{\mathrm{H}_h'} := \sup_{0 \neq v_h \in \mathrm{H}_h} \frac{a(w_h, v_h) - a_h(w_h, v_h)}{\|v_h\|_{\mathrm{H}}}.$$

Then, inspired by [15, Lemma 6.1], we notice that

$$\begin{aligned} &\frac{a(w_h, v_h) - a_h(w_h, v_h)}{\|v_h\|_{\mathrm{H}}} \\ &= \frac{a(w_h, v_h) - a(u, v_h) + a(u, v_h) - a_h(u, v_h) + a_h(u, v_h) - a_h(w_h, v_h)}{\|v_h\|_{\mathrm{H}}} \\ &\le \big(\|\mathcal{A}\| + \|\mathcal{A}_h\|\big) \|u - w_h\|_{\mathrm{H}} + \frac{a(u, v_h) - a_h(u, v_h)}{\|v_h\|_{\mathrm{H}}}, \end{aligned}$$

which implies

$$\inf_{w_h \in \mathrm{H}_h} \|a(w_h, \cdot) - a_h(w_h, \cdot)\|_{\mathrm{H}_h'} \le \big(\|\mathcal{A}\| + \|\mathcal{A}_h\|\big) \operatorname{dist}(u, \mathrm{H}_h) + \|a(u, \cdot) - a_h(u, \cdot)\|_{\mathrm{H}_h'}. \tag{5.6}$$

Hence, replacing (5.6) back into (5.5) we obtain (5.3) and conclude the proof. □

In order to apply Lemma 5.1, we now observe that the problems (2.22) and (4.2) can be rewritten as two pairs of corresponding continuous and discrete formulations, namely

$$\begin{aligned} (A + B_{\mathbf{u}})((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \mathbf{v})) &= (F_{\mathrm{D}} + F_{\underline{\boldsymbol{\phi}}})(\boldsymbol{\tau}, \mathbf{v}) \quad \forall (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}_0(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega), \\ (A + B_{\mathbf{u}_h})((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)) &= (F_{\mathrm{D}} + F_{\underline{\boldsymbol{\phi}}_h})(\boldsymbol{\tau}_h, \mathbf{v}_h) \ \forall (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}, \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \mathcal{A}_{j,\mathbf{u}}((\boldsymbol{\rho}_j, \phi_j), (\boldsymbol{\eta}_j, \psi_j)) &= G_j(\boldsymbol{\eta}_j) \quad \forall (\boldsymbol{\eta}_j, \psi_j) \in \mathbf{H}(\mathrm{div}_{4/3}; \Omega) \times \mathrm{L}^4(\Omega), \\ \mathcal{A}_{j,\mathbf{u}_h}((\boldsymbol{\rho}_{j,h}, \phi_{j,h}), (\boldsymbol{\eta}_{j,h}, \psi_{j,h})) &= G_j(\boldsymbol{\eta}_{j,h}) \ \forall (\boldsymbol{\eta}_{j,h}, \psi_{j,h}) \in \mathbf{H}_h^{\boldsymbol{\rho}} \times \mathrm{H}_h^{\phi}, \end{aligned} \tag{5.8}$$

where the forms $\mathcal{A}_{j,\mathbf{u}}$ and $\mathcal{A}_{j,\mathbf{u}_h}$ are defined as in (3.30).

The following lemma provides a preliminary estimate for the error $\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|$.

Lemma 5.2 *There exists a positive constant $\widetilde{C}_{S,1}$, independent of h, such that*

$$\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \le \widetilde{C}_{S,1} \left\{ \operatorname{dist}(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}) + \operatorname{dist}(\mathbf{u}, \mathbf{H}_h^{\mathbf{u}}) \right\} + \frac{1}{r_1} \|\mathbf{u}\|_{1,\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + C_{\mathbf{S}} \|\mathbf{g}\|_{0,4;\Omega} \|\underline{\boldsymbol{\phi}} - \underline{\boldsymbol{\phi}}_h\|_{0,4;\Omega}, \tag{5.9}$$

where r_1 and $C_{\mathbf{S}}$ are defined in (3.24) and (3.40), respectively.

Proof From (3.25) we know that the bounded bilinear forms $A + B_{\mathbf{u}}$ and $A + B_{\mathbf{u}_h}$ are elliptic with the same constant $\frac{\alpha_A}{2}$. In addition, it is clear that $F_D + F_{\underline{\boldsymbol{\phi}}}$ and $F_D + F_{\underline{\boldsymbol{\phi}}_h}$ are bounded linear functionals in $\mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$ and $\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$, respectively. Then, applying Lemma 5.1 to the context given by (5.7), we find that

$$\begin{aligned} \|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \,\le\, & \widetilde{C}_{S,1} \left\{ \operatorname{dist}(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}) + \operatorname{dist}(\mathbf{u}, \mathbf{H}_h^{\mathbf{u}}) \right\} \\ & + \widetilde{C}_{S,2} \left\{ \|F_{\underline{\boldsymbol{\phi}}} - F_{\underline{\boldsymbol{\phi}}_h}\|_{(\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}})'} + \|B_{\mathbf{u}}((\boldsymbol{\sigma}, \mathbf{u}), \cdot) - B_{\mathbf{u}_h}((\boldsymbol{\sigma}, \mathbf{u}), \cdot)\|_{(\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}})'} \right\}, \end{aligned} \tag{5.10}$$

where, according to (5.4), the estimates (3.3), (3.4), and the fact that $\mathbf{u} \in \mathbf{W}_r$, $\mathbf{u}_h \in \mathbf{W}_{\widetilde{r}}$ (cf. (3.37), (4.19)), there holds

$$\widetilde{C}_{S,1} := 1 + \frac{2}{\alpha_A} \left(3\, C_A + \mathrm{F}\, \|\mathbf{K}\|_\infty \, \|\mathbf{i}_4\|^2 \left(2\, r + \widetilde{r} \right) \right) \quad \text{and} \quad \widetilde{C}_{S,2} := \frac{2}{\alpha_A}. \tag{5.11}$$

Next, using the definition of $F_{\underline{\boldsymbol{\varphi}}}$ (cf. (2.17)) and its continuity bound (cf. (3.9)), and applying Hölder's inequality, we readily get

$$\|F_{\underline{\boldsymbol{\phi}}} - F_{\underline{\boldsymbol{\phi}}_h}\|_{(\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}})'} \,\le\, C_F\, \|\mathbf{g}\|_{0,4;\Omega}\, \|\underline{\boldsymbol{\phi}} - \underline{\boldsymbol{\phi}}_h\|_{0,4;\Omega}. \tag{5.12}$$

