A Mixed Method for Maxwell Eigenproblem

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Abstract

We propose a mixed method for the computation of the eigenvalues of the Maxwell eigenproblem, in terms of the electric field and a multiplier. The method allows the Lagrange elements of any order greater than or equal to two for the electric field, while a piecewise constant element always for the multiplier. We show that optimal error estimates yield for singular as well as smooth solutions. For the Maxwell eigenproblem in L-shaped domain which has singular and smooth eigenfunctions, we present numerical results for illustrating the effectiveness of the proposed method and for confirming the theoretical results.

Keywords Maxwell eigenproblem · Mixed finite element · Error estimates

Mathematics Subject Classification 65N30

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a simply-connected polygonal domain with boundary $\partial\Omega$. Let ω^2 be the unknown eigenvalue, $\mathbf{u}$ the electric field(the unknown eigenfunction solution) and $\mathbf{t}$ the unit tangential vector which orients anticlockwise along $\partial\Omega$. This paper is concerned with the following Maxwell eigenproblem:

$$\begin{cases} \mathbf{curl}\ \text{curl}\ \mathbf{u} = \omega^2\mathbf{u} & \text{in } \Omega, \\ \text{div}\ \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{t} \cdot \mathbf{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

To develop the finite element method, in addition to the electric field $\mathbf{u}$, we introduce an additional new unknown variable p, which is called the multiplier, in order to relax the div equation (Gauss law) so that it can hold in some weak other than pointwise sense. The Gauss law of the div equation relates to the charge conservation or to the fact that the magnetic induction field is solenoidal in nature. Thus, the above Maxwell eigenproblem is naturally

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recast into the saddle-point problem (cf. [35]) in terms of the electric field $\mathbf{u}$ and the multiplier p:

$$\begin{cases} \mathbf{curl}\,\mathrm{curl}\,\mathbf{u} - \nabla p = \omega^2 \mathbf{u} & \text{in } \Omega, \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{t} \cdot \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Note that the multiplier p is in fact a *dummy variable*, i.e., $p = 0$ identically. The saddle-point problem is itself actually a type of Hodge-decomposition (cf., [32]). Such decomposition has been fundamental in the mathematical and numerical studies of Maxwell equations. Moreover, (1.2) allows the classical Babuška–Osborn theory ([4,36]), since it provides a compact operator.

Finite element computations of the Maxwell equations are important in understanding electromagnetic phenomena. The main task is to well approximate the electric field $\mathbf{u}$ and the eigenvalue ω^2. However, unlike other problems, the Maxwell eigenproblem may have *infinite singular solutions* and meanwhile, may also have *infinite smooth solutions*. The singular solution usually means that

$$\mathbf{u} \in (H^r(\Omega))^2, \quad \mathrm{curl}\,\mathbf{u} \in H^r(\Omega), \quad 0 \le r < 1, \tag{1.3}$$

while the smooth solution means that

$$\mathbf{u} \in (H^{1+t}(\Omega))^2, \quad t \ge 0. \tag{1.4}$$

Whenever Ω is nonconvex carrying reentrant corners, as is well-known (1.3) is commonplace in electromagnetism (cf., [2,6,32]). There are many difficult issues in the finite element computations, but this paper is only concerned with two. One is that the low regularity solution $\mathbf{u}$ could not be well approximated, and even may be wrongly approximated. The other is that there may exist so-called spurious eigenmodes, i.e., there are some nonphysical eigenvalues which pollute the physically true eigenvalues so that the latter could not be selected out. See [34,37] and references therein. As is well-known, the Nédélec elements ([38,39]) which are only $H_0(\mathrm{curl}\,;\Omega)$-conforming have been popular for solving Maxwell equations (cf. [28,34,37] and references therein).

As alternatives, the Lagrange elements have been increasingly interesting, and over the last decades, there have been a number of such methods, see [5,9–12,14,18,20,23–27,29–31], etc.

In this paper, we propose a new approach for solving (1.2), with the use of Lagrange elements, different from the methods in the above cited literature.

In fact, we shall use the standard Lagrange elements of polynomials P_ℓ on barycentric refinements of total degree not greater than the integer $\ell \ge 2$ to approximate each component of the electric field, while for the multiplier, we always use *the piecewise constant P_0 element* no matter what $\ell \ge 2$ is. If denoting the finite element spaces of the electric field and the multiplier by $\mathbf{U}_h$ and Q_h, respectively, which are subspaces of

$$\mathbf{U} := H_0(\mathrm{curl}\,;\Omega) \cap H(\mathrm{div}\,;\Omega), \quad Q := L^2(\Omega), \tag{1.5}$$

the pair $(\mathbf{U}, Q)$ and the pair $(\mathbf{U}_h, Q_h)$ obviously differ from the usual choices: $H_0(\mathbf{curl}\,;\Omega)$ and $H_0^1(\Omega)$ and Nédélec elements, and consequently, the finite element variational formulation is different. In addition, the finite element variational formulation and the finite element spaces of this paper are different from the ones in the literature as cited in the above in the

context of Lagrange elements. As a matter of fact, with the above $(\mathbf{U}_h, Q_h)$ at hand, we propose the following finite element method: Find $(\omega_h^2, \mathbf{u}_h) \in \mathbb{R} \times \mathbf{U}_h$, $p_h \in Q_h$, such that

$$\begin{cases} (\text{curl } \mathbf{u}_h, \text{curl } \mathbf{v}_h) + (\text{div } \mathbf{v}_h, p_h) = \omega_h^2(\mathbf{u}_h, \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{U}_h, \\ (\text{div } \mathbf{u}_h, q_h) = 0, & \forall q_h \in Q_h. \end{cases} \quad (1.6)$$

For these reasons, all the issues of the well-posedness, the convergence, the error bounds, and whether the method is spectral-correct and spurious-free are far from known. We must develop a new theory. Such theory is different from the one for the Nédélec element methods which involve a pair of finite element spaces $(\mathbf{N}_h, M_h)$. The latter theory is based on the so-called de Rham complex exact sequence ([34,37]) which simply reads $\nabla M_h \subset H_0^1(\Omega) \xrightarrow{\nabla} \mathbf{N}_h \subset H_0(\text{curl} ; \Omega)$. Here, $(\mathbf{U}_h, Q_h)$ do not satisfy such sequence. Unlike the Nédélec elements, the Lagrange elements are scalar-oriented in the definitions of degrees of freedom, and are thus more favorable in coding and implementation. In the regime of Lagrange elements, to the authors' knowledge, the formulation (1.6) is new and the simplest, and the theory here relies on the key Lemma 4.2 of this paper, while such lemma is the first time witnessed for the Maxwell eigenproblem. The use of this lemma and the theory developed open a new way for the spectral-correct and spurious-free approximations for Lagrange elements. Some numerical results show that the Lagrange elements in three-dimensions also work. Like some Lagrange element methods cited as above, the disadvantage here is the saddle-point structure; on the other hand, since Q_h is the constant element, the saddle-point system may be realized by the classical penalty algorithm (cf., [13,32]) decoupling the computations of the electric field and the multiplier.

With the pairs $(\mathbf{U}, Q)$ and $(\mathbf{U}_h, Q_h)$, together with (1.6), the issues mentioned in the above will be investigated with the help of the abstract Babuška–Osborn theory ([4,36]) for mixed methods of eigenvalue problems of compact operators. Such abstract theory mainly consists of the uniform convergence of the discrete solution operator with respect to the L^2 norm, because we are dealing with the compact operator associated with (1.2).

Since the crucial property is the uniform convergence, how to establish this convergence will be the main goal of this paper. For that goal, we establish inf-sup conditions which link the *curl* operator and the *div* operator through three finite element spaces(i.e., $\mathbf{U}_h$, Q_h and an auxiliary finite element space W_h, where W_h is only for theoretical purpose and does not involve any numerical implementations). With these inf-sup conditions, we can construct two Fortin-type interpolations. Then, we can show the well-posedness of the method, including the kernel-ellipticity and the inf-sup condition of the pair $(\mathbf{U}_h, Q_h)$, and we can obtain the uniform and optimal convergence of the discrete solution operator. Consequently, applying the Babuška–Osborn theory of mixed methods, we show that the method is spectrally-correct, spurious-free, and obtain the error bounds for eigenvalues and eigenfunctions, optimal relative to both the regularity and the order of approximation. Here we would like to remark that although the multiplier is always approximated by the piecewise constant P_0 element, the optimum in the error bounds with respect to the order of approximation ℓ is not affected. This is because the continuous multiplier p is identically zero. Numerical results are presented to illustrate the proposed method and confirm the theoretical results.

The remaining part of this paper is organized as follows. Section 2 is preliminaries. The mixed finite element problem is set up in Sect. 3. The inf-sup condition and Fortin-type interpolations are established in Sect. 4. A general theory of stability and error estimates is reasoned in Sect. 5 for any order Lagrange elements. Numerical results in non-convex domains are presented in the last section.

2 Preliminaries

In this section, we review the curl and div Hilbert spaces.

Given a simply-connected polygon $\Omega \subset \mathbb{R}^2$, with a connected Lipschitz boundary $\Gamma := \partial\Omega$(a planar curve). Let $\mathbf{n}$ denote the unit outward normal vector to Γ. Introduce the usual Hilbert spaces [1]: $H^1(\Omega) = \{q \in L^2(\Omega) : \nabla q \in (L^2(\Omega))^2\}$, $H_0^1(\Omega) = \{q \in H^1(\Omega) : q|_\Gamma = 0\}$, $H^1(\Omega)/\mathbb{R} = \{q \in H^1(\Omega) : \int_\Omega q = 0\}$. Let $H^1(\Omega)$, $H_0^1(\Omega)$, and $H^1(\Omega)/\mathbb{R}$ be equipped with norm $||q||_1^2 = ||q||_0^2 + ||\nabla q||_0^2$ and semi-norm $|q|_1 = ||\nabla q||_0$. We also need Hilbert space $H^s(\Omega)$ with norm $||q||_s$ for $s \in \mathbb{R}$, where $H^0(\Omega) = L^2(\Omega)$. In addition, introduce Hilbert spaces $H(\text{curl} ; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 : \text{curl } \mathbf{v} \in L^2(\Omega)\}$, $H_0(\text{curl} ; \Omega) = \{\mathbf{v} \in H(\text{curl} ; \Omega) : \mathbf{v} \cdot \mathbf{t}|_\Gamma = 0\}$, $H(\text{div} ; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 : \text{div } \mathbf{v} \in L^2(\Omega)\}$, $H(\text{div}^0 ; \Omega) = \{\mathbf{v} \in H(\text{div} ; \Omega) : \text{div } \mathbf{v} = 0\}$, $H_0(\text{div} ; \Omega) = \{\mathbf{v} \in H(\text{div} ; \Omega) : \mathbf{v} \cdot \mathbf{n}|_\Gamma = 0\}$. The norm for $H(\text{curl} ; \Omega)$ is $||\mathbf{v}||_{0,\text{curl}}^2 = ||\mathbf{v}||_0^2 + ||\text{curl } \mathbf{v}||_0^2$ and the norm for $H(\text{div} ; \Omega)$ is $||\mathbf{v}||_{0,\text{div}}^2 = ||\mathbf{v}||_0^2 + ||\text{div } \mathbf{v}||_0^2$. The Hilbert space $H(\text{curl} ; \Omega) \cap H(\text{div} ; \Omega)$ is equipped with the norm $||\mathbf{v}||_{0,\text{curl},\text{div}}^2 = ||\mathbf{v}||_0^2 + ||\text{curl } \mathbf{v}||_0^2 + ||\text{div } \mathbf{v}||_0^2$. As a result of the following Proposition 2.1, the Hilbert space $H_0(\text{curl} ; \Omega) \cap H(\text{div} ; \Omega)$ can be equipped with the norm $||\mathbf{v}||_{\text{curl},\text{div}}^2 := ||\text{curl } \mathbf{v}||_0^2 + ||\text{div } \mathbf{v}||_0^2$, which is equivalent to the norm $||\mathbf{v}||_{0,\text{curl},\text{div}}$. We in addition introduce $L_0^2(\Omega) = \{w \in L^2(\Omega) : \int_\Omega w = 0\}$. The norm of $L_0^2(\Omega)$ is still denoted by $|| \cdot ||_0$.

Proposition 2.1 [2,32,42] *For any* $\mathbf{v} \in H_0(\text{curl} ; \Omega) \cap H(\text{div} ; \Omega)$*, we have*

$$||\text{curl } \mathbf{v}||_0^2 + ||\text{div } \mathbf{v}||_0^2 \geq C||\mathbf{v}||_0^2.$$

Proposition 2.2 [2] *We have the following Hodge-decomposition(L^2 orthogonal):*

$$(L^2(\Omega))^2 = \mathbf{curl}\,(H^1(\Omega)/\mathbb{R}) + \nabla H_0^1(\Omega).$$

Proposition 2.3 [2] *The following continuous embedding hold for some* $1/2 < r \leq 1$*:*

$$H_0(\text{curl} ; \Omega) \cap H(\text{div} ; \Omega) \hookrightarrow (H^r(\Omega))^2,$$

where for any $\mathbf{v} \in H_0(\text{curl} ; \Omega) \cap H(\text{div} ; \Omega)$*, we have*

$$||\mathbf{v}||_r \leq C||\mathbf{v}||_{\text{curl},\text{div}}.$$

The value of the regularity r comes from the Poisson equation of Dirichlet boundary condition [2,33], and roughly, $1/2 < r \leq \pi/\kappa$, for the largest opening angle $\pi \leq \kappa < 2\pi$ at the reentrant corners.

3 The Finite Element Method

In this section, we define the finite element method.

Denote by $\mathcal{T}_h$ the conforming triangulation of Ω into shape-regular triangles [17], i.e., $\mathcal{T}_h = \{T\}$, $h := \max_{T \in \mathcal{T}_h} h_T$ and h_T the diameter of $T \in \mathcal{T}_h$. Let $\mathcal{T}_{h/2}$ denote the barycentric refinement of $\mathcal{T}_h$ in the following way: for every $T \in \mathcal{T}_h$, we divide T into three sub-triangles by connecting the barycentre to the three vertices. Let $D \subset \Omega$ be an open subset. Denote by $P_\ell(D)$ the space of polynomials on D of total degree not greater than ℓ, where $\ell \geq 0$ is an integer. Denote either of $\mathcal{T}_h$ or $\mathcal{T}_{h/2}$ by the notation $\mathscr{C}_h$. Introduce

$$V_h^{(\ell)}(\mathscr{C}_h) = \{v_h \in L^2(\Omega) : v_h|_T \in P_\ell(T), \forall T \in \mathscr{C}_h\}. \tag{3.1}$$

For $\ell \geq 2$, we define the Lagrange finite element space $\mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2})$ for the solution

$$\mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2}) = (V_h^{(\ell)}(\mathcal{T}_{h/2}) \cap H^1(\Omega))^2 \cap H_0(\text{curl} ; \Omega). \tag{3.2}$$

In addition, we introduce

$$Q_h^{(0)}(\mathcal{T}_h) = V_h^{(0)}(\mathcal{T}_h). \tag{3.3}$$

The finite element method we propose is to find $(\omega_h^2, \mathbf{u}_h \neq \mathbf{0}, p_h) \in \mathbb{R} \times \mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2}) \times Q_h^{(0)}(\mathcal{T}_h)$ such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \omega_h^2(\mathbf{u}_h, \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2}), \\ b(\mathbf{u}_h, q_h) = 0, & \forall q_h \in Q_h^{(0)}(\mathcal{T}_h), \end{cases} \tag{3.4}$$

where

$$a(\mathbf{u}, \mathbf{v}) = (\text{curl}\, \mathbf{u}, \text{curl}\, \mathbf{v}), \tag{3.5}$$

$$b(\mathbf{v}, q) = (\text{div}\, \mathbf{v}, q). \tag{3.6}$$

4 The Dual Fortin-Type Interpolation

In this section we shall show an inf-sup condition and dual Fortin interpolation associated with the following trilinear form, which will be the main tools for analyzing the stability and the error bounds. We introduce a trilinear form over $(H(\text{curl} ; \Omega) \cap H(\text{div} ; \Omega)) \times L_0^2(\Omega) \times L^2(\Omega)$ as follows:

$$c(\mathbf{v}, (\chi, q)) = (\text{curl}\, \mathbf{v}, \chi) + (\text{div}\, \mathbf{v}, q) : (H(\text{curl} ; \Omega) \cap H(\text{div} ; \Omega))$$
$$\times L_0^2(\Omega) \times L^2(\Omega) \to \mathbb{R}. \tag{4.1}$$

This trilinear form links the curl operator and the div operator. The main ingredients of the theory for stability and error bounds are the dual Fortin-type interpolation and the inf-sup condition of $c(\cdot, (\cdot, \cdot))$ and the primal Fortin-type interpolation which will be established in this section.

4.1 Auxiliary Finite Element Spaces

We shall introduce three auxiliary finite element spaces for theoretical purpose.

