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A Source of Uncertainty in Computed Discontinuous Flows

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Abstract
It is speculated that some discontinuous weak solutions of boundary-value problems for nonlinear systems of conservation laws are computed, however routinely, with prescribed boundary data insufficient to uniquely determine such a solution. Stationary, transonic fluid flow exemplifies applications of present concern. A supplemental, a posteriori computation is described, which can potentially resolve this issue in any specific case.

Keywords Conservation laws · Admissible solutions · Discontinuous flow

1 Introduction

Absence of an established underlying well-posed problem exposes numerically obtained solutions of a mathematical model to a source of uncertainty. The form of input data, required a priori for a numerical investigation to proceed, is necessarily determined ad hoc, choice thereof arguably justified by subsequently obtained results. Such results may well be accepted as proof beyond reasonable doubt of existence of an acceptable solution corresponding to prescribed data, and of continuous dependence of obtained solutions on the data.

Such does not imply that sufficient input to uniquely determine an acceptable solution has been prescribed. Indeed, such input may not be available a priori. Some of the input data needed to distinguish an obtained solution may be generated artificially, by the adopted approximation scheme. If so, an alternative scheme might produce equally convincing results with materially different solutions for the same prescribed data.

In this context, we address the solution of boundary-value problems (including Cauchy problems and initial boundary-value problems) for first order systems of conservation laws,

$$\sum_{i=1}^{m} v_{ij}(z)_{x_i} = 0, \quad j = 1, \ldots, n. \tag{1.1}$$

In (1.1), the dependent variable x assumes values in an open, connected set $\Omega \subset \mathbb{R}^m$, and on the boundary $\partial\Omega$, where a piecewise continuous, exterior unit normal ν is almost everywhere defined. The dependent variable z, depending on x, assumes values in "phase

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space", a designated open set $D \subseteq \mathbb{R}^n$. The system-specific "flux functions" v_{ij} are smooth functions on D.

Numerical investigation of such a system typically derives from identification of a qualitatively specified solution set $\mathcal{S}$, of interest in a specific application. Stationary, transonic fluid flow problems in a specific domain Ω illustrate the examples of concern here.

Adoption of a boundary-value problem for (1.1) is based on the assumption that the elements $z \in \mathcal{S}$ are uniquely distinguished by partial specification of the corresponding boundary flux $v_\nu(z)$,

$$v_\nu(z)_j(x) \stackrel{def}{=} \sum_{i=1}^{m} \nu_i(x) v_{ij}(z(x)), \quad j = 1, \ldots, n, \tag{1.2}$$

for almost all $x \in \partial\Omega$. Subscript ν is defined analogously throughout.

Specifically, we assume the existence of a $1-1$ mapping

$$\mathcal{B} : \mathcal{S} \to \mathcal{D} \tag{1.3}$$

relating elements $z \in \mathcal{S}$ to elements $b \in \mathcal{D}$ by

$$b = (I - P) v_\nu(z \underset{\partial\Omega}{|}). \tag{1.4}$$

In (1.4), P is a projection map on the space of n-vector functions on $\partial\Omega$, designating the form of the boundary conditions for (1.1) associated with $\mathcal{S}$. In particular, P is independent of $z \in \mathcal{S}$. Without loss of generality, we take P symmetric with respect to the $L_2(\partial\Omega)^n$ inner product. For simplicity, here we assume P specified pointwise, so that for $x \in \partial\Omega$, $P(x)$ is a given, symmetric, idempotent $n \times n$ matrix. Typically P is continuous in x, if not constant, within segments of $\partial\Omega$ where ν is continuous. Throughout I is the identity operator on whatever spaces.

From (1.3), (1.4), it follows that $\mathcal{D}$ is the projection of $v_\nu(\mathcal{S})$ onto the subspace $\ker P$ (of the space of n-vector functions on $\partial\Omega$). Here we assume

$$\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2, \tag{1.5}$$

corresponding to

$$\ker P = (\ker P)_1 \oplus (\ker P)_2, \tag{1.6}$$

with

$$(\ker P)_1 \cap (\ker P)_2 = \{0\}. \tag{1.7}$$

In (1.5), $\mathcal{D}_1$ is assumed an open subset of a Banach space $\mathcal{D}'$, tacitly assuming that in practice, elements of $\mathcal{D}_1$ can be approximated with respect to the norm $\|\cdot\|_{\mathcal{D}'}$. The elements of $\mathcal{D}_2$ are assumed isolated and known precisely; typically $\mathcal{D}_2$ is trivial. Thus $\mathcal{D}'$ is identified as the tangent space of $\mathcal{D}$, at any point $b \in \mathcal{D}$.

Limits for $\mathcal{S}$ are understood as determined implicitly, from (1.3), (1.4) and whatever adopted limits for $\mathcal{D}$.

In a fluid flow problem we anticipate identification of $\mathcal{D}_1$, $(\ker P)_1$ with an inflow boundary segment, and $\mathcal{D}_2$, $(\ker P)_2$ with the remainder of the boundary $\partial\Omega$.

We seek to recover elements of $\mathcal{S}$ as images of a mapping $\mathcal{A} : \mathcal{D} \to \mathcal{S}$ satisfying

$$\mathcal{A} \in C(\mathcal{D} \to \mathcal{G}) \tag{1.8}$$

with respect to the norms $\|\cdot\|_{\mathcal{D}'}$, $\|\cdot\|_{\mathcal{G}}$. Here and throughout, $\mathcal{G}$ is a Banach space containing $\mathcal{S}$ as a subset, such that approximation of elements of $\mathcal{S}$ in the norm $\|\cdot\|_{\mathcal{G}}$ suffices in practice.

Anticipating discontinuous, weak solutions, the mapping $\mathcal{A}$ is based on the weak form of (1.1), (1.4)

$$\int_{\partial\Omega} b \cdot \theta = \iint_{\Omega} \sum_{i=1}^{m} \sum_{j=1}^{n} v_{ij}(z)\theta_{j,x_i} \tag{1.9}$$

for all $\theta \in X$,

$$X \stackrel{def}{=} \left\{ \theta \in (C^{\infty}(\Omega) \cap C(\bar{\Omega}))^n | P\theta \underset{\partial\Omega}{|} = 0 \right\}. \tag{1.10}$$

In (1.9) and throughout, single intervals are over $(m-1)$-manifolds and double integrals over m-manifolds. Dots denote either the $\ell_2(\mathbb{R}^m)$ or $\ell_2(\mathbb{R}^n)$ inner product.

Given a system (1.1) and a qualitatively described solution set $\mathcal{S}$, we seek to recover a mapping $\mathcal{A}$ as the limit of a sequence of approximations. The mapping P corresponding to $\mathcal{S}$ is typically determined, using (1.4), (1.6), on "physical grounds", so that each element of $\mathcal{D}$ contains all of the boundary data available a priori to the anticipated corresponding element of $\mathcal{S}$.

With $\delta > 0$ designating whatever discretization parameters, δ assuming a sequence of value decreasing to zero, we construct an approximation scheme, a sequence of mappings

$$\mathcal{A}_\delta \in C(\mathcal{D} \to \mathcal{G}), \quad \delta > 0, \tag{1.11}$$

such that the images

$$z_\delta(b) = \mathcal{A}_\delta(b), \quad \delta > 0, \ b \in \mathcal{D} \tag{1.12}$$

satisfy

$$\iint_{\Omega} \sum_{i=1}^{m} \sum_{j=1}^{n} v_{ij}(z_\delta)\theta_{j,x_i} \xrightarrow{\delta\downarrow 0} \int_{\partial\Omega} b \cdot \theta \tag{1.13}$$

for any $\theta \in X$, $b \in \mathcal{D}$ independent of δ.

We seek $\{\mathcal{A}_\delta\}$ such that (1.11) holds uniformly with respect to δ, and such that for any $b \in \mathcal{D}$

$$z_\delta(b) \xrightarrow{\delta\downarrow 0} z_0(b) \tag{1.14}$$

strongly in $\mathcal{G}$, with

$$z_0(b) \in \mathcal{S}. \tag{1.15}$$

Examples of schemes with such results supported by strong empirical evidence have been well-known for some time, in a wide variety of applications [5,7,10,13].

Assuming such, we tacitly identify $\mathcal{A}$ as the weak limit of $\{\mathcal{A}_\delta\}$ as $\delta \downarrow 0$, implicitly by

$$\mathcal{A}(b) = z_0(b), \quad b \in \mathcal{D}. \tag{1.16}$$

From (1.11) (uniformly with respect to δ) and (1.14), we have A satisfying (1.8).

From (1.13), (1.14), (1.15), with $\mathcal{B}$ given in (1.3), (1.4), we have

$$\mathcal{B} \circ \mathcal{A} = I \tag{1.17}$$

on $\mathcal{D}$.

Remaining unanswered is whether $\mathcal{B}$ is $1-1$, equivalently whether

$$\mathcal{A} \circ \mathcal{B} = I \tag{1.18}$$

on $\mathcal{S}$.

Indeed, establishment of (1.8), (1.18) is veritable determination of a well-posed boundary-value problem for the system (1.1), with $\mathcal{S}, \mathcal{D}, P$ related by (1.4). Unsurprisingly, the conditions (1.8), (1.18) severely restrict the choice of P, with (1.8) requiring ker P sufficiently small and (1.18) requiring ker P sufficiently large. For example, if ker P' is a proper subspace of ker P, then (1.18) holding for P precludes such for P', while if (1.8) holds for P', such cannot be expected for P. There is no assurance of either existence or uniqueness of P such that (1.8), (1.18) hold.

Below we obtain partial results for the question of whether (1.18) holds for some $\mathcal{A}$ obtained from (1.11), (1.12), (1.13), (1.14), (1.15), (1.16). Our discussion is directed by results gleaned from two very special cases. Making indicated restrictions, we obtain conditions approximating (1.18) locally which can be investigated numerically, given a sequence $\{z_\delta(b)\}$ for some $b \in \mathcal{D}$. With the usual qualifications, such investigation will either materially increase our confidence that (1.18) holds or else provide information on where ker P is too small.

2 Basic Assumptions: The Liner Case

Even for one space-dimensional Cauchy problems for hyperbolic systems (1.1), existence theorems remain restricted by the assumption of "small data" [1,2,6], or to very special conditions [12,15]. In this context, we address here a linear approximation of (1.8), (1.18), tacitly assuming that a mapping $\mathcal{A}$ satisfying (1.8), (1.17), (1.18) is pointwise Frechet differentiable. With $\mathcal{D}$ of the form (1.5), we consider the existence of a linear map at each $z \in \mathcal{S}$

$$d\mathcal{A}(z) : \mathcal{D}' \to \mathcal{S}'(z) \tag{2.1}$$

with $\mathcal{D}', \mathcal{S}'(z)$ respectively the tangent spaces of $\mathcal{D}, \mathcal{S}$ at z.

The condition (1.8) is associated with boundedness of $d\mathcal{A}(z)$, suitable norms for $\mathcal{D}'$ and $\mathcal{S}'(z)$ to be determined. With $\mathcal{A}$ satisfying (1.17) by convention, the condition (1.18) is associated with uniqueness of the images $d\mathcal{A}(z)b^{\cdot}$, within $\mathcal{S}'(z)$, for arbitrary $b^{\cdot} \in \mathcal{D}'$. Thus z for which such $d\mathcal{A}(z)$ hold can be made admissible in the sense of [14].

Here we avoid the assumption that the system (1.1) is everywhere hyperbolic for several reasons. Such is not the case in typical applications of present concern. Investigation of whether P is such that (1.11), (1.12), (1.13), (1.14), (1.15), (1.16) all hold, with (1.18) failing, is inappropriate for hyperbolic systems, for which P is largely if not entirely known a priori. Finally, the treatment in [14] suggest that in higher dimensions ($m \geq 3$) , hyperbolicity is a disadvantage in establishing (2.1).

Nonetheless we assume here systems (1.1), admitting an entropy extension [9], an m-vector entropy flux

$$q \in C^3(D \to \mathbb{R}^n) \tag{2.2}$$

satisfying

$$q_i(z) = \tilde{q}_i(v_{i1}(z), \ldots, v_{in}(z)), \quad i = 1, \ldots, m \tag{2.3}$$

with

$$\frac{\partial \tilde{q}_i}{\partial v_{ij}}(z) = z_j \tag{2.4}$$

for all $i = 1, \ldots, m$, $j = 1, \ldots, n$ such that v_{ij} is not identically constant (zero by convention).

From (2.1), (2.4), the Lagrange dual

$$\psi \in C^3(D \to \mathbb{R}^m) \tag{2.5}$$

obtained from

$$\psi_i(z) = \sum_{j=1}^{n} z_j v_{ij}(z) - q_i(z), \quad i = 1, \dots, m \tag{2.6}$$

satisfies [8,11]

$$\frac{\partial \psi_i}{\partial z_j}(z) = v_{ij}(z), \quad i = 1, \dots, m, \quad j = 1, \dots, n_j, \tag{2.7}$$

emphasizing that a convexity assumption is not made [4]. From (2.7), continuous solutions of (1.1) satisfy

$$\sum_{i=1}^{m} \sum_{k=1}^{n} \psi_{i,z_j z_k}(z) z_{k,x_i} = 0, \quad j = 1, \dots, n \tag{2.8}$$

illustrating a resemblance to symmetric linear systems (1.1), of the form

$$\sum_{i=1}^{m} V_i z_{x_i} = 0 \tag{2.9}$$

with each V_i a symmetric $n \times n$ matrix. Indeed, (2.9) is a special case of (1.1), corresponding to

$$v_{ij}(z) = (V_i z)_j, \quad i = 1, \dots, m, \quad j = 1, \dots, n. \tag{2.10}$$

Existence and uniqueness results for hyperbolic systems (2.9) are well-known [3]. The following discussion of boundary-value problems for nonhyperbolic systems (2.9) displays several features extending to nonlinear systems equipped with an entropy extension, and appearing in subsequent discussion of (2.1). We postulate a strong correlation between suitable P for linear systems and for nonlinear systems, for example by replacing V_ν by $\psi_{\nu,zz}(z)$ in (2.25), (2.36) below. Whether such can be done in practice, with $z \in \mathcal{S}$ unavailable a priori, is unclear.

