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Asymptotic Analysis and Numerical Methods for Oscillatory Infinite Generalized Bessel Transforms with an Irregular Oscillator

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Abstract
In this work, we perform a complete asymptotic analysis and the construction of affordable quadrature rules for a class of oscillatory infinite Bessel transform with a general oscillator. Especially in the presence of critical points, e.g., endpoints, zeros and stationary points, we first derive a series of useful asymptotic expansions in inverse powers of the frequency parameter ω. The resulting asymptotic expansions clarify the large ω behavior of the transform and provide powerful tools for designing quadrature rules and conducting error analysis. As a consequence, efficient and affordable new modified Filon-type methods for computing the transform numerically are proposed. Particularly, we carry out the rigorous error analysis and obtain asymptotic error estimates in inverse powers of ω. Numerical examples can confirm our analysis. The accuracy can be improved greatly by either adding more derivatives interpolation at endpoints or adding more interior nodes. Moreover, only using a small number of nodes and multiplicities, we can obtain the required accuracy level. For fixed number of nodes and multiplicities, the higher accuracy can be achieved with the larger values of ω.

Keywords Oscillatory infinite Bessel transforms · Stationary points · Asymptotic expansions · New modified Filon-type methods · Error analysis

Mathematics Subject Classification 65D30 · 65D32 · 65R10 · 41A60

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1 Introduction

Highly oscillatory integrals arise frequently in many fields such as electromagnetics, acoustics scattering, electrodynamics, computerized tomography [2,4,6,7,12,13]. Numerical evaluation of such integrals is challenging in scientific computing. In this paper we mainly focus on quadrature rules and asymptotic expansions of oscillatory infinite Bessel transforms of the form

$$I[f]=\int_a^{+\infty} x^{\alpha} f(x)J_m(\omega g(x))dx, \tag{1.1}$$

where $\alpha<0, a>0$, and ω denotes the frequency of the oscillation. Moreover, $f(x)$ and $g(x)$ are sufficiently smooth functions on $[a,+\infty)$. Here, $J_m(z)$ [1, p. 358] is the Bessel function of the first kind and of order m with $\Re(m)>-1$. In most of the cases, such integrals cannot be calculated analytically, and then one has to resort to numerical methods. Traditionally one would have to resolve the oscillations by taking several sub-intervals for each period, resulting in a scheme whose complexity would grow linearly with the frequency of the oscillations. Consequently, the classical integration methods like Gauss quadrature are inapplicable for high frequency ω, since they often require many function evaluations which make them highly time consuming.

In the following, we first mention several related articles for computing infinite range oscillatory integrals. As early as in 1976, Blakemore et al. [5] made comparison of some numerical methods for computing infinite range oscillatory integrals. However, those methods presented in [5] converge slowly, and have to use an extrapolation technique to accelerate convergence. Wang et al. [34] investigated the asymptotics and fast computation of one-sided oscillatory infinite Hilbert transforms. Xu et al. [42] studied the fast computation of a class of oscillatory infinite Bessel Hilbert transform. Hascelik [15] gave an asymptotic Filon-type method for calculating the infinite oscillatory integral $\int_a^{+\infty} f(x)e^{i\omega g(x)}dx$. On the basis of Hascelik's ideas [15], Chen [9,10] proposed efficient numerical methods for approximating $\int_a^{+\infty} f(x)J_m(\omega x)dx$. Recently, numerical methods for computing these oscillatory infinite integrals $\int_1^{+\infty} x^{\alpha} f(x)G(\omega x)dx$ were studied in [23,24], where $G(\omega x)$ denote many different oscillatory kernel functions, such as $e^{i\omega x}$, $J_m(\omega x)$, $Y_m(\omega x)$, $H_m^{(1)}(\omega x)$, $H_m^{(2)}(\omega x)$, $Ai(-\omega x)$, respectively. More recently, Chen [11] proposed an asymptotic rule for computing the infinite Bessel transform $\int_0^{+\infty} f(x)J_m(\omega x)dx$. In addition, there has been tremendous interest in developing numerical methods for singular or nonsingular oscillatory Bessel transforms on finite intervals (see [8,21,22,25,26,32,35,37–41,43,44]). Moreover, it is noteworthy that asymptotic analysis and computation of finite generalized Fourier transform $\int_a^b f(x)e^{i\omega g(x)}dx$ [19,20] and finite Bessel transform $\int_a^b f(x)J_m(\omega g(x))dx$ [35,40] were extensively studied by Iserles, Nørsett, Wang and Xiang.

Let us go back to observe the problem to be solved. In particular, it should be noted that transforms (1.1) are oscillatory infinite Bessel integrals with an irregular oscillator $g(x)$. The infinite integration interval $[a,+\infty)$ further complicates the evaluation of the integrals and makes the asymptotic analysis and computation of (1.1) more difficult than that of the oscillatory finite Bessel transform $\int_a^b f(x)J_m(\omega g(x))dx$ investigated in [35,40]. In addition, the irregular oscillator $g(x)$ may have either zero points or stationary points. The presence of zero points or stationary points can change the nature of oscillatory infinite integrals (1.1), and make the asymptotic analysis and computation of (1.1) much more difficult than that of oscillatory infinite integrals $\int_0^{+\infty} f(x)J_m(\omega x)dx$ and $\int_1^{+\infty} x^{\alpha} f(x)J_m(\omega x)dx$ studied in [11,24], respectively. To the best of our knowledge, so far little research has been done for asymptotic analysis and numerical computation of the infinite Bessel transform (1.1) with an irregular oscillator $g(x)$. This motivates us to develop asymptotic expansions and efficient quadrature rules for computing these integrals.

Consequently, we aim to introduce and analyze asymptotic expansions and a modified Filon-type quadrature rule for the integrals (1.1). The modified Filon-type quadrature rule and its error analysis are based on Hermite interpolation polynomials on the bounded interval. In the following, we briefly outline the steps of solution as follows. By change of variable $x = \frac{2a}{t+1}$, we first transform (1.1) into singular integrals of the type

$$I[f] = (2a)^{\alpha+1} \int_{-1}^{1} (t+1)^{-\alpha-2} H(t) J_m(\omega g(2a/(t+1)))dt := \tilde{I}[H], \tag{1.2}$$

where

$$H(t) = \begin{cases} f(\frac{2a}{t+1}) = f(x), & t \in (-1, 1], \\ \lim_{t\to -1^+} f(\frac{2a}{t+1}) = \lim_{x\to+\infty} f(x) = C \text{ (constant)}, & t = -1. \end{cases}$$

Here, we make a change of variables from $[a, +\infty)$ to $(-1, 1]$. There are two main reasons. On the one hand, we can construct a modified Filon-type quadrature rule based on Hermite interpolation polynomials of $H(t)$ on the bounded interval $(-1, 1]$. On the other hand, we can conveniently conduct error analysis for the modified Filon-type quadrature rule by asymptotic expansions of (1.2) on the bounded interval $(-1, 1]$. In fact, we can also keep $[a, +\infty)$ and directly derive asymptotic expansions, which seems to us that it simplifies the calculations, but it is not easy to use the resulting asymptotic expansions for performing error analysis. In Sects. 2 and 3 we shall derive asymptotic expansions in inverse powers of ω for (1.2), both with and without stationary points of the oscillator $g(x)$, which clarify the large ω behavior of the integrals (1.1) and (1.2). The resulting asymptotic expansions can provide powerful tools for constructing quadrature rules and conducting error analysis. Further, based on these asymptotic expansions, we devise some efficient and affordable quadrature rules such as a new modified Filon-type method, and more particularly, perform their error analysis in inverse powers of ω. Additionally, some numerical examples in Sect. 4 are listed to show the high accuracy and efficiency of these quadrature rules. We conclude this paper with some final remarks in Sect. 5.

2 Asymptotic Analysis and Quadrature Rules of the Case Without Stationary Points

The case without stationary points can be divided into the two types, i.e., type I: $g(x) \neq 0$ and type II: $g(x)$ having zeros on $x \in [a, +\infty)$.

2.1 Asymptotic Analysis for the Type I: $g(x) \neq 0$ for $x \in [a, +\infty)$

Here we begin our analysis from the simple case that $g(x) \neq 0$ for $x \in [a, +\infty)$.

Theorem 2.1 *Let*

$$R_0[H](t) = H(t),$$
$$R_{k+1}[H](t) = -\frac{\alpha(t+1)R_k[H](t)}{g'(\frac{2a}{t+1})} + (t+1)^2 \left[g\left(\frac{2a}{t+1}\right)\right]^{m+k+1}$$

$$\frac{d}{dt}\left\{\frac{R_k[H](t)}{[g(\frac{2a}{t+1})]^{m+k+1}g'(\frac{2a}{t+1})}\right\},\ k=0,1,2,\ldots.$$

Suppose that $f(x), g(x) \in C^{\infty}[a,+\infty)$, $\lim_{x\to+\infty} g'(x) \neq 0$ *and* $g(x) \neq 0$, $g'(x) \neq 0$ *for arbitrary* $x \in [a,+\infty)$, *(which implies that* $\lim_{x\to+\infty} g(x) = \infty$*). Moreover, assume that* $\frac{R_s[H](t)}{g'(\frac{2a}{t+1})}$ *for* $s = 0, 1, 2, \ldots,$ *converge as* $t \to -1^+$*. Then for* $a > 0$ *and* $\alpha < 0$ *it follows that*

$$I[f] \sim -2a^{\alpha+1}\sum_{k=0}^{\infty}\frac{R_k[H](1)}{(2a\omega)^{k+1}g'(a)}J_{m+k+1}(\omega g(a)),\ \text{as}\ \omega\to\infty. \tag{2.1}$$

Proof By induction on $s \geq 1$, we can prove the following identity

$$\begin{aligned} I[f] &= -2a^{\alpha+1}\sum_{k=0}^{s-1}\frac{R_k[H](1)}{(2a\omega)^{k+1}g'(a)}J_{m+k+1}(\omega g(a)) \\ &\quad + \frac{(2a)^{\alpha-s+1}}{\omega^s}\int_{-1}^{1}(t+1)^{-\alpha-2}R_s[H](t)J_{m+s}(\omega g(2a/(t+1)))dt. \end{aligned} \tag{2.2}$$

Based on the derivative formula [1, p. 361]

$$\frac{d}{dx}[x^{m+1}J_{m+1}(x)] = x^{m+1}J_m(x), \tag{2.3}$$

we have

$$\begin{aligned} &\frac{d}{dt}\left\{\left[g\left(\frac{2a}{t+1}\right)\right]^{m+s}J_{m+s}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right\} \\ &= -\frac{2a\omega}{(t+1)^2}g'\left(\frac{2a}{t+1}\right)\left[g\left(\frac{2a}{t+1}\right)\right]^{m+s}J_{m+s-1}\left(\omega g\left(\frac{2a}{t+1}\right)\right). \end{aligned} \tag{2.4}$$

When m is fixed, x is a real number and $x \to \infty$, it follows from [1, p. 364] and [28],

$$J_m(x) = \sqrt{2/(\pi x)}(\cos(x - \frac{1}{2}m\pi - \frac{1}{4}\pi) + O(x^{-1})) = O(x^{-\frac{1}{2}}). \tag{2.5}$$

From [1, p. 362] and [29], we obtain

$$|J_\nu(x)| \leq 1, \nu \geq 0, x \in \Re. \tag{2.6}$$

Suppose that $s = 1$. Using (2.4)–(2.6) together with known conditions, and by integration by parts we have

$$\begin{aligned} I[f] &= -\frac{(2a)^{\alpha}}{\omega}\int_{-1}^{1}(t+1)^{-\alpha}\frac{R_0[H](t)}{[g(\frac{2a}{t+1})]^{m+1}g'(\frac{2a}{t+1})}d\left\{\left[g\left(\frac{2a}{t+1}\right)\right]^{m+1}J_{m+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right\} \\ &= -\frac{(2a)^{\alpha}}{\omega}\left[(t+1)^{-\alpha}\frac{R_0[H](t)}{g'(\frac{2a}{t+1})}J_{m+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right]_{-1}^{1} \\ &\quad + \frac{(2a)^{\alpha}}{\omega}\int_{-1}^{1}(t+1)^{-\alpha-2}R_1[H](t)J_{m+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt \\ &= -\frac{a^{\alpha}R_0[H](1)}{\omega g'(a)}J_{m+1}(\omega g(a)) \\ &\quad + \frac{(2a)^{\alpha}}{\omega}\int_{-1}^{1}(t+1)^{-\alpha-2}R_1[H](t)J_{m+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt. \end{aligned} \tag{2.7}$$

So the identity (2.2) holds for $s = 1$.

For $\alpha < 0$ and $s \geq 1$, we can easily prove from (2.5) and (2.6) together with known conditions that

$$\lim_{t\to -1^+} (t+1)^{-\alpha} \frac{R_s[H](t)}{g'(\frac{2a}{t+1})} J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right) = 0. \tag{2.8}$$

From the construction form of $R_{s+1}[H](t)$,

$$R_{s+1}[H](t) = -\frac{\alpha(t+1)R_k[H](t)}{g'(\frac{2a}{t+1})} + (t+1)^2\left[g\left(\frac{2a}{t+1}\right)\right]^{m+k+1} \frac{d}{dt}\left\{\frac{R_k[H](t)}{[g(\frac{2a}{t+1})]^{m+k+1}g'(\frac{2a}{t+1})}\right\},\ k=0,1,2,\ldots,$$

and with the given condition that $\frac{R_s[H](t)}{g'(\frac{2a}{t+1})}$ for $s = 0, 1, 2, \ldots$, converge as $t \to -1^+$, we can see that each $(1+t)^{-\alpha-2}R_{s+1}[H](t)$ is a combination in powers of $1+t$ of degree more than -1 (that is, $(1+t)^{\lambda-1}$, $\lambda > 0$), which is expressed by

$$(1+t)^{-\alpha-2}R_{s+1}[H](t) = (1+t)^{\lambda-1}\varphi(t),$$

with $\varphi(t) \in C[-1, 1]$. Therefore, it follows from (2.6) that

$$\begin{aligned}
&\left|\int_{-1}^{1}(t+1)^{-\alpha-2}R_{s+1}[H](t)J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt\right| \\
&\leq \int_{-1}^{1}\left|(t+1)^{-\alpha-2}R_{s+1}[H](t)J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right|dt \\
&\leq \int_{-1}^{1}\left|(t+1)^{-\alpha-2}R_{s+1}[H](t)\right|dx \\
&\leq \int_{-1}^{1}\left|(1+t)^{\lambda-1}\varphi(t)\right|dx < \infty.
\end{aligned} \tag{2.9}$$

For $s \geq 1$, integration by parts on the last formula of (2.2) by using (2.5) and (2.6) together with known conditions yields

$$\begin{aligned}
&\frac{(2a)^{\alpha-s+1}}{\omega^s}\int_{-1}^{1}(t+1)^{-\alpha-2}R_s[H](t)J_{m+s}(\omega g(2a/(t+1)))dt \\
&= -\frac{(2a)^{\alpha-s}}{\omega^{s+1}}\int_{-1}^{1}(t+1)^{-\alpha}\frac{R_s[H](t)}{[g(\frac{2a}{t+1})]^{m+s+1}g'(\frac{2a}{t+1})} \\
&\qquad d\left\{\left[g\left(\frac{2a}{t+1}\right)\right]^{m+s+1}J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right\} \\
&= -\frac{(2a)^{\alpha-s}}{\omega^{s+1}}\left[(t+1)^{-\alpha}\frac{R_s[H](t)}{g'(\frac{2a}{t+1})}J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right]_{-1}^{1} \\
&\quad + \frac{(2a)^{\alpha-s}}{\omega^{s+1}}\int_{-1}^{1}(t+1)^{-\alpha-2}R_{s+1}[H](t)J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt \\
&= -\frac{2a^{\alpha+1}R_s[H](1)}{(2a\omega)^{s+1}g'(a)}J_{m+s+1}(\omega g(a))
\end{aligned}$$

$$+\frac{(2a)^{\alpha-s}}{\omega^{s+1}}\int_{-1}^{1}(t+1)^{-\alpha-2}R_{s+1}[H](t)J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt. \quad (2.10)$$

Combining (2.7) and (2.10) leads to the desired result (2.2). By using (2.9) and letting $s\to\infty$ for (2.2), we complete the proof of (2.1). □

By truncating after the first s terms of the asymptotic expansion (2.1), we obtain the s-step *asymptotic methods* as follows

$$Q_s^A[H] = -2a^{\alpha+1}\sum_{k=0}^{s-1}\frac{R_k[H](1)}{(2a\omega)^{k+1}g'(a)}J_{m+k+1}(\omega g(a)), \quad (2.11)$$

which represent the efficiency of the approximations to $I[f]$, once $2a\omega$ is sufficiently large.
In the sequel we give the error estimate of the s-step *asymptotic methods*.