In turn, from the continuity bound of $B_{\mathbf{w}}$ (cf. (3.4)) and the fact that $\big| |\mathbf{u}| - |\mathbf{u}_h| \big| \le |\mathbf{u} - \mathbf{u}_h|$, it follows that

$$\|B_{\mathbf{u}}((\boldsymbol{\sigma}, \mathbf{u}), \cdot) - B_{\mathbf{u}_h}((\boldsymbol{\sigma}, \mathbf{u}), \cdot)\|_{(\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}})'} \,\le\, \mathrm{F}\, \|\mathbf{K}\|_\infty \, \|\mathbf{i}_4\|^2 \, \|\mathbf{u}\|_{1,\Omega}\, \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}. \tag{5.13}$$

Thus, replacing (5.12) and (5.13) back into (5.10), and using the explicit expression of $\widetilde{C}_{S,2}$ (cf. (5.11)), we find that

$$\begin{aligned} \|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \,\le\, & \widetilde{C}_{S,1} \left\{ \operatorname{dist}(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}) + \operatorname{dist}(\mathbf{u}, \mathbf{H}_h^{\mathbf{u}}) \right\} \\ & + \frac{2\, \mathrm{F}\, \|\mathbf{K}\|_\infty \, \|\mathbf{i}_4\|^2}{\alpha_A} \, \|\mathbf{u}\|_{1,\Omega}\, \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \frac{2\, C_F}{\alpha_A}\, \|\mathbf{g}\|_{0,4;\Omega}\, \|\underline{\boldsymbol{\phi}} - \underline{\boldsymbol{\phi}}_h\|_{0,4;\Omega}, \end{aligned}$$

which, together with the definition of r_1 and $C_{\mathbf{S}}$ (cf. (3.24) and (3.40)), implies (5.9) and concludes the proof. □

Next, we have the following result concerning $\|(\boldsymbol{\rho}_j, \phi_j) - (\boldsymbol{\rho}_{j,h}, \phi_{j,h})\|$.

Lemma 5.3 *There exists a positive constant $\widehat{C}_{S,1}$, independent of h, such that*

$$\begin{aligned} \sum_{j=1}^{2} \|(\boldsymbol{\rho}_j, \phi_j) - (\boldsymbol{\rho}_{j,h}, \phi_{j,h})\| \le \widehat{C}_{S,1} & \sum_{j=1}^{2} \left(\operatorname{dist}(\boldsymbol{\rho}_j, \mathbf{H}_h^{\boldsymbol{\rho}}) + \operatorname{dist}(\phi_j, \mathrm{H}_h^{\phi}) \right) \\ & + \frac{c_{\widetilde{\mathbf{S}}}}{\widetilde{r}_2}\, \|\underline{\boldsymbol{\phi}}_D\|_{1/2,\Gamma}\, \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}, \end{aligned} \tag{5.14}$$

where $c_{\widetilde{\mathbf{S}}}$ is defined at the end of the proof of Lemma 3.3, and $\widetilde{r}_2$ is given by (4.16).

Proof We first recall from Sects. 3.3 and 4.2 that the forms $\mathcal{A}_{j,\mathbf{u}}$ and $\mathcal{A}_{j,\mathbf{u}_h}$, $j \in \{1, 2\}$, satisfy the hypotheses of Lemma 5.1 on $\mathbf{H} := \mathbf{H}(\mathrm{div}_{4/3};\Omega) \times \mathrm{L}^4(\Omega)$ and $\mathbf{H}_h := \mathbf{H}_h^{\boldsymbol{\rho}} \times \mathrm{H}_h^{\phi}$, respectively, the latter with constant $\widetilde{\alpha} = \frac{\widetilde{\gamma}_j}{2}$ (cf. (4.14)). Therefore, applying Lemma 5.1 to the context (5.8), we arrive at

$$\begin{aligned} \|(\boldsymbol{\rho}_j, \phi_j) - (\boldsymbol{\rho}_{j,h}, \phi_{j,h})\| \le\, & \widehat{C}_{S,1}^{j} \left(\operatorname{dist}(\boldsymbol{\rho}_j, \mathbf{H}_h^{\boldsymbol{\rho}}) + \operatorname{dist}(\phi_j, \mathrm{H}_h^{\phi}) \right) \\ & + \widehat{C}_{S,2}^{j}\, \|\mathcal{A}_{j,\mathbf{u}}((\boldsymbol{\rho}_j, \phi_j), \cdot) - \mathcal{A}_{j,\mathbf{u}_h}((\boldsymbol{\rho}_j, \phi_j), \cdot)\|_{\mathbf{H}_h'}, \end{aligned} \tag{5.15}$$

where, according to (5.4), the definition of the form $\mathcal{A}_{j,\mathbf{w}}$ (cf. (3.30)), the estimates (3.5), (3.6), and (3.7), and the fact that $\mathbf{u} \in \mathbf{W}_r$, $\mathbf{u}_h \in \mathbf{W}_{\widetilde{r}}$ (cf. (3.37), (4.19)), there holds

$$\widehat{C}^j_{S,1} := 1 + \frac{2}{\widetilde{\gamma}_j}\left(6 + \|\mathbf{Q}_j^{-1}\|_\infty\left(3 + \mathrm{R}_j\,\|\mathbf{i}_4\|\left(2\,r + \widetilde{r}\right)\right)\right) \quad \text{and} \quad \widehat{C}^j_{S,2} := \frac{2}{\widetilde{\gamma}_j}, \quad j \in \{1,2\}. \tag{5.16}$$

Next, in order to bound the last term on the right-hand side of (5.15), we notice that the definition of the form $\mathcal{A}_{j,\mathbf{w}}$ (cf. (3.30)) and the continuity bound of c_j (cf. (3.7)), give

$$\begin{aligned} &\left|\mathcal{A}_{j,\mathbf{u}}((\boldsymbol{\rho}_j,\phi_j),(\boldsymbol{\eta}_{j,h},\psi_{j,h})) - \mathcal{A}_{j,\mathbf{u}_h}((\boldsymbol{\rho}_j,\phi_j),(\boldsymbol{\eta}_{j,h},\psi_{j,h}))\right| = \left|c_j(\mathbf{u}-\mathbf{u}_h;\phi_j,\boldsymbol{\eta}_{j,h})\right| \\ &\quad \le \mathrm{R}_j\,\|\mathbf{Q}_j^{-1}\|_\infty\,\|\mathbf{i}_4\|\,\|\phi_j\|_{0,4;\Omega}\,\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}\,\|(\boldsymbol{\eta}_{j,h},\psi_{j,h})\| \quad \forall\,(\boldsymbol{\eta}_{j,h},\psi_{j,h}) \in \mathbf{H}_h, \end{aligned}$$

