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A Viscosity-Independent Error Estimate of a Pressure-Stabilized Lagrange–Galerkin Scheme for the Oseen Problem

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Abstract
We consider a pressure-stabilized Lagrange–Galerkin scheme for the transient Oseen problem with small viscosity. In the scheme we use the equal-order approximation of order k for both the velocity and pressure, and add a symmetric pressure stabilization term. We show an error estimate for the velocity with a constant independent of the viscosity if the exact solution is sufficiently smooth. We also show an error estimate of a discrete primitive of the pressure. Numerical examples show high accuracy of the scheme for problems with small viscosity.

Keywords Transient Oseen problem · Lagrange–Galerkin scheme · Finite element method · Equal-order elements · Symmetric pressure stabilization · Dependence on viscosity

Mathematics Subject Classification 65M12 · 65M25 · 65M60 · 76D07 · 76M10

1 Introduction

We consider a finite element scheme for the transient Oseen problem, known as linearization of the Navier–Stokes problem, with small viscosity. In this paper we construct a pressure-stabilized Lagrange–Galerkin (LG) scheme with higher-order elements, and show an error estimate independent of the viscosity.

When the viscosity is small, the finite element method suffers from two kinds of instabilities. We begin with the issue of the material derivative. In such case the convection is dominated and it is important to put weight on information in the upwind direction to make schemes stable. We here focus on the LG method, e.g. [31,32,34,37,39], which is a combination of the characteristics method and the finite element method. One of the advantages of it is that the resultant matrix is symmetric, which allows us to use efficient linear solvers

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[3]. Recently a LG method with a locally linearized velocity [40] has been developed [39] and convergence has been shown. The locally linearized velocity overcomes the difficulty in computing composite function term that appears in LG schemes. In [39] inf-sup stable elements [6] were used.

Besides the inf-sup stable elements, P_1/P_1-element with a pressure stabilization term has been also used in LG methods, where P_k/P_l shows that we use the conforming triangular or tetrahedral element of order k for the velocity and order l for the pressure. Notsu and Tabata have been proposed a LG scheme using the stabilization term of Brezzi and Pitkäranta [8] for the Navier–Stokes problem [29,30], and analyzed the scheme for the Oseen problem and Navier–Stokes problem [31,32]. Jia et al. [24] have been proposed and analyzed a LG scheme using the stabilization term of Bochev et al. [5].

Here we extend the P_1/P_1 pressure-stabilized LG scheme to higher-order elements. Simple symmetric stabilization terms for higher-order elements have been presented and applied to stationary problems in, e.g., [2,7,9,16,36] and to the transient Stokes problem in [11]. On the other hand, classical stabilization terms based on the residual of the momentum equations also have been studied for stationary problems in, e.g., [18,19,23] and for the transient Stokes problem in [27]. These terms are, however, rather complicated to implement compared to the symmetric stabilization especially for transient problems.

Apart from the issue of the material derivative in the Oseen or Navier–Stokes problems, dependence on the viscosity appears even in the Stokes problems. Numerical solutions of the velocities contain approximation errors of the pressures multiplied by the inverse of the viscosity in standard finite element methods (e.g. [26]). The grad-div stabilization [20] is a choice to improve stability. Error analyses independent of the viscosity were performed in [33] for the Stokes problem and in [14] for the transient Oseen problem relying on this term. In [4] a LG scheme was developed for the Navier–Stokes problem with local projection stabilization that includes the grad-div term.

In this paper we use P_k/P_k-element, $k \geq 1$, and pressure-stabilization in the LG scheme for the transient Oseen problem, and show an error estimate independent of the viscosity. In the scheme the symmetric pressure stabilization of Burman [9] is used and symmetry of the LG method is inherited. Although a pressure stabilized scheme for the transient Stokes problem has been analyzed by Burman and Fernández [11], we here take the constant of the stabilization term in a different way such that the constant does not depend on the viscosity. We consider the case where the viscosity ν is small and the exact solution is sufficiently smooth. The error bound presented here is of order $\Delta t + h^2 + h^k$ in the L^2-norm for the velocity and for $\nu^{1/2}$ times the gradient of the velocity, with constants independent of ν. Here, Δt is a time increment, h is a spatial mesh size. This scheme is essentially unconditionally stable, that is, we can take Δt and h independently. The grad-div stabilization is not needed in the analysis as noted by de Frutos et al. [15]. The technique used in our estimate is a projection of the exact solution of the velocity with the error independent of ν. The same projection was used by de Frutos et al. [14]. Following [15], we also derive an error estimate of a discrete primitive of the pressure.

This paper is organized as follows. In the next section, after preparing notation, we state the Oseen problem and a pressure-stabilized LG scheme. In Sect. 3 we show error estimates with constants independent of the viscosity and give proofs. In Sect. 4 we give some numerical results that show high accuracy for small viscosity and large pressures, and additionally show results of the Navier–Stokes problem. In Sect. 5 we give conclusions. In the "Appendix" section we recall some lemmas used in the LG methods.

2 A Pressure-Stabilized LG Scheme for the Oseen Problem

We prepare notation used throughout this paper, state the Oseen problem and then introduce our scheme.

Let Ω be a polygonal or polyhedral domain of $\mathbb{R}^d$ $(d = 2, 3)$. We use the Sobolev spaces $W^{m,p}(\Omega)$ equipped with the norm $\|\cdot\|_{m,p}$ and the semi-norm $|\cdot|_{m,p}$ for $p \in [1, \infty]$ and a non-negative integer m. We denote $W^{0,p}(\Omega)$ by $L^p(\Omega)$. $W_0^{1,p}(\Omega)$ is the subspace of $W^{1,p}(\Omega)$ consisting of functions whose traces vanish on the boundary of Ω. When $p = 2$, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and drop the subscript 2 in the corresponding norm and semi-norm. For the vector-valued function $w \in W^{1,\infty}(\Omega)^d$ we define the semi-norm $|w|_{1,\infty}$ by

$$\left\| \left[\sum_{i,j=1}^{d} \left(\frac{\partial w_i}{\partial x_j} \right)^2 \right]^{1/2} \right\|_{0,\infty}.$$

The pair of parentheses $(\cdot, \cdot)$ shows the $L^2(\Omega)^i$-inner product for $i = 1, d$ or $d \times d$. $L_0^2(\Omega)$ is the space of functions $q \in L^2(\Omega)$ satisfying $(q, 1) = 0$. We also use the notation $|\cdot|_{m,K}$ and $(\cdot, \cdot)_K$ for the semi-norm and the inner product on a set K.

Let $T > 0$ be a time. For a Sobolev space $X(\Omega)^i$, $i = 1, d$, we use the abbreviations $H^m(X) = H^m(0, T; X(\Omega)^i)$ and $C(X) = C([0, T]; X(\Omega)^i)$. We define the function space Z^m by

$$\begin{aligned} Z^m &:= \{v \in H^j(0, T; H^{m-j}(\Omega)^d); \quad j = 0, \ldots, m, \|v\|_{Z^m} < \infty\}, \\ \|v\|_{Z^m} &:= \left(\sum_{j=0}^{m} \|v\|^2_{H^j(0,T;H^{m-j}(\Omega)^d)} \right)^{1/2}. \end{aligned}$$

We also use the notation $H^m(t_1, t_2; X)$ and $Z^m(t_1, t_2)$ for spaces on a time interval (t_1, t_2).

We consider the Oseen problem: find $(u, p) : \Omega \times (0, T) \to \mathbb{R}^d \times \mathbb{R}$ such that

$$\begin{aligned} \frac{\partial u}{\partial t} + (w \cdot \nabla)u - \nu \Delta u + \nabla p = f \quad &\text{in} \quad \Omega \times (0, T), \\ \nabla \cdot u = 0 \quad &\text{in} \quad \Omega \times (0, T), \\ u = 0 \quad &\text{on} \quad \partial\Omega \times (0, T), \\ u(\cdot, 0) = u^0 \quad &\text{in} \quad \Omega, \end{aligned} \tag{1}$$

where $\partial\Omega$ represents the boundary of Ω, the constant $0 < \nu \leq 1$ represents a viscosity, and $w, f : \Omega \times (0, T) \to \mathbb{R}^d$ and $u^0 : \Omega \to \mathbb{R}^d$ are given functions.