Even in the linear system case, the obtained uniqueness results are incomplete. In this generality, we cannot preclude the existence of nontrivial solutions corresponding to zero boundary data, making it impossible to recover uniqueness simply by suitable choice of P.

Using (2.10), (1.4) assumes the form

$$b = (I - P) V_\nu z \tag{2.11}$$

on $\partial\Omega$, and (1.9) the form

$$\begin{aligned} \int_{\partial\Omega} b \cdot \theta &= \iint_{\Omega} \sum_{i=1}^{m} (V_i z) \cdot \theta_{x_i} \\ &= \iint_{\Omega} z \cdot \sum_{i=1}^{m} V_i \theta_{x_i} \end{aligned} \tag{2.12}$$

We denote

$$\|\theta\|_V \stackrel{def}{=} \left\| \sum_{i=1}^{m} V_i \theta_{x_i} \right\|_{L_2(\Omega)} \tag{2.13}$$

and Z the completion of X in the norm $\| \cdot \|_V$.

Theorem 2.1 *Assume* ker P *sufficiently small that there exists a constant* c *such that for all* $\theta \in X$,

$$\|\theta\|_{L_2(\partial\Omega)} \leq c\|\theta\|_V. \tag{2.14}$$

Then for any $b \in L_2(\partial\Omega)^n$ *of the form* (2.11)*, there exists* $z \in L_2(\Omega)^n$ *satisfying* (2.12).

Remarks Uniqueness is not claimed.

Hereafter c is a (sufficiently large positive) generic constant, relevant dependences thereof denoted by subscripts.

Proof Using (2.14), $Z \underset{\partial\Omega}{|}$ is complete with respect to the norm $\|\theta\|_{L_2(\partial\Omega)}$. For any fixed b, the functional on Z given by

$$J_b(\theta) = -\int_{\partial\Omega} b\theta + \tfrac{1}{2}\|\theta\|_V^2$$

is bounded below from (2.14), and achieves a (local) minimum at some point $\xi \in Z$, satisfying

$$\int_{\partial\Omega} b \cdot \theta = \iint_{\Omega} \left(\sum_{i=1}^{m} V_i \xi_{x_i}\right) \cdot \left(\sum_{j=1}^{m} V_j \theta_{x_j}\right) \tag{2.15}$$

for all $\theta \in Z$. Thus

$$z = \sum_{i=1}^{m} V_i \xi_{x_i} \tag{2.16}$$

satisfies (2.12). □

Solutions of (2.9) conserve energy; this is where the symmetry of the V_i is essential. The m-vector energy flux $\mathcal{E}$, determined from

$$\mathcal{E}_i(z) = \tfrac{1}{2} z \cdot V_i z, \quad i = 1, \ldots, m \tag{2.17}$$

satisfies

$$\nabla \cdot \mathcal{E}(z) = 0 \tag{2.18}$$

weakly within Ω. For linear systems (1.1), the functions q, ψ, $\mathcal{E}$ coincide.

Without loss of generality, we have

$$\mathcal{E}(z) = \nabla\Phi(z) + \Psi(z) \tag{2.19}$$

with a scalar function Φ and m-vector Ψ satisfying

$$\nabla \cdot \Psi(z) = 0 \tag{2.20}$$

weakly within Ω and

$$\nu \cdot \Psi(z) = 0 \tag{2.21}$$

almost everywhere on $\partial\Omega$.

From (2.18), (2.19), (2.20), (2.21), we have

$$\triangle\Phi(z) = 0 \tag{2.22}$$

weakly within Ω, and

$$\int_{\partial\Omega} \mathcal{E}_\nu(z)\Theta = \iint_{\Omega} \nabla\Theta \cdot \nabla\Phi(z) \tag{2.23}$$

for any smooth scalar function Θ. (In subsequent sections, Θ, Φ, Ψ are used generically, not necessarily the same in any two places.)

From (2.19), (2.21), almost everywhere on $\partial\Omega$,

$$\mathcal{E}_\nu(z) = \nu \cdot \nabla\Phi(z). \tag{2.24}$$

Almost everywhere on $\partial\Omega$, we denote by V_ν^{-1} the inverse of V_ν on range V_ν, and the identity map on ker V_ν.

Lemma 2.2 *Assume P such that almost everywhere on $\partial\Omega$,*

$$P V_\nu P \geq 0. \tag{2.25}$$

Then for any z, b satisfying (2.11), almost everywhere on $\partial\Omega$,

$$\mathcal{E}_\nu(z) \geq -c|b|^2. \tag{2.26}$$

Proof From (2.25), for any $a \in \mathbb{R}^n$

$$Pa \cdot V_\nu^{-1} Pa \geq \tfrac{1}{c}|Pa|^2. \tag{2.27}$$

From (2.6), using (2.11)

$$\begin{aligned}
\mathcal{E}_\nu &= \tfrac{1}{2} z \cdot V_\nu z \\
&= \tfrac{1}{2} V_\nu z \cdot V_\nu^{-1} V_\nu z \\
&= \tfrac{1}{2}(b + P V_\nu z) \cdot V_\nu^{-1}(b + P V_\nu z) \\
&= \tfrac{1}{2} b \cdot V_\nu^{-1} b + b \cdot V_\nu^{-1} P V_\nu z + \tfrac{1}{2} P V_\nu z \cdot V_\nu^{-1} P V_\nu z
\end{aligned} \tag{2.28}$$

and (2.26) follows from (2.27), (2.28). □

Choosing $\nabla\Theta = -\nabla\Phi$ in (2.23) and adding a constant to Θ as necessary, we obtain an estimate for z.

Corollary 2.3 *Assume (2.25); then*

$$\|\nabla\Phi(z)\|_{L_2(\Omega)} \leq c\||b|^2\|_{L_2(\partial\Omega)}. \tag{2.29}$$

The condition (2.25) also implies a partial uniqueness result.

Theorem 2.4 *Assume (2.25); then $V_\nu z$ (on $\partial\Omega$) and Φ (up to an additive constant) are unique for a given b.*

Proof Denote by z, z' two solutions of (2.9), (2.12), (2.11). Then $z - z'$ satisfies (2.9), (2.12), (2.11) with b vanishing identically, so from (2.26)

$$\mathcal{E}_\nu(z - z') \geq 0. \tag{2.30}$$

From (2.24), almost everywhere on $\partial\Omega$

$$\nu \cdot \nabla\Phi(z - z') \geq 0 \tag{2.31}$$

and as $\Phi(z - z')$ satisfies (2.22), (2.31) must hold with equality.

Thus from (2.17), (2.11)

$$\begin{aligned}
0 &= \mathcal{E}_\nu(z - z') \\
&= \tfrac{1}{2}(z - z') \cdot V_\nu(z - z')
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2} V_\nu (z - z') \cdot V_\nu^{-1} V_\nu (z - z') \\
&= \tfrac{1}{2} P V_\nu (z - z') \cdot V_\nu^{-1} P V_\nu (z - z')
\end{aligned} \tag{2.32}
$$

so using (2.27)

$$
P V_\nu z = P V_\nu z' \tag{2.33}
$$

and from (2.11), (2.33)

$$
V_\nu z = V_\nu z'. \tag{2.34}
$$

From (2.6), (2.34), using (2.24)

$$
\begin{aligned}
\mathcal{E}_\nu(z) &= \mathcal{E}_\nu(z') \\
&= \nu \cdot \nabla \Phi(z) \\
&= \nu \cdot \nabla \Phi(z')
\end{aligned} \tag{2.35}
$$

and as $\Phi(z)$, $\Phi(z')$ both satisfy (2.22), they must coincide. □

The mapping P is uniquely determined by (2.25) simultaneously with a condition

$$
(I - P)a \cdot V_\nu (I - P)a \le -\tfrac{1}{c} |(I - P)a|^2 \tag{2.36}
$$

almost everywhere on $\partial\Omega$, for all $a \in \mathbb{R}^n$. Use of (2.36), (1.10), obtaining

$$
\begin{aligned}
-\iint_\Omega \theta \cdot \sum_{i=1}^m V_i \theta_{x_i} &= -\tfrac{1}{2} \int_{\partial\Omega} \theta \cdot V_\nu \theta \\
&\ge \tfrac{1}{c} \|\theta\|^2_{L_2(\partial\Omega)},
\end{aligned} \tag{2.37}
$$

reduces verification of (2.14) to verification of an apparently simpler condition

$$
\|\theta\|_{L_2(\Omega)} \le c \left(\left\| \sum_{i=1}^m V_i \theta_{x_i} \right\|_{L_2(\Omega)} + \|\theta\|_{L_2(\partial\Omega)} \right). \tag{2.38}
$$

Theorem 2.5 *If $m = 2$, the condition (2.38) necessarily holds.*

Proof We consider the mixed eigenvalue problem

$$
V_1 a_j = \lambda_j V_2 a_j, \tag{2.39}
$$

notwithstanding that neither V_1 nor V_2 is positive definite (in the absence of hyperbolicity).

Without loss of generality, the integers $1, \dots, n$ may be divided into values of j where (2.39) holds with $a_j \in \mathbb{R}^n,\ |a_j| = 1$ and λ_j real; values of j where

$$
V_1 a_j = 0 \tag{2.40}
$$

with $a_j \in \mathbb{R}^n,\ |a_j| = 1$; and pairs $j, j+1$, such that

$$
V_1 a_j = \lambda_j V_2 a_j - \lambda_{j+1} V_2 a_{j+1}, \tag{2.41}
$$

$$
V_1 a_{j+1} = \lambda_j V_2 a_{j+1} + \lambda_{j+1} V_2 a_j, \tag{2.42}
$$

with $a_j, a_{j+1} \in \mathbb{R}^n,\ |a_{j+1}| = 1,\ \lambda_j, \lambda_{j+1} \in \mathbb{R},\ \lambda_{j+1} \ne 0$.

As the set $\{V_2 a_j\}$ spans $\mathbb{R}^n$, it will suffice to verify (2.38) for each scalar function $a_j \cdot V_2 \theta$.

In the "hyperbolic" case where (2.39) holds with nonzero real λ_j, we have

$$\begin{aligned} a_j \cdot \sum_{i=1}^{n} V_i\theta_{x_i} &= a_j(V_1\theta_{x_1} + V_2\theta_{x_2}) \\ &= a_j \cdot (\lambda_j V_2\theta_{x_1} + V_2\theta_{x_2}) \\ &= \lambda_j(a_j \cdot V_2\theta)_{x_1} + (a_j V_2\theta)_{x_2}, \end{aligned} \tag{2.43}$$

an ordinary differential equation for $a_j \cdot V_2\theta$, from which an estimate analogous to (2.38),

$$\|a_j \cdot V_2\theta\|_{L_2(\Omega)} \le c(\|a_j(V_1\theta_{x_1} + V_2\theta_{x_2})\|_{L_2(\Omega)} + \|a_j \cdot V_2\theta\|_{L_2(\partial\Omega)}) \tag{2.44}$$

is immediate.

In the "degenerate" case (2.40), $a_j \cdot V_2\theta$ vanishes identically and there is nothing to prove.

In the "elliptic" case (2.41), (2.42), we obtain

$$a_j \cdot (V_1\theta_{x_1} + V_2\theta_{x_2}) = \lambda_j(a_j \cdot V_2\theta)_{x_1} - \lambda_{j+1}(a_{j+1} \cdot V_2\theta)_{x_1} + (a_j \cdot V_2\theta)_{x_2}, \tag{2.45}$$

$$a_{j+1} \cdot (V_1\theta_{x_1} + V_2\theta_{x_2}) = \lambda_j(a_{j+1} \cdot V_2\theta)_{x_1} + \lambda_{j+1}(a_j \cdot V_2\theta)_{x_1} + (a_{j+1} \cdot V_2\theta)_{x_2}. \tag{2.46}$$

Application of the standard "cross differentiation" technique to (2.45), (2.46) gives

$$\begin{aligned} \left(\left(\lambda_j \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2 + \lambda_{j+1}^2 \frac{\partial^2}{\partial x_1^2}\right)(a_j \cdot V_2\theta) &= \left(\lambda_j \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) a_j \cdot (V_1\theta_{x_1} + V_2\theta_{x_2}) \\ &\quad + \lambda_{j+1} \frac{\partial}{\partial x_1} a_{j+1} \cdot (V_1\theta_{x_1} + V_2\theta_{x_2}); \end{aligned} \tag{2.47}$$

$$\begin{aligned} \left(\left(\lambda_j \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2 + \lambda_{j+1}^2 \frac{\partial^2}{\partial x_1^2}\right)(a_{j+1} \cdot V_2\theta) &= \left(\lambda_j \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) a_{j+1} \cdot (V_1\theta_{x_1} + V_2\theta_{x_2}) \\ &\quad - \lambda_{j+1} \frac{\partial}{\partial x_1} a_j \cdot (V_1\theta_{x_1} + V_2\theta_{x_2}); \end{aligned} \tag{2.48}$$

holding weakly, in $H^{-1}(\Omega)$.

