



A Decoupled, Linear and Unconditionally Energy Stable Scheme with Finite Element Discretizations for Magneto-Hydrodynamic Equations

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Abstract

In this paper, we consider numerical approximations for solving the nonlinear magnetohydrodynamical system, that couples the Navier–Stokes equations and Maxwell equations together. By combining the projection method and some subtle implicit–explicit treatments for nonlinear coupling terms, we develop a fully decoupled, linear and unconditionally energy stable scheme for solving this system, where a new auxiliary velocity field is specifically introduced in order to decouple the computations of the magnetic field from the velocity field. We further prove that the fully discrete scheme with finite element approximations is unconditional energy stable. By deriving the L^∞ bound of the numerical solution and the relation between the new auxiliary velocity field and the velocity field, and using negative norm technique, we obtain the optimal error estimates rigorously. Various numerical experiments are implemented to demonstrate the stability and the accuracy in simulating some benchmark problems, including the Kelvin–Helmholtz shear instability and the magnetic-frozen phenomenon in the lid-driven cavity.

Keywords Magneto-hydrodynamics · Linear · Decoupled · Unconditional energy stability · First order · Error estimates

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1 Introduction

The hydrodynamical behaviors of conducting fluids (plasmas, liquid metals, and electrolytes, etc.) under external electromagnetic field are usually modeled by the so-called magnetohydrodynamical (MHD) system, see [12,23,25]. The fundamental image behind the MHD system is that magnetic fields can induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. Thus the governing equations, which describe the MHD system, couple the Navier–Stokes equations for hydrodynamics and Maxwell’s equations for electromagnetism. Concerning the corresponding extensive theoretical modeling/numerical analysis of the MHD system, we refer to [2–4,10,15,16,18–22,24,28,29] and the references therein.

For solving the nonlinear coupled MHD system numerically, one challenging issue is to design linear and decoupled schemes while preserving the energy stability, i.e., the energy dissipation laws hold in the discrete level. The main associated difficulty is to decouple nonlinear couplings between the magnetic field and the velocity field appearing in the convection and Lorentz forces. Simple discretizations, like fully explicit or implicit type schemes to handle these terms can lead to considerable instabilities or suffer from costly time expense. We recall there are many attempts that have been made in this direction recently. For examples, in [18], the authors developed two implicit–explicit type methods where the first order method is shown to be unconditionally stable and the second order method is shown to be conditionally stable. However, the model considered in [18] is the reduced version, namely, the magnetic field is assumed to be a fixed function. In [32,33], the authors developed a decoupled type scheme for the full MHD system, but it is conditionally energy stable with a time step constraint. In [7], the authors developed some unconditionally energy stable schemes based on the projection type methods for the Navier–Stokes equations. However, the velocity field and the magnetic field are coupled together. In [31], the authors presented optimal error estimates for the first order energy stable projection scheme where the velocity field and the magnetic field are still coupled together (partially decoupled scheme). In [21], the authors developed a totally decoupled scheme where the computations of Navier–Stokes equations are based on the commutator of Laplacian and Leray projection, and all nonlinear and coupling terms are treated explicitly. However, the scheme is still conditionally stable. In [26], the author had developed a fully decoupled scheme by treating the coupled terms explicitly, however the scheme does not allow the discrete energy stability. Furthermore, the error estimate for pressure is suboptimal.

In this paper, we develop a fully decoupled, linear, and unconditionally energy stable scheme for solving the MHD system, and carry out the corresponding rigorous stability and error analysis for the fully discrete scheme in the context of finite element approximations. We adopt the first order projection method to solve the Navier–Stokes equation and introduce an auxiliary intermediate velocity variable to decouple the computation of the magnetic field from velocity. Meanwhile, we make some subtle implicit–explicit treatments for nonlinear coupling terms to linearize the system and ensure the energy stability, that is proved rigorously. Our decoupled idea is somewhat similar to matrix splitting algorithms in [4]. The key ingredient are all the introduction of an auxiliary velocity variable to decouple the computations of velocity from the magnetic field. Furthermore, the error estimates and numerical studies of unconditional stabilities are not presented in [4].

It is remarkable that the error analysis for the fully discrete finite element scheme we propose is rather challenging while its energy stability is quite straightforward once the scheme is appropriately designed. In [31], the error bound is derived directly by using the

standard energy approach since the scheme is skew-symmetric hence many challenging terms are cancelled. However, there are some essential difficulties to derive the error bound in the developed scheme in this paper. More precisely, the introduction of the auxiliary velocity variable actually leads to some essential hurdles in establishing the L^∞ bound of the numerical solutions that is the key ingredient to derive error estimates, and obtaining optimal error estimates for the pressure. To this end, the quantitative relations between the auxiliary velocity and velocity field is needed to be verified. By using the mathematical induction method combined with the convergence analysis, we further prove the L^∞ stabilities of the numerical solution, and obtain the optimal error estimate for the pressure by means of the negative norm estimate method. It is shown that the velocity and magnetic field are of optimal error orders when $\delta t |\log h| \leq 1$, and the pressure can attain the optimal convergence order when $\delta t |\log h|^{\frac{1}{2}} \lesssim h$. Finally, several benchmark numerical experiments, including the Kelvin–Helmholtz shear instability and the magnetic-frozen phenomenon in the lid-driven cavity, are implemented to confirm the stability and the accuracy of the scheme.

The rest of paper is organized as follows. In Sect. 2, we present the model and derive the associated energy dissipation law. In Sect. 3, we develop the fully discrete finite element scheme. We prove the associated energy dissipation law in the fully discrete level. In Sect. 4, we derive the error estimates of the fully discrete scheme. In Sect. 5, various numerical experiments are presented to demonstrate the stability and effectiveness of the scheme. Finally, some concluding remarks are given in Sect. 6.

2 The MHD Model and Energy Law

The incompressible MHD equations reads as follows,

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p + s \mathbf{B} \times \nabla \times \mathbf{B} = 0, \quad (2.1)$$

$$\mathbf{B}_t + \eta \nabla \times \nabla \times \mathbf{B} - \nabla \times (\mathbf{u} \times \mathbf{B}) = 0, \quad (2.2)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.4)$$

for $(\mathbf{x}, t) \in \Omega \times [0, T)$ with $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, where \mathbf{u} denotes the velocity field, p is the pressure, and \mathbf{B} is the magnetic field. For the physical parameters, $\nu^{-1} = R_e$ (fluid Reynolds number), $\eta^{-1} = R_m$ (magnetic Reynolds number), and s is the coupling coefficient, which are given by

$$R_e = \frac{UL}{\mu_f}, \quad R_m = \mu_m \sigma UL, \quad s = \frac{B^2}{\rho \mu_m U^2},$$

where U is the characteristic velocity, L is the characteristic length, μ_f is the kinematic viscosity, μ_m is the magnetic permeability, σ is the electric conductivity, B is the characteristic magnetic field, and ρ is the fluid density. The system is equipped with the following boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{B} \times \mathbf{n}|_{\partial\Omega} = 0, \quad (2.5)$$

and initial conditions

$$\mathbf{u}|_{(t=0)} = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{B}|_{(t=0)} = \mathbf{B}_0(\mathbf{x}), \quad (2.6)$$

with $\nabla \cdot \mathbf{u}_0 = 0$, $\nabla \cdot \mathbf{B}_0 = 0$, where \mathbf{n} denotes the outward unit normal of $\partial\Omega$.

We fix some notations here. In this paper, we focus on the numerical analysis of the MHD system in two dimensional case ($d = 2$). The corresponding results in three dimensional case ($d = 3$) are listed in subsequent remarks. Here and after, for two vector functions \mathbf{x} , \mathbf{y} , we denote the L^2 inner product as $(\mathbf{x}, \mathbf{y}) = \int_{\Omega} \mathbf{x} \cdot \mathbf{y} dx$ and L^2 norm $\|\mathbf{x}\|^2 = (\mathbf{x}, \mathbf{x})$. Let $W^{k,r}(\Omega)$ stands for the standard Sobolev spaces equipped with the standard Sobolev norms $\|\cdot\|_{k,r}$. For $r = 2$, we write $H^k(\Omega)$ for $W^{k,2}(\Omega)$ and its corresponding norm is $\|\cdot\|_k$. We let $L^r(\Omega)$ denote the usual Lebesgue space on Ω with the norm $\|\cdot\|_{L^r}$.

For appropriate function setting of the MHD equations, we define

$$H_0^1(\Omega) = \{\phi \in H^1(\Omega) : \phi|_{\partial\Omega} = 0\}, \quad V = \{\mathbf{u} \in H_0^1(\Omega)^2 : \nabla \cdot \mathbf{u} = 0\}$$

$$L_0^2(\Omega) = \left\{ \phi \in L^2(\Omega) : \int_{\Omega} \phi dx = 0 \right\}, \quad H_{\tau}^1(\Omega) = \{\mathbf{w} \in H^1(\Omega)^2 : \mathbf{w} \times \mathbf{n}|_{\partial\Omega} = 0\}.$$

The following Poincaré type inequalities are well-known, see [11], which will be frequently used in our proof,

$$\|\mathbf{w}\|_1 \leq C_{\Omega} \|\nabla \mathbf{w}\|, \quad \mathbf{w} \in H_0^1(\Omega)^d, \tag{2.7}$$

$$\|\mathbf{w}\|_1 \leq C_{\Omega} (\|\nabla \times \mathbf{w}\| + \|\nabla \cdot \mathbf{w}\|), \quad \mathbf{w} \in H_{\tau}^1(\Omega), \tag{2.8}$$

$$\|\phi\|_1 \leq C_{\Omega} \|\nabla \phi\|, \quad \phi \in H^1(\Omega) \cap L_0^2(\Omega). \tag{2.9}$$

The model (2.1)–(2.6) follows the energy dissipation law. By taking the L^2 inner product of (2.1) with \mathbf{u} , and of (2.2) with $s\mathbf{B}$, using (2.3)–(2.4) and $((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{u}) = 0$ and integration by parts, we have

$$(\mathbf{u}_t, \mathbf{u}) + \nu \|\nabla \mathbf{u}\|^2 + s(\mathbf{B} \times \nabla \times \mathbf{B}, \mathbf{u}) = 0, \tag{2.10}$$

$$s(\mathbf{B}_t, \mathbf{B}) + s\eta \|\nabla \times \mathbf{B}\|^2 + s\eta \|\nabla \cdot \mathbf{B}\|^2 - s(\mathbf{u} \times \mathbf{B}, \nabla \times \mathbf{B}) = 0. \tag{2.11}$$

By taking the summation of the two equalities, we obtain

$$\frac{d}{dt} E(\mathbf{u}, \mathbf{B}) = -\nu \|\nabla \mathbf{u}\|^2 - s\eta \|\nabla \times \mathbf{B}\|^2 - s\eta \|\nabla \cdot \mathbf{B}\|^2, \tag{2.12}$$

where $E(\mathbf{u}, \mathbf{B}) = \frac{1}{2} \|\mathbf{u}\|^2 + \frac{s}{2} \|\mathbf{B}\|^2$ is the total energy of the MHD system (2.1)–(2.6).

3 Numerical Scheme and Energy Stability

In this section, we develop a fully decoupled, linearized, and unconditionally energy stable scheme in the fully discrete forms for solving the system (2.1)–(2.6), and show the unconditional energy stability in the fully discrete form with finite element discretizations for the spatial direction.

For $l \geq 1, r \geq 1$, we define the conforming finite element spaces $\mathbf{V}_h \subset H_0^1(\Omega)^2, \mathbf{U}_h \subset H^1(\Omega)^2$ with $\mathbf{V}_h \subset \mathbf{U}_h$, which consist of continuous piecewise polynomials of degree l , and $\mathbf{C}_h \subset H_{\tau}^1(\Omega), M_h \subset H^1(\Omega) \cap L_0^2(\Omega)$, which consist of continuous piecewise polynomials of degree r , and $l - 1$, respectively. Moreover, the \mathbf{V}_h and M_h satisfy the *inf-sup* condition:

$$\beta \|q_h\| \leq \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\|}, \quad \forall q_h \in M_h. \tag{3.1}$$

A well known mixed finite element pair for V_h and M_h satisfying (3.1) is the Taylor-Hood element [11]. For $w_h \in U_h$ or C_h , the following inverse inequalities hold

$$\begin{aligned} \|w_h\|_{L^q} &\leq ch^{(\frac{1}{q}-\frac{1}{p})^2} \|w_h\|_{L^p}, \quad p \leq q, \\ \|w_h\|_{L^\infty} &\leq c|\log h|^{\frac{1}{2}} \|w_h\|_1, \\ \|w_h\|_1 &\leq ch^{-1} \|w_h\|. \end{aligned} \tag{3.2}$$

In order to derive the error estimates in spatial discretization, we recall the well-known Stokes projection and Maxwell projection, see [11,32], i.e., given $u \in V$, $p \in L^2_0(\Omega)$, find $R(u,p) \in V_h$ and $Q(u,p) \in M_h$, such that

$$\begin{cases} \nu(\nabla R(u,p), \nabla v_h) - (Q(u,p), \nabla \cdot v_h) = \nu(\nabla u, \nabla v_h) - (p, \nabla \cdot v_h), \quad \forall v_h \in V_h, \\ (\nabla \cdot R(u,p), q_h) = (\nabla \cdot u, q_h), \quad \forall q_h \in M_h; \end{cases}$$

and given $B \in H^1_\tau(\Omega)$, find $J_B \in C_h$, such that

$$(\nabla \times J_B, \nabla \times C_h) + (\nabla \cdot J_B, \nabla \cdot C_h) = (\nabla \times B, \nabla \times C_h) + (\nabla \cdot B, \nabla \cdot C_h).$$

It is obvious that $R(u,p)$, $Q(u,p)$, J_B are stable in the sense that

$$\|\nabla R(u,p)\| + \|Q(u,p)\| \leq c(\|\nabla u\| + \|p\|), \quad \|J_B\|_1 \leq c\|B\|_1. \tag{3.3}$$

Besides, if assuming $(u, p, B) \in H^{l+1}(\Omega)^2 \times H^l(\Omega) \times H^{r+1}(\Omega)^2$, then the following approximations hold

$$\begin{aligned} \|u - R(u,p)\| + h(\|\nabla(u - R(u,p))\| + \|p - Q(u,p)\|) &\leq C_\Omega h^{l+1}(\|u\|_{l+1} + \|p\|_l), \\ \|B - J_B\| + h\|B - J_B\|_1 &\leq C_\Omega h^{r+1} \|B\|_{r+1}. \end{aligned} \tag{3.4}$$

Since $Q(u,p)$ is in $H^1(\Omega)$, we can derive its H^1 stability as follows, which will be used in our analysis.

Theorem 3.1 *The Stokes projection $Q(u,p)$ is H^1 stable in the sense that*

$$\|Q(u,p)\|_1 \leq c(\|u\|_2 + \|p\|_1).$$

Proof We define H^1 projection $P_1 : H^1(\Omega) \cap L^2_0(\Omega) \rightarrow M_h$ by

$$(\nabla p - \nabla P_1 p, \nabla q_h) = 0, \quad \forall q_h \in M_h,$$

and it is easy to find

$$\|p - P_1 p\| + h\|p - P_1 p\|_1 \leq ch\|p\|_1.$$

Using the above property, (3.4) and inverse inequality $\|q_h\|_1 \leq ch^{-1}\|q_h\|$, we derive

$$\begin{aligned} \|Q(u,p)\|_1 &\leq \|p - Q(u,p)\|_1 + \|p\|_1 \leq \|p - P_1 p\|_1 + \|P_1 p - Q(u,p)\|_1 + \|p\|_1 \\ &\leq c\|p\|_1 + ch^{-1}\|P_1 p - Q(u,p)\| \\ &\leq c\|p\|_1 + ch^{-1}\|P_1 p - p\| + ch^{-1}\|p - Q(u,p)\| \\ &\leq c\|p\|_1 + ch^{-1}h(\|u\|_2 + \|p\|_1) \leq c(\|u\|_2 + \|p\|_1). \end{aligned}$$

The proof is finished. □

We assume the solution to (2.1)–(2.6) has the following regularity conditions

$$(A0) : \mathbf{u} \in L^\infty(0, T; H^{l+1}(\Omega)^2), \quad p \in L^\infty(0, T; H^l(\Omega)), \quad \mathbf{B} \in L^\infty(0, T; H^{r+1}(\Omega)^2), \tag{3.5}$$

such that the following approximate properties hold

$$\begin{aligned} & \| \mathbf{u}(t_n) - R(\mathbf{u}(t_n), p(t_n)) \| + h(\| \nabla(\mathbf{u}(t_n) - R(\mathbf{u}(t_n), p(t_n))) \| \\ & + \| p(t_n) - Q(\mathbf{u}(t_n), p(t_n)) \|) \leq ch^{l+1}, \end{aligned} \tag{3.6}$$

$$\| \mathbf{B}(t_n) - J_{\mathbf{B}(t_n)} \| + h\| \mathbf{B}(t_n) - J_{\mathbf{B}(t_n)} \|_1 \leq ch^{r+1}. \tag{3.7}$$

Now, we develop the fully discrete scheme based on the finite element discretization. It reads as follows.

Given the initial data $\mathbf{B}_h^0 = J_{\mathbf{B}0}, (\mathbf{u}_h^0, p_h^0) = (R(\mathbf{u}_0, p_0), Q(\mathbf{u}_0, p_0))$, where p_0 is obtained by $\mathbf{u}_0, \mathbf{B}_0$ and (2.1) following the method in [14], we compute $\mathbf{B}_h^{n+1}, \tilde{\mathbf{u}}_h^{n+1}, p_h^{n+1}, \mathbf{u}_h^{n+1}$ by the following steps.

Step 1 Find $\mathbf{B}_h^{n+1} \in \mathbf{C}_h, \mathbf{u}_{h*}^n \in \mathbf{U}_h$ such that

$$\begin{aligned} & \frac{1}{\delta t}(\mathbf{B}_h^{n+1} - \mathbf{B}_h^n, \mathbf{c}_h) + \eta(\nabla \times \mathbf{B}_h^{n+1}, \nabla \times \mathbf{c}_h) + \eta(\nabla \cdot \mathbf{B}_h^{n+1}, \nabla \cdot \mathbf{c}_h) \\ & - (\mathbf{u}_{h*}^n \times \mathbf{B}_h^n, \nabla \times \mathbf{c}_h) = 0, \quad \mathbf{c}_h \in \mathbf{C}_h, \end{aligned} \tag{3.8}$$

$$\frac{1}{\delta t}(\mathbf{u}_{h*}^n - \mathbf{u}_h^n, \mathbf{w}_h) + s(\mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^{n+1}, \mathbf{w}_h) = 0, \quad \mathbf{w}_h \in \mathbf{U}_h. \tag{3.9}$$

Step 2 Find $\tilde{\mathbf{u}}_h^{n+1} \in \mathbf{V}_h$ such that

$$\begin{aligned} & \frac{1}{\delta t}(\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_{h*}^n, \mathbf{v}_h) + \nu(\nabla \tilde{\mathbf{u}}_h^{n+1}, \nabla \mathbf{v}_h) + b(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}, \mathbf{v}_h) \\ & - (p_h^n, \nabla \cdot \mathbf{v}_h) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h. \end{aligned} \tag{3.10}$$

Step 3 Find $p_h^{n+1} \in M_h$ such that

$$(\nabla p_h^{n+1}, \nabla q_h) = -\frac{1}{\delta t}(\nabla \cdot \tilde{\mathbf{u}}_h^{n+1}, q_h) + (\nabla p_h^n, \nabla q_h), \quad q_h \in M_h. \tag{3.11}$$

Step 4 Update $\mathbf{u}_h^{n+1} \in \mathbf{V}_h + \nabla M_h$ from

$$\mathbf{u}_h^{n+1} = \tilde{\mathbf{u}}_h^{n+1} - \delta t \nabla(p_h^{n+1} - p_h^n). \tag{3.12}$$

Remark 3.2 To avoid some technical difficulties in the error estimates at the initial time, here we set the initial data \mathbf{u}_h^0, p_h^0 by the Stokes projection of $\mathbf{u}(0)$ and $p(0)$. The difficulties at the initial time can be also overcome by making some approximation assumptions as (5.18) in [13]. In the numerical implementations, we can obtain the initial data through solving two equations at $t = 0$ like (3.7) and (3.8) in [1].