Theorem 2.2 *Under the same conditions as those of Theorem 2.1, it is true that*

$$Q_s^A[H]-I[f]\sim O(\omega^{-s-\frac{3}{2}}),\ \ as\ \ \omega\to\infty, \quad (2.12)$$

where $Q_s^A[H]$ is given in (2.11).

Proof On the basis of (2.5) and (2.8)–(2.10) , we can directly obtain from (2.2) and (2.11)

$$\begin{aligned}|Q_s^A[H]-I[f]| &= \left|\frac{(2a)^{\alpha-s+1}}{\omega^s}\int_{-1}^{1}(t+1)^{-\alpha-2}R_s[H](t)J_{m+s}(\omega g(2a/(t+1)))dt\right|\\ &= \left|-\frac{2a^{\alpha+1}R_s[H](1)}{(2a\omega)^{s+1}g'(a)}J_{m+s+1}(\omega g(a))\right.\\ &\quad\left.+\frac{(2a)^{\alpha-s}}{\omega^{s+1}}\int_{-1}^{1}(t+1)^{-\alpha-2}R_{s+1}[H](t)J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt\right|\\ &= O(\omega^{-s-\frac{3}{2}}),\quad \omega\to\infty.\end{aligned}$$

This completes the proof of (2.12). □

From the above asymptotic expansion (2.1) and the error estimate (2.12), we can immediately observe that the error order can be improved by adding derivative information of $H(t)$ at the endpoint $t=1$. Similarly, we can see that, asymptotically, the values of (1.2) depend on the behavior of $H(t)$ and $g(2a/(t+1))$ around $t=1$, which are independent of the behavior of $H(t)$ and $g(2a/(t+1))$ and its derivatives as t tends to -1. For convenience, in the following all modified Filon-type methods, the multiplicities m_0 associated with node point $t=-1$ is set to be 0. This also implies that the behavior of $f(x)$, $g(x)$ and their derivatives around $x=a$ completely determines the values of (1.1), which are independent of the behavior of $f(x)$, $g(x)$ and their derivatives when x tends to infinity.

2.2 Asymptotic Analysis for the Type II: $g(x)$ Having Zeros on $x\in[a,+\infty)$

In this subsection we only consider the case that $g(x)$ has one zero on $[a,+\infty)$. If $g(x)$ has a few zeros on $[a,+\infty)$, we have to split the whole interval into some subintervals such that $g(x)$ has only one zero on each subinterval.

Theorem 2.3 *Define that*

$$\sigma_0[H](t) = H(t),$$

$$\sigma_{k+1}[H](t) = -\frac{\alpha(t+1)\{\sigma_k[H](t)-\sigma_k[H](\zeta)\}}{g'(\frac{2a}{t+1})} + (t+1)^2\left[g\left(\frac{2a}{t+1}\right)\right]^{m+k+1}$$
$$\frac{d}{dt}\left\{\frac{\sigma_k[H](t)-\sigma_k[H](\zeta)}{[g(\frac{2a}{t+1})]^{m+k+1}g'(\frac{2a}{t+1})}\right\},\ k=0,1,2,\ldots.$$

Let $f(x), g(x) \in C^\infty[a,+\infty)$, $g'(x) \neq 0$ *for arbitrary* $x \in [a,+\infty)$, *and* $\lim_{x\to+\infty} g'(x) \neq 0$, *(which implies that* $\lim_{x\to+\infty} g(x) = \infty$*). Moreover, assume that* $g(2a/(1+\zeta)) = 0$ *for* $\zeta \in (-1,1]$ *and* $g(2a/(1+t)) \neq 0$ *for* $t \in (-1,1]\setminus\{\zeta\}$. *If* $\frac{\sigma_s[H](t)-\sigma_s[H](\zeta)}{g'(\frac{2a}{t+1})}$ *for* $s=0,1,2,\ldots$, *converge as* $t \to -1^+$, *then for* $a>0$ *and* $\alpha<0$ *it follows that*

$$I[f] \sim \sum_{k=0}^{\infty}\frac{\sigma_k[H](\zeta)}{(2a\omega)^k}\widetilde{M}(m+k,\omega) - 2a^{\alpha+1}\sum_{k=0}^{\infty}\frac{\sigma_k[H](1)-\sigma_k[H](\zeta)}{(2a\omega)^{k+1}g'(a)}J_{m+k+1}(\omega g(a)), \tag{2.13}$$

as $\omega \to \infty$, *where*

$$\widetilde{M}(m+k,\omega) = (2a)^{\alpha+1}\int_{-1}^{1}(t+1)^{-\alpha-2}J_{m+k}(\omega g(2a/(t+1)))dt.$$

Proof By induction on $s \geq 1$, we can prove the following identity

$$I[f] = \sum_{k=0}^{s-1}\frac{\sigma_k[H](\zeta)}{(2a\omega)^k}\widetilde{M}(m+k,\omega) - 2a^{\alpha+1}\sum_{k=0}^{s-1}\frac{\sigma_k[H](1)-\sigma_k[H](\zeta)}{(2a\omega)^{k+1}g'(a)}J_{m+k+1}(\omega g(a))$$
$$+\frac{(2a)^{\alpha-s+1}}{\omega^s}\int_{-1}^{1}(t+1)^{-\alpha-2}\sigma_s[H](t)J_{m+s}(\omega g(2a/(t+1)))dt. \tag{2.14}$$

Suppose that $s=1$. Similarly, we obtain

$$\begin{aligned}
I[f] =\ & (2a)^{1+\alpha}\int_{-1}^{1}\sigma_0[H](\zeta)(t+1)^{-\alpha-2}J_m\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt \\
&+(2a)^{1+\alpha}\int_{-1}^{1}(t+1)^{-\alpha-2}(\sigma_0[H](t)-\sigma_0[H](\zeta))J_m\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt \\
=\ & \sigma_0[H](\zeta)\widetilde{M}(m,\omega) \\
&-\frac{(2a)^\alpha}{\omega}\int_{-1}^{1}(t+1)^{-\alpha}\frac{\sigma_0[H](t)-\sigma_0[H](\zeta)}{[g(\frac{2a}{t+1})]^{m+1}g'(\frac{2a}{t+1})} \\
&\quad d\left\{\left[g\left(\frac{2a}{t+1}\right)\right]^{m+1}J_{m+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right\} \\
=\ & \sigma_0[H](\zeta)\widetilde{M}(m,\omega) - \frac{(2a)^\alpha}{\omega}\left[(t+1)^{-\alpha}\frac{\sigma_0[H](t)-\sigma_0[H](\zeta)}{g'(\frac{2a}{t+1})}J_{m+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right]_{-1}^{1} \\
&+\frac{(2a)^\alpha}{\omega}\int_{-1}^{1}(t+1)^{-\alpha-2}\sigma_1[H](t)J_{m+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt \\
=\ & \sigma_0[H](\zeta)\widetilde{M}(m,\omega) - \frac{a^\alpha(\sigma_0[H](1)-\sigma_0[H](\zeta))}{\omega g'(a)}J_{m+1}(\omega g(a)) \\
&+\frac{(2a)^\alpha}{\omega}\int_{-1}^{1}(t+1)^{-\alpha-2}\sigma_1[H](t)J_{m+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt.
\end{aligned} \tag{2.15}$$

So the identity (2.14) holds for $s=1$.

For $s \geq 1$, we find from the last formula of (2.14) that

$$
\begin{aligned}
&\frac{(2a)^{\alpha-s+1}}{\omega^s}\int_{-1}^{1}(t+1)^{-\alpha-2}\sigma_s[H](t)J_{m+s}(\omega g(2a/(t+1)))dt\\
&=\frac{(2a)^{\alpha-s+1}}{\omega^s}\sigma_s[H](\zeta)\int_{-1}^{1}(t+1)^{-\alpha-2}J_{m+s}(\omega g(2a/(t+1)))dt\\
&\quad+\frac{(2a)^{\alpha-s+1}}{\omega^s}\int_{-1}^{1}(t+1)^{-\alpha-2}(\sigma_s[H](t)-\sigma_s[H](\zeta))J_{m+s}(\omega g(2a/(t+1)))dt\\
&=\frac{1}{(2a\omega)^s}\sigma_s[H](\zeta)\widetilde{M}(m+s,\omega)\\
&\quad-\frac{(2a)^{\alpha-s}}{\omega^{s+1}}\int_{-1}^{1}(t+1)^{-\alpha}\frac{\sigma_s[H](t)-\sigma_s[H](\zeta)}{[g(\frac{2a}{t+1})]^{m+s+1}g'(\frac{2a}{t+1})}\\
&\quad d\left\{\left[g\left(\frac{2a}{t+1}\right)\right]^{m+s+1}J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right\}\\
&=\frac{1}{(2a\omega)^s}\sigma_s[H](\zeta)\widetilde{M}(m+s,\omega)\\
&\quad-\frac{(2a)^{\alpha-s}}{\omega^{s+1}}\left[(t+1)^{-\alpha}\frac{\sigma_s[H](t)-\sigma_s[H](\zeta)}{g'(\frac{2a}{t+1})}J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right]_{-1}^{1}\\
&\quad+\frac{(2a)^{\alpha-s}}{\omega^{s+1}}\int_{-1}^{1}(t+1)^{-\alpha-2}\sigma_{s+1}[H](t)J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt\\
&=\frac{1}{(2a\omega)^s}\sigma_s[H](\zeta)\widetilde{M}(m+s,\omega)-\frac{2a^{\alpha+1}(\sigma_s[H](1)-\sigma_s[H](\zeta))}{(2a\omega)^{s+1}g'(a)}J_{m+s+1}(\omega g(a))\\
&\quad+\frac{(2a)^{\alpha-s}}{\omega^{s+1}}\int_{-1}^{1}(t+1)^{-\alpha-2}\sigma_{s+1}[H](t)J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt. \qquad (2.16)
\end{aligned}
$$

Combining (2.15) and (2.16) leads to the desired result (2.14). By letting $s \to \infty$ for (2.14), we complete the proof of (2.13). □

By truncating after the first s terms of the asymptotic expansion (2.13), we obtain

$$
Q_s^A[H]=\sum_{k=0}^{s-1}\frac{\sigma_k[H](\zeta)}{(2a\omega)^k}\widetilde{M}(m+k,\omega)-2a^{\alpha+1}\sum_{k=0}^{s-1}\frac{\sigma_k[H](1)-\sigma_k[H](\zeta)}{(2a\omega)^{k+1}g'(a)}J_{m+k+1}(\omega g(a)), \qquad (2.17)
$$

which represent the efficiency of the approximations to $I[f]$, once $2a\omega$ is sufficiently large.

2.3 Modified Filon-Type Methods and Their Error Analysis for the Case Without Stationary Points

The error of the asymptotic method is uncontrolled due to the divergence of the asymptotic series for a fixed ω. In order to overcome this weakness, based on the above observation we now design an alternative method, i.e., a more accurate and convergent *modified Filon-type method*, which requires identical information and produce the same rate of asymptotic decay. And this new method can be regarded as a modification of the standard method presented in [19,20]. To achieve this, we first introduce a valuable result.