We define the bilinear forms a on $H_0^1(\Omega)^d \times H_0^1(\Omega)^d$ and b on $H_0^1(\Omega)^d \times L_0^2(\Omega)$ by

$$a(u, v) := \nu(\nabla u, \nabla v), \quad b(v, q) := -(\nabla \cdot v, q).$$

Then, we can write the weak form of (1) as follows: find $(u, p) : (0, T) \to H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that for $t \in (0, T)$,

$$\left(\left(\frac{\partial u}{\partial t} + (w \cdot \nabla)u \right)(t), v \right) + a(u(t), v) + b(v, p(t)) = (f(t), v), \quad \forall v \in H_0^1(\Omega)^d, \tag{2a}$$

$$b(u(t), q) = 0, \quad \forall q \in L_0^2(\Omega), \tag{2b}$$

with $u(0) = u^0$.

We introduce time discretization. Let $\Delta t > 0$ be a time increment, $N_T := \lfloor T/\Delta t \rfloor$ the number of time steps, $t^n := n\Delta t$, and $\psi^n := \psi(\cdot, t^n)$ for a function ψ defined in $\Omega \times (0, T)$. For a set of functions $\psi = \{\psi^n\}_{n=0}^{N_T}$ we use two norms $\|\cdot\|_{\ell^\infty(L^2)}$ and $\|\cdot\|_{\ell^2(L^2)}$ defined by

$$\|\psi\|_{\ell^\infty(L^2)} := \max\left\{\|\psi^n\|_0; n = 0, \dots, N_T\right\},$$
$$\|\psi\|_{\ell^2(L^2)} := \left(\Delta t \sum_{n=1}^{N_T} \|\psi^n\|_0^2\right)^{1/2}.$$

Let w be smooth. The characteristic curve $X(t; x, s)$ is defined by the solution of the system of the ordinary differential equations,

$$\frac{dX}{dt}(t; x, s) = w(X(t; x, s), t), \quad t < s,$$
$$X(s; x, s) = x. \tag{3}$$

Then, we can write the material derivative term $\frac{\partial u}{\partial t} + (w \cdot \nabla)u$ as follows:

$$\left(\frac{\partial u}{\partial t} + (w \cdot \nabla)u\right)(X(t), t) = \frac{d}{dt}u(X(t), t).$$

For $w^* : \Omega \to \mathbb{R}^d$ we define the mapping $X_1(w^*) : \Omega \to \mathbb{R}^d$ by

$$(X_1(w^*))(x) := x - w^*(x)\Delta t. \tag{4}$$

Remark 1 The image of x by $X_1(w(\cdot, t))$ is nothing but the approximate value of $X(t - \Delta t; x, t)$ obtained by solving (3) by the backward Euler method.

Then, it holds that

$$\frac{\partial u^n}{\partial t} + (w^n \cdot \nabla)u^n = \frac{u^n - u^{n-1} \circ X_1(w^{n-1})}{\Delta t} + O(\Delta t),$$

where the symbol $\circ$ stands for the composition of functions, e.g., $(g \circ f)(x) := g(f(x))$.

We next introduce spatial discretization. Let $\{\mathcal{T}_h\}_{h\downarrow 0}$ be a regular family of triangulations of $\overline{\Omega}$ [12], $h_K := \mathrm{diam}(K)$ for an element $K \in \mathcal{T}_h$, and $h := \max_{K\in\mathcal{T}_h} h_K$. For a positive integer m, the finite element space of order m is defined by

$$W_h^{(m)} := \{\psi_h \in C(\overline{\Omega});\ \psi_{h|K} \in \mathrm{P}_m(K),\ \forall K \in \mathcal{T}_h\},$$

where $\mathrm{P}_m(K)$ is the set of polynomials on K whose degrees are equal to or less than m. Let $\Pi_h^{(m)} : C(\overline{\Omega}) \to W_h^{(m)}$ be the Lagrange interpolation operator, which is naturally extended to vector-valued functions.

We begin with a scheme using the standard $\mathrm{P}_k/\mathrm{P}_{k-1}$-finite element, which is called (generalized) Taylor–Hood element. Let

$$V_h \times \overline{Q}_h := ((W_h^{(k)})^d \cap H_0^1(\Omega)^d) \times (W_h^{(k-1)} \cap L_0^2(\Omega)) \tag{5}$$

be the $\mathrm{P}_k/\mathrm{P}_{k-1}$-finite element space for $k \geq 2$. The LG scheme with a locally linearized velocity and this Taylor–Hood element for the Oseen problem (OsTH) is stated as follows:

Scheme OsTH Let $u_h^0 \in V_h$ be an approximation of u^0. Find $\{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times \overline{Q}_h$ such that

$$\left(\frac{u_h^n - u_h^{n-1} \circ X_1(\Pi_h^{(1)} w^{n-1})}{\Delta t}, v_h\right) + a(u_h^n, v_h) + b(v_h, p_h^n)$$
$$= (f^n, v_h), \qquad \forall v_h \in V_h, \tag{6a}$$
$$b(u_h^n, q_h) = 0, \qquad \forall q_h \in \overline{Q}_h. \tag{6b}$$

When $k = 2$, this type of scheme for the Navier–Stokes problem has already been introduced and analyzed in [39]. In the mapping $X_1(\cdot)$, a locally linearized velocity $\Pi_h^{(1)} w^{n-1}$ is used instead of the original velocity w^{n-1}. If the original velocity is used, it is difficult to evaluate the exact value of integration. The next proposition assures that the scheme with the locally linearized velocity is exactly computable.

Proposition 1 ([38,39]) *Let $u_h, v_h \in (W_h^{(m)})^d$ for a positive integer m. Suppose that*

$$w^* \in W_0^{1,\infty}(\Omega)^d \quad \textit{and} \quad \alpha_* \Delta t |w^*|_{1,\infty} < 1, \tag{7}$$

where α_* is the constant defined in (11) below. Then, $\int_\Omega (u_h \circ X_1(\Pi_h^{(1)} w^*)) \cdot v_h dx$ is exactly computable.

With the Assumption (7) for $w^* = w^{n-1}$ at each step n, (6) is exactly computable thanks to Proposition 1. The condition $w^* = 0$ on $\partial\Omega$ is assumed so that the inclusion $(X_1(\Pi_h^{(1)} w^*))(\Omega) \subset \Omega$ holds, which is necessary in our analysis. For example, the condition also appears in Lemmas 7–11 below.

In [39] the authors have analyzed the scheme to show the estimates

$$\|\nabla(u_h - u)\|_{\ell^\infty(L^2)}, \ \|p_h - p\|_{\ell^2(L^2)} \le c(\nu^{-1})(\Delta t + h^2),$$

where the constant c depends on ν^{-1} exponentially.

Here we use the equal-order element with pressure stabilization. Let

$$V_h \times Q_h := ((W_h^{(k)})^d \cap H_0^1(\Omega)^d) \times (W_h^{(k)} \cap L_0^2(\Omega))$$

be the equal-order $\mathrm{P}_k/\mathrm{P}_k$-finite element space for $k \ge 1$. We define a pressure stabilization term $\mathcal{C}_h : Q_h \times Q_h \to \mathbb{R}$, which enables us to use the equal-order element, by

$$\mathcal{C}_h(p_h, q_h) := \sum_{K \in \mathcal{T}_h} h_K^{2k} \sum_{|\alpha|=k} (D^\alpha p_h, D^\alpha q_h)_K,$$

where α is the multi-index and D^α is the partial differential operator. We define the corresponding semi-norm on Q_h by

$$|q_h|_h := \mathcal{C}_h(q_h, q_h)^{1/2} = \left(\sum_{K \in \mathcal{T}_h} h_K^{2k} |q_h|_{k,K}^2\right)^{1/2}. \tag{8}$$

Remark 2 The term $\mathcal{C}_h$ introduced by Burman [9] is an extension of that by Brezzi and Pitkäranta [8] for the $\mathrm{P}_1/\mathrm{P}_1$-element to higher order elements. For the stabilization term, instead of $\mathcal{C}_h$, we can also choose another positive semi-definite bilinear form whose corresponding semi-norm is equivalent to (8). Examples include the terms in [2,16,36], as pointed out in [9].