Remark 3.3 For the convective term in (3.10), we use the trilinear form as follows (cf. [30]),

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}) - \frac{1}{2}((\mathbf{u} \cdot \nabla)\mathbf{w}, \mathbf{v}). \tag{3.13}$$

Note when $\mathbf{u} \in H^1(\Omega)^2, \mathbf{v}, \mathbf{w} \in H_0^1(\Omega)^2$, by using the integration by parts, we derive

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = ((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}) + \frac{1}{2}((\nabla \cdot \mathbf{u})\mathbf{v}, \mathbf{w}). \tag{3.14}$$

Therefore, when $\nabla \cdot \mathbf{u} = 0$, we have $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = ((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w})$.

Remark 3.4 For the fully discrete scheme (3.8), the exact divergence free for the magnetic field \mathbf{B}_h^{n+1} is not valid any more. Therefore a common practice is to add a penalty term $(\nabla \cdot \mathbf{B}_h^{n+1}, \nabla \cdot \mathbf{c}_h)$ in the momentum equation in order to verify the coercivity of the magnetic equation, and approximate the divergence free condition of the magnetic field for the spatial discrete case, cf. [15].

Remark 3.5 The final velocity field \mathbf{u}_h^{n+1} satisfies the discrete divergence free condition. This can be deduced by taking the L^2 inner product of (3.12) with ∇q_h , we obtain

$$(\mathbf{u}_h^{n+1}, \nabla q_h) = -(\nabla \cdot \tilde{\mathbf{u}}_h^{n+1}, q_h) - \delta t (\nabla (p_h^{n+1} - p_h^n), \nabla q_h).$$

After combining with (3.11), we arrive at

$$(\mathbf{u}_h^{n+1}, \nabla q_h) = 0, \quad \forall q_h \in M_h. \tag{3.15}$$

There are two advantages in the scheme (3.8)–(3.12): *linear* and *decoupled*, where the magnetic field \mathbf{B} , pressure p and velocity \mathbf{u} can be solved linearly and independently at each time step. In addition, the unconditional energy stability holds, which is shown as follows.

Theorem 3.6 *The scheme (3.8)–(3.12) is unconditionally energy stable in the sense that*

$$\begin{aligned} & s \|\mathbf{B}_h^{n+1}\|^2 + \|\mathbf{u}_h^{n+1}\|^2 + \delta t^2 \|\nabla p_h^{n+1}\|^2 \\ & + 2\delta t \left(s\eta \|\nabla \times \mathbf{B}_h^{n+1}\|^2 + s\eta \|\nabla \cdot \mathbf{B}_h^{n+1}\|^2 + \nu \|\nabla \tilde{\mathbf{u}}_h^{n+1}\|^2 \right) \\ & \leq s \|\mathbf{B}_h^n\|^2 + \|\mathbf{u}_h^n\|^2 + \delta t^2 \|\nabla p_h^n\|^2. \end{aligned} \tag{3.16}$$

Proof By taking $\mathbf{c}_h = s \mathbf{B}_h^{n+1}$ in (3.8) and $\mathbf{w}_h = \mathbf{u}_{h*}^n$ in (3.9), and using the following identity

$$2a(a - b) = a^2 - b^2 + (a - b)^2, \tag{3.17}$$

we obtain

$$\begin{aligned} & \frac{s}{2\delta t} \left(\|\mathbf{B}_h^{n+1} - \mathbf{B}_h^n\|^2 + \|\mathbf{B}_h^{n+1}\|^2 - \|\mathbf{B}_h^n\|^2 \right) + s\eta \|\nabla \times \mathbf{B}_h^{n+1}\|^2 + s\eta \|\nabla \cdot \mathbf{B}_h^{n+1}\|^2 \\ & + s(\mathbf{B}_h^n \times \mathbf{u}_{h*}^n, \nabla \times \mathbf{B}_h^{n+1}) = 0, \end{aligned} \tag{3.18}$$

and

$$\frac{1}{2\delta t} \left(\|\mathbf{u}_{h*}^n - \mathbf{u}_h^n\|^2 + \|\mathbf{u}_{h*}^n\|^2 - \|\mathbf{u}_h^n\|^2 \right) - s(\nabla \times \mathbf{B}_h^{n+1} \times \mathbf{B}_h^n, \mathbf{u}_{h*}^n) = 0. \tag{3.19}$$

By taking $\mathbf{v}_h = \tilde{\mathbf{u}}_h^{n+1}$ in (3.10) and using the skew-symmetric property:

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad \mathbf{u} \in L^2(\Omega)^2, \quad \mathbf{v} \in H^1(\Omega)^2, \tag{3.20}$$

we derive

$$\frac{1}{2\delta t} \left(\|\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_{h*}^n\|^2 + \|\tilde{\mathbf{u}}_h^{n+1}\|^2 - \|\mathbf{u}_{h*}^n\|^2 \right) + \nu \|\nabla \tilde{\mathbf{u}}_h^{n+1}\|^2 + (\nabla p_h^n, \tilde{\mathbf{u}}_h^{n+1}) = 0. \tag{3.21}$$

We rewrite (3.12) as

$$\frac{1}{\delta t} \mathbf{u}_h^{n+1} + \nabla p_h^{n+1} = \frac{1}{\delta t} \tilde{\mathbf{u}}_h^{n+1} + \nabla p_h^n.$$

By taking the L^2 inner product of the above with itself on both sides, and using (3.15), we obtain

$$\frac{1}{2\delta t} \|\mathbf{u}_h^{n+1}\|^2 - \frac{1}{2\delta t} \|\tilde{\mathbf{u}}_h^{n+1}\|^2 + \frac{\delta t}{2} \|\nabla p_h^{n+1}\|^2 - \frac{\delta t}{2} \|\nabla p_h^n\|^2 = (\tilde{\mathbf{u}}_h^{n+1}, \nabla p_h^n). \tag{3.22}$$

Finally, by taking the summations (3.18), (3.19), (3.21) and (3.22), we obtain

$$\begin{aligned} & \frac{s}{2\delta t} (\|\mathbf{B}_h^{n+1}\|^2 - \|\mathbf{B}_h^n\|^2 + \|\mathbf{B}_h^{n+1} - \mathbf{B}_h^n\|^2) + s\eta \|\nabla \times \mathbf{B}_h^{n+1}\|^2 + s\eta \|\nabla \cdot \mathbf{B}_h^{n+1}\|^2 \\ & + \frac{1}{2\delta t} (\|\mathbf{u}_h^{n+1}\|^2 - \|\mathbf{u}_h^n\|^2 + \|\mathbf{u}_h^n - \mathbf{u}_h^{n*}\|^2) + \frac{1}{2\delta t} \|\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^{n*}\|^2 \\ & + \nu \|\nabla \tilde{\mathbf{u}}_h^{n+1}\|^2 + \frac{\delta t}{2} \|\nabla p_h^{n+1}\|^2 - \frac{\delta t}{2} \|\nabla p_h^n\|^2 = 0. \end{aligned} \tag{3.23}$$

After multiplying with $2\delta t$ and dropping some positive terms, we obtain (3.16). □

4 Error Estimates

In this section, we provide the convergence analysis for the fully discrete scheme (3.8)–(3.12). We first develop the error estimates for velocity and magnetic field in Sect. 4.1, then improve the error estimate for pressure in Sect. 4.2. Without loss of generality, we denote by C a generic constant that is independent of δt and h but possibly depends on the data and the solution, and use $f \lesssim g$ to denote that there is a generic constant C such that $f \leq Cg$.

4.1 Error Estimates for the Velocity and Magnetic Field

For convenience, we denote $d_t \phi^n = \frac{\phi^n - \phi^{n-1}}{\delta t}$, $d_t \phi(t_n) = \frac{\phi(t_n) - \phi(t_{n-1})}{\delta t}$ for any variable ϕ , $\phi(t)$. The model system (2.1)–(2.2) at t_{n+1} can be written as the truncation forms:

$$\begin{aligned} & (d_t \mathbf{B}(t_{n+1}), \mathbf{c}) + \eta(\nabla \times \mathbf{B}(t_{n+1}), \nabla \times \mathbf{c}) + \eta(\nabla \cdot \mathbf{B}(t_{n+1}), \nabla \cdot \mathbf{c}) \\ & - (\mathbf{u}(t_n) \times \mathbf{B}(t_n), \nabla \times \mathbf{c}) = (R_b^{n+1}, \mathbf{c}), \quad \forall \mathbf{c} \in \mathbf{H}_t^1(\Omega), \end{aligned} \tag{4.1}$$

$$\begin{aligned} & (d_t \mathbf{u}(t_{n+1}), \mathbf{v}) + \nu(\nabla \mathbf{u}(t_{n+1}), \nabla \mathbf{v}) + b(\mathbf{u}(t_n), \mathbf{u}(t_{n+1}), \mathbf{v}) - (p(t_n), \nabla \cdot \mathbf{v}) \\ & + s(\mathbf{B}(t_n) \times \nabla \times \mathbf{B}(t_{n+1}), \mathbf{v}) = (R_u^{n+1}, \mathbf{v}), \quad \forall \mathbf{v} \in H_0^1(\Omega)^2, \end{aligned} \tag{4.2}$$

$$\frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_{n+1})}{\delta t} + \nabla(p(t_{n+1}) - p(t_n)) = R_p^{n+1}, \tag{4.3}$$

where R_b^{n+1} , R_u^{n+1} , R_p^{n+1} are the truncation terms.

We denote the error functions as

$$\left\{ \begin{aligned} \theta_b^n &= \mathbf{J}_{\mathbf{B}(t_n)} - \mathbf{B}_h^n, \rho_b^n = \mathbf{B}(t_n) - \mathbf{J}_{\mathbf{B}(t_n)}, \\ \tilde{\theta}_u^n &= R(\mathbf{u}(t_n), p(t_n)) - \tilde{\mathbf{u}}_h^n, \rho_u^n = \mathbf{u}(t_n) - R(\mathbf{u}(t_n), p(t_n)), \theta_u^n = R(\mathbf{u}(t_n), p(t_n)) - \mathbf{u}_h^n, \\ \theta_p^n &= Q(\mathbf{u}(t_n), p(t_n)) - p_h^n, \rho_p^n = p(t_n) - Q(\mathbf{u}(t_n), p(t_n)), \\ e_b^n &= \mathbf{B}(t_n) - \mathbf{B}_h^n = \theta_b^n + \rho_b^n, \tilde{e}_u^n = \mathbf{u}(t_n) - \tilde{\mathbf{u}}_h^n = \tilde{\theta}_u^n + \rho_u^n, \\ e_u^n &= \mathbf{u}(t_n) - \mathbf{u}_h^n = \theta_u^n + \rho_u^n, e_p^n = p(t_n) - p_h^n = \theta_p^n + \rho_p^n. \end{aligned} \right. \tag{4.4}$$

By subtracting (3.8) from (4.1), (3.10) from (4.2) and applying (3.9), and (3.12) from (4.3), we obtain the following error equations,

$$(d_t e_b^{n+1}, \mathbf{c}_h) + \eta(\nabla \times e_b^{n+1}, \nabla \times \mathbf{c}_h) + \eta(\nabla \cdot e_b^{n+1}, \nabla \cdot \mathbf{c}_h)$$

$$\begin{aligned}
 & + (\mathbf{B}(t_n) \times \mathbf{u}(t_n), \nabla \times \mathbf{c}_h) - (\mathbf{B}_h^n \times \mathbf{u}_{h*}^n, \nabla \times \mathbf{c}_h) = (R_b^{n+1}, \mathbf{c}_h), \quad \mathbf{c}_h \in \mathbf{C}_h, \quad (4.5) \\
 & \frac{1}{\delta t} (\tilde{e}_u^{n+1} - e_u^n, \mathbf{v}_h) + \nu (\nabla \tilde{e}_u^{n+1}, \nabla \mathbf{v}_h) + b(\mathbf{u}(t_n), \mathbf{u}(t_{n+1}), \mathbf{v}_h) \\
 & - b(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}, \mathbf{v}_h) - (e_p^n, \nabla \cdot \mathbf{v}_h) \\
 & + s(\mathbf{B}(t_n) \times \nabla \times \mathbf{B}(t_{n+1}), \mathbf{v}_h) - s(\mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^{n+1}, \mathbf{v}_h) = (R_u^{n+1}, \mathbf{v}_h), \quad \mathbf{v}_h \in \mathbf{V}_h, \quad (4.6)
 \end{aligned}$$

$$\frac{e_u^{n+1}}{\delta t} + \nabla e_p^{n+1} = \frac{\tilde{e}_u^{n+1}}{\delta t} + \nabla e_p^n + R_p^{n+1}. \quad (4.7)$$

We assume the exact solution $(\mathbf{u}, \mathbf{B}, p)$ to the system (2.1)–(2.6) possesses the following regularity conditions,

$$\begin{aligned}
 \text{(A1):} \quad & \mathbf{u}_t, \mathbf{B}_t \in L^\infty(0, T; H^1(\Omega)^2), \quad p_t \in L^\infty(0, T; H^1(\Omega)), \\
 & \mathbf{u}_{tt}, \mathbf{B}_{tt} \in L^\infty(0, T; L^2(\Omega)^2).
 \end{aligned}$$

One can easily establish the following estimates for the truncation errors, provided that the exact solutions are sufficiently smooth or satisfy the assumptions (A0), (A1).

Lemma 4.1 *Under the assumptions (A0) and (A1), the truncation errors satisfy*

$$\|\mathbf{R}_u^n\| + \|\mathbf{R}_b^n\| + \|\mathbf{R}_p^n\| \lesssim \delta t, \quad 0 \leq n \leq \lceil \frac{T}{\delta t} \rceil.$$

Proof Due to the page limit, we omit the proof since it is rather standard. □

There are two essential challenges in deriving the error analysis for velocity and magnetic field. One is from the auxiliary intermediate function \mathbf{u}_{h*}^n since there is no error equation corresponding to it. The other is how to obtain the L^∞ bound of the numerical solution \mathbf{B}_h^n directly from the Eq. (3.8). To this end, we first build the relation between \mathbf{u}_{h*}^n and \mathbf{u}_h^n in Lemma 4.2. Then, by using the mathematical induction method, we establish the L^∞ bound for \mathbf{B}_h^n in Lemma 4.3. The final error estimates are derived in Theorem 4.4.

Lemma 4.2 *Under the assumption (A0), the following estimate holds*

$$\|\mathbf{u}_{h*}^n - \mathbf{u}_h^n\|^2 \lesssim \|\theta_u^n\|^2 + \delta t^2 \|\mathbf{B}_h^n\|_{L^\infty}^2 + \delta t^2 \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\nabla \times \theta_b^{n+1}\|^2.$$

Proof By taking $\mathbf{w}_h = \mathbf{u}_{h*}^n - R(\mathbf{u}(t_n), \mathbf{p}(t_n))$ in (3.9) and using (3.17), we get

$$\begin{aligned}
 & \frac{1}{2\delta t} (\|\mathbf{u}_{h*}^n - \mathbf{u}_h^n\|^2 + \|\mathbf{u}_{h*}^n - R(\mathbf{u}(t_n), \mathbf{p}(t_n))\|^2 - \|\theta_u^n\|^2) \\
 & = -s \left(\mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^{n+1}, \mathbf{u}_{h*}^n - R(\mathbf{u}(t_n), \mathbf{p}(t_n)) \right) \\
 & \leq s \|\mathbf{B}_h^n\|_{L^\infty} \|\nabla \times \mathbf{B}_h^{n+1}\| \|\mathbf{u}_{h*}^n - R(\mathbf{u}(t_n), \mathbf{p}(t_n))\| \\
 & \lesssim \frac{1}{2\delta t} \|\mathbf{u}_{h*}^n - R(\mathbf{u}(t_n), \mathbf{p}(t_n))\|^2 + \delta t \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\nabla \times \mathbf{B}_h^{n+1}\|^2, \quad (4.8)
 \end{aligned}$$

which implies

$$\begin{aligned}
 \|\mathbf{u}_{h*}^n - \mathbf{u}_h^n\|^2 & \lesssim \|\theta_u^n\|^2 + \delta t^2 \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\nabla \times \mathbf{B}_h^{n+1}\|^2 \\
 & \lesssim \|\theta_u^n\|^2 + \delta t^2 \|\mathbf{B}_h^n\|_{L^\infty}^2 (\|\nabla \times \mathbf{B}(t_{n+1})\|^2 + \|\nabla \times \rho_b^{n+1}\|^2 + \|\nabla \times \theta_b^{n+1}\|^2) \\
 & \lesssim \|\theta_u^n\|^2 + \delta t^2 \|\mathbf{B}_h^n\|_{L^\infty}^2 + \delta t^2 \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\nabla \times \theta_b^{n+1}\|^2, \quad (4.9)
 \end{aligned}$$

where we use (3.7). □

In order to demonstrate the L^∞ stability of \mathbf{B}_h^n , we first show the L^∞ bound of $\mathbf{J}_{\mathbf{B}(t_n)}$. We denote $\Pi : H^{r+1}(\Omega)^2 \cap \mathbf{H}_t^1(\Omega) \rightarrow C_h$ the interpolation operator, then we have the approximation properties (cf. [11])

$$\|\mathbf{B} - \Pi\mathbf{B}\|_{L^\infty} \lesssim h^r \|\mathbf{B}\|_{r+1}, \quad \|\mathbf{B} - \Pi\mathbf{B}\| \lesssim h^{r+1} \|\mathbf{B}\|_{r+1}. \tag{4.10}$$

Suppose the assumption (A0) holds, from (4.10), (3.7) and the inverse inequality (3.2), we derive

$$\begin{aligned} \|\mathbf{J}_{\mathbf{B}(t_n)}\|_{L^\infty} &\leq \|\mathbf{B}(t_n) - \mathbf{J}_{\mathbf{B}(t_n)}\|_{L^\infty} + \|\mathbf{B}(t_n)\|_{L^\infty} \\ &\leq \|\mathbf{B}(t_n) - \Pi\mathbf{B}(t_n)\|_{L^\infty} + \|\Pi\mathbf{B}(t_n) - \mathbf{J}_{\mathbf{B}(t_n)}\|_{L^\infty} + \|\mathbf{B}(t_n)\|_{L^\infty} \\ &\leq C_\Omega h^r \|\mathbf{B}(t_n)\|_{r+1} + C_\Omega h^{-1} \|\Pi\mathbf{B}(t_n) - \mathbf{J}_{\mathbf{B}(t_n)}\| + \|\mathbf{B}(t_n)\|_{L^\infty} \\ &\leq C_\Omega h^r \|\mathbf{B}(t_n)\|_{r+1} + C_\Omega h^{-1} (\|\Pi\mathbf{B}(t_n) - \mathbf{B}(t_n)\| \\ &\quad + \|\mathbf{B}(t_n) - \mathbf{J}_{\mathbf{B}(t_n)}\|) + \|\mathbf{B}(t_n)\|_{L^\infty} \\ &\leq C_\Omega h^r \|\mathbf{B}(t_n)\|_{r+1} + \|\mathbf{B}(t_n)\|_{L^\infty} \leq C_0, \quad 0 \leq n \leq \left\lceil \frac{T}{\delta t} \right\rceil. \end{aligned} \tag{4.11}$$

Furthermore, (4.11) also implies that the $l^2(L^\infty)$ bound of $\mathbf{J}_{\mathbf{B}(t_n)}$ satisfies

$$\delta t \sum_{n=0}^{\lceil \frac{T}{\delta t} \rceil} \|\mathbf{J}_{\mathbf{B}(t_n)}\|_{L^\infty}^2 \leq TC_0^2. \tag{4.12}$$

Let $\kappa_0 = 2TC_0^2 + 1$, and assume the following regularity conditions hold

$$(A2) : \mathbf{u}_t \in L^2(0, T; H^{l+1}(\Omega)^2), \quad p_t \in L^2(0, T; H^1(\Omega)), \quad \mathbf{B}_t \in L^2(0, T; H^{r+1}(\Omega)^2).$$

For the error estimates of velocity and magnetic field, we need the $l^2(L^\infty)$ bound of \mathbf{B}_h^n . Meanwhile for the error estimate of pressure, we further need the $l^\infty(L^\infty)$ bound of \mathbf{B}_h^n . Now, we state the $l^2(L^\infty)$ stability of \mathbf{B}_h^n in the following lemma.