The operator in the left side of (2.47), (2.48) is uniformly elliptic, and we obtain

$$\begin{aligned} &\|a_j \cdot V_2\theta\|_{L_2(\Omega)} + \|a_{j+1} \cdot V_2\theta\|_{L_2(\Omega)} \le c(\|a_j \cdot (V_1\theta_{x_1} + V_2\theta_{x_2})\|_{L_2(\Omega)} \\ &\quad + \|a_{j+1} \cdot (V_1\theta_{x_1} + V_2\theta_{x_2})\|_{L_2(\Omega)} + \|a_j \cdot V_2\theta\|_{L_2(\partial\Omega)} + \|a_{j+1} \cdot V_2\theta\|_{L_2(\partial\Omega)}). \end{aligned} \tag{2.49}$$

□

Lemma 2.2 and the a priori estimate Corollary 2.3 extend to nonlinear systems satisfying (2.7), albeit in weakened form.

For fixed b at each $x \in \partial\Omega$ where $\nu(x)$ is defined, we denote

$$D_x \stackrel{def}{=} \{a \in D \mid (I - P(x))\psi^{\dagger}_{\nu(x),z}(a) = b(x)\}. \tag{2.50}$$

For any $a \in D_x$, the tangent space $D'_x(a)$ to D_x at a satisfies

$$D'_x(a) \subseteq \ker\Big((I - P(x))\psi_{\nu(x),zz}(a)\Big). \tag{2.51}$$

However, for any x such that for all $a \in D_x$

$$\ker \psi_{\nu(x),zz}(a) \subseteq \operatorname{range} P(x), \tag{2.52}$$

as anticipated,

$$D_x'(a) = \operatorname{range} P(x), \tag{2.53}$$

independent of $a \in D_x$.

The following result is pointwise with respect to $x \in \partial\Omega$. For simplicity of notation, we drop (x) from ν, P, b, a_0, $\bar{a}$ as no ambiguity arises.

Theorem 2.6 *At a point $x \in \partial\Omega$ where ν, P are defined, assume the existence of $a_0 \in D_x$ such that for all $a \in D_x$, there exists a trajectory*

$$\bar{a} \in H^1\big((0,1) \to D_x\big), \quad \bar{a}(0) = a_0, \; \bar{a}(1) = a, \tag{2.54}$$

such that for all $t \in (0,1)$

$$P\psi^\dagger_{\nu,z}(\bar{a}(t)) = (1-t)P\psi^\dagger_{\nu,z}(a_0) + tP\psi^\dagger_{\nu,z}(a). \tag{2.55}$$

Assume in addition that

$$a' \cdot \psi_{\nu,zz}(a'')a' \geq 0 \tag{2.56}$$

for all $a' \in D_x'(a'')$, $\;a'' \in D_x$.

Then for any $a \in D_x$,

$$q_\nu(a) \geq a_0 \cdot P(\psi^\dagger_{\nu,z}(a) - \psi^\dagger_{\nu,z}(a_0)) + q_\nu(a_0). \tag{2.57}$$

Remarks In general, from (2.50) the point a_0 depends on b. But for any x such that $P\psi^\dagger_{\nu,z}(D_x)$ is a convex subset of $\mathbb{R}^n$, we can satisfy (2.55) with a_0 satisfying

$$P\psi^\dagger_{\nu,z}(a_0) = 0 \tag{2.58}$$

independent of b, and independent of x within any segment of $\partial\Omega$ where ν, P are constant. But (2.58) does not generally determine a_0 uniquely, (2.57) notwithstanding.

The condition (1.18) is understood as requiring $\ker P$ sufficiently large, using (2.51). Indeed, (2.56) holds trivially at a point x where P vanishes. At a point x where (2.53) holds, (2.56) simplifies to

$$P\psi_{\nu,zz}(a'')P \geq 0 \tag{2.59}$$

for all $a'' \in D_x$, analogous to (2.25) for linear systems.

The conclusion (2.57) provides a lower bound on $q_\nu(D_x)$ at any point where

$$a_0 \in \ker P \tag{2.60}$$

or at any point where

$$a_0 \cdot P\psi^\dagger_{\nu,z}(a) \geq -c_b + \frac{1}{c'}q_\nu(a), \quad c' > 1, \tag{2.61}$$

for all $a \in D_x$.

Proof For any fixed $a \in D_x$, from (2.54), (2.55)

$$\frac{d}{dt}P\psi^\dagger_{\nu,z}(\bar{a}(t)) = P(\psi^\dagger_{\nu,z}(a) - \psi^\dagger_{\nu,z}(a_0)), \tag{2.62}$$

and

$$\frac{d}{dt} P\psi^{\dagger}_{\nu,z}(\bar{a}(t)) = P\psi_{\nu,zz}(\bar{a}(t))\bar{a}_t(t), \tag{2.63}$$

with

$$\bar{a}(t) \in D_x, \tag{2.64}$$

$$\bar{a}_t(t) \in D'_x(\bar{a}(t)), \tag{2.65}$$

for all $t \in (0, 1)$, and

$$\bar{a}_t \in L_2(0, 1)^n. \tag{2.66}$$

Successively using (2.3), (2.7), (1.4), (2.54), (2.4), (2.62)

$$\begin{aligned} q_\nu(a) - q_\nu(a_0) &= \tilde{q}_\nu(\psi^{\dagger}_{\nu,z}(a)) - \tilde{q}_\nu(\psi^{\dagger}_{\nu,z}(a_0)) \\ &= \tilde{q}_\nu(b + P\psi^{\dagger}_{\nu,z}(a)) - \tilde{q}_\nu(b + P\psi^{\dagger}_{\nu,z}(a_0)) \\ &= \int_0^1 \frac{d}{dt}\tilde{q}_\nu(b + P\psi^{\dagger}_{\nu,z}(\bar{a}(t)))dt \\ &= \int_0^1 \bar{a}(t) \cdot \frac{d}{dt} P\psi^{\dagger}_{\nu,z}(\bar{a}(t))dt \\ &= P(\psi^{\dagger}_{\nu,z}(a) - \psi^{\dagger}_{\nu,z}(a_0)) \cdot \int_0^1 \bar{a}(t)dt. \end{aligned} \tag{2.67}$$

Partial integration in (2.67), subsequently using (2.62), (2.63), gives

$$\begin{aligned} q_\nu(a) - q_\nu(a_0) &= P(\psi^{\dagger}_{\nu,z}(a) - \psi^{\dagger}_{\nu,z}(a_0)) \cdot (a_0 + \int_0^1 (1-t)\bar{a}_t(t)dt) \\ &= a_0 \cdot P(\psi^{\dagger}_{\nu,z}(a) - \psi^{\dagger}_{\nu,z}(a_0)) + \int_0^1 (1-t)\bar{a}_t(t) \cdot \frac{d}{dt} P\psi^{\dagger}_{\nu,z}(\bar{a}(t))dt \\ &= a_0 \cdot P(\psi^{\dagger}_{\nu,z}(a) - \psi^{\dagger}_{\nu,z}(a_0)) + \int_0^1 (1-t)\bar{a}_t(t) \cdot P\psi_{\nu,zz}(\bar{a}(t))\bar{a}_t(t)dt \end{aligned} \tag{2.68}$$

which is defined in view of (2.66).

Using (2.64), (2.65), (2.51), for any $0 < t < 1$,

$$(I - P)\psi_{\nu,zz}(\bar{a}(t))\bar{a}_t(t) = 0 \tag{2.69}$$

and the integral in (2.68) is

$$\int_0^1 (1-t)\bar{a}_t(t) \cdot \psi_{\nu,zz}(\bar{a}(t))\bar{a}_t(t)dt \geq 0 \tag{2.70}$$

using (2.56) with $a' = \bar{a}_t(t),\ a'' = \bar{a}(t)$ at each t.

The conclusion (2.57) is immediate from (2.68), (2.70). □

An analog of Corollary 2.3 is obtained as follows. Assume, presumably using Theorem 2.5, that there exists $\partial\Omega_I \leq \partial\Omega$ such that almost everywhere in $\partial\Omega_I$, for all $z \in \mathcal{S}$

$$q_\nu(z) \geq -c_b. \tag{2.71}$$

Assume that the familiar entropy inequality [9]

$$\nabla \cdot q(z) \leq 0 \tag{2.72}$$

is part of the specification of $\mathcal{S}$.

Then for smooth nonnegative scalar Θ satisfying

$$\mathrm{supp}\Theta \cap \partial\Omega \subseteq \partial\Omega_I, \tag{2.73}$$

the moment of (2.72) with Θ, after partial integration, gives

$$\begin{aligned} -\iint_\Omega \nabla\Theta \cdot q(z) &\leq -\int\limits_{\partial\Omega} q_\nu(z)\Theta \\ &= -\int\limits_{\partial\Omega_i} q_\nu(z)\Theta \\ &\leq c_b\|\Theta\|_{L_1(\partial\Omega_I)} \end{aligned} \tag{2.74}$$

for all $z \in \mathcal{S}$, using (2.73), (2.71).

3 The Existence and Boundedness Conditions

We now address solution of (2.1).

Using (2.7), the relation (1.9) becomes

$$\int\limits_{\partial\Omega} b \cdot \theta = \iint_\Omega \sum_{i=1}^{m}\sum_{v=1}^{n} \psi_{i,z_j}(z)\theta_{j,x_i} \tag{3.1}$$

for all $\theta \in X$ given in (1.10).

Linearization of (3.1), to obtain expression of $\mathcal{S}'(z)$ and $d\mathcal{A}(z)$, requires assumption of the form of $z \in \mathcal{S}$ and the sense in which (1.1) is satisfied.

Here we assume

$$z \in C(\bar{\Omega}\backslash\Gamma)^n \tag{3.2}$$

satisfying (1.1) in the sense of n-vector measures on Ω.

In (3.2) and throughout, the discontinuity locus Γ, which depends on $z \in \mathcal{S}$, is assumed a finite union of $(m-1)$-manifolds, with unit normal $\hat{\mu}$ defined continuously almost everywhere on Γ. (The manifolds comprising Γ may intersect.) We assume one-sided limiting values of z almost everywhere on Γ, and denote by $[\cdot]$ jumps on Γ in the direction $\hat{\mu}$.

Such determines the form of an assumed Frechet derivative of each term in (3.1). With P assumed independent of $z \in \mathcal{S}$, almost everywhere on $\partial\Omega$, from (1.4), (2.7), we have

$$\begin{aligned} db(z) &= (I - P)\psi_{\nu,zz}(z)\dot{z}|_{\partial\Omega} \\ &\stackrel{def}{=} \dot{b} \end{aligned} \tag{3.3}$$

with some $\dot{z}\,|_{\partial\Omega}(x) \in \mathbb{R}^n$, for almost all $x \in \partial\Omega$.

The corresponding Frechet derivative of $\psi_{i,z_j}(z)$ includes a regular part (with respect to Lebesgue measure in Ω)

$$d\psi_{i,z_j}(z)|_{\Omega\backslash\Gamma} = \psi_{i,z_j z}\dot{z} \tag{3.4}$$

with $\dot{z}(x) \in \mathbb{R}^n,\ x \in \Omega\backslash\Gamma$, and a singular part with support on Γ,

$$d\psi_{i,z_j}(z)\underset{\Gamma}{|} = -[\psi_{i,z_j}(z)]\hat{\mu}\cdot\dot{x}|_\Gamma \tag{3.5}$$

almost everywhere on Γ with $\dot{x}\underset{\Gamma}{|}(x) \in \mathbb{R}^m$ for almost all $x \in \Gamma$.

From (1.10), the test space X is independent of $z \in \mathcal{S}$, so the linearization of (3.1) is

$$\begin{aligned}\underset{\partial\Omega}{|}\dot{b}\cdot\theta &= \iint\limits_{\Omega\backslash\Gamma}\sum_{i=1}^{m}\sum_{j,k=1}^{n}\psi_{i,z_j z_k}(z)\dot{z}_k\theta_{j,x_i} - \int\limits_\Gamma \hat{\mu}\cdot\dot{x}\underset{\Gamma}{|}\sum_{i=1}^{m}\sum_{j=1}^{n}[\psi_{i,z_j}(z)]\theta_{j,x_i}\\ &\overset{def}{=} \iint\limits_{\Omega\backslash\Gamma}\dot{z}\cdot R\theta + \int\limits_\Gamma \sigma S\theta\end{aligned} \tag{3.6}$$

with

$$(R\theta)_k \overset{def}{=} \sum_{i=1}^{m}\sum_{j=1}^{n}\psi_{i,z_j z_k}(z)\theta_{j,k_i},\quad k = 1,\ldots,n, \tag{3.7}$$

in $\Omega\backslash\Omega$;

$$\sigma \overset{def}{=} -\hat{\mu}\cdot\dot{x}\underset{\Gamma}{|}, \tag{3.8}$$

$$S\theta \overset{def}{=} \sum_{i=1}^{m}\sum_{j=1}^{n}[\psi_{i,z_j}(z)]\theta_{j,x_i}, \tag{3.9}$$

almost everywhere on Γ. Here and throughout, we adapt the notation employed in [14] as much as possible for consistency. For simplicity of notation, the dependence of $R\theta$, $S\theta$ on z is suppressed throughout.