which, together with (5.15), the explicit expression of $\widehat{C}^j_{S,2}$ (cf. (5.16)) and the bound of $\|\phi_j\|_{0,4;\Omega}$ (cf. (3.35)), yields

$$\begin{aligned} \|(\boldsymbol{\rho}_j,\phi_j) - (\boldsymbol{\rho}_{j,h},\phi_{j,h})\| &\le \widehat{C}^j_{S,1}\left(\operatorname{dist}(\boldsymbol{\rho}_j,\mathbf{H}_h^{\boldsymbol{\rho}}) + \operatorname{dist}(\phi_j,\mathrm{H}_h^{\phi})\right) \\ &+ \frac{2\,\mathrm{R}_j\,\|\mathbf{Q}_j^{-1}\|_\infty\,\|\mathbf{i}_4\|}{\widetilde{\gamma}_j}\,c_{\widetilde{\mathbf{S}}_j}\,\|\phi_{j,\mathrm{D}}\|_{1/2,\Gamma}\,\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}. \end{aligned} \tag{5.17}$$

Then, recalling the definitions of $c_{\widetilde{\mathbf{S}}}$ and $\widetilde{r}_2$, it is easy to see that (5.17) implies (5.14) with $\widehat{C}_{S,1} := \max\left\{\widehat{C}^1_{S,1}, \widehat{C}^2_{S,1}\right\}$, thus concluding the proof. □

The required Céa estimate will now follow from (5.9) and (5.14). In fact, we first bound $\|\underline{\boldsymbol{\phi}} - \underline{\boldsymbol{\phi}}_h\|_{0,4;\Omega}$ in (5.9) by the right hand side of (5.14). Next, in order to obtain an explicit expression in terms of data, we bound $\|\mathbf{u}\|_{1,\Omega}$ as in (3.44) instead of directly by r, that is

$$\|\mathbf{u}\|_{1,\Omega} \le c_{\mathbf{T}}\left\{\|\mathbf{u}_\mathrm{D}\|_{0,\Gamma} + \|\mathbf{u}_\mathrm{D}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega}\left(\|\underline{\boldsymbol{\phi}}_\mathrm{D}\|_{1/2,\Gamma} + \|\underline{\boldsymbol{\phi}}_\mathrm{r}\|_{0,4;\Omega}\right)\right\},$$

which, together with the fact that $C_{\mathbf{S}}c_{\widetilde{\mathbf{S}}}$ and $1/r_1$, $1/\widetilde{r}_2$ are bounded by $c_{\mathbf{T}}$ and $1/\widetilde{r}_0$, respectively, allows us to deduce from (5.9) that

$$\begin{aligned} \|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\| &\le \widetilde{C}_{S,1}\left\{\operatorname{dist}(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}) + \operatorname{dist}(\mathbf{u},\mathbf{H}_h^{\mathbf{u}})\right\} \\ &+ \widehat{C}_{S,1}\,C_{\mathbf{S}}\,\|\mathbf{g}\|_{0,4;\Omega}\sum_{j=1}^{2}\left(\operatorname{dist}(\boldsymbol{\rho}_j,\mathbf{H}_h^{\boldsymbol{\rho}}) + \operatorname{dist}(\phi_j,\mathrm{H}_h^{\phi})\right) \\ &+ \frac{c_{\mathbf{T}}}{\widetilde{r}_0}\left\{\|\mathbf{u}_\mathrm{D}\|_{0,\Gamma} + \|\mathbf{u}_\mathrm{D}\|_{1/2,\Gamma} + \|\mathbf{g}\|_{0,4;\Omega}\left(2\,\|\underline{\boldsymbol{\phi}}_\mathrm{D}\|_{1/2,\Gamma} + \|\underline{\boldsymbol{\phi}}_\mathrm{r}\|_{0,4;\Omega}\right)\right\}\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}. \end{aligned} \tag{5.18}$$

Thus, imposing the term that multiplies $\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$ in (5.18) to be sufficiently small, say $\le 1/2$, we derive the a priori error estimate for $\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|$, which, employed then to bound the last term on the right-hand side of (5.14), provides the corresponding upper bound for $\sum_{j=1}^{2}\|(\boldsymbol{\rho}_j,\phi_j) - (\boldsymbol{\rho}_{j,h},\phi_{j,h})\|$. More precisely, we have proved the following result.

Theorem 5.4 *Assume that the data* $\mathbf{u}_{\mathrm{D}}$*,* $\underline{\boldsymbol{\phi}}_{\mathrm{D}}$*, and* $\underline{\boldsymbol{\phi}}_{\mathrm{r}}$ *satisfy*

$$\frac{c_{\mathbf{T}}}{\widetilde{r}_0}\left\{\|\mathbf{u}_{\mathrm{D}}\|_{0,\Gamma}+\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}+\|\mathbf{g}\|_{0,4;\Omega}\left(2\,\|\underline{\boldsymbol{\phi}}_{\mathrm{D}}\|_{1/2,\Gamma}+\|\underline{\boldsymbol{\phi}}_{\mathrm{r}}\|_{0,4;\Omega}\right)\right\}\leq\frac{1}{2}.$$

Then, there exists a positive constant C, independent of h, but depending on ν, $\mathtt{F}$, $\mathtt{R}_j$, $\|\mathbf{i}_4\|$, $\|\mathbf{K}\|_\infty$, $\|\mathbf{Q}_j^{-1}\|_\infty$, $\|\mathbf{g}\|_{0,4;\Omega}$, α_A, γ_j, $\widetilde{\gamma}_j$, $j\in\{1,2\}$*, and the datum* $\underline{\boldsymbol{\phi}}_{\mathrm{D}}$*, such that*

$$\begin{aligned}&\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|+\sum_{j=1}^{2}\|(\boldsymbol{\rho}_j,\phi_j)-(\boldsymbol{\rho}_{j,h},\phi_{j,h})\|\\&\quad\leq C\left\{\mathrm{dist}\,(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}})+\mathrm{dist}\,(\mathbf{u},\mathbf{H}_h^{\mathbf{u}})+\sum_{j=1}^{2}\Big(\mathrm{dist}\,(\boldsymbol{\rho}_j,\mathbf{H}_h^{\boldsymbol{\rho}})+\mathrm{dist}\,(\phi_j,\mathrm{H}_h^{\phi})\Big)\right\}.\end{aligned}\tag{5.19}$$

Now, in order to approximate the pressure, the velocity gradient, and the heat and diffusive vectors, we propose, motivated by (2.7) and (2.9), the expressions

$$p_h=-\frac{1}{n}\,\mathrm{tr}(\boldsymbol{\sigma}_h),\quad \big(\nabla\mathbf{u}\big)_h=\frac{1}{\nu}\,\boldsymbol{\sigma}_h^{\mathrm{d}},\quad\text{and}\quad \widetilde{\boldsymbol{\rho}}_{j,h}=\boldsymbol{\rho}_{j,h}+\mathtt{R}_j\,\phi_{j,h}\,\mathbf{u}_h,\ j\in\{1,2\}\,,\tag{5.20}$$

respectively, with $(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}}$ and $(\boldsymbol{\rho}_{j,h},\phi_{j,h})\in\mathbf{H}_h^{\boldsymbol{\rho}}\times\mathrm{H}_h^{\phi}$, $j\in\{1,2\}$, being the unique solution of the discrete problem (4.2). The corresponding error estimates are established in the following lemma.