Introduce a third finite element space

$$W_h^{(\ell-1)}(\mathcal{T}_{h/2}) = V_h^{(\ell-1)}(\mathcal{T}_{h/2}) \cap L_0^2(\Omega), \tag{4.2}$$

which is only used for theoretical analysis and will not be involved with the implementation of the finite element method. The norm for $W_h^{(\ell-1)}(\mathcal{T}_{h/2})$ is the same as that of $L_0^2(\Omega)$, i.e., $\inf_{c \in \mathbb{R}} ||v + c||_0$, still denoted by the same notation $||v||_0$. Let the kernel set of $c(\cdot, (\cdot, \cdot))$ be defined by

$$\mathbf{K}_h(c) = \Big\{ \mathbf{v}_{0,h} \in \mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2}) : c(\mathbf{v}_{0,h}, (\chi_h, q_h))$$
$$= 0, \quad \forall q_h \in Q_h^{(0)}(\mathcal{T}_h), \forall \chi_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2}) \Big\}. \tag{4.3}$$

Lemma 4.1 *The kernel set* $\mathbf{K}_h(c)$ *can also be equivalently defined as follows:*

$$\mathbf{K}_h(c) = \{\mathbf{v}_h \in \mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2}) : a(\mathbf{v}_h, \mathbf{v}_h) = 0, b(\mathbf{v}_h, q_h) = 0, \forall q_h \in Q_h^{(0)}(\mathcal{T}_h)\}. \quad (4.4)$$

Proof From (4.1), (4.3), (3.5) and (3.6), it follows that (4.4) holds. $\qquad \square$

For the finite dimensional space $\mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2})$, we can have the following orthogonal decomposition with respect to the L^2 inner product $(\cdot, \cdot)$:

$$\mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2}) = \mathbf{K}_h(c) + \mathbf{K}_h(c)^\perp, \quad (4.5)$$

where

$$\mathbf{K}_h(c)^\perp = \{\mathbf{v}_{0,h}^\perp \in \mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2}) : (\mathbf{v}_{0,h}, \mathbf{v}_{0,h}^\perp) = 0 \quad \forall \mathbf{v}_{0,h} \in \mathbf{K}_h(c)\}. \quad (4.6)$$

Introduce a fourth finite element space

$$\mathbf{X}_h^{(\ell)}(\mathcal{T}_{h/2}) = \mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2}) \cap (H_0^1(\Omega))^2, \quad (4.7)$$

which, together with $W_h^{(\ell-1)}(\mathcal{T}_{h/2})$, is also only used for theoretical analysis for some inf-sup condition relating to the curl operator.

Lemma 4.2 [40,41,44,45] *The following relation between* $\mathbf{X}_h^{(\ell)}(\mathcal{T}_{h/2})$ *and* $W_h^{(\ell-1)}(\mathcal{T}_{h/2})$ *holds:*

$$\operatorname{curl} \mathbf{X}_h^{(\ell)}(\mathcal{T}_{h/2}) \equiv W_h^{(\ell-1)}(\mathcal{T}_{h/2}).$$

The conclusion is originally stated in term of the 'div ' operator in the cited references, but in two-dimensions, with $\pi/2$ rotation, the conclusion is the same for the scalar 'curl ' operator. For $\ell = 2, 3$, this lemma was shown in [40, Theorem 4.6.1, Theorem 6.4.1] through the following inf-sup condition(in terms of the scalar curl operator and the finite element spaces here):

$$\sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{X}_h^{(\ell)}(\mathcal{T}_{h/2})} \frac{(\operatorname{curl} \mathbf{v}_h, w_h)}{||\mathbf{v}_h||_1} \geq C||w_h||_0, \quad \forall w_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2}).$$

This inf-sup condition and the trivial fact $\operatorname{curl} \mathbf{X}_h^{(\ell)}(\mathcal{T}_{h/2}) \subset W_h^{(\ell-1)}(\mathcal{T}_{h/2})$ lead to the conclusion as stated. For $\ell = 2$, a different argument for showing this conclusion can be found in [44]. For general ℓ, this conclusion is essentially due to [40], see the remarks after Theorem 6.4.1 therein. In addition, the three-dimensional results in [45] are applicable to obtain the two-dimensional conclusion for general $\ell \geq 2$. For $\ell \geq 4$, from [41], if we do not use the barycentric refinements, i.e., $\mathcal{T}_{h/2} := \mathcal{T}_h$, Lemma 4.2 still holds, but under some very restrictive conditions on the meshes. We also refer the readers to [16, sections 4 and 6] for the conclusion(stated for the div operator).

Obviously, we also have

$$\operatorname{curl} \mathbf{X}_h^{(\ell)}(\mathcal{T}_{h/2}) \equiv \operatorname{curl} \mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2}) \equiv W_h^{(\ell-1)}(\mathcal{T}_{h/2}).$$

We finally introduce a fifth finite element space, which is a C^1 element,

$$Y_h^{(\ell+1)}(\mathcal{T}_{h/2}) = V_h^{(\ell+1)}(\mathcal{T}_{h/2}) \cap H_0^1(\Omega) \cap H^2(\Omega), \quad (4.8)$$

which is only used for theoretical analysis for discrete analogs of the Hodge-decomposition and the regular-singular decomposition, and we have

$$\nabla Y_h^{(\ell+1)}(\mathcal{T}_{h/2}) \subset \mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2}). \quad (4.9)$$

For a later theoretical use, below we review the local degrees of freedom of $Y_h^{(\ell+1)}(\mathcal{T}_{h/2})$ from [21]. Given a triangle T of $\mathcal{T}_h$, whose subdivision consists of three sub-triangles K_1, K_2, K_3, all of which belong to $\mathcal{T}_{h/2}$. The three vertices of T are called *exterior vertices* and the three edges of T *exterior edges*, and the barycentre of T is called *interior vertex* and the three edges in the interior of T are called *interior edges*. Thus, T has three exterior vertices, three exterior edges, three interior edges, one interior vertex, and three sub-triangles. The local degrees of freedom are listed below from (1)–(7):

(1) The value and gradient (i.e., $\partial/\partial_{x_1}, \partial/\partial_{x_2}$) at the exterior vertices of T,
(2) The value at $\ell + 1 - 3$ distinct points in the interior of each exterior edge of T,
(3) The normal derivative(i.e., $\partial/\partial n$) at $\ell+1-2$ distinct points in the interior of each exterior edge of T,
  and if $\ell + 1 \geq 4$,
(4) The value and gradient at the interior vertex of T,
(5) The value at $\ell + 1 - 4$ distinct points in the interior of each interior edge of T,
(6) The normal derivative at $\ell + 1 - 4$ distinct points in the interior of each interior edge of T,
  and
(7) The value at $(1/2)(\ell+1-4)(\ell+1-5)$ distinct points in the interior of each K_i chosen so that if a polynomial of degree $\ell + 1 - 6$ vanishes at those points, then it vanishes identically.

All the above degrees of freedom who locate on the exterior and interior edges and in the interior of element and sub-elements can be replaced by the *average quantities* such as *edge integrals* and *volume integrals*, e.g., $\int_F \partial q / \partial n$, $\int_F q$ and $\int_T q$ for a given interpolated function q.

We emphasize that all these three auxiliary finite element spaces $W_h^{(\ell-1)}(\mathcal{T}_{h/2})$, $\mathbf{X}_h^{(\ell)}(\mathcal{T}_{h/2})$, $Y_h^{(\ell+1)}(\mathcal{T}_{h/2})$ are merely technicalities of our proofs for establishing inf-sup conditions, Fortin-type interpolations. They have no influences on the finite element discretization which only uses the Lagrange elements $\mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2})$ we have defined in Sect. 3.

4.2 Mesh-Dependent Norms

Let $\Pi_h^{(0)}$ denote the L^2 projection operator onto $Q_h^{(0)}(\mathcal{T}_h)$. Introduce mesh-dependent norms as follows:

$$||\mathbf{v}||_{h,\text{curl},\text{div}}^2 := ||\text{curl } \mathbf{v}||_0^2 + ||\Pi_h^{(0)}\text{div } \mathbf{v}||_0^2, \tag{4.10}$$

$$||\mathbf{v}||_{h,0,\text{curl},\text{div}}^2 = ||\mathbf{v}||_0^2 + ||\mathbf{v}||_{h,\text{curl},\text{div}}^2 = ||\mathbf{v}||_{0,\text{curl}}^2 + ||\Pi_h^{(0)}\text{div } \mathbf{v}||_0^2, \tag{4.11}$$

$$|||\mathbf{v}|||_h^2 = ||\mathbf{v}||_{h,0,\text{curl},\text{div}}^2 + \mathcal{D}_h(\mathbf{v}), \tag{4.12}$$

$$\mathcal{D}_h(\mathbf{v}) = \sum_{T \in \mathcal{T}_{h/2}} h_T^{-2r} \frac{\sum_{i=1}^{m_\ell} (\mathbf{v}, \nabla(\psi_{i,T} b_T))_{0,T} (\mathbf{v}, \nabla(\psi_{i,T} b_T))_{0,T}}{\sum_{i=1}^{m_\ell} ||\nabla(\psi_{i,T} b_T)||_{0,T}^2}, \tag{4.13}$$

where $m_\ell = \ell(\ell+1)/2$, and $\{\psi_{i,T}, 1 \leq i \leq m_\ell\}$ is chosen so that each $\psi_{i,T} \in P_{\ell-1}(T)$ for $T \in \mathcal{T}_{h/2}$ and the following *local inclusion* holds:

$$\text{div }(\mathbf{v}_h|_T) \in \Psi_T := \text{span}\{\psi_{i,T}, 1 \leq i \leq m_\ell\}, \quad \forall \mathbf{v}_h \in \mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2}), \forall T \in \mathcal{T}_{h/2},$$

$$b_T = \lambda_1^T \lambda_2^T \lambda_3^T \tag{4.14}$$

is the usual element-bubble function, where λ_j^T, $j = 1, 2, 3$, are the three basis functions of the linear element $P_1(T)$. Note that once $\mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2})$ is defined, (4.14) can be accordingly defined.

Proposition 4.1 *The $\mathcal{D}_h(\cdot)$ has the following properties:*

$$\mathcal{D}_h(\mathbf{v}) \le C \sum_{T \in \mathscr{T}_{h/2}} h_T^{-2r} ||\mathbf{v}||_{0,T}^2, \quad \forall \mathbf{v} \in (L^2(\Omega))^2, \tag{4.15}$$

$$\mathcal{D}_h(\mathbf{v}) \le C \sum_{T \in \mathscr{T}_{h/2}} h_T^{2-2r} ||\text{div } \mathbf{v}||_{0,T}^2, \quad \forall \mathbf{v} \in H(\text{div}; \Omega), \tag{4.16}$$

$$C \sum_{T \in \mathscr{T}_{h/2}} h_T^{2-2r} ||\text{div } \mathbf{v}_h||_{0,T}^2 \le \mathcal{D}_h(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2}). \tag{4.17}$$

Proof In fact, by the Cauchy–Schwarz inequality,

$$\sum_{i=1}^{m_\ell} |(\mathbf{v}, \nabla(\psi_{i,T} b_T))_{0,T}|^2 \le ||\mathbf{v}||_{0,T}^2 \sum_{i=1}^{m_\ell} ||\nabla(\psi_{i,T} b_T)||_{0,T}^2,$$

and by the definition (4.13), we immediately obtain (4.15). From (4.13), by the formula of integration by parts,

$$(\mathbf{v}, \nabla(\psi_{i,T} b_T))_{0,T} = -(\text{div } \mathbf{v}, \psi_{i,T} b_T)_{0,T},$$

and by the Cauchy–Schwarz inequality,

$$|(\text{div } \mathbf{v}, \psi_{i,T} b_T)_{0,T}| \le ||\text{div } \mathbf{v}||_{0,T} ||\psi_{i,T} b_T||_{0,T},$$

but, by the Poincaré–Friedrichs' inequality (cf., Lemma 4.1 in [22]),

$$||\psi_{i,T} b_T||_{0,T} \le C h_T ||\nabla(\psi_{i,T} b_T)||_{0,T},$$

and we have

$$\sum_{i=1}^{m_\ell} |(\mathbf{v}, \nabla(\psi_{i,T} b_T))_{0,T}|^2 = \sum_{i=1}^{m_\ell} |(\text{div } \mathbf{v}, \psi_{i,T} b_T)_{0,T}|^2 \le C h_T^2 ||\text{div } \mathbf{v}||_{0,T}^2 \sum_{i=1}^{m_\ell} ||\nabla(\psi_{i,T} b_T)||_{0,T}^2,$$

from which and the definition (4.13) it follows that (4.16) holds. From the local inclusion (4.14), following the same argument in Lemma 4.3 of [24], we can show (4.17). $\square$

We also emphasize that the mesh-dependent norm $|||\cdot|||_h$ is merely for theoretical purpose for estimating the error bounds of the dual Fortin interpolation. We only use the mesh-dependent norms $||\cdot||_{h,\text{curl},\text{div}}$ and $||\cdot||_{h,0,\text{curl},\text{div}}$ in the finite element method which has been defined in Sect. 3.

4.3 The Inf-Sup Condition

We are now in a position to establish the inf-sup conditions associated with $c(\cdot, (\cdot, \cdot))$.

Lemma 4.3 *For any given* $p_h \in Q_h^{(0)}$, *there exists a* $\mathbf{v}_{0,h}^{\star}{}^{\perp} \in \mathbf{K}_h(c)^{\perp}$, *with* curl $\mathbf{v}_{0,h}^{\star}{}^{\perp} = 0$, *satisfies*

$$
\begin{cases}
c(\mathbf{v}_{0,h}^{\star}{}^{\perp}, (w_h, p_h)) = b(\mathbf{v}_{0,h}^{\perp}, p_h) = ||p_h||_0^2, \quad \forall w_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2}), \\
|||\mathbf{v}_{0,h}^{\star}{}^{\perp}|||_h \le C||p_h||_0;
\end{cases}
\tag{4.18}
$$

$$
\sup_{\substack{0 \ne \mathbf{v}_h \in \mathbf{K}_h(c)^{\perp} \\ \text{curl } \mathbf{v}_h = 0}} \frac{c(\mathbf{v}_h, (w_h, p_h))}{|||\mathbf{v}_h|||_h} \ge C||p_h||_0, \quad \forall p_h \in Q_h^{(0)}(\mathcal{T}_h), \forall w_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2}).
\tag{4.19}
$$

Proof For given $p_h \in Q_h^{(0)}(\mathcal{T}_h)$, consider the Poisson equation of homogeneous Dirichlet boundary condition: Find $\theta \in H_0^1(\Omega)$ such that

$$
-\Delta\theta = p_h \quad \text{in } \Omega, \quad \theta = 0 \quad \text{on } \Gamma.
$$

It is known from [33] (Theorem 4.4.4.13 and Corollary 4.4.4.14 for polygon) that, for some $1/2 < r \le 1$, $\theta \in H^{1+r}(\Omega)$ and $\Delta\theta \in L^2(\Omega)$,

$$
||\Delta\theta||_0 + ||\theta||_{1+r} \le C||p_h||_0.
$$

Letting

$$
\mathbf{v}^\sharp := -\nabla\theta \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega),
$$

we have

$$
\text{curl } \mathbf{v}^\sharp = 0, \quad \text{div } \mathbf{v}^\sharp = p_h, \quad \mathbf{v}^\sharp \in (H^r(\Omega))^2, \quad ||\mathbf{v}^\sharp||_r + ||\mathbf{v}^\sharp||_{0,\text{div}} \le C||p_h||_0.
$$

Denote by $\mathscr{F}_h(\mathcal{T}_h)$ denote the set of all edges of $\mathcal{T}_h$. Define

$$
\mathbf{v}_h^\sharp := -\nabla\theta_h \in \mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2}), \quad \theta_h \in Y_h^{(\ell+1)}(\mathcal{T}_{h/2}),
$$

where θ_h is the finite element interpolation of θ, satisfying

$$
\int_F \nabla(\theta - \theta_h) \cdot \mathbf{n} = 0, \quad \forall F \in \mathscr{F}_h(\mathcal{T}_h), \quad ||\theta - \theta_h||_0 + h|\theta - \theta_h|_1 \le Ch^{1+r}||\theta||_{1+r}.
$$

We remark that the finite element interpolation $\theta_h \in Y_h^{(\ell+1)}(\mathcal{T}_{h/2})$ of $\theta \in H_0^1(\Omega) \cap H^2(\Omega)$ can be found in [21]; in particular, therein $\partial\theta_h/\partial n(a_F) = \partial\theta/\partial n(a_F)$ at all the mid-points a_F of $F \in \mathscr{F}_h(\mathcal{T}_h)$. Such degree of freedom $\partial\theta/\partial n(a_F)$ can be replaced by the edge average $|F|^{-1}\int_F \partial\theta/\partial n$, where $|F|$ is the length of $F \in \mathscr{F}_h(\mathcal{T}_h)$, and consequently, the regularity of the interpolated function can be relaxed to $\theta \in H_0^1(\Omega) \cap H^{1+r}(\Omega)$ for $1/2 < r \le 1$ and $\Delta\theta \in L^2(\Omega)$.