Lemma 4.3 *Assuming that the solution to (2.1)–(2.6) satisfies the regularity assumptions (A0)–(A2), then there exist two positive constants δt_0 and h_0 such that, when $\delta t \leq \delta t_0$, $h \leq h_0$ and $\delta t |\log h| \leq 1$, the solution \mathbf{B}_h^n of scheme (3.8)–(3.12) satisfies*

$$\delta t \sum_{n=0}^{\lceil \frac{T}{\delta t} \rceil} \|\mathbf{B}_h^n\|_{L^\infty}^2 \leq \kappa_0, \quad \max_{0 \leq n \leq \lceil \frac{T}{\delta t} \rceil} \delta t \|\mathbf{B}_h^n\|_{L^\infty}^2 \leq \kappa_1 \delta t + \kappa_2 h^{2\min\{l,r\}+1}, \tag{4.13}$$

where $\kappa_1 = 2C_0^2 + C_\Omega C(\kappa_0)$, $\kappa_2 = C_\Omega C(\kappa_0)$, and the constants δt_0 , h_0 , $C(\kappa_0)$ will be verified in the proof.

Proof We use the mathematical induction method here.

When $n = 0$, we have $\delta t \|\mathbf{B}_h^0\|_{L^\infty}^2 = \delta t \|\mathbf{J}_{\mathbf{B}_0}\|_{L^\infty}^2 \leq C_0^2 \delta t \leq \kappa_0$, and $\delta t \|\mathbf{B}_h^0\|_{L^\infty}^2 \leq C_0^2 \delta t \leq \kappa_1 \delta t + \kappa_2 h^{2\min\{l,r\}+1}$ ($\kappa_1 \geq C_0^2$).

Assuming that

$$\delta t \sum_{n=0}^N \|\mathbf{B}_h^n\|_{L^\infty}^2 \leq \kappa_0, \quad \max_{0 \leq n \leq N} \delta t \|\mathbf{B}_h^n\|_{L^\infty}^2 \leq \kappa_1 \delta t + \kappa_2 h^{2\min\{l,r\}+1} \tag{4.14}$$

are valid, we will show

$$\delta t \sum_{n=0}^{N+1} \|\mathbf{B}_h^n\|_{L^\infty}^2 \leq \kappa_0, \quad \max_{0 \leq n \leq N+1} \delta t \|\mathbf{B}_h^n\|_{L^\infty}^2 \leq \kappa_1 \delta t + \kappa_2 h^{2\min\{l,r\}+1} \tag{4.15}$$

are also valid via the following two steps. In *Step i*, we use the induction assumptions to derive the convergence analysis. In *Step ii*, using the convergence results obtained in *Step i* and inverse inequalities, we get the $l^2(L^\infty)$ bounds of \mathbf{B}_h^{N+1} .

(*Step i*). By taking $\mathbf{c}_h = \theta_b^{n+1}$ in (4.5), we obtain

$$\begin{aligned} & \frac{1}{2\delta t} (\|\theta_b^{n+1}\|^2 - \|\theta_b^n\|^2 + \|\theta_b^{n+1} - \theta_b^n\|^2) + \eta (\|\nabla \times \theta_b^{n+1}\|^2 + \|\nabla \cdot \theta_b^{n+1}\|^2) \\ & + (d_t \rho_b^{n+1}, \theta_b^{n+1}) + (\mathbf{B}(t_n) \times \mathbf{u}(t_n) - \mathbf{B}_h^n \times \mathbf{u}_{h*}^n, \nabla \times \theta_b^{n+1}) = (R_b^{n+1}, \theta_b^{n+1}). \end{aligned} \tag{4.16}$$

By taking $\mathbf{v}_h = \tilde{\theta}_u^{n+1}$ in (4.6), we derive

$$\begin{aligned} & \frac{1}{2\delta t} (\|\tilde{\theta}_u^{n+1}\|^2 - \|\theta_u^n\|^2 + \|\tilde{\theta}_u^{n+1} - \theta_u^n\|^2) + \nu \|\nabla \tilde{\theta}_u^{n+1}\|^2 \\ & + \nu (\nabla \rho_u^{n+1}, \nabla \tilde{\theta}_u^{n+1}) + (d_t \rho_u^{n+1}, \tilde{\theta}_u^{n+1}) \\ & + b(\mathbf{u}(t_n), \mathbf{u}(t_{n+1}), \tilde{\theta}_u^{n+1}) - b(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}, \tilde{\theta}_u^{n+1}) + (\nabla \rho_p^n, \tilde{\theta}_u^{n+1}) + (\nabla \theta_p^n, \tilde{\theta}_u^{n+1}) \\ & + s(\mathbf{B}(t_n) \times \nabla \times \mathbf{B}(t_{n+1}), \tilde{\theta}_u^{n+1}) - s(\mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^{n+1}, \tilde{\theta}_u^{n+1}) = (R_u^{n+1}, \tilde{\theta}_u^{n+1}). \end{aligned} \tag{4.17}$$

We rewrite (4.7) to obtain

$$\frac{1}{\delta t} \theta_u^{n+1} + \nabla \theta_p^{n+1} = \frac{1}{\delta t} \tilde{\theta}_u^{n+1} + \nabla \theta_p^n + \rho^{n+1}, \tag{4.18}$$

where $\rho^{n+1} = R_p^{n+1} - \delta t \nabla d_t \rho_p^{n+1}$. By taking the L^2 inner product of (4.18) with itself on both sides, we obtain

$$\begin{aligned} (\tilde{\theta}_u^{n+1}, \nabla \theta_p^n) &= \frac{1}{2\delta t} (\|\theta_u^{n+1}\|^2 - \|\tilde{\theta}_u^{n+1}\|^2) + \frac{\delta t}{2} (\|\nabla \theta_p^{n+1}\|^2 - \|\nabla \theta_p^n\|^2) \\ & - (\tilde{\theta}_u^{n+1}, \rho^{n+1}) - \delta t (\nabla \theta_p^n, \rho^{n+1}) - \frac{\delta t}{2} \|\rho^{n+1}\|^2. \end{aligned} \tag{4.19}$$

We combine (4.16), (4.17) and (4.19) to obtain

$$\begin{aligned} & \frac{1}{2\delta t} (\|\theta_b^{n+1} - \theta_b^n\|^2 + \|\theta_b^{n+1}\|^2 - \|\theta_b^n\|^2) \\ & + \eta (\|\nabla \times \theta_b^{n+1}\|^2 + \|\nabla \cdot \theta_b^{n+1}\|^2) + \nu \|\nabla \tilde{\theta}_u^{n+1}\|^2 \\ & + \frac{\delta t}{2} (\|\nabla \theta_p^{n+1}\|^2 - \|\nabla \theta_p^n\|^2) + \frac{1}{2\delta t} (\|\tilde{\theta}_u^{n+1} - \theta_u^n\|^2 + \|\theta_u^{n+1}\|^2 - \|\theta_u^n\|^2) \\ & = (\mathbf{u}(t_n) \times \mathbf{B}(t_n) - \mathbf{u}_{h*}^n \times \mathbf{B}_h^n, \nabla \times \theta_b^{n+1}) \tag{: term A} \\ & - b(\mathbf{u}(t_n), \mathbf{u}(t_{n+1}), \tilde{\theta}_u^{n+1}) + b(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}, \tilde{\theta}_u^{n+1}) \tag{: term B} \\ & - s(\mathbf{B}(t_n) \times \nabla \times \mathbf{B}(t_{n+1}) - \mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^{n+1}, \tilde{\theta}_u^{n+1}) \tag{: term C} \\ & + (\tilde{\theta}_u^{n+1}, \rho^{n+1}) + \delta t (\nabla \theta_p^n, \rho^{n+1}) \tag{: term D} \\ & + (\rho_p^n - \rho_p^{n+1}, \nabla \cdot \tilde{\theta}_u^{n+1}) - (d_t \rho_b^{n+1}, \theta_b^{n+1}) - (d_t \rho_u^{n+1}, \tilde{\theta}_u^{n+1}) \tag{: term E} \\ & + (R_b^{n+1}, \theta_b^{n+1}) + (R_u^{n+1}, \tilde{\theta}_u^{n+1}) \tag{: term F} \\ & + \frac{\delta t}{2} \|\rho^{n+1}\|^2. \end{aligned} \tag{4.20}$$

We estimate the terms (A–F) one by one as follows.

For term A, using the Lemma 4.2, (3.6), (3.7) and regularity assumptions, we derive

$$\begin{aligned}
 (\text{term A}) &\leq |(\mathbf{u}(t_n) \times \mathbf{B}(t_n) - \mathbf{u}_{h^*}^n \times \mathbf{B}_h^n, \nabla \times \theta_b^{n+1})| \\
 &= |(\mathbf{u}(t_n) \times e_b^n + (\mathbf{u}(t_n) - \mathbf{u}_{h^*}^n) \times \mathbf{B}_h^n, \nabla \times \theta_b^{n+1})| \\
 &\leq (\|e_b^n\| \|\mathbf{u}(t_n)\|_{L^\infty} + \|\mathbf{B}_h^n\|_{L^\infty} \|\mathbf{u}(t_n) - \mathbf{u}_{h^*}^n\|) \|\nabla \times \theta_b^{n+1}\| \\
 &\lesssim \frac{\eta}{8} \|\nabla \times \theta_b^{n+1}\|^2 + \|\theta_b^n\|^2 + \|\rho_b^n\|^2 \\
 &\quad + \|\mathbf{B}_h^n\|_{L^\infty}^2 (\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|^2 + \|\mathbf{u}_h^n - \mathbf{u}_{h^*}^n\|^2) \\
 &\lesssim \frac{\eta}{8} \|\nabla \times \theta_b^{n+1}\|^2 + \|\theta_b^n\|^2 + h^{2r+2} + \|\mathbf{B}_h^n\|_{L^\infty}^2 (h^{2l+2} + \|\theta_u^n\|^2) + \delta t^2 \|\mathbf{B}_h^n\|_{L^\infty}^4 \\
 &\quad + \delta t^2 \|\mathbf{B}_h^n\|_{L^\infty}^4 \|\nabla \times \theta_b^{n+1}\|^2.
 \end{aligned}$$

For term B, by using (3.20), regularity assumptions, the inverse inequalities (3.2) and (3.6), we derive

$$\begin{aligned}
 (\text{term B}) &\leq |b(\mathbf{u}(t_n), \mathbf{u}(t_{n+1}), \tilde{\theta}_u^{n+1}) - b(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}, \tilde{\theta}_u^{n+1})| \\
 &= |b(e_u^n, \mathbf{u}(t_{n+1}), \tilde{\theta}_u^{n+1}) - b(\mathbf{u}_h^n, \tilde{\theta}_u^{n+1}, \rho_u^{n+1})| \\
 &= |b(e_u^n, \mathbf{u}(t_{n+1}), \tilde{\theta}_u^{n+1}) + b(e_u^n, \tilde{\theta}_u^{n+1}, \rho_u^{n+1}) - b(\mathbf{u}(t_n), \tilde{\theta}_u^{n+1}, \rho_u^{n+1})| \\
 &\lesssim \|e_u^n\| \|\nabla \tilde{\theta}_u^{n+1}\| \|\mathbf{u}(t_{n+1})\|_2 + \|e_u^n\| \|\nabla \tilde{\theta}_u^{n+1}\|_{L^3} \|\nabla \rho_u^{n+1}\| + \|e_u^n\| \|\tilde{\theta}_u^{n+1}\|_{L^\infty} \|\nabla \rho_u^{n+1}\| \\
 &\quad + \|\mathbf{u}(t_n)\|_{L^\infty} \|\nabla \tilde{\theta}_u^{n+1}\| \|\rho_u^{n+1}\| \\
 &\lesssim \|e_u^n\| \|\nabla \tilde{\theta}_u^{n+1}\| + \|e_u^n\| \|\nabla \tilde{\theta}_u^{n+1}\| h^{-\frac{1}{3}} h^l + \|\nabla \tilde{\theta}_u^{n+1}\| \|\rho_u^{n+1}\| \\
 &\lesssim \frac{\nu}{10} \|\nabla \tilde{\theta}_u^{n+1}\|^2 + \|\theta_u^n\|^2 + \|\rho_u^n\|^2 + \|\rho_u^{n+1}\|^2 \lesssim \frac{\nu}{10} \|\nabla \tilde{\theta}_u^{n+1}\|^2 + \|\theta_u^n\|^2 + h^{2l+2}.
 \end{aligned}$$

For term C, from the equality of $\nabla \times (a \times b) = b \nabla a - a \nabla b + a \nabla \cdot b - b \nabla \cdot a$, the inverse inequality (3.2), regularity conditions and (3.7), we derive

$$\begin{aligned}
 (\text{term C}) &\leq s |(\mathbf{B}(t_n) \times \nabla \times \mathbf{B}(t_{n+1}) - \mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^{n+1}, \tilde{\theta}_u^{n+1})| \\
 &= s |(e_b^n \times \nabla \times \mathbf{B}(t_{n+1}), \tilde{\theta}_u^{n+1}) + (\mathbf{B}_h^n \times \nabla \times e_b^{n+1}, \tilde{\theta}_u^{n+1})| \\
 &\leq s |(e_b^n \times \nabla \times \mathbf{B}(t_{n+1}), \tilde{\theta}_u^{n+1})| + s |(\mathbf{B}_h^n \times \nabla \times \theta_b^{n+1}, \tilde{\theta}_u^{n+1})| \\
 &\quad + s |(\mathbf{B}_h^n \times \nabla \times \rho_b^{n+1}, \tilde{\theta}_u^{n+1})| \\
 &\leq s |(e_b^n \times \nabla \times \mathbf{B}(t_{n+1}), \tilde{\theta}_u^{n+1})| + s |(\mathbf{B}_h^n \times \nabla \times \theta_b^{n+1}, \tilde{\theta}_u^{n+1} - \theta_u^n)| \\
 &\quad + s |(\mathbf{B}_h^n \times \nabla \times \theta_b^{n+1}, \theta_u^n)| \\
 &\quad + s |(e_b^n \times \nabla \times \rho_b^{n+1}, \tilde{\theta}_u^{n+1})| + s |(\mathbf{B}(t_n) \times \nabla \times \rho_b^{n+1}, \tilde{\theta}_u^{n+1})| \\
 &\leq s |(e_b^n \times \nabla \times \mathbf{B}(t_{n+1}), \tilde{\theta}_u^{n+1})| + s |(\mathbf{B}_h^n \times \nabla \times \theta_b^{n+1}, \tilde{\theta}_u^{n+1} - \theta_u^n)| \\
 &\quad + s |(\mathbf{B}_h^n \times \nabla \times \theta_b^{n+1}, \theta_u^n)| + s |(e_b^n \times \nabla \times \rho_b^{n+1}, \tilde{\theta}_u^{n+1})| \\
 &\quad + s |(\mathbf{B}(t_n) \nabla \tilde{\theta}_u^{n+1} - \tilde{\theta}_u^{n+1} \nabla \mathbf{B}(t_n) - \mathbf{B}(t_n) \nabla \cdot \tilde{\theta}_u^{n+1}, \rho_b^{n+1})| \\
 &\lesssim \|e_b^n\| \|\nabla \times \mathbf{B}(t_{n+1})\|_{L^3} \|\tilde{\theta}_u^{n+1}\|_{L^6} + \|\mathbf{B}_h^n\|_{L^\infty} \|\nabla \times \theta_b^{n+1}\| \|\tilde{\theta}_u^{n+1} - \theta_u^n\| \\
 &\quad + \|\mathbf{B}_h^n\|_{L^\infty} \|\nabla \times \theta_b^{n+1}\| \|\theta_u^n\| + \|e_b^n\| \|\nabla \times \rho_b^{n+1}\| \|\tilde{\theta}_u^{n+1}\|_{L^\infty} \\
 &\quad + \|\mathbf{B}(t_n)\|_{L^\infty} \|\nabla \tilde{\theta}_u^{n+1}\| \|\rho_b^{n+1}\| + \|\tilde{\theta}_u^{n+1}\|_{L^6} \|\nabla \mathbf{B}(t_n)\|_{L^3} \|\rho_b^{n+1}\| \\
 &\lesssim \|e_b^n\| \|\nabla \tilde{\theta}_u^{n+1}\| + \|\mathbf{B}_h^n\|_{L^\infty} \|\nabla \times \theta_b^{n+1}\| \|\tilde{\theta}_u^{n+1} - \theta_u^n\| \\
 &\quad + \|\mathbf{B}_h^n\|_{L^\infty} \|\nabla \times \theta_b^{n+1}\| \|\theta_u^n\| + \|e_b^n\| h^r h^{-\frac{1}{3}} \|\nabla \tilde{\theta}_u^{n+1}\| + \|\nabla \tilde{\theta}_u^{n+1}\| h^{r+1}
 \end{aligned}$$

$$\begin{aligned} &\lesssim \frac{\nu}{10} \|\nabla \tilde{\theta}_u^{n+1}\|^2 + \frac{\eta}{8} \|\nabla \times \theta_b^{n+1}\|^2 + \|\theta_b^n\|^2 + \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\tilde{\theta}_u^{n+1} - \theta_u^n\|^2 \\ &\quad + \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\theta_u^n\|^2 + h^{2r+2}. \end{aligned}$$

For term D, term E and term F, using (2.7), (2.8), and Lemma 4.1, we derive

$$\begin{aligned} \text{(term D)} &\leq \|\tilde{\theta}_u^{n+1}\| \|\rho^{n+1}\| + \delta t \|\nabla \theta_p^n\| \|\rho^{n+1}\| \lesssim \frac{\nu}{10} \|\nabla \tilde{\theta}_u^{n+1}\|^2 + \|\rho^{n+1}\|^2 + \delta t^2 \|\nabla \theta_p^n\|^2 \\ &\lesssim \frac{\nu}{10} \|\nabla \tilde{\theta}_u^{n+1}\|^2 + \delta t^2 + \delta t^2 \|\nabla d_t \rho_p^{n+1}\|^2 + \delta t^2 \|\nabla \theta_p^n\|^2, \end{aligned}$$

and

$$\begin{aligned} \text{(term E)} &\lesssim \delta t \|d_t \rho_p^{n+1}\| \|\nabla \tilde{\theta}_u^{n+1}\| + \|d_t \rho_b^{n+1}\| (\|\nabla \times \theta_b^{n+1}\| \\ &\quad + \|\nabla \cdot \theta_b^{n+1}\|) + \|d_t \rho_u^{n+1}\| \|\nabla \tilde{\theta}_u^{n+1}\| \\ &\lesssim \frac{\nu}{10} \|\nabla \tilde{\theta}_u^{n+1}\|^2 + \frac{\eta}{8} (\|\nabla \times \theta_b^{n+1}\|^2 + \|\nabla \cdot \theta_b^{n+1}\|^2) \\ &\quad + \delta t^2 \|d_t \rho_p^{n+1}\|^2 + \|d_t \rho_b^{n+1}\|^2 + \|d_t \rho_u^{n+1}\|^2, \end{aligned}$$

and

$$\begin{aligned} \text{(term F)} &\leq \|R_b^{n+1}\| \|\theta_b^{n+1}\| + \|R_u^{n+1}\| \|\tilde{\theta}_u^{n+1}\| \\ &\lesssim \frac{\nu}{10} \|\nabla \tilde{\theta}_u^{n+1}\|^2 + \frac{\eta}{8} (\|\nabla \times \theta_b^{n+1}\|^2 + \|\nabla \cdot \theta_b^{n+1}\|^2) + \|R_b^{n+1}\|^2 + \|R_u^{n+1}\|^2 \\ &\lesssim \frac{\nu}{10} \|\nabla \tilde{\theta}_u^{n+1}\|^2 + \frac{\eta}{8} (\|\nabla \times \theta_b^{n+1}\|^2 + \|\nabla \cdot \theta_b^{n+1}\|^2) + \delta t^2. \end{aligned}$$