Lemma 5.5 *Assume that the hypotheses of Theorem 5.4 hold. Let* p, $\nabla\mathbf{u}$*, and* $\widetilde{\boldsymbol{\rho}}_j$, $j\in\{1,2\}$*, be given by (2.7)–(2.9). In addition, let* p_h, $\big(\nabla\mathbf{u}\big)_h$*, and* $\widetilde{\boldsymbol{\rho}}_{j,h}$, $j\in\{1,2\}$*, be the discrete counterparts introduced in (5.20). Then, there exists a positive constant C, independent of h, but depending on* ν, $\mathtt{F}$, $\mathtt{R}_j$, $\|\mathbf{i}_4\|$, $\|\mathbf{K}\|_\infty$, $\|\mathbf{Q}_j^{-1}\|_\infty$, $\|\mathbf{g}\|_{0,4;\Omega}$, α_A, γ_j, $\widetilde{\gamma}_j$, $j\in\{1,2\}$*, and the datum* $\underline{\boldsymbol{\phi}}_{\mathrm{D}}$*, such that*

$$\begin{aligned}&\|p-p_h\|_{0,\Omega}+\left\|\nabla\mathbf{u}-\big(\nabla\mathbf{u}\big)_h\right\|_{0,\Omega}+\sum_{j=1}^{2}\|\widetilde{\boldsymbol{\rho}}_j-\widetilde{\boldsymbol{\rho}}_{j,h}\|_{0,\Omega}\\&\quad\leq C\left\{\mathrm{dist}\,(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}})+\mathrm{dist}\,(\mathbf{u},\mathbf{H}_h^{\mathbf{u}})+\sum_{j=1}^{2}\Big(\mathrm{dist}\,(\boldsymbol{\rho}_j,\mathbf{H}_h^{\boldsymbol{\rho}})+\mathrm{dist}\,(\phi_j,\mathrm{H}_h^{\phi})\Big)\right\}.\end{aligned}$$

Proof It follows from (2.9) and (5.20), adding and subtracting $\mathtt{R}_j\,\phi_j\,\mathbf{u}_h$, that

$$\widetilde{\boldsymbol{\rho}}_j-\widetilde{\boldsymbol{\rho}}_{j,h}=(\boldsymbol{\rho}_j-\boldsymbol{\rho}_{j,h})+\mathtt{R}_j\,\phi_j\,(\mathbf{u}-\mathbf{u}_h)+\mathtt{R}_j\,(\phi_j-\phi_{j,h})\,\mathbf{u}_h,\quad j\in\{1,2\}.$$

Next, employing the triangle and Hölder inequalities, and the continuous injection $\mathbf{i}_4$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, we find that

$$\begin{aligned}\|\widetilde{\boldsymbol{\rho}}_j-\widetilde{\boldsymbol{\rho}}_{j,h}\|_{0,\Omega}&\leq\|\boldsymbol{\rho}_j-\boldsymbol{\rho}_{j,h}\|_{0,\Omega}\\&\quad+\mathtt{R}_j\,\|\mathbf{i}_4\|\Big(\|\phi_j\|_{0,4;\Omega}\,\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}+\|\mathbf{u}_h\|_{1,\Omega}\,\|\phi_j-\phi_{j,h}\|_{0,4;\Omega}\Big),\end{aligned}$$

which, together with the estimate (3.35) bounding $\|\phi_j\|_{0,4;\Omega}$ and the fact that $\mathbf{u}_h\in\mathbf{W}_{\widetilde{r}}$ (cf. (4.19)), implies that there exists a positive constant C, depending only on

ν, $\mathtt{F}$, $\mathtt{R}_j$, $\|\mathbf{i}_4\|$, $\|\mathbf{K}\|_\infty$, $\|\mathbf{Q}_j^{-1}\|_\infty$, α_A, $\widetilde{\gamma}_j$, $j \in \{1,2\}$, and the datum $\underline{\boldsymbol{\phi}}_{\mathrm{D}}$, all of them independent of h, such that

$$\|p - p_h\|_{0,\Omega} + \left\|\nabla\mathbf{u} - \left(\nabla\mathbf{u}\right)_h\right\|_{0,\Omega} + \sum_{j=1}^{2} \|\widetilde{\boldsymbol{\rho}}_j - \widetilde{\boldsymbol{\rho}}_{j,h}\|_{0,\Omega}$$
$$\leq C\left\{\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\| + \sum_{j=1}^{2}\|(\boldsymbol{\rho}_j,\phi_j) - (\boldsymbol{\rho}_{j,h},\phi_{j,h})\|\right\}.$$

Then, the result is a straightforward consequence of the foregoing inequality and Theorem 5.4. □

Finally we complete our a priori error analysis with the corresponding rates of convergence of our Galerkin scheme (4.2), for which we supose in the sequel that the finite element subspaces specified in Sect. 4.3 are employed.

Theorem 5.6 *In addition to the hypotheses of Theorems 3.9, 4.7 and 5.4, assume that there exists $l \in (0, k+1]$ such that $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div};\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{H}^l(\Omega)$, $\mathbf{u} \in \mathbf{H}^{l+1}(\Omega)$, and such that for each $j \in \{1,2\}$, $\boldsymbol{\rho}_j \in \mathbf{H}^l(\Omega)$, $\mathrm{div}(\boldsymbol{\rho}_j) \in \mathrm{W}^{l,4/3}(\Omega)$, and $\phi_j \in \mathrm{W}^{l,4}(\Omega)$. Then, there exists $C > 0$, independent of h, such that*

$$\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\| + \sum_{j=1}^{2}\|(\boldsymbol{\rho}_j,\phi_j) - (\boldsymbol{\rho}_{j,h},\phi_{j,h})\|$$
$$\leq C\,h^l\left\{\|\boldsymbol{\sigma}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,\Omega} + \|\mathbf{u}\|_{l+1,\Omega} + \sum_{j=1}^{2}\left(\|\boldsymbol{\rho}_j\|_{l,\Omega} + \|\mathrm{div}(\boldsymbol{\rho}_j)\|_{l,4/3;\Omega} + \|\phi_j\|_{l,4;\Omega}\right)\right\}.$$

Proof It follows from the Céa estimate (5.19) and the approximation properties ($\mathbf{AP}_h^{\boldsymbol{\sigma}}$), ($\mathbf{AP}_h^{\mathbf{u}}$), ($\mathbf{AP}_h^{\boldsymbol{\rho}}$) and ($\mathbf{AP}_h^{\phi}$) specified in Sect. 4.3. □

Consequently, from Lemma 5.5 and Theorem 5.6 we obtain the optimal convergence of the post-processed unknowns introduced in (5.20).