In terms of $\mathbf{v}^\sharp$ and $\mathbf{v}_h^\sharp$,

$$
\int_F (\mathbf{v}^\sharp - \mathbf{v}_h^\sharp) \cdot \mathbf{n} = 0, \quad \forall F \in \mathscr{F}_h(\mathcal{T}_h), \quad ||\mathbf{v}^\sharp - \mathbf{v}_h^\sharp||_0 \le Ch^r||\mathbf{v}^\sharp||_r.
$$

We also have

$$
\begin{aligned}
||\Pi_h^{(0)}\text{div}(\mathbf{v}^\sharp - \mathbf{v}_h^\sharp)||_0^2 &= (\Pi_h^{(0)}\text{div}(\mathbf{v}^\sharp - \mathbf{v}_h^\sharp), \Pi_h^{(0)}\text{div}(\mathbf{v}^\sharp - \mathbf{v}_h^\sharp)) \\
&= (\text{div}(\mathbf{v}^\sharp - \mathbf{v}_h^\sharp), \Pi_h^{(0)}\text{div}(\mathbf{v}^\sharp - \mathbf{v}_h^\sharp)) \\
&= \sum_{F \in \mathscr{F}_h(\mathcal{T}_h)} \int_F (\mathbf{v}^\sharp - \mathbf{v}_h^\sharp) \cdot \mathbf{n}[\![\Pi_h^{(0)}\text{div}(\mathbf{v}^\sharp - \mathbf{v}_h^\sharp))]\!] = 0,
\end{aligned}
$$

i.e.,

$$\Pi_h^{(0)} \text{div } (\mathbf{v}^\sharp - \mathbf{v}_h^\sharp) = 0,$$

and

$$\Pi_h^{(0)} \text{div } \mathbf{v}_h^\sharp = \Pi_h^{(0)} \text{div } \mathbf{v}^\sharp = p_h.$$

For $Y_h^{(\ell+1)}(\mathcal{T}_{h/2})$, let $Y_h^\star$ denote the subspace which is spanned by all the basis function each of which corresponds to the degree of freedom of the zeroth-order normal derivative(i.e., $\int_F \partial q / \partial n$) at the mid-point of the exterior edge of $T \in \mathcal{T}_h$. We can split $Y_h^{(\ell+1)}(\mathcal{T}_h)$ in the following orthogonal decomposition with respect to the scalar product $(\nabla \cdot, \nabla \cdot)$:

$$Y_h^{(\ell+1)}(\mathcal{T}_{h/2}) = N_h + Y_h^\star,$$

where N_h denotes the orthogonal complement of $Y_h^\star$. Every $\xi_h^\star \in Y_h^\star$ takes a zero value at every other 'node' corresponding to other degrees of freedom, and particularly,

$$\xi_h^\star|_{\partial T} = 0, \quad \forall T \in \mathcal{T}_h.$$

On the other hand, the zeroth-order normal derivatives of every $\varphi_h \in N_h$ must take zero values at all the mid-points of the exterior edges of T for all $T \in \mathcal{T}_h$, and consequently,

$$\int_{\partial T} \partial \varphi_h / \partial n = 0, \quad \forall T \in \mathcal{T}_h.$$

With $\theta_h \in Y_h^{(\ell+1)}(\mathcal{T}_h)$ as constructed in the above, we split θ_h with respect to the scalar product $(\nabla \cdot, \nabla \cdot)$ as follows:

$$\theta_h = \varphi_h + \theta_h^\star, \quad \varphi_h \in N_h, \quad \theta_h^\star \in Y_h^\star,$$

For such $\theta_h^\star \in Y_h^\star$, it satisfies

$$\theta_h^\star|_{\partial T} = 0, \quad \forall T \in \mathcal{T}_h,$$

while $\varphi_h \in N_h$ satisfies

$$\int_{\partial T} \partial \varphi_h / \partial n = 0, \quad \forall T \in \mathcal{T}_h.$$

Hence, by the Poincaré–Friedrichs' inequality,

$$||\theta_h^\star||_{0,T} \le C h_T |\theta_h^\star|_{1,T}, \quad \forall T \in \mathcal{T}_h,$$

and we have

$$||\theta_h^\star||_0 \le C h |\theta_h^\star|_1.$$

Now,

$$|\theta_h^\star|_1^2 = (\nabla \theta_h^\star, \nabla \theta_h^\star) = (\nabla \theta_h^\star, \nabla \theta_h) = (\nabla \theta_h^\star, \nabla \theta_h - \nabla \theta) + (\nabla \theta_h^\star, \nabla \theta),$$

where

$$(\nabla \theta_h^\star, \nabla \theta_h - \nabla \theta) \le C |\theta_h^\star|_1 |\theta - \theta_h|_1 \le C h^r |\theta_h^\star|_1 ||\theta||_{1+r},$$
$$(\nabla \theta_h^\star, \nabla \theta) = -(\theta_h^\star, \Delta \theta) \le ||\theta_h^\star||_0 ||\Delta \theta||_0 \le C h |\theta_h^\star|_1 ||\Delta \theta||_0,$$

we have

$$|\theta_h^\star|_1 \le C h^r (||\theta||_{1+r} + ||\Delta \theta||_0).$$

Let

$$\mathbf{v}_h^\star := -\nabla\theta_h^\star, \quad \mathbf{v}_h^1 := -\nabla\varphi_h,$$

where $\mathbf{v}_h^1 \in \mathbf{K}_h(c)$, since curl $\mathbf{v}_h^1 = 0$ and $\int_T \mathrm{div}\, \mathbf{v}_h^1 = \int_{\partial T} \mathbf{v}_h^1 \cdot \mathbf{n} = -\int_{\partial T} \partial\varphi_h/\partial n = 0$, i.e., $\Pi_h^{(0)}\mathrm{div}\, \mathbf{v}_h^1 = 0$. Thus, since

$$\mathbf{v}_h^\sharp = \mathbf{v}_h^1 + \mathbf{v}_h^\star,$$

we have

$$\Pi_h^{(0)}\mathrm{div}\, \mathbf{v}_h^\star = \Pi_h^{(0)}\mathrm{div}\, (\mathbf{v}_h^\sharp - \mathbf{v}_h^1) = \Pi_h^{(0)}\mathrm{div}\, \mathbf{v}_h^\sharp = p_h.$$

Now splitting $\mathbf{v}_h^\star$ with respect to the L^2 inner product $(\cdot, \cdot)$:

$$\mathbf{v}_h^\star = \mathbf{v}_{0,h}^\star + {\mathbf{v}_{0,h}^\star}^\perp, \quad \mathbf{v}_{0,h}^\star \in \mathbf{K}_h(c), \quad {\mathbf{v}_{0,h}^\star}^\perp \in \mathbf{K}_h(c)^\perp,$$

we have

$$||\mathbf{v}_h^\star||_0^2 = ||\mathbf{v}_{0,h}^\star||_0^2 + ||{\mathbf{v}_{0,h}^\star}^\perp||_0^2,$$
$$\Pi_h^{(0)}\mathrm{div}\, {\mathbf{v}_{0,h}^\star}^\perp = \Pi_h^{(0)}\mathrm{div}\, \mathbf{v}_h^\star = p_h.$$

We next estimate $|||{\mathbf{v}_{0,h}^\star}^\perp|||_h$. By (4.16) and the inverse estimates, we have

$$
\begin{aligned}
|||{\mathbf{v}_{0,h}^\star}^\perp|||_h^2 &= ||{\mathbf{v}_{0,h}^\star}^\perp||_{0,\mathrm{curl}}^2 + ||\Pi_h^{(0)}\mathrm{div}\, {\mathbf{v}_{0,h}^\star}^\perp||_0^2 + \mathcal{D}_h({\mathbf{v}_{0,h}^\star}^\perp) \\
&= ||{\mathbf{v}_{0,h}^\star}^\perp||_0^2 + ||p_h||_0^2 + \mathcal{D}_h({\mathbf{v}_{0,h}^\star}^\perp) \\
&\le ||\mathbf{v}_h^\star||_0^2 + ||p_h||_0^2 + Ch^{2-2r}||\mathrm{div}\, {\mathbf{v}_{0,h}^\star}^\perp||_0^2 \\
&\le ||\mathbf{v}_h^\star||_0^2 + ||p_h||_0^2 + Ch^{-2r}||{\mathbf{v}_{0,h}^\star}^\perp||_0^2 \\
&\le ||\mathbf{v}_h^\star||_0^2 + ||p_h||_0^2 + Ch^{-2r}||\mathbf{v}_h^\star||_0^2 \\
&= C|\theta_h^\star|_1^2 + ||p_h||_0^2 + Ch^{-2r}|\theta_h^\star|_1^2 \\
&\le C(||\theta||_{1+r}^2 + ||\Delta\theta||_0^2) + ||p_h||_0^2 \\
&\le C||p_h||_0^2.
\end{aligned}
$$

That is,

$$|||{\mathbf{v}_{0,h}^\star}^\perp|||_h \le C||p_h||_0.$$

We also have

$$\mathrm{curl}\, {\mathbf{v}_{0,h}^\star}^\perp = \mathrm{curl}\, (\mathbf{v}_h^\star - \mathbf{v}_{0,h}^\star) = \mathrm{curl}\, \mathbf{v}_h^\star = -\mathrm{curl}\, \nabla\theta_h^\star = 0,$$
$$c({\mathbf{v}_{0,h}^\star}^\perp, (w_h, p_h)) = (\mathrm{div}\, {\mathbf{v}_{0,h}^\star}^\perp, p_h) = (\Pi_h^{(0)}\mathrm{div}\, {\mathbf{v}_{0,h}^\star}^\perp, p_h) = ||p_h||_0^2, \quad \forall w_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2}),$$

and we have

$$\sup_{\substack{0 \ne \mathbf{v}_h \in \mathbf{K}_h(c)^\perp \\ \mathrm{curl}\, \mathbf{v}_h = 0}} \frac{c(\mathbf{v}_h, (w_h, p_h))}{|||\mathbf{v}_h|||_h} \ge C||p_h||_0, \quad \forall p_h \in Q_h^{(0)}(\mathcal{T}_h), \forall w_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2}).$$

$\square$

Lemma 4.4 *For all $w_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2})$ and for all $p_h \in Q_h^{(0)}(\mathcal{T}_h)$, we have*

$$\sup_{0 \ne \mathbf{v}_h \in \mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2})} \frac{c(\mathbf{v}_h, (w_h, p_h))}{|||\mathbf{v}_h|||_h} \ge C(||w_h||_0 + ||p_h||_0). \tag{4.20}$$

Proof For any $w_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2})$, from Lemma 4.2 there exists

$$\mathbf{v}_h^\diamond \in \mathbf{X}_h^{(\ell)}(\mathcal{T}_{h/2}), \quad (\text{curl }\mathbf{v}_h^\diamond, w_h) = ||w_h||_0^2, \quad ||\mathbf{v}_h^\diamond||_1 \leq C||w_h||_0.$$

Note that Lemma 4.2 holds with the following inf-sup condition:

$$\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{X}_h^{(\ell)}(\mathcal{T}_{h/2})} \frac{(\text{curl }\mathbf{v}_h, w_h)}{||\mathbf{v}_h||_1} \geq C||w_h||_0 \quad \forall w_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2}),$$

where as defined earlier, $||w_h||_0$ is understood as the norm of $L_0^2(\Omega)$.

From Proposition 4.1, we have

$$|||\mathbf{v}_h^\diamond|||_h^2 = ||\mathbf{v}_h^\diamond||_{0,\text{curl}}^2 + ||\Pi_h^{(0)}\text{div }\mathbf{v}_h^\diamond||_0^2 + \mathcal{D}_h(\mathbf{v}_h^\diamond) \leq C||\mathbf{v}_h^\diamond||_1^2 \leq C||w_h||_0^2. \quad (4.21)$$

On the other hand, by the Cauchy-Schwarz inequality and the Young inequality we have

$$\begin{aligned}
c(\mathbf{v}_h^\diamond, (w_h, p_h)) &= ||w_h||_0^2 + (\text{div }\mathbf{v}_h^\diamond, p_h) \\
&\geq ||w_h||_0^2 - ||\text{div }\mathbf{v}^\diamond||_0||p_h||_0 \\
&\geq ||w_h||_0^2 - C||\mathbf{v}_h^\diamond||_1||p_h||_0 \\
&\geq ||w_h||_0^2 - C||w_h||_0||p_h||_0 \\
&\geq C_1||w_h||_0^2 - C_2||p_h||_0^2.
\end{aligned} \quad (4.22)$$

From Lemma 4.3, taking

$$\mathbf{v}_h = \mathbf{v}_{0,h}^{\star}{}^{\perp} + \delta\mathbf{v}_h^\diamond, \quad \delta = \frac{1}{2C_2},$$

from (4.18), (4.21) and (4.22), we have

$$\begin{aligned}
c(\mathbf{v}_h, (w_h, p_h)) &= c(\mathbf{v}_{0,h}^{\star}{}^{\perp}, (w_h, p_h)) + \delta c(\mathbf{v}_h^\diamond, (w_h, p_h)) \\
&\geq \delta C_1||w_h||_0^2 + (1 - \delta C_2)||p_h||_0^2 \\
&\geq C(||w_h||_0^2 + ||p_h||_0^2), \\
|||\mathbf{v}_h|||_h &\leq C(||w_h||_0 + ||p_h||_0).
\end{aligned}$$

Directly, it follows that the inf-sup condition (4.20) holds. □

We note that, from the fact that all norms are equivalent in a finite dimensional space, the definition (4.3) of $\mathbf{K}_h(c)$, Lemma 4.1, (4.20) and the split of (4.5)–(4.6), there exists a constant $C_h > 0$ which might depend on h such that, for all $w_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2})$ and for all $p_h \in Q_h^{(0)}(\mathcal{T}_h)$,

$$\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{K}_h(c)^\perp} \frac{c(\mathbf{v}_h, (w_h, p_h))}{|||\mathbf{v}_h|||_h} \geq C_h(||w_h||_0 + ||p_h||_0). \quad (4.23)$$

The constant $C_h > 0$ for any $h > 0$, but, at the moment, it is not clear whether C_h can be independent of h. However, (4.23) is sufficient to ensure the existence of the dual Fortin-type interpolation in the next subsection, and Lemmas 4.3 and 4.4 are sufficient for the error estimates of the dual Fortin-type interpolation under suitable hypotheses. Later on, we shall improve Lemma 4.4 by showing that (4.20) can hold on $\mathbf{K}_h(c)^\perp$, and then conclude that (4.23) holds uniformly in h, i.e., C_h is indeed independent of h.

By the way, we have the following boundedness:

$$|c(\mathbf{v}, (w, p))| \leq C ||\mathbf{v}||_{h,\text{curl ,div}} \, (||w||_0 + ||p||_0) \leq C |||\mathbf{v}|||_h (||w||_0 + ||p||_0), \quad (4.24)$$

for all $\mathbf{v} \in H(\text{curl}\,;\,\Omega) \cap H(\text{div}\,;\,\Omega)$, $\forall w \in L_0^2(\Omega)$, $\forall p \in L^2(\Omega)$.

4.4 The Dual Fortin-Type Interpolation

In this subsection, with the inf-sup conditions, following the same idea in [25], we show that there is a 'dual' Fortin-type interpolation over $Q_h^{(0)}(\mathcal{T}_h) \times W_h^{(\ell-1)}(\mathcal{T}_{h/2})$ through the trilinear form $c(\cdot, (\cdot, \cdot))$.

We consider the problem: Given $p \in L^2(\Omega)$ and $w \in L_0^2(\Omega)$, to find $\widetilde{p} \in Q_h^{(0)}(\mathcal{T}_h)$ and $\widetilde{w} \in W_h^{(\ell-1)}(\mathcal{T}_{h/2})$ such that

$$c(\mathbf{v}_h, (\widetilde{w}, \widetilde{p})) = c(\mathbf{v}_h, (w, p)), \quad \forall \mathbf{v}_h \in \mathbf{K}_h(c)^{\perp}. \quad (4.25)$$

With the choice of $\mathbf{K}_h(c)^{\perp}$, the established inf-sup inequality (4.23) and the boundedness (4.24), from [3] we know that the stated problem has a unique solution. We denote $(\widetilde{p}, \widetilde{w})$ as the dual Fortin-type interpolation of (p, w).

Theorem 4.1 *Let $p \in H_0^1(\Omega)$ and $w \in L_0^2(\Omega)$, and let $(\widetilde{p}, \widetilde{w}) \in Q_h^{(0)}(\mathcal{T}_h) \times W_h^{(\ell-1)}(\mathcal{T}_{h/2})$ be constructed as in problem (4.25). We have*

$$||\widetilde{p}||_0 + ||\widetilde{w}||_0 \leq C(||\nabla p||_0 + ||w||_0). \quad (4.26)$$

If additionally

$$(\nabla p, \mathbf{v}_{0,h}) = 0, \quad \forall \mathbf{v}_{0,h} \in \mathbf{K}_h(c), \quad (4.27)$$

then,

$$
||p - \widetilde{p}||_0 + ||w - \widetilde{w}||_0
$$
$$
\leq C \inf_{\substack{q_h \in Q_h^{(0)}(\mathcal{T}_h) \\ \chi_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2})}} \left(||p - q_h||_0 + \left(\sum_{T \in \mathcal{T}_h} h_T^{2r-2} ||p - q_h||_{0,T}^2 \right)^{1/2} + ||w - \chi_h||_0 \right)
$$
$$
\quad (4.28)
$$

Proof From the inf-sup condition (4.20)

$$C(||\widetilde{p}||_0 + ||\widetilde{w}||_0) \leq \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{U}_h^{(\ell)}(\mathcal{T}_{h/2})} \frac{c(\mathbf{v}_h, (\widetilde{w}, \widetilde{p}))}{|||\mathbf{v}_h|||_h},$$

writing $\mathbf{v}_h = \mathbf{v}_{0,h} + \mathbf{v}_{0,h}^{\perp}$, $\mathbf{v}_{0,h} \in \mathbf{K}_h(c)$, $\mathbf{v}_{0,h}^{\perp} \in \mathbf{K}_h(c)^{\perp}$, we have

$$c(\mathbf{v}_h, (\widetilde{w}, \widetilde{p})) = c(\mathbf{v}_{0,h}^{\perp}, (\widetilde{w}, \widetilde{p})) = c(\mathbf{v}_{0,h}^{\perp}, (w, p)) = (\text{curl } \mathbf{v}_{0,h}^{\perp}, w) + (\text{div } \mathbf{v}_{0,h}^{\perp}, p),$$

but

$$|(\text{curl } \mathbf{v}_{0,h}^{\perp}, w)| = |(\text{curl } \mathbf{v}_h, w)| \leq ||\text{curl } \mathbf{v}_h||_0 ||w||_0,$$
$$|(\text{div } \mathbf{v}_{0,h}^{\perp}, p)| = |(\mathbf{v}_{0,h}^{\perp}, -\nabla p)| \leq ||\mathbf{v}_{0,h}^{\perp}||_0 ||\nabla p||_0 \leq ||\mathbf{v}_h||_0 ||\nabla p||_0,$$

where we have used the fact that $||\mathbf{v}_{0,h}^{\perp}||_0^2 = ||\mathbf{v}_h||_0^2 - ||\mathbf{v}_{0,h}||_0^2 \leq ||\mathbf{v}_h||_0^2$. It follows that (4.26) holds.