Combining the above estimates with (4.20), we obtain

$$\begin{aligned} &\|\theta_b^{n+1}\|^2 - \|\theta_b^n\|^2 + \|\theta_u^{n+1}\|^2 - \|\theta_u^n\|^2 + \|\tilde{\theta}_u^{n+1} - \theta_u^n\|^2 + \delta t^2 (\|\nabla \theta_p^{n+1}\|^2 - \|\nabla \theta_p^n\|^2) \\ &\quad + \delta t \eta (\|\nabla \times \theta_b^{n+1}\|^2 + \|\nabla \cdot \theta_b^{n+1}\|^2) + \delta t \nu \|\nabla \tilde{\theta}_u^{n+1}\|^2 \\ &\lesssim \delta t \|\theta_b^n\|^2 + \delta t \|\theta_u^n\|^2 + \delta t \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\theta_u^n\|^2 + \delta t^3 \|\nabla \theta_p^n\|^2 \\ &\quad + \delta t \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\tilde{\theta}_u^{n+1} - \theta_u^n\|^2 + \delta t^3 \|\mathbf{B}_h^n\|_{L^\infty}^4 \\ &\quad + \delta t^3 \|\mathbf{B}_h^n\|_{L^\infty}^4 \|\nabla \times \theta_b^{n+1}\|^2 + \delta t^3 \|d_t \rho_p^{n+1}\|_1^2 + \delta t (\|d_t \rho_b^{n+1}\|^2 + \|d_t \rho_u^{n+1}\|^2) \\ &\quad + \delta t^3 + \delta t (h^{2l+2} + h^{2r+2} + \|\mathbf{B}_h^n\|_{L^\infty}^2 h^{2l+2}). \end{aligned} \tag{4.21}$$

We choose any two positive constants δt_1 and h_1 that satisfy $\kappa_1 \delta t_1 + \kappa_2 h_1^{2 \min\{l,r\}+1} \leq \min(1, \sqrt{\frac{\eta}{2}}, \frac{\eta}{4})$. Then when $\delta t \leq \delta t_1$ and $h \leq h_1$, from the induction assumptions (4.14) for $n \leq N$, we have

$$\begin{aligned} \delta t \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\tilde{\theta}_u^{n+1} - \theta_u^n\|^2 &\leq (\kappa_1 \delta t_1 + \kappa_2 h_1^{2 \min\{l,r\}+1}) \|\tilde{\theta}_u^{n+1} - \theta_u^n\|^2 \leq \|\tilde{\theta}_u^{n+1} - \theta_u^n\|^2, \\ \delta t^3 \|\mathbf{B}_h^n\|_{L^\infty}^4 \|\nabla \times \theta_b^{n+1}\|^2 &\leq (\kappa_1 \delta t_1 + \kappa_2 h_1^{2 \min\{l,r\}+1})^2 \delta t \|\nabla \times \theta_b^{n+1}\|^2 \leq \frac{\eta}{2} \delta t \|\nabla \times \theta_b^{n+1}\|^2. \end{aligned}$$

Besides, from the inverse inequality (3.2), (4.14), and the assumption $\delta t |\log h| \leq 1$, $\delta t \leq \delta t_1$ and $h \leq h_1$, we get

$$\begin{aligned} \delta t^3 \|\mathbf{B}_h^n\|_{L^\infty}^4 &= \delta t^3 \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\mathbf{B}_h^n\|_{L^\infty}^2 \lesssim \delta t^3 \|\mathbf{B}_h^n\|_{L^\infty}^2 (\|\theta_b^n\|_{L^\infty}^2 + \|\mathbf{J}_{\mathbf{B}(t_n)}\|_{L^\infty}^2) \\ &\lesssim \delta t^3 \|\mathbf{B}_h^n\|_{L^\infty}^2 |\log h| (\|\nabla \times \theta_b^n\|^2 + \|\nabla \cdot \theta_b^n\|^2) + \delta t^3 \|\mathbf{B}_h^n\|_{L^\infty}^2 \\ &\lesssim (\kappa_1 \delta t + \kappa_2 h^{2 \min\{l,r\}+1}) \delta t (\|\nabla \times \theta_b^n\|^2 + \|\nabla \cdot \theta_b^n\|^2) + \delta t^3 \|\mathbf{B}_h^n\|_{L^\infty}^2 \\ &\lesssim \frac{\eta}{4} \delta t (\|\nabla \times \theta_b^n\|^2 + \|\nabla \cdot \theta_b^n\|^2) + \delta t^3 \|\mathbf{B}_h^n\|_{L^\infty}^2. \end{aligned}$$

Therefore, (4.21) becomes

$$\begin{aligned} & \|\theta_b^{n+1}\|^2 - \|\theta_b^n\|^2 + \|\theta_u^{n+1}\|^2 - \|\theta_u^n\|^2 + \delta t^2 (\|\nabla \theta_p^{n+1}\|^2 - \|\nabla \theta_p^n\|^2) \\ & + \delta t \frac{\eta}{2} (\|\nabla \times \theta_b^{n+1}\|^2 + \|\nabla \cdot \theta_b^{n+1}\|^2) - \frac{\eta}{4} \delta t (\|\nabla \times \theta_b^n\|^2 + \|\nabla \cdot \theta_b^n\|^2) + \delta t \nu \|\nabla \tilde{\theta}_u^{n+1}\|^2 \\ & \lesssim \delta t \|\theta_b^n\|^2 + \delta t \|\theta_u^n\|^2 + \delta t \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\theta_u^n\|^2 + \delta t^3 \|\nabla \theta_p^n\|^2 + \delta t^3 \|\mathbf{B}_h^n\|_{L^\infty}^2 + \delta t^3 \|d_t \rho_p^{n+1}\|_1^2 \\ & + \delta t (\|d_t \rho_b^{n+1}\|^2 + \|d_t \rho_u^{n+1}\|^2) + \delta t^3 + \delta t (h^{2l+2} + h^{2r+2} + \|\mathbf{B}_h^n\|_{L^\infty}^2 h^{2l+2}). \end{aligned}$$

Summing up the above inequality from $n = 0$ to N , and using the induction assumptions (4.14) and $\theta_b^0 = \theta_u^0 = \theta_p^0 = 0$, we obtain

$$\begin{aligned} & \|\theta_b^{N+1}\|^2 + \|\theta_u^{N+1}\|^2 + \delta t^2 \|\nabla \theta_p^{N+1}\|^2 \\ & + \delta t \sum_{n=0}^N \left(\frac{\eta}{4} \|\nabla \times \theta_b^{n+1}\|^2 + \frac{\eta}{4} \|\nabla \cdot \theta_b^{n+1}\|^2 + \nu \|\nabla \tilde{\theta}_u^{n+1}\|^2 \right) \\ & \lesssim \delta t \sum_{n=0}^N \left(\|\theta_b^n\|^2 + \|\theta_u^n\|^2 + \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\theta_u^n\|^2 + \delta t^2 \|\nabla \theta_p^n\|^2 \right) + \delta t^3 \sum_{n=0}^N \|d_t \rho_p^{n+1}\|_1^2 \tag{4.22} \\ & + \delta t \sum_{n=0}^N (\|d_t \rho_b^{n+1}\|^2 + \|d_t \rho_u^{n+1}\|^2) + \delta t^2 + h^{2l+2} + h^{2r+2} + \kappa_0 (\delta t^2 + h^{2l+2}). \end{aligned}$$

In addition, using (3.6), (3.7), Theorem 3.1 and regularity condition (A2), we obtain

$$\begin{aligned} \delta t \sum_{n=0}^N \|d_t \rho_b^{n+1}\|^2 + \|d_t \rho_u^{n+1}\|^2 & \lesssim h^{2r+2} \delta t \sum_{n=0}^N \|d_t \mathbf{B}(t_{n+1})\|_{r+1}^2 \\ & + h^{2l+2} \delta t \sum_{n=0}^N (\|d_t \mathbf{u}(t_{n+1})\|_{l+1}^2 + \|d_t p(t_n)\|_1^2) \\ & \lesssim h^{2r+2} + h^{2l+2}, \end{aligned}$$

and

$$\begin{aligned} \delta t^3 \sum_{n=0}^N \|d_t \rho_p^{n+1}\|_1^2 & \lesssim \delta t^3 \sum_{n=0}^N \|d_t p(t_{n+1})\|_1^2 + \delta t^3 \sum_{n=0}^N \|Q(d_t \mathbf{u}(t_{n+1}), d_t p(t_{n+1}))\|_1^2 \\ & \lesssim \delta t^3 \sum_{n=0}^N \|d_t p(t_{n+1})\|_1^2 + \delta t^3 \sum_{n=0}^N (\|d_t \mathbf{u}(t_{n+1})\|_2^2 + \|d_t p(t_{n+1})\|_1^2) \lesssim \delta t^2. \end{aligned}$$

Then, (4.22) becomes

$$\begin{aligned} & \|\theta_b^{N+1}\|^2 + \|\theta_u^{N+1}\|^2 + \delta t^2 \|\nabla \theta_p^{N+1}\|^2 \\ & + \delta t \sum_{n=0}^N \left(\eta \|\nabla \times \theta_b^{n+1}\|^2 + \eta \|\nabla \cdot \theta_b^{n+1}\|^2 + \nu \|\nabla \tilde{\theta}_u^{n+1}\|^2 \right) \tag{4.23} \\ & \lesssim \delta t \sum_{n=0}^N \left(\|\theta_b^n\|^2 + \|\theta_u^n\|^2 + \|\mathbf{B}_h^n\|_{L^\infty}^2 \|\theta_u^n\|^2 + \delta t^2 \|\nabla \theta_p^n\|^2 \right) \\ & + (\kappa_0 + 1) \delta t^2 + (\kappa_0 + 1) h^{2l+2} + h^{2r+2}. \end{aligned}$$

By applying the Gronwall’s inequality to (4.23), we obtain

$$\begin{aligned} & \|\theta_b^{N+1}\|^2 + \|\theta_u^{N+1}\|^2 + \delta t^2 \|\nabla \theta_p^{N+1}\|^2 \\ & + \delta t \sum_{n=0}^N \left(\eta \|\nabla \times \theta_b^{n+1}\|^2 + \eta \|\nabla \cdot \theta_b^{n+1}\|^2 + \nu \|\nabla \tilde{\theta}_u^{n+1}\|^2 \right) \\ & \leq C(\kappa_0)(\delta t^2 + h^{2l+2} + h^{2r+2}), \end{aligned} \tag{4.24}$$

where $C(\kappa_0)$ is a constant depending on κ_0 .

(Step ii). We assume that the condition $\delta t |\log h| \leq 1$ holds. By using the inverse inequality (3.2), (2.8), (4.11), (4.12) and (4.24), we derive

$$\begin{aligned} \delta t \sum_{n=0}^{N+1} \|\mathbf{B}_h^n\|_{L^\infty}^2 & \leq 2\delta t \sum_{n=0}^{N+1} \|J_{\mathbf{B}(t_n)}\|_{L^\infty}^2 + 2\delta t \sum_{n=0}^{N+1} \|\theta_b^n\|_{L^\infty}^2 \\ & \leq 2TC_0^2 + C_\Omega |\log h| \delta t \sum_{n=0}^{N+1} (\|\nabla \times \theta_b^n\|^2 + \|\nabla \cdot \theta_b^n\|^2) \\ & \leq 2TC_0^2 + C_\Omega C(\kappa_0) |\log h| (\delta t^2 + h^{2l+2} + h^{2r+2}) \\ & \leq 2TC_0^2 + C_\Omega C(\kappa_0) (\delta t + |\log h| h^{2 \min\{l,r\}+2}) \\ & \leq 2TC_0^2 + 1 = \kappa_0, \end{aligned}$$

where we assume $\delta t \leq \delta t_2$, $h \leq h_2$ and δt_2 and h_2 satisfy $C_\Omega C(\kappa_0)(\delta t_2 + |\log h_2| h_2^{2 \min\{l,r\}+2}) < 1$.

By setting $\kappa_1 = 2C_0^2 + C_\Omega C(\kappa_0)$, $\kappa_2 = C_\Omega C(\kappa_0)$, we derive

$$\begin{aligned} \delta t \|\mathbf{B}_h^{N+1}\|_{L^\infty}^2 & \leq 2\delta t \|J_{\mathbf{B}(t_{N+1})}\|_{L^\infty}^2 + 2\delta t \|\theta_b^{N+1}\|_{L^\infty}^2 \\ & \leq 2C_0^2 \delta t + C_\Omega |\log h| \delta t (\|\nabla \times \theta_b^{N+1}\|^2 + \|\nabla \cdot \theta_b^{N+1}\|^2) \\ & \leq 2C_0^2 \delta t + C_\Omega C(\kappa_0) |\log h| (\delta t^2 + h^{2l+2} + h^{2r+2}) \\ & \leq 2C_0^2 \delta t + C_\Omega C(\kappa_0) (\delta t + h^{2 \min\{l,r\}+1}) \\ & = \kappa_1 \delta t + \kappa_2 h^{2 \min\{l,r\}+1}. \end{aligned}$$

The proof is completed if we simply let $\delta t \leq \delta t_0$ ($:= \min(\delta t_1, \delta t_2)$) and $h \leq h_0$ ($:= \min(h_1, h_2)$). □

Based on the above Lemma, now it is ready to obtain the error estimates for the velocity field and magnetic field, as follows.

Theorem 4.4 *Under the conditions of Lemma 4.3, the following estimates hold*

$$\begin{aligned} \max_{0 \leq m \leq \lfloor \frac{T}{\delta t} \rfloor} \left(\|e_b^m\| + \|e_u^m\| + \delta t \|\nabla e_p^m\| \right) & \lesssim \delta t + h^{\min\{l,r\}+1}, \\ \delta t \sum_{n=0}^{\lfloor \frac{T}{\delta t} \rfloor} \left(\eta \|\nabla \times e_b^{n+1}\|^2 + \eta \|\nabla \cdot e_b^{n+1}\|^2 + \nu \|\nabla \tilde{e}_u^{n+1}\|^2 \right) & \lesssim \delta t^2 + h^{2 \min\{l,r\}}. \end{aligned} \tag{4.25}$$

Proof By Lemma 4.3, we already obtained

$$\delta t \sum_{n=0}^{\lfloor \frac{T}{\delta t} \rfloor} \|\mathbf{B}_h^n\|_{L^\infty}^2 \leq \kappa_0, \quad \max_{0 \leq n \leq \lfloor \frac{T}{\delta t} \rfloor} \delta t \|\mathbf{B}_h^n\|_{L^\infty}^2 \leq \kappa_1 \delta t + \kappa_2 h^{2 \min\{l,r\}+1}. \tag{4.26}$$

By following the same procedures of *Step i* of the proof in Lemma 4.3, for $m \leq \lfloor \frac{T}{\delta t} \rfloor$, we obtain

$$\begin{aligned} & \|\theta_b^{m+1}\|^2 + \|\theta_u^{m+1}\|^2 + \delta t^2 \|\nabla \theta_p^{m+1}\|^2 \\ & + \delta t \sum_{n=0}^m \left(\eta \|\nabla \times \theta_b^{n+1}\|^2 + \eta \|\nabla \cdot \theta_b^{n+1}\|^2 + \nu \|\nabla \tilde{\theta}_u^{n+1}\|^2 \right) \lesssim \delta t^2 + h^{2l+2} + h^{2r+2}. \end{aligned} \tag{4.27}$$

By using the triangle inequality, (3.6) and (3.7), the desired results (4.25) are concluded. \square

Remark 4.5 The convergence results in Theorem 4.4 are still valid for the three dimensional case ($d = 3$), but the condition between the time step and grid size becomes more restrictive. More precisely, the condition is $\delta t \lesssim h$ for $d = 3$ instead of $\delta t |\log h| < 1$ for $d = 2$. This is because a different inverse inequality will be used for $d = 3$ [11], namely

$$\|\mathbf{B}_h\|_{L^\infty} \lesssim h^{-\frac{1}{2}} \|\mathbf{B}_h\|_{L^6} \lesssim h^{-\frac{1}{2}} \|\mathbf{B}_h\|_1. \tag{4.28}$$

We omit the details due to the page limit and leave this to the interested readers.

4.2 Optimal Error Estimate for Pressure

Note the convergence order for the pressure term in Theorem 4.4 is not optimal. In this subsection, we aim at proving the optimal error estimate for the pressure. Due to the introduction of the auxiliary velocity field \mathbf{u}_{h*}^n , there exist two specific difficulties to improve the order for the pressure, including (i) how to derive the error between $d_t \mathbf{u}(t_n)$ and $d_t \mathbf{u}_{h*}^n$ since there does not exist an error equation for $d_t \mathbf{u}_{h*}^n$; and (ii) how to derive the optimal error estimate for $\|d_t \theta_b^n\|$ which is needed for the estimate of the pressure.

For (i), we solve it by developing a relation between $\|d_t \mathbf{u}_h^n - d_t \mathbf{u}_{h*}^n\|$ and other error functions in Lemma 4.7. For (ii), we build the error estimate for $\|d_t \theta_b^{m+1}\|_{-1}^2 + \delta t \sum_{n=1}^m \|d_t \theta_b^{n+1}\|^2$ by means of negative norm estimate technique in Lemmas 4.9 and 4.10. Based on these estimates, by using the inf–sup condition, the optimal convergence order of the pressure is finally obtained in Theorem 4.11.

For the error estimate of the pressure, we need the $l^\infty(L^\infty)$ stability of \mathbf{B}_h^n that is derived in the following Lemma.

Lemma 4.6 *Assuming that the solution to (2.1)–(2.6) satisfies assumptions (A0)–(A2), then when $\delta t \leq h$, the following estimates hold*

$$\max_{0 \leq n \leq \lfloor \frac{T}{\delta t} \rfloor} \|\mathbf{B}_h^n\|_{L^\infty} \leq \kappa_3, \quad \max_{0 \leq n \leq \lfloor \frac{T}{\delta t} \rfloor} \|\mathbf{B}_h^n\|_1 \leq \kappa_4,$$

where κ_3, κ_4 will be verified in the proof.

Proof From (4.27), if $\delta t |\log h| \leq 1$, we have

$$\max_{0 \leq n \leq \lfloor \frac{T}{\delta t} \rfloor} \|\theta_b^n\| \lesssim \delta t + h^{l+1} + h^{r+1}.$$

We further assume a more restrictive condition for the time step: $\delta t \leq h$, then by the inverse inequality (3.2), we derive

$$\begin{aligned} \|\mathbf{B}_h^n\|_{L^\infty} &\leq \|J_{\mathbf{B}(t_n)}\|_{L^\infty} + \|\theta_b^n\|_{L^\infty} \leq \|J_{\mathbf{B}(t_n)}\|_{L^\infty} + C_\Omega h^{-1} \|\theta_b^n\| \\ &\leq C_0 + Ch^{-1}(\delta t + h^{l+1} + h^{r+1}) = \kappa_3; \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{B}_h^n\|_1 &\leq \|e_b^n\|_1 + \|\mathbf{B}(t_n)\|_1 \leq \|\theta_b^n\|_1 + \|\rho_b^n\|_1 + \|\mathbf{B}(t_n)\|_1 \\ &\leq C_\Omega h^{-1} \|\theta_b^n\| + \|\rho_b^n\|_1 + \|\mathbf{B}(t_n)\|_1 \\ &\leq Ch^{-1}(\delta t + h^{l+1} + h^{r+1}) + C = \kappa_4. \end{aligned}$$

The proof is finished. □

We assume the solution $\mathbf{B}(t)$ to (2.1)–(2.6) satisfies:

$$(A3) : \mathbf{B}_t \in L^\infty(0, T; H^1(\Omega)^2) \cap L^\infty(0, T; L^\infty(\Omega)^2).$$

In the next Lemma, we find the L^2 relation between $d_t \mathbf{u}_h^n$ and $d_t \mathbf{u}_{h*}^n$ with other error functions which will be used in the error estimate for pressure.