Lemma 5.7 *Let $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega)$ and $(\boldsymbol{\rho}_j,\phi_j) \in \mathbf{H}(\mathrm{div}_{4/3};\Omega) \times \mathrm{L}^4(\Omega)$, $j \in \{1,2\}$, be the unique solution of the continuous problem (2.22), and let p, $\nabla\mathbf{u}$, and $\widetilde{\boldsymbol{\rho}}_j$ given by (2.7)–(2.9). In addition, let p_h, $\left(\nabla\mathbf{u}\right)_h$, and $\widetilde{\boldsymbol{\rho}}_{j,h}$ be the discrete counterparts introduced in (5.20). Assume that the hypotheses of Theorem 5.6 hold. Then, there exists $C > 0$, independent of h, such that*

$$\|p - p_h\|_{0,\Omega} + \left\|\nabla\mathbf{u} - \left(\nabla\mathbf{u}\right)_h\right\|_{0,\Omega} + \sum_{j=1}^{2} \|\widetilde{\boldsymbol{\rho}}_j - \widetilde{\boldsymbol{\rho}}_{j,h}\|_{0,\Omega}$$
$$\leq C\,h^l\left\{\|\boldsymbol{\sigma}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,\Omega} + \|\mathbf{u}\|_{l+1,\Omega} + \sum_{j=1}^{2}\left(\|\boldsymbol{\rho}_j\|_{l,\Omega} + \|\mathrm{div}(\boldsymbol{\rho}_j)\|_{l,4/3;\Omega} + \|\phi_j\|_{l,4;\Omega}\right)\right\}.$$

6 Numerical Results

In this section we present some examples illustrating the performance of our augmented fully-mixed finite element scheme (4.2), and confirming the rates of convergence provided by Theorem 5.6 and Lemma 5.7. Our implementation is based on a `FreeFem++` code [27], in conjunction with the direct linear solver `UMFPACK` [19]. A Newton–Raphson algorithm with a fixed tolerance $\mathsf{tol} = 1\mathrm{E}-6$ is used for the resolution of the nonlinear problem (4.2). As usual, the iterative method is finished when the relative error between two consecutive iterations of the complete coefficient vector, namely $\mathbf{coeff}^{m+1}$ and $\mathbf{coeff}^{m}$, is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1}-\mathbf{coeff}^{m}\|_{\ell^2}}{\|\mathbf{coeff}^{m+1}\|_{\ell^2}} \le \mathsf{tol}$$

where $\|\cdot\|_{\ell^2}$ is the standard ℓ^2-norm in $\mathrm{R}^{\mathtt{DOF}}$ with `DOF` denoting the total number of degrees of freedom generated by the finite element subspaces. The condition of zero-average pressure (translated in terms of the trace of $\boldsymbol{\sigma}$) is imposed through a real Lagrange multiplier.

Errors between exact and approximate solutions are denoted as

$$\mathsf{e}(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div};\Omega}, \quad \mathsf{e}(\mathbf{u}) = \|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}, \quad \mathsf{e}(p) = \|p-p_h\|_{0,\Omega},$$

$$\mathsf{e}(\nabla\mathbf{u}) = \left\|\nabla\mathbf{u}-\big(\nabla\mathbf{u}\big)_h\right\|_{0,\Omega}, \quad \mathsf{e}(\boldsymbol{\rho}_j) = \|\boldsymbol{\rho}_j-\boldsymbol{\rho}_{j,h}\|_{\mathrm{div}_{4/3};\Omega},$$

$$\mathsf{e}(\phi_j) = \|\phi_j-\phi_{j,h}\|_{0,4;\Omega}, \quad \mathsf{e}(\widetilde{\boldsymbol{\rho}}_j) = \|\widetilde{\boldsymbol{\rho}}_j-\widetilde{\boldsymbol{\rho}}_{j,h}\|_{0,\Omega}, \quad j\in\{1,2\}.$$

In turn, we let $\mathsf{r}(\cdot)$ be their corresponding rates of convergence, that is

$$\mathsf{r}(\diamond) := \frac{\log(\mathsf{e}(\diamond)/\mathsf{e}'(\diamond))}{\log(h/h')}, \quad \text{for each } \diamond\in\left\{\boldsymbol{\sigma},\mathbf{u},p,\nabla\mathbf{u},\boldsymbol{\rho}_j,\phi_j,\widetilde{\boldsymbol{\rho}}_j\right\},$$

where h and h' denote two consecutive meshsizes with errors $\mathsf{e}(\diamond)$ and $\mathsf{e}'(\diamond)$, respectively.

The examples to be considered in this section are described next. In all of them, for the sake of simplicity, we choose the parameters $\nu = 1$, $\varrho = 1$, $\mathtt{R}_1 = 1$, $\mathtt{R}_2 = 1$, and $\underline{\boldsymbol{\phi}}_{\mathrm{r}} = (0,0)$. In turn, and according to (3.36), the stabilization parameters are taken as $\kappa_1 = \nu$ and $\kappa_2 = \nu/2$. In addition, in the first two examples the tensors $\mathbf{K}$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ are taken as the identity matrix $\mathbb{I}$, which satisfy (2.3).