From the inf-sup condition (4.20) we have, for all $q_h \in Q_h^{(0)}(\mathscr{T}_h)$ and for all $\chi_h \in W_h^{(\ell-1)}(\mathscr{T}_{h/2})$,

$$C(||\widetilde{p} - q_h||_0 + ||\widetilde{w} - \chi_h||_0) \leq \sup_{0 \neq \mathbf{v}_h \in \mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2})} \frac{c(\mathbf{v}_h, (\widetilde{w} - \chi_h, \widetilde{p} - q_h))}{|||\mathbf{v}_h|||_h},$$

where, writing $\mathbf{v}_h = \mathbf{v}_{0,h} + \mathbf{v}_{0,h}^\perp$, $\mathbf{v}_{0,h} \in \mathbf{K}_h(c)$ and $\mathbf{v}_{0,h}^\perp \in \mathbf{K}_h(c)^\perp$, by (4.25) we have

$$c(\mathbf{v}_h, (\widetilde{w} - \chi_h, \widetilde{p} - q_h)) = c(\mathbf{v}_h, (\widetilde{w} - w, \widetilde{p} - p)) + c(\mathbf{v}_h, (w - \chi_h, p - q_h))$$
$$= c(\mathbf{v}_{0,h}, (\widetilde{w} - w, \widetilde{p} - p)) + c(\mathbf{v}_h, (w - \chi_h, p - q_h)),$$

and by $\mathbf{v}_{0,h} \in \mathbf{K}_h(c)$, we have curl $\mathbf{v}_{0,h} = 0$, $\Pi_h^{(0)}$div $\mathbf{v}_{0,h} = 0$, while by the assumption (4.27),

$$(\nabla p, \mathbf{v}_{0,h}) = 0,$$

and we have

$$c(\mathbf{v}_{0,h}, (\widetilde{w} - w, \widetilde{p} - p)) = (\text{curl } \mathbf{v}_{0,h}, \widetilde{w} - w) + (\text{div } \mathbf{v}_{0,h}, \widetilde{p} - p)$$
$$= -(\text{div } \mathbf{v}_{0,h}, p) = (\nabla p, \mathbf{v}_{0,h}) = 0.$$

On the other hand,

$$c(\mathbf{v}_h, (w - \chi_h, p - q_h)) = (\text{curl } \mathbf{v}_h, w - \chi_h) + (\text{div } \mathbf{v}_h, p - q_h)$$
$$\leq ||\text{curl } \mathbf{v}_h||_0 ||w - \chi_h||_0 + \left(\sum_{T \in \mathscr{T}_h} h_T^{2-2r} ||\text{div } \mathbf{v}_h||_{0,T}^2 \right)^{1/2}$$
$$\times \left(\sum_{T \in \mathscr{T}_h} h_T^{2r-2} ||p - q_h||_{0,T}^2 \right)^{1/2}$$
$$\leq ||\text{curl } \mathbf{v}_h||_0 ||w - \chi_h||_0 + C \left(\sum_{K \in \mathscr{T}_{h/2}} h_K^{2-2r} ||\text{div } \mathbf{v}_h||_{0,K}^2 \right)^{1/2}$$
$$\times \left(\sum_{T \in \mathscr{T}_h} h_T^{2r-2} ||p - q_h||_{0,T}^2 \right)^{1/2},$$

where we have used the fact that $h_T \leq Ch_K$ for all $K \subset T$ with $K \in \mathscr{T}_{h/2}$, for all $T \in \mathscr{T}_h$ and that $||\text{div } \mathbf{v}_h||_{0,T}^2 = \sum_{K \subset T} ||\text{div } \mathbf{v}_h||_{0,K}^2$. We thus straightforwardly obtain the stated error estimates (4.28). □

Corollary 4.1 *For $p \in H_0^1(\Omega)$ and $w \in H^1(\Omega)/\mathbb{R}$, under the same conditions as in Theorem* 4.1,

$$||p - \widetilde{p}||_0 + ||w - \widetilde{w}||_0 \leq Ch^r(||p||_1 + ||w||_1), \tag{4.29}$$

where r comes from the norm $||| \cdot |||_h$ as defined by (4.12) and (4.13), the same as the r in Proposition 2.3. *If additionally $p = 0$ and $w \in H^{1+s}(\Omega)$ for any $s \in \mathbb{R}$, then, $\widetilde{p} = 0$, and*

$$||w - \widetilde{w}||_0 \leq Ch^{\min(\ell, 1+s)} ||w||_{1+s}. \tag{4.30}$$

Proof Choosing a $q_h \in Q_h^{(0)}(\mathscr{T}_h)$ as the L^2 projection of p such that

$$||p - q_h||_{0,T} \leq Ch_T||p||_{1,T}, \quad \forall T \in \mathscr{T}_h$$

and $\chi_h \in W_h^{(\ell-1)}(\mathscr{T}_{h/2})$ as the L^2 projection of w such that

$$||w - \chi_h||_{0,T} \leq Ch_T||w||_{1,T}, \quad \forall T \in \mathscr{T}_{h/2},$$

we obtain (4.29) from (4.28).

From problem (4.25), choosing $\mathbf{v}_h \in \mathbf{K}_h(c)^\perp$ with curl $\mathbf{v}_h = 0$, we have

$$(\text{div } \mathbf{v}_h, \widetilde{p}) = (\text{div } \mathbf{v}_h, p),$$

and from the inf-sup condition (4.19) in Lemma 4.3, we have

$$C||\widetilde{p}||_0 \leq \sup_{\substack{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{K}_h(c)^\perp \\ \text{curl } \mathbf{v}_h = 0}} \frac{(\text{div } \mathbf{v}_h, \widetilde{p})}{|||\mathbf{v}_h|||_h} = \sup_{\substack{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{K}_h(c)^\perp \\ \text{curl } \mathbf{v}_h = 0}} \frac{(\text{div } \mathbf{v}_h, p)}{|||\mathbf{v}_h|||_h},$$

but $p = 0$, we have $\widetilde{p} = 0$. From (4.28) we have therefore

$$||w - \widetilde{w}||_0 \leq C \inf_{\chi_h \in W_h^{(\ell-1)}(\mathscr{T}_{h/2})} ||w - \chi_h||_0,$$

and we have no difficulty in finding a suitable $\chi_h \in W_h^{(\ell-1)}(\mathscr{T}_{h/2})$ such that (4.30) holds.
□

However, we want to study a general p which does not necessarily satisfy (4.27). We can improve Lemma 4.4 to achieve that goal. Such improvement is stated in Theorem 4.2 below.

Lemma 4.5 *For any given* $w_h \in W_h^{(\ell-1)}(\mathscr{T}_{h/2})$*, there exists a* $\mathbf{v}_{0,h}^{*\perp} \in \mathbf{K}_h(c)^\perp$*, with* $\Pi_h^{(0)} \text{div } \mathbf{v}_{0,h}^{*\perp} = 0$*, such that*

$$\begin{cases} c(\mathbf{v}_{0,h}^{*\perp}, (w_h, p_h)) = (\text{curl } \mathbf{v}_{0,h}^{*\perp}, w_h) = ||w_h||_0^2, & \forall p_h \in Q_h^{(0)}(\mathscr{T}_h), \\ |||\mathbf{v}_{0,h}^{*\perp}|||_h \leq C||w_h||_0; \end{cases} \tag{4.31}$$

$$\sup_{\substack{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{K}_h(c)^\perp \\ \Pi_h^{(0)} \text{div } \mathbf{v}_h = 0}} \frac{c(\mathbf{v}_h, (w_h, p_h))}{|||\mathbf{v}_h|||_h} \geq C||w_h||_0, \quad \forall w_h \in W_h^{(\ell-1)}(\mathscr{T}_{h/2}). \tag{4.32}$$

Proof Inequality (4.32) follows directly from (4.31). To show (4.31), we first construct a general 'primal' Fortin-type interpolation, which will also be used for error estimates, and then apply this to obtain (4.31).

Let ρ_h denote the L^2 projection onto $W_h^{(\ell-1)}(\mathscr{T}_{h/2})$, i.e., for a $w \in L_0^2(\Omega)$, define $\rho_h w \in W_h^{(\ell-1)}(\mathscr{T}_{h/2})$ by

$$(\rho_h w, \chi_h) = (w, \chi_h), \quad \forall \chi_h \in W_h^{(\ell-1)}(\mathscr{T}_{h/2}). \tag{4.33}$$

Such ρ_h has the following approximation properties:

$$||\rho_h w||_0 \leq ||w||_0, \quad ||\rho_h w - w||_0 \leq Ch^{\min(\ell, 1+s)}||w||_{1+s}, \quad \text{for } w \in H^{1+s}(\Omega) \cap L_0^2(\Omega). \tag{4.34}$$

Given $\mathbf{u} \in H_0(\mathrm{curl}\,; \Omega) \cap H(\mathrm{div}\,^0; \Omega)$. We shall show that there exists a 'primal' Fortin-type interpolation $\widetilde{\mathbf{u}} \in \mathbf{K}_h(c)^\perp$ such that

$$\begin{cases} \widetilde{\mathbf{u}} \in \mathbf{K}_h(c)^\perp, \\ \mathrm{curl}\,\widetilde{\mathbf{u}} = \rho_h \mathrm{curl}\,\mathbf{u}, \\ \Pi_h^{(0)} \mathrm{div}\,\widetilde{\mathbf{u}} = 0, \end{cases} \tag{4.35}$$

and $\widetilde{\mathbf{u}}$ satisfies the following approximation properties:

$$||\mathbf{u} - \widetilde{\mathbf{u}}||_0 \leq Ch^r ||\mathrm{curl}\,\mathbf{u}||_0, \tag{4.36}$$

$$||\mathrm{curl}\,(\mathbf{u} - \widetilde{\mathbf{u}})||_0 = \inf_{\chi_h \in W_h^{(\ell-1)}(\mathscr{T}_{h/2})} ||\mathrm{curl}\,\mathbf{u} - \chi_h||_0, \tag{4.37}$$

$$|||\mathbf{u} - \widetilde{\mathbf{u}}|||_h \leq C ||\mathrm{curl}\,\mathbf{u}||_0, \tag{4.38}$$

$$||\widetilde{\mathbf{u}}||_{h,0,\mathrm{curl}\,,\mathrm{div}} \leq C ||\mathrm{curl}\,\mathbf{u}||_0, \tag{4.39}$$

$$|||\widetilde{\mathbf{u}}|||_h \leq C ||\mathbf{u}||_{0,\mathrm{curl}}. \tag{4.40}$$

In fact, firstly, for $\rho_h \mathrm{curl}\,\mathbf{u} \in W_h^{(\ell-1)}(\mathscr{T}_{h/2})$, by Lemma 4.2, there exists $\mathbf{v}_h\dagger \in \mathbf{X}_h^{(\ell)}(\mathscr{T}_{h/2}) \subset \mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2})$ such that

$$\mathrm{curl}\,\mathbf{v}_h^\dagger = \rho_h \mathrm{curl}\,\mathbf{u}, \quad ||\mathbf{v}_h^\dagger||_1 \leq C ||\rho_h \mathrm{curl}\,\mathbf{u}||_0.$$

From the proof of Lemma 4.3, for $\Pi_h^{(0)} \mathrm{div}\,\mathbf{v}_h^\dagger \in Q_h^{(0)}(\mathscr{T}_h)$, we can find a $\vartheta_h \in Y_h^{(\ell+1)}(\mathscr{T}_{h/2})$ such that, $\mathbf{v}_h^0 := -\nabla \vartheta_h \in \mathbf{K}_h(c)^\perp$,

$$\Pi_h^{(0)} \mathrm{div}\,\mathbf{v}_h^0 = \Pi_h^{(0)} \mathrm{div}\,\mathbf{v}_h^\dagger.$$

Define

$$\mathbf{v}_h^\ddagger := \mathbf{v}_h^\dagger - \mathbf{v}_h^0,$$

which satisfies

$$\mathrm{curl}\,\mathbf{v}_h^\ddagger = \mathrm{curl}\,\mathbf{v}_h^\dagger = \rho_h \mathrm{curl}\,\mathbf{u}, \quad \Pi_h^{(0)} \mathrm{div}\,\mathbf{v}_h^\ddagger = 0.$$

Split

$$\mathbf{v}_h^\ddagger = \mathbf{v}_{0,h}^\ddagger + \mathbf{v}_{0,h}^\ddagger{}^\perp, \quad \mathbf{v}_{0,h}^\ddagger \in \mathbf{K}_h(c), \quad \mathbf{v}_{0,h}^\ddagger{}^\perp \in \mathbf{K}_h(c)^\perp,$$

and define

$$\widetilde{\mathbf{u}} := \mathbf{v}_{0,h}^\ddagger{}^\perp.$$

We shall show that such $\widetilde{\mathbf{u}}$ indeed fulfills (4.35)–(4.40). By definition, (4.35) holds. Now, to show (4.36)–(4.40), we consider the problem:

$$\mathbf{curl}\,\mathrm{curl}\,\mathbf{z} - \nabla \mathrm{div}\,\mathbf{z} = \mathbf{u} - \widetilde{\mathbf{u}} \quad \text{in } \Omega, \quad \mathbf{z} \cdot \mathbf{t} = 0, \quad \mathrm{div}\,\mathbf{z} = 0 \quad \text{on } \Gamma, \tag{4.41}$$

which is a type of Hodge-decomposition, putting

$$p := \mathrm{div}\,\mathbf{z} \in H_0^1(\Omega), \quad w := \mathrm{curl}\,\mathbf{z} \in H^1(\Omega)/\mathbb{R}.$$

Here we need the form (4.41) in order to use the dual Fortin-type interpolation. Since $\mathrm{div}\,\mathbf{u} = 0$ and $\widetilde{\mathbf{u}} \in \mathbf{K}_h(c)^\perp$, we have $\mathbf{u} = \mathbf{curl}\,\psi$ for a scalar function ψ, $(\mathbf{u}, \mathbf{v}_{0,h}) = (\psi, \mathrm{curl}\,\mathbf{v}_{0,h}) = 0$ for all $\mathbf{v}_{0,h} \in \mathbf{K}_h(c)$, and we have

$$(\mathbf{u} - \widetilde{\mathbf{u}}) \perp \mathbf{K}_h(c).$$

As a consequence, from (4.41) we see that

$$(\nabla p, \mathbf{v}_{0,h}) = 0, \quad \forall \mathbf{v}_{0,h} \in \mathbf{K}_h(c).$$

In addition,

$$||p||_1 \le C||\mathbf{u} - \widetilde{\mathbf{u}}||_0, \quad ||w||_1 \le C|w|_1 = C||\mathbf{curl}\, w||_0 = C||\mathbf{curl}\, \mathbf{curl}\, \mathbf{z}||_0 \le C||\mathbf{u} - \widetilde{\mathbf{u}}||_0.$$

From (4.25), let $\widetilde{p} \in Q_h^{(0)}(\mathcal{T}_h)$ and $\widetilde{w} \in W_h^{(\ell-1)}(\mathcal{T}_{h/2})$ be the dual Fortin-type interpolation of p and w, respectively. Then, from (4.41), (4.25), (4.35), (4.33), (4.34), and Corollary 4.1, we find that

$$
\begin{aligned}
||\mathbf{u} - \widetilde{\mathbf{u}}||_0^2 &= (\mathbf{curl}\, \mathbf{curl}\, \mathbf{z}, \mathbf{u} - \widetilde{\mathbf{u}}) - (\nabla \mathrm{div}\, \mathbf{z}, \mathbf{u} - \widetilde{\mathbf{u}}) \\
&= (\mathrm{curl}\, \mathbf{z}, \mathrm{curl}\, (\mathbf{u} - \widetilde{\mathbf{u}})) - (p, \mathrm{div}\, \widetilde{\mathbf{u}}) \\
&= (\mathrm{curl}\, \mathbf{z}, \mathrm{curl}\, \mathbf{u} - \rho_h \mathrm{curl}\, \mathbf{u})) - (\widetilde{p}, \mathrm{div}\, \widetilde{\mathbf{u}}) + (\mathrm{curl}\, \widetilde{\mathbf{u}}, w - \widetilde{w}) \\
&= (\mathrm{curl}\, \mathbf{z} - \rho_h \mathrm{curl}\, \mathbf{z}, \mathrm{curl}\, \mathbf{u} - \rho_h \mathrm{curl}\, \mathbf{u}) + (\mathrm{curl}\, \widetilde{\mathbf{u}}, w - \widetilde{w}) \\
&= (\mathrm{curl}\, \mathbf{z} - \rho_h \mathrm{curl}\, \mathbf{z}, \mathrm{curl}\, \mathbf{u}) + (\rho_h \mathrm{curl}\, \mathbf{u}, w - \widetilde{w}) \\
&\le ||\mathrm{curl}\, \mathbf{z} - \rho_h \mathrm{curl}\, \mathbf{z}||_0 ||\mathrm{curl}\, \mathbf{u}||_0 + ||\mathrm{curl}\, \mathbf{u}||_0 ||w - \widetilde{w}||_0 \\
&\le Ch ||\mathrm{curl}\, \mathbf{z}||_1 ||\mathrm{curl}\, \mathbf{u}||_0 + Ch^r (||p||_1 + ||w||_1) ||\mathrm{curl}\, \mathbf{u}||_0 \\
&\le Ch^r ||\mathrm{curl}\, \mathbf{u}||_0 ||\mathbf{u} - \widetilde{\mathbf{u}}||_0,
\end{aligned}
$$

that is, (4.36) holds. From (4.35), since $\mathrm{curl}\, \widetilde{\mathbf{u}}$ is the L^2 projection of $\mathrm{curl}\, \mathbf{u}$, (4.37) holds obviously. From (4.35), (4.36), (4.37) and Proposition 4.1, it follows that (4.38) holds, since

$$
\begin{aligned}
|||\mathbf{u} - \widetilde{\mathbf{u}}|||_h^2 &= ||\mathbf{u} - \widetilde{\mathbf{u}}||_{0,\mathrm{curl}}^2 + ||\Pi_h^{(0)} \mathrm{div}\, (\mathbf{u} - \widetilde{\mathbf{u}})||_0^2 + \mathcal{D}_h(\mathbf{u} - \widetilde{\mathbf{u}}) \\
&\le Ch^{2r} ||\mathrm{curl}\, \mathbf{u}||_0^2 + C||\mathrm{curl}\, \mathbf{u}||_0^2 + Ch^{-2r} ||\mathbf{u} - \widetilde{\mathbf{u}}||_0^2 \\
&\le C||\mathrm{curl}\, \mathbf{u}||_0^2.
\end{aligned}
$$

Both (4.38) and (4.39) then follow easily.