We are now in position to state a pressure-stabilized LG scheme for the Oseen problem (OsPstab).

Scheme OsPstab Let $u_h^0 \in V_h$ be an approximation of u^0. Find $\{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that

$$\left(\frac{u_h^n - u_h^{n-1} \circ X_1(\Pi_h^{(1)} w^{n-1})}{\Delta t}, v_h\right) + a(u_h^n, v_h) + b(v_h, p_h^n)$$

$$= (f^n, v_h), \ \forall v_h \in V_h, \tag{9a}$$

$$b(u_h^n, q_h) - \delta_0 \mathcal{C}_h(p_h^n, q_h) = 0, \ \forall q_h \in Q_h, \tag{9b}$$

where $\delta_0 > 0$ is a stabilization parameter.

With the Assumption (7) for $w^* = w^{n-1}$ at each step n, (9) is exactly computable and has a unique solution (u_h^n, p_h^n) thanks to Proposition 1 and the stabilization term $\mathcal{C}_h$ [9]. The error introduce by the locally linearized velocity is properly estimated in (21) below.

Remark 3 1. In Scheme OsPstab, the resultant matrix to be solved is symmetric and remains unchanged at each time step, which enables us to use efficient linear solvers and preconditioners [3].

2. The inequality (7) is related to the stability condition of the Scheme OsPstab. This scheme is essentially unconditionally stable, that is, the condition on Δt is not affected by the mesh size h. It is known that the backward Euler, BDF2, Crank–Nicolson schemes (e.g. [14]) are also unconditionally stable. However, these schemes contain unsymmetric matrices that originate from the convective terms.
3. When $k = 1$, Notsu and Tabata [31] proposed and analyzed a pressure-stabilized LG scheme, where the locally linearized velocity was not introduced.
4. Burman and Fernández [11] proposed and analyzed a scheme for the transient Stokes problem using the same type of pressure stabilization. Since in their choice the stabilization parameter δ_0 is proportional to $1/\nu$, it seems to be difficult to get error estimates independent of ν, which we will show in the next section.

3 An Error Estimate Focused on the Viscosity for the Oseen Problem

3.1 An Error Estimate for the Velocity

Before stating the result we introduce hypotheses.

Hypothesis 1 The velocity w and the exact solution (u, p) of the Oseen problem (1) satisfies

$$w \in C(W_0^{1,\infty} \cap W^{2,\infty}) \cap H^1(L^\infty), \ u \in Z^2 \cap H^1(H^{k+1}), \ p \in C(H^{k+1}).$$

Hypothesis 2 The time increment Δt satisfies $0 < \Delta t \leq \Delta t_0$, where

$$\Delta t_0 := \frac{1}{4\alpha_* |w|_{C(W^{1,\infty})}},$$

and α_* is the constant defined in (11) below.

Hypothesis 3 (*Triangulation*) Every element $K \in \mathcal{T}_h$ has at least one internal vertex.

Hypothesis 4 (*Choice of the initial value*) There exists a positive constant c independent of h such that

$$\|u_h^0 - u^0\|_0 \leq c h^k |u^0|_k.$$

Remark 4 1. Hypothesis 1 implies that $u \in C(H^{k+1})$ and $u^0 \in H^{k+1}(\Omega)^d$.
2. Under Hypotheses 1 and 2, the property (7) for $w^* = w^{n-1}$ at each step n is clearly satisfied.
3. Hypothesis 4 is satisfied if we take u_h^0 as the Lagrange interpolation of u^0, for example. Herein, Hypothesis 4 is enough to derive the estimate for the velocity (Theorem 1) and a discrete primitive of the pressure (Theorem 2). However, to derive estimates for the pressure in a norm, a special choice of the initial value is needed. In the analysis of schemes with symmetric pressure stabilization [11], they took u_h^0 as the Ritz-projection of the initial value u^0 to derive the optimal estimate $O(h^k)$ for the pressure in L^2-norm. In the analysis of pressure-stabilized Petrov–Galerkin schemes [27], they proposed a special choice of u_h^0 to remove the effect of the pressure error at the initial time. In the present scheme with a special choice of the initial value, we can derive the estimate for the pressure of the optimal order as in [31] but the constant depends on the viscosity.

Theorem 1 *Let $V_h \times Q_h$ be the $\mathrm{P}_k/\mathrm{P}_k$-finite element space for $k \geq 1$. Suppose Hypotheses 1–4. Let $(u_h, p_h) := \{(u_h^n, p_h^n)\}_{n=0}^{N_T}$ be the solution of Scheme OsPstab. Then it holds that*

$$\|u_h - u\|_{\ell^\infty(L^2)}, \sqrt{\nu}\|\nabla(u_h - u)\|_{\ell^2(L^2)} \leq c_*(\Delta t + h^2 + h^k)\Big[\|w\|_{H^1(L^\infty)} + \|u\|_{Z^2} + \|u\|_{H^1(H^{k+1})} + (1+\delta_0^{-1})\|u\|_{\ell^\infty(H^{k+1})} (1+\delta_0)\|p\|_{\ell^2(H^{k+1})}\Big], \tag{10}$$

where c_* is a positive constant independent of ν, h, Δt but depends on T, $\|u\|_{C(H^1)}$ and $\|w\|_{C(W^{2,\infty})}$.

Remark 5 1. Note that we assumed that $\nu \leq 1$.
2. The parameter δ_0 should not depend on ν from the viewpoint of this estimate. In [11] they took $\delta_0 = 1/\nu$. Indeed, if we consider the physical units, the dimension of δ_0 is same as that of $1/\nu$, which is seen, e.g., in (25) below. However, to minimize the terms in the right hand side of (25) and (26), the choice $\delta_0 \sim 1/\nu$ does not seem always optimal.
3. If $\mathrm{P}_k/\mathrm{P}_{k-1}$-element is employed, we have an estimate of the same order $\Delta t + h^2 + h^k$, but it seems to be difficult to remove the dependence on the viscosity, which is observed in the numerical experiments in Sect. 4.
4. The term h^2 appears in (10) because of the introduction of the locally linearized velocity.
5. It seems to be difficult to derive the estimate of order $O(h^{k+1})$ for the spatial discretization in $\ell^\infty(L^2)$ independent of the viscosity. Although another type of Stokes projection yields an estimate of order $O(h^{k+1})$, e.g. [31], the projection error contains the dependence. According to [25], it is open that whether such viscosity-independent “semi-robust” estimate can be proved for some method. The estimate $O(h^{k+1/2})$ can be found for the continuous interior penalty method [10] and the local projection stabilization method [15] for the Navier–Stokes problem. De Frutos et al. [14] derived the same order $O(h^k)$ as ours independent of the viscosity for the backward Euler method or the BDF2 formula with the grad-div stabilization.
6. When $k = 1$, Notsu and Tabata [31] analyzed the pressure-stabilized LG scheme without the locally linearized velocity. They derived the estimates

$$\|\nabla(u_h - u)\|_{\ell^\infty(L^2)}, \ \|p_h - p\|_{\ell^2(L^2)} \leq c(\nu^{-1})(\Delta t + h),$$

where the constant c depends on ν^{-1} exponentially.

Before the proof we prepare some lemmas. First we recall a discrete version of the Gronwall inequality.