Example 1: 2D Domain with Different Values of the Parameter F

In this first example, we corroborate the rates of convergence in a two dimensional domain and also study the performance of the numerical method with respect to the number of Newton iterations required to achieve certain tolerance when different values of the parameter $\mathtt{F}$ are given. The domain is the square $\Omega = (-1,1)^2$. We consider the potential type gravitational acceleration $\mathbf{g} = (0,-1)^{\mathrm{t}}$, and the data $\mathbf{f}(\underline{\boldsymbol{\phi}})$ given in (2.2) is adjusted so that the exact solution is given by the smooth functions

$$\mathbf{u}(x_1,x_2) = \begin{pmatrix} \sin(\pi x_1)\,\cos(\pi x_2) \\ -\cos(\pi x_1)\,\sin(\pi x_2)\end{pmatrix}, \quad p(x_1,x_2) = \cos(\pi x_1)\,\exp(x_2),$$

$$\phi_1(x_1,x_2) = 0.5+0.5\,\cos(x_1x_2), \quad \phi_2(x_1,x_2) = 0.1+0.3\,\exp(x_1x_2).$$

The model problem is then complemented with the appropriate Dirichlet boundary conditions. Tables 1 and 2 show the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations when $\mathtt{F} = 10$. Notice that we are able not only to approximate the original unknowns but also the pressure field, the velocity gradient, and the heat and diffusive vectors through the formulae (5.20). The results confirm that the optimal rates of convergence $O(h^{k+1})$, provided by Theorem 5.6 and Lemma 5.7 are

Table 1 Example 1, Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the fully mixed $\mathbb{RT}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation for the coupling of the Brinkman–Forchheimer and double-diffusion equations with $\mathtt{F} = 10$

DOF	h	iter	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(p)	r(p)	e($\nabla\mathbf{u}$)	r($\nabla\mathbf{u}$)
374	0.7454	5	15.8084	–	3.8883	–	1.9479	–	2.8877	–
1556	0.3667	5	7.0304	1.142	1.8380	1.056	0.6651	1.515	1.3648	1.056
5666	0.1971	5	3.5383	1.106	0.9590	1.048	0.3186	1.186	0.6954	1.086
22022	0.1036	5	1.7561	1.090	0.4756	1.091	0.1528	1.143	0.3522	1.058
87692	0.0554	5	0.8814	1.099	0.2385	1.101	0.0759	1.115	0.1754	1.112
353456	0.0284	5	0.4373	1.051	0.1182	1.052	0.0367	1.089	0.0865	1.059

e($\boldsymbol{\rho}_1$)	r($\boldsymbol{\rho}_1$)	e(ϕ_1)	r(ϕ_1)	e($\boldsymbol{\rho}_2$)	r($\boldsymbol{\rho}_2$)	e(ϕ_2)	r(ϕ_2)	e($\widetilde{\boldsymbol{\rho}}_1$)	r($\widetilde{\boldsymbol{\rho}}_1$)	e($\widetilde{\boldsymbol{\rho}}_2$)	r($\widetilde{\boldsymbol{\rho}}_2$)
0.9428	–	0.0453	–	0.5352	–	0.0652	–	0.5526	–	0.3243	–
0.4170	1.150	0.0200	1.152	0.2376	1.145	0.0324	0.986	0.3640	0.589	0.1859	0.784
0.2173	1.050	0.0109	0.976	0.1200	1.100	0.0177	0.975	0.2056	0.920	0.1010	0.983
0.1076	1.094	0.0057	1.007	0.0603	1.071	0.0094	0.991	0.1038	1.064	0.0512	1.059
0.0544	1.086	0.0029	1.090	0.0302	1.103	0.0049	1.036	0.0528	1.076	0.0259	1.084
0.0270	1.051	0.0014	1.046	0.0149	1.057	0.0024	1.087	0.0263	1.049	0.0129	1.048

Table 2 Example 1, Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the fully mixed $\mathbb{RT}_1 - \mathbf{P}_2 - \mathbf{RT}_1 - \mathrm{P}_1^{\mathrm{dc}} - \mathbf{RT}_1 - \mathrm{P}_1^{\mathrm{dc}}$ approximation for the coupling of the Brinkman–Forchheimer and double-diffusion equations with $\mathtt{F} = 10$

DOF	h	iter	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(p)	r(p)	e($\nabla\mathbf{u}$)	r($\nabla\mathbf{u}$)
1178	0.7454	5	4.1840	–	1.1413	–	0.3184	–	0.5832	–
5054	0.3667	5	0.8514	2.244	0.2352	2.226	0.0613	2.322	0.1160	2.277
18626	0.1971	5	0.2264	2.134	0.0641	2.094	0.0159	2.173	0.0320	2.073
72890	0.1036	5	0.0576	2.130	0.0158	2.181	0.0039	2.191	0.0080	2.164
291278	0.0554	5	0.0143	2.221	0.0041	2.163	0.0010	2.166	0.0020	2.169
1176134	0.0284	5	0.0035	2.120	0.0010	2.105	0.0002	2.101	0.0005	2.105

e($\boldsymbol{\rho}_1$)	r($\boldsymbol{\rho}_1$)	e(ϕ_1)	r(ϕ_1)	e($\boldsymbol{\rho}_2$)	r($\boldsymbol{\rho}_2$)	e(ϕ_2)	r(ϕ_2)	e($\widetilde{\boldsymbol{\rho}}_1$)	r($\widetilde{\boldsymbol{\rho}}_1$)	e($\widetilde{\boldsymbol{\rho}}_2$)	r($\widetilde{\boldsymbol{\rho}}_2$)
0.2572	–	0.0083	–	0.1203	–	0.0082	–	0.2187	–	0.0930	–
0.0545	2.188	0.0014	2.508	0.0264	2.140	0.0018	2.112	0.0518	2.030	0.0224	2.009
0.0146	2.124	0.0004	2.090	0.0072	2.101	0.0005	2.025	0.0141	2.094	0.0063	2.044
0.0037	2.150	0.0001	1.917	0.0018	2.115	0.0002	1.948	0.0036	2.143	0.0016	2.121
0.0009	2.218	3 E-05	2.260	0.0005	2.185	4 E-05	2.111	0.0009	2.215	0.0004	2.172
0.0002	2.112	6 E-06	2.155	0.0001	2.115	9 E-06	2.150	0.0002	2.111	0.0001	2.120

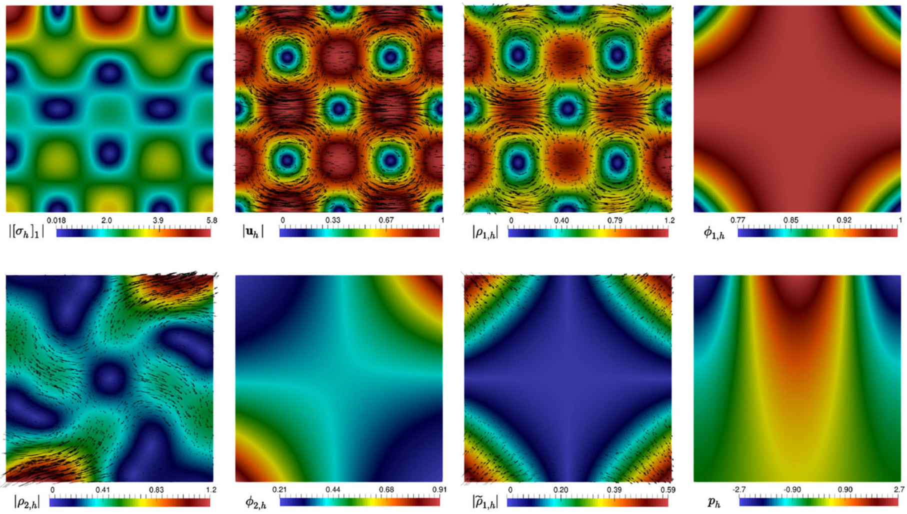

Fig. 1 Example 1, Computed magnitude of the pseudostress tensor component, velocity, and pseudoheat vector, and temperature field (top plots); magnitude of the pseudodiffusive vector, concentration field, magnitude of the heat vector and pressure field (bottom plots)