Lemma 4.7 *Assuming that the solution to (2.1)–(2.6) satisfies assumptions (A0)–(A3), when $\delta t \lesssim h$, the following estimate holds*

$$\|d_t \mathbf{u}_h^n - d_t \mathbf{u}_{h*}^n\|^2 \lesssim \delta t^2 + \|d_t \theta_u^n\|^2 + \|\nabla \times (e_b^{n+1} - e_b^n)\|^2 + \frac{\eta}{8} \|d_t \theta_b^n\|^2 + \|d_t \rho_b^n\|^2.$$

Proof By applying d_t to (3.9), we obtain

$$\left(\frac{d_t \mathbf{u}_{h*}^n - d_t \mathbf{u}_h^n}{\delta t}, \mathbf{w}_h \right) + s(\mathbf{B}_h^n \times \nabla \times d_t \mathbf{B}_h^{n+1}, \mathbf{w}_h) + s(d_t \mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^n, \mathbf{w}_h) = 0.$$

By taking $\mathbf{w}_h = 2\delta t d_t \mathbf{u}_{h*}^n - 2\delta t d_t R(u(t_n), p(t_n))$ and using Lemma 4.6, regularity condition (A3) and inverse inequality (3.2), we derive

$$\begin{aligned} \|\mathbf{w}_h\|^2 + \|d_t \mathbf{u}_{h*}^n - d_t \mathbf{u}_h^n\|^2 &= \|d_t \theta_u^n\|^2 \\ &\quad - 2\delta t s(\mathbf{B}_h^n \times \nabla \times d_t \mathbf{B}_h^{n+1}, \mathbf{w}_h) - 2\delta t s(d_t \mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^n, \mathbf{w}_h) \\ &\lesssim \|d_t \theta_u^n\|^2 + \delta t \|\mathbf{B}_h^n\|_{L^\infty} \|\nabla \times d_t \mathbf{B}_h^{n+1}\| \|\mathbf{w}_h\| + \delta t \|d_t \mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^n\| \|\mathbf{w}_h\| \\ &\lesssim \|d_t \theta_u^n\|^2 + \|\mathbf{w}_h\|^2 + \delta t^2 \|\nabla \times d_t \mathbf{B}_h^{n+1}\|^2 + \delta t^2 \|d_t \mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^n\|^2 \\ &\lesssim \|d_t \theta_u^n\|^2 + \|\mathbf{w}_h\|^2 + \delta t^2 \|\nabla \times d_t \mathbf{B}(t_{n+1})\|^2 + \delta t^2 \|\nabla \times d_t e_b^{n+1}\|^2 \\ &\quad + \delta t^2 \|d_t \mathbf{B}(t_n) \times \nabla \times \mathbf{B}_h^n\|^2 + \delta t^2 \|d_t e_b^n \times \nabla \times \mathbf{B}_h^n\|^2 \\ &\lesssim \|d_t \theta_u^n\|^2 + \|\mathbf{w}_h\|^2 + \delta t^2 + \|\nabla \times (e_b^{n+1} - e_b^n)\|^2 \\ &\quad + \delta t^2 \|d_t \mathbf{B}(t_n)\|_{L^\infty}^2 \|\nabla \times \mathbf{B}_h^n\|^2 + \delta t^2 \|d_t e_b^n\|^2 \|\nabla \times \mathbf{B}_h^n\|_{L^\infty}^2 \\ &\lesssim \|d_t \theta_u^n\|^2 + \|\mathbf{w}_h\|^2 + \delta t^2 + \|\nabla \times (e_b^{n+1} - e_b^n)\|^2 + \delta t^2 \|d_t e_b^n\|^2 h^{-2} \|\nabla \times \mathbf{B}_h^n\|^2 \\ &\lesssim \|d_t \theta_u^n\|^2 + \|\mathbf{w}_h\|^2 + \delta t^2 + \|\nabla \times (e_b^{n+1} - e_b^n)\|^2 + \frac{\eta}{8} \|d_t \theta_b^n\|^2 + \|d_t \rho_b^n\|^2, \end{aligned}$$

where we assume $\delta t^2 \lesssim \frac{\eta}{8} h^2$. After eliminating $\|\mathbf{w}_h\|^2$, we arrive at the conclusion. □

We apply d_t to (4.5)–(4.7) to obtain the following error equations:

$$\begin{aligned} &\left(\frac{d_t \tilde{e}_u^{n+1} - d_t e_u^n}{\delta t}, \mathbf{v}_h \right) + \nu(\nabla d_t \tilde{e}_u^{n+1}, \nabla \mathbf{v}_h) + b(\mathbf{u}(t_n), d_t \mathbf{u}(t_{n+1}), \mathbf{v}_h) + b(d_t \mathbf{u}(t_n), \mathbf{u}(t_n), \mathbf{v}_h) \\ &\quad - b(\mathbf{u}_h^n, d_t \tilde{\mathbf{u}}_h^{n+1}, \mathbf{v}_h) - b(d_t \mathbf{u}_h^n, \tilde{\mathbf{u}}_h^n, \mathbf{v}_h) + (\nabla d_t e_p^n, \mathbf{v}_h) \end{aligned}$$

$$\begin{aligned}
 &+ s(\mathbf{B}(t_n) \times \nabla \times d_t \mathbf{B}(t_{n+1}), \mathbf{v}_h) + s(d_t \mathbf{B}(t_n) \times \nabla \times \mathbf{B}(t_n), \mathbf{v}_h) \\
 &- s(\mathbf{B}_h^n \times \nabla \times d_t \mathbf{B}_h^{n+1}, \mathbf{v}_h) - s(d_t \mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^n, \mathbf{v}_h) \\
 &= (d_t R_u^{n+1}, \mathbf{v}_h).
 \end{aligned} \tag{4.29}$$

$$\frac{d_t e_u^{n+1}}{\delta t} + \nabla d_t e_p^{n+1} = \frac{d_t \tilde{e}_u^{n+1}}{\delta t} + \nabla d_t e_p^n + d_t R_p^{n+1}. \tag{4.30}$$

$$\begin{aligned}
 &\left(\frac{d_t e_b^{n+1} - d_t e_b^n}{\delta t}, \mathbf{c}_h \right) + \eta(\nabla \times d_t e_b^{n+1}, \nabla \times \mathbf{c}_h) + \eta(\nabla \cdot d_t e_b^{n+1}, \nabla \cdot \mathbf{c}_h) \\
 &+ (d_t \mathbf{B}(t_n) \times \mathbf{u}(t_n), \nabla \times \mathbf{c}_h) + (\mathbf{B}(t_{n-1}) \times d_t \mathbf{u}(t_n), \nabla \times \mathbf{c}_h) \\
 &- (d_t \mathbf{B}_h^n \times \mathbf{u}_{h*}^n, \nabla \times \mathbf{c}_h) - (\mathbf{B}_h^{n-1} \times d_t \mathbf{u}_{h*}^n, \nabla \times \mathbf{c}_h) \\
 &= (d_t R_b^{n+1}, \mathbf{c}_h).
 \end{aligned} \tag{4.31}$$

We assume the solution $\mathbf{u}, p, \mathbf{B}$ to (2.1)–(2.6) also satisfy the following regularities:

$$\begin{aligned}
 \text{(A4)} : & \mathbf{u}_{tt}, \mathbf{B}_{tt} \in L^2(0, T; H^1(\Omega)^2), \\
 & p_{tt} \in L^2(0, T; H^1(\Omega)), \quad \mathbf{u}_{ttt}, \mathbf{B}_{ttt} \in L^2(0, T; L^2(\Omega)^2), \\
 \text{(A5)} : & \mathbf{u}_t \in L^\infty(0, T; H^1(\Omega)^2 \cap H^2(\Omega)^2), \\
 & p_t \in L^\infty(0, T; H^1 \cap H^{l-1}), \quad \mathbf{B}_t \in L^\infty(0, T; H^r(\Omega)^2).
 \end{aligned}$$

For the truncation errors in (4.29)–(4.31), we have the following Lemma.

Lemma 4.8 *Under the assumptions (A0), (A1) and (A4), the truncation errors satisfy*

$$\delta t \sum_{n=0}^{\lceil \frac{T}{\delta t} \rceil} (\|d_t R_u^n\|^2 + \|d_t R_b^n\|^2 + \|d_t R_p^n\|^2) \lesssim \delta t^2. \tag{4.32}$$

Proof Due to the page limit, we omit the proof since it is rather standard. □

We define a symmetric positive definite operator $A_h : \mathbf{C}_h \rightarrow \mathbf{C}_h$ through

$$(A_h \mathbf{B}_h, \mathbf{c}_h) = (\nabla \times \mathbf{B}_h, \nabla \times \mathbf{c}_h) + (\nabla \cdot \mathbf{B}_h, \nabla \cdot \mathbf{c}_h).$$

Moreover, we denote

$$\|\mathbf{B}_h\|_{-1}^2 = (A_h^{-1} \mathbf{B}_h, \mathbf{B}_h),$$

and it is easy to find $\|A_h^{-1} \mathbf{B}_h\| \lesssim \|\mathbf{B}_h\|_{-1} \lesssim \|\mathbf{B}_h\|$.

We will estimate the errors of $\|d_t \theta_b^1\|_{-1}$ and $\|d_t \theta_u^1\|$ at the initial time step in the following Lemma, that are needed to estimate $\|d_t \theta_b^n\|_{-1}$ and $\|d_t \theta_u^n\|$, both of which are crucial to derive the optimal error estimate for pressure.

Lemma 4.9 *Under the conditions of Lemma 4.3 and (A5), the following three estimates hold*

$$\begin{aligned}
 \text{(i)} & \|d_t \theta_b^1\|_{-1}^2 + \delta t \|d_t \theta_b^1\|^2 \lesssim h^{2l} + h^{2r} + \delta t^2, \\
 \text{(ii)} & \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \frac{1}{\delta t} \|\nabla \tilde{\theta}_u^1\|^2 \lesssim h^{2l} + h^{2r} + \delta t^2, \\
 \text{(iii)} & \|d_t \theta_u^1\|^2 + \delta t^2 \|\nabla d_t \theta_p^1\|^2 \lesssim h^{2l} + h^{2r} + \delta t^2.
 \end{aligned}$$

Proof (i). By taking $n = 0$ in (4.5) and setting $c_h = A_h^{-1} \frac{\theta_b^1}{\delta t}$, from $\theta_b^0 = 0$, we obtain

$$\begin{aligned} & \left\| \frac{\theta_b^1}{\delta t} \right\|_{-1}^2 + \frac{\eta}{\delta t} \|\theta_b^1\|^2 + \left(d_t \rho_b^1, A_h^{-1} \frac{\theta_b^1}{\delta t} \right) \\ & + \left(\mathbf{B}_0 \times \mathbf{u}_0, \nabla \times A_h^{-1} \frac{\theta_b^1}{\delta t} \right) - \left(\mathbf{B}_h^0 \times \mathbf{u}_{h*}^0, \nabla \times A_h^{-1} \frac{\theta_b^1}{\delta t} \right) = \left(\mathbf{R}_b^1, A_h^{-1} \frac{\theta_b^1}{\delta t} \right). \end{aligned} \tag{4.33}$$

We estimate the inner product terms in (4.33) as follows.

$$\begin{aligned} \left| \left(d_t \rho_b^1, A_h^{-1} \frac{\theta_b^1}{\delta t} \right) \right| & \leq \|d_t \rho_b^1\| \left\| A_h^{-1} \frac{\theta_b^1}{\delta t} \right\| \lesssim \|d_t \rho_b^1\| \left\| \frac{\theta_b^1}{\delta t} \right\|_{-1} \lesssim \frac{1}{6} \left\| \frac{\theta_b^1}{\delta t} \right\|_{-1}^2 + \|d_t \rho_b^1\|^2. \\ \left| \left(\mathbf{R}_b^1, A_h^{-1} \frac{\theta_b^1}{\delta t} \right) \right| & \leq \|\mathbf{R}_b^1\| \left\| A_h^{-1} \frac{\theta_b^1}{\delta t} \right\| \lesssim \|\mathbf{R}_b^1\| \left\| \frac{\theta_b^1}{\delta t} \right\|_{-1} \lesssim \frac{1}{6} \left\| \frac{\theta_b^1}{\delta t} \right\|_{-1}^2 + \|\mathbf{R}_b^1\|^2. \end{aligned}$$

From the Lemma 4.2 and (4.27), we have $\|\mathbf{u}_0 - \mathbf{u}_{h*}^0\|^2 \leq \delta t^2 + h^{2l+2} + h^{2r+2}$. Thus, we obtain

$$\begin{aligned} & \left| \left(\mathbf{B}_0 \times \mathbf{u}_0, \nabla \times A_h^{-1} \frac{\theta_b^1}{\delta t} \right) - \left(\mathbf{B}_h^0 \times \mathbf{u}_{h*}^0, \nabla \times A_h^{-1} \frac{\theta_b^1}{\delta t} \right) \right| \\ & \leq \left| \left(\mathbf{e}_b^0 \times \mathbf{u}_0, \nabla \times A_h^{-1} \frac{\theta_b^1}{\delta t} \right) \right| + \left| \left(\mathbf{B}_h^0 \times (\mathbf{u}_0 - \mathbf{u}_{h*}^0), \nabla \times A_h^{-1} \frac{\theta_b^1}{\delta t} \right) \right| \\ & \leq \|e_b^0\| \|\mathbf{u}_0\|_{L^\infty} \left\| \frac{\theta_b^1}{\delta t} \right\|_{-1} + \|\mathbf{B}_h^0\|_{L^\infty} \|\mathbf{u}_0 - \mathbf{u}_{h*}^0\| \left\| \frac{\theta_b^1}{\delta t} \right\|_{-1} \\ & \lesssim \|e_b^0\| \left\| \frac{\theta_b^1}{\delta t} \right\|_{-1} + \|\mathbf{u}_0 - \mathbf{u}_{h*}^0\| \left\| \frac{\theta_b^1}{\delta t} \right\|_{-1} \lesssim \frac{1}{6} \left\| \frac{\theta_b^1}{\delta t} \right\|_{-1}^2 + \delta t^2 + h^{2l+2} + h^{2r+2}. \end{aligned}$$

Combining the above estimates with (4.33) and using Lemma 4.1 and (3.7), we deduce

$$\frac{1}{2} \left\| \frac{\theta_b^1}{\delta t} \right\|_{-1}^2 + \frac{\eta}{\delta t} \|\theta_b^1\|^2 \lesssim \|d_t \rho_b^1\|^2 + \delta t^2 + h^{2l+2} + h^{2r+2} + \|\mathbf{R}_b^1\|^2 \lesssim \delta t^2 + h^{2l} + h^{2r}, \tag{4.34}$$

which leads to the conclusion (i).

(ii). By setting $n = 0$ in (4.6) and taking $\mathbf{v}_h = \frac{1}{\delta t} \tilde{\theta}_u^1$, since $\theta_u^0 = \theta_p^0 = 0$, we obtain

$$\begin{aligned} & \left(d_t \rho_u^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) + \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \frac{\nu}{\delta t} \|\nabla \tilde{\theta}_u^1\|^2 + b \left(\mathbf{u}_0, \mathbf{u}(t_1), \frac{\tilde{\theta}_u^1}{\delta t} \right) - b \left(\mathbf{u}_h^0, \tilde{\mathbf{u}}_h^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) \\ & + s \left(\mathbf{B}_0 \times \nabla \times \mathbf{B}(t_1), \frac{\tilde{\theta}_u^1}{\delta t} \right) - s \left(\mathbf{B}_h^0 \times \nabla \times \mathbf{B}_h^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) \\ & - \left(\nabla(\rho_p^1 - \rho_p^0), \frac{\tilde{\theta}_u^1}{\delta t} \right) = \left(\mathbf{R}_u^1, \frac{\tilde{\theta}_u^1}{\delta t} \right). \end{aligned} \tag{4.35}$$

The inner product terms in (4.35) can be estimated as follows. By using (3.6) and regularity conditions, we get

$$\left| \left(d_t \rho_u^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) \right| \leq \|d_t \rho_u^1\| \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\| \lesssim \frac{1}{10} \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \|d_t \rho_u^1\|^2 \lesssim \frac{1}{10} \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + h^{2l}.$$

By using (3.20) and (3.6), we get

$$\begin{aligned} & \left| b \left(\mathbf{u}_0, \mathbf{u}(t_1), \frac{\tilde{\theta}_u^1}{\delta t} \right) - b \left(\mathbf{u}_h^0, \tilde{\mathbf{u}}_h^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) \right| \leq \left| b \left(\mathbf{e}_u^0, \mathbf{u}(t_1), \frac{\tilde{\theta}_u^1}{\delta t} \right) \right| + \left| b \left(\mathbf{u}_h^0, \rho_u^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) \right| \\ & \lesssim \|\nabla \mathbf{e}_u^0\| \|\mathbf{u}(t_1)\|_2 \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\| + (\|\mathbf{u}_h^0\|_{L^\infty} + \|\nabla \cdot \mathbf{u}_h^0\|_{L^3}) \|\nabla \rho_u^1\| \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\| \\ & \lesssim \frac{1}{10} \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \|\nabla \mathbf{e}_u^0\|^2 + \|\nabla \rho_u^1\|^2 \lesssim \frac{1}{10} \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + h^{2l}. \end{aligned}$$

By using the integration by parts and (3.7), we obtain

$$\begin{aligned} & s \left| \left(\mathbf{B}_0 \times \nabla \times \mathbf{B}(t_1), \frac{\tilde{\theta}_u^1}{\delta t} \right) - \left(\mathbf{B}_h^0 \times \nabla \times \mathbf{B}_h^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) \right| \\ & = s \left| \left(\mathbf{e}_b^0 \times \nabla \times \mathbf{B}(t_1), \frac{\tilde{\theta}_u^1}{\delta t} \right) + \left(\mathbf{B}_h^0 \times \nabla \times \mathbf{e}_b^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) \right| \\ & = s \left| \left(\mathbf{e}_b^0 \times \nabla \times \mathbf{B}(t_1), \frac{\tilde{\theta}_u^1}{\delta t} \right) + \left(\mathbf{B}_h^0 \times \nabla \times \theta_b^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) + \left(\mathbf{B}_h^0 \times \nabla \times \rho_b^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) \right| \\ & = s \left| \left(\mathbf{e}_b^0 \times \nabla \times \mathbf{B}(t_1), \frac{\tilde{\theta}_u^1}{\delta t} \right) + \left(\mathbf{B}_h^0 \nabla \frac{\tilde{\theta}_u^1}{\delta t} - \frac{\tilde{\theta}_u^1}{\delta t} \nabla \mathbf{B}_h^0, \theta_b^1 \right) + \left(\mathbf{B}_h^0 \times \nabla \times \rho_b^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) \right| \\ & \lesssim \|\nabla \mathbf{e}_b^0\| \|\mathbf{B}(t_1)\|_2 \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\| + \frac{1}{\delta t} (\|\mathbf{B}_h^0\|_{L^\infty} + \|\nabla \mathbf{B}_h^0\|_{L^3}) \|\nabla \tilde{\theta}_u^1\| \|\theta_b^1\| \\ & \quad + \|\mathbf{B}_h^0\|_{L^\infty} \|\nabla \times \rho_b^1\| \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\| \\ & \lesssim \frac{1}{10} \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \|\nabla \mathbf{e}_b^0\|^2 + \frac{\nu}{2\delta t} \|\nabla \tilde{\theta}_u^1\|^2 + \frac{1}{\delta t} \|\theta_b^1\|^2 + \|\nabla \times \rho_b^1\|^2, \\ & \lesssim \frac{1}{10} \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \frac{\nu}{2\delta t} \|\nabla \tilde{\theta}_u^1\|^2 + \frac{1}{\delta t} \|\theta_b^1\|^2 + h^{2r}. \end{aligned}$$

From the Theorem 3.1 and (A5), we get

$$\begin{aligned} \delta t \left| \left(\nabla d_t \rho_p^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) \right| & \leq \delta t \|\nabla d_t \rho_p^1\| \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\| \lesssim \frac{1}{10} \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \delta t^2 \|\nabla d_t \rho_p^1\|^2 \\ & \lesssim \frac{1}{10} \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \delta t^2 (\|d_t p(t_1)\|_1^2 + \|d_t \mathbf{u}(t_1)\|_2^2) \lesssim \frac{1}{10} \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \delta t^2. \end{aligned}$$

From the Lemma 4.1, we have

$$\left(R_u^1, \frac{\tilde{\theta}_u^1}{\delta t} \right) \leq \|R_u^1\| \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\| \lesssim \|R_u^1\|^2 + \frac{1}{10} \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 \lesssim \frac{1}{10} \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \delta t^2.$$

Combining these estimates with (4.35) and using (4.34), we derive

$$\left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \frac{\nu}{\delta t} \|\nabla \tilde{\theta}_u^1\|^2 \lesssim h^{2l} + h^{2r} + \frac{1}{\delta t} \|\theta_b^1\|^2 + \delta t^2 \lesssim h^{2l} + h^{2r} + \delta t^2, \tag{4.36}$$

which is the conclusion (ii).