Lemma 1 (discrete Gronwall inequality) *Let γ_0 and γ_1 be non-negative numbers, $\Delta t \in (0, \frac{1}{2\gamma_0}]$ be a real number, and $\{x^n\}_{n\ge 0}$, $\{y^n\}_{n\ge 1}$ and $\{b^n\}_{n\ge 1}$ be non-negative sequences. Suppose*

$$\frac{x^n - x^{n-1}}{\Delta t} + y^n \le \gamma_0 x^n + \gamma_1 x^{n-1} + b^n,\ \forall n \ge 1.$$

Then, it holds that

$$x^n + \Delta t \sum_{i=1}^{n} y^i \le \exp[(2\gamma_0 + \gamma_1)n\Delta t]\left(x^0 + \Delta t \sum_{i=1}^{n} b^i\right), \quad \forall n \ge 1.$$

Lemma 1 is shown by using the inequalities

$$\begin{aligned} x^n + y^n\Delta t &\le (1-\gamma_0\Delta t)^{-1}[(1-\gamma_0\Delta t)x^n + y^n\Delta t] \\ &\le (1-\gamma_0\Delta t)^{-1}[(1+\gamma_1\Delta t)x^{n-1} + b^n\Delta t] \\ &\le \exp[(2\gamma_0+\gamma_1)\Delta t](x^{n-1} + b^n\Delta t), \end{aligned}$$

where we have used simple inequalities

$$\begin{aligned} &1 \le \frac{1}{1-\gamma_0\Delta t} \le 1 + 2\gamma_0\Delta t, \\ &1 \le 1 + \gamma\Delta t \le \exp(\gamma\Delta t), \quad \gamma \ge 0. \end{aligned}$$

Instead of the well-know summation form of the discrete Gronwall inequality, e.g., in [22], we use this form because the condition on Δt does not include γ_1, which will make the proof simpler.

In Lemmas 2–4 below, the constants c are independent of h.

We recall the fundamental properties of Lagrange and Clément interpolations [12,13].

Lemma 2 *Suppose that $\{\mathcal{T}_h\}_{h\downarrow 0}$ is a regular family of triangulations of $\overline{\Omega}$.*
(i) Let $\Pi_h^{(m)} : C(\overline{\Omega})^i \to (W_h^{(m)})^i$, $i = 1, d$, be the Lagrange interpolation operator to P_m-finite element space for a positive integer m. Then there exist positive constants $\alpha_* \ge 1$ and c independent of h such that

$$\begin{aligned} |\Pi_h^{(1)} w|_{1,\infty} &\le \alpha_* |w|_{1,\infty}, \quad \forall w \in W^{1,\infty}(\Omega)^d, \\ \|\Pi_h^{(1)} w - w\|_{0,\infty} &\le c h^2 |w|_{2,\infty}, \quad \forall w \in W^{2,\infty}(\Omega)^d, \\ \|\Pi_h^{(m)} w - w\|_{0,K} &\le c h_K^{m+1} |w|_{m+1,K},\ \forall K \in \mathcal{T}_h, \forall w \in H^{m+1}(K)^i, i = 1, d. \end{aligned} \tag{11}$$

(ii) Let $\Pi_{h,C}^{(m)} : L^2(\Omega) \to W_h^{(m)}$ be the Clément interpolation operator to P_m-finite element space for a positive integer m. Then there exists a positive constants c such that

$$\begin{aligned} |\Pi_{h,C}^{(m)}\psi - \psi|_1 &\le c h^m |\psi|_{m+1}, && \forall \psi \in H^{m+1}(\Omega), \\ \left(\sum_{K\in\mathcal{T}_h} |\Pi_{h,C}^{(m)}\psi|_{m,K}^2\right)^{1/2} &\le c|\psi|_m, && \forall \psi \in H^m(\Omega). \end{aligned}$$

In our analysis, we need an approximation z_h of a divergence-free velocity and a bound for $b(z_h, q_h)$ for $q_h \in Q_h$. We choose z_h as the Lagrange interpolation or the first component of a modified Stokes projection. These are examples of various projection considered in [11, inequalities (3.5) and (3.6)].

When $k \geq 2$, we use the auxiliary P_{k-1}-pressure space $\overline{Q}_h$ defined in (5), and $(\widehat{z}_h, \widehat{r}_h) \in V_h \times \overline{Q}_h$ be the Stokes projection of $(z, r) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ for the fixed viscosity $\nu = 1$ defined by

$$(\nabla \widehat{z}_h, \nabla v_h) - (\nabla \cdot v_h, \widehat{r}_h) = (\nabla z, \nabla v_h) - (\nabla \cdot v_h, r), \qquad \forall v_h \in V_h, \tag{12a}$$

$$-(\nabla \cdot \widehat{z}_h, \overline{q}_h) = -(\nabla \cdot z, \overline{q}_h), \qquad \forall \overline{q}_h \in \overline{Q}_h. \tag{12b}$$

In the case $(z, r) = (u, 0)$ this is the modified Stokes projection introduced by de Frutos et al. [14].

Lemma 3 *Suppose that $\{\mathcal{T}_h\}_{h \downarrow 0}$ is a regular family of triangulations of $\overline{\Omega}$ and Hypothesis 3. Let $V_h \times \overline{Q}_h$ be the $\mathrm{P}_k/\mathrm{P}_{k-1}$-finite element space for $k \geq 2$. Then, there exists a positive constant c such that*

$$\|\widehat{z}_h - z\|_1, \|\widehat{r}_h - r\|_0 \leq ch^k(|z|_{k+1} + |r|_k), \tag{13}$$

where $(\widehat{z}_h, \widehat{r}_h) \in V_h \times \overline{Q}_h$ is the Stokes projection of $(z, r) \in (H_0^1(\Omega)^d \cap H^{k+1}(\Omega)^d) \times (L_0^2(\Omega) \cap H^k(\Omega))$ defined in (12).

This estimate is a direct consequence of the inf-sup stability for the $\mathrm{P}_k/\mathrm{P}_{k-1}$-element [6]. Since in (12) the fixed viscosity is used, we have the estimate of the projection independent of the viscosity.

Lemma 4 *Suppose that $z \in H^{k+1}(\Omega)^d$ satisfies $\nabla \cdot z = 0$, $\{\mathcal{T}_h\}_{h \downarrow 0}$ is a regular family of triangulations of $\overline{\Omega}$ and Hypothesis 3. Let $V_h \times Q_h$ be the $\mathrm{P}_k/\mathrm{P}_k$-finite element space for $k \geq 1$. Let $z_h \in V_h$ be the Lagrange interpolation of z when $k = 1$, or the first component of the Stokes projection of $(z, 0)$ defined in (12) when $k \geq 2$. Then, there exists a positive constant c such that*

$$b(z_h, q_h) \leq ch^k|z|_{k+1}|q_h|_h, \ \forall q_h \in Q_h, \tag{14}$$

where the semi-norm $|\cdot|_h$ is defined in (8)

Proof When $k = 1$, by using $\nabla \cdot z = 0$, the integration by part and Lemma 2, we get the estimate (14) as follows:

$$\begin{aligned} b(z_h, q_h) = b(z_h - z, q_h) = (z_h - z, \nabla q_h) &\leq c \sum_{K \in \mathcal{T}_h} h_K^2 |z|_{2,K} \|\nabla q_h\|_{0,K} \\ &\leq ch|z|_2|q_h|_h. \end{aligned}$$

When $k \geq 2$, it holds that from (12b)

$$b(z_h, \overline{q}_h) = b(z, \overline{q}_h) = 0, \ \forall \overline{q}_h \in \overline{Q}_h.$$

Let $\overline{\Pi}_h^{(k-1)} q_h \in \overline{Q}_h$ be the Lagrange interpolation of q_h with the correction of the constant so that $\overline{\Pi}_h^{(k-1)} q_h \in L_0^2(\Omega)$. Since $q_h - \overline{\Pi}_h^{(k-1)} q_h \in L_0^2(\Omega)$ and $L_0^2(\Omega)$ is orthogonal to constants, it holds that from Lemma 2

$$\|q_h - \overline{\Pi}_h^{(k-1)} q_h\|_0 \leq \|q_h - \Pi_h^{(k-1)} q_h\|_0 \leq c \left(\sum_{K \in \mathcal{T}_h} h_K^{2k} |q_h|_{k,K}^2 \right)^{1/2}.$$

By Lemma 3 we get the estimate (14) as follows:

$$
\begin{aligned}
b(z_h, q_h) &= b(z_h, q_h - \overline{\Pi}_h^{(k-1)} q_h) \\
&\le \|\nabla \cdot (z_h - z)\|_0 \|q_h - \overline{\Pi}_h^{(k-1)} q_h\|_0 \le c h^k |z|_{k+1} |q_h|_h .
\end{aligned}
$$

□

We now begin the proof of Theorem 1, where we also refer to Lemmas 7–10 in the “Appendix” section for properties of the mapping $X_1(\cdot)$.