Table 3 Example 1, performance of the iterative method (number of iterations) upon variations of the parameter F with polynomial degree $k = 0$

	h					
F	0.7454	0.3667	0.1971	0.1036	0.0554	0.0284
10^0	4	4	4	4	4	4
10^1	5	5	5	5	5	5
10^2	6	7	7	7	7	7
10^3	8	8	8	8	8	8
10^4	10	30	30	8	8	8
10^5	12	30	30	30	30	8

attained for $k = 0, 1$. The Newton method exhibits a behavior independent of the meshsize, converging in five iterations in all cases. In Fig. 1 we display the solution obtained with the fully-mixed $\mathbb{RT}_1 - \mathbf{P}_2 - \mathbf{RT}_1 - \mathrm{P}_1^{\mathrm{dc}} - \mathbf{RT}_1 - \mathrm{P}_1^{\mathrm{dc}}$ approximation with 1, 176, 134 DOF, where $\mathrm{P}_1^{\mathrm{dc}}$ denotes the piecewise linear discontinuous finite element.

On the other hand, in Table 3 we show the behaviour of the iterative method with polynomial degree $k = 0$, as a function of the parameter $\mathtt{F} \in \{10^0, 10^1, 10^2, 10^3, 10^4, 10^5\}$, considering different meshsizes h, and a tolerance $\mathsf{tol} = 1\mathrm{E} - 06$. Here we observe that the higher the parameter F the higher the number of Newton iterations required.

Example 2: Convergence Against Smooth Exact Solutions in a 3D Domain

In our second example, we consider the cube domain $\Omega = (0, 1)^3$. The solution is given by

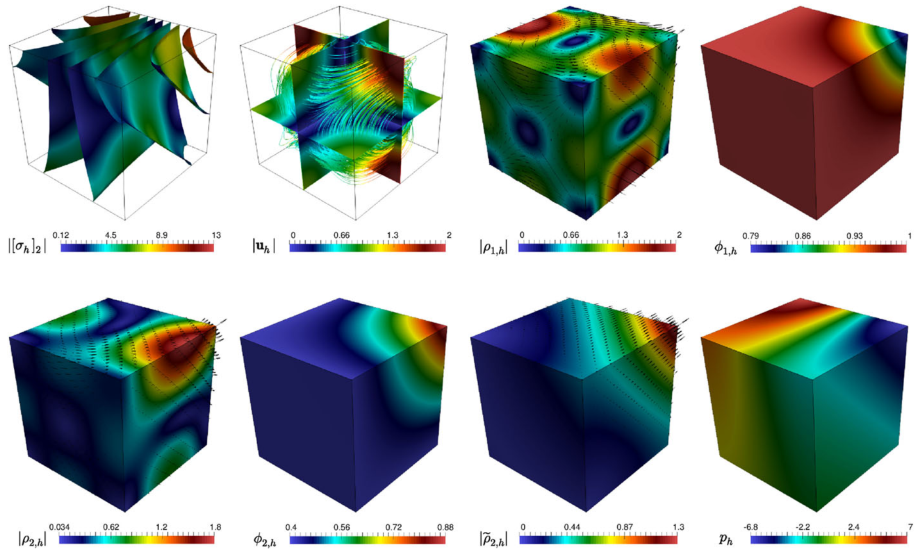

Fig. 2 Example 2, Computed magnitude of the pseudostress tensor component, velocity, and pseudoheat vector, and temperature field (top plots); magnitude of the pseudodiffusive vector, concentration field, magnitude of the diffusive vector and pressure field (bottom plots)

$$\mathbf{u}(x_1, x_2, x_3) = \begin{pmatrix} \sin(\pi x_1)\ \cos(\pi x_2)\ \cos(\pi x_3) \\ -2\cos(\pi x_1)\ \sin(\pi x_2)\ \cos(\pi x_3) \\ \cos(\pi x_1)\ \cos(\pi x_2)\ \sin(\pi x_3) \end{pmatrix},$$

$$p(x_1, x_2, x_3) = \cos(\pi x_1)\ \exp(x_2 + x_3),$$

$$\phi_1(x_1, x_2, x_3) = 0.5 + 0.5\ \cos(x_1 x_2 x_3), \quad \phi_2(x_1, x_2, x_3) = 0.1 + 0.3\ \exp(x_1 x_2 x_3).$$

Similarly to the first example, we consider $\mathtt{F} = 10$ and the potential type gravitational acceleration $\mathbf{g} = (0, 0, -1)^{\mathrm{t}}$, whereas the data $\mathbf{f}(\boldsymbol{\phi})$ is computed from (2.2) using the above solution. The numerical solutions are shown in Fig. 2, which were built using the fully-mixed $\mathbb{RT}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation with 2, 497, 827 DOF. The convergence history for a set of quasi-uniform mesh refinements using $k = 0$ is shown in Table 4. Again, the mixed finite element method converges optimally with order $\mathcal{O}(h)$, as it was proved by Theorem 5.6 and Lemma 5.7.