(iii) By setting $n = 0$ in (4.7), taking the L^2 inner product of the obtained equation with itself on both sides, and using (4.36), we obtain

$$\left\| \frac{\theta_u^1}{\delta t} \right\|^2 + \|\nabla \theta_p^1\|^2 = \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \|\rho^1\|^2 + 2 \left(\frac{\tilde{\theta}_u^1}{\delta t}, \rho^1 \right) \lesssim \left\| \frac{\tilde{\theta}_u^1}{\delta t} \right\|^2 + \|\rho^1\|^2 \lesssim h^{2l} + h^{2r} + \delta t^2, \tag{4.37}$$

where $\rho^1 = R_p^1 - \delta t \nabla d_t \rho_p^1$, and $\|\rho^1\| \lesssim \delta t \|d_t \rho_p^1\|_1 + \|R_p^1\| \lesssim \delta t + \delta t (\|d_t p(t_1)\|_1 + \|d_t u(t_1)\|_2) \lesssim \delta t$ by using Theorem 3.1 and Lemma 4.1. Since $\theta_u^0 = 0, \theta_p^0 = 0$, we obtain the conclusion (iii). \square

We assume the solution to (2.1)–(2.6) has the following regularity:

$$\begin{aligned} \text{(A6)} : \mathbf{B}_{tt} &\in L^2(0, T; H^r(\Omega)^2), \quad \mathbf{u}_{tt} \in L^2(0, T; H^2(\Omega)^2 \cap H^1(\Omega)^2), \\ p_{tt} &\in L^2(0, T; H^{l-1}(\Omega)). \end{aligned}$$

In the following Lemma, we will obtain the error estimate for $\|d_t \theta_u^n\|$.

Lemma 4.10 Under assumptions (A0)–(A6), when $\delta t |\log h|^{\frac{1}{2}} \lesssim h, \delta t \leq \delta t_0$ and $h \leq h_0$, the following estimate holds for $1 \leq m \leq \lceil \frac{T}{\delta t} \rceil - 1$,

$$\|d_t \theta_u^{m+1}\|^2 + \delta t^2 \|\nabla d_t \theta_p^{m+1}\|^2 + \delta t \sum_{n=1}^m \nu \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 \lesssim \delta t^2 + h^{2l} + h^{2r}.$$

Proof In Step i, we first prove the error estimate of $\|d_t \theta_b^m\|_{-1}^2 + \delta t \sum_{n=1}^m \|d_t \theta_b^n\|^2$ using the negative norm technique; and in Step ii, we establish the error estimate for $\|d_t \theta_u^n\|$ using the result from Step i.

(Step i). By taking $\mathbf{c}_h = A_h^{-1} d_t \theta_b^{n+1}$ in (4.31), we have

$$\begin{aligned} &\frac{1}{2\delta t} (\|d_t \theta_b^{n+1}\|_{-1}^2 - \|d_t \theta_b^n\|_{-1}^2 + \|d_t \theta_b^{n+1} - d_t \theta_b^n\|_{-1}^2) + \eta \|d_t \theta_b^{n+1}\|^2 \\ &= -(d_t \mathbf{B}(t_n) \times \mathbf{u}(t_n), \nabla \times A_h^{-1} d_t \theta_b^{n+1}) + (d_t \mathbf{B}_h^n \times \mathbf{u}_{h^*}^n, \nabla \times A_h^{-1} d_t \theta_b^{n+1}) \\ &\quad - (\mathbf{B}(t_{n-1}) \times d_t \mathbf{u}(t_n), \nabla \times A_h^{-1} d_t \theta_b^{n+1}) + (\mathbf{B}_h^{n-1} \times d_t \mathbf{u}_{h^*}^n, \nabla \times A_h^{-1} d_t \theta_b^{n+1}) \\ &\quad - (d_{tt} \rho_b^{n+1}, A_h^{-1} d_t \theta_b^{n+1}) + (d_t R_b^{n+1}, A_h^{-1} d_t \theta_b^{n+1}). \end{aligned} \tag{4.38}$$

We estimate the terms on the right hand side as follows.

By using Lemmas 4.2 and 4.6 and Theorem 4.4, we obtain $\|\mathbf{u}(t_n) - \mathbf{u}_{h^*}^n\|^2 \lesssim \delta t^2 + h^{2l+2} + h^{2r+2}$. Hence from the inverse inequality (3.2) and (3.7), we derive

$$\begin{aligned} & |(d_t \mathbf{B}(t_n) \times \mathbf{u}(t_n), \nabla \times A_h^{-1} d_t \theta_b^{n+1}) - (d_t \mathbf{B}_h^n \times \mathbf{u}_{h^*}^n, \nabla \times A_h^{-1} d_t \theta_b^{n+1})| \\ &= |(d_t e_b^n \times \mathbf{u}(t_n), \nabla \times A_h^{-1} d_t \theta_b^{n+1}) + (d_t \mathbf{B}_h^n \times (\mathbf{u}(t_n) - \mathbf{u}_{h^*}^n), \nabla \times A_h^{-1} d_t \theta_b^{n+1})| \\ &= |(d_t e_b^n \times \mathbf{u}(t_n), \nabla \times A_h^{-1} d_t \theta_b^{n+1}) + ((d_t \mathbf{B}(t_n) - d_t e_b^n) \times (\mathbf{u}(t_n) - \mathbf{u}_{h^*}^n), \nabla \times A_h^{-1} d_t \theta_b^{n+1})| \\ &\lesssim \|d_t e_b^n\| \|\mathbf{u}(t_n)\|_{L^\infty} \|d_t \theta_b^{n+1}\|_{-1} + \|d_t \mathbf{B}(t_n)\|_{L^\infty} \|\mathbf{u}(t_n) - \mathbf{u}_{h^*}^n\| \|d_t \theta_b^{n+1}\|_{-1} \\ &\quad + \|d_t e_b^n\| \|\mathbf{u}(t_n) - \mathbf{u}_{h^*}^n\| \|\nabla \times A_h^{-1} d_t \theta_b^{n+1}\|_{L^\infty} \\ &\lesssim \frac{\eta}{8} \|d_t e_b^n\|^2 + \|d_t \theta_b^{n+1}\|_{-1}^2 + \|\mathbf{u}(t_n) - \mathbf{u}_{h^*}^n\|^2 + \|\mathbf{u}(t_n) - \mathbf{u}_{h^*}^n\|^2 \|d_t \theta_b^{n+1}\|_{-1}^2 h^{-2} \\ &\lesssim \frac{\eta}{8} \|d_t \theta_b^n\|^2 + \|d_t \rho_b^n\|^2 + \|d_t \theta_b^{n+1}\|_{-1}^2 + \delta t^2 + h^{2l+2} + h^{2r+2} \\ &\lesssim \frac{\eta}{8} \|d_t \theta_b^n\|^2 + \|d_t \theta_b^{n+1}\|_{-1}^2 + \delta t^2 + h^{2l+2} + h^{2r}. \end{aligned}$$

By using Lemmas 4.7, 4.6, Theorem 4.4, (3.6), and (3.7), we derive the following three inequalities:

$$\begin{aligned} & |(\mathbf{B}(t_{n-1}) \times d_t \mathbf{u}(t_n), \nabla \times A_h^{-1} d_t \theta_b^{n+1}) - (\mathbf{B}_h^{n-1} \times d_t \mathbf{u}_{h^*}^n, \nabla \times A_h^{-1} d_t \theta_b^{n+1})| \\ &= |(e_b^{n-1} \times d_t \mathbf{u}(t_n), \nabla \times A_h^{-1} d_t \theta_b^{n+1}) + (\mathbf{B}_h^{n-1} \times d_t (\mathbf{u}(t_n) - \mathbf{u}_{h^*}^n), \nabla \times A_h^{-1} d_t \theta_b^{n+1})| \\ &\leq \|e_b^{n-1}\| \|d_t \mathbf{u}(t_n)\|_{L^\infty} \|d_t \theta_b^{n+1}\|_{-1} + \|\mathbf{B}_h^{n-1}\|_{L^\infty} \|d_t (\mathbf{u}(t_n) - \mathbf{u}_{h^*}^n)\| \|d_t \theta_b^{n+1}\|_{-1} \\ &\lesssim \|e_b^{n-1}\|^2 + \|d_t \theta_b^{n+1}\|_{-1}^2 + (\|d_t e_u^n\| + \|d_t (\mathbf{u}_h^n - \mathbf{u}_{h^*}^n)\|) \|d_t \theta_b^{n+1}\|_{-1} \\ &\lesssim \|e_b^{n-1}\|^2 + \|d_t \theta_b^{n+1}\|_{-1}^2 + \|d_t \theta_u^n\|^2 + \|d_t \rho_u^n\|^2 + \delta t^2 + \|\nabla \\ &\quad \times (e_b^{n+1} - e_b^n)\|^2 + \frac{\eta}{8} \|d_t \theta_b^n\|^2 + \|d_t \rho_b^n\|^2 \\ &\lesssim \|d_t \theta_b^{n+1}\|_{-1}^2 + \|d_t \theta_u^n\|^2 + \|\nabla \\ &\quad \times (e_b^{n+1} - e_b^n)\|^2 + \frac{\eta}{8} \|d_t \theta_b^n\|^2 + \delta t^2 + h^{2r} + h^{2l}, \\ & (d_t R_b^{n+1}, A_h^{-1} d_t \theta_b^{n+1}) \leq \|d_t R_b^{n+1}\| \|A_h^{-1} d_t \theta_b^{n+1}\| \\ &\lesssim \|d_t R_b^{n+1}\| \|d_t \theta_b^{n+1}\| \lesssim \frac{\eta}{8} \|d_t \theta_b^{n+1}\|^2 + \|d_t R_b^{n+1}\|^2, \end{aligned}$$

and

$$|(d_{tt} \rho_b^{n+1}, A_h^{-1} d_t \theta_b^{n+1})| \leq \|d_{tt} \rho_b^{n+1}\| \|A_h^{-1} d_t \theta_b^{n+1}\| \lesssim \|d_{tt} \rho_b^{n+1}\|^2 + \frac{\eta}{8} \|d_t \theta_b^{n+1}\|^2.$$

Combining the above estimates with (4.38), we obtain

$$\begin{aligned} & \|d_t \theta_b^{n+1}\|_{-1}^2 - \|d_t \theta_b^n\|_{-1}^2 + \eta \delta t \|d_t \theta_b^{n+1}\|^2 + \frac{\eta}{2} \delta t (\|d_t \theta_b^{n+1}\|^2 - \|d_t \theta_b^n\|^2) \\ &\lesssim \delta t (\delta t^2 + h^{2l} + h^{2r} + \|d_{tt} \rho_b^{n+1}\|^2) \\ &\quad + \delta t (\|\nabla \times e_b^{n+1}\|^2 + \|\nabla \times e_b^n\|^2) \\ &\quad + \delta t \|d_t \theta_b^{n+1}\|_{-1}^2 + \delta t \|d_t \theta_u^n\|^2 + \delta t \|d_t R_b^{n+1}\|^2. \end{aligned}$$

Summing above inequality from $n = 1$ to m , using Theorem 4.4, Lemma 4.8, and (i) of Lemma 4.9, we derive

$$\begin{aligned} & \|d_t \theta_b^{m+1}\|_{-1}^2 + \eta \delta t \sum_{n=1}^m \|d_t \theta_b^{n+1}\|^2 \lesssim \delta t \sum_{n=1}^m \|d_{tt} \rho_b^{n+1}\|^2 \\ & + \delta t \sum_{n=1}^m \|d_t \theta_b^{n+1}\|_{-1}^2 + \delta t \sum_{n=1}^m \|d_t \theta_u^n\|^2 \\ & + \|d_t \theta_b^1\|_{-1}^2 + \delta t \|d_t \theta_b^1\|^2 + \delta t^2 + h^{2l} + h^{2r} \\ & \lesssim \delta t \sum_{n=1}^m \|d_t \theta_b^{n+1}\|_{-1}^2 + \delta t \sum_{n=1}^m \|d_t \theta_u^n\|^2 + \delta t^2 + h^{2l} + h^{2r}. \end{aligned}$$

Therefore, by Gronwall’s inequality, we obtain if $\delta t \leq \delta t_0$,

$$\|d_t \theta_b^{m+1}\|_{-1}^2 + \eta \delta t \sum_{n=1}^m \|d_t \theta_b^{n+1}\|^2 \lesssim \delta t \sum_{n=1}^m \|d_t \theta_u^n\|^2 + \delta t^2 + h^{2l} + h^{2r}. \tag{4.39}$$

(Step ii). By taking $\mathbf{v}_h = d_t \tilde{\theta}_u^{n+1}$ in (4.29), we have

$$\begin{aligned} & \frac{1}{2\delta t} (\|d_t \tilde{\theta}_u^{n+1}\|^2 - \|d_t \theta_u^n\|^2 + \|d_t \tilde{\theta}_u^{n+1} - d_t \theta_u^n\|^2) + \nu \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 + b(\mathbf{u}(t_n), d_t \mathbf{u}(t_{n+1}), d_t \tilde{\theta}_u^{n+1}) \\ & - b(\mathbf{u}_h^n, d_t \tilde{\mathbf{u}}_h^{n+1}, d_t \tilde{\theta}_u^{n+1}) + b(d_t \mathbf{u}(t_n), \mathbf{u}(t_n), d_t \tilde{\theta}_u^{n+1}) - b(d_t \mathbf{u}_h^n, \tilde{\mathbf{u}}_h^n, d_t \tilde{\theta}_u^{n+1}) \\ & + s(\mathbf{B}(t_n) \times \nabla \times d_t \mathbf{B}(t_{n+1}), d_t \tilde{\theta}_u^{n+1}) - s(\mathbf{B}_h^n \times \nabla \times d_t \mathbf{B}_h^{n+1}, d_t \tilde{\theta}_u^{n+1}) \\ & + s(d_t \mathbf{B}(t_n) \times \nabla \times \mathbf{B}(t_n), d_t \tilde{\theta}_u^{n+1}) - s(d_t \mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^n, d_t \tilde{\theta}_u^{n+1}) + (d_{tt} \rho_u^{n+1}, d_t \tilde{\theta}_u^{n+1}) \\ & + (\nabla d_t \theta_p^n, d_t \tilde{\theta}_u^{n+1}) + (\nabla d_t \rho_p^n - \nabla d_t \rho_p^{n+1}, d_t \tilde{\theta}_u^{n+1}) = (d_t \mathbf{R}_u^{n+1}, d_t \tilde{\theta}_u^{n+1}). \end{aligned} \tag{4.40}$$

From (4.30), we derive

$$\frac{d_t \theta_u^{n+1}}{\delta t} + \nabla d_t \theta_p^{n+1} = \frac{d_t \tilde{\theta}_u^{n+1}}{\delta t} + \nabla d_t \theta_p^n + d_t \rho^{n+1}, \tag{4.41}$$

where $d_t \rho^{n+1} = \nabla d_t \rho_p^n - \nabla d_t \rho_p^{n+1} + d_t \mathbf{R}_p^{n+1}$.

By taking the L^2 inner product of (4.41) with itself on both sides, we obtain

$$\begin{aligned} & \frac{1}{2\delta t} (\|d_t \theta_u^{n+1}\|^2 - \|d_t \tilde{\theta}_u^{n+1}\|^2) + \frac{\delta t}{2} (\|\nabla d_t \theta_p^{n+1}\|^2 - \|\nabla d_t \theta_p^n\|^2) - (d_t \tilde{\theta}_u^{n+1}, d_t \rho^{n+1}) \\ & - \delta t (\nabla d_t \theta_p^n, d_t \rho^{n+1}) - \frac{\delta t}{2} \|d_t \rho^{n+1}\|^2 = (d_t \tilde{\theta}_u^{n+1}, \nabla d_t \theta_p^n). \end{aligned} \tag{4.42}$$

Combining (4.40) and (4.42), we get

$$\begin{aligned}
 & \frac{1}{2\delta t} (\|d_t \theta_u^{n+1}\|^2 - \|d_t \theta_u^n\|^2 + \|d_t \tilde{\theta}_u^{n+1} - d_t \theta_u^n\|^2) + \nu \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 \\
 & + \frac{\delta t}{2} (\|\nabla d_t \theta_p^{n+1}\|^2 - \|\nabla d_t \theta_p^n\|^2) \\
 & + b(\mathbf{u}(t_n), d_t \mathbf{u}(t_{n+1}), d_t \tilde{\theta}_u^{n+1}) - b(\mathbf{u}_h^n, d_t \tilde{\mathbf{u}}_h^{n+1}, d_t \tilde{\theta}_u^{n+1}) \tag{I} \\
 & + b(d_t \mathbf{u}(t_n), \mathbf{u}(t_n), d_t \tilde{\theta}_u^{n+1}) - b(d_t \mathbf{u}_h^n, \tilde{\mathbf{u}}_h^n, d_t \tilde{\theta}_u^{n+1}) \tag{II} \\
 & + s(\mathbf{B}(t_n) \times \nabla \times d_t \mathbf{B}(t_{n+1}), d_t \tilde{\theta}_u^{n+1}) - s(\mathbf{B}_h^n \times \nabla \times d_t \mathbf{B}_h^{n+1}, d_t \tilde{\theta}_u^{n+1}) \tag{III} \\
 & + s(d_t \mathbf{B}(t_n) \times \nabla \times \mathbf{B}(t_n), d_t \tilde{\theta}_u^{n+1}) - s(d_t \mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^n, d_t \tilde{\theta}_u^{n+1}) \tag{IV} \\
 & + (d_{tt} \rho_u^{n+1}, d_t \tilde{\theta}_u^{n+1}) + (\nabla d_t \rho_p^n - \nabla d_t \rho_p^{n+1}, d_t \tilde{\theta}_u^{n+1}) \tag{V} \\
 & = (d_t R_u^{n+1}, d_t \tilde{\theta}_u^{n+1}) + \delta t (\nabla d_t \theta_p^n, d_t \rho^{n+1}) \\
 & + \frac{\delta t}{2} \|d_t \rho^{n+1}\|^2 + (d_t \tilde{\theta}_u^{n+1}, d_t \rho^{n+1}). \tag{VI}
 \end{aligned}
 \tag{4.43}$$

We estimate the terms I–VI as follows.