Identifying

$$\dot{b} \in \mathcal{D}',\quad (\dot{z},\sigma) \in \mathcal{S}'(z), \tag{3.10}$$

we have an explicit statement of (2.1) in (3.6).

In a neighborhood of Γ, we use orthogonal coordinates $\hat{\mu}, \alpha_1,\ldots,\alpha_{m-1}$, with the unit vectors $\hat{\alpha}_i$ tangential to Γ. Almost everywhere on Γ, elements $z \in \mathcal{S}$ satisfy the Rankine-Hugoniot condition.

$$[\psi_{\mu,z}(z)] = 0. \tag{3.11}$$

The expression (3.9) is invariant under rotation of the space coordinates, so using (3.11) we have

$$S\theta = \sum_{i=1}^{m-1}\sum_{j=1}^{n}[\psi_{\alpha_i,z_j}(z)]\theta_{j,\alpha_i} \tag{3.12}$$

depending only on the tangential derivatives of θ on Γ.

Norms for $\mathcal{D}'$, $\mathcal{S}'(z)$ induce seminorms on X,

$$\|\theta\|_{\partial\Omega} \overset{def}{=} \underset{\dot{b}\in\mathcal{D}'\backslash\{0\}}{lub}\ \frac{\int_{\partial\Omega}\dot{b}\cdot\theta}{\|\dot{b}\|_{\mathcal{D}'}}, \tag{3.13}$$

$$\|\theta\| \overset{def}{=} \underset{(\dot{z},\sigma)\in\mathcal{S}'\setminus\{0\}}{lub} \frac{\iint\limits_{\partial\setminus\Omega} \dot{z}\cdot R\theta + \int\limits_{\Gamma} \sigma S\theta}{\|(\dot{z},\sigma)\|_{\mathcal{S}'(z)}}. \tag{3.14}$$

Using (3.13), (3.14), the boundedness condition for $d\mathcal{A}(z)$.

$$\|(\dot{z},\sigma)\|_{\mathcal{S}'(z)} \le c_z\|\dot{b}\|_{\mathcal{D}'} \tag{3.15}$$

is equivalent to a condition on X analogous to (2.14)

$$\|\theta\|_{\partial\Omega} \le c_z\|\theta\|. \tag{3.16}$$

We note that $\|\theta\|_{\partial\Omega}$ is independent of $z \in \mathcal{S}$, whereas $\|\theta\|$ depends only on $R\theta$, $S\theta$, and not explicitly on P. The condition (3.16) depends on z, P, and we recover (as expected) the condition that if (3.16) is satisfied for all $z \in \mathcal{S}$ with some P, it is satisfied with any P' such that ker $P' \subset$ ker P. Using (3.7), (3.14), we observe that (3.16) is impossible unless (2.52) holds.

For a fixed $z \in \mathcal{S}$, denote by Z the completion of X in the norm $\|\theta\|$. Unlike X, the space Z depends on z. If (3.16) holds, then $P\theta \underset{\partial\Omega}{|} = 0$ for all $\theta \in Z$.

An easy extension of Theorem 2.1 applies to (3.6). Throughout we denote by $(Z \underset{\partial\Omega}{|})^*$, $(RZ)^*$, $(SZ)^*$ the dual spaces of $Z \underset{\partial\Omega}{|}$, RZ, SZ determined by integrability of products over $\partial\Omega$, $\Omega\setminus\Gamma$, Γ respectively.

Theorem 3.1 *Assume that* (3.16) *holds, that*

$$\mathcal{D}' \subseteq (Z \underset{\partial\Omega}{|})^* \tag{3.17}$$

and that the norm $\|\theta\|$ *given in* (3.14) *is Frechet differentiable (with respect to* $\theta \in Z$*).*

Then for any $\dot{b} \in \mathcal{D}'$*, there exists* $\dot{z} \in (RZ)^*$*,* $\sigma \in (SZ)^*$ *satisfying* (3.6)*,* (3.15)*.*

Remarks Uniqueness is not claimed.

Proof For any fixed $\dot{b} \in \mathcal{D}'$, using (3.16) the functional on Z determined by

$$J_{\dot{b}}(\theta) \overset{def}{=} -\int\limits_{\partial\Omega} \dot{b}\cdot\theta + \frac{1}{2}\|\theta\|^2 \tag{3.18}$$

is bounded below. At any stationary point $\xi \in Z$ of $J_{\dot{b}}$ (for example the global minimum), by hypothesis of Frechet differentiability of $\|\theta\|$, for any $\theta \in Z$, necessarily

$$\begin{aligned} \int\limits_{\partial\Omega} \dot{b}\cdot\theta &= \|\xi\| d\|\xi\|(\theta) \\ &= \|\xi\|\Big(\iint\limits_{\Omega\setminus\Gamma} \zeta(\xi)\cdot R\theta + \int\limits_{\Gamma} \zeta_\Gamma(\xi) S\theta\Big) \end{aligned} \tag{3.19}$$

with $\zeta(\xi) \in (RZ)^*$, $\zeta_\Gamma(\xi) \in (SZ)^*$, as R, S are given respectively in disjoint regions $\Omega\setminus\Gamma$, Γ.

Then

$$\dot{z} = \|\xi\|\zeta(\xi), \quad \sigma = \|\xi\|\zeta_\Gamma(\xi) \tag{3.20}$$

satisfy (3.6), (3.15). □

We note that the condition (3.16), equivalent to the existence of a lower bound for the functional

$$J(\theta) = -\|\theta\|_{\partial\Omega} + \frac{1}{2}\|\theta\|^2$$
$$\geq -\frac{1}{2}c_z^2 \tag{3.21}$$

for all $\theta \in Z$, c_z the constant in (3.16).

4 The Uniqueness Condition

From (3.6), uniqueness of $(\dot{z}, \sigma)$ within $\mathcal{S}'(z)$ is equivalent to the statement that $(f, g) \in \mathcal{S}'(z)$ satisfying

$$\iint\limits_{\Omega\backslash\Gamma} f \cdot R\theta + \int\limits_{\Gamma} gS\theta = 0 \tag{4.1}$$

for all $\theta \in Z$ implies

$$f = g = 0. \tag{4.2}$$

For a fixed $\dot{b} \in \mathcal{S}'$, such requires that $J_{\dot{b}}$ given in (3.18) has only the global minimum as a stationary point, and that equivalent norms $\|\theta\|$, $\|\theta\|'$, determining the same space Z but different functionals $J_{\dot{b}}$ determine the same $(\dot{z}, \sigma)$ from (3.20).

Furthermore the space $\mathcal{S}'(z)$ remains ambiguous, restricted so far only by $\mathcal{D}'$ and (3.3).

We denote

$$\Lambda = \text{ span } \{(\dot{z}, \sigma)|\dot{b} \in \mathcal{D}'\} \tag{4.3}$$

$\dot{b}, \dot{z}, \sigma$ satisfying (3.20). From (3.6), necessarily

$$\Lambda \subseteq \mathcal{S}'(z) \tag{4.4}$$

and

$$\mathcal{S}'(z) \subseteq (RZ)^* \times (SZ)^*. \tag{4.5}$$

From (4.5), we note that for z continuous and Γ nonexistent in (3.2), $S\theta$ vanishing identically, the condition (4.1), (4.2) is trivially satisfied.

The selection of (4.5) holding with equality, made in [14], avoids precise specification of $\mathcal{D}'$. Such makes (4.1), (4.2) equivalent to a representation for Z of the form

$$Z = Z_{\Omega\backslash\Gamma} \oplus Z_{\Gamma}, \tag{4.6}$$

with

$$SZ_{\Omega\backslash\Gamma} = 0 \tag{4.7}$$

on Γ, and

$$RZ_{\Gamma} = 0 \tag{4.8}$$

in $\Omega\backslash\Gamma$. We note that in general, using (3.12) $Z_{\Omega\backslash\Gamma}$ contains all elements of Z vanishing identically on Γ, whereas Z_{Γ} may be trivial. Holding for a specific Z, the representation (4.6), (4.7), (4.8) necessarily holds for X and thus for any Z obtained with some norm $\|\theta\|$.

The statement (4.6), (4.7), (4.8) is a statement of ker P sufficiently large.

Theorem 4.1 *Assume* (4.6)*,* (4.7)*,* (4.8) *for some P. The such holds for any P' with* ker $P \subset$ ker P'.

Proof Denote by Z' the completion of X' in the norm $\|\theta\|$, X' obtained from (1.10) with P' replacing P. By hypothesis

$$Z \subset Z' \tag{4.9}$$

implying

$$(RZ')^* \times (SZ')^* \subset (RZ)^* \times (SZ)^*. \tag{4.10}$$

If (4.6), (4.7), (4.8) fails for Z', then there exists nontrivial

$$(f, g) \in (RZ')^* \times (SZ')^* \tag{4.11}$$

so (4.6), (4.7), (4.8) would fail for Z. □

A more precise statement of the required size of ker P is possible. Denote by X_0 the space X determined from (1.10) with $P = 0$, satisfying no boundary conditions on $\partial\Omega$, and Z_0 the completion of X_0 in the norm $\|\theta\|$. (We observe that (3.16) cannot hold for X_0, Z_0; constant θ is a counterexample.)

We assume, however, that (4.6), (4.7), (4.8) holds for X_0, Z_0, and denote

$$Z_{0,\cap} = Z_{0,\Omega\backslash\Gamma} \cap Z_{0,\Gamma}, \tag{4.12}$$

the functions $\theta_{0,\cap}$ satisfying

$$R\theta_{0,\cap} = 0, \; S\theta_{0,\cap} = 0 \tag{4.13}$$

with no boundary conditions on $\partial\Omega$.

Theorem 4.2 *Assume (4.6),(4.7), (4.8) for Z_0. Then such holds for any Z, with P such that*

$$\{PZ_{0,\Gamma}\} = \{PZ_{0,\cap}\}. \tag{4.14}$$

Remarks Failure of (4.6), (4.7), (4.8) for Z_0 implies nontrivial (f, g) satisfying (4.1) for all $\theta \in Z_0$. Such is regarded as highly unlikely for z obtained as z_0 from (1.14), (1.15).

Proof As $Z \subset Z_0$, for any $\theta \in Z$ by assumption there exist $\theta_{0,\Omega\backslash\Gamma} \in Z_{0,\Omega\backslash\Gamma}, \theta_{0,\Gamma} \in Z_{0,\Gamma}$ such that

$$\theta = \theta_{0,\Omega\backslash\Gamma} + \theta_{0,\Gamma}. \tag{4.15}$$

Using (4.14), there exists $\theta_{0,\cap} \in Z_{0,\cap}$ such that almost everywhere on $\partial\Omega$,

$$P\theta_{0,\cap} = P\theta_{0,\Gamma}. \tag{4.16}$$

From (1.10), $P\theta$ vanishes almost everywhere on $\partial\Omega$, so from (4.15), (4.16)

$$P\theta_{0,\Omega\backslash\Gamma} = -P\theta_{0,\Gamma} = -P\theta_{0,\cap}. \tag{4.17}$$

Now using (4.15), (4.13)

$$\theta = (\theta_{0,\Omega\backslash\Gamma} + \theta_{0,\cap}) + (\theta_{0,\Gamma} - \theta_{0,\cap}) \tag{4.18}$$

is the required expression for θ satisfying (4.6), (4.7), (4.8). □

Investigation of (4.6), (4.7), (4.8) for a given z, P is made by judiciary choice of equivalent norms

$$\|\theta\|_\lambda^2 = \iint\limits_{\Omega\backslash\Gamma} w|R\theta|^2 + \lambda \int\limits_\Gamma w_\Gamma (S\theta)^2, \quad \lambda = 1, 2. \tag{4.19}$$

In (4.19), w, w_Γ are scalar functions of x in $\Omega\backslash\Gamma$, Γ, respectively, generally depending on z and unbounded above, satisfying

$$w(x), w_\Gamma(x) \geq 1. \tag{4.20}$$

Such corresponds to

$$\|(\dot{z}, \sigma)\|^2_{\mathcal{S}'(z),\lambda} = \iint\limits_{\Omega\backslash\Gamma} \frac{1}{w}|\dot{z}|^2 + \frac{1}{\lambda}\int\limits_{\Gamma} \frac{1}{w_\Gamma}\sigma^2 \tag{4.21}$$

in (3.14);

$$\zeta_\lambda(\xi) = \frac{1}{\|\xi\|_\lambda} wR\xi, \zeta_{\Gamma,\lambda}(\xi) = \frac{\lambda}{\|\xi\|_\lambda} w_\Gamma S\xi \tag{4.22}$$

in (3.19);

$$\dot{z}_\lambda = wR\xi_\lambda, \sigma_\lambda = \lambda w_\Gamma S\xi_\lambda \tag{4.23}$$

in (3.20), with $\xi_\lambda \in Z$ satisfying

$$\int\limits_{\partial\lambda} \dot{b}\cdot\theta = \iint\limits_{\omega\backslash\Gamma} wR\xi_\lambda \cdot R\theta + \lambda \int\limits_{\Gamma} w_\Gamma(S\xi_\lambda)(S\theta) \tag{4.24}$$

for all $\theta \in Z$; and

$$(RZ)^* = wRZ, \quad (SZ)^* = w_\Gamma SZ \tag{4.25}$$

throughout.