Lemma 2.1 *Suppose that* $g(\frac{2a}{t+1}) \in C^\infty(-1,1]$ *and* $g'(\frac{2a}{t+1}) \neq 0$ *for* $t \in (-1,1]$. *Let*

$$\varphi_k(t) = 2a(t+1)^\alpha g'\left(\frac{2a}{t+1}\right)\left[g\left(\frac{2a}{t+1}\right)\right]^k, \quad k \geq 0, \quad a > 0, \quad \alpha \leq 0,$$

and

$$E_n = \left\{ v(t) | v(t) = \sum_{k=0}^{n} c_k \varphi_k(t), \quad c_k \in \mathbb{R} \right\}.$$

Then E_n *is an extended complete Chebyshev system.*

Proof Based on that $g(\frac{2a}{t+1}) \in C^\infty(-1,1]$, we get that $\varphi_k(t) \in C^\infty(-1,1]$. Since $g'(\frac{2a}{t+1}) \neq 0$ for $t \in (-1,1]$, it is obvious that $g(\frac{2a}{t+1})$ is a monotonic function on $(-1,1]$. This implies that the change of variable $y = g(\frac{2a}{t+1})$ is a bijection. Then for any $v(t) \in E_l$ and $l \in \{0,1,\ldots,n\}$, by setting $y = g(\frac{2a}{t+1})$ we obtain that

$$v(t) = \sum_{k=0}^{l} c_k \varphi_k(t) = 2a(t+1)^\alpha g'\left(\frac{2a}{t+1}\right) \sum_{k=0}^{l} c_k y^k.$$

For one thing, we observe that $v(t) \equiv 0$ implies that $c_k = 0$, $k = 0,\ldots,l$, and therefore $\{\varphi_0(t),\ldots,\varphi_l(t)\}$ is a basis of E_l. For another thing, we can see that $v(t)$ has at most l zeros in $(-1,1]$ counting multiplicities. From [33, Theorem 2.33 on page 34], it follows that E_l is an extended Chebyshev system. Therefore, for arbitrary $l \in \{0,1,\ldots,n\}$, E_n is an extended complete Chebyshev system. This completes the proof. □

Theorem 2.4 *Let* $\{t_k\}_0^\eta$ *be a set of node points such that* $-1 = t_0 < t_1 < \cdots < t_\eta = 1$ *and* $\{m_k\}_0^\eta$ *be a set of multiplicities associated with the above node points such that* $m_0, m_1, \ldots, m_\eta \geq s$, *where* s *is a nonnegative integer. Suppose that*

$$P(t) = \Sigma_{k=0}^{n} 2ac_k(t+1)^\alpha g'\left(\frac{2a}{t+1}\right)\left[g\left(\frac{2a}{t+1}\right)\right]^k,$$

where $n = \Sigma_{k=0}^{\eta} m_k - 1$. *From Lemma* 2.1 *and* [33, Theorem 9.9 on page 370], *the interpolation polynomial* $P(t) \in E_n$ *is the unique solution to the system of equations*

$$P^{(j-1)}(t_k) = H^{(j-1)}(t_k), \quad j = 1,2,\ldots,m_k,$$

for every integer $0 \leq k \leq \eta$. *Then the modified Filon-type method for* (1.2) *of the type I:* $g(x) \neq 0$ *for* $x \in [a,+\infty)$, *is defined by the following quadrature formula*

$$Q_s^F[H] \equiv \tilde{I}[P] = (2a)^{\alpha+1} \sum_{k=0}^{n} c_k M_k, \tag{2.18}$$

where the modified moments

$$M_k = 2a \int_{-1}^{1} (t+1)^{-2} g'(2a/(t+1))[g(2a/(t+1))]^k J_m(\omega g(2a/(t+1)))dt, \tag{2.19}$$

can be explicitly computed by (2.28). *Moreover, for* $\omega \gg 1$, *the absolute error of the quadrature formula* (2.18) *behaves asymptotically as*

$$Q_s^F[H] - I[f] \sim O(\omega^{-s-\frac{3}{2}}). \tag{2.20}$$

Proof We now substitute $H(t) - P(t)$ for $H(t)$ in the above asymptotic expansion (2.1). Since $P^{(j)}(1) = H^{(j)}(1), j = 0, 1, \ldots, s-1$, it follows that $R_k[H - P](1) = 0, k = 0, 1, \ldots, s-1$. Therefore, by using (2.5), we can obtain the desired results (2.20) directly from the above asymptotic expansion (2.1). □

Following the idea of Theorem 2.4, we devise a modified Filon-type method for the case of $g(x)$ having zeros. From the asymptotic expansion (2.13), we can see that the value of $I[f]$ also depends on the zero point ζ of $g(2a/(t+1))$ except endpoints. Therefore, we need to make ζ as an interpolating point. Suppose that $t_\nu = \zeta$ for some $\nu \in \{0, 1, \ldots, \eta\}$. By using the same interpolation polynomial $P(t) \in E_n$ as that of Theorem 2.4, we obtain the similar modified Filon-type method (2.18). Here, if $g(a) \neq 0$, the required moments M_k can be computed by (2.28). If $g(a) = 0$, the required moments M_k can be computed by (2.29).

In order to give the error estimate of the modified Filon-type method for the case of $g(x)$ having zeros, we now analyze the asymptotics of the above moments $\widetilde{M}(m, \omega)$ from Theorem 2.3.

Lemma 2.2 *Under the assumption of g in Theorem 2.3, it follows that*

$$\widetilde{M}(m, \omega) = O\left(\frac{1}{\omega}\right), \text{ as } \omega \to \infty. \tag{2.21}$$

Proof From the above given conditions that $g'(x) \neq 0$ for arbitrary $x \in [a, +\infty)$, it follows that $g(x)$ is strictly monotonic on $[a, +\infty)$. Hence, $g(x)$ possesses an inverse function, i.e., $g^{-1}(t)$.

By change of variables $u = \frac{2a}{1+\zeta}$ and $t = \frac{2a}{x} - 1$, together with the given conditions and then using asymptotic analysis similar to that of Theorem 2.4, for $\alpha < 0$ we have

$$\begin{aligned}
\widetilde{M}(m, \omega) =& \ (2a)^{\alpha+1} \int_{-1}^{\zeta} (t+1)^{-\alpha-2} J_m(\omega g(2a/(t+1)))dt \\
& + (2a)^{\alpha+1} \int_{\zeta}^{1} (t+1)^{-\alpha-2} J_m(\omega g(2a/(t+1)))dt \\
=& \int_a^u x^\alpha J_m(\omega g(x))dx + \int_u^{+\infty} x^\alpha J_m(\omega g(x))dx \\
=& -\int_0^{g(a)} \frac{[g^{-1}(t)]^\alpha}{g'(g^{-1}(t))} J_m(\omega t)dt + \int_0^{+\infty} \frac{[g^{-1}(t)]^\alpha}{g'(g^{-1}(t))} J_m(\omega t)dt \\
\sim & \frac{(\frac{2a}{1+\zeta})^\alpha}{g'(\frac{2a}{1+\zeta})} \left[\int_0^{+\infty} J_m(\omega t)dt - \int_0^{g(a)} J_m(\omega t)dt \right] \\
& - \frac{1}{\omega} \left[\frac{a^\alpha}{g'(a)} - \frac{(\frac{2a}{1+\zeta})^\alpha}{g'(\frac{2a}{1+\zeta})} \right] J_{m+1}(\omega g(a)),
\end{aligned} \tag{2.22}$$

where

$$\lim_{x \to +\infty} g(x) = +\infty.$$

When $\lim_{x\to+\infty} g(x) = -\infty$, the similar conclusion can be obtained.

Since, by setting $\omega t = y$, and from $\int_0^\infty J_m(t)dt = 1$ [1, p. 486], we have

$$\int_0^b J_m(\omega t)dt = \frac{1}{\omega} \int_0^{\omega b} J_m(y)dy = O\left(\frac{1}{\omega}\right), \ \ \omega \to +\infty,$$

the two integrals in the fifth line of (2.22) behave asymptotically as $O(\frac{1}{\omega})$ for fixed m and $\Re(m) > -1$. If $g(a) \neq 0$, the last line of (2.22) behaves asymptotically as $O\left(\frac{1}{\omega^{\frac{3}{2}}}\right)$ for fixed m and $\Re(m) > -1$. If $g(a) = 0$, both the second integral in the fifth line of (2.22) and the last line of (2.22) vanish. Therefore, this leads to the desired results. □

In the following, based on Lemma 2.2 we start to derive error estimates.

Theorem 2.5 *Under the same conditions as those of Theorem 2.3, it is true that, as $\omega \to \infty$,*

$$Q_s^A[H] - I[f] \sim \begin{cases} O(\frac{1}{\omega^{s+\frac{3}{2}}}), & \text{if } \sigma_s[H](\zeta) = 0, \\ O(\frac{1}{\omega^{s+1}}), & \text{if } \sigma_s[H](\zeta) \neq 0, \end{cases} \tag{2.23}$$

where $Q_s^A[H]$ is given in (2.17).

Proof On the basis of (2.5), (2.14), (2.16) and (2.17), together with (2.21) from Lemma 2.2, we can obtain for $\omega \to \infty$,

$$\begin{aligned} |Q_s^A[H] - I[f]| &= \left| \frac{(2a)^{\alpha-s+1}}{\omega^s} \int_{-1}^{1} (t+1)^{-\alpha-2} \sigma_s[H](t) J_{m+s}(\omega g(2a/(t+1)))dt \right| \\ &= \left| \frac{1}{(2a\omega)^s} \sigma_s[H](\zeta) \widetilde{M}(m+s,\omega) \right. \\ &\quad - \frac{2a^{\alpha+1}(\sigma_s[H](1) - \sigma_s[H](\zeta))}{(2a\omega)^{s+1} g'(a)} J_{m+s+1}(\omega g(a)) \\ &\quad \left. + \frac{(2a)^{\alpha-s}}{\omega^{s+1}} \int_{-1}^{1} (t+1)^{-\alpha-2} \sigma_{s+1}[H](t) J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right) dt \right| \\ &= \begin{cases} O(\frac{1}{\omega^{s+\frac{3}{2}}}), \text{ if } \sigma_s[H](\zeta) = 0, \\ O(\frac{1}{\omega^{s+1}}), \text{ if } \sigma_s[H](\zeta) \neq 0. \end{cases} \end{aligned}$$

This completes the proof of (2.23). □

Theorem 2.6 *Under the same conditions as those of Theorem 2.4, and assume that multiplicities $m_\nu, m_\eta \geq s$, then for the case of $g(x)$ with zeros and without stationary points, and as $\omega \to \infty$, the absolute error of the the modified Filon-type method behaves asymptotically as,*

$$Q_s^F[H] - I[f] \sim \begin{cases} O(\frac{1}{\omega^{s+\frac{3}{2}}}), & \text{if } \sigma_s[H-P](\zeta) = 0, \\ O(\frac{1}{\omega^{s+1}}), & \text{if } \sigma_s[H-P](\zeta) \neq 0. \end{cases} \tag{2.24}$$

Proof We now replace $H(t)$ with $H(t) - P(t)$ in the above asymptotic expansion (2.13). Since $P^{(j)}(\zeta) = H^{(j)}(\zeta)$, $P^{(j)}(1) = H^{(j)}(1)$, $j = 0, 1, \ldots, s-1$, it follows that $\sigma_k[H-P](1) = \sigma_k[H-P](\zeta) = 0$, $k = 0, 1, \ldots, s-1$. Therefore, by using (2.5) together with (2.21) from Lemma 2.2, we can obtain the desired results (2.24) directly from both the above asymptotic expansion (2.13) and the proof of Theorem 2.5. □

2.4 Computation of the Modified Moments M_k Required in the Above Modified Filon-Type Methods

From [14, p. 851], we have

$$\int_1^{+\infty} y^{\tau-1}(y-1)^{\lambda-1} G_{p,q}^{m,n}\left(\begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix} \middle| \sigma y \right) dy$$

$$= \Gamma(\lambda) G^{m+1,n}_{p+1,q+1}\left(\begin{matrix} a_1,\ldots,a_n,a_{n+1},\ldots,a_p,1-\tau, \\ 1-\tau-\lambda,b_1,\ldots,b_m,b_{m+1},\ldots,b_q \end{matrix}\middle|\sigma\right),$$

where the restrictions on the parameters are listed as follows:
either $p+q \le 2(m+n)$, $|\arg\sigma| \le (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, $\Re(1-\tau-\lambda-a_j) > -1$, $j = 1,2,\ldots,n$, $\Re(\lambda) > 0$, $\Re[\sum_{j=1}^{p} a_j - \sum_{k=1}^{q} b_k + (q-p)(1-\tau-\lambda+\frac{1}{2})] > -\frac{1}{2}$, or $q < p$ (or $q \le p$ for $|\sigma| > 1$), $\Re(1-\tau-\lambda-a_j) > -1$, $j = 1,2,\ldots,n$, $\Re(\lambda) > 0$.

By using changes of the variable, and some recurrence relations and identities of the Meijer G-function $G^{m,n}_{p,q}$ [16,31], the reference [17] gives an important generalized formula

$$\int_a^{+\infty} x^{\mu-1}(x-a)^{\nu-1} G^{m,n}_{p,q}\left(\begin{matrix} a_1,\ldots,a_n,a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m,b_{m+1},\ldots,b_q \end{matrix}\middle|\sigma x^t\right) dx$$
$$= \frac{\Gamma(\nu)}{t^{\nu} a^{1-\mu-\nu}} G^{m+t,n}_{p+t,q+t}\left(\begin{matrix} a_1,\ldots,a_n,a_{n+1},\ldots,a_p,\frac{1-\mu}{t},\ldots,\frac{t-\mu}{t} \\ \frac{1-\mu-\nu}{t},\ldots,\frac{t-\mu-\nu}{t},b_1,\ldots,b_m,b_{m+1},\ldots,b_q \end{matrix}\middle|\sigma a^t\right), \quad t \in N^+, \tag{2.25}$$

where the restrictions on the parameters $a, \mu, \nu, \sigma, m, n, p, q$ are similar to the above and hence are omitted. Throughout this paper, we only need to employ the special case of $a = \nu = 1$ in (2.25) for deriving the explicit formulae of the desired modified moments. Moreover, the Meijer G-function $G^{m,n}_{p,q}$ can be expressed via the Mellin–Barnes integral in the complex plane [3, pp. 206, 207]

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1,\ldots,a_n,a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m,b_{m+1},\ldots,b_q \end{matrix}\middle| z\right)$$
$$= \frac{1}{2\pi i}\int_L \frac{\prod_{k=1}^{m}\Gamma(b_k-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{k=m+1}^{q}\Gamma(1-b_k+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)} z^s ds. \tag{2.26}$$

There are three different paths L of integration (for details one can refer to [3, pp. 206, 207] and [14, p. 1032]).