For term I, by using (3.20), Theorem 4.4, and the inverse inequality (3.2), we derive

$$\begin{aligned}
 |\text{I}| &= |b(\mathbf{u}(t_n), d_t \mathbf{u}(t_{n+1}), d_t \tilde{\theta}_u^{n+1}) - b(\mathbf{u}_h^n, d_t \tilde{\mathbf{u}}_h^{n+1}, d_t \tilde{\theta}_u^{n+1})| \\
 &= |b(e_u^n, d_t \mathbf{u}(t_{n+1}), d_t \tilde{\theta}_u^{n+1}) + b(\mathbf{u}_h^n, d_t \rho_u^{n+1}, d_t \tilde{\theta}_u^{n+1})| \\
 &= |b(e_u^n, d_t \mathbf{u}(t_{n+1}), d_t \tilde{\theta}_u^{n+1}) - b(e_u^n, d_t \rho_u^{n+1}, d_t \tilde{\theta}_u^{n+1}) + b(\mathbf{u}(t_n), d_t \rho_u^{n+1}, d_t \tilde{\theta}_u^{n+1})| \\
 &\lesssim \|e_u^n\| \|\nabla d_t \tilde{\theta}_u^{n+1}\| (\|d_t \mathbf{u}(t_{n+1})\|_{L^\infty} + \|\nabla d_t \mathbf{u}(t_{n+1})\|_{L^3}) \\
 &\quad + \|e_u^n\| \|\nabla d_t \rho_u^{n+1}\| (\|d_t \tilde{\theta}_u^{n+1}\|_{L^\infty} \\
 &\quad + \|\nabla d_t \tilde{\theta}_u^{n+1}\|_{L^3}) + \|\mathbf{u}(t_n)\|_{L^\infty} \|d_t \rho_u^{n+1}\| \|\nabla d_t \tilde{\theta}_u^{n+1}\| \\
 &\lesssim \frac{\nu}{12} \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 + \|e_u^n\|^2 \|d_t \mathbf{u}(t_{n+1})\|_2^2 + h^{-\frac{2}{3}} \|e_u^n\|^2 \|d_t \rho_u^{n+1}\|_1^2 + \|d_t \rho_u^{n+1}\|^2 \\
 &\lesssim \frac{\nu}{12} \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 + \delta t^2 + h^{2r} + h^{2l} + \|d_t \rho_u^{n+1}\|_1^2.
 \end{aligned}
 \tag{4.44}$$

For term II, by applying the inverse inequality (3.2) and (3.6), we get

$$\begin{aligned}
 |\text{II}| &= |b(d_t \mathbf{u}(t_n), \mathbf{u}(t_n), d_t \tilde{\theta}_u^{n+1}) - b(d_t \mathbf{u}_h^n, \tilde{\mathbf{u}}_h^n, d_t \tilde{\theta}_u^{n+1})| \\
 &= |b(d_t e_u^n, \mathbf{u}(t_n), d_t \tilde{\theta}_u^{n+1}) + b(d_t \mathbf{u}_h^n, \tilde{e}_u^n, d_t \tilde{\theta}_u^{n+1})| \\
 &= |b(d_t e_u^n, \mathbf{u}(t_n), d_t \tilde{\theta}_u^{n+1}) - b(d_t e_u^n, \tilde{e}_u^n, d_t \tilde{\theta}_u^{n+1}) + b(d_t \mathbf{u}(t_n), \tilde{e}_u^n, d_t \tilde{\theta}_u^{n+1})| \\
 &= |b(d_t e_u^n, \mathbf{u}(t_n), d_t \tilde{\theta}_u^{n+1}) - b(d_t e_u^n, \tilde{\theta}_u^n + \rho_u^n, d_t \tilde{\theta}_u^{n+1}) + b(d_t \mathbf{u}(t_n), \tilde{e}_u^n, d_t \tilde{\theta}_u^{n+1})| \\
 &\lesssim \|d_t e_u^n\| \|\mathbf{u}(t_n)\|_2 \|\nabla d_t \tilde{\theta}_u^{n+1}\| + \|d_t e_u^n\| (\|\nabla \tilde{\theta}_u^n\| \|d_t \tilde{\theta}_u^{n+1}\|_{L^\infty} + \|\tilde{\theta}_u^n\|_{L^\infty} \|\nabla d_t \tilde{\theta}_u^{n+1}\|) \\
 &\quad + \|d_t e_u^n\| (\|\nabla \rho_u^n\| \|d_t \tilde{\theta}_u^{n+1}\|_{L^\infty} + \|\rho_u^n\| \|\nabla d_t \tilde{\theta}_u^{n+1}\|_{L^\infty}) \\
 &\quad + \|d_t \mathbf{u}(t_n)\|_1 \|\nabla \tilde{e}_u^n\| \|\nabla d_t \tilde{\theta}_u^{n+1}\| \\
 &\lesssim \|d_t e_u^n\| \|\nabla d_t \tilde{\theta}_u^{n+1}\| + |\log h|^{\frac{1}{2}} \|d_t e_u^n\| \|\nabla \tilde{\theta}_u^n\| \|\nabla d_t \tilde{\theta}_u^{n+1}\| \\
 &\quad + \|d_t e_u^n\| (h^l h^{-\frac{1}{3}} \|d_t \tilde{\theta}_u^{n+1}\|_{L^6} + h^{l+1} h^{-1} \|\nabla d_t \tilde{\theta}_u^{n+1}\|) + \|\nabla \tilde{e}_u^n\| \|\nabla d_t \tilde{\theta}_u^{n+1}\| \\
 &\lesssim \frac{\nu}{12} \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 + \|d_t e_u^n\|^2 + |\log h| \|\nabla \tilde{\theta}_u^n\|^2 \|d_t e_u^n\|^2 + \|\nabla \tilde{e}_u^n\|^2 \\
 &\lesssim \frac{\nu}{12} \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 + \|d_t \theta_u^n\|^2 + h^{2l} + \|\nabla \tilde{\theta}_u^n\|^2 + |\log h| \|\nabla \tilde{\theta}_u^n\|^2 \|d_t \theta_u^n\|^2 + \|\nabla \tilde{e}_u^n\|^2.
 \end{aligned}
 \tag{4.45}$$

For term III, by using the integration by parts, the inverse inequality (3.2), Theorem 4.4 and (3.7), we derive

$$\begin{aligned}
 |(\text{III})| &= s|(\mathbf{B}(t_n) \times \nabla \times d_t \mathbf{B}(t_{n+1}), d_t \tilde{\theta}_u^{n+1}) - (\mathbf{B}_h^n \times \nabla \times d_t \mathbf{B}_h^{n+1}, d_t \tilde{\theta}_u^{n+1})| \\
 &= s|(e_b^n \times \nabla \times d_t \mathbf{B}(t_{n+1}), d_t \tilde{\theta}_u^{n+1}) + (\mathbf{B}_h^n \times \nabla \times d_t e_b^{n+1}, d_t \tilde{\theta}_u^{n+1})| \\
 &= s|(e_b^n \times \nabla \times d_t \mathbf{B}(t_{n+1}), d_t \tilde{\theta}_u^{n+1}) + ((\mathbf{B}(t_n) - e_b^n) \times \nabla \times d_t e_b^{n+1}, d_t \tilde{\theta}_u^{n+1})| \\
 &\lesssim |(e_b^n \times \nabla \times d_t \mathbf{B}(t_{n+1}), d_t \tilde{\theta}_u^{n+1})| + |(\mathbf{B}(t_n) \nabla d_t \tilde{\theta}_u^{n+1} - d_t \tilde{\theta}_u^{n+1} \nabla \mathbf{B}(t_n), d_t e_b^{n+1})| \\
 &\quad + |(e_b^n \times \nabla \times d_t \theta_b^{n+1}, d_t \tilde{\theta}_u^{n+1})| + |(e_b^n \times \nabla \times d_t \rho_b^{n+1}, d_t \tilde{\theta}_u^{n+1})| \\
 &\lesssim \|e_b^n\|_1 \|d_t \mathbf{B}(t_{n+1})\|_1 \|\nabla d_t \tilde{\theta}_u^{n+1}\| + (\|\mathbf{B}(t_n)\|_{L^\infty} + \|\nabla \mathbf{B}(t_n)\|_{L^3}) \|\nabla d_t \tilde{\theta}_u^{n+1}\| \|d_t e_b^{n+1}\| \\
 &\quad + \|e_b^n\| \|\nabla \times d_t \theta_b^{n+1}\| \|d_t \tilde{\theta}_u^{n+1}\|_{L^\infty} + \|e_b^n\|_1 \|d_t \rho_b^{n+1}\|_1 \|\nabla d_t \tilde{\theta}_u^{n+1}\| \tag{4.46} \\
 &\lesssim \|e_b^n\|_1 \|\nabla d_t \tilde{\theta}_u^{n+1}\| + \|\nabla d_t \tilde{\theta}_u^{n+1}\| \|d_t e_b^{n+1}\| + \|e_b^n\| h^{-1} |\log h|^{\frac{1}{2}} \|d_t \theta_b^{n+1}\| \|\nabla d_t \tilde{\theta}_u^{n+1}\| \\
 &\quad + \|e_b^n\|_1 \|d_t \rho_b^{n+1}\|_1 \|\nabla d_t \tilde{\theta}_u^{n+1}\| \\
 &\lesssim \frac{\nu}{12} \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 + \|e_b^n\|_1^2 + \|d_t e_b^{n+1}\|^2 + |\log h| h^{-2} \|e_b^n\|^2 \|d_t \theta_b^{n+1}\|^2 \\
 &\lesssim \frac{\nu}{12} \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 + \|e_b^n\|_1^2 + \|d_t \theta_b^{n+1}\|^2 + \|d_t \rho_b^{n+1}\|^2 \\
 &\lesssim \frac{\nu}{12} \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 + \|e_b^n\|_1^2 + \|d_t \theta_b^{n+1}\|^2 + h^{2r},
 \end{aligned}$$

where we also use $\|d_t \rho_b^{n+1}\|_1 \lesssim \|d_t \mathbf{B}(t_{n+1})\|_1$ and $\delta t |\log h|^{\frac{1}{2}} \lesssim h$.

For term IV, by using the integration by parts, the inverse inequality (3.2), Theorem 4.4 and (3.7), we derive

$$\begin{aligned}
 |(\text{IV})| &= s|(d_t \mathbf{B}(t_n) \times \nabla \times \mathbf{B}(t_n), d_t \tilde{\theta}_u^{n+1}) - (d_t \mathbf{B}_h^n \times \nabla \times \mathbf{B}_h^n, d_t \tilde{\theta}_u^{n+1})| \\
 &= s|(d_t e_b^n \times \nabla \times \mathbf{B}(t_n), d_t \tilde{\theta}_u^{n+1}) + (d_t \mathbf{B}_h^n \times \nabla \times e_b^n, d_t \tilde{\theta}_u^{n+1})| \\
 &\lesssim |(d_t e_b^n \times \nabla \times \mathbf{B}(t_n), d_t \tilde{\theta}_u^{n+1})| + |(d_t \mathbf{B}(t_n) \times \nabla \times e_b^n, d_t \tilde{\theta}_u^{n+1})| \\
 &\quad + |(d_t \rho_b^n \times \nabla \times e_b^n, d_t \tilde{\theta}_u^{n+1})| + |(d_t \theta_b^n \times \nabla \times e_b^n, d_t \tilde{\theta}_u^{n+1})| \\
 &\lesssim |(d_t e_b^n \times \nabla \times \mathbf{B}(t_n), d_t \tilde{\theta}_u^{n+1})| + |(d_t \mathbf{B}(t_n) \times \nabla \times e_b^n, d_t \tilde{\theta}_u^{n+1})| \\
 &\quad + |(d_t \rho_b^n \times \nabla \times e_b^n, d_t \tilde{\theta}_u^{n+1})| + |(d_t \theta_b^n \nabla d_t \tilde{\theta}_u^{n+1} - d_t \tilde{\theta}_u^{n+1} \nabla d_t \theta_b^n, e_b^n)| \\
 &\lesssim \|d_t e_b^n\| \|\nabla \times \mathbf{B}(t_n)\|_{L^3} \|d_t \tilde{\theta}_u^{n+1}\|_{L^6} + \|d_t \mathbf{B}(t_n)\|_{L^3} \|\nabla \times e_b^n\| \|d_t \tilde{\theta}_u^{n+1}\|_{L^6} \\
 &\quad + \|d_t \rho_b^n\|_{L^3} \|\nabla \times e_b^n\| \|d_t \tilde{\theta}_u^{n+1}\|_{L^6} + (\|d_t \theta_b^n\|_{L^\infty} \|\nabla d_t \tilde{\theta}_u^{n+1}\| \\
 &\quad + \|d_t \tilde{\theta}_u^{n+1}\|_{L^\infty} \|\nabla d_t \theta_b^n\|) \|e_b^n\| \tag{4.47} \\
 &\lesssim \|d_t e_b^n\| \|\nabla d_t \tilde{\theta}_u^{n+1}\| + \|\nabla \times e_b^n\| \|\nabla d_t \tilde{\theta}_u^{n+1}\| + \|d_t \rho_b^n\|_1 \|\nabla \times e_b^n\| \|\nabla d_t \tilde{\theta}_u^{n+1}\| \\
 &\quad + (h^{-1} \|d_t \theta_b^n\| \|\nabla d_t \tilde{\theta}_u^{n+1}\| + |\log h|^{\frac{1}{2}} h^{-1} \|\nabla d_t \tilde{\theta}_u^{n+1}\| \|d_t \theta_b^n\|) \|e_b^n\| \\
 &\lesssim \frac{\nu}{12} \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 + \|d_t e_b^n\|^2 + \|\nabla \times e_b^n\|^2 + |\log h| h^{-2} \|e_b^n\|^2 \|d_t \theta_b^n\|^2 \\
 &\lesssim \frac{\nu}{12} \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 + \|d_t \rho_b^n\|^2 + \|\nabla \times e_b^n\|^2 + \|d_t \theta_b^n\|^2 \\
 &\lesssim \frac{\nu}{12} \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 + \|\nabla \times e_b^n\|^2 + \|d_t \theta_b^n\|^2 + h^{2r}.
 \end{aligned}$$

For terms V and VI, by (2.7), we derive

$$\begin{aligned}
 |(\text{term V})| &\lesssim |(d_{tt}\rho_u^{n+1}, d_t\tilde{\theta}_u^{n+1})| + \delta t |(d_{tt}\rho_p^{n+1}, \nabla \cdot d_t\tilde{\theta}_u^{n+1})| \\
 &\lesssim \|d_{tt}\rho_u^{n+1}\| \|d_t\tilde{\theta}_u^{n+1}\| + \delta t \|d_{tt}\rho_p^{n+1}\| \|\nabla \cdot d_t\tilde{\theta}_u^{n+1}\| \\
 &\lesssim \frac{\nu}{12} \|\nabla d_t\tilde{\theta}_u^{n+1}\|^2 + \|d_{tt}\rho_u^{n+1}\|^2 + \delta t^2 \|d_{tt}\rho_p^{n+1}\|^2,
 \end{aligned}
 \tag{4.48}$$

and

$$\begin{aligned}
 (\text{VI}) &\leq |(d_t R_u^{n+1}, d_t\tilde{\theta}_u^{n+1})| + \delta t |(\nabla d_t\theta_p^n, d_t\rho^{n+1})| + \frac{\delta t}{2} \|d_t\rho^{n+1}\|^2 + |(d_t\tilde{\theta}_u^{n+1}, d_t\rho^{n+1})| \\
 &\lesssim \|d_t R_u^{n+1}\| \|\nabla d_t\tilde{\theta}_u^{n+1}\| + \delta t \|\nabla d_t\theta_p^n\| \|d_t\rho^{n+1}\| \\
 &\quad + \frac{\delta t}{2} \|d_t\rho^{n+1}\|^2 + \|\nabla d_t\tilde{\theta}_u^{n+1}\| \|d_t\rho^{n+1}\| \\
 &\lesssim \frac{\nu}{12} \|\nabla d_t\tilde{\theta}_u^{n+1}\|^2 + \|d_t R_u^{n+1}\|^2 + \delta t^2 \|\nabla d_t\theta_p^n\|^2 + \|d_t\rho^{n+1}\|^2.
 \end{aligned}
 \tag{4.49}$$

Combining (4.44)–(4.49) with (4.43), we obtain

$$\begin{aligned}
 &\|d_t\theta_u^{n+1}\|^2 - \|d_t\theta_u^n\|^2 + \delta t \nu \|\nabla d_t\tilde{\theta}_u^{n+1}\|^2 + \delta t^2 (\|\nabla d_t\theta_p^{n+1}\|^2 - \|\nabla d_t\theta_p^n\|^2) \\
 &\lesssim \delta t (1 + |\log h| \|\nabla\tilde{\theta}_u^n\|^2) \|d_t\theta_u^n\|^2 + \delta t^3 \|\nabla d_t\theta_p^n\|^2 \\
 &\quad + \delta t (\|d_t\theta_b^n\|^2 + \|d_t\theta_b^{n+1}\|^2) \\
 &\quad + \delta t (\|d_t\rho_u^{n+1}\|_1^2 + \delta t^2 + h^{2l} + h^{2r} + \|d_{tt}\rho_u^{n+1}\|^2 + \|d_t\rho^{n+1}\|^2) \\
 &\quad + \delta t^3 \|d_{tt}\rho_p^{n+1}\|^2 + \delta t \|d_t R_u^{n+1}\|^2 + \delta t (\|\nabla\tilde{\theta}_u^n\|^2 + \|\nabla\tilde{e}_u^n\|^2 + \|e_b^n\|_1^2).
 \end{aligned}
 \tag{4.50}$$

By taking the summation of (4.50) from $n = 1$ to m , and using Lemma 4.8, Theorem 4.4, Lemma 4.9, (4.39), and (3.6), (3.3), we obtain

$$\begin{aligned}
 &\|d_t\theta_u^{m+1}\|^2 + \delta t \sum_{n=1}^m \nu \|\nabla d_t\tilde{\theta}_u^{n+1}\|^2 + \delta t^2 \|\nabla d_t\theta_p^{m+1}\|^2 \\
 &\lesssim \delta t \sum_{n=1}^m (1 + |\log h| \|\nabla\tilde{\theta}_u^n\|^2) \|d_t\theta_u^n\|^2 + \delta t^3 \sum_{n=1}^m \|\nabla d_t\theta_p^n\|^2 \\
 &\quad + \delta t \sum_{n=1}^m (\|d_t\theta_b^n\|^2 + \|d_t\theta_b^{n+1}\|^2) \\
 &\quad + \delta t \sum_{n=1}^m (\|d_t\rho_u^{n+1}\|_1^2 + \|d_{tt}\rho_u^{n+1}\|^2 + \|d_t\rho^{n+1}\|^2) \\
 &\quad + \delta t^3 \sum_{n=1}^m \|d_{tt}\rho_p^{n+1}\|^2 + \delta t^2 + h^{2l} + h^{2r} \\
 &\lesssim \delta t \sum_{n=1}^m (1 + |\log h| \|\nabla\tilde{\theta}_u^n\|^2) \|d_t\theta_u^n\|^2 + \delta t^3 \sum_{n=1}^m \|\nabla d_t\theta_p^n\|^2 \\
 &\quad + \delta t \sum_{n=1}^m \|d_t\rho^{n+1}\|^2 + \delta t^2 + h^{2l} + h^{2r}.
 \end{aligned}$$

In addition, by applying the Theorem 3.1 and Lemma 4.8,

$$\begin{aligned} \delta t \sum_{n=1}^m \|d_t \rho^{n+1}\|^2 &\leq \delta t^3 \sum_{n=1}^m \|d_{tt} \rho_p^{n+1}\|_1^2 + \delta t \sum_{n=1}^m \|d_t R_p^{n+1}\|^2 \\ &\lesssim \delta t^3 \sum_{n=1}^m (\|d_{tt} p(t_{n+1})\|_1^2 + \|d_{tt} \mathbf{u}(t_{n+1})\|_2^2) + \delta t^2 \\ &\lesssim \delta t^2. \end{aligned}$$

Therefore, by using (4.27) and the Gronwall’s inequality, we obtain

$$\begin{aligned} \|d_t \theta_u^{m+1}\|^2 + \delta t^2 \|\nabla d_t \theta_p^{m+1}\|^2 + \delta t \sum_{n=1}^m v \|\nabla d_t \tilde{\theta}_u^{n+1}\|^2 \\ \lesssim \delta t \sum_{n=1}^m (1 + |\log h| \|\nabla \tilde{\theta}_u^n\|^2) \|d_t \theta_u^n\|^2 \\ + \delta t^3 \sum_{n=1}^m \|\nabla d_t \theta_p^n\|^2 + \delta t^2 + h^{2l} + h^{2r} \\ \lesssim (\delta t^2 + h^{2l} + h^{2r}) \exp\left(\delta t \sum_{n=1}^m (1 + |\log h| \|\nabla \tilde{\theta}_u^n\|^2)\right) \lesssim \delta t^2 + h^{2l} + h^{2r}, \end{aligned} \tag{4.51}$$

that is the desired result. □

Finally, we are ready to obtain the optimal error estimate for pressure in the next Theorem.