For any given $\dot{b} \in \mathcal{D}$, we solve (3.6) for $(\dot{z}_\lambda, \sigma_\lambda)$, $\lambda = 1, 2$. (Nothing is gained by use of additional values of λ.) If

$$(\dot{z}_1, \sigma_1) \neq (\dot{z}_2, \sigma_2) \tag{4.26}$$

then

$$f = \dot{z}_1 - \dot{z}_2, \quad g = \sigma_1 - \sigma_2 \tag{4.27}$$

is a nontrivial solution of (4.1) and (4.6), (4.7), (4.8) fails.

As against this:

Theorem 4.3 *Assume that for each $\dot{b} \in \mathcal{D}'$, that either*

$$\dot{z}_1 = \dot{z}_2 \tag{4.28}$$

almost everywhere in $\Omega\backslash\Gamma$, or else

$$\sigma_1 = \sigma_2 \tag{4.29}$$

almost everywhere on Γ.

Then the subspaces $Z_{\Omega\backslash\Gamma}$, Z_Γ satisfying (4.7), (4.8), respectively, are sufficiently large that

$$\Lambda \subseteq wRZ_{\Omega\backslash\Gamma} \times w_\Gamma SZ_\Gamma. \tag{4.30}$$

Remarks The conclusion (4.30) implies that (4.1),(4.2) holds with $\mathcal{S}'(z)$ obtained from (4.4) with equality.

This result is independent of the choice of w, w_Γ, subject to (4.2) and that w, w_Γ are sufficiently large that (3.16) holds and (3.6) is solvable.

Proof Either of (4.28) or (4.29) implies the other, as $\dot{b}$, $\dot{z}$ (respectively $\dot{b}$, σ) uniquely determine $\sigma \in w_\Gamma SZ$ (respectively $\dot{z} \in wRZ$) satisfying (3.6).

For an arbitrary $\dot{b} \in \mathcal{D}'$, assume (4.28), (4.29) hold.

From (4.28), (4.23)

$$\begin{aligned} \dot{z}_\lambda &= \dot{z}_1 = \dot{z}_2 \\ &= wR\xi_1 \\ &= wR\xi_2, \end{aligned} \tag{4.31}$$

so from (4.31), (4.8)

$$\xi_1 - \xi_2 \in Z_\Gamma. \tag{4.32}$$

Similarly from (4.29), (4.23)

$$\begin{aligned} \sigma_\lambda &= \sigma_1 = \sigma_2 \\ &= w_\Gamma S\xi_1 \\ &= 2w_\Gamma S\xi_2, \end{aligned} \tag{4.33}$$

so from (4.7)

$$\xi_1 - 2\xi_2 \in Z_{\Omega\setminus\Gamma}. \tag{4.34}$$

Using (4.23), (4.29)

$$\begin{aligned} w_\Gamma S(\xi_1 - \xi_2) &= w_\Gamma S\xi_1 - \frac{1}{2}w_\Gamma S(2\xi_2) \\ &= \sigma_1 - \frac{1}{2}\sigma_2 \\ &= \frac{1}{2}\sigma_\lambda \end{aligned} \tag{4.35}$$

so using (4.32), (4.35)

$$\sigma_\lambda \in w_\Gamma SZ_\Gamma. \tag{4.36}$$

Similarly using (4.23), (4.28)

$$\begin{aligned} wR(\xi_1 - 2\xi_2) &= wR\xi_1 - 2wR\xi_2 \\ &= \dot{z}_1 - 2\dot{z}_2 \\ &= -\dot{z}_1, \end{aligned} \tag{4.37}$$

so from (4.34), (4.37)

$$\dot{z}_\lambda \in wRZ_{\Omega\setminus\Gamma}. \tag{4.38}$$

The conclusion (4.30) is immediate from (4.3), (4.36), (4.38). □

5 A Posteriori Investigation

For δ assuming a sequence of values decreasing to zero, we are given a sequence $\{z_\delta\}$ arguably satisfying (1.12), (1.13), (1.14), (1.15) for some given $b \in \mathcal{D}$. We seek corroboration that a mapping $\mathcal{A}$ determined from (1.16) satisfies (1.8), (1.17), (1.18) at least locally. Such has been tacitly associated with the conditions (3.16) and (4.28), (4.29) for whatever $\dot{b} \in \mathcal{D}'$.

A specific P is necessarily associated with $\{z_\delta\}$ through (1.13), (1.10). We assume ker P sufficiently small that (2.52) holds almost everywhere on $\partial\Omega$ for the limit z_0. Specifically, for any $x \in \partial\Omega,\ e_x \in \mathbb{R}^n$ such that

$$\psi_{\nu(x),zz}(z_0(x))e_x = 0, \tag{5.1}$$

necessarily

$$e_x \in \operatorname{range} P(x). \tag{5.2}$$

Then it appears very unlikely that convergence (1.14) would be obtained with overspecified boundary data. We anticipate ker P sufficiently small that (3.16) can be satisfied, with suitable $w, w_\Gamma,\ \|\cdot\|_{\partial\Omega}$ to be determined empirically.

Our principal concern is that (4.28), (4.29) will fail for some $\dot{b} \in \mathcal{D}'$, determining nontrivial (f, g) from (4.27) satisfying (4.1). Hereafter we understand R, S, Z, Γ associated with the specific limit z_0, replacing z by z_0 in (3.2), (3.7), (3.9), (3.12).

Choosing θ with support in $\bar{\Omega}\backslash\bar{\Gamma}$, partial integration in (4.1) determines f satisfying

$$R^\dagger f = 0 \tag{5.3}$$

in $\Omega\backslash\Gamma,\ R^\dagger$ the transpose operator of R, and using (1.10)

$$(I - P)\psi_{\nu,zz}(z_0)f = 0 \tag{5.4}$$

almost everywhere on $\partial\Omega$, rewritten

$$\psi_{\nu,zz}(z_0)f|_{\partial\Omega} \in \operatorname{range} P. \tag{5.5}$$

The condition (5.5) is symptomatic of insufficient prescribed boundary data, ker P too small in (1.4).

We may investigate the conditions (3.16), (4.28), (4.29), by solving (4.24), empirically through discrete, unconstrained minimization of the functionals J, $J_{\dot{b}}$ given in (3.21), (3.18), over a nested sequence of finite-dimensional test spaces $\{X_\delta\}$. We assume each

$$X_\delta \subset (W^{1,\infty}(\Omega) \cap C(\bar{\Omega}))^n, \tag{5.6}$$

satisfying (1.10), becoming dense in X as $\delta \downarrow 0$ in $W^{1,\infty}(\Omega)^n$ and with respect to the norm $\|\cdot\|_{\partial\Omega}$ on $\partial\Omega$.

For simplicity, we shall assume below that each

$$z_\delta \in (W^{1,\infty}(\Omega) \cap C(\bar{\Omega}))^n, \tag{5.7}$$

noting that analogous results are obtained using point-values and partial summations.

From (3.21), (3.18), it will suffice to find suitable discretizations of four terms

$$\iint\limits_{\Omega\backslash\Gamma} w|R\theta|^2, \quad \int\limits_{\Gamma} w_\Gamma(S\theta)^2, \quad \int\limits_{\partial\Omega} \dot{b}\cdot\theta, \quad \|\theta\|_{\partial\Omega}. \tag{5.8}$$

The two boundary terms in (5.8) are regarded as straightforward.

We assume a pointwise bound

$$\|z_0\|_{L_\infty(\Omega)} \le c \tag{5.9}$$

uniformly with respect to δ, and (1.14) strongly in $L_p(\Omega)$ for any finite p.

For the first term in (5.8), we approximate the operator R in (3.7) by

$$(R_\delta\theta)_k \stackrel{def}{=} \sum_{i=1}^{m}\sum_{j=1}^{n} \psi_{i,z_jz_k}(z_\delta)\theta_{j,x_i}, \quad k = 1,\ldots,n. \tag{5.10}$$

Considerable freedom exists in the choice of the weight functions w, w_Γ. The condition (3.16) is unaltered by multiplication of w, w_Γ by positive functions of x bounded uniformly above and below. The condition (4.6) (as applied to X, equivalent to that for Z) is independent of w, w_Γ; we tacitly anticipate the same for the closely related condition (4.30).

Here w, w_Γ are determined indirectly, as limits of approximations.

Empirically we seek to find a sequence $\{w_\delta\}$ satisfying

$$\|w_\delta\|_{L_{p'(\Omega)}} \leq c \tag{5.11}$$

for some $p' > 1$, uniformly with respect to δ, and

$$w_\delta \xrightarrow{\delta \downarrow 0} w \tag{5.12}$$

strongly in $L_1(\Omega)$. Then from (5.9), (5.10), (5.11), (5.12),

$$\|w_\delta^{\frac{1}{2}} R_\delta \theta - w^{\frac{1}{2}} R\theta\|_{L_2(\Omega)} \xrightarrow{\delta \downarrow 0} 0 \tag{5.13}$$

for any $\theta \in (W^{1,\infty}(\Omega))^n$.

Discretization of the second term in (5.8) if far more problematic, as Γ is unavailable and the z_δ do not satisfy (3.2). We introduce a second discretization parameter ε, also assuming a sequence of values decreasing to zero, understanding δ sufficiently small depending on ε throughout, $\lim \varepsilon, \delta \downarrow 0$ understood with

$$\delta = o(\varepsilon). \tag{5.14}$$

For $\theta \in (W^{1,\infty}(\Omega))^n$, we "approximate" $S\theta$ by an m-vector function on Ω, using a mapping

$$S_{\varepsilon\delta} = S_\varepsilon(z_\delta) \in C((W^{1,\infty}(\Omega))^n \to (L_\infty(\Omega))^m). \tag{5.15}$$

We "approximate" w_Γ by a nonnegative scalar function w_{Γ_ε} in Ω,

$$\|w_{\Gamma_\varepsilon}\|_{Lp'(\Omega)} \leq c \tag{5.16}$$

for some $p' > 1$, uniformly with respect to ε.

We abbreviate L_2-norms weighted using w, w_δ, w_Γ, w_{Γ_ε}

$$\|\mathcal{T}\|_w \stackrel{def}{=} \|w^{\frac{1}{2}}\mathcal{T}\|_{L_2(\Omega)}, \quad \|\mathcal{T}\|_{w_\delta} \stackrel{def}{=} \|w_\delta^{\frac{1}{2}}\mathcal{T}\|_{L_2(\Omega)},$$
$$\|\mathcal{T}\|_{w_\Gamma} \stackrel{def}{=} \|w_\Gamma^{\frac{1}{2}}\mathcal{T}\|_{L_2(\Gamma)}, \quad \|\mathcal{T}\|_{w_{\Gamma_\varepsilon}} \stackrel{def}{=} \|w_{\Gamma_\varepsilon}^{\frac{1}{2}}\mathcal{T}\|_{L_2(\Omega)}$$

with $\mathcal{T}$ generic.

Then for $\theta_\delta \in X_\delta$, we discretize (3.21) by

$$J_{\varepsilon\delta}(\theta_\delta) \stackrel{def}{=} -\|\theta_\delta\|_{\partial\Omega} + \tfrac{1}{2}\|R_\delta\theta_\delta\|_{w_\delta}^2 + \tfrac{1}{2}\|S_{\varepsilon\delta}\theta_\delta\|_{w_{\Gamma_\varepsilon}}^2, \tag{5.17}$$

and (3.18), (4.19) by

$$J_{\dot{b}\varepsilon\delta\lambda}(\theta_\delta) \stackrel{def}{=} -\int_{\partial\Omega} \dot{b}\cdot\theta_\delta + \tfrac{1}{2}\|R_\delta\theta_\delta\|_{w_\delta}^2 + \tfrac{\lambda}{2}\|S_{\varepsilon\delta}\theta_\delta\|_{w_{\Gamma_\varepsilon}}^2, \quad \lambda = 1, 2. \tag{5.18}$$

For any bounded Ω, from (5.15), (5.16)

$$w_{\Gamma_\varepsilon}^{\frac{1}{2}} S_{\varepsilon 0} \in C\Big((W^{1,\infty}(\Omega))^n \to L_2(\Omega)^m\Big); \tag{5.19}$$

we assume also, for any fixed $\varepsilon > 0$, $\theta \in (W^{1,\infty}(\Omega))^n$

$$\lim_{\delta\downarrow 0} \|S_{\varepsilon\delta}\theta - S_{\varepsilon 0}\theta\|_{w_{\Gamma\varepsilon}} = 0. \tag{5.20}$$

Three conditions, discussed in detail in the following section, relate $S_{\varepsilon\delta}$, w_{Γ_ε} to S, w_Γ:

$$\limsup_{\varepsilon\downarrow 0} \|S_{\varepsilon 0}\theta\|_{w_{\Gamma\varepsilon}} \leq c_+\|S\theta\|_{w_\Gamma}, \quad \text{for any } \theta \in X; \tag{5.21}$$

$$\liminf_{\varepsilon\downarrow 0} \|S_{\varepsilon 0}\theta\|_{w_{\Gamma\varepsilon}} \geq \frac{1}{c_-}\|S\theta\|_{w_\Gamma}, \quad \text{for any } \theta \in X; \tag{5.22}$$

any sequence $\{\theta_\delta\}$, $\theta_\delta \in X_\delta$, such that

$$\|S_{\varepsilon\delta}\theta_\delta\|_{w_{\Gamma\varepsilon}} \leq c, \tag{5.23}$$

contains a subsequence with a weak limit $\theta_0 \in Z$ satisfying

$$\iint\limits_{\Omega} w_{\Gamma_\varepsilon} S_{\varepsilon\delta}\theta_\delta \cdot S_{\varepsilon_0}\theta \xrightarrow{\varepsilon,\delta\downarrow 0} \int\limits_{\Gamma} w_\Gamma(S\theta_0)(S\theta) \tag{5.24}$$

for all $\theta \in X$.