Following the identity in [3, p. 219] and [14, p. 1034],

$$J_m(x) = G^{1,0}_{0,2}\left(\begin{matrix} - \\ \frac{1}{2}m, -\frac{1}{2}m \end{matrix}\middle|\frac{1}{4}x^2\right),$$

together with (2.25), we have

$$\int_1^{+\infty} x^{\lambda} J_m(\omega x)dx$$
$$= \int_1^{+\infty} x^{\lambda} G^{1,0}_{0,2}\left(\begin{matrix} - \\ \frac{1}{2}m, -\frac{1}{2}m \end{matrix}\middle|\frac{1}{4}\omega^2 x^2\right)dx$$
$$= \frac{1}{2} G^{3,0}_{2,4}\left(\begin{matrix} -\frac{\lambda}{2}, -\frac{\lambda-1}{2} \\ -\frac{\lambda+1}{2}, -\frac{\lambda}{2}, \frac{1}{2}m, -\frac{1}{2}m \end{matrix}\middle|\frac{1}{4}\omega^2\right). \tag{2.27}$$

(1) If $\lim_{x\to+\infty} g(x) = +\infty$ and $g(a) \neq 0$, by using (2.27), we have

$$M_k = \int_{g(a)}^{+\infty} y^k J_m(\omega y)dy$$
$$= (g(a))^{k+1}\int_1^{+\infty} y^k J_m(\omega g(a) y)dy$$
$$= \frac{1}{2}(g(a))^{k+1} G^{3,0}_{2,4}\left(\begin{matrix} -\frac{k}{2}, -\frac{k-1}{2} \\ -\frac{k+1}{2}, -\frac{k}{2}, \frac{1}{2}m, -\frac{1}{2}m \end{matrix}\middle|\frac{1}{4}(\omega g(a))^2\right). \tag{2.28}$$

(2) If $\lim_{x\to+\infty} g(x) = +\infty$ and $g(a) = 0$, by using (2.27), we have

$$M_k = \int_0^{+\infty} y^k J_m(\omega y)dy = \int_0^1 y^k J_m(\omega y)dy + \int_1^{+\infty} y^k J_m(\omega y)dy = G(\omega, m, k) + \frac{1}{2} G^{3,0}_{2,4}\left(\begin{matrix} -\frac{k}{2}, -\frac{k-1}{2} \\ -\frac{k+1}{2}, -\frac{k}{2}, \frac{1}{2}m, -\frac{1}{2}m \end{matrix} \middle| \frac{1}{4}\omega^2 \right), \tag{2.29}$$

where $G(\omega, m, k)$ can be expressed by the following formulas (see [14, p. 676], [27, p. 44] and [1, p. 480]), for $\Re(k + m) > -1$, $\omega > 0$,

$$G(\omega, m, k) = \int_0^1 x^k J_m(\omega x)dx = \frac{2^u \Gamma(\frac{m+k+1}{2})}{\omega^{k+1}\Gamma(\frac{m-k+1}{2})} + \frac{1}{\omega^k}[(k + m - 1)J_m(\omega)S_{k-1,m-1}(\omega) - J_{m-1}(\omega)S_{k,m}(\omega)] \tag{2.30}$$

$$= \frac{\omega^m}{2^m(k + m + 1)\Gamma(m + 1)} {}_1F_2(\frac{k + m + 1}{2}; \frac{k + m + 3}{2}, m + 1; -\frac{\omega^2}{4}) \tag{2.31}$$

$$= \frac{\Gamma(\frac{m+k+1}{2})}{\omega\Gamma(\frac{m-k+1}{2})} \sum_{j=0}^{\infty} \frac{(m + 2j + 1)\Gamma(\frac{m-k+1}{2} + j)}{\Gamma(\frac{m+k+3}{2} + j)} J_{m+2j+1}(\omega). \tag{2.32}$$

When $\lim_{x\to+\infty} g(x) = -\infty$, the modified moments M_k can be similarly obtained. Here, $S_{\mu,\nu}(z)$, $\Gamma(z)$, ${}_1F_2(\mu; \nu, \lambda; z)$ denote a Lommel function of the second kind, the Gamma function, a class of generalized hypergeometric function, respectively. Moreover, ${}_1F_2(\mu; \nu, \lambda; z)$ converges for all $|z|$. From [36, p. 346], $S_{\mu,\nu}(z)$ can be expressed in terms of ${}_1F_2(\mu; \nu, \lambda; z)$, namely,

$$S_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu + \nu + 1)(\mu - \nu + 1)} {}_1F_2(1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 3}{2}; -\frac{z^2}{4}) - \frac{2^{\mu-1}\Gamma(\frac{\mu+\nu+1}{2})}{\pi\Gamma(\frac{\nu-\mu}{2})}(J_\nu(z) - \cos(\pi(\mu - \nu)/2)Y_\nu(z)), \tag{2.33}$$

where $Y_\nu(z)$ is a Bessel function of the second kind of order ν. The efficient implementation of the moments is based on the fast computation of the above-mentioned special functions. Obviously, when programming the proposed algorithm in a language like Matlab, we can calculate the values of $\Gamma(z)$, $J_m(z)$, $Y_m(z)$, ${}_1F_2(\mu; \nu, \lambda; z)$ and the Meijer G-function $G^{m,n}_{p,q}$ by invoking the built-in functions 'gamma(z)', 'besselj(m, z)','bessely(m, z)', 'hypergeom($\mu, [\nu, \lambda], z$)', 'meijerG($[a_1, \ldots, a_n]$, $[a_{n+1}, \ldots, a_p]$,$[b_1, \ldots, b_m]$, $[b_{m+1}, \ldots, b_q]$, z)', respectively.

3 Asymptotic Analysis and Quadrature Rules of the Case with Stationary Points

In this section we only consider the case that $g(x)$ has one stationary point on $[a, +\infty)$. If $g(x)$ has a finite number of stationary points on $[a, +\infty)$, we have to split the whole interval into some subintervals such that $g(x)$ has only one stationary point on each subinterval. Moreover, there are also two types, i.e., type I: $g(x) = 0$ or type II: $g(x) \neq 0$ at this stationary point. Here we mainly focus on the analysis and computation of type I. It can be similarly processed for the latter case. Additionally, it should be noted that when the only one stationary point may be different from its only one zero point, we need to divide the whole interval into two subintervals such that one includes the zero point and the other contains the stationary point.

3.1 Asymptotic Analysis of the Case with Stationary Points

For the case of $g(x)$ with only one stationary point at the interior point $x = \varsigma \in (a, +\infty)$, we derive the asymptotic expansion as follows.

Theorem 3.1 *Let*

$$\begin{aligned} \tau_0[H](t) &= H(t), \\ \tau_{k+1}[H](t) &= -\frac{\alpha(t+1)\{\tau_k[H](t) - C_r[\tau_k](t,\xi)\}}{g'(\frac{2a}{t+1})} \\ &\quad + (t+1)^2\left[g\left(\frac{2a}{t+1}\right)\right]^{m+k+1}\frac{d}{dt}\left\{\frac{\tau_k[H](t) - C_r[\tau_k](t,\xi)}{[g(\frac{2a}{t+1})]^{m+k+1}g'(\frac{2a}{t+1})}\right\},\ k = 0, 1, 2, \ldots, \\ C_r[\tau_k](t,\xi) &= \sum_{j=0}^{r}\frac{\tau_k[H]^{(j)}(\xi)}{j!}(t-\xi)^j. \end{aligned}$$

Suppose that $f(x), g(x) \in C^\infty[a, +\infty)$*. Moreover, assume that* $\frac{\tau_s[H](t) - C_r[\tau_s](t,\xi)}{g'(\frac{2a}{t+1})}$ *for* $s = 0, 1, 2, \ldots,$ *converge as* $t \to -1^+$*. If* $g(\varsigma) = g'(\varsigma) = \cdots = g^{(r)}(\varsigma) = 0$, $g^{(r+1)}(\varsigma) \neq 0$*, where* $x = \varsigma = \frac{2a}{1+\xi} \in (a, +\infty)$ *for* $\xi \in (-1, 1)$*, and positive integer* $r \geq 1$*, but* $g(x) \neq 0$ *and* $g'(x) \neq 0$ *for* $x \in [a, +\infty) \setminus \{\varsigma\}$, $\lim_{x\to+\infty} g'(x) \neq 0$*, (which implies that* $\lim_{x\to+\infty} g(x) = \infty$*), then for* $a > 0$ *and* $\alpha < 0$ *it follows that as* $\omega \to \infty$*,*

$$\begin{aligned} I[f] \sim &\sum_{k=0}^{\infty}\frac{1}{(2a\omega)^k}\sum_{j=0}^{r}\frac{\tau_k[H]^{(j)}(\xi)}{j!}M_j(\xi, m+k, \omega) \\ &- 2a^{\alpha+1}\sum_{k=0}^{\infty}\frac{\tau_k[H](1) - C_r[\tau_k](1,\xi)}{(2a\omega)^{k+1}g'(a)}J_{m+k+1}(\omega g(a)), \end{aligned} \tag{3.1}$$

where

$$M_j(\xi, m+k, \omega) = (2a)^{\alpha+1}\int_{-1}^{1}(t+1)^{-\alpha-2}(t-\xi)^j J_{m+k}(\omega g(2a/(t+1)))dt.$$

Proof By induction on $s \geq 1$, we can prove the following identity

$$\begin{aligned} I[f] = &\sum_{k=0}^{s-1}\frac{1}{(2a\omega)^k}\sum_{j=0}^{r}\frac{\tau_k[H]^{(j)}(\xi)}{j!}M_j(\xi, m+k, \omega) \\ &- 2a^{\alpha+1}\sum_{k=0}^{s-1}\frac{\tau_k[H](1) - C_r[\tau_k](1,\xi)}{(2a\omega)^{k+1}g'(a)}J_{m+k+1}(\omega g(a)) \\ &+ \frac{(2a)^{\alpha-s+1}}{\omega^s}\int_{-1}^{1}(t+1)^{-\alpha-2}\tau_s[H](t)J_{m+s}(\omega g(2a/(t+1)))dt. \end{aligned} \tag{3.2}$$

For $s = 1$, we have

$$
\begin{aligned}
I[f] = &\ (2a)^{1+\alpha} \int_{-1}^{1} (t+1)^{-\alpha-2} C_r[\tau_0](t,\xi) J_m\left(\omega g\left(\frac{2a}{t+1}\right)\right) dt \\
&+ (2a)^{1+\alpha} \int_{-1}^{1} (t+1)^{-\alpha-2} (\tau_0[H](t) - C_r[\tau_0](t,\xi)) J_m\left(\omega g\left(\frac{2a}{t+1}\right)\right) dt \\
= &\sum_{j=0}^{r} \frac{\tau_0[H]^{(j)}(\xi)}{j!} M_j(\xi, m, \omega) \\
&- \frac{(2a)^\alpha}{\omega} \int_{-1}^{1} (t+1)^{-\alpha} \frac{\tau_0[H](t) - C_r[\tau_0](t,\xi)}{[g(\frac{2a}{t+1})]^{m+1} g'(\frac{2a}{t+1})} \\
&\qquad d\left\{\left[g\left(\frac{2a}{t+1}\right)\right]^{m+1} J_{m+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right\} \\
= &\sum_{j=0}^{r} \frac{\tau_0[H]^{(j)}(\xi)}{j!} M_j(\xi, m, \omega) \\
&- \frac{(2a)^\alpha}{\omega} \left[(t+1)^{-\alpha} \frac{\tau_0[H](t) - C_r[\tau_0](t,\xi)}{g'(\frac{2a}{t+1})} J_{m+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right]_{-1}^{1} \\
&+ \frac{(2a)^\alpha}{\omega} \int_{-1}^{1} (t+1)^{-\alpha-2} \tau_1[H](t) J_{m+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right) dt \\
= &\sum_{j=0}^{r} \frac{\tau_0[H]^{(j)}(\xi)}{j!} M_j(\xi, m, \omega) - \frac{a^\alpha(\tau_0[H](1) - C_r[\tau_0](1,\xi))}{\omega g'(a)} J_{m+1}(\omega g(a)) \\
&+ \frac{(2a)^\alpha}{\omega} \int_{-1}^{1} (t+1)^{-\alpha-2} \tau_1[H](t) J_{m+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right) dt. \qquad (3.3)
\end{aligned}
$$

So the identity (3.2) holds for $s = 1$.

For $s \geq 1$, we find from the last formula of (3.2) that

$$
\begin{aligned}
&\frac{(2a)^{\alpha-s+1}}{\omega^s} \int_{-1}^{1} (t+1)^{-\alpha-2} \tau_s[H](t) J_{m+s}(\omega g(2a/(t+1))) dt \\
&= \frac{(2a)^{\alpha-s+1}}{\omega^s} \int_{-1}^{1} (t+1)^{-\alpha-2} C_r[\tau_s](t,\xi) J_{m+s}(\omega g(2a/(t+1))) dt \\
&\quad + \frac{(2a)^{\alpha-s+1}}{\omega^s} \int_{-1}^{1} (t+1)^{-\alpha-2} (\tau_s[H](t) - C_r[\tau_s](t,\xi)) J_{m+s}(\omega g(2a/(t+1))) dt \\
&= \frac{1}{(2a\omega)^s} \sum_{j=0}^{r} \frac{\tau_s[H]^{(j)}(\xi)}{j!} M_j(\xi, m+s, \omega) \\
&\quad - \frac{(2a)^{\alpha-s}}{\omega^{s+1}} \int_{-1}^{1} (t+1)^{-\alpha} \frac{\tau_s[H](t) - C_r[\tau_s](t,\xi)}{[g(\frac{2a}{t+1})]^{m+s+1} g'(\frac{2a}{t+1})} \\
&\qquad d\left\{\left[g\left(\frac{2a}{t+1}\right)\right]^{m+s+1} J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right\} \\
&= \frac{1}{(2a\omega)^s} \sum_{j=0}^{r} \frac{\tau_s[H]^{(j)}(\xi)}{j!} M_j(\xi, m+s, \omega)
\end{aligned}
$$

$$
-\frac{(2a)^{\alpha-s}}{\omega^{s+1}}\left[(t+1)^{-\alpha}\frac{\tau_s[H](t)-C_r[\tau_s](t,\xi)}{g'(\frac{2a}{t+1})}J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)\right]_{-1}^{1}
+\frac{(2a)^{\alpha-s}}{\omega^{s+1}}\int_{-1}^{1}(t+1)^{-\alpha-2}\tau_{s+1}[H](t)J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt
$$

$$
=\frac{1}{(2a\omega)^s}\sum_{j=0}^{r}\frac{\tau_s[H]^{(j)}(\xi)}{j!}M_j(\xi,m+s,\omega)-\frac{2a^{\alpha+1}(\tau_s[H](1)-C_r[\tau_s](1,\xi))}{(2a\omega)^{s+1}g'(a)}J_{m+s+1}(\omega g(a))
+\frac{(2a)^{\alpha-s}}{\omega^{s+1}}\int_{-1}^{1}(t+1)^{-\alpha-2}\tau_{s+1}[H](t)J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt. \tag{3.4}
$$

Combining (3.3) and (3.4) leads to the desired result (3.2). By letting $s\to\infty$ for (3.2), we complete the proof of (3.1). □

For the case of $g(x)$ with only one stationary point at the endpoint $x=\varsigma=a$, i.e., $\xi=1$, by L'Hôpital's rule r-times, we have

$$
\lim_{t\to 1^-}\frac{\tau_k[H](t)-C_r[\tau_k](t,1)}{g'(2a/(t+1))}=0,
$$

and then by the method analogous to that used in the proof of Theorem 3.1, we can derive the asymptotic expansion as follows.

Corollary 3.1 *Suppose that $f(x), g(x)\in C^\infty[a,+\infty)$. Moreover, assume that $f^{(k)}(x)$ and $g^{(k)}(x)$ for $k=0,1,2,\ldots,$ converge as $x\to+\infty$. If $g(a)=g'(a)=\cdots=g^{(r)}(a)=0$, $g^{(r+1)}(a)\neq 0$, where $x=a=\frac{2a}{1+\xi}$ for $\xi=1$ and positive integer $r\geq 1$, but $g(x)\neq 0$ and $g'(x)\neq 0$ for $x\in(a,+\infty)$, and $\lim_{x\to+\infty}g'(x)\neq 0$, (which implies that $\lim_{x\to+\infty}g(x)=\infty$), then for $a>0$ and $\alpha<0$ it follows that as $\omega\to\infty$,*

$$
I[f]\sim\sum_{k=0}^{\infty}\frac{1}{(2a\omega)^k}\sum_{j=0}^{r}\frac{\tau_k[H]^{(j)}(\xi)}{j!}M_j(\xi,m+k,\omega), \tag{3.5}
$$

where $\tau_k[H](t)$ and $M_j(\xi,m+k,\omega)$ are given in Theorem 3.1.