Now, we show (4.31). For $w_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2})$, we consider the problem: Find $\mathbf{u} \in H_0(\mathrm{curl}\,; \Omega) \cap H(\mathrm{div}\,^0; \Omega)$ such that

$$\mathrm{curl}\, \mathbf{u} = w_h, \quad \mathrm{div}\, \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u} \cdot \mathbf{t} = 0 \quad \text{on } \Gamma.$$

Take

$$\mathbf{v}_{0,h}^* {}^\perp := \widetilde{\mathbf{u}},$$

which satisfies (4.35)–(4.40). Immediately,

$$
\begin{aligned}
c(\mathbf{v}_{0,h}^* {}^\perp, (w_h, p_h)) &= (\mathrm{curl}\, \mathbf{v}_{0,h}^* {}^\perp, w_h) = ||w_h||_0^2, \quad |||\mathbf{v}_{0,h}^* {}^\perp|||_h \\
&\le C||\mathbf{u}||_{0,\mathrm{curl}} \le C||\mathrm{curl}\, \mathbf{u}||_0 = C||w_h||_0.
\end{aligned}
$$

$\square$

Theorem 4.2 *For all $w_h \in W_h^{(\ell-1)}(\mathcal{T}_{h/2})$ and for all $p_h \in Q_h^{(0)}(\mathcal{T}_h)$, we have*

$$\sup_{\mathbf{0} \ne \mathbf{v}_h \in \mathbf{K}_h(c)^\perp} \frac{c(\mathbf{v}_h, (w_h, p_h))}{|||\mathbf{v}_h|||_h} \ge C(||w_h||_0 + ||p_h||_0). \tag{4.42}$$

For any given $p \in H_0^1(\Omega)$ and $w \in H^1(\Omega)/\mathbb{R}$, let $\widetilde{p} \in Q_h^{(0)}(\mathcal{T}_h)$ and $\widetilde{w} \in W_h^{(\ell-1)}(\mathcal{T}_{h/2})$ be the dual Fortin-type interpolation of p and w, respectively, defined by (4.25). Then,

$$||\widetilde{p}||_0 + ||\widetilde{w}||_0 \le C(||\nabla p||_0 + ||w||_0), \tag{4.43}$$

which was established in Theorem 4.1, *satisfying*

$$||p - \widetilde{p}||_0 + ||w - \widetilde{w}||_0$$

$$\leq C \inf_{\substack{q_h \in Q_h^{(0)}(\mathscr{T}_h) \\ \chi_h \in W_h^{(\ell-1)}(\mathscr{T}_{h/2})}} \left(||p - q_h||_0 + \left(\sum_{T \in \mathscr{T}_h} h_T^{2r-2} ||p - q_h||_{0,T}^2 \right)^{1/2} + ||w - \chi_h||_0 \right),$$

(4.44)

$$||p - \widetilde{p}||_0 + ||w - \widetilde{w}||_0 \leq Ch^r (||p||_1 + ||w||_1). \tag{4.45}$$

Proof Combining Lemmas 4.3 and 4.5, following a similar argument in proving Lemma 4.4, we have (4.42), while (4.43)–(4.45) follow similarly as in Theorem 4.1 and Corollary 4.1. □

We should emphasize that no condition (4.27) is required in Theorem 4.2. In addition, (4.42) confirms that the constant C_h in (4.23) actually does not depend on h. Theorem 4.2 plays a key for the uniform convergence in the spectral approximation.

Introduce the kernel space

$$\mathbf{K}_h(b) = \{\mathbf{v}_h \in \mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2}) : b(\mathbf{v}_h, q_h) = 0, \forall q_h \in Q_h^{(0)}(\mathscr{T}_h)\}$$
$$= \{\mathbf{v}_h \in \mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2}) : \Pi_h^{(0)} \text{div } \mathbf{v}_h = 0\}. \tag{4.46}$$

Corollary 4.2 *We have the following kernel-coercivity:*

$$a(\mathbf{v}_h, \mathbf{v}_h) = ||\text{curl } \mathbf{v}_h||_0^2 \geq C ||\mathbf{v}_h||_0^2, \quad \forall \mathbf{v}_h \in \mathbf{K}_h(b) \cap \mathbf{K}_h(c)^{\perp}.$$

Proof From Proposition 2.2 we write $\mathbf{v}_h$ as the following L^2 orthogonal Hodge-decomposition:

$$\mathbf{v}_h = \mathbf{curl } w - \nabla p, \quad p \in H_0^1(\Omega), \quad w \in H^1(\Omega)/\mathbb{R},$$
$$||\mathbf{v}_h||_0^2 = ||\nabla p||_0^2 + ||\mathbf{curl } w||_0^2,$$

and let $\widetilde{p} \in Q_h^{(0)}(\mathscr{T}_h)$, $\widetilde{w} \in W_h^{(\ell-1)}(\mathscr{T}_{h/2})$ be the dual Fortin-type interpolations of p, w, respectively, i.e.

$$c(\mathbf{v}_h, (\widetilde{w}, \widetilde{p})) - c(\mathbf{v}_h, (w, p)) = 0 \quad \forall \mathbf{v}_h \in \mathbf{K}_h(c)^{\perp},$$

such that (see Theorem 4.2)

$$||\widetilde{p}||_0 + ||\widetilde{w}||_0 \leq C (||p||_1 + ||w||_0) \leq C (||p||_1 + |w|_1) \leq C ||\mathbf{v}_h||_0.$$

where we have used the Poincaré inequality $||w||_0 \leq C|w|_1 = C||\mathbf{curl } w||_0$. Since $\mathbf{v}_h \in \mathbf{K}_h(b)$, we have (div $\mathbf{v}_h, \widetilde{p}) = 0$, and we have

$$||\mathbf{v}_h||_0^2 = (\mathbf{v}_h, \mathbf{v}_h) = (\mathbf{v}_h, \mathbf{curl } w - \nabla p)$$
$$= c(\mathbf{v}_h, (w, p)) = c(\mathbf{v}_h, (\widetilde{w}, \widetilde{p})) = (\text{curl } \mathbf{v}_h, \widetilde{w})$$
$$\leq ||\text{curl } \mathbf{v}_h||_0 ||\widetilde{w}||_0 \leq C ||\text{curl } \mathbf{v}_h||_0 ||\mathbf{v}_h||_0,$$

from which the stated coercivity follows. □

5 Error Estimates

In this section, we shall give the error estimates of the solution operators and then give the error estimates of the eigenvalues and the eigenfunctions.

5.1 Properties of Discrete Eigenvalue and Eigenfunction

Put

$$\lambda_h := \omega_h^2 + 1.$$

Problem (3.4) can be restated as follows: Find $(\lambda_h, \mathbf{u}_h \neq \mathbf{0}, p_h) \in \mathbb{R} \times \mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2}) \times Q_h^{(0)}(\mathscr{T}_h)$ such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + (\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \lambda_h(\mathbf{u}_h, \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2}), \\ b(\mathbf{u}_h, q_h) = 0, & \forall q_h \in Q_h^{(0)}(\mathscr{T}_h). \end{cases} \quad (5.1)$$

Lemma 5.1 *Problem* (5.1) *is well-posed. There exists an eigenvalue* $\lambda_h^0 = 1$ *which has been priori identified and corresponds to the eigenfunction space* $\mathbf{K}_h(c)$. *All the other eigenvalues* $\lambda_h \neq 1$ *correspond to eigenfunction spaces belonging to* $\mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$ *and*

$$\lambda_h \geq 1 + C, \quad (5.2)$$

where the constant C *comes from Corollary* 4.2

Proof The well-posedness easily follows from the classical theory of saddle-point problems, because

$$a(\mathbf{v}_h, \mathbf{v}_h) + (\mathbf{v}_h, \mathbf{v}_h) = ||\mathbf{v}_h||_{0,\text{curl}}^2 = ||\mathbf{v}_h||_{h,0,\text{curl},\text{div}}^2, \quad \forall \mathbf{v}_h \in \mathbf{K}_h(b)$$

and Lemma 4.3 already indicates the inf-sup condition

$$\sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2})} \frac{b(\mathbf{v}_h, q_h)}{||\mathbf{v}_h||_{h,0,\text{curl},\text{div}}} \geq C||q_h||_0, \quad \forall q_h \in Q_h^{(0)}(\mathscr{T}_h). \quad (5.3)$$

It can also be easily seen that $\lambda_h^0 = 1$ is an eigenvalue and its eigenfunctions belong to $\mathbf{K}_h(c)$. Now, let $\lambda_h \neq 1$ be any eigenvalue, corresponding to eigenfunctions $\mathbf{u}_h$. Split

$$\mathbf{u}_h = \mathbf{u}_{0,h} + \mathbf{u}_{0,h}^\perp, \quad \mathbf{u}_{0,h} \in \mathbf{K}_h(c), \quad \mathbf{u}_{0,h}^\perp \in \mathbf{K}_h(c)^\perp.$$

Taking $\mathbf{v}_h := \mathbf{u}_{0,h}$ in the first equation of (5.1), from (4.4) we have

$$||\mathbf{u}_{0,h}||_0^2 = \lambda_h ||\mathbf{u}_{0,h}||_0^2,$$

but $\lambda_h \neq 1$, we immediately have $\mathbf{u}_{0,h} = \mathbf{0}$, and $\mathbf{u}_h = \mathbf{u}_{0,h}^\perp \in \mathbf{K}_h(c)^\perp$. On the other hand, by the second equation of (5.1), it is obvious that $\mathbf{u}_h \in \mathbf{K}_h(b)$. In other words, any eigenfunction $\mathbf{u}_h$ of $\lambda_h \neq 1$ must belong to $\mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$. Next, taking $\mathbf{v}_h = \mathbf{u}_h$(the eigenfunction of $\lambda_h \neq 1$) in the first equation of (5.1), we have

$$||\mathbf{u}_h||_{0,\text{curl}}^2 = \lambda_h ||\mathbf{u}_h||_0^2,$$

but, from Corollary 4.2, we have

$$||\mathbf{u}_h||_{0,\text{curl}}^2 \geq (1 + C)||\mathbf{u}_h||_0^2,$$

and as a result, we have $\lambda_h \geq 1 + C$. □

Remark 5.1 According to Lemma 5.1, the eigenvalue $\lambda_h^0 = 1$, whose eigenfunction space is $\mathbf{K}_h(c)$ and which has been a priori identified, should be abandoned. Such a situation also exists in the Nédélec element method and discontinuous Galerkin method, e.g., see [15]. In other words, problem (3.4) have zero eigenvalues which should be abandoned. All what we need

are the nonzero eigenvalues $\omega_h^2 \neq 0$ of problem (3.4), or equivalently, all what we need are the eigenvalues $\lambda_h \neq 1$ of problem (5.1). All the eigenvalues $\lambda_h \neq 1$ of problem (5.1), with eigenfunctions belonging to $\mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$, have been *well separated* from the eigenvalue $\lambda_h^0 = 1$ according to (5.2), while the eigenvalue $\lambda_h^0 = 1$ is already a priori identified. Equivalently, all the eigenvalues $\omega_h^2 \neq 0$ of problem (3.4), with eigenfunctions belonging to $\mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$, have been *well separated* from the eigenvalue '0' by $\omega_h^2 \geq C > 0$ from (5.2), i..e, the eigenvalue '0' which is already a priori identified cannot pollute any other nonzero eigenvalues.

5.2 Error Estimates

This section is devoted to the error estimates.

Firstly, let A denote the solution operator mapping from $(L^2(\Omega))^2$ onto $\mathbf{U}$, which is well-defined by the mixed problem (5.4) below, i.e., for any $\mathbf{f} \in (L^2(\Omega))^2$, we have $A\mathbf{f} \in \mathbf{U}$; we also define the multiplier as $B\mathbf{f} \in Q$, where the solution operator B maps $(L^2(\Omega))^2$ onto Q. In other words, for any given $\mathbf{f} \in (L^2(\Omega))^2$, $A\mathbf{f} \in \mathbf{U}$ and $B\mathbf{f} \in Q$ are determined by

$$\begin{cases} a(A\mathbf{f}, \mathbf{v}) + b(\mathbf{v}, B\mathbf{f}) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathbf{U}, \\ b(A\mathbf{f}, q) = 0, & \forall q \in Q. \end{cases} \tag{5.4}$$

The solution operator A is compact from $(L^2(\Omega))^2$ to $\mathbf{U}$, since $\mathbf{U} = H_0(\mathbf{curl}\,; \Omega) \cap H(\mathrm{div}\,; \Omega)$ is compactly embedded in $(L^2(\Omega))^2$ (cf., [42]). Here we have seen that the saddle-point problem is advantageous because it can provide a compact solution operator. In addition, the multiplier which is denoted by $B\mathbf{f} \in Q$ satisfies

$$B\mathbf{f} = 0, \quad \text{if } \mathbf{f} \in H(\mathrm{div}\, 0; \Omega). \tag{5.5}$$

Note that we study $A\mathbf{u}$ for the eigenfunction $\mathbf{u}$, while $\mathbf{u} \in H(\mathrm{div}\, 0; \Omega)$. In other words, the multiplier $p \in Q = L^2(\Omega)$ in the eigenvalue problem (1.2) satisfies

$$p = \omega^2 B\mathbf{u} = 0. \tag{5.6}$$

In terms of A, the eigenvalue problem (1.2) can be written in the equivalent form

$$A\mathbf{u} = (\omega^2)^{-1}\mathbf{u}, \quad p = \omega^2 B\mathbf{u} = 0. \tag{5.7}$$

On the other hand, for any nonzero eigenvalues $\omega_h^2 \neq 0$, from the finite element eigenvalue problem (3.4), we introduce the discrete solution operator $A_h : (L^2(\Omega))^2 \to \mathbf{K}_h(c)^\perp$ and the discrete operator $B_h : (L^2(\Omega))^2 \to Q_h^{(0)}(\mathscr{T}_h)$, i.e., for any given $\mathbf{f} \in (L^2(\Omega))^2$, $A_h\mathbf{f} \in \mathbf{K}_h(c)^\perp$ and $B_h\mathbf{f} \in Q_h^{(0)}(\mathscr{T}_h)$ are determined by

$$\begin{cases} a(A_h\mathbf{f}, \mathbf{v}_h) + b(\mathbf{v}_h, B_h\mathbf{f}) = (\mathbf{f}, \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{K}_h(c)^\perp, \\ b(A_h\mathbf{f}, q_h) = 0, & \forall q_h \in Q_h^{(0)}(\mathscr{T}_h). \end{cases} \tag{5.8}$$

Note that $A_h\mathbf{f} \in \mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$, in fact. In terms of A_h, the finite element eigenvalue problem (3.4) can be written in the form

$$A_h\mathbf{u}_h = (\omega_h^2)^{-1}\mathbf{u}_h, \quad \text{where } \omega_h^2 \neq 0, \quad p_h = \omega_h^2 B_h\mathbf{u}_h. \tag{5.9}$$

For convenience, put

$$\mathbf{z} := A\mathbf{f}, \quad \rho := B\mathbf{f}. \tag{5.10}$$

We consider the source problem: for any given $\mathbf{f} \in (L^2(\Omega))^2$, find $\mathbf{z}$ and ρ such that

$$\begin{cases} \mathbf{curl} \operatorname{curl} \mathbf{z} - \nabla \rho = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{z} = 0 & \text{in } \Omega, \\ \mathbf{t} \cdot \mathbf{z} = 0 & \text{on } \Gamma, \\ \rho = 0 & \text{on } \Gamma. \end{cases} \tag{5.11}$$

The variational problem of (5.11) is to find $\mathbf{z} \in \mathbf{U}$ and $\rho \in Q$ such that

$$\begin{cases} a(\mathbf{z}, \mathbf{v}) + b(\mathbf{v}, \rho) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathbf{U}, \\ b(\mathbf{z}, q) = 0, & \forall q \in Q. \end{cases} \tag{5.12}$$

The finite element problem reads as follows: Find $\mathbf{z}_h \in \mathbf{K}_h(c)^\perp$ and $\rho_h \in Q_h^{(0)}(\mathscr{T}_h)$ such that

$$\begin{cases} a(\mathbf{z}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \rho_h) = (\mathbf{f}, \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{K}_h(c)^\perp, \\ b(\mathbf{z}_h, q_h) = 0, & \forall q_h \in Q_h^{(0)}(\mathscr{T}_h). \end{cases} \tag{5.13}$$

With the exact solution pair $(\mathbf{z}, \rho)$ of problem (5.12) and with the finite element solution $(\mathbf{z}_h, \rho_h)$ of problem (5.13), we find that

$$\begin{cases} a(\mathbf{z} - \mathbf{z}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \rho - \rho_h) = 0, & \forall \mathbf{v}_h \in \mathbf{K}_h(c)^\perp, \\ b(\mathbf{z} - \mathbf{z}_h, q_h) = 0, & \forall q_h \in Q_h^{(0)}(\mathscr{T}_h). \end{cases} \tag{5.14}$$

We note that

$$|a(\mathbf{u}, \mathbf{v})| \leq ||\mathbf{u}||_{h,0,\operatorname{curl},\operatorname{div}} ||\mathbf{v}||_{h,0,\operatorname{curl},\operatorname{div}}, \quad \forall \mathbf{u}, \mathbf{v} \in H(\operatorname{curl};\Omega) \cap H(\operatorname{div};\Omega),$$

$$|b(\mathbf{v}, q_h)| = |(\operatorname{div} \mathbf{v}, q_h)| = |(\Pi_h^{(0)} \operatorname{div} \mathbf{v}, q_h)|$$
$$\leq ||\mathbf{v}||_{h,0,\operatorname{curl},\operatorname{div}} ||q_h||_0, \quad \forall \mathbf{v} \in H(\operatorname{curl};\Omega) \cap H(\operatorname{div};\Omega), \forall q_h \in Q_h^{(0)}(\mathscr{T}_h),$$

$$|(\mathbf{f}, \mathbf{v})| \leq ||\mathbf{f}||_0 ||\mathbf{v}||_{h,0,\operatorname{curl},\operatorname{div}} \quad \forall \mathbf{v} \in H(\operatorname{curl};\Omega) \cap H(\operatorname{div};\Omega).$$