To establish (3.16), it will suffice to determine w_δ, w_{Γ_ε}, $\|\cdot\|_{\partial\Omega}$ such that

$$\underset{\varepsilon,\delta\downarrow 0}{glb}\,\{J_{\varepsilon\delta}(X_\delta)\} \geq -c_0 \tag{5.25}$$

for some finite c_0.

Theorem 5.1 *Assume that* (5.25)*,* (5.21)*,* (5.13)*,* (5.19)*,* (5.20) *hold.*
Then (3.16)*,* (3.21) *hold with constants satisfying*

$$c_z = c_+(2c_0)^{\frac{1}{2}}. \tag{5.26}$$

Proof For any $\theta \in X$, from (3.21), using (5.21), (5.13), (5.19)

$$\begin{aligned} J(\theta) &= -\|\theta\|_{\partial\Omega} + \tfrac{1}{2}\|R\theta\|_w^2 + \tfrac{1}{2}\|S\theta\|_{w_\Gamma}^2 \\ &\geq -\|\theta\|_{\partial\Omega} + \tfrac{1}{2}\|R\theta\|_w^2 + \frac{1}{2c_+^2}\|S_{\varepsilon 0}\theta\|_{w_{\Gamma\varepsilon}}^2 - o(1) \\ &\geq -\|\theta_\delta\|_{\partial\Omega} + \tfrac{1}{2}\|R_\delta\theta_\delta\|_{w_\delta}^2 + \frac{1}{2c_+^2}\|S_{\varepsilon_0}\theta_\delta\|_{w_{\Gamma\varepsilon}}^2 - o(1), \end{aligned} \tag{5.27}$$

for suitable $\theta_\delta \in X_\delta$ approximating θ. Here and throughout, $o(1)$ is generic, understood as $\varepsilon, \delta \downarrow 0$.

From (5.20), with $\delta' > 0$ sufficiently small, depending on θ_δ, ε,

$$\|S_{\varepsilon\delta'}\theta_\delta - S_{\varepsilon 0}\theta_\delta\|_{w_{\Gamma_\varepsilon}} = o(1). \tag{5.28}$$

Use of (5.13), (5.28) in (5.27) gives

$$J(\theta) \geq -\|\theta_\delta\|_{\partial\Omega} + \tfrac{1}{2}\|R_{\delta'}\theta_\delta\|_{w_{\delta'}}^2 + \frac{1}{2c_+^2}\|S_{\varepsilon\delta'}\theta_\delta\|_{w_{\Gamma\varepsilon}}^2 - o(1). \tag{5.29}$$

From the assumption of nested test spaces, $\theta_\delta \in X_{\delta'}$, so (5.29) implies

$$J(\theta) \geq c_+^2 glb\{J_{\varepsilon,\delta'}(X_{\delta'})\} - o(1) \tag{5.30}$$

and (3.16) (3.21), (5.26) follow from (5.25), (5.30). □

The converse statement is materially weaker; the conditions (3.21), (5.25) are not equivalent.

Theorem 5.2 *Assume that* (3.16), (3.21), (5.22), (5.13), (5.20) *hold.*
Then for any $\theta \in X$

$$\liminf_{\varepsilon,\delta\downarrow 0} J_{\varepsilon\delta}(\theta) \geq -\tfrac{1}{2}c_-^2 c_z^2. \tag{5.31}$$

Proof From (5.17), using (5.13), (5.20), (5.22)

$$\begin{aligned} J_{\varepsilon\delta}(\theta) &= -\|\theta\|_{\partial\Omega} + \tfrac{1}{2}\|R_\delta\theta\|_{w_\delta}^2 + \tfrac{1}{2}\|S_{\varepsilon\delta}\theta\|_{w_{\Gamma\varepsilon}}^2 \\ &\geq -\|\theta\|_{\partial\Omega} + \tfrac{1}{2}\|R\theta\|_w^2 + \tfrac{1}{2}\|S_{\varepsilon\delta}\theta\|_{w_{\Gamma\varepsilon}}^2 - o(1), \end{aligned} \tag{5.32}$$

so

$$\begin{aligned} \liminf_{\varepsilon,\delta\downarrow 0} J_{\varepsilon\delta}(\theta) &\geq -\|\theta\|_{\partial\Omega}^2 + \tfrac{1}{2}\|R\theta\|_w^2 + \frac{1}{2c_-^2}\|S\theta\|_{w_\Gamma}^2 \\ &\geq c_-^2 \operatorname*{glb}_{\theta\in X} J(\theta), \end{aligned} \tag{5.33}$$

from which (5.31) follows using (3.21). □

We now assume w_δ, w_{Γ_ε}, $\|\cdot\|_{\partial\Omega}$ such that (5.25) holds, deterring discussion of this. For $\dot{b} \in \mathcal{D}'$, $\varepsilon, \delta > 0$, $\lambda = 1, 2$, a point $\xi_{\dot{b}\varepsilon\delta\lambda} \in X_\delta$ uniquely minimizes $J_{\dot{b}\varepsilon\delta\lambda}$ given in (5.18) over X_δ. Thus for any $\theta_\delta \in X_\delta$, the discretization of (4.24) is

$$\int_{\partial\Omega} \dot{b}\cdot\theta_\delta = \iint_\Omega w_\delta R_\delta \xi_{\dot{b}\varepsilon\delta\lambda}\cdot R_\delta\theta_\delta + \lambda \iint_\Omega w_{\Gamma_\varepsilon} S_{\varepsilon\delta}\xi_{\dot{b}\varepsilon\delta\lambda}\cdot S_{\varepsilon\delta}\theta_\delta. \tag{5.34}$$

Analogously with (3.16), (3.21), from (5.25), (5.18)

$$\|\xi_{\dot{b}\varepsilon\delta\lambda}\|_{\partial\Omega} \leq (2c_0)^{\frac{1}{2}}(\|R_\delta\xi_{\dot{b}\varepsilon\delta\lambda}\|_{w_\delta}^2 + \|S_{\varepsilon\delta}\xi_{\dot{b}\varepsilon\delta\lambda}\|_{w_{\Gamma_\varepsilon}}^2)^{\frac{1}{2}}; \tag{5.35}$$

setting $\theta_\delta = \xi_{\dot{b}\varepsilon\delta\lambda}$ in (5.34) and using (5.35), (3.13),

$$\|R_\delta\xi_{\dot{b}\varepsilon\delta\lambda}\|_{w_\delta}, \|S_{\varepsilon\delta}\xi_{\dot{b}\varepsilon\delta\lambda}\|_{w_{\Gamma\varepsilon}}, \quad \frac{1}{(2c_0)^{\frac{1}{2}}}\|\xi_{\dot{b}\varepsilon\delta\lambda}\|_{\partial\Omega} \leq (2c_0)^{\frac{1}{2}}\|\dot{b}\|_{\mathcal{D}'}. \tag{5.36}$$

For arbitrary $\theta \in X$, we take θ_δ approximating θ in (5.34). Using (5.13), (5.36), (5.20),

$$\int_{\partial\Omega} \dot{b}\cdot\theta = \iint_\Omega w_\delta^{\frac{1}{2}} R_\delta\xi_{\dot{b}\varepsilon\delta\lambda}\cdot w^{\frac{1}{2}} R\theta + \lambda \iint_\Omega w_{\Gamma_\varepsilon} S_{\varepsilon\delta}\xi_{\dot{b}\varepsilon\delta\lambda}\cdot S_{\varepsilon 0}\theta + o(1). \tag{5.37}$$

From (5.36), taking a subsequence as necessary,

$$w_\delta^{\frac{1}{2}} R_\delta \xi_{\dot{b}\varepsilon\delta\lambda} \overset{\varepsilon,\delta\downarrow 0}{\rightharpoonup} w^{\frac{1}{2}} R\xi_{\dot{b}\lambda} + \Xi_{\dot{b}\lambda} \tag{5.38}$$

weakly in $L_2(\Omega)^n$, with

$$\xi_{\dot{b}\lambda} \in Z, \tag{5.39}$$

$$\iint_\Omega w^{\frac{1}{2}} R\theta'\cdot\Xi_{\dot{b}\lambda} = 0 \tag{5.40}$$

for all $\theta' \in Z$.

We use (5.36), (5.23), (5.24) in (5.37) to obtain, again taking a subsequence as necessary,

$$\iint\limits_{\Omega} w_{\Gamma_\varepsilon} S_{\varepsilon\delta}\xi_{\dot b\varepsilon\delta\lambda} \cdot S_{\varepsilon 0}\theta \overset{\varepsilon,\delta\downarrow 0}{\longrightarrow} \int\limits_{\Gamma} w_\Gamma (S\xi'_{\dot b\lambda})(S\theta) \tag{5.41}$$

with some $\xi'_{\dot b\lambda} \in Z$ independent of θ.

From (5.37), (5.38), (5.39), (5.40), (5.41), for all $\theta \in X$ and $\xi_{\dot b\lambda}, \xi'_{\dot b\lambda} \in Z$ so obtained, we have

$$\int\limits_{\partial\Omega} \dot b \cdot \theta = \iint\limits_{\Omega\backslash\Gamma} wR\xi_{\dot b\lambda} \cdot R\theta + \lambda \int\limits_{\Gamma} w_\Gamma (S\xi'_{\dot b\lambda})(S\theta). \tag{5.42}$$

Theorem 5.3 *The pair of conditions* (4.28), (4.29) *is equivalent to the condition that*

$$R\xi_{\dot b 1} = R\xi_{\dot b 2} \tag{5.43}$$

is unique in (5.38).

Remarks No use of $\xi'_{\dot b\lambda}$ is required.

Proof Using (5.42), we satisfy (3.6) with

$$\dot z_\lambda = wR\xi_{\dot b\lambda}, \ \ \sigma_\lambda = w_\Gamma S\xi'_{\dot b\lambda} \tag{5.44}$$

for any such $\xi_{\dot b,\lambda}, \xi'_{\dot b\lambda}$.

If multiple $R\xi_{\dot b\lambda}$ are obtained from different subsequences in (5.38), or if (5.43) fails, we have multiple solutions of (3.6) for $\dot b$ as selected. Then nontrivial (f, g) satisfying (4.1) follows from (4.27), and (4.28), (4.29) cannot hold.

As against this, we observe that $\dot b, \dot z$ uniquely determine $\sigma \in w_\Gamma\{SZ\}$ satisfying (3.6). Thus if $\xi_{\dot b\lambda}$ is unique in (5.43), from (4.23), (5.44), necessarily

$$S\xi'_{\dot b\lambda} = \lambda S\xi_{\dot b\lambda}, \quad \lambda = 1, 2. \tag{5.45}$$

Now (4.28) is immediate from (5.43), (5.44), and (4.29) follows from (5.44), (5.45) by the same argument used to prove Theorem 4.3. □

Using (5.38), (5.40), the condition (5.43) is implied by

$$w_\delta^{\frac{1}{2}} R_\delta(\xi_{\dot b\varepsilon\delta 1} - \xi_{\dot b\varepsilon\delta 2}) \overset{\varepsilon,\delta\downarrow 0}{\rightharpoonup} 0 \tag{5.46}$$

weakly in $L_2(\Omega)^n$.

6 An Expression for $S_{\varepsilon\delta}$

An expression for $S_{\varepsilon\delta}$ is given here such that the conditions (5.19), (5.20), (5.21) and (5.23), (5.24) can be satisfied with some w_{Γ_ε} satisfying (5.16). The m-vector form of $S_{\varepsilon\delta}$ given in (5.15) is convenient here but by no means necessary. Alternative expressions for $S_{\varepsilon\delta}$ based on expedience or experience with computation schemes $\mathcal{A}_\delta$ are anticipated, particularly in the case where (6.8) below fails.

We employ orthogonal coordinates $\alpha_1, \ldots, \alpha_{m-1}, \mu$ (introduced in Sect. 3) in an open neighborhood of each segment of Γ within which $\hat\mu$ is continuously defined. For each such segment of Γ, by convention

$$x(\alpha, 0) \in \Gamma, \tag{6.1}$$

$$[z_0](x(\alpha,0)) \stackrel{def}{=} z_0(x(\alpha,0+)) - z_0(x(\alpha,0-)), \tag{6.2}$$

$$\frac{1}{c} \le \frac{\partial(\alpha_1,\ldots,\alpha_{m-1},\mu)}{\partial(x_1,\ldots,x_m)} \le c, \tag{6.3}$$

the last condition within an open neighborhood. We note that near intersection points in Γ, where $\hat{\mu}$ is not continuous, the local coordinates α, μ for a given x relate to a specific segment of Γ, and may not be unique.

From results in [14], we anticipate that unbounded w_Γ, likely

$$w_\Gamma(x) = O(|[z_0]|(x)^{-2}), \quad x \in \Gamma \tag{6.4}$$

will be needed to satisfy (3.16) and thus (5.25). Such makes (5.16) problematical, in view of (5.22). In particular we need large values of w_{Γ_ε} confined to a small subset of Ω.