By truncating after the first s terms of the asymptotic expansions (3.1) and (3.5), we can obtain the following asymptotic formulae.

(1) For the stationary point $x=\varsigma\in(a,+\infty)$, we have

$$
Q_m^A[f]=\sum_{k=0}^{s-1}\frac{1}{(2a\omega)^k}\sum_{j=0}^{r}\frac{\tau_k[H]^{(j)}(\xi)}{j!}M_j(\xi,m+k,\omega)
-2a^{\alpha+1}\sum_{k=0}^{s-1}\frac{\tau_k[H](1)-C_r[\tau_k](1,\xi)}{(2a\omega)^{k+1}g'(a)}J_{m+k+1}(\omega g(a)). \tag{3.6}
$$

(2) For the stationary point $x=a$, we have

$$
Q_m^A[f]=\sum_{k=0}^{s-1}\frac{1}{(2a\omega)^k}\sum_{j=0}^{r}\frac{\tau_k[H]^{(j)}(\xi)}{j!}M_j(\xi,m+k,\omega). \tag{3.7}
$$

3.2 Modified Filon-Type Method and Its Error Analysis for the Case with Stationary Points

Now, we consider the modified Filon-type method and its error analysis for the case with stationary points. Before proceeding, we first establish a helpful lemma for the case of $g(\frac{2a}{t+1})$ with a stationary point of type I and of order r at $t=\xi=1$. When $g(\frac{2a}{t+1})$ has a stationary point of type I and of order r at $t=\xi\in(-1,1)$, a similar result can be obtained.

Lemma 3.1 *Suppose that* $g(\frac{2a}{t+1})\in C^{\infty}(-1,1]$ *and it has a stationary point of type I and of order* r *at* $t=\xi=1$ *and* $g^{(r+1)}(\frac{2a}{t+1})\neq 0$ *for* $t\in(-1,1]$, $a>0,\alpha<0$. *Let*

$$\psi_k(t)=\begin{cases}2a(t+1)^{\alpha}g'\left(\frac{2a}{t+1}\right)\left[g\left(\frac{2a}{t+1}\right)\right]^{\frac{k-r}{r+1}}, & \text{if } g^{(r+1)}\left(\frac{2a}{t+1}\right)>0,\\ 2a(t+1)^{\alpha}g'\left(\frac{2a}{t+1}\right)\left[-g\left(\frac{2a}{t+1}\right)\right]^{\frac{k-r}{r+1}}, & \text{if } g^{(r+1)}\left(\frac{2a}{t+1}\right)<0,\end{cases}$$

and

$$\widetilde{E}_n=\left\{v(t)|v(t)=\sum_{k=0}^{n}c_k\psi_k(t),\quad c_k\in\mathbb{R}\right\}.$$

Then $\widetilde{E}_n$ *is an extended complete Chebyshev system.*

Proof Here we only focus on the case of $g^{(r+1)}(\frac{2a}{t+1})>0$ for $t\in(-1,1]$. When $g^{(r+1)}(\frac{2a}{t+1})<0$, we have the similar result. With the assumption, we can derive that $g^{(j)}(\frac{2a}{t+1})>0$ for $t\in(-1,1)$ and $j=1,\ldots,r$ and thus $g(\frac{2a}{t+1})$ is strictly increasing in (-1,1). For any $v(t)\in\widetilde{E}_l$ and $l\in\{0,1,\ldots,n\}$, letting $y^{r+1}=g(\frac{2a}{t+1})$, we obtain

$$v(t)=\sum_{k=0}^{l}c_k\psi_k(t)=\phi(t)\sum_{k=0}^{l}c_ky^k,$$

where $\phi(t)=2a(t+1)^{\alpha}g'(\frac{2a}{t+1})g(\frac{2a}{t+1})^{\frac{-r}{r+1}}$. It is easily verified that

$$\lim_{t\to 1}\phi(t)=2^{1+\alpha}a(r+1)\left[\frac{g^{(r+1)}(a)}{(r+1)!}\right]^{\frac{1}{r+1}}>0,$$

and then $\phi(t)$ has a removable discontinuity point at $t=1$. Next, define that

$$\phi(1)=2^{1+\alpha}a(r+1)\left[\frac{g^{(r+1)}(a)}{(r+1)!}\right]^{\frac{1}{r+1}}.$$

This implies that $\phi(t)\in C^{\infty}(-1,1]$ and $\phi(t)>0$ for $t\in(-1,1]$. The remaining part of the proof is similar to that of Lemma 2.1. □

Here, under the assumption $g(x)$ in Theorem 3.1 or Corollary 3.1, we consider a modified Filon-type method for the case that $g^{(r+1)}(2a/(t+1))>0$ for $t\in(-1,1]$. Once that $g^{(r+1)}(2a/(t+1))<0$, by making use of $J_m(x)=e^{-m\pi i}J_m(-x)$, we can rewrite (1.2) as

$$I[f]=e^{-m\pi i}(2a)^{\alpha+1}\int_{-1}^{1}(t+1)^{-\alpha-2}H(t)J_m(-\omega g(2a/(t+1)))dt,$$

with $-g^{(r+1)}(2a/(t+1)) > 0$. From the asymptotic expansions (3.1) and (3.5), we can see that the value of $I[f]$ also depends on the stationary point ξ of $g(2a/(t+1))$ except endpoints, where $\xi \in (-1, 1]$. Therefore, we need to impose ξ as an interpolating point. Suppose that $t_\ell = \xi$ for some $\ell \in \{0, 1, \ldots, \eta\}$. Then, define that

$$\psi_k(t) = 2a(t+1)^\alpha g'\left(\frac{2a}{t+1}\right)\left[g\left(\frac{2a}{t+1}\right)\right]^{\frac{k-r}{r+1}}.$$

From Lemma 3.1 and [33, Theorem 9.9 on page 370], we know that there exists a unique function $\tilde{P}(t) = \sum_{k=0}^{n} c_k \psi_k(t) \in \widetilde{E}_n$ such that

$$\tilde{P}^{(j)}(t_k) = H^{(j)}(t_k), \quad j = 0, 1, 2, m_k - 1,$$

for every integer $0 \le k \le \eta$ and $n = \Sigma_{k=0}^{\eta} m_k - 1$. Then the modified Filon-type method for the case with stationary points is defined by

$$Q_s^F[H] \equiv \tilde{I}[\tilde{P}] = (2a)^{\alpha+1} \sum_{k=0}^{n} c_k \widetilde{M}_k. \tag{3.8}$$

Here, if $\lim_{x\to+\infty} g(x) = +\infty$ and $g(a) \neq 0$, from (2.27), the modified moments $\widetilde{M}_k$ can be expressed by

$$\begin{aligned}\widetilde{M}_k &= \int_{g(a)}^{+\infty} y^{\frac{k-r}{r+1}} J_m(\omega y) dy \\ &= (g(a))^{\frac{k+1}{r+1}} \int_1^{+\infty} y^{\frac{k-r}{r+1}} J_m(\omega g(a) y) dy \\ &= \frac{1}{2}(g(a))^{\frac{k+1}{r+1}} G_{2,4}^{3,0}\left(\begin{matrix} \frac{r-k}{2(r+1)}, \frac{2r-k+1}{2(r+1)} \\ -\frac{k+1}{2(r+1)}, \frac{r-k}{2(r+1)}, \frac{1}{2}m, -\frac{1}{2}m \end{matrix} \middle| \frac{1}{4}(\omega g(a))^2\right).\end{aligned} \tag{3.9}$$

If $\lim_{x\to+\infty} g(x) = +\infty$ and $g(a) = 0$, from (2.27), the modified moments $\widetilde{M}_k$ can be rewritten as

$$\begin{aligned}\widetilde{M}_k &= \int_0^{+\infty} y^{\frac{k-r}{r+1}} J_m(\omega y) dy \\ &= \int_0^1 y^{\frac{k-r}{r+1}} J_m(\omega y) dy + \int_1^{+\infty} y^{\frac{k-r}{r+1}} J_m(\omega y) dy \\ &= G\left(\omega, m, \frac{k-r}{r+1}\right) + \frac{1}{2} G_{2,4}^{3,0}\left(\begin{matrix} \frac{r-k}{2(r+1)}, \frac{2r-k+1}{2(r+1)} \\ -\frac{k+1}{2(r+1)}, \frac{r-k}{2(r+1)}, \frac{1}{2}m, -\frac{1}{2}m \end{matrix} \middle| \frac{1}{4}\omega^2\right),\end{aligned} \tag{3.10}$$

where $G\left(\omega, m, \frac{k-r}{r+1}\right)$ can be explicitly computed by (2.30)–(2.32).

Remark 3.1 On the one hand, it should be noted from (2.28), (2.29), (3.9) and (3.10) that the desired modified moments M_k and $\widetilde{M}_k$ can be explicitly expressed and computed by some special functions, such as the Meijer G-function, the Lommel function and the generalized hypergeometric function (see Tables 1, 2, 3). On the other hand, those numerical examples in the Sect. 4 show that the presented numerical method can produce quite accurate approximations for computing the considered integral (1.1), which in turn verifies that the values of the required modified moments can be of great precision. In fact, from those numerical examples in the Sect. 4, we can also see that the required accuracy level can be obtained only by using a small number of nodes $N = 2, 3, 4, 5, 6$ and multiplicities $s = 1, 2, 3$ at the endpoint $t = 1$ with the Hermite interpolation problem. This implies that the degree of the required Hermite interpolation polynomials is very small, in general, no more than 10. This leads to the fact that the values of n in (2.18) of Theorem 2.4 and (3.8), are no more than 10. Moreover, from (2.18) and (3.8), we can see that $0 \le k \le n$. Hence, the values of k in the modified moments (2.28), (2.29), (3.9) and (3.10) required in those numerical examples of the Sect. 4, are in general very small (≤ 10). This implies that we do not need to consider the case of large k in practical applications. The numerical experiments in Tables 1 and 2, show that the modified moments formulae (2.28) and (2.29) can be used for small k with moderate or large ω. Actually, the modified moments formulae (2.28) and (2.29) may be also available for some large k with large ω, which are shown in the numerical example (see Table 3). However, the modified moments formulae (2.28), (2.29), (3.9) and (3.10) may be not applicable for some large k with small ω, e.g., $k = 200$, $\omega = 10$, $m = 2$. Fortunately, this paper mainly focuses on the case of moderate or large ω and small k (see those numerical examples of the Sect. 4 and Tables 1, 2). In [30], the modified moments (2.29) can be also explicitly represented by the Gamma function under the given conditions. In fact, for the other case of small k with $\Re(m + k) > -1$ and moderate or large ω, the modified moments (2.29) can be explicitly expressed and computed by some special functions, such as the Meijer G-function, the Lommel function and the generalized hypergeometric function (see Table 2). The related convergence analysis for all parameters ω, k, m of the modified moments formulae (2.28), (2.29), (3.9) and (3.10) will be discussed theoretically in our future research. In addition, because of only using the small number of interpolation node points and multiplicities, the system of linear equations obtained in the related Hermite interpolation is usually very small and can be easily solved. Hence, the coefficients c_k of the related interpolation polynomials can be efficiently and accurately computed by solving the linear system of equations of small size. Furthermore, the calculation of the modified moments using the Meijer G-function looks complicated. In the future, we shall seek a simple and feasible method for simplifying the computation of the modified moments.

We now derive the asymptotics of the above moments $M_j(\xi, m, \omega)$ from Theorem 3.1 and Corollary 3.1, for performing the error estimate of the above quadrature rules.

Lemma 3.2 *Under the assumption of g in Theorem 3.1, for $0 \le j \le r$, it follows that*

$$M_j(\xi, m, \omega) = O\left(\frac{1}{\omega^{\frac{j+1}{r+1}}}\right), \quad as \;\; \omega \to \infty. \tag{3.11}$$

Proof Change of variable $t = \frac{2a}{x} - 1$ yields

$$\begin{aligned} M_j(\xi, m, \omega) &= (2a)^{\alpha+1} \int_{-1}^{1} (t+1)^{-\alpha-2} (t-\xi)^j J_m(\omega g(2a/(t+1))) dt \\ &= \int_a^{+\infty} x^{\alpha-j} [2a - (1+\xi)x]^j J_m(\omega g(x)) dx. \end{aligned} \tag{3.12}$$

For the case of $\lim_{x\to+\infty} g(x) = +\infty$, by subdividing the integration interval at $x = \varsigma = \frac{2a}{1+\xi}$ and then change of variable $g(x) = y^{r+1}$, the integral formula in the last line of (3.12) is rewritten as

$$\begin{aligned} M_j(\xi, m, \omega) = &\int_a^{\varsigma} x^{\alpha-j} [2a - (1+\xi)x]^j J_m(\omega g(x)) dx \\ &+ \int_{\varsigma}^{+\infty} x^{\alpha-j} [2a - (1+\xi)x]^j J_m(\omega g(x)) dx \end{aligned}$$

Table 1 The values of M_k, $k = 0, 2, 4, 8$, explicitly computed by (2.28), for $g(a) = 1$, $m = 1$, with $\omega = 50, 100, 200, 400$

$\omega \backslash$k	0	2	4	8
50	$1.116246553385037 \times 10^{-3}$	$1.194256015885177 \times 10^{-3}$	$1.268443859134484 \times 10^{-3}$	$1.402592924092711 \times 10^{-3}$
100	$1.998585030422312 \times 10^{-4}$	$2.152875734450537 \times 10^{-4}$	$2.305444137891201 \times 10^{-4}$	$2.603959410967644 \times 10^{-4}$
200	$-7.718719965282545 \times 10^{-5}$	$-7.447197274370655 \times 10^{-5}$	$-7.174185144003891 \times 10^{-5}$	$-6.624349384777230 \times 10^{-5}$
400	$-9.706295382695991 \times 10^{-5}$	$-9.694767809660256 \times 10^{-5}$	$-9.682755498234040 \times 10^{-5}$	$-9.657284012477980 \times 10^{-5}$

Table 2 The values of M_k, $k = 0, 2, 4, 6$, explicitly computed by (2.29), for $g(a) = 0$, $m = 2$, with $\omega = 25, 50, 100, 200$