Proof (Theorem 1) Here we simply write $X_{1h}^{n-1} = X_1(\Pi_h^{(1)} w^{n-1})$. We use c to represent a generic positive constant that is independent of ν, Δt and h but depends on Sobolev norms $\|u\|_{C(H^1)}$ and $\|w\|_{C(W^{2,\infty})}$, and may take a different value at each occurrence.

Let $z_h(t) \in V_h$ be, as in Lemma 4, the Lagrange interpolation of $u(t)$ when $k = 1$, or the first component of the Stokes projection of $(u(t), 0)$ defined in (12) when $k \ge 2$, and let $r_h(t) \in Q_h$ be the Clément interpolation of $p(t)$ with the correction of the constant so that $r_h(t) \in L_0^2(\Omega)$. We define the error terms by

$$
(e_h^n, \varepsilon_h^n) := (u_h^n - z_h^n, p_h^n - r_h^n), \ \ \eta(t) := u(t) - z_h(t).
$$

From (9a), (2a) with $t = t^n$ and $v = v_h$, and (9b), we have an error equations in (e_h^n, ε_h^n):

$$
\begin{aligned}
&\left(\frac{e_h^n - e_h^{n-1} \circ X_{1h}^{n-1}}{\Delta t}, v_h \right) + a(e_h^n, v_h) + b(v_h, \varepsilon_h^n) \\
&\quad = (R^n, v_h) + a(\eta^n, v_h) + b(v_h, p^n - r_h^n), \qquad \forall v_h \in V_h, \qquad (15a)\\
&b(e_h^n, q_h) - \delta_0 \mathcal{C}_h(\varepsilon_h^n, q_h) = -b(z_h^n, q_h) + \delta_0 \mathcal{C}_h(r_h^n, q_h), \qquad \forall q_h \in Q_h, \qquad (15b)
\end{aligned}
$$

for $n = 1, \ldots, N_T$, where $R^n := R_1^n + R_2^n + R_3^n$,

$$
\begin{aligned}
R_1^n &:= \frac{\partial u^n}{\partial t} + (w^n \cdot \nabla) u^n - \frac{u^n - u^{n-1} \circ X_1(w^{n-1})}{\Delta t}, \\
R_2^n &:= \frac{u^{n-1} \circ X_{1h}^{n-1} - u^{n-1} \circ X_1(w^{n-1})}{\Delta t}, \\
R_3^n &:= \frac{\eta^n - \eta^{n-1} \circ X_{1h}^{n-1}}{\Delta t}.
\end{aligned} \tag{16}
$$

Substituting $(v_h, q_h) = (e_h^n, \varepsilon_h^n)$ in (15) and using the identity $(a-b)a = (1/2)(a^2 - b^2 + (a-b)^2)$ yields

$$
\begin{aligned}
&\frac{1}{2\Delta t}\left(\|e_h^n\|_0^2 - \|e_h^{n-1} \circ X_{1h}^{n-1}\|_0^2 + \|e_h^n - e_h^{n-1} \circ X_{1h}^{n-1}\|_0^2\right) + \nu \|\nabla e_h^n\|_0^2 + \delta_0 |\varepsilon_h^n|_h^2 \\
&\quad = (R^n, e_h^n) + a(\eta^n, e_h^n) + b(e_h^n, p^n - r_h^n) + b(z_h^n, \varepsilon_h^n) - \delta_0 \mathcal{C}_h(r_h^n, \varepsilon_h^n).
\end{aligned} \tag{17}
$$

We now estimate the terms in (17). With Hypothesis 2 and the properties

$$
\Pi_h^{(1)} w^{n-1} \in W_0^{1,\infty}(\Omega)^d \quad \text{and} \quad |\Pi_h^{(1)} w^{n-1}|_{1,\infty} \Delta t \le \alpha_* |w^{n-1}|_{1,\infty} \Delta t \le 1/4, \tag{18}
$$

we use Lemma 7 in “Appendix” to have

$$
\|e_h^{n-1} \circ X_{1h}^{n-1}\|_0^2 \le (1 + c\Delta t) \|e_h^{n-1}\|_0^2. \tag{19}
$$

To apply the discrete Gronwall inequality (Lemma 1) with $x^n := \|e_h^n\|_0^2$, we fix a γ_0 such that $\Delta t_0 \leq \frac{1}{2\gamma_0}$. From the Schwarz's inequality, we obtain

$$(R_i^n, e_h^n) \leq \frac{2}{\gamma_0}\|R_i^n\|_0^2 + \frac{\gamma_0}{8}\|e_h^n\|_0^2, \ i = 1, 2, 3.$$

We estimate $\|R_i^n\|_0$, $i = 1, 2, 3$. By Lemma 8 in "Appendix",

$$\|R_1^n\|_0 \leq c\sqrt{\Delta t}\left(\|u\|_{Z^2(t^{n-1},t^n)} + \left\|\frac{\partial w}{\partial t}\right\|_{L^2(t^{n-1},t^n;L^\infty)}\right). \tag{20}$$

By Lemma 9 in "Appendix" with $q = 2$, $p = \infty$, $p' = 1$, $w_1 = \Pi_h^{(1)} w^{n-1}$, $w_2 = w^{n-1}$ and $v = u^{n-1}$, and by Lemma 2,

$$\|R_2^n\|_0 \leq c\|\Pi_h^{(1)} w^{n-1} - w^{n-1}\|_{0,\infty} \leq ch^2. \tag{21}$$

By Lemma 10 in "Appendix" with $v = \eta$ and $w^* = \Pi_h^{(1)} w^{n-1}$, and by Lemmas 2 or 3,

$$\begin{aligned}\|R_3^n\|_0 &\leq \frac{c}{\sqrt{\Delta t}}\left(\left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(t^{n-1},t^n;L^2)} + \|\nabla\eta\|_{L^2(t^{n-1},t^n;L^2)}\right)\\ &\leq \frac{ch^k}{\sqrt{\Delta t}}\left(\|u\|_{H^1(t^{n-1},t^n;H^{k+1})}\right).\end{aligned} \tag{22}$$

An estimate for a is easily obtained by Lemmas 2 or 3:

$$a(\eta^n, e_h^n) \leq \frac{\nu}{2}\|\nabla\eta^n\|_0^2 + \frac{\nu}{2}\|\nabla e_h^n\|_0^2 \leq ch^{2k}|u^n|_{k+1}^2 + \frac{\nu}{2}\|\nabla e_h^n\|_0^2, \tag{23}$$

where we note that we assumed $\nu \leq 1$. The integration by part and Lemma 2-(ii) yields

$$\begin{aligned}b(e_h^n, p^n - r_h^n) = (e_h^n, \nabla(p^n - r_h^n)) &\leq \frac{\gamma_0}{8}\|e_h^n\|_0^2 + \frac{2}{\gamma_0}\|\nabla(p^n - r_h^n)\|_0^2\\ &\leq \frac{\gamma_0}{8}\|e_h^n\|_0^2 + ch^{2k}|p^n|_{k+1}^2.\end{aligned} \tag{24}$$

By Lemma 4,

$$b(z_h^n, \varepsilon_h^n) \leq ch^k|u^n|_{k+1}|\varepsilon_h^n|_h \leq \frac{c}{\delta_0}h^{2k}|u^n|_{k+1}^2 + \frac{\delta_0}{4}|\varepsilon_h^n|_h^2. \tag{25}$$

By using stability of Clément interpolation (Lemma 2-(ii)),

$$-\delta_0\mathcal{C}_h(r_h^n, \varepsilon_h^n) \leq \delta_0|r_h^n|_h|\varepsilon_h^n|_h \leq \delta_0|r_h^n|_h^2 + \frac{\delta_0}{4}|\varepsilon_h^n|_h^2 \leq c\delta_0 h^{2k}|p^n|_k^2 + \frac{\delta_0}{4}|\varepsilon_h^n|_h^2. \tag{26}$$