Theorem 4.11 *Under the assumptions of Lemma 4.10, the following estimate holds*

$$\|\nabla \tilde{e}_u^n\| + \|e_p^n\| \lesssim \delta t + h^{\min\{l,r\}}, \quad 1 \leq n \leq \left\lfloor \frac{T}{\delta t} \right\rfloor. \tag{4.52}$$

Proof From the inequality $\|\nabla \tilde{\theta}_u^{n+1}\| - \|\nabla \tilde{\theta}_u^n\| \leq \|\nabla \tilde{\theta}_u^{n+1} - \nabla \tilde{\theta}_u^n\|$ and Lemma 4.10, we derive

$$\begin{aligned} \|\nabla \tilde{\theta}_u^{m+1}\| - \|\nabla \tilde{\theta}_u^1\| &\leq \sum_{n=1}^m \|\nabla \tilde{\theta}_u^{n+1} - \nabla \tilde{\theta}_u^n\| \\ &\leq \left(\sum_{n=1}^m \|\nabla \tilde{\theta}_u^{n+1} - \nabla \tilde{\theta}_u^n\|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^m 1\right)^{\frac{1}{2}} \lesssim \delta t + h^l + h^r. \end{aligned}$$

By applying (ii) of Lemma 4.9, we derive

$$\|\nabla \tilde{\theta}_u^m\| \lesssim \delta t + h^l + h^r, \quad \text{for all } 1 \leq m \leq \left\lfloor \frac{T}{\delta t} \right\rfloor.$$

From the triangle inequality, we have

$$\|\nabla \tilde{e}_u^m\| \leq \|\nabla \tilde{\theta}_u^m\| + \|\nabla \rho_u^m\| \lesssim \delta t + h^l + h^r, \quad \text{for all } 1 \leq m \leq \left\lfloor \frac{T}{\delta t} \right\rfloor. \tag{4.53}$$

By taking the summation of (4.6) and (4.7), we obtain the error equation with respective to pressure p

$$\begin{aligned}
 (\theta_p^{n+1}, \nabla \cdot \mathbf{v}_h) &= (d_t e_u^{n+1}, \mathbf{v}_h) + \nu(\nabla \tilde{e}_u^{n+1}, \nabla \mathbf{v}_h) + b(e_u^n, \mathbf{u}(t_{n+1}), \mathbf{v}_h) + b(\mathbf{u}_h^n, \tilde{e}_u^{n+1}, \mathbf{v}_h) \\
 &\quad + s(e_b^n \times \nabla \times \mathbf{B}(t_{n+1}), \mathbf{v}_h) + s(\mathbf{B}_h^n \times \nabla \times e_b^{n+1}, \mathbf{v}_h) \\
 &\quad - (R_u^{n+1} + R_p^{n+1}, \mathbf{v}_h) - (\rho_p^{n+1}, \nabla \cdot \mathbf{v}_h).
 \end{aligned} \tag{4.54}$$

For the four trilinear terms in (4.54), we estimate them one by one as follows.

$$b(e_u^n, \mathbf{u}(t_{n+1}), \mathbf{v}_h) \lesssim \|e_u^n\| \|\mathbf{u}(t_{n+1})\|_2 \|\nabla \mathbf{v}_h\| \lesssim \|e_u^n\| \|\nabla \mathbf{v}_h\|. \tag{4.55}$$

By applying the inverse inequality (3.2) and Theorem 4.4, we obtain

$$\begin{aligned}
 b(\mathbf{u}_h^n, \tilde{e}_u^{n+1}, \mathbf{v}_h) &= b(\mathbf{u}(t_n), \tilde{e}_u^{n+1}, \mathbf{v}_h) - b(e_u^n, \tilde{e}_u^{n+1}, \mathbf{v}_h) \lesssim \|\nabla \mathbf{u}(t_n)\| \|\nabla \tilde{e}_u^{n+1}\| \|\nabla \mathbf{v}_h\| \\
 &\quad + \|e_u^n\| \|\nabla \tilde{e}_u^{n+1}\| (\|\nabla \mathbf{v}_h\|_{L^3} + \|\mathbf{v}_h\|_{L^\infty}) \\
 &\lesssim \|\nabla \tilde{e}_u^{n+1}\| \|\nabla \mathbf{v}_h\| + h^{-\frac{1}{3}} \|e_u^n\| \|\nabla \tilde{e}_u^{n+1}\| \|\nabla \mathbf{v}_h\| \lesssim \|\nabla \tilde{e}_u^{n+1}\| \|\nabla \mathbf{v}_h\|,
 \end{aligned} \tag{4.56}$$

and

$$s(e_b^n \times \nabla \times \mathbf{B}(t_{n+1}), \mathbf{v}_h) \leq s \|e_b^n\| \|\nabla \times \mathbf{B}(t_{n+1})\|_{L^3} \|\mathbf{v}_h\|_{L^6} \lesssim \|e_b^n\| \|\nabla \mathbf{v}_h\|. \tag{4.57}$$

By using the integration by parts and the inverse inequality (3.2), (3.7) and Theorem 4.4, we get

$$\begin{aligned}
 s(\mathbf{B}_h^n \times \nabla \times e_b^{n+1}, \mathbf{v}_h) &= s(\mathbf{B}(t_n) \times \nabla \times e_b^{n+1}, \mathbf{v}_h) - s(e_b^n \times \nabla \times e_b^{n+1}, \mathbf{v}_h) \\
 &= s(\mathbf{B}(t_n) \nabla \mathbf{v}_h - \mathbf{v}_h \nabla \mathbf{B}(t_n), e_b^{n+1}) - s(e_b^n \times \nabla \times \rho_b^{n+1}, \mathbf{v}_h) - s(e_b^n \times \nabla \times \theta_b^{n+1}, \mathbf{v}_h) \\
 &\lesssim (\|\mathbf{B}(t_n)\|_{L^\infty} + \|\nabla \mathbf{B}(t_n)\|_{L^3}) \|\nabla \mathbf{v}_h\| \|e_b^{n+1}\| + \|e_b^n\| \|\nabla \times \rho_b^{n+1}\| \|\mathbf{v}_h\|_{L^\infty} \\
 &\quad + \|e_b^n\| \|\nabla \times \theta_b^{n+1}\| \|\mathbf{v}_h\|_{L^\infty} \\
 &\lesssim \|\nabla \mathbf{v}_h\| \|e_b^{n+1}\| + \|e_b^n\| \|\nabla \times \rho_b^{n+1}\| |\log h|^{\frac{1}{2}} \|\nabla \mathbf{v}_h\| + \|e_b^n\| \|\theta_b^{n+1}\| h^{-1} |\log h|^{\frac{1}{2}} \|\nabla \mathbf{v}_h\| \\
 &\lesssim \|\nabla \mathbf{v}_h\| \|e_b^{n+1}\| + \|e_b^n\| \|\nabla \mathbf{v}_h\|.
 \end{aligned} \tag{4.58}$$

Combining (4.55)–(4.58) with (4.54), we obtain

$$\begin{aligned}
 (\theta_p^{n+1}, \nabla \cdot \mathbf{v}_h) &\lesssim \|d_t e_u^{n+1}\| \|\nabla \mathbf{v}_h\| + \|\nabla \tilde{e}_u^{n+1}\| \|\nabla \mathbf{v}_h\| \\
 &\quad + \|e_u^n\| \|\nabla \mathbf{v}_h\| + \|e_b^n\| \|\nabla \mathbf{v}_h\| + \|\nabla \mathbf{v}_h\| \|e_b^{n+1}\| \\
 &\quad + \|R_u^{n+1}\| \|\nabla \mathbf{v}_h\| + \|R_p^{n+1}\| \|\nabla \mathbf{v}_h\| + \|\rho_p^{n+1}\| \|\nabla \mathbf{v}_h\|.
 \end{aligned}$$

By applying the inf-sup condition (3.1), Lemma 4.10, (4.53), Theorem 4.4, Lemma 4.1 and (3.6), we obtain

$$\begin{aligned}
 \beta \|\theta_p^{n+1}\| &\lesssim \|d_t e_u^{n+1}\| + \|\nabla \tilde{e}_u^{n+1}\| + \|e_u^n\| + \|e_b^n\| + \|e_b^{n+1}\| \\
 &\quad + \|R_u^{n+1}\| + \|R_p^{n+1}\| + \|\rho_p^{n+1}\| \lesssim \delta t + h^l + h^r.
 \end{aligned}$$

The proof is completed by triangle inequality and (3.6). □

Remark 4.12 For the three dimensional case ($d = 3$), since the inverse inequalities are more restrictive: $\|\mathbf{w}_h\|_{L^\infty} \lesssim h^{-\frac{3}{2}} \|\mathbf{w}_h\|$, $\|\mathbf{w}_h\|_{L^\infty} \lesssim h^{-\frac{1}{2}} \|\mathbf{w}_h\|_{L^6} \lesssim h^{-\frac{1}{2}} \|\mathbf{w}_h\|_1$, Theorem 4.11 is still valid but provided that $\delta t \lesssim h^{\frac{3}{2}}$.

5 Numerical Examples

We now implement various 2D numerical experiments to validate the stability and accuracy of the scheme. We use the inf-sup stable P2/P1 element [6] for the velocity and pressure, and linear element for the magnetic field.

5.1 Accuracy Test

We first perform numerical simulations to test the convergence rates of the proposed scheme. The computational domain is $\Omega = [0, 1] \times [0, 1]$. We assume the following functions

$$\mathbf{u} = (y^5 + t^2, x^5 + t^2), \quad p = 10(2x - 1)(2y - 1)(1 + t^2), \quad \mathbf{B} = (\sin(y) + t^2, \sin(x) + t^2)$$

to be the exact solution, and impose some suitable force fields such that the given solution can satisfy the system. Choose $\nu = \eta = s = 1$. In Table 1, we list the numerical errors between the numerical solution and the exact solution at $T = 1$ with different time step sizes. Since the relation between the spatial grid size h and temporal step size δt to be $\delta t = h^2$, we observe the second order accuracy asymptotically for $\|e_u\|_{L^2}$, $\|e_u\|_{H^1}$, $\|e_p\|_{L^2}$, $\|e_b\|_{L^2}$ and the first order accuracy for $\|e_b\|_{H^1}$, as predicted theoretically.

5.2 Stability Test

We show the evolution of the total free energy in this example. We set the computed domain to be $\Omega = [0, 1]^2$, and the initial conditions for \mathbf{u} , p , \mathbf{B} are

$$\begin{cases} \mathbf{u}^0 = (x^2(x-1)^2y(y-1)(2y-1), -y^2(y-1)^2x(x-1)(2x-1)), & p^0 = 0, \\ \mathbf{B}^0 = (\sin(\pi x) \cos(\pi y), -\sin(\pi y) \cos(\pi x)). \end{cases} \quad (5.1)$$

We test the energy stability over matching time of the proposed scheme under varying physical parameters of $R_e = R_m = 10, 50$. The coupling parameter is fixed as $s = 1$, and mesh size is $h = 1/64$. In Fig. 1, we present the time evolution of the total free energy for four different time steps of $\delta t = 0.05, 0.01, 0.001, 0.0001$ until $T = 5$. We observe that all four energy curves decay monotonically for all time step sizes, which numerically confirms that our algorithm is unconditionally energy stable.

5.3 Hydromagnetic Kelvin–Helmholtz instability

The Kelvin–Helmholtz (K–H) instability in sheared flow configurations is an efficient mechanism to initiate mixing of fluids, transport of momentum and energy, and the development of turbulence. This phenomenon is of interest in investigating a variety of space, astrophysical, and geophysical situations involving sheared plasma flows. Configurations where it is relevant include the interface between the solar wind and the magnetosphere, coronal streamers moving through the solar wind, etc. Since most astrophysical environments are electrically conducting and relevant fluids are likely to be magnetized, it is thus of prime importance to understand the role of magnetic fields in the K–H instability. About the theoretical and numerical study of Hydromagnetic K–H instability, we refer to [5,8,9,12,17,24,27] and the references therein.

We study the occurring of the K–H instability in a single shear flow configuration that is embedded in a uniform flow-aligned magnetic field. The simulation is performed in the

Table 1 The numerical errors and convergence order for $\|e_u\|_{L^2}$, $\|e_u\|_{H^1}$, $\|e_p\|_{L^2}$, $\|e_b\|_{L^2}$, and $\|e_b\|_{H^1}$ at $t = 1$. Here we set $\nu = \eta = s = 1$ and $\delta t = h^2$

δt	h	$\ e_u\ _{L^2}$	Order	$\ e_u\ _{H^1}$	Order	$\ e_p\ _{L^2}$	Order	$\ e_b\ _{L^2}$	Order	$\ e_b\ _{H^1}$	Order
$1/4^2$	1/4	4.03e-2	–	5.43e-1	–	1.53	–	9.67e-3	–	7.17e-2	–
$1/8^2$	1/8	3.56e-3	3.50	8.22e-2	2.72	2.03e-1	2.91	3.50e-3	1.47	3.19e-2	1.16
$1/16^2$	1/16	2.47e-4	3.85	1.01e-2	3.02	2.63e-2	2.94	9.50e-4	1.88	1.41e-2	1.17
$1/32^2$	1/32	2.25e-5	3.45	1.60e-3	2.65	5.30e-3	2.31	2.42e-4	1.97	6.77e-3	1.06
$1/64^2$	1/64	4.30e-6	2.39	3.55e-4	2.17	1.30e-3	2.02	6.09e-05	1.99	3.34e-3	1.01
$1/128^2$	1/128	1.05e-6	2.02	8.73e-5	2.02	3.25e-4	2.00	1.52e-5	2.00	1.66e-3	1.00

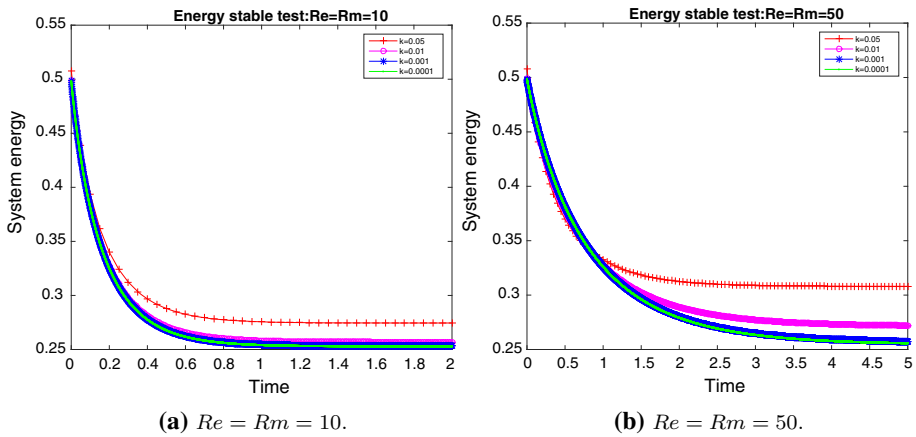


Fig. 1 Time evolution of the free energy functional till $T = 5$ for four different time steps and two sets of order parameters **a** $R_e = R_m = 10$ and **b** $R_e = R_m = 50$. The energy curves show the decays for all time steps $\delta t = 0.05, 0.01, 0.001, 0.0001$

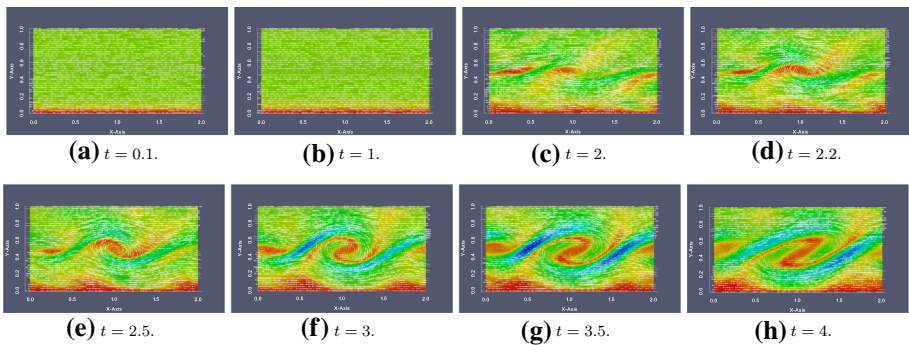


Fig. 2 The dynamical behaviors of the magnetic field together with the velocity field that shows the hydromagnetic K–H instability. Snapshots of the numerical approximation are taken at $t = 0.1, 1, 2, 2.2, 2.5, 3, 3.5, 4$

computed domain of $[0, 2] \times [0, 1]$. The initial velocity field is $\mathbf{u}_0 = (1.5, 0)$ in the top half domain, and $\mathbf{u}_0 = (-1.5, 0)$ in the bottom half domain. The sheared initial magnetic field is $\mathbf{B}_0 = (\tanh(y/\epsilon), 0)$ where $\epsilon = 0.07957747154595$ (cf. [9]). The velocity \mathbf{u} , magnetic field \mathbf{B} and pressure p are periodic boundary conditions on left and right boundaries. On the top and bottom boundary, the second component v of the velocity field $\mathbf{u} = (u, v)$ is imposed. The boundary conditions for \mathbf{B} are $\mathbf{B} \times \mathbf{n} = \mathbf{B}_0 \times \mathbf{n}$ for the top boundary and $-\mathbf{B}_0 \times \mathbf{n}$ for the bottom. We let $R_e = R_m = 1000, s = 0.09$, and use the time step $\delta t = 0.01$, grid size $h = \frac{1}{40}$ to compute it.

In Fig. 2, we show snapshots of the magnitude of \mathbf{B}_1 that is the first component of $\mathbf{B} = (B_1, B_2)$ together with the velocity field \mathbf{u} at $t = 0.1, 1, 2, 2.2, 2.5, 3, 3.5, 4$. When time evolves, we can observe the vortices start to form around $t = 2$. After $t = 2.5$, the profiles of vortices and the magnetic field show the typical structure of K-H instability, and soon it deforms and rotates along with the flow. The obtained numerical results coincide well with the numerical/experimental results discussed in [5,8,17,27], qualitatively.

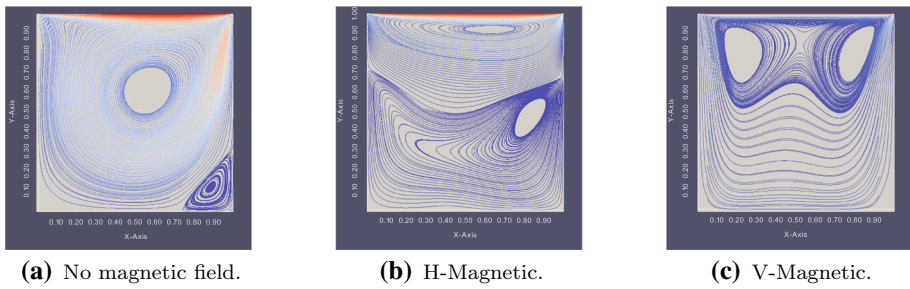


Fig. 3 The steady state of the streamlines for the velocity field for three cases, where **a** No magnetic field, **b** H-magnetic, and **c** V-magnetic

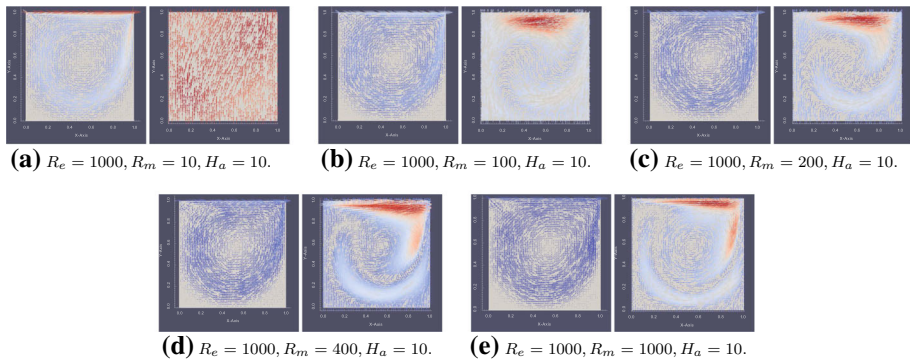


Fig. 4 The steady state of the velocity field \mathbf{u} and the magnetic field \mathbf{B} for five set of order parameters, shown in the caption of each subfigure, in which, the left one is the velocity field and the right one is the magnetic field

5.4 Lid Driven Cavity

Finally we perform the lid driven cavity flow simulation of a conducting fluid. The computed domain is $(x, y) \in \Omega = [0, 1] \times [0, 1]$. The boundary condition of the magnetic field is either horizontal case $\mathbf{B}|_{\partial\Omega} = (1, 0)$ (denoted by H-magnetic) or vertical case $\mathbf{B}|_{\partial\Omega} = (0, 1)$ (denoted by V-magnetic). The no-slip boundary conditions are imposed on the bottom, left, and right sides of the cavity and the lid moves from left to right with the constant speed $\mathbf{u}|_{y=1} = (1, 0)$.

First, we set $R_e = 400, R_m = 40$ and Hartmann number $H_a (= \sqrt{sR_e R_m}) = 100$. In Fig. 3, we show the steady state for three cases: no magnetic field, H-magnetic and V-magnetic, respectively. For the case of no magnetic field, this situation is actually a pure hydrodynamic problem, in which we observe the cavity is dominated by one large primary eddy in the center, together with a smaller secondary eddy in the right corner. For the H-magnetic case, two large eddies appear in the upper and lower part. For the V-magnetic case, two large eddies are located in the upper left and upper right. Similar features had been observed in [3] as well.

Second, we perform the so-called “magnetic frozen” simulations by varying the magnetic Reynolds number R_m . We set the fluid Reynolds number and Hartmann number to be $R_e = 1000, H_a = 10$, and five different values of $R_m = 10, 100, 200, 400, 1000$ in Fig. 4. We observe that the magnetic field bends gradually as R_m grows, which means the convection of

magnetic field yields more and more notable influences. When the magnetic field is strong enough, for instance when $R_m = 100, 200, 400, 1000$, the magnetic field form a big eddy at the center of the cavity which is quite similar to the corresponding fluid flow. This feature is called as the magnetic field is *frozen* inside the velocity field. Similar phenomenon had been reported in [28] as well.

6 Concluding Remarks

In this paper, we develop a fully decoupled, linear and unconditionally energy stable scheme for solving the incompressible MHD system, and prove the unconditional energy stability for the fully discrete scheme with the finite element approximations. We further rigorously establish the optimal error estimates for this scheme, and implement ample benchmark numerical experiments to demonstrate the stability and the accuracy in simulating some benchmark problems, including the Kelvin–Helmholtz shear instability and the magnetic-frozen phenomenon in the lid-driven cavity.

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