$\omega \backslash$k	0	2	4	6
25	$4.000000000000000 \times 10^{-2}$	$1.920000000000003 \times 10^{-4}$	$-1.535999999999031 \times 10^{-6}$	$5.160960000029705 \times 10^{-8}$
50	$2.000000000000000 \times 10^{-2}$	$2.399999999999993 \times 10^{-5}$	$-4.799999999988841 \times 10^{-8}$	$4.032000000802477 \times 10^{-10}$
100	$1.000000000000000 \times 10^{-2}$	$3.000000000000290 \times 10^{-6}$	$-1.499999999860988 \times 10^{-9}$	$3.150000117517482 \times 10^{-12}$
200	$5.000000000000000 \times 10^{-3}$	$3.750000000000498 \times 10^{-7}$	$-4.687499997532707 \times 10^{-11}$	$2.460938354140185 \times 10^{-14}$

Table 3 The values of M_k, $k = 100, 200, 400, 800$, explicitly computed by (2.28), for $g(a) = 1$, $m = 1$, with $\omega = 50, 100, 200, 400$

$\omega\backslash$k	100	200	400	800
50	$1.006151397552692 \times 10^{-3}$	$5.245012820336750 \times 10^{-4}$	$2.571566687745625 \times 10^{-4}$	$1.257505160157230 \times 10^{-4}$
100	$4.895521320536341 \times 10^{-4}$	$3.493449768696509 \times 10^{-4}$	$1.933261299129579 \times 10^{-4}$	$9.802271874013719 \times 10^{-5}$
200	$4.740755042680808 \times 10^{-5}$	$9.784996830332475 \times 10^{-5}$	$9.338219172086505 \times 10^{-5}$	$5.937431681218065 \times 10^{-5}$
400	$-8.601201929780201 \times 10^{-5}$	$-6.849261066532220 \times 10^{-5}$	$-3.697501194984548 \times 10^{-5}$	$-1.015985660724051 \times 10^{-5}$

$$
\begin{aligned}
&= \int_{\sqrt[r+1]{g(a)}}^{0} x^{\alpha-j}[2a-(1+\xi)x]^j \frac{(r+1)y^r}{g'(x)} J_m(\omega y^{r+1})dy \\
&\quad + \int_0^{+\infty} x^{\alpha-j}[2a-(1+\xi)x]^j \frac{(r+1)y^r}{g'(x)} J_m(\omega y^{r+1})dy \\
&= (r+1)\left(\frac{2a}{\varsigma}\right)^j \left[\int_{\sqrt[r+1]{g(a)}}^{0} x^{\alpha-j}(\varsigma-x)^j \frac{y^r}{g'(x)} J_m(\omega y^{r+1})dy \right. \\
&\quad \left. + \int_0^{+\infty} x^{\alpha-j}(\varsigma-x)^j \frac{y^r}{g'(x)} J_m(\omega y^{r+1})dy \right] \\
&= (r+1)\left(\frac{2a}{\varsigma}\right)^j \left[-\int_0^{\sqrt[r+1]{g(a)}} y^j \varphi(y) J_m(\omega y^{r+1})dy \right. \\
&\quad \left. + \int_0^{+\infty} y^j \varphi(y) J_m(\omega y^{r+1})dy \right], \qquad (3.13)
\end{aligned}
$$

where

$$
\varphi(y) = \frac{x^{\alpha-j} y^{r-j} (\varsigma - x)^j}{g'(x)},
$$

and it follows from the given conditions on $g(x)$ and $g(x) = y^{r+1}$ that $\varphi(y)$ is a C^∞ function and $\varphi(0) \neq 0$. When $g(a) = 0$, the first integral in the last line of (3.13) vanishes.

Next, for $g(a) \neq 0$ we set $\phi(y) = y^j \varphi(y)$ and $\sqrt[r+1]{g(a)} = A \neq 0$ in the last line of (3.13). This implies from $\varphi(0) \neq 0$ that $\phi^{(n)}(0) = 0$ for $0 \leq n \leq j-1$ and $\phi^{(n)}(0) \neq 0$ for $j \leq n \leq r$. By similar asymptotic analysis to that of Theorem 3.1 and Corollary 3.1, the two integrals in the last line of (3.13) employ the asymptotic expansions as follows,

$$
\begin{aligned}
&-\int_0^{\sqrt[r+1]{g(a)}} y^j \varphi(y) J_m(\omega y^{r+1})dy \\
&= -\int_0^A \phi(y) J_m(\omega y^{r+1})dy \\
&\sim -\sum_{n=0}^{r} \frac{\phi^{(n)}(0)}{n!} \int_0^A y^n J_m(\omega y^{r+1})dy - \frac{1}{\omega} \frac{\phi(A) - \sum_{n=0}^{r} \frac{\phi^{(n)}(0)}{n!} A^n}{(r+1)A^r} J_{m+1}(\omega A^{r+1}) \\
&= -\sum_{n=j}^{r} \frac{\phi^{(n)}(0)}{n!} \int_0^A y^n J_m(\omega y^{r+1})dy - \frac{1}{\omega} \frac{\phi(A) - \sum_{n=j}^{r} \frac{\phi^{(n)}(0)}{n!} A^n}{(r+1)A^r} J_{m+1}(\omega A^{r+1}),
\end{aligned} \qquad (3.14)
$$

$$
\begin{aligned}
&\int_0^{+\infty} y^j \varphi(y) J_m(\omega y^{r+1})dy \\
&\sim \sum_{n=0}^{r} \frac{\phi^{(n)}(0)}{n!} \int_0^{+\infty} y^n J_m(\omega y^{r+1})dy \\
&= \sum_{n=j}^{r} \frac{\phi^{(n)}(0)}{n!} \int_0^{+\infty} y^n J_m(\omega y^{r+1})dy. \qquad (3.15)
\end{aligned}
$$

By setting $\omega y^{r+1} = u$, for $0 \le j \le n \le r$, the integrals in the last lines of (3.14) and (3.15) can be rewritten as

$$\begin{aligned}
\int_0^A y^n J_m(\omega y^{r+1}) dy &= \frac{1}{\omega^{\frac{n+1}{r+1}}} \int_0^{\omega g(a)} u^{\frac{n-r}{r+1}} J_m(u) du \\
&= \frac{1}{\omega^{\frac{n+1}{r+1}}} \left[\int_0^1 u^{\frac{n-r}{r+1}} J_m(u) du + \int_1^{\omega g(a)} u^{\frac{n-r}{r+1}} J_m(u) du \right] \\
&= O\left(\frac{1}{\omega^{\frac{n+1}{r+1}}} \right), \quad \omega \to +\infty, \qquad (3.16)\\
\int_0^{+\infty} y^n J_m(\omega y^{r+1}) dy &= \frac{1}{\omega^{\frac{n+1}{r+1}}} \int_0^{+\infty} u^{\frac{n-r}{r+1}} J_m(u) du \\
&= \frac{1}{\omega^{\frac{n+1}{r+1}}} \left[\int_0^1 u^{\frac{n-r}{r+1}} J_m(u) du + \int_1^{+\infty} u^{\frac{n-r}{r+1}} J_m(u) du \right] \\
&= O\left(\frac{1}{\omega^{\frac{n+1}{r+1}}} \right), \quad \omega \to +\infty. \qquad (3.17)
\end{aligned}$$

Here, for $n = r$, since $\int_0^\infty J_m(t) dt = 1$ [1, p. 486], the right-hand integrals in the first line of (3.16) and (3.17) are convergent as $\omega \to +\infty$. For $0 \le j \le n < r$, i.e., $-1 < \frac{n-r}{r+1} < 0$, on the one hand, from [1, p. 360] and [28], when m is fixed and $m > -1$,

$$J_m(u) \sim \frac{u^m}{2^m \Gamma(m+1)}, \quad \text{as } u \to 0^+,$$

which leads to that

$$u^{\frac{n-r}{r+1}} J_m(u) \sim \frac{u^{m+\frac{n-r}{r+1}}}{2^m \Gamma(m+1)}, \quad \text{as } u \to 0^+.$$

Note that $u^{m+\frac{n-r}{r+1}} \ge 0$ and $J_m(u) \ge 0$ for $0 < u \le 1$. Hence, the integrand $u^{m+\frac{n-r}{r+1}} J_m(u)$ is non-negative for $0 < u \le 1$. Since the integral $\int_0^1 u^{m+\frac{n-r}{r+1}} du$ is convergent for $m+\frac{n-r}{r+1} > -1$, by comparison test, the first integral $\int_0^1 u^{\frac{n-r}{r+1}} J_m(u) du$ in the second line of (3.16) and (3.17) is convergent for $m + \frac{n-r}{r+1} > -1$. On the other hand, as $\omega \to +\infty$, the second integral $\int_1^{\omega g(a)} u^{\frac{n-r}{r+1}} J_m(u) du$ in the second line of (3.16) and the second integral $\int_1^{+\infty} u^{\frac{n-r}{r+1}} J_m(u) du$ in the second line of (3.17) have the same convergence. Moreover, when $u \in [1, +\infty)$, the Bessel function $J_m(u)$ is a sign changing function. Since $F(t) = \int_1^t J_m(u) du$ is bounded on $[1, +\infty)$ and $\int_1^{+\infty} J_m(u) du$ is convergent, by Dirichlet's test or Abel's test, it is easily verified that the infinite integral $\int_1^{+\infty} u^{\frac{n-r}{r+1}} J_m(u) du$ is convergent for $-1 < \frac{n-r}{r+1} < 0$. Therefore, this leads to the two estimates in the last lines of (3.16) and (3.17) for all $0 \le n < r$, $m + \frac{n-r}{r+1} > -1$ (that is, $m > -\frac{1}{r+1} > -1$). Hence, the first term in the last line of (3.14) and the last line of (3.15) behave asymptotically as $O\left(\frac{1}{\omega^{\frac{j+1}{r+1}}} \right)$. For $A \ne 0$, the second term in the last line of (3.14) behaves asymptotically as $O\left(\frac{1}{\omega^{\frac{3}{2}}} \right)$. Therefore, the two integrals in the last line of (3.13) behave asymptotically as $O\left(\frac{1}{\omega^{\frac{j+1}{r+1}}} \right)$. When $\lim_{x \to +\infty} g(x) = -\infty$, the similar conclusion can be obtained. Therefore we complete this proof. □

In the sequel, we show error estimate of (3.6) and (3.7).

Theorem 3.2 *Under the same conditions as those of Theorem 3.1 or Corollary 3.1, it is true that, as $\omega \to \infty$,*

$$Q_s^A[H] - I[f] \sim O\left(\frac{1}{\omega^{s+\frac{1}{r+1}}}\right), \tag{3.18}$$

where $Q_s^A[H]$ is given in (3.6) and (3.7), respectively.

Proof When $Q_s^A[H]$ is defined as in (3.6), by combining (3.2), (3.4), (3.6) and (3.11) from Lemma 3.2, we have

$$\begin{aligned}|Q_s^A[H] - I[f]| &= \left|\frac{(2a)^{\alpha-s+1}}{\omega^s}\int_{-1}^{1}(t+1)^{-\alpha-2}\tau_s[H](t)J_{m+s}(\omega g(2a/(t+1)))dt\right| \\ &= \left|\frac{1}{(2a\omega)^s}\sum_{j=0}^{r}\frac{\tau_s[H]^{(j)}(\xi)}{j!}M_j(\xi, m+s, \omega)\right. \\ &\quad - \frac{2a^{\alpha+1}(\tau_s[H](1) - C_r[\tau_s](1,\xi))}{(2a\omega)^{s+1}g'(a)}J_{m+s+1}(\omega g(a)) \\ &\quad \left.+ \frac{(2a)^{\alpha-s}}{\omega^{s+1}}\int_{-1}^{1}(t+1)^{-\alpha-2}\tau_{s+1}[H](t)J_{m+s+1}\left(\omega g\left(\frac{2a}{t+1}\right)\right)dt\right| \\ &= O\left(\frac{1}{\omega^{s+\frac{1}{r+1}}}\right), \omega \to \infty.\end{aligned}$$

This completes the proof of (3.18). When $Q_s^A[H]$ is given in (3.7), the similar proof can be performed. □

With the above Lemma 3.2, we start to derive error estimate of the modified Filon-type method (3.8).

Theorem 3.3 *Under the same conditions as those of Theorem 3.1 (or Corollary 3.1), and assume that $g^{(r+1)}(2a/(t+1)) > 0$ for $t \in (-1, 1]$ and multiplicities $m_\ell \geq s(r+1)$, $m_\eta \geq s$ (or $m_\eta \geq s(r+1)$), then it is true that, as $\omega \to \infty$,*

$$Q_s^F[H] - I[f] \sim O\left(\frac{1}{\omega^{s+\frac{1}{r+1}}}\right), \tag{3.19}$$

where $Q_s^F[H]$ is given in (3.8).

Proof We now replace $H(t)$ with $H(t) - \tilde{P}(t)$ in the above asymptotic expansion (3.1). Since $\tilde{P}^{(n)}(\xi) = H^{(n)}(\xi)$, $\tilde{P}^{(n)}(1) = H^{(n)}(1)$, $n = 0, 1, \ldots, s-1$, it follows that $\tau_k[H-\tilde{P}](1) = \tau_k[H-\tilde{P}]^{(j)}(\xi) = 0$, $k = 0, 1, \ldots, s-1$, $j = 0, 1, \ldots, r$. Therefore, by using (2.5) together with (3.11) from Lemma 3.2, we can obtain the desired results (3.19) directly from the above asymptotic expansion (3.1) and the proof of Theorem 3.2. □

4 Numerical Examples

We now test numerical examples to illustrate the quality of the approximations obtained by the proposed quadrature rules. Since the presented asymptotic method is divergent for fixed ω, we only consider numerical examples of the proposed modified Filon-type methods.