We choose ε sufficiently small, depending on a positive quantifier τ, denoting

$$\Gamma_\tau \stackrel{def}{=} \{x \in \Gamma \,\Big|\, |[z_0]|(x) \ge \tau\}. \tag{6.5}$$

The segments of Γ_τ are known only approximately, but with δ sufficiently small depending on ε, we can determine regions $\Gamma_{\tau\varepsilon} \subset \Omega$ satisfying

$$\left\{x(\alpha,\mu) \in \Omega \,\Big|\, x(\alpha,0) \in \Gamma_\tau, \;\; |\mu| < \frac{\varepsilon}{2}\right\} \subseteq \Gamma_{\tau\varepsilon}, \tag{6.6}$$

and

$$\Gamma_{\tau\varepsilon} \subseteq \left\{x(\alpha,\mu) \in \Omega \,\Big|\, x(\alpha,0) \in \Gamma_{\tau/2}, |\mu| < \varepsilon\right\}. \tag{6.7}$$

Making an obvious abuse of notation, throughout we use

$$\iint\limits_{\Gamma_{\tau\varepsilon}} = \int\limits_{\Gamma_\tau} d\alpha \int\limits_{-\varepsilon}^{\varepsilon} d\mu,$$

employing (6.3).

In the special case where

$$|[z_0]| \ge \underline{\tau} > 0 \tag{6.8}$$

uniformly on Γ, the parameter τ is unnecessary; the condition (6.4) is vacuous, and we understand Γ_τ as all of Γ.

For $x \in \Omega \backslash \Gamma$ we denote

$$\eta_\varepsilon(x) \stackrel{def}{=} \underset{x' \in \Omega\backslash\Gamma}{lub} |z_0(x) - z_0(x')| \tag{6.9}$$

for x' connected to x by a trajectory of length not exceeding $\frac{3}{2}\varepsilon$ and not intersecting $\Gamma_{2\tau}$. For $x \in \Gamma$,

$$\eta_\varepsilon(x(\alpha,0)) \stackrel{def}{=} \text{maximum}\left(\eta_\varepsilon(x(\alpha,0+)), \eta_\varepsilon(x(\alpha,0-))\right). \tag{6.10}$$

Then from (6.9), (6.10), (3.2), (6.5)

$$\eta_\varepsilon \xrightarrow{\tau,\varepsilon\downarrow 0} 0 \tag{6.11}$$

uniformly in Ω.

For a conveniently chosen open set

$$\omega \subseteq \{x \in \mathbb{R}^m,\ |x| < \frac{1}{4}\} \tag{6.12}$$

with volume $|\omega|$, $\ i, l = 1, \ldots, m, \quad j = 1, \ldots, n$, we denote

$$h_{ijl}^{\varepsilon\delta}(x) \overset{def}{=} \frac{1}{|\omega|} \iint_{\omega} \Big(\psi_{i,z_j}(z_\delta(x + \varepsilon y + \frac{\varepsilon}{4}\hat{x}_l(x, y))) - \psi_{i,z_j}(z_\delta(x + \varepsilon y - \frac{\varepsilon}{4}\hat{x}_l(x, y)))\Big) dy \tag{6.13}$$

with the understood restriction

$$x + \varepsilon y \pm \frac{\varepsilon}{4}\hat{x}_l(x, y) \in \bar{\Omega}$$

and with the convention that the unit vectors $\hat{x}_l(x, y)$ satisfy

$$\hat{x}_l(x, y) \overset{def}{=} \hat{x}_l(x(\alpha, \mu) + \varepsilon y) \tag{6.14}$$

$$\hat{x}_l(x(\alpha, \mu) + \varepsilon y) \cdot \hat{\mu}(x(\alpha, 0)) \geq 0 \tag{6.15}$$

for $x \in \Gamma_{\tau\varepsilon}$, $y \in \omega$.

From (6.13), (6.9), (6.12)

$$|h_{ijl}^{\varepsilon 0}(x)| \leq c\eta_\varepsilon(x), \quad x \in \Omega \backslash \Gamma_{\tau\varepsilon}. \tag{6.16}$$

Within $\Gamma_{\tau\varepsilon}$, by inspection of (6.13), $h_{ij\cdot}^{\cdot\delta}$ is an approximation of $[\psi_{i,z_j}(z_\delta)]$. Such is made precise as follows.

Lemma 6.1 *For $x = x(\alpha, \mu) \in \Gamma_{\tau\varepsilon}$, $\ i, l = 1, \ldots, m, \quad j = 1, \ldots, n$,*

$$h_{ijl}^{\varepsilon 0}(x(\alpha, \mu)) = [\psi_{i,z_j}(z_0)](x(\alpha, 0)\gamma_{\varepsilon l}(x(\alpha, \mu)) + \gamma'_{\varepsilon l}(x(\alpha, \mu)), \tag{6.17}$$

with $\gamma_{\varepsilon l}$ satisfying

$$\|\gamma_{\varepsilon l}\|_{L_\infty(\Gamma_{\tau\varepsilon})} \leq 1, \tag{6.18}$$

$$\sum_{l=1}^{m} \int_{-\varepsilon/2}^{\varepsilon/2} \gamma_{\varepsilon l}(x(\alpha, \mu'))^2 d\mu' \geq \frac{\varepsilon}{4} - o(\varepsilon), \tag{6.19}$$

as $\varepsilon \downarrow 0$, uniformly with respect to α; and $\gamma'_{\varepsilon,l}$ satisfying

$$\|\gamma'_{\varepsilon l}\|_{L_\infty(\{x(\alpha,\mu')||\mu'|<\varepsilon\})} \leq c\eta_\varepsilon(x(\alpha, 0)). \tag{6.20}$$

Proof For $x \in \Gamma_{\tau\varepsilon}$, $\ y \in \omega$, $\ l = 1, \ldots, m$, $\ \hat{x}_l$ satisfying (6.14), (6.15),

$$\chi_{\varepsilon l}(x, y) = \begin{cases} 1, & \text{for } x + \varepsilon y \pm \frac{\varepsilon}{4}\hat{x}_l \text{ on opposite sides of } \Gamma \\ 0, & \text{otherwise} \end{cases}. \tag{6.21}$$

From (6.13), with $x = x(\alpha, \mu)$, $\hat{x}_l$ satisfying (6.14), (6.15),

$$h_{ijl}^{\varepsilon 0}(x) = \frac{1}{|\omega|} \iint_{\omega} \chi_{\varepsilon l}(x, y)[\psi_{i,z_j}(z_0)](x(\alpha, 0))dy$$

$$+ \frac{1}{|\omega|} \iint_{\omega} \chi_{\varepsilon l}(x, y)\Big(\psi_{i,z_j}(z_0(x + \varepsilon y + \frac{\varepsilon}{4}\hat{x}_l)) - \psi_{i,z_j}(z_0(x(\alpha, 0+))$$

$$+\psi_{i,z_j}(z_0(x(\alpha,0-))-\psi_{i,z_j}(z_0(x+\varepsilon y-\frac{\varepsilon}{4}\hat{x}_l))\Big)dy$$
$$+\frac{1}{|\omega|}\iint\limits_{\omega}(1-\chi_{\varepsilon l}(x,y))\Big(\psi_{i,z_j}(z_0(x+\varepsilon y+\frac{\varepsilon}{4}\hat{x}_l))-\psi_{i,z_j}(z_0(x+\varepsilon y-\frac{\varepsilon}{4}\hat{x}_l))\Big)dy. \tag{6.22}$$

Using (6.21), (6.9), (6.10) the last two terms in (6.22) determine $\gamma'_{\varepsilon l}$ satisfying (6.20). Thus (6.17) holds with

$$\gamma_{\varepsilon l}(x)=\frac{1}{|\omega|}\iint\limits_{\omega}\chi_{\varepsilon l}(x,y)dy \tag{6.23}$$

from which (6.18) follows immediately from (6.21).

For $l=1,\ldots,m$, $\hat{x}_l$ satisfying (6.14), (6.15), we denote

$$\tilde{\chi}_{\varepsilon l}(x(\alpha,\mu),y)\overset{def}{=}\begin{cases}1, & |\mu+\varepsilon y\cdot\hat{\mu}(x(\alpha,0)|<\frac{\varepsilon}{4}\hat{\mu}(x(\alpha,0))\cdot\hat{x}_l(x(\alpha,0))\\ 0, & \text{otherwise,}\end{cases} \tag{6.24}$$

and

$$\tilde{\gamma}_{\varepsilon l}(x)=\frac{1}{|\omega|}\iint\limits_{\omega}\tilde{\chi}_{\varepsilon l}(x,y)dy. \tag{6.25}$$

From (6.24), (6.25),

$$\|\tilde{\gamma}_{\varepsilon l}\|_{L_\infty(\Gamma_{\tau\varepsilon})}\le 1. \tag{6.26}$$

Comparing (6.21), (6.24) and (6.23), (6.25), we observe that were $\hat{\mu}(x(\alpha,0))$ independent of α, $\tilde{\chi}_{\varepsilon l}$, $\tilde{\gamma}_{\varepsilon l}$ would coincide with $\chi_{\varepsilon l}$, $\gamma_{\varepsilon l}$. With $\hat{\mu}(x(\alpha,0))$ continuous with respect to α, we have

$$\|\gamma_{\varepsilon l}-\tilde{\gamma}_{\varepsilon l}\|_{L_\infty(\Gamma_{\tau\varepsilon})}=o(1) \tag{6.27}$$

as $\varepsilon\downarrow 0$, so using (6.18), (6.26), it will suffice to verify (6.19) with $\gamma_{\varepsilon l}$ replaced by $\tilde{\gamma}_{\varepsilon l}$.

From (6.25), (6.24)

$$\begin{aligned}\int\limits_{-\varepsilon/2}^{\varepsilon/2}\tilde{\gamma}_{\varepsilon l}(x(\alpha,\mu'))d\mu' &= \frac{1}{|\omega|}\int\limits_{-\varepsilon/2}^{\varepsilon/2}\iint\limits_{\omega}\tilde{\chi}_{\varepsilon l}(x(\alpha\mu'),y)dyd\mu'\\ &= \frac{1}{|\omega|}\iint\limits_{\omega}\int\limits_{-\varepsilon/2}^{\varepsilon/2}\tilde{\chi}_{\varepsilon l}(x(\alpha,\mu'))d\mu'dy\\ &= \frac{1}{|\omega|}\iint\limits_{\omega}\frac{\varepsilon}{2}\hat{\mu}(x(\alpha,0))\cdot\hat{x}_l(x(\alpha,0))dy\\ &= \frac{\varepsilon}{2}\hat{\mu}(x(\alpha,0))\cdot\hat{x}_\ell(x(\alpha,0)).\end{aligned} \tag{6.28}$$

From (6.28)

$$\sum_{l=1}^{m}\left(\int\limits_{-\varepsilon/2}^{\varepsilon/2}\tilde{\gamma}(x(\alpha,\mu'))d\mu'\right)^2=\frac{\varepsilon^2}{4} \tag{6.29}$$

so

$$\sum_{l=1}^{m} \int_{-\varepsilon/2}^{\varepsilon/2} \tilde{\gamma}\left(x(\alpha,\mu')\right)^2 d\mu' \geq \frac{\varepsilon}{4}. \tag{6.30}$$

Now (6.19) follows from (6.30), (6.27). □

An expression for $S_{\varepsilon\delta}\theta$ is obtained analogously with (3.9), but using $h^{\varepsilon\delta}_{\dots}$,

$$(S_{\varepsilon\delta}\theta)_l(x) \stackrel{def}{=} \varepsilon^{-\frac{1}{2}} \sum_{i=1}^{m}\sum_{j=1}^{n} h^{\varepsilon\delta}_{ijl}(x)\theta_{j,x_i}(x), \quad x \in \Omega, l = 1,\dots,m, \quad \theta \in (W^{1,\infty}(\Omega))^n. \tag{6.31}$$

We assume τ (unless (6.8) holds), ε, δ each assuming a sequence of values decreasing to zero, with Γ_τ determined from (6.5), $\Gamma_{\tau\varepsilon}$ satisfying (6.6), (6.7).

Theorem 6.2 *For each x such that $x(\alpha,0) \in \Gamma_\tau$, assume w_{Γ_ε}, w_Γ related by*

$$\liminf_{\substack{|\mu|<\varepsilon \\ \varepsilon\downarrow 0}} w_{\Gamma_\varepsilon}(x(\alpha,\mu)) \geq \frac{1}{c} w_\Gamma(x(\alpha,0)), \tag{6.32}$$

and

$$\limsup_{\substack{|\mu|<\varepsilon \\ \varepsilon\downarrow 0}} w_{\Gamma_\varepsilon}(x(\alpha,\mu)) \leq c w_\Gamma(x(\alpha,0)). \tag{6.33}$$

Assume δ sufficiently small, depending on ε, that

$$\|h^{\varepsilon\delta}_{ijl} - h^{\varepsilon 0}_{ijl}\|_{w_{\Gamma_\varepsilon}} = o(\varepsilon^{\frac{1}{2}}) \tag{6.34}$$

for all $i, l = 1,\dots,m, \ \ j = 1,\dots,n$.

Assume ε sufficiently small, depending on τ, that

$$\int_{\Gamma_\tau} w_\Gamma \eta_\varepsilon^2 = o(1) \tag{6.35}$$

with $\eta_\varepsilon|_{\Gamma_\tau}$ given in (6.10), and

$$\varepsilon^{1/p'} \|w_\Gamma\|_{L_{p'}(\Gamma_\tau)} \leq c \tag{6.36}$$

for some $p' > 1$.

Outside of $\Gamma_{\tau\varepsilon}$, assume

$$\iint_{\Omega\backslash\Gamma_{\tau\varepsilon}} w_{\Gamma_\varepsilon}\eta_\varepsilon^2 = o(\varepsilon) \tag{6.37}$$

and

$$\|w_{\Gamma_\varepsilon}\|_{L_{p'}}(\Omega\backslash\Gamma_{\tau\varepsilon}) \leq c \tag{6.38}$$

with the same p' as in (6.36).