Gathering the estimates (19)–(26), from (17) we obtain

$$\begin{aligned}&\frac{1}{2\Delta t}(\|e_h^n\|_0^2 - \|e_h^{n-1}\|_0^2) + \frac{\nu}{2}\|\nabla e_h^n\|_0^2 + \frac{\delta_0}{2}|\varepsilon_h^n|_h^2 \leq c\|e_h^{n-1}\|_0^2 + \frac{\gamma_0}{2}\|e_h^n\|_0^2\\ &\quad + c\left\{\Delta t\left(\|u\|_{Z^2(t^{n-1},t^n)}^2 + \left\|\frac{\partial w}{\partial t}\right\|_{L^2(t^{n-1},t^n;L^\infty)}^2\right)\right.\\ &\qquad \left. + \frac{h^{2k}}{\Delta t}\|u\|_{H^1(t^{n-1},t^n;H^{k+1})}^2 + h^4 + h^{2k}[(1+\delta_0^{-1})\|u^n\|_{k+1}^2 + (1+\delta_0)\|p^n\|_{k+1}^2]\right\},\end{aligned}$$

We now apply Lemma 1 to obtain the following for $1 \le n \le N_T$

$$
\begin{aligned}
&\|e_h^n\|_0^2 + \nu\Delta t \sum_{j=1}^{n} \|\nabla e_h^j\|_0^2 + \delta_0 \Delta t \sum_{j=1}^{n} |\varepsilon_h^j|_h^2 \\
&\quad \le c \exp\{c' n \Delta t\}(\Delta t^2 + h^{2k} + h^4)\Bigg[\left\|\frac{\partial w}{\partial t}\right\|^2_{L^2(0,t^n;L^\infty)} + \|u\|^2_{Z^2(0,t^n)} + \|u\|^2_{H^1(0,t^n;H^{k+1})} \\
&\quad + (1+\delta_0^{-1})\Delta t \sum_{j=1}^{n} \|u^j\|_{k+1}^2 + (1+\delta_0)\Delta t \sum_{j=1}^{n} \|p^j\|_{k+1}^2 + \|u^0\|_{k+1}^2\Bigg],
\end{aligned} \tag{27}
$$

where we have used Hypothesis 4 for the initial value. We have the conclusion by the triangle inequalities,

$$
\begin{aligned}
\|u_h - u\|_{\ell^\infty(L^2)} &\le \|e_h\|_{\ell^\infty(L^2)} + \|\eta\|_{\ell^\infty(L^2)} \\
&\le \|e_h\|_{\ell^\infty(L^2)} + ch^k \|u\|_{\ell^\infty(H^{k+1})}, \\
\|\nabla(u_h - u)\|_{\ell^2(L^2)} &\le \|\nabla e_h\|_{\ell^2(L^2)} + \|\nabla \eta\|_{\ell^2(L^2)} \\
&\le \|\nabla e_h\|_{\ell^2(L^2)} + ch^k \|u\|_{\ell^2(H^{k+1})}.
\end{aligned}
$$

□

Remark 6 Our analysis need that Q_h is P_k-finite element space in the estimate (24) to have $O(h^k)$ in H^1-norm.

3.2 An Error Estimate of a Discrete Primitive of the Pressure

Following [15] we derive an error estimate of a discrete in time primitive of the pressure instead of the $\ell^2(L^2)$ norm of the pressure. We also use the estimate in [17] to bound a term in the LG scheme.

First we recall the inf-sup stability of the stabilized method [9].

Lemma 5 *There exists a positive constant c such that*

$$
\|q_h\|_0 \le c \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_1} + c|q_h|_h, \;\; q_h \in Q_h.
$$

We now define the discrete in time primitive of the pressure:

$$
P^n := \Delta t \sum_{j=1}^{n} p^j, \;\; P_h^n := \Delta t \sum_{j=1}^{n} p_h^j, \;\; n = 1, \ldots, N_T.
$$

Theorem 2 *Under the same assumption as in Theorem 1, it holds that*

$$
\begin{aligned}
&\|P_h - P\|_{\ell^\infty(L^2)} \\
&\quad \le c_{**}(\Delta t + h^2 + h^k) \max\{1, \delta_0^{-1/2}\}\Big[\|w\|_{H^1(L^\infty)} + \|u\|_{Z^2} + \|u\|_{H^1(H^{k+1})} \\
&\quad + (1+\delta_0^{-1})\|u\|_{\ell^\infty(H^{k+1})} + (1+\delta_0)\|p\|_{\ell^2(H^{k+1})}\Big],
\end{aligned} \tag{28}
$$

where c_{**} is a positive constant independent of ν, h, Δt but depends on T, $\|u\|_{C(H^1)}$ and $\|w\|_{C(W^{2,\infty})}$.

Proof We use same notation in the proof of Theorem 1. We define E_h^n by

$$E_h^n := \Delta t \sum_{j=1}^{n} \varepsilon_h^j.$$

By Lemma 5 it holds that

$$\|E_h^n\|_0 \le c \sup_{v_h \in V_h} \frac{b(v_h, E_h^n)}{\|v_h\|_1} + c|E_h^n|_h. \tag{29}$$

From (15a) we obtain for $n = 1, \ldots, N_T$,

$$\begin{aligned} b(v_h, E_h^n) =& -\sum_{j=1}^{n}(e_h^j - e_h^{j-1} \circ X_{1h}^{j-1}, v_h) - \Delta t \sum_{j=1}^{n} a(u_h^n - u^n, v_h) \\ &+ \Delta t \sum_{j=1}^{n}(R^j, v_h) + \Delta t \sum_{j=1}^{n} b(v_h, p^n - r_h^n) \\ =:& I_1 + I_2 + I_3 + I_4. \end{aligned}$$

For I_1, it holds that

$$|I_1| = \left|(e_h^n - e_h^0, v_h) + \sum_{j=1}^{n}(e_h^{j-1} - e_h^{j-1} \circ X_{1h}^{j-1}, v_h)\right|,$$

and

$$|(e_h^{j-1} - e_h^{j-1} \circ X_{1h}^{j-1}, v_h)| \le \|e_h^{j-1} - e_h^{j-1} \circ X_{1h}^{j-1}\|_{-1}\|v_h\|_1 \le c\Delta t\|e_h^{j-1}\|_0\|v_h\|_1,$$

where $\|\cdot\|_{-1}$ is the norm in H^{-1} and we have used Lemma 11 in the “Appendix” section. We then obtain

$$|I_1| \le (\|e_h^n\|_0 + \|e_h^0\|_0 + cT\|e_h\|_{\ell^\infty(L^2)})\|v_h\|_1.$$

Estimates of $|I_i|$, $i = 2, 3, 4$, are easily obtained as follows:

$$\begin{aligned} |I_2| &\le \nu\Delta t \sum_{j=1}^{n} \|\nabla(u_h^j - u^j)\|_0\|\nabla v_h\|_0 \\ &\le \sqrt{\nu T}(\sqrt{\nu}\|\nabla(u_h - u)\|_{\ell^2(L^2)})\|v_h\|_1, \\ |I_3| &\le \Delta t \sum_{j=1}^{n} \|R^j\|_0\|v_h\|_0 \le \sqrt{T}\|R\|_{\ell^2(L^2)}\|v_h\|_1, \\ |I_4| &\le \Delta t \sum_{j=1}^{n} \|p^n - r_h^n\|_0\|v_h\|_1. \end{aligned}$$

For $|E_h^n|_h$, it holds that

$$|E_h^n|_h \le \sqrt{T}\delta_0^{-1/2}\left(\delta_0\Delta t \sum_{j=1}^{n} |\varepsilon_h^j|_h^2\right)^{1/2}.$$

Gathering these estimates, from (29) we have

$$\|E_h^n\|_0 \le c\Bigg[(1+T)\|e_h\|_{\ell^\infty(L^2)} + \sqrt{\nu T}(\sqrt{\nu}\|\nabla(u_h-u)\|_{\ell^2(L^2)}) + \sqrt{T}\|R\|_{\ell^2(L^2)} + \Delta t\sum_{j=1}^n \|p^n - r_h^n\|_0 + \sqrt{T}\delta_0^{-1/2}\Big(\delta_0\Delta t\sum_{j=1}^n |\varepsilon_h^j|_h^2\Big)^{1/2}\Bigg].$$

The bound for $\delta_0\Delta t\sum_{j=1}^n |\varepsilon_h^j|_h^2$ was already obtained in (27) in the proof of Theorem 1. Gathering these estimate and using (10), (20)–(22) and Lemma 2, we have we have the conclusion (28).