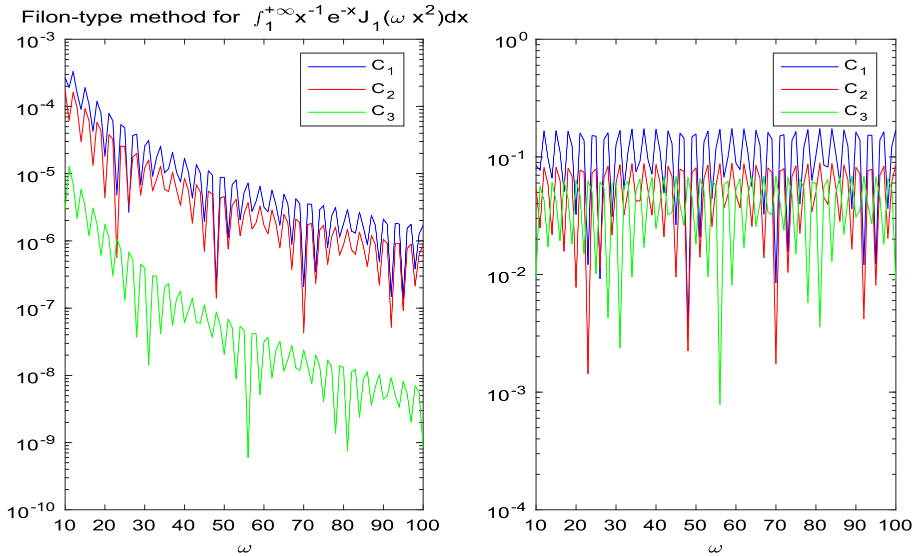

Fig. 1 The left panel shows the absolute error of $Q^F[H]$ with nodes $\{-\frac{4}{5}, 0, 1\}$ and multiplicities all one (left upper blue C_1), $Q^F[H]$ with nodes $\{-\frac{4}{5}, -\frac{2}{5}, 0, \frac{2}{5}, 1\}$ and multiplicities all one (left middle red C_2) and $Q^F[H]$ with nodes $\{-\frac{4}{5}, -\frac{2}{5}, 0, \frac{2}{5}, 1\}$ and multiplicities all two (left lower green C_3). The right panel shows the absolute error of the first two methods scaled by $\omega^{\frac{5}{2}}$ and the absolute error of the last method scaled by $\omega^{\frac{7}{2}}$

4.1 The Case Without Stationary Points: Type I: $g(x) \neq 0$ for $x \in [a, +\infty)$

Example 1 Let us consider an example where $f(x) = e^{-x}$, $g(x) = x^2$, $a = 1$, $\alpha = -1$ and $m = 1$. By using different modified Filon-type methods: $Q^F[H]$ with nodes $\{-\frac{4}{5}, 0, 1\}$ and multiplicities all one, $Q^F[H]$ with nodes $\{-\frac{4}{5}, -\frac{2}{5}, 0, \frac{2}{5}, 1\}$ and multiplicities all one and $Q^F[H]$ with nodes $\{-\frac{4}{5}, -\frac{2}{5}, 0, \frac{2}{5}, 1\}$ and multiplicities all two, we show numerical results in Fig. 1.

Example 2 Let us consider an example where $f(x) = \sin\frac{1}{x}$, $g(x) = x$, $a = 1$, $\alpha = -2$ and $m = 2$. By using different modified Filon-type methods: $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{1}{8}, 1\}$ and multiplicities all one, $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{3}{8}, \frac{1}{8}, \frac{1}{2}, 1\}$ and multiplicities all one and $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{3}{8}, \frac{1}{8}, \frac{1}{2}, 1\}$ and multiplicities all two, we show numerical results in Fig. 2.

In Examples 1 and 2, the first two modified Filon-type methods correspond to $s = 1$ and thus the decay rate of their absolute error is $O(\omega^{-\frac{5}{2}})$. Similarly, the last one corresponds to $s = 2$ and therefore the decay rate of its absolute error is $O(\omega^{-\frac{7}{2}})$. It is easy to see from Figs. 1 and 2 that this coincides with error estimate (2.20) in Theorem 2.4. As shown in Figs. 1 and 2, the accuracy of the produced numerical results can be improved greatly by either adding more derivatives interpolation at endpoint 1 or adding more interior nodes. Moreover, only using a small number of nodes and multiplicities, we can obtain the required accuracy level. Particularly, for fixed number of nodes and multiplicities, the higher accuracy can be achieved with the larger values of ω.

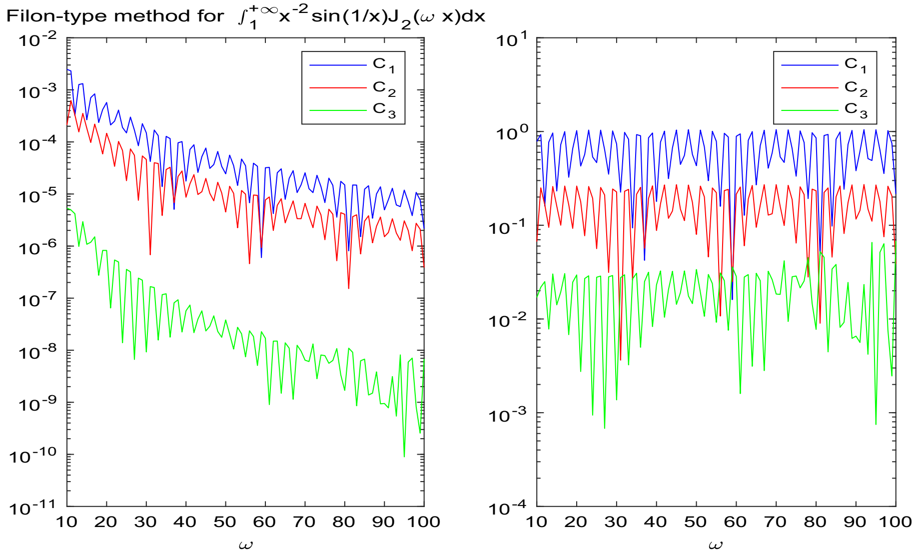

Fig. 2 The left panel shows the absolute error of $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{1}{8}, 1\}$ and multiplicities all one (left upper blue C_1), $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{3}{8}, \frac{1}{8}, \frac{1}{2}, 1\}$ and multiplicities all one (left middle red C_2) and $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{3}{8}, \frac{1}{8}, \frac{1}{2}, 1\}$ and multiplicities all two (left lower green C_3). The right panel shows the absolute error of the first two methods scaled by $\omega^{\frac{5}{2}}$ and the absolute error of the last method scaled by $\omega^{\frac{7}{2}}$

4.2 The Case Without Stationary Points: Type II: $g(x)$ Having Zeros on $x \in [a, +\infty)$

Example 3 We consider an example with $f(x) = \frac{1}{x}$, $g(x) = x - 2$, $a = 2$, $\alpha = -2$ and $m = 2$. Here, $g(x) = x - 2$ has an unique zero at $x = 2$, i.e., $g(\frac{2a}{1+t})$ has an unique zero at $t = \zeta = 1$. In Fig. 3 we present the accuracy of modified Filon-type methods: $Q^F[H]$ with nodes $\{-\frac{4}{5}, 1\}$ and multiplicities all one, $Q^F[H]$ with nodes $\{-\frac{4}{5}, -\frac{2}{5}, \frac{1}{5}, 1\}$ and multiplicities all one and $Q^F[H]$ with nodes $\{-\frac{4}{5}, -\frac{2}{5}, \frac{1}{5}, 1\}$ and multiplicities $\{1, 1, 1, 2\}$. Actually, we can regard this example as the case of $\alpha = 0$ in the integral $\int_2^{+\infty} x^{\alpha} x^{-3} J_2(\omega(x-2))dx$. Therefore, the modified Filon-type method is also available for the case of $\alpha = 0$ with some fast decreasing functions $f(x)$.

Example 4 We consider an example with $f(x) = e^{-x^2}$, $g(x) = x^2 - 4$, $a = 2$, $\alpha = -2$ and $m = 3$. Evidently, $g(x) = x^2 - 4$ has an unique zero at $x = 2$, i.e., $g(\frac{2a}{1+t})$ has an unique zero at $t = \zeta = 1$. In Fig. 4 we compare the accuracy of three modified Filon-type methods: $Q^F[H]$ with nodes $\{-\frac{7}{8}, 1\}$ and multiplicities all one, $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{3}{8}, \frac{1}{2}, 1\}$ and multiplicities all one and $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{3}{8}, \frac{1}{2}, 1\}$ and multiplicities $\{1, 1, 1, 2\}$.

Obviously, the decay rates obtained in Figs. 3 and 4 are consistent with the error analysis of the modified Filon-type methods in Theorem 2.6. The produced accuracy improves greatly as ω increases for fixed node points. The proposed methods can achieve an error that decays faster for increasing frequency ω.

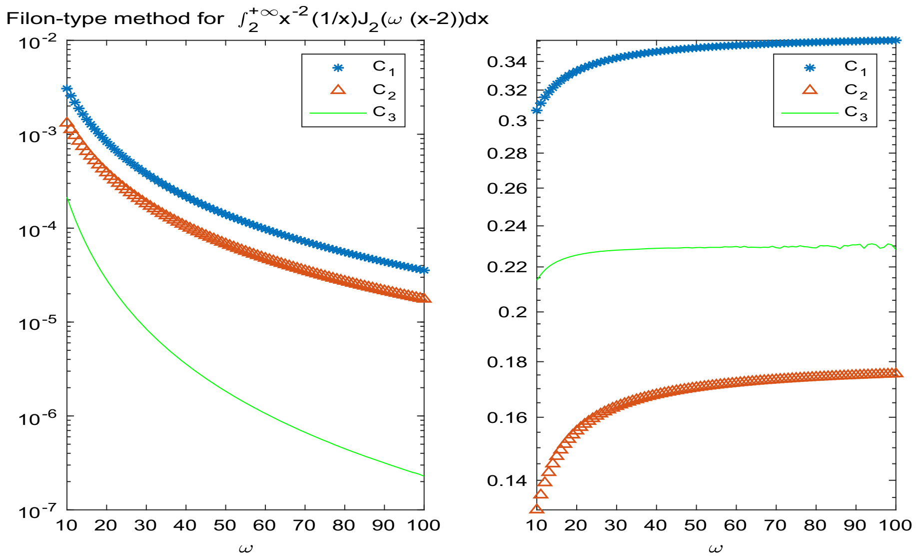

Fig. 3 The left panel shows the absolute error of $Q^F[H]$ with nodes $\{-\frac{4}{5}, 1\}$ and multiplicities all one (C_1), $Q^F[H]$ with nodes $\{-\frac{4}{5}, -\frac{2}{5}, \frac{2}{5}, 1\}$ and multiplicities all one (C_2) and $Q^F[H]$ with nodes $\{-\frac{4}{5}, -\frac{2}{5}, \frac{2}{5}, 1\}$ and multiplicities $\{1, 1, 1, 2\}$ (C_3). The right panel shows the absolute error of the first two methods scaled by ω^2 and the absolute error of the last method scaled by ω^3

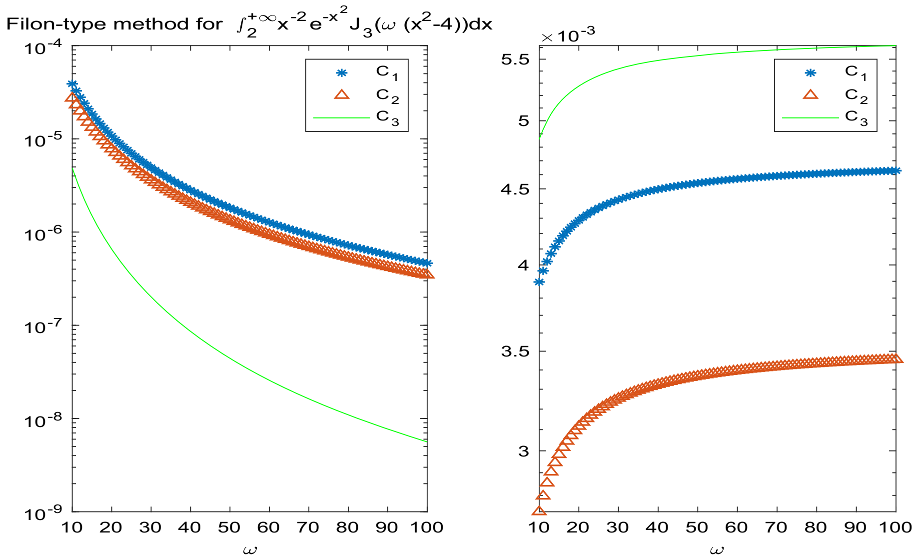

Fig. 4 The left panel shows the absolute error of $Q^F[H]$ with nodes $\{-\frac{7}{8}, 1\}$ and multiplicities all one (C_1), $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{3}{8}, \frac{1}{2}, 1\}$ and multiplicities all one (C_2) and $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{3}{8}, \frac{1}{2}, 1\}$ and multiplicities $\{1, 1, 1, 2\}$ (C_3). The right panel shows the absolute error of the first two methods scaled by ω^2 and the absolute error of the last method scaled by ω^3

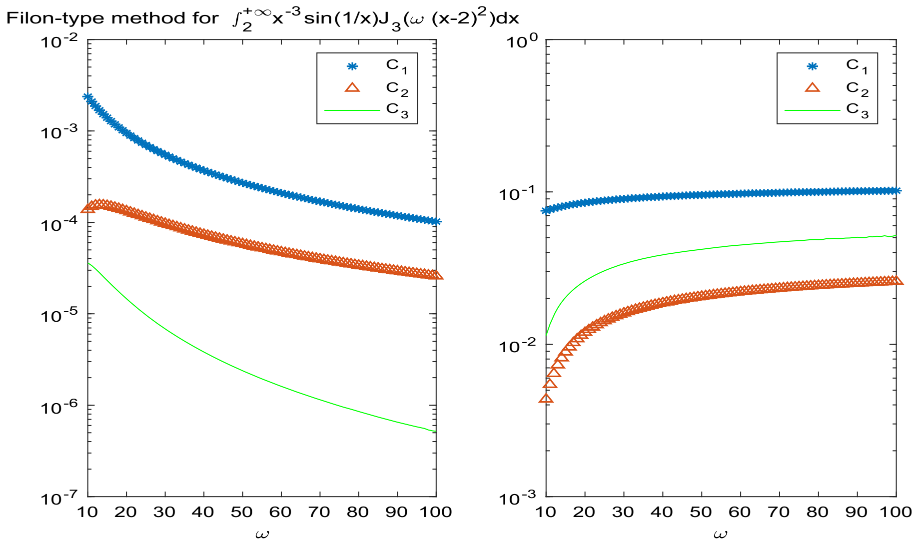

Fig. 5 The left panel shows the absolute error of $Q^F[H]$ with nodes $\{-\frac{5}{6}, 1\}$ and multiplicities $\{1, 2\}$ (C_1), $Q^F[H]$ with nodes $\{-\frac{5}{6}, -\frac{1}{2}, \frac{1}{2}, 1\}$ and multiplicities $\{1, 1, 1, 2\}$ (C_2) and $Q^F[H]$ with nodes $\{-\frac{5}{6}, -\frac{1}{2}, \frac{1}{2}, 1\}$ and multiplicities $\{1, 1, 1, 4\}$ (C_3). The right panel shows the absolute error of the first two methods scaled by $\omega^{\frac{3}{2}}$ and the absolute error of the last method scaled by $\omega^{\frac{5}{2}}$

4.3 The Case with Stationary Points

Example 5 Let us consider this example with $f(x) = \sin\frac{1}{x}$, $g(x) = (x-2)^2$, $a = 2$, $\alpha = -3$ and $m = 3$, where $r = 1$. By comparing the accuracy of three modified Filon-type methods: $Q^F[H]$ with nodes $\{-\frac{5}{6}, 1\}$ and multiplicities $\{1, 2\}$, $Q^F[H]$ with nodes $\{-\frac{5}{6}, -\frac{1}{2}, \frac{1}{2}, 1\}$ and multiplicities $\{1, 1, 1, 2\}$ and $Q^F[H]$ with nodes $\{-\frac{5}{6}, -\frac{1}{2}, \frac{1}{2}, 1\}$ and multiplicities $\{1, 1, 1, 4\}$, we give the desired numerical results in Fig. 5.