Consequently, $a(\cdot, \cdot)$, $b(\cdot, \cdot)$, $(\mathbf{f}, \cdot)$ are all bounded over

$$\mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2}) \times \mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2}), \quad \mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2}) \times Q_h^{(0)}(\mathscr{T}_h), \quad \mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2}),$$

respectively, with respect to the solution spaces

$$(\mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2}), ||\cdot||_{h,0,\operatorname{curl},\operatorname{div}}), \quad (Q_h^{(0)}(\mathscr{T}_h), ||\cdot||_0).$$

In addition, Lemma 4.3 already provides the following inf-sup condition:

$$\sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{K}_h(c)^\perp} \frac{b(\mathbf{v}_h, q_h)}{||\mathbf{v}_h||_{h,0,\operatorname{curl},\operatorname{div}}} \geq C ||q_h||_0, \quad \forall q_h \in Q_h. \tag{5.15}$$

Theorem 5.1 *For any* $\mathbf{f} \in (L^2(\Omega))^2$, *problem* (5.13) *admits a unique solution* $(\mathbf{z}_h, \rho_h) \in \mathbf{K}_h(c)^\perp \times Q_h^{(0)}(\mathscr{T}_h)$, *satisfying*

$$||\mathbf{z}_h||_{h,0,\operatorname{curl},\operatorname{div}} + ||\rho_h||_0 \leq c ||\mathbf{f}||_0.$$

Proof With the kernel-coercivity on $\mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$ in Corollary 4.2 and (5.15) the inf-sup condition, we find that the conclusion is just a simple consequence of the classical theory ([13,32]) of saddle-point problems applied to problem (5.13). $\square$

Remark 5.2 With Corollary 4.2 the $\mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$-ellipticity and (5.15) the inf-sup condition again, we can also have the well-posedness of the general mixed problem with any given $\mathbf{j} \in (L^2(\Omega))^2$ and $g \in L^2(\Omega)$: Find $\mathbf{z}_h \in \mathbf{K}_h(c)^\perp$ and $\xi_h \in Q_h^{(0)}(\mathscr{T}_h)$ such that for all $\mathbf{v}_h \in \mathbf{K}_h(c)^\perp$ and $q_h \in Q_h^{(0)}(\mathscr{T}_h)$,

$$\begin{cases} a(\mathbf{z}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \xi_h) = (\mathbf{j}, \mathbf{v}_h), \\ b(\mathbf{z}_h, q_h) = (g, q_h). \end{cases} \tag{5.16}$$

As a consequence, for any given $g \in L^2(\Omega)$,

$$\mathbf{G}_h(b; g) = \{\mathbf{v}_h \in \mathbf{K}_h(c)^\perp : b(\mathbf{v}_h, q_h) = (g, q_h), \forall q_h \in Q_h^{(0)}(\mathscr{T}_h)\} \neq \emptyset. \tag{5.17}$$

When $g = 0$, the set $\mathbf{G}_h(b; 0)$ is none other than $\mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$. This remark, i.e., the result (5.17), is only for adapting the routine in [13] to establish the error estimates in Theorem 5.2 of the related source problem.

Now, we are in a position to give the first main result of error estimates. Since we are dealing with the eigenvalue problem (1.2), where $\mathbf{u}$ is solenoidal, i.e., div $\mathbf{u} = 0$, we first consider the case

$$\mathbf{f} \in H(\text{div } 0; \Omega). \tag{5.18}$$

With this right-hand side, the source problem (5.11) has a dummy multiplier variable, i.e.,

$$\rho = 0. \tag{5.19}$$

Theorem 5.2 *Let $(\mathbf{z}, \rho)$ denote the exact solution of the source problem (5.11) with (5.18) and (5.19), and $(\mathbf{z}_h, \rho_h)$ the finite element solution of (5.13). Then,*

$$||\mathbf{z} - \mathbf{z}_h||_{h,0,\text{curl},\text{div}} \leq c \inf_{\mathbf{w}_h \in \mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)} ||\mathbf{z} - \mathbf{w}_h||_{h,0,\text{curl},\text{div}}. \tag{5.20}$$

Additionally, under the inf-sup condition (5.15),

$$||\mathbf{z} - \mathbf{z}_h||_{h,0,\text{curl},\text{div}} \leq c \inf_{\mathbf{v}_h \in \mathbf{K}_h(c)^\perp} ||\mathbf{z} - \mathbf{v}_h||_{h,0,\text{curl},\text{div}}, \tag{5.21}$$

$$||\rho_h||_0 \leq c ||\mathbf{z} - \mathbf{z}_h||_{h,0,\text{curl},\text{div}}. \tag{5.22}$$

Before proving this theorem, we should note that unlike Theorem 5.1, the classical theory of saddle-point problems cannot immediately lead to the above error estimates. We would rather give the details, but still following the procedure outlined in [13] to prove (5.20)–(5.22).

Proof of Theorem 5.2 Firstly, we show (5.20). Let $\mathbf{w}_h$ be any element of $\mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$. Since $\mathbf{w}_h - \mathbf{z}_h \in \mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$, from Corollary 4.2, we have

$$C||\mathbf{w}_h - \mathbf{z}_h||_{h,0,\text{curl},\text{div}} \leq \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)} \frac{a(\mathbf{w}_h - \mathbf{z}_h, \mathbf{v}_h)}{||\mathbf{v}_h||_{h,0,\text{curl},\text{div}}},$$

where

$$a(\mathbf{w}_h - \mathbf{z}_h, \mathbf{v}_h) = a(\mathbf{w}_h - \mathbf{z}, \mathbf{v}_h) + a(\mathbf{z} - \mathbf{z}_h, \mathbf{v}_h),$$

but, for $(\mathbf{z}, \rho)$ the exact solution and $(\mathbf{z}_h, \rho_h)$ the finite element solution, from (5.14) and (5.19), we find that

$$a(\mathbf{z} - \mathbf{z}_h, \mathbf{v}_h) = -b(\mathbf{v}_h, \rho - \rho_h) = b(\mathbf{v}_h, \rho_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b). \tag{5.23}$$

From the Cauchy-Schwarz inequality, we have

$$||\mathbf{w}_h - \mathbf{z}_h||_{h,0,\text{curl},\text{div}} \leq C \inf_{\mathbf{w}_h \in \mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)} ||\mathbf{z} - \mathbf{w}_h||_{h,0,\text{curl},\text{div}},$$

and (5.20) follows from the triangle inequality.

Secondly, we use the inf-sup condition (5.15) to show that

$$\inf_{\mathbf{w}_h \in \mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)} ||\mathbf{z} - \mathbf{w}_h||_{h,0,\text{curl},\text{div}} \leq C \inf_{\mathbf{v}_h \in \mathbf{K}_h(c)^\perp} ||\mathbf{z} - \mathbf{v}_h||_{h,0,\text{curl},\text{div}}. \tag{5.24}$$

Note that from the inf-sup condition (5.15), for any $\mathbf{t}_h \in \mathbf{K}_h(c)^\perp/\mathbf{K}_h(b)$(the quotient subspace), we have

$$C||\mathbf{t}_h||_{h,0,\text{curl},\text{div}} \leq \sup_{0 \neq \mu_h \in Q_h^{(0)}(\mathcal{T}_h)} \frac{b(\mathbf{t}_h, \mu_h)}{||\mu_h||_0}. \tag{5.25}$$

Let $\mathbf{v}_h$ be any element of $\mathbf{K}_h(c)^\perp$. We look for $\mathbf{r}_h \in \mathbf{K}_h(c)^\perp$ such that

$$b(\mathbf{r}_h, \mu_h) = b(\mathbf{z} - \mathbf{v}_h, \mu_h), \quad \forall \mu_h \in Q_h^{(0)}(\mathcal{T}_h). \tag{5.26}$$

From Remark 5.2, we conclude that (5.26) has at least one solution. From (5.25), we can in fact find the solution in $\mathbf{K}_h(c)^\perp/\mathbf{K}_h(b)$ to satisfy

$$\begin{aligned}
||\mathbf{r}_h||_{h,0,\text{curl},\text{div}} &\leq C \sup_{0 \neq \mu_h \in Q_h^{(0)}(\mathcal{T}_h)} \frac{b(\mathbf{r}_h, \mu_h)}{||\mu_h||_0} \\
&= C \sup_{0 \neq \mu_h \in Q_h^{(0)}(\mathcal{T}_h)} \frac{b(\mathbf{z} - \mathbf{v}_h, \mu_h)}{||\mu_h||_0} \\
&= C||\Pi_h^{(0)}\text{div}\,(\mathbf{z} - \mathbf{v}_h)||_0.
\end{aligned} \tag{5.27}$$

From (5.26), we also know that $\mathbf{w}_h := \mathbf{r}_h + \mathbf{v}_h \in \mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$, since $b(\mathbf{z}, \mu_h) = 0$. Thus, writing

$$||\mathbf{z} - \mathbf{w}_h||_{h,0,\text{curl},\text{div}} = ||\mathbf{z} - \mathbf{v}_h - \mathbf{r}_h||_{h,0,\text{curl},\text{div}} \leq ||\mathbf{z} - \mathbf{v}_h||_{h,0,\text{curl},\text{div}} + ||\mathbf{r}_h||_{h,0,\text{curl},\text{div}}$$

we get directly (5.24) from (5.27) and the definition (4.11) of the norm $||\cdot||_{h,0,\text{curl},\text{div}}$, and (5.21) follows.

Thirdly, from the inf-sup condition (5.15), (5.14), and (5.19), we have no difficulty in obtaining the estimate (5.22) for ρ_h. □

Remark 5.3 If $\mathbf{f} \in (L^2(\Omega))^2$ only and div $\mathbf{f} \neq 0$(in the distributional sense), then from the source problem (5.11), $\rho \neq 0$ and $\rho \in H_0^1(\Omega)$ only, satisfying

$$||\rho||_1 \leq C||\mathbf{f}||_0. \tag{5.28}$$

Note that, from problem (5.11), it can be seen that the multiplier is determined as follows:

$$-\Delta\rho = \text{div }\mathbf{f} \quad \text{in } \Omega, \quad \rho = 0 \quad \text{on } \Gamma. \tag{5.29}$$

We now state the second main result of the error estimates.

Theorem 5.3 *Let* $(\mathbf{z}, \rho)$ *denote the exact solution of problem* (5.11) *with* $\mathbf{f} \in (L^2(\Omega))^2$, *and* $(\mathbf{z}_h, \rho_h)$ *the finite element solution of* (5.13) *in* $\mathbf{K}_h(c)^\perp \times Q_h^{(0)}(\mathcal{T}_h)$. *Then,*

$$||\mathbf{z} - \mathbf{z}_h||_{h,0,\text{curl},\text{div}} + ||\rho - \rho_h||_0 \leq Ch^r||\mathbf{f}||_0. \tag{5.30}$$

Proof Under the $\mathbf{K}_h(c)^{\perp} \cap \mathbf{K}_h(b)$-coercivity in Corollary 4.2 and the inf-sup condition (5.15), following the same argument but with obvious modifications for proving Theorem 5.2, we can obtain

$$
\begin{aligned}
C||\mathbf{z} - \mathbf{z}_h||_{h,0,\mathrm{curl},\mathrm{div}} + ||\rho - \rho_h||_0 &\leq \inf_{\mathbf{v}_h \in \mathbf{K}_h(c)^{\perp}} ||\mathbf{z} - \mathbf{v}_h||_{h,0,\mathrm{curl},\mathrm{div}} \\
&+ \inf_{q_h \in Q_h^{(0)}(\mathscr{T}_h)} \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{K}_h(c)^{\perp}} \frac{b(\mathbf{v}_h, \rho - q_h)}{||\mathbf{v}_h||_{h,0,\mathrm{curl},\mathrm{div}}}.
\end{aligned} \tag{5.31}
$$

Let q_h and w_h be the dual Fortin-type interpolation of ρ and $w := \mathrm{curl}\,\mathbf{z}$, from (4.25) and Theorem 4.2, we have

$$
b(\mathbf{v}_h, \rho - q_h) = (\mathrm{curl}\,\mathbf{v}_h, w_h - w) \leq Ch^r ||\mathrm{curl}\,\mathbf{v}_h||_0 ||\mathrm{curl}\,\mathbf{z}||_1 \leq Ch^r ||\mathrm{curl}\,\mathbf{v}_h||_0 ||\mathbf{curl}\,\mathrm{curl}\,\mathbf{z}||_0,
$$

and we have

$$
\inf_{q_h \in Q_h^{(0)}(\mathscr{T}_h)} \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{K}_h(c)^{\perp}} \frac{b(\mathbf{v}_h, \rho - q_h)}{||\mathbf{v}_h||_{h,0,\mathrm{curl},\mathrm{div}}} \leq Ch^r ||\mathbf{curl}\,\mathrm{curl}\,\mathbf{z}||_0. \tag{5.32}
$$

On the other hand, for $\mathbf{z} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$, taking $\widetilde{\mathbf{z}}$ as constructed by (4.35)–(4.40), we have

$$
\begin{aligned}
\inf_{\mathbf{v}_h \in \mathbf{K}_h(c)^{\perp}} ||\mathbf{z} - \mathbf{v}_h||_{h,0,\mathrm{curl},\mathrm{div}} &\leq Ch^r (||\mathrm{curl}\,\mathbf{z}||_0 + ||\mathrm{curl}\,\mathbf{z}||_1) \\
&= Ch^r (||\mathrm{curl}\,\mathbf{z}||_0 + ||\mathbf{curl}\,\mathrm{curl}\,\mathbf{z}||_0). \tag{5.33}
\end{aligned}
$$

But,

$$
||\mathrm{curl}\,\mathbf{z}||_0 + ||\mathbf{curl}\,\mathrm{curl}\,\mathbf{z}||_0 + ||\rho||_1 \leq C||\mathbf{f}||_0, \tag{5.34}
$$

the proof is then done. □

Remark 5.4 The key role of (5.30) is that it provides a uniform convergence with respect to $\mathbf{f} \in (L^2(\Omega))^2$, i.e.,

$$
\sup_{\mathbf{0} \neq \mathbf{f} \in (L^2(\Omega))^2} \frac{||\mathbf{z} - \mathbf{z}_h||_{h,0,\mathrm{curl},\mathrm{div}} + ||\rho - \rho_h||_0}{||\mathbf{f}||_0} \to 0, \quad \text{as } h \to 0. \tag{5.35}
$$

From Theorem 5.3 and Remark 5.4, in terms of the continuous solution operator A and the discrete solution operator A_h, we have the following theorem.

Theorem 5.4 *For any* $\mathbf{f} \in (L^2(\Omega))^2$, *letting* $A\mathbf{f} \in \mathbf{U}$ *and* $\mathbf{A}_h \mathbf{f} \in \mathbf{K}_h(c)^{\perp}$ *be defined by* (5.11) *and* (5.13), *respectively,*

$$
||(A - A_h)\mathbf{f}||_{h,0,\mathrm{curl},\mathrm{div}} \leq Ch^r ||\mathbf{f}||_0, \tag{5.36}
$$

$$
||(A - A_h)\mathbf{f}||_0 \leq Ch^r ||\mathbf{f}||_0. \tag{5.37}
$$

Theorem 5.4 ensures that the finite element eigenvalue problem (3.4) provides the spectral-correct, spurious-free approximations of the eigenvalue problem (1.2). In other words, all the nonzero discrete eigenvalues $\omega_h^2 \neq 0$ correctly converge to the exact eigenvalues ω^2 with correct multiplicities.

In what follows, we study the order of convergence.

Let ω^2 be an eigenvalue of (1.2) of multiplicity k and let $\mathbf{E} \subset \mathbf{U}$ be the corresponding eigenspace. Obviously,

$$
\mathbf{E} \subset \mathbf{K}(b) := H_0(\mathbf{curl};\Omega) \cap H(\mathrm{div}^0;\Omega). \tag{5.38}
$$

We denote by $\omega_{1,h}^2, \cdots, \omega_{k,h}^2$ the discrete eigenvalues converging to ω^2 and by $\mathbf{E}_h$ the direct sum of the corresponding eigenspaces. Introduce the gaps between the spaces of continuous and discrete eigenfunctions:

$$\hat{\delta}(\mathbf{E}, \mathbf{E}_h) = \max(\delta(\mathbf{E}, \mathbf{E}_h), \delta(\mathbf{E}_h, \mathbf{E})),$$

$$\delta(\mathbf{E}, \mathbf{E}_h) = \sup_{\mathbf{u} \in \mathbf{E}, ||\mathbf{u}||_0 = 1} \inf_{\mathbf{v}_h \in \mathbf{E}_h} ||\mathbf{u} - \mathbf{v}_h||_0,$$

$$\delta(\mathbf{E}_h, \mathbf{E}) = \sup_{\mathbf{u}_h \in \mathbf{E}_h, ||\mathbf{u}_h||_0 = 1} \inf_{\mathbf{v} \in \mathbf{E}} ||\mathbf{u}_h - \mathbf{v}||_0.$$

Applying the abstract theory from ([36], Section 5), from Theorem 5.2, together with the primal Fortin-type interpolation (4.35)–(4.40), or from Theorem 5.3, we have the following theorem about the error estimates of the gap $\hat{\delta}(\mathbf{E}, \mathbf{E}_h)$.

Theorem 5.5 *Under the same hypotheses of Theorem 5.4, there exists a constant C such that*

$$\hat{\delta}(\mathbf{E}, \mathbf{E}_h) \leq C||(A - A_h)_{|\mathbf{E}}||_0 \leq Ch^r. \tag{5.39}$$

$\square$

We can also define the gaps in the norm $|| \cdot ||_{h,0,\text{curl},\text{div}}$, we have the same convergence order as (5.39).

Moreover, noting $B\mathbf{u} = 0$ for any $\mathbf{u} \in \mathbf{E}$. Applying the abstract theory from([36], Section 5), from Theorem 5.2 or from Theorem 5.3, we have the convergence order for the eigenvalues.