4 Numerical Results

We consider test problems given by manufactured solutions in $d = 2$. We compare Schemes OsTH and OsPstab with $k = 2$ for the Oseen problem (1) to show higher accuracy of Scheme OsPstab for small viscosity and large pressures. We additionally show numerical results of the Navier–Stokes problem, which is given by replacing w by the unknown u in (1). The corresponding Schemes NSTH and NSPstab are given by replacing w^{n-1} by u_h^{n-1} in Schemes OsTH and OsPstab.

Scheme NSTH Let $u_h^0 \in V_h$ be an approximation of u^0. Find $\{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times \overline{Q}_h$ such that

$$\left(\frac{u_h^n - u_h^{n-1}\circ X_1(\Pi_h^{(1)}u_h^{n-1})}{\Delta t}, v_h\right) + a(u_h^n, v_h) + b(v_h, p_h^n) = (f^n, v_h), \quad \forall v_h \in V_h,$$
$$b(u_h^n, q_h) = 0, \quad \forall q_h \in \overline{Q}_h.$$

Scheme NSPstab Let $u_h^0 \in V_h$ be an approximation of u^0. Find $\{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that

$$\left(\frac{u_h^n - u_h^{n-1}\circ X_1(\Pi_h^{(1)}u_h^{n-1})}{\Delta t}, v_h\right) + a(u_h^n, v_h) + b(v_h, p_h^n) = (f^n, v_h),\ \forall v_h \in V_h,$$
$$b(u_h^n, q_h) - \delta_0\mathcal{C}_h(p_h^n, q_h) = 0,\ \forall q_h \in Q_h.$$

In the four schemes we set the initial value as $u_h^0 = \Pi_h^{(2)}u^0$, where $\Pi_h^{(2)}$ is the interpolation operator to the P_2-element.

Example 1 We consider the Oseen problem and the Navier–Stokes problem. Let $\Omega = (0, 1)^2$, $T = 1$. The functions f and u^0 are defined so that the exact solution is

$$\begin{aligned} u_1(x,t) &= \phi(x_1, x_2, t), \\ u_2(x,t) &= -\phi(x_2, x_1, t), \\ p(x,t) &= C_p\sin(\pi(x_1+2x_2)+1+t), \end{aligned} \tag{30}$$

where

$$\phi(a,b,t) = -\sin(\pi a)^2\sin(\pi b)\{\sin(\pi(a+t)) + 3\sin(\pi(a+2b+t))\}.$$

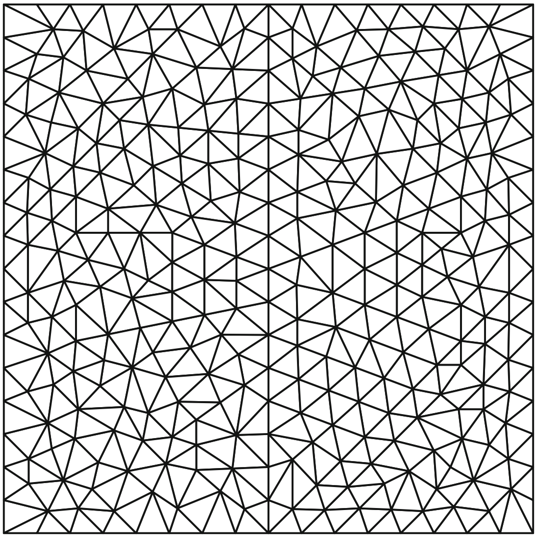

Fig. 1 The triangulation of $\overline{\Omega}$ for $N = 16$ used in Examples 1 and 2

Table 1 Symbols used in Example 1

ϕ X	u $\ell^\infty(L^2)$	u $\ell^2(H_0^1)$	p $\ell^2(L^2)$
TH	▲	●	■
Pstab	△	○	□

For the Oseen problem we set $w := u$. We consider the six cases $\nu = 10^{-2}, 10^{-4}, 10^{-6}$, $C_p = 1, 10$.

For triangulations of domains FreeFem++ [21] is used. Let $N = 16, 23, 32, 45$ and 64 be the division number of each side of $\overline{\Omega}$, and we set $h = 1/N$. When $\nu = 10^{-6}$, we also performed experiments for $N = 90, 108$ and 128. Figure 1 shows the triangulation of $\overline{\Omega}$ when $N = 16$. The time increment Δt is set to be $\Delta t = h^2$ so that we can observe the convergence behavior of order h^2. The purpose of the choice $\Delta t = O(h^2)$ is to examine the theoretical convergence order, but it is not based on the stability condition. We set the stabilization parameter $\delta_0 = 10^{-1}$ for Schemes OsPstab and NSPstab.

The relative error E_X is defined by

$$E_X(\phi) = \frac{\|\phi - \phi_h\|_{X,h}}{\|\phi\|_{X,h}},$$

for $\phi = u$ in $X = \ell^\infty(L^2)$ and $\ell^2(H_0^1)$, and for $\phi = p$ in $X = \ell^2(L^2)$. Here $\|\cdot\|_{X,h}$ means that the spatial norm is computed approximately by numerical quadrature of order nine [28]. Table 1 shows the symbols used in graphs. Since every graph of the relative error E_X versus h is depicted in the logarithmic scale, the slope corresponds to the convergence order.

Case (a) Let $C_p = 1$ in (30). We consider the Oseen problem and compare Schemes OsTH and OsPstab.

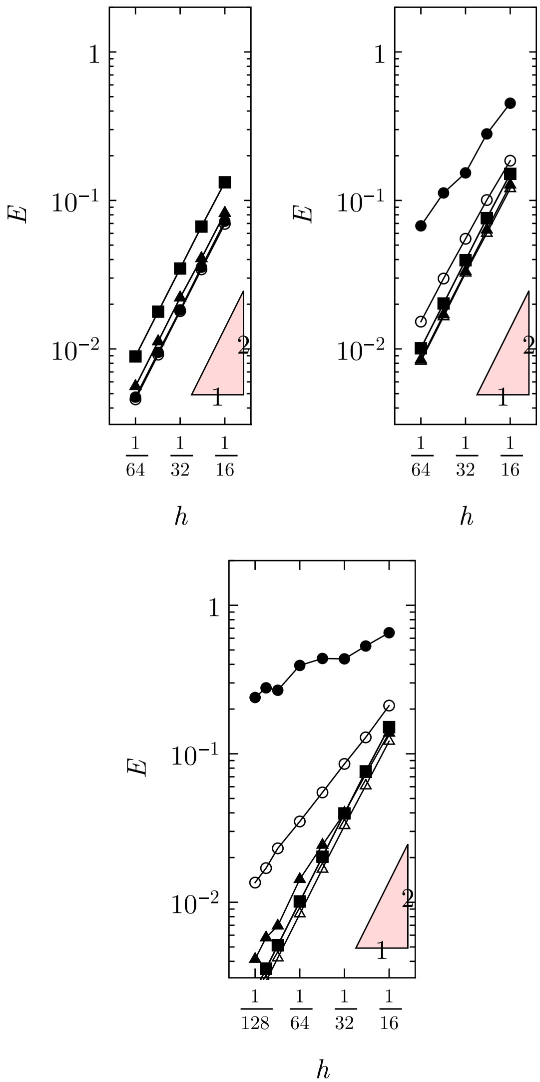

Fig. 2 Case (a). Relative errors versus h for $\nu = 10^{-2}$ (top left), $\nu = 10^{-4}$ (top right) and $\nu = 10^{-6}$ (bottom)

Figure 2 shows the graphs of the errors $E_{\ell^\infty(L^2)}(u)$ (▲,△), $E_{\ell^2(H_0^1)}(u)$ (●,○) and $E_{\ell^2(L^2)}(p)$ (■,□) versus h. When $\nu = 10^{-2}$, all convergence orders are almost two and there are no significant differences in both schemes.