Example 6 Let us consider this example with $f(x) = e^{-x^2}$, $g(x) = x(x-1)^2$, $a = 1$, $\alpha = -1$ and $m = 1$ where $r = 1$. By comparing the accuracy of three modified Filon-type methods: $Q^F[H]$ with nodes $\{-\frac{7}{8}, 1\}$ and multiplicities $\{1, 2\}$, $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{1}{4}, \frac{3}{8}, 1\}$ and multiplicities $\{1, 1, 1, 2\}$ and $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{1}{4}, \frac{3}{8}, 1\}$ and multiplicities $\{1, 1, 1, 3\}$, we give the desired numerical results in Fig. 6.

In Examples 5 and 6, the first two modified Filon-type methods hold for $s = 1$ and thus the decay rate of their absolute error is $O(\omega^{-\frac{3}{2}})$. Similarly, the last one in Examples 5 corresponds to $s = 2$ and therefore the decay rate of its absolute error is $O(\omega^{-\frac{5}{2}})$. In Examples 6, the last one corresponds to $s = \frac{3}{2}$ and thus the decay rate of its absolute error is $O(\omega^{-2})$. This implies that Figs. 5 and 6 can confirm our error analysis in Theorem 3.3. Additionally, Examples 5 and 6 show the high accuracy and efficiency of the proposed methods. All these presented methods share an advantageous property that the error decreases greatly as ω increases.

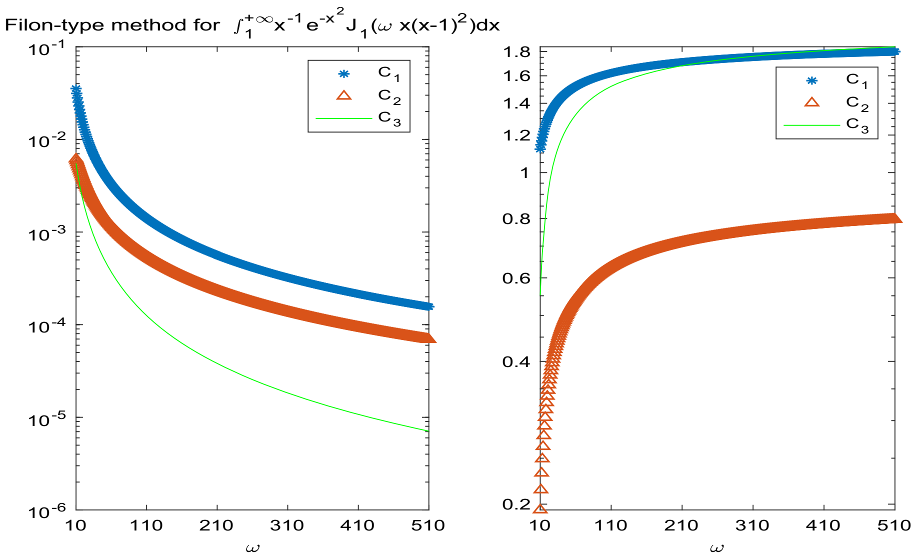

Fig. 6 The left panel shows the absolute error of $Q^F[H]$ with nodes $\{-\frac{7}{8}, 1\}$ and multiplicities $\{1, 2\}$ (C_1), $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{1}{4}, \frac{3}{8}, 1\}$ and multiplicities $\{1, 1, 1, 2\}$ (C_2) and $Q^F[H]$ with nodes $\{-\frac{7}{8}, -\frac{1}{4}, \frac{3}{8}, 1\}$ and multiplicities $\{1, 1, 1, 3\}$ (C_3). The right panel shows the absolute error of the first two methods scaled by $\omega^{\frac{3}{2}}$ and the absolute error of the last method scaled by ω^2

Table 4 Relative errors of the modified Filon-type method $Q^F[H]$ for the integral $\int_1^{+\infty} x^{-1}e^{-x}J_1(\omega x^2)dx$, with nodes $\{-\frac{4}{5}, 0, 1\}$ and multiplicities $\{1, 1, 1\}$, $\{1, 1, 2\}$, and $\{1, 1, 3\}$, for $\omega = 100, 200, 500, 1000$

ω	s = 1	s = 2	s = 3
100	4.91×10^{-2}	2.26×10^{-4}	3.58×10^{-5}
200	2.03×10^{-2}	6.12×10^{-5}	6.40×10^{-6}
500	7.49×10^{-4}	1.07×10^{-5}	2.16×10^{-8}
1000	2.22×10^{-4}	2.68×10^{-6}	1.61×10^{-9}

4.4 Relative Errors

Here we show the relative errors of the modified Filon-type method by two examples in Tables 4 and 5.

Tables 4 and 5 present the relative errors of the modified Filon-type method for the above two integrals. We can see from Table 4 that the Filon-type method exhibits the fast convergence as the multiplicities increase at the endpoint $t = 1$ for fixed ω. Table 5 illustrates that adding the more nodes for fixed ω, the approximate accuracy can be enhanced. Furthermore, for fixed number of nodes and multiplicities, the higher accuracy can be achieved with the larger values of ω.

Remark 4.1 The choice of the extreme point 1 as interpolation point for the highly oscillatory integral (1.2) is not only a technical necessity but also can improve asymptotic order to obtain the required high accuracy. The main approximative power of the modified Filon-type method comes from matching function values and derivatives at the end-points (or, with

Table 5 Relative errors of the modified Filon-type method $Q^F[H]$ for the integral $\int_2^{+\infty} x^{-3}\sin\frac{1}{x}J_3(\omega(x-2)^2)dx$, with nodes $\{-\frac{5}{6},1\}$, $\{-\frac{5}{6},-\frac{1}{2},\frac{1}{2},1\}$ and $\{-\frac{5}{6},-\frac{2}{3},-\frac{1}{2},\frac{1}{2},\frac{2}{3},1\}$ and multiplicities $\{1,2\}$, $\{1,1,1,2\}$, $\{1,1,1,1,1,2\}$, for $\omega = 10, 50, 100, 200$

ω	n = 2	n = 4	n = 6
10	2.38×10^{-3}	1.38×10^{-4}	1.57×10^{-5}
50	2.70×10^{-4}	5.83×10^{-5}	5.09×10^{-6}
100	1.02×10^{-4}	2.59×10^{-5}	3.37×10^{-6}
200	3.77×10^{-5}	1.06×10^{-5}	1.71×10^{-6}

Here, n is the number of nodes

greater generality, at critical points), and the addition of internal nodes in real number field is intended merely to decrease the error 'constant', rather than improve the asymptotic order. Indeed, computing one extra derivative at the end-points can improve the asymptotic order. The above method is usually known as the real Filon-type method based on polynomial interpolation at real number points. In particular, Huybrechs and Olver [18] presented an important consequence that the asymptotic order of the complex Filon-type method can be doubled by replacing evaluations of derivatives at critical points by evaluations at certain points in the complex plane. The complex Filon-type method enjoys the same high asymptotic order of accuracy as the numerical steepest descent method. In our future work we shall consider how to use the complex Filon-type method for computing the highly oscillatory Bessel transform (1.1).

5 Conclusions

In this work, we provide a complete asymptotic analysis and the construction of affordable quadrature rules for a class of infinite Bessel transform with a general oscillator. For each type of critical points including zeros and stationary points, we first derive a series of useful asymptotic expansions in inverse powers of ω. On the basis of the resulting useful asymptotic expansions, we then present the modified Filon-type method. Particularly, owing to the resulting asymptotic expansions, we give the rigorous error analysis and obtain asymptotic error estimates in inverse powers of ω. It is worth noting that for fixed nodes the accuracy increases when oscillation becomes faster. Additionally, for fixed frequency ω, the accuracy can be also improved by adding either the derivatives of $H(t)$ at $t = 1$ or the number of interpolation nodes. The presented numerical examples can confirm our numerical analysis.

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References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. National Bureau of Standards, Washington, D.C. (1964)
2. Arfken, G.: Mathematical Methods for Physicists, 3rd edn. Academic Press, Orlando (1985)

3. Bateman, H., Erdélyi, A.: Higher Transcendental Functions, vol. I. McGraw-Hill, New York (1953)
4. Bao, G., Sun, W.: A fast algorithm for the electromagnetic scattering from a large cavity. SIAM J. Sci. Comput. **27**, 553–574 (2005)
5. Blakemore, M., Evans, G.A., Hyslop, J.: Comparison of some methods for evaluating infinite range oscillatory integrals. J. Comput. Phys. **22**, 352–376 (1976)
6. Brunner, H.: Open problems in the computational solution of Volterra functional equations with highly oscillatory kernels. In: Effective Computational Methods for Highly Oscillatory Solutions, HOP. Isaac Newton Institute (2007)
7. Brunner, H.: On the numerical solution of first-kind Volterra integral equations with highly oscillatory kernels. HOP: Highly Oscillatory Problems: From Theory to Applications, pp. 13–17. Isaac Newton Institute, September (2010)
8. Chen, R.: Numerical approximations to integrals with a highly oscillatory Bessel kernel. Appl. Numer. Math. **62**, 636–648 (2012)
9. Chen, R.: On the implementation of the asymptotic Filon-type method for infinite integrals with oscillatory Bessel kernels. Appl. Math. Comput. **228**, 477–488 (2014)
10. Chen, R., An, C.: On the evaluation of infinite integrals involving Bessel functions. Appl. Math. Comput. **235**, 212–220 (2014)
11. Chen, R., Xiang, S., Kuang, X.: On evaluation of oscillatory transforms from position to momentum space. Appl. Math. Comput. **344**, 183–190 (2019)
12. Davies, P.J., Duncan, D.B.: Stability and convergence of collocation schemes for retarded potential integral equations. SIAM J. Numer. Anal. **42**, 1167–1188 (2004)
13. Domínguez, V., Graham, I.G., Smyshlyaev, V.P.: A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering. Numer. Math. **106**, 471–510 (2007)
14. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products, 7th edn. Academic Press, New York (2007)
15. Hascelik, A.: An asymptotic Filon-type method for infinite range highly oscillatory integrals with exponential kernel. Appl. Numer. Math. **63**, 1–13 (2013)
16. http://functions.wolfram.com/HypergeometricFunctions/MeijerG/17/01/02/0001/
17. http://functions.wolfram.com/HypergeometricFunctions/MeijerG/21/02/07/0002/
18. Huybrechs, D., Olver, S.: Superinterpolation in highly oscillatory quadrature. Found. Comput. Math. **12**(2), 203–228 (2012)
19. Iserles, A., Nørsett, S.P.: On quadrature methods for highly oscillatory integrals and their implementation. BIT Numer. Math. **44**, 755–772 (2004)
20. Iserles, A., Nørsett, S.P.: Efficient quadrature of highly oscillatory integrals using derivatives. Proc. R. Soc. A **461**, 1383–1399 (2005)
21. Kang, H., Ling, C.: Computation of integrals with oscillatory singular factors of algebraic and logarithmic type. J. Comput. Appl. Math. **285**, 72–85 (2015)
22. Kang, H., Ma, J.: Quadrature rules and asymptotic expansions for two classes of oscillatory Bessel integrals with singularities of algebraic or logarithmic type. Appl. Numer. Math. **118**, 277–291 (2017)
23. Kang, H.: Numerical integration of oscillatory Airy integrals with singularities on an infinite interval. J. Comput. Appl. Math. **333**, 314–326 (2018)
24. Kang, H.: Efficient calculation and asymptotic expansions of many different oscillatory infinite integrals. Appl. Math. Comput. **346**, 305–318 (2019)
25. Levin, D.: Fast integration of rapidly oscillatory functions. J. Comput. Appl. Math. **67**, 95–101 (1996)
26. Lewanowicz, S.: Evaluation of Bessel function integrals with algebraic singularities. J. Comput. Appl. Math. **37**, 101–112 (1991)
27. Luke, Y.L.: Integrals of Bessel Functions. McGraw-Hill, New York (1962)
28. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W.: NIST digital library of mathematical functions. http://dlmf.nist.gov/10.7
29. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W.: NIST digital library of mathematical functions. http://dlmf.nist.gov/10.14
30. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W.: NIST digital library of mathematical functions. https://dlmf.nist.gov/10.22
31. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W.: NIST digital library of mathematical functions. https://dlmf.nist.gov/16.19
32. Piessens, R., Branders, M.: Modified Clenshaw–Curtis method for the computation of Bessel function integrals. BIT Numer. Math. **23**, 370–381 (1983)
33. Schumaker, L.L.: Spline Functions: Basic Theory, 3rd edn. Cambridge University Press, Cambridge (2007)

34. Wang, H., Zhang, L., Huybrechs, D.: Asymptotic expansions and fast computation of oscillatory Hilbert transforms. Numer. Math. **123**, 709–743 (2013)
35. Wang, H.: A unified framework for asymptotic analysis and computation of finite Hankel transform. J. Math. Anal. Appl. **483**, Article ID 123640 (2019)
36. Watson, G.N.: A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge (1952)
37. Xiang, S.: On quadrature of Bessel transformations. J. Comput. Appl. Math. **177**, 231–239 (2005)
38. Xiang, S.: Numerical analysis of a fast integration method for highly oscillatory functions. BIT Numer. Math. **47**, 469–482 (2007)
39. Xiang, S., Gui, W., Mo, P.: Numerical quadrature for Bessel transformation. Appl. Numer. Math. **28**, 1247–1261 (2008)
40. Xiang, S., Wang, H.: Fast integration of highly oscillatory integrals with exotic oscillators. Math. Comput. **79**, 829–844 (2010)
41. Xiang, S., Cho, Y., Wang, H., Brunner, H.: Clenshaw–Curtis–Filon-type methods for highly oscillatory bessel transforms and applications. IMA J. Numer. Anal. **31**(4), 1281–1314 (2011)
42. Xu, Z., Xiang, S., He, G.: Efficient evaluation of oscillatory Bessel Hilbert transforms. J. Comput. Appl. Math. **258**, 57–66 (2014)
43. Xu, Z., Xiang, S.: Computation of highly oscillatory Bessel transforms with algebraic singularities (2016). arXiv:1605.08568
44. Xu, Z., Milovanovic, G.: Efficient method for the computation of oscillatory Bessel transform and Bessel Hilbert transform. J. Comput. Appl. Math. **308**, 117–137 (2016)