Theorem 5.6 *Under the same hypotheses of Theorem 5.4, there exists a constant C such that, for $i = 1, 2, \cdots, k$,*

$$|\omega^2 - \omega_{i,h}^2| \leq C \left(||(A - A_h)_{|\mathbf{E}}||_{h,0,\text{curl},\text{div}} \right)^2 \leq Ch^{2r}. \tag{5.40}$$

$\square$

Remark 5.5 If the eigenfunctions are smooth enough, say belonging to $(H^{1+\ell}(\Omega))^2$, we can obtain from Theorem 5.2

$$\hat{\delta}(\mathbf{E}, \mathbf{E}_h) \leq Ch^\ell, \tag{5.41}$$

$$|\omega^2 - \omega_{i,h}^2| \leq Ch^{2\ell}, \quad i = 1, 2, \cdots, k. \tag{5.42}$$

For singular solution, we obtain the optimal error bounds in Theorems 5.5 and 5.6 relative to the regularity of the solution; for smooth solutions, we obtain the optimal error bounds in (5.41) and (5.42) relative to the both the regularity of the solution and the order of approximation.

Remark 5.6 So far, the approximations from the proposed mixed method has been proved to be spectrally-correct and spurious-free. It would be interesting to know whether the so-called *discrete compactness property* holds. Such property is well-known as the key in the computation of the Maxwell eigenproblem, which was originally due to [35]. Unexpectedly, the estimate (5.32), which we have established as one part of the argument for proving Theorem 5.3, indeed implies the discrete compactness property which reads as follows:

For any uniformly bounded sequence $\{\mathbf{v}_h\} \subset \mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$, i.e., $\mathbf{v}_h \in \mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$, satisfies $||\mathbf{v}_h||_{h,0,\text{curl},\text{div}} \leq 1$ for all h, there exists a $\mathbf{v} \in (L^2(\Omega))^2$(which is actually in $H(\text{div}^0; \Omega)$) such that $||\mathbf{v}_h - \mathbf{v}||_0 \to 0$ as $h \to 0$.

Below is a proof. From Theorem 5.3, putting $\mathbf{f} := \mathbf{v}_h$. From (5.11) we have

$$\mathbf{v}_h = \mathbf{v} - \nabla\rho, \quad \mathbf{v} := \mathbf{curl}\ \mathrm{curl}\ \mathbf{z} \in H(\mathrm{div}\ ^0; \Omega), \quad \rho \in H_0^1(\Omega),$$

where

$$||\mathbf{v}_h||_0^2 = ||\mathbf{v}||_0^2 + ||\nabla\rho||_0^2.$$

But, since $||\mathbf{v}_h||_{h,0,\mathrm{curl}\ ,\mathrm{div}} \leq 1$, we have

$$||\mathbf{v}||_0 + ||\rho||_1 \leq C||\mathbf{v}_h||_0 \leq C||\mathbf{v}_h||_{h,0,\mathrm{curl}\ ,\mathrm{div}} \leq C,$$

where the constant C is independent of h. Since $\mathbf{v}_h \in \mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$, letting q_h denote the dual Fortin-type interpolation of ρ as in the proof of Theorem 5.3, we have

$$||\mathbf{v}_h - \mathbf{v}||_0^2 = (\mathbf{v}_h - \mathbf{v}, \mathbf{v}_h - \mathbf{v}) = (\mathbf{v}_h - \mathbf{v}, -\nabla\rho) = -(\mathbf{v}_h, \nabla\rho) = b(\mathbf{v}_h, \rho) = b(\mathbf{v}_h, \rho - q_h).$$

On the other hand, from (4.35)–(4.40), we observe that (5.32) and (5.33) actually hold on $\mathbf{K}_h(c)^\perp \cap \mathbf{K}_h(b)$. Then, from (5.32),

$$|b(\mathbf{v}_h, \rho - q_h)| \leq Ch^r ||\mathbf{v}_h||_{h,0,\mathrm{curl}\ ,\mathrm{div}} ||\mathbf{curl}\ \mathrm{curl}\ \mathbf{z}||_0 = Ch^r ||\mathbf{v}_h||_{h,0,\mathrm{curl}\ ,\mathrm{div}} ||\mathbf{v}||_0 \leq Ch^r,$$

and we conclude that the stated discrete compactness property holds. Therefore, by Theorem 5.3, within the theory for which the key condition is the discrete compactness property in [35], the mixed method here is spectral-correct and spurious-free.

Remark 5.7 One referee of this paper pointed out the theory by Boffi et al. (see [7]) for mixed finite element methods of eigenproblems. Within that theory, two key properties for spectral-correct and spurious-free approximations are the so-called *weak approximability* and *strong approximability*. Unexpectedly, the two estimates (5.32) and (5.33) in the proof Theorem 5.3 correspond to the weak approximability and the strong approximability, respectively. Then, from [7, Theorem 1, page 3676], we still conclude the uniform convergence, the same as stated in Theorem 5.4. In other words, the theory in [7] can also be used to show that the mixed method here is spectral-correct and spurious-free.

Remark 5.8 Here we would like to comment on the fact that the singular solution of not being H^1 function can be correctly approximated by the H^1-conforming Lagrange elements, with optimum in convergence with respect to the regularity of the solution, i.e., $\mathbf{u} \in (H^r(\Omega))^2$ and curl $\mathbf{u} \in H^r(\Omega)$ (in two-dimensions, curl $\mathbf{u} \in H^1(\Omega)$). The correct approximation is ensured by the primal Fortin-type interpolation $\widetilde{\mathbf{u}}$ constructed by (4.35)–(4.40). From the construction, the interpolation includes the gradient of a function in $Y_h^{(\ell+1)}(\mathscr{T}_{h/2}) \subset H^2(\Omega)$. Such inclusion (via the primal Fortin-type interpolation) helps to establish the correct approximation for the div operator; actually, since $\mathbf{u} \in H_0(\mathrm{curl}\ ; \Omega) \cap H(\mathrm{div}\ ^0; \Omega)$, we have $\Pi_h^{(0)}\mathrm{div}\ (\widetilde{\mathbf{u}} - \mathbf{u}) = \Pi_h^{(0)}\mathrm{div}\ \widetilde{\mathbf{u}} = 0$, i.e., $b(\widetilde{\mathbf{u}} - \mathbf{u}, q_h) = 0$ for all $q_h \in Q_h^{(0)}(\mathscr{T}_h)$. From the second equation of (4.35), (4.37) and Lemma 4.2, the curl $\mathbf{u}$ is correctly approximated by curl $\widetilde{\mathbf{u}}$. The L^2 estimates of $\mathbf{u} - \widetilde{\mathbf{u}}$ is obtained from the dual Fortin-type interpolation. Therefore, for $\mathbf{u} \in H_0(\mathrm{curl}\ ; \Omega) \cap H(\mathrm{div}\ ^0; \Omega)$ with the regularity $\mathbf{u} \in (H^r(\Omega))^2$ and curl $\mathbf{u} \in H^r(\Omega)$, via the primal Fortin-type interpolation, the mixed method yields a correct approximation of the singular solution. The convergence is measured in the norm $|||\cdot|||_{h,0,\mathrm{curl}\ ,\mathrm{div}}$ which is defined by (4.10) and (4.11). We should note that as is well-known in the literature (e.g., cf. [19]), the convergence in the natural norm $||\cdot||_{0,\mathrm{curl}\ ,\mathrm{div}}$ which is defined in Sect. 2 cannot be expected for any Lagrange element methods, *because $(H^1(\Omega))^2 \cap H_0(\mathrm{curl}\ ; \Omega)$ is not dense in $H_0(\mathrm{curl}\ ; \Omega) \cap H(\mathrm{div}\ ; \Omega)$ with respect to the natural norm* $||\cdot||_{0,\mathrm{curl}\ ,\mathrm{div}}$, unless Ω is

convex or smooth enough. All the Lagrange element methods such as cited in the Introduction section use some weaker norm than the natural norm $|| \cdot ||_{0,\text{curl},\text{div}}$ and the common main strategy is to weaken the div operator in some sense, e.g., the weighted div method in [20]. Here, the div operator is weakened in the finite element problem by the multiplier with the finite element space $Q_h^{(0)}(\mathscr{T}_h)$.

Remark 5.9 One referee pointed out a Lagrange element method in [43] (see also [8]) but using the classical variational formulation which reads as follows: Find ω^2 and $\mathbf{u} \in H_0(\text{curl};\Omega)$ such that

$$(\text{curl}\,\mathbf{u}, \text{curl}\,\mathbf{v}) = \omega^2(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in H_0(\text{curl};\Omega). \tag{5.43}$$

This is a different method. It is not clear whether the discrete counterpart of problem (5.43) relates to the discrete problem (3.4) here, and the analysis for (5.43) still keeps open as it has been so far.

Remark 5.10 The finite element method of this paper can be in principle straightforwardly used for the three-dimensional eigenproblem, with the three-dimensional vector '**curl** $= \nabla \times$' operator replacing the scalar 'curl' operator and with a suitably defined finite element space $\mathbf{U}_h$ for the electric field(e.g., the P_4 element on the barycentric refinements) and Q_h for the multiplier is still the constant element. We have presented some numerical results for an eigenproblem in three-dimensions. But, it is not clear whether the two-dimensional theory could be generalized to the three-dimensional eigenproblem, because the establishment of Lemma 4.2 for the three-dimensional **curl** operator seems to be unknown.

Remark 5.11 It would be interesting to investigate the dimension of the kernel space $\mathbf{K}_h(c)$, actually $\mathbf{K}_h(c) \cap \mathbf{K}_h(b)$, of the a priori identified and abandoned eigenvalue '0' or $\lambda_h^0 = 1$ in Lemma 5.1. Below, we give the dimension. Let $\mathbf{u}_h^0 \in \mathbf{K}_h(c) \cap \mathbf{K}_h(b)$ denote any eigenfunction relating to the eigenvalue $\lambda_h^0 = 1$. From (4.1), (3.6), (4.3) and (4.46), we have the following two equations:

$$(\text{curl}\,\mathbf{u}_h^0, w_h) = 0, \quad \forall w_h \in W_h^{(\ell-1)}(\mathscr{T}_{h/2}), \quad (\text{div}\,\mathbf{u}_h^0, q_h) = 0, \quad \forall q_h \in Q_h^{(0)}(\mathscr{T}_h).$$

Then, the dimension of $\mathbf{K}_h(c) \cap \mathbf{K}_h(b)$ is equal to the number of the degrees of freedom of $\mathbf{U}_h^{(\ell)}(\mathscr{T}_{h/2})$ subtracting the sum of the number of the degrees of freedom of $W_h^{(\ell-1)}(\mathscr{T}_{h/2})$ and the number of the degrees of freedom of $Q_h^{(0)}(\mathscr{T}_h)$. This is confirmed by calculating the number of the discrete zero eigenvalues from the numerical results for $\ell = 2$ in L-shaped domain.

Remark 5.12 As a concluding remark, within the theory by Babuška–Osborn [4,36], we have established two estimates (5.32) and (5.33) for the mixed method we have presented in this paper. From these estimates, we conclude Theorem 5.3, i.e., the uniform convergence in Theorem 5.4 which is the key condition in the theory by Babuška–Osborn; we also conclude the discrete compactness property in the theory by Kikuchi [35] and both the weak approximability and the strong approximability in the theory by Boffi et al. [7]. Therefore, whichever theory is used either the theory by Babuška–Osborn, or the theory by Kikuchi, or the theory by Boffi et al, the proposed mixed method is spectral-correct and spurious-free. For the spectral approximation of the compact operator, the theory by Babuška–Osborn is thus more general.

Fig. 1 The barycentric
refinement mesh of $h = 1/2$ for
L-shaped domain
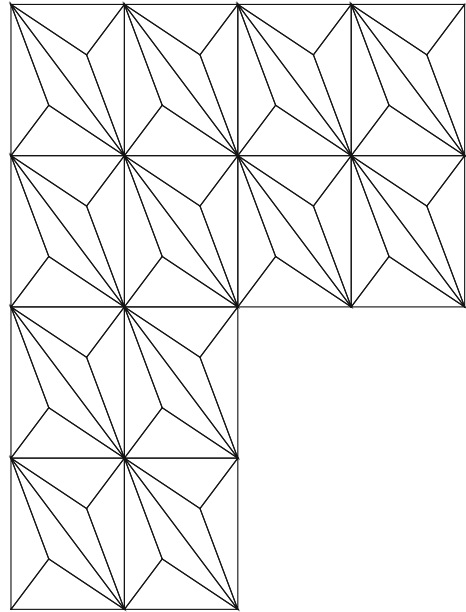
6 Numerical Results

In this section, we provide numerical results of the Maxwell eigenproblem in L-shaped domain. Since this problem has singular eigenfunctions and smooth eigenfunctions, it suffices to use such problem to illustrate the performance of the proposed mixed method and the obtained theoretical results. We also provide some three-dimensional numerical results. Additionally, numerical results are given for the criss-cross mesh.

The L-shaped domain is $(-1, 1)^2 \setminus ([0, 1] \times [-1, 0])$. The reentrant corner is at the origin, with the opening angle $3\pi/2$. The domain is non-convex. The first five eigenvalues are available as follows (https://perso.univ-rennes1.fr/monique.dauge/benchmax.html):

$$1.47562182408,$$
$$3.53403136678,$$
$$9.86960440109,$$
$$9.86960440109,$$
$$11.3894793979.$$

The regularity for the eigenfunctions of the above five eigenvalues are as follows: for any $\epsilon > 0$, the 1rst Maxwell eigenfunction has the strong unbounded singularity, belonging to $(H^{2/3-\epsilon}(\Omega))^2$, the 2nd one belongs to $(H^{4/3-\epsilon}(\Omega))^2$, the 3rd and 4th ones are analytic (exact value of the eigenvalue $\pi^2 = 9.86960440108936$), and the 5th one seems belonging to $(H^{4/3-\epsilon}(\Omega))^2$.

The numerical results are obtained for which the uniform triangle meshes are generated by the barycentric refinement as shown by Fig. 1.

The numerical results show that the proposed mixed method can effectively solve the Maxwell eigenproblem, optimally approximating the eigenvalues whose eigenfunctions are singular or smooth. We use the P_2 element, P_3 element and P_4 element. All the results are accordingly reported in Tables 1, 2 and 3, where the convergence rate is computed by

Table 1 Finite element eigenvalues in L-shaped domain using P_2 element

ω^2	$1/h$	ω_h^2	$|\omega^2 - \omega_h^2|/|\omega^2|$	Conv. rate
1.47562182408	2	1.24859256624436	1.5385E−01	–
	4	1.38024538295140	6.4635E−02	1.25
	8	1.43681673052291	2.6297E−02	1.30
	16	1.46007155583090	1.0538E−02	1.32
	32	1.46942737341303	4.1979E−03	1.33
	64	1.47315994943214	1.6684E−03	1.33
3.53403136678	2	3.52461498343833	2.6645E−03	–
	4	3.53246614272948	4.4290E−04	2.59
	8	3.53377820993822	7.1634E−05	2.63
	16	3.53399107280163	1.1402E−05	2.65
	32	3.53402499439782	1.8031E−06	2.66
	64	3.53403036143924	2.8447E−07	2.66
9.86960440109	2	9.88658420199062	1.7204E−03	–
	4	9.87079892329209	1.2103E−04	3.83
	8	9.86968257422809	7.9206E−06	3.93
	16	9.86960935601154	5.0204E−07	3.98
	32	9.86960471192718	3.1494E−08	3.99
	64	9.86960442019671	1.9359E−09	4.02
9.86960440109	2	9.88687299566249	1.7497E−03	–
	4	9.87080184900686	1.2133E−04	3.85
	8	9.86968260179590	7.9234E−06	3.94
	16	9.86960935638135	5.0208E−07	3.98
	32	9.86960471195824	3.1498E−08	3.99
	64	9.86960442043108	1.9597E−09	4.01
11.38947939790	2	11.39585047754120	5.5938E−04	–
	4	11.38916368873260	2.7719E−05	4.33
	8	11.38924892781950	2.0235E−05	0.45
	16	11.38943011987580	4.3266E−06	2.23
	32	11.38947079670730	7.5519E−07	2.52
	64	11.38947799035300	1.2358E−07	2.61

$\log_2 \left(\dfrac{|\omega^2 - \omega_h^2|/|\omega^2|}{|\omega^2 - \omega_{h/2}^2|/|\omega^2|} \right)$ for the two consecutive mesh sizes h and $h/2$. From these results, we can see that the theoretical error bounds are confirmed.

From Table 1, we see that corresponding to the five eigenvalues, respectively, the convergence rate is about $4/3 \approx 2r$ for the first eigenvalue with $r = 2/3 - \epsilon$, $8/3 \approx 2r$ for the second eigenvalue with $r = 4/3 - \epsilon$, $4 = 2\ell$ for $\ell = 2$ for the third and fourth eigenvalues with analytical eigenfunctions, $8/3 \approx 2r$ for the fifth eigenvalue with $r = 4/3 - \epsilon$. These are consistent with the theoretical results.