When $\nu = 10^{-4}$, the convergence orders of $E_{\ell^\infty(L^2)}(u)$ (▲,△) are almost two in both schemes and there are no significant differences. The values of them are almost 1.5 times larger than those for $\nu = 10^{-2}$. The convergence order of $E_{\ell^2(H_0^1)}(u)$ in Scheme OsTH (●) is less than two, while the convergence order is almost two in OsPstab (○) and the value for $N = 64$ is four times smaller than that in Scheme OsTH. In order to obtain the convergence order two in Scheme OsTH, finer meshes seem to be necessary. The convergence order of the error $E_{\ell^2(L^2)}(p)$ (■,□) is almost two in both schemes and the values are almost same as those for $\nu = 10^{-4}$. However, we do not have theoretical estimates for p independent of the viscosity in the $\ell^2(L^2)$-norm.

We observe that, although in Case (a) there are no significant differences between the both schemes in the errors $E_{\ell^\infty(L^2)}(u)$ (▲,△) for $\nu = 10^{-2}$ and $\nu = 10^{-4}$, Scheme OsPstab shows higher accuracy for $\nu = 10^{-4}$ in the errors $E_{\ell^2(H_0^1)}(u)$ (○). When $\nu = 10^{-6}$, the increase in the errors $E_{\ell^\infty(L^2)}(u)$ of Scheme OsPstab (△) as compared with $\nu = 10^{-4}$ is less than 2%. We cannot observe monotonic convergence in $E_{\ell^2(H_0^1)}(u)$ of Scheme OsTH (●), whereas we can observe the convergence of Scheme OsPstab (○) but the order is less than two. We note that in the theoretical error bound of $E_{\ell^2(H_0^1)}(u)$ the constant includes $\sqrt{\nu}$. To observe the convergence order $O(h^2)$, more finer meshes will be necessary.

We consider the problem where the pressure value is larger.

Case (b) Let $C_p = 10$ in (30). We consider the Oseen problem and compare Schemes OsTH and OsPstab.

Figure 3 shows the graphs of the errors. When $\nu = 10^{-2}$, the values of $E_{\ell^\infty(L^2)}(u)$ (▲,△) are almost same as Case (a). We observe differences in $E_{\ell^2(H_0^1)}(u)$ in the two schemes. The values of errors in Scheme OsTH (●) are about 1.5 times as large as those in Scheme OsPstab (○), and the values in the both schemes are about two to three times as large as in Case (a). The values of relative errors $E_{\ell^2(L^2)}(p)$ (■,□) are, conversely, smaller than those in Case (a).

When $\nu = 10^{-4}$, differences of the schemes appear more clearly in $E_{\ell^\infty(L^2)}(u)$ and $E_{\ell^2(H_0^1)}(u)$ than Case (a). The values of $E_{\ell^\infty(L^2)}(u)$ in Scheme OsTH (▲) are almost two to three times as large as those in Scheme OsPstab (△). The values in Scheme OsPstab (△) are almost 1.5 times larger than those for $\nu = 10^{-2}$. For $N = 16$ and 23 the values of $E_{\ell^2(H_0^1)}(u)$ in Scheme OsTH (●) are too large to be plotted in the graph, and for $N = 32, 45$ and 64 the values are almost four to seven times as large as those in Scheme OsPstab (○). The values of relative errors $E_{\ell^2(L^2)}(p)$ (■,□) are, conversely, smaller than those in Case (a).

When $\nu = 10^{-6}$, for $E_{\ell^\infty(L^2)}(u)$ in Scheme OsPstab (△), we observe less than 15% increase compared with $\nu = 10^{-4}$. The values of $E_{\ell^2(H_0^1)}(u)$ in Scheme OsTH (●) are too large to be plotted on the graph.

We additionally consider the Navier–Stokes problem.

Case (c) Let $C_p = 1$ in (30). We consider the Navier–Stokes problem and compare Schemes NSTH and NSPstab.

Figure 4 shows the graphs of the errors. We observe almost the same behavior of the errors $E_{\ell^\infty(L^2)}(u)$ (▲,△) and $E_{\ell^2(H_0^1)}(u)$ (●,○) as in Case (a) while the values of $E_{\ell^2(L^2)}(p)$ (■,□) are almost 1.5 to 2 times as large as in Case (a).

Case (d) Let $C_p = 10$ in (30). We consider the Navier–Stokes problem and compare Schemes NSTH and NSPstab.

Figure 5 shows the graphs of the errors. When $\nu = 10^{-2}$, we observe the almost same behavior as in Case (b). When $\nu = 10^{-4}$, the values of $E_{\ell^\infty(L^2)}(u)$ (▲,△) and $E_{\ell^2(L^2)}(p)$ (■,□) are almost two to four times as large as in Case (b), while the values of $E_{\ell^2(H_0^1)}(u)$ (●,○) are almost same as in Case (b). When $\nu = 10^{-6}$, the errors of Scheme NSTH (▲, ●, ■) are not shown because the values $E_{\ell^2(H_0^1)}(u)$ of Scheme NSTH (●) at $N = 16, 23, 32$ are larger than 7.0 and thus not solved property.

Example 2 In the Navier–Stokes problem we set

$$\Omega = (0,1)^2, \quad T = 40, \quad \nu = 10^{-4}, \quad f(x,t) = (0, 10\sin(2\pi x_2))^T, \quad u^0 = 0,$$

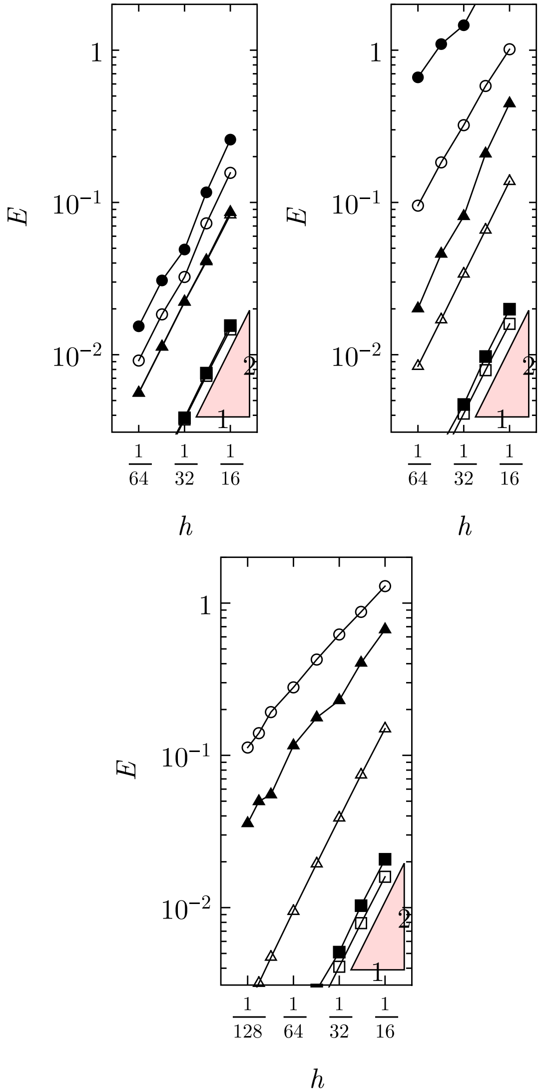

Fig. 3 Case (b). Relative errors versus h for $\nu = 10^{-2}$ (top left), $\nu = 10^{-4}$ (top right) and $\nu = 10^{-6}$ (bottom)

and compare Schemes NSTH and NSPstab.

We can easily check that the solution is $(u, p)(x, t) = (0, -