Example 3: Flow Through Porous Media with Channel Network

In our last example, inspired by [2, Sect. 5.2.4], we focus on flow through porous media with channel network. We consider the square domain $\Omega = (-1, 1)^2$ with an internal channel network denoted as Ω_c, which is described in the first plot of Fig. 3. We consider the coupling of the Brinkman–Forchheimer and double-diffusion equations (2.8) in the whole domain Ω with $\mathbf{Q}_1 = 0.5\,\mathbb{I}$, $\mathbf{Q}_2 = 0.125\,\mathbb{I}$, but with different values of the parameters $\mathtt{F}$ and $\mathbf{K} = \alpha\,\mathbb{I}$ for the interior and the exterior of the channel, that is,

$$\mathtt{F} = \begin{cases} 10 & \text{in } \Omega_c \\ 1 & \text{in } \overline{\Omega} \setminus \Omega_c \end{cases} \quad \text{and} \quad \alpha = \begin{cases} 1 & \text{in } \Omega_c \\ 0.001 & \text{in } \overline{\Omega} \setminus \Omega_c \end{cases}.$$

Table 4 Example 2, Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the fully mixed $\mathbb{RT}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation for the coupling of the Brinkman–Forchheimer and double-diffusion equations with $\mathtt{F} = 10$

DOF	h	iter	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(p)	r(p)	e($\nabla\mathbf{u}$)	r($\nabla\mathbf{u}$)
777	0.7071	5	14.5717	–	3.4300	–	1.2399	–	2.3432	–
5463	0.3536	5	7.8985	0.884	2.1333	0.685	0.6761	0.875	1.2356	0.923
40971	0.1768	5	3.8238	1.047	1.1576	0.882	0.3099	1.125	0.6305	0.971
317331	0.0884	4	1.8673	1.034	0.5925	0.966	0.1335	1.215	0.3183	0.986
2497827	0.0442	4	0.9260	1.012	0.2980	0.991	0.0606	1.141	0.1597	0.995

e($\boldsymbol{\rho}_1$)	r($\boldsymbol{\rho}_1$)	e(ϕ_1)	r(ϕ_1)	e($\boldsymbol{\rho}_2$)	r($\boldsymbol{\rho}_2$)	e(ϕ_2)	r(ϕ_2)	e($\widetilde{\boldsymbol{\rho}}_1$)	r($\widetilde{\boldsymbol{\rho}}_1$)	e($\widetilde{\boldsymbol{\rho}}_2$)	$r(\widetilde{\boldsymbol{\rho}}_2)$
0.5715	–	0.0282	–	0.2928	–	0.0557	–	0.2650	–	0.1502	–
0.3195	0.839	0.0157	0.849	0.1604	0.868	0.0311	0.894	0.2198	0.270	0.1092	0.461
0.1588	1.008	0.0080	0.968	0.0799	1.006	0.0158	0.972	0.1391	0.660	0.0668	0.709
0.0783	1.020	0.0040	1.000	0.0395	1.016	0.0080	0.994	0.0752	0.888	0.0358	0.898
0.0390	1.006	0.0020	1.002	0.0197	1.005	0.0040	0.999	0.0384	0.968	0.0183	0.970

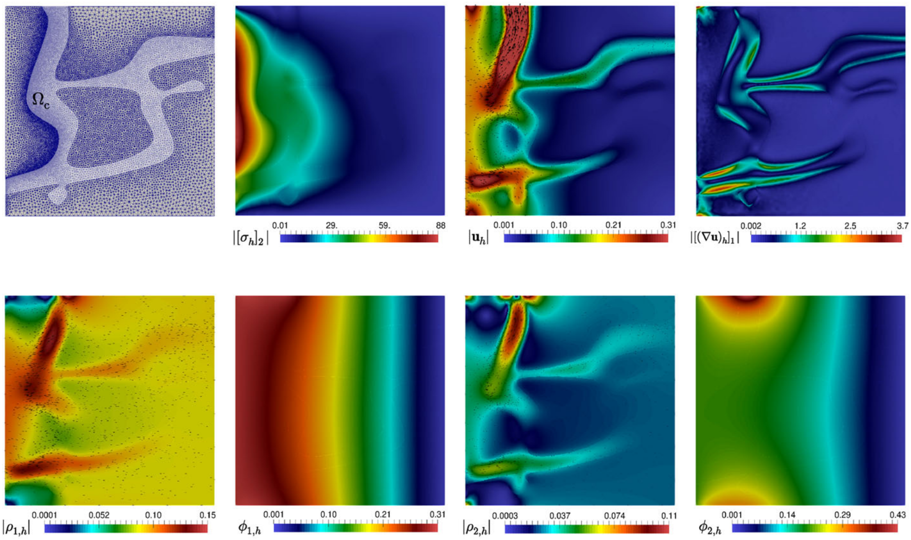

Fig. 3 Example 3, Domain configuration, computed magnitude of the pseudostress tensor component, velocity, and velocity gradient component (top plots); magnitude of the pseudoheat vector, temperature field, magnitude of the pseudodiffusive vector, and concentration field (bottom plots)

The parameter choice corresponds to increased inertial effect ($\mathtt{F} = 10$) in the channel and a high permeability ($\alpha = 1$), compared to reduced inertial effect ($\mathtt{F} = 1$) in the porous media and low permeability ($\alpha = 0.001$). In addition, the boundaries conditions are

$$\mathbf{u}\cdot\mathbf{n} = 0.2, \quad \mathbf{u}\cdot\mathbf{t} = 0 \quad \text{on} \quad \Gamma_{\text{left}}, \quad \boldsymbol{\sigma}\,\mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega\setminus\Gamma_{\text{left}},$$

$$\phi_1 = 0.3 \quad \text{on} \quad \Gamma_{\text{left}}, \quad \phi_1 = 0 \quad \text{on} \quad \Gamma_{\text{right}}, \quad \boldsymbol{\rho}_1\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\text{top}}\cup\Gamma_{\text{bottom}},$$

$$\phi_2 = 0.2 \quad \text{on} \quad \Gamma_{\text{left}}, \quad \phi_2 = 0 \quad \text{on} \quad \Gamma_{\text{right}}, \quad \boldsymbol{\rho}_2\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\text{top}}\cup\Gamma_{\text{bottom}}.$$

In particular, the first row of boundary equations corresponds to inflow on the left boundary and zero stress outflow on the rest of the boundary. We stress here that, using similar arguments to those employed in [12], we are able to extended our analysis to the present case of mixed boundary conditions for the double-diffusion equations. In Fig. 3 we display the computed magnitude of the pseudostress tensor component, velocity, velocity gradient component, pseudoheat and pseudodiffusive vectors, and the temperature and concentration fields, which were built using the fully-mixed $\mathbb{RT}_0 - \mathbf{P}_1 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation on a mesh with 27, 287 triangle elements (actually representing 257, 284 $\mathtt{DOF}$). As expected, we observe faster flow through the channel network, with a significant velocity gradient across the interface between the channel and the porous media. The pseudostress is more diffused, since it includes the pressure field. The temperature is higher on the left of the boundary and goes decaying to the right of the domain. Notice that both the pseudoheat and pseudodiffusive vectors are higher in the channel. This example illustrates the ability of the coupling of the Brinkman–Forchheimer and double-diffusion equations to handle heterogeneous media using spatially varying parameters. The example is particularly challenging, due to the strong jump discontinuity of the parameters across the two regions, which are handled very well by our numerical method.

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