From Table 2, we see that corresponding to the five eigenvalues, respectively, the convergence rate is about $4/3 \approx 2r$ for the first eigenvalue with $r = 2/3 - \epsilon$, $8/3 \approx 2r$ for the second eigenvalue with $r = 4/3 - \epsilon$, $6 = 2\ell$ for $\ell = 3$ for the third and fourth eigenvalues with analytical eigenfunctions, $8/3 \approx 2r$ for the fifth eigenvalue with $r = 4/3 - \epsilon$. These are

Table 2 Finite element eigenvalues in L-shaped domain using P_3 element

ω^2	$1/h$	ω_h^2	$|\omega^2 - \omega_h^2|/|\omega^2|$	Conv. rate
1.47562182408	2	1.37260019340807	6.9816E−02	–
	4	1.43374377209089	2.8380E−02	1.30
	8	1.45884306380164	1.1371E−02	1.32
	16	1.46893848295231	4.5292E−03	1.33
	32	1.47296571601359	1.8000E−03	1.33
	64	1.47456715406756	7.1473E−04	1.33
3.53403136678	2	3.53178763377640	6.3489E−04	–
	4	3.53367670946601	1.0035E−04	2.66
	8	3.53397547706695	1.5815E−05	2.67
	16	3.53402256212760	2.4914E−06	2.67
	32	3.53402997994109	3.9242E−07	2.67
	64	3.53403114834904	6.1808E−08	2.67
9.86960440109	2	9.86981129391765	2.0963E−05	–
	4	9.86960805422315	3.7014E−07	5.82
	8	9.86960445962785	5.9311E−09	5.96
	16	9.86960440198589	9.0773E−11	6.03
	32	9.86960440216369	1.0879E−10	− 0.26
	64	9.86960440106915	2.1125E−12	5.69
9.86960440109	2	9.86989008500269	2.8946E−05	–
	4	9.86960909179044	4.7527E−07	5.93
	8	9.86960447522193	7.5111E−09	5.98
	16	9.86960440355411	2.4967E−10	4.91
	32	9.86960441066704	9.7036E−10	− 1.96
	64	9.86960440108235	7.7500E−13	10.29
11.38947939790	2	11.38699143022030	2.1844E−04	–
	4	11.38897522454420	4.4267E−05	2.30
	8	11.38939885234620	7.0719E−06	2.65
	16	11.38946673950950	1.1114E−06	2.67
	32	11.38947740541000	1.7494E−07	2.67
	64	11.38947908466560	2.7502E−08	2.67

consistent with the theoretical results. For the 3rd and 4th eigenvalues, when $h = 1/16$, the computed are already very accurate close to the exact solutions where the relative errors are of magnitude about 1.0E−10 so that the computed convergence rates for successive meshes after $h = 1/16$ do not truly reflect the predicted.

From Table 3, we see that corresponding to the five eigenvalues, respectively, the convergence rate is about $4/3 \approx 2r$ for the first eigenvalue with $r = 2/3 - \epsilon$, $8/3 \approx 2r$ for the second eigenvalue with $r = 4/3 - \epsilon$, $8 = 2\ell$ for $\ell = 4$ for the third and fourth eigenvalues with analytical eigenfunctions, $8/3 \approx 2r$ for the fifth eigenvalue with $r = 4/3 - \epsilon$. These are consistent with the theoretical results. For the 3rd and 4th eigenvalues, when $h = 1/4$, the computed are already very accurate close to the exact solutions where the relative errors are

Table 3 Finite element eigenvalues in L-shaped domain using P_4 element

ω^2	$1/h$	ω_h^2	$|\omega^2 - \omega_h^2|/|\omega^2|$	Conv. rate
1.47562182408	2	1.41810442831540	3.8978E−02	–
	4	1.45245960371947	1.5697E−02	1.31
	8	1.46637545161301	6.2661E−03	1.32
	16	1.47194398287482	2.4924E−03	1.33
	32	1.47416097311807	9.8999E−04	1.33
	64	1.47504188486512	3.9301E−04	1.33
3.53403136678	2	3.53334748574238	1.9351E−04	–
	4	3.53392368368660	3.0470E−05	2.67
	8	3.53401440740811	4.7989E−06	2.67
	16	3.53402869626236	7.5566E−07	2.67
	32	3.53403094673058	1.1886E−07	2.67
	64	3.53403130045075	1.8769E−08	2.66
9.86960440109	2	9.86960641952633	2.0451E−07	–
	4	9.86960440936467	8.3840E−10	7.93
	8	9.86960440112122	3.1634E−12	8.05
	16	9.86960440062396	4.7220E−11	-3.90
	32	9.86960440065085	4.4495E−11	0.09
	64	9.86960440093985	1.5213E−11	1.55
9.86960440109	2	9.86960642425001	2.0499E−07	–
	4	9.86960440936846	8.3878E−10	7.93
	8	9.86960440112782	3.8320E−12	7.77
	16	9.86960440743662	6.4305E−10	-7.39
	32	9.86960440660897	5.5919E−10	0.20
	64	9.86960440100940	8.1663E−12	6.10
11.38947939790	2	11.38848458010000	8.7345E−05	–
	4	11.38932332537450	1.3703E−05	2.67
	8	11.38945498348860	2.1436E−06	2.68
	16	11.38947556276420	3.3673E−07	2.67
	32	11.38947879428710	5.2997E−08	2.67
	64	11.38947930283520	8.3467E−09	2.67

of magnitude about 1.0E−10 so that the computed convergence rates for successive meshes after $h = 1/4$ do not truly reflect the predicted.

The following Table 4 shows the relationship among the degrees of freedom(dof) of $\mathbf{U}_h^{(2)}(\mathscr{T}_{h/2})$, $W_h^{(1)}(\mathscr{T}_{h/2})$ and $Q_h^{(0)}(\mathscr{T}_h)$ for P_2 element on the barycentric refinements in L-shaped domain. This table confirms the conclusion given in Remark 5.11.

Below, we present a three-dimensional(3D) numerical test for the Maxwell eigenproblem in the thick L-shaped domain $\Omega = ((-1, 1)^2/([0, 1] \times [-1, 0])) \times (0, 1) \subset \mathbb{R}^3$. The first nine eigenvalues are available as follows (https://perso.univ-rennes1.fr/monique.dauge/benchmax.html):

Table 4 Number of discrete zero eigenvalues and dofs of $\mathbf{U}_h^{(2)}(\mathcal{T}_{h/2})$, $W_h^{(1)}(\mathcal{T}_{h/2})$ and $Q_h^{(0)}(\mathcal{T}_h)$

$1/h$	1	2	4	8	16	32
dof of $\mathbf{U}_h^{(2)}(\mathcal{T}_{h/2})$	68	284	1148	4604	18428	73724
dof of $W_h^{(1)}(\mathcal{T}_{h/2})$	53	215	863	3455	13823	55295
dof of $Q_h^{(0)}(\mathcal{T}_h)$	6	24	96	384	1536	6144
Number of zero eigenvalues	9	45	189	765	3069	12285

$$9.63972384472,$$
$$11.34522622520,$$
$$13.40363576790,$$
$$15.19725192650,$$
$$19.50932824580,$$
$$19.73920880220,$$
$$19.73920880220,$$
$$19.73920880220,$$
$$21.25908379900.$$

No regularity results are available for the corresponding eigenfunctions. The meshes of tetrahedra are barycentric refinements. We employ the mixed $(P_4)^3 - P_0$ element in the following way. The mesh $\mathcal{T}_h$ is composed of tetrahedra, while the sub-mesh $\mathcal{T}_{h/2}$ is obtained from $\mathcal{T}_h$ by connecting the barycentre of each tetrahedron $T \in \mathcal{T}_h$ to its four vertices. On each $K \subset T \in \mathcal{T}_h$ and $K \in \mathcal{T}_{h/2}$, we use the $(P_4(K))^3$ element for the electric field, while on each $T \in \mathcal{T}_h$, we use the constant element $P_0(T)$ for the multiplier. Such $(P_4)^3 - \mathcal{P}_0$ element is the lowest-order element on the barycentric refinements that may be possibly theoretically proven to be spectral-correct and spurious-free. The three-dimensional computation for the Maxwell eigenproblem is rather time-consuming, because there are a huge number of degrees of freedom for the high order $(P_4)^3$ element on the barycentric refinements. Thus, we only present the numerical results for a few mesh sizes: $h = 1, 1/2, 1/4$. Here $h = 1$ means that we partition the sub-domain $(0, 1)^3$ in the following way: it is first partitioned into two prisms, and each prism is partitioned into three tetrahedra, and then each tetrahedron is barycentric-refinement-based partitioned into four sub-tetrahedra, i.e., there are six tetrahedra in the mesh $\mathcal{T}_h$ in the cube $(0, 1)^3$, and there are twenty-four sub-tetrahedra in the sub-mesh $\mathcal{T}_{h/2}$ in the cube $(0, 1)^3$; other sub-domains(cubes) are similarly partitioned. The numerical results are reported in Table 5 using the proposed mixed method in 3D: Find $(\omega_h^2, \mathbf{u}_h \neq \mathbf{0}, p_h) \in \mathbb{R} \times \mathbf{U}_h^{(4)}(\mathcal{T}_{h/2}) \times Q_h^{(0)}(\mathcal{T}_h)$ such that

$$\begin{cases} (\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + (\text{div}\,\mathbf{v}_h, p_h) = \omega_h^2(\mathbf{u}_h, \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{U}_h^{(4)}(\mathcal{T}_{h/2}), \\ (\text{div}\,\mathbf{u}_h, q_h) = 0, & \forall q_h \in Q_h^{(0)}(\mathcal{T}_h), \end{cases}$$

From Table 5, for these few mesh sizes, although the computed eigenvalues have low accuracy, the asymptotic convergence rates appear to be good. We moreover compute the eigenvalues using a stabilized method: Find $(\omega_h^2, \mathbf{u}_h \neq \mathbf{0}, p_h) \in \mathbb{R} \times \mathbf{U}_h^{(4)}(\mathcal{T}_{h/2}) \times Q_h^{(0)}(\mathcal{T}_h)$ such that

$$\begin{cases} (\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + \sum_{T \in \mathcal{T}_h} h_T^2 (\text{div}\,\mathbf{u}_h, \text{div}\,\mathbf{v}_h)_{0,T} + (\text{div}\,\mathbf{v}_h, p_h) = \omega_h^2(\mathbf{u}_h, \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{U}_h^{(4)}(\mathcal{T}_{h/2}), \\ (\text{div}\,\mathbf{u}_h, q_h) = 0, & \forall q_h \in Q_h^{(0)}(\mathcal{T}_h), \end{cases}$$

Table 5 Finite element eigenvalues in 3D thick L-shaped domain using P_4 element

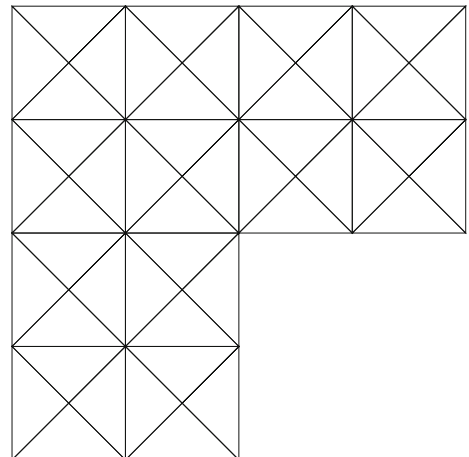| ω^2	$1/h$	ω_h^2	$|\omega^2 - \omega_h^2|/|\omega^2|$	Conv. rate
9.63972384472	1	0.31104858475161	9.6773E−01	–
	2	2.09967443631374	7.8219E−01	0.31
	4	8.34301225053444	1.3452E−01	2.54
11.34522622520	1	0.43360129202145	9.6178E−01	–
	2	2.12232656664357	8.1293E−01	0.24
	4	8.34528478013753	2.6442E−01	1.62
13.40363576790	1	1.02924660320636	9.2321E−01	–
	2	2.66370404225194	8.0127E−01	0.20
	4	9.64664199864069	2.8030E−01	1.52
15.19725192650	1	2.07849852817239	8.6323E−01	–
	2	2.70578431001422	8.2196E−01	0.07
	4	10.11413548043320	3.3448E−01	1.30
19.50932824580	1	3.65814616727680	8.1249E−01	–
	2	2.73482863097847	8.5982E−01	− 0.08
	4	10.15650024722180	4.7940E−01	0.84
19.73920880220	1	4.32185174288536	7.8105E−01	–
	2	2.85888938185043	8.5517E−01	− 0.13
	4	10.22237096302830	4.8213E−01	0.83
19.73920880220	1	6.00283896555682	6.9589E−01	–
	2	3.06961232920241	8.4449E−01	− 0.28
	4	10.25836450276590	4.8031E−01	0.81
19.73920880220	1	7.74444918373253	6.0766E−01	–
	2	3.07494493221733	8.4422E−01	− 0.47
	4	10.38898991430580	4.7369E−01	0.83
21.25908379900	1	9.26449977505675	5.6421E−01	–
	2	3.24421410173137	8.4740E−01	− 0.59
	4	10.66407096028730	4.9838E−01	0.77

Interesting, with the same few mesh sizes, the computed eigenvalues have much better accuracy, as is seen from Table 6. The analysis and theory for these 3D computations are not yet known.

We additionally consider the criss-cross mesh of the L-shaped domain(see Fig. 2). This mesh is generated as follows: firstly generating a macro square mesh and then dividing each macro square into four triangles by connecting the two diagonal lines. Following the same idea for the mixed method in the barycentric refinement mesh, the electric field is approximated by the P_2 element in the criss-cross mesh while the multiplier is approximated by the P_0 element in the macro square, see Fig. 3. The numerical results, which are reported in Table 7, show that the eigenvalues are correctly approximated and the convergence rates are quite like those for the P_2 element in the barycentric refinement mesh (cf. Table 1). On the one hand, the *Fraeijs de Veubeke-Sander quadrilateral* C^1 element in the criss-cross refinement in [17](page 286) should play an important role in the convergence, since the gradient of this C^1 element is a subspace of the P_2 element(C^0 element) in the criss-cross mesh; but, on the

Table 6 Finite element eigenvalues in 3D thick L-shaped domain using P_4 element and stabilization

ω^2	$1/h$	ω_h^2	$|\omega^2 - \omega_h^2|/|\omega^2|$	Conv. rate
9.63972384472	1	9.69340657194835	5.5689E−03	–
	2	9.66123370875044	2.2314E−03	1.32
	4	9.64788754440316	8.4688E−04	1.40
11.34522622520	1	13.71186339921360	2.0860E−01	–
	2	13.52266775131840	1.9193E−01	0.12
	4	13.43752354621980	1.8442E−01	0.06
13.40363576790	1	14.61079867196800	9.0062E−02	–
	2	14.08967398979470	5.1183E−02	0.82
	4	13.50116243049290	7.2761E−03	2.81
15.19725192650	1	15.23031858529790	2.1758E−03	–
	2	15.19797576739960	4.7630E−05	5.51
	4	15.19737284519040	7.9566E−06	2.58
19.50932824580	1	19.78434970959670	1.4097E−02	–
	2	19.74004299584520	1.1826E−02	0.25
	4	19.73921250368070	1.1783E−02	0.01
19.73920880220	1	19.78463506833080	2.3013E−03	–
	2	19.74007009636080	4.3634E−05	5.72
	4	19.73921257603510	1.9118E−07	7.83
19.73920880220	1	19.78484230182280	2.3118E−03	–
	2	19.74007040555270	4.3649E−05	5.73
	4	19.73921258427230	1.9160E−07	7.83
19.73920880220	1	21.12981193732730	7.0449E−02	–
	2	20.78838826867910	5.3152E−02	0.41
	4	20.55576662476000	4.1367E−02	0.36
21.25908379900	1	21.55879382984100	1.4098E−02	–
	2	21.31180126498760	2.4798E−03	2.51
	4	21.27385804331540	6.9496E−04	1.84

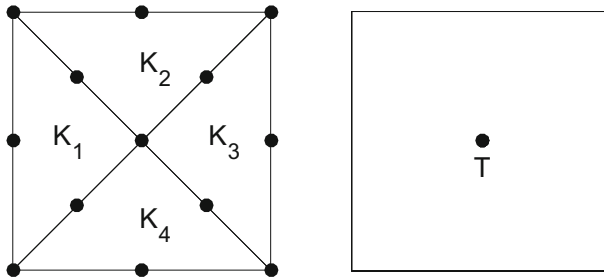**Fig. 2** The criss-cross mesh of $h = 1/2$ for L-shaped domain

Fig. 3 Left: P_2 element in criss-cross mesh for $\mathbf{u}_h$. Right: P_0 element for p_h

Table 7 Finite element eigenvalues in L-shaped domain using P_2 element in criss-cross mesh

ω^2	$1/h$	ω_h^2	$|\omega^2 - \omega_h^2|/|\omega^2|$	Conv. rate
1.47562182408	2	1.30682686532695	1.1439E−01	–
	4	1.40270226938387	4.9416E−02	1.21
	8	1.44573555936858	2.0253E−02	1.29
	16	1.46360943008524	8.1406E−03	1.31
	32	1.47083052368678	3.2470E−03	1.33
	64	1.47371657111986	1.2912E−03	1.33
3.53403136678	2	3.52218810422874	3.3512E−03	–
	4	3.53207798654169	5.5273E−04	2.60
	8	3.53371979642336	8.8163E−05	2.65
	16	3.53398206047050	1.3952E−05	2.66
	32	3.53402358624609	2.2016E−06	2.66
	64	3.53403014044250	3.4701E−07	2.67
9.86960440109	2	9.88488611352695	1.5484E−03	–
	4	9.87062368484707	1.0328E−04	3.91
	8	9.86966921748706	6.5673E−06	3.98
	16	9.86960846998371	4.1227E−07	3.99
	32	9.86960465567572	2.5795E−08	4.00
	64	9.86960441700123	1.6121E−09	4.00
9.86960440109	2	9.88488611352698	1.5484E−03	–
	4	9.87062368484719	1.0328E−04	3.91
	8	9.86966921748786	6.5673E−06	3.98
	16	9.86960846998471	4.1227E−07	3.99
	32	9.86960465568457	2.5796E−08	4.00
	64	9.86960441701599	1.6136E−09	4.00
11.38947939790	2	11.38577287200070	3.2543E−04	–
	4	11.38782849132890	1.4495E−04	1.17
	8	11.38911099034530	3.2346E−05	2.16
	16	11.38941373516850	5.7652E−06	2.49
	32	11.38946856106180	9.5148E−07	2.60
	64	11.3894776598212	1.5260E−07	2.64

other hand, it is not clear whether the present analysis could be applied or not, while these C^0 and C^1 elements and these meshes are beyond the scope of this paper.

7 Concluding Remarks

We have studied a new mixed finite element method for solving the Maxwell eigenproblem in two-dimensions, with the Lagrange elements in barycentric refinements of any order $\ell \geq 2$. Optimum in the error bounds is obtained. Spectral-correct and spurious-free approximations are theoretically and numerically proven. The method can accommodate singular solutions as well as smooth solutions. The extension to the three-dimensional problem can be done straightforwardly, if Lemma 4.2 holds. So far, however, the proof of Lemma 4.2 in three-dimensions remains open.

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