Then (5.16), (5.20), (5.21), (5.21), (5.22) and the conditions (5.23), (5.24) hold.

Proof The bound (5.16) follows from (6.38), (6.36), (6.33), (6.7).

The bound (5.20) follows from (6.31), (6.34).

Within $\Gamma_{\tau\varepsilon}$, for arbitrary fixed $\theta \in (W^{1,\infty}(\Omega))^n$, $l = 1, \ldots, m$, from (6.31), (6.17)

$$\begin{aligned} &(S_{\varepsilon 0}\theta)_l(x(\alpha,\mu)) \\ &= \varepsilon^{-\frac{1}{2}} \sum_{i=1}^{m}\sum_{j=1}^{n}\Big([\psi_{i,z_j}(z_0)](x(\alpha,0))\gamma_{\varepsilon l}(x(\alpha,\mu)) + \gamma'_{\varepsilon l}(x(\alpha,\mu))\Big)\theta_{j,x_i}(x(\alpha,\mu)). \end{aligned} \tag{6.39}$$

From (6.20), (6.33), (6.35)

$$\begin{aligned} \frac{1}{\varepsilon}\iint\limits_{\Gamma_{\tau\varepsilon}} \sum_{l=1}^{m} \gamma'^2_{\varepsilon l} w_{\Gamma_\varepsilon} &\le c \int\limits_{\Gamma_\tau} w_\Gamma \eta_\varepsilon^2 \\ &= o(1), \end{aligned} \tag{6.40}$$

so from (6.39), (6.40)

$$\begin{aligned} \iint\limits_{\Gamma_{\tau\varepsilon}} w_{\Gamma_\varepsilon}|S_{\varepsilon 0}\theta|^2 &= (1+o(1)) \int\limits_{\Gamma_\tau} \left(\sum_{i=1}^{m}\sum_{j=1}^{n}[\psi_{i,z_j}(z_0)]\theta_{j,x_i}\right)^2 \frac{1}{\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon}\sum_{l=1}^{m} w_{\Gamma_\varepsilon}\gamma_{\varepsilon l}^2 + o(1) \\ &= (1+o(1)) \int\limits_{\Gamma_\tau} (S\theta)^2 \frac{1}{\varepsilon}\int\limits_{-\varepsilon}^{\varepsilon}\sum_{l=1}^{m} w_{\Gamma_\varepsilon}\gamma_{\varepsilon l}^2 + o(1) \end{aligned} \tag{6.41}$$

using (3.9).

Now (5.21) follows from (6.41) using (6.33), (6.18), and (5.22) follows from (6.41) using (6.32), (6.19).

From (6.31), (6.13), (6.9), (6.37)

$$\|w_{\Gamma_\varepsilon}^{\frac{1}{2}} S_{\varepsilon 0}\theta\|_{L_2(\Omega\setminus\Gamma_{\tau\varepsilon})} = o(1). \tag{6.42}$$

For a sequence $\{\theta_\delta\}$ satisfying (5.23), using (6.42), (5.19), then (6.39), (6.40), (3.9)

$$\begin{aligned} \iint\limits_{\Omega} w_{\Gamma_\varepsilon} S_{\varepsilon\delta}\theta_\delta \cdot S_{\varepsilon 0}\theta &= \iint\limits_{\Gamma_{\tau\varepsilon}} w_{\Gamma_\varepsilon} S_{\varepsilon\delta}\theta_\delta \cdot S_{\varepsilon 0}\theta + o(1) \\ &= \iint\limits_{\Gamma_{\tau\varepsilon}} w_{\Gamma_\varepsilon} \sum_{l=1}^{m}(S_{\varepsilon\delta}\theta_\delta)_l \varepsilon^{-\frac{1}{2}}\gamma_{\varepsilon l}(S\theta) + o(1) \\ &= \int\limits_{\Gamma_\tau} w_\Gamma^{\frac{1}{2}}(S\theta)\int\limits_{-\varepsilon}^{\varepsilon} \frac{w_{\Gamma_\varepsilon}}{w_\Gamma^{\frac{1}{2}}}\sum_{l=1}^{m}\varepsilon^{-\frac{1}{2}}\gamma_{\varepsilon l}(S_{\varepsilon\delta}\theta_\delta)_l + o(1) \\ &\stackrel{def}{=} \int\limits_{\Gamma} w_\Gamma^{\frac{1}{2}}(S\theta)\beta_{\varepsilon\delta} + o(1), \end{aligned} \tag{6.43}$$

with

$$\|\beta_{\varepsilon\delta}\|_{L_2(\Gamma_\tau)} \le c \tag{6.44}$$

obtained from (6.33), (6.18), (5.23), $\beta_{\varepsilon\delta}$ vanishing in $\Gamma\setminus\Gamma_\tau$ by convention.

Using (6.44), taking a subsequence as necessary, we have

$$\beta_{\varepsilon\delta} \overset{\varepsilon,\delta\downarrow 0}{\rightharpoonup} w^{\frac{1}{2}} S\theta_0 + \beta' \tag{6.45}$$

with some $\theta_0 \in Z$ and β' satisfying

$$\int_\Gamma w_\Gamma^{\frac{1}{2}} (S\theta')\beta' = 0 \tag{6.46}$$

for all $\theta' \in Z$.

Now (5.24) follows from (6.43), (6.45), (6.46). □

7 Application, Limitations and Open Issues

Convergence of a sequence of approximations (1.14) is regarded here as compelling evidence that the boundary data (1.4) is not over-specified. In present language, such implies ker P not too large and (3.16) holding for some suitable $\|\cdot\|_{\partial\Omega}$, w, w_Γ, c_z.

A subsequent computation presumably seeks to either corroborate (1.8), (1.18), using (4.30) if not (4.6), (4.7), (4.8), or else determine where ker P is too small, where the boundary date is under-specified, using (5.5).

In this context, an empirical conclusion of whether (5.46) holds, obtained by solution of (5.34) with various $\dot{b} \in \mathcal{D}'$, will suffice. Using (5.38), (5.44), if (5.46) holds then (4.28) holds and (4.30) follows from Theorem 4.3. Alternatively, if (5.46) fails for some $\dot{b}$ we have nontrivial (f, g) satisfying (4.1) from (4.27). It is unnecessary to consider the weak limit in (5.46). With (5.36) holding, (5.46) follows, for example, from

$$\|(1-\Delta)^{-1} w_\delta^{\frac{1}{2}} R_\delta(\xi_{\dot{b}\varepsilon\delta 1} - \xi_{\dot{b}\varepsilon\delta 2})\|_{L_2(\Omega)} \overset{\varepsilon,\delta\downarrow 0}{\longrightarrow} 0. \tag{7.1}$$

Expedient simplification of the procedure is anticipated. For example, the averages over ω in (6.13) are needed only to obtain (5.20), as uniform convergence in $\Omega\backslash\Gamma$ cannot be realistically assumed in (1.14). In particular, Lemma 6.1 survives disregarding this issue and replacing $h_{ijl}^{\varepsilon\delta}$ by

$$\tilde{h}_{ijl}^{\varepsilon\delta}(x) \overset{def}{=} \psi_{i,z_j}\left(z_\delta\left(x + \frac{\varepsilon}{4}\hat{x}_l\right)\right) - \psi_{i,z_j}\left(z_\delta\left(x - \frac{\varepsilon}{4}\hat{x}_l\right)\right) \tag{7.2}$$

in (6.31). The convention (6.14), (6.15) is unnecessary computationally.

In practice, we have available only a finite sequence $\{z_\delta\}$ of approximations with

$$\delta \geq \underline{\delta} > 0, \tag{7.3}$$

for some $\underline{\delta}$. The conditions (1.14), (1.15), (5.9) are necessarily subjective conclusions, as must be (5.46) or (7.1).

Failing to satisfy (3.2), the approximations z_δ are not elements of $\mathcal{S}$. Interpolating as necessary, we may recover (5.7), but not uniformly with respect to δ.

For each z_δ, the discontinuity locus Γ is approximated by a "transition region". For any τ, identification of $\Gamma_{\tau\varepsilon}$ satisfying (6.5), (6.6), (6.7) requires ε sufficiently large, depending on z_δ and possibly τ. In particular, from (7.3) we are necessarily restricted to values

$$\varepsilon \geq \underline{\varepsilon} > 0 \tag{7.4}$$

for some $\underline{\varepsilon}$.

If (3.16) requires w_Γ unbounded, as anticipated in (6.4), the conditions (6.35), (7.4) are problematic. In contrast, if (6.8) holds, identifying $\Gamma_{\boldsymbol{\tau}}$ with Γ, we expect $[S]$ to satisfy (3.16) with

$$w_\Gamma = 1, \tag{7.5}$$

and (6.32), (6.33) with

$$w_{\Gamma_\varepsilon}(x) = \begin{cases} 1, & x \in \Gamma_{\boldsymbol{\tau}\varepsilon} \\ 0, & \text{otherwise.} \end{cases} \tag{7.6}$$

In any event, the essential open issue is the condition (5.25), needed to assure solvability of (5.34), obtaining the bounds (5.36), and to determine the space $\mathcal{D}'$ from (3.13) from which $\dot{b}$ will be selected. Theorem 5.2 may not suffice here; the condition (5.22) used to obtain (5.33) from (5.32), requires test functions θ of class C^1 in a neighborhood of Γ. In principle, such permits $\theta_{\varepsilon,\delta} \in X$ satisfying (3.16) with

$$\|\theta_{\varepsilon\delta}\|_{\partial\Omega} = 1, \tag{7.7}$$

$$\|S\theta_{\varepsilon\delta}\|_{w_\Gamma} \geq c_z, \tag{7.8}$$

but

$$\|R_\delta\theta_{\varepsilon\delta}\|_{w\delta,} \|S_{\varepsilon\delta}\theta_{\varepsilon\delta}\|_{w_{\Gamma\varepsilon}} \xrightarrow{\varepsilon,\delta\downarrow 0} 0. \tag{7.9}$$

Then $\tilde{\theta}_{\varepsilon,\delta} \in X_\delta$ approximating a constant multiple of $\theta_{\varepsilon\delta}$ contradicts (5.25).

It is unclear when or whether such will occur, and we have potential means to suppress such. First, the spaces X_δ need not correspond to the text spaces implicitly or explicitly associated with z_δ, and may be suitably restricted; the condition (5.6) suffices. Second, dissipation terms may be included in (5.17), (5.18). Third, we may choose $\|\cdot\|_{\partial\Omega}$, w_δ and perhaps w_{Γ_ε} conveniently, subject to (5.11), (5.12).

Having chosen space X_δ and whatever (if any) dissipation is to be added to (5.17), a trial-and-error procedure with regard to $\|\cdot\|_{\partial\Omega}$, w_δ (assuming (6.8), (7.6) for simplicity) is anticipated to establish (5.25). Previous results suggest an initial choice

$$w_\delta^{(0)} = 1, \|\cdot\|_{\partial\Omega}^{(0)} = \|\cdot\|_{L_2(\partial\Omega)}, \quad \|\cdot\|_{\mathcal{D}'}^{(0)} = \|\cdot\|_{L_2(\partial\Omega)} \tag{7.10}$$

using (3.13).

For $l = 0, 1, \ldots,$ we seek a minimum of $J_{\varepsilon\delta}$ over X_δ, using $w_\delta^{(l)}$, $\|\cdot\|_{\partial\Omega}^{(l)}$ in (5.17). Such may fail, implying existence of a sequence $\theta_{\varepsilon\delta} \in X_\delta$ satisfying

$$J_{\varepsilon\delta}(\theta'_{\varepsilon\delta}) \xrightarrow{\varepsilon,\delta\downarrow 0} -\infty. \tag{7.11}$$

Then

$$\theta_{\varepsilon\delta} = \frac{\theta'_{\varepsilon\delta}}{\|\theta'_{\varepsilon\delta}\|_{\partial\Omega}^{(l)}} \tag{7.12}$$

satisfying(7.7), (7.9) (and (7.8) if (3.16) holds with this $w_\delta^{(l)}$, $\|\cdot\|_{\partial\Omega}^{(l)}$).

We choose a bounded, invertible mapping Q_l on the space $X_\delta \underset{\partial\Omega}{|}$ such that the sequence $\{Q_l\theta_{\varepsilon\delta}\}$ converges in the norm $\|\cdot\|_{\partial\Omega}^{(l)}$. Typically such Q_l is a smoothing or weighting operator. Then we set

$$\|\theta\|_{\partial\Omega}^{(l+1)} = \|Q_l\theta\|_{\partial\Omega}^{(l)}, \quad \|\dot{b}\|_{\mathcal{D}'}^{(l+1)} = \|Q_l^{-1}\dot{b}\|_{\mathcal{D}'}^{(l)}, \tag{7.13}$$

and choose $w_\delta^{(l+1)}$ satisfying (5.11), (5.12), and such that

$$\|Q_l\theta_{\varepsilon\delta}\|_{\partial\Omega}^{(l)} \leq c\|R_\delta\theta_{\varepsilon\delta}\|_{w_\delta(l+1)}. \tag{7.14}$$

We attempt to minimize $J_{\varepsilon\delta}$ over $X\delta$ using $w_\delta^{(l+1)}$, $\|\cdot\|_{\partial\Omega}^{(l+1)}$, expecting success for some finite value of l. Failure to achieve such suggests that (3.16) cannot be satisfied for z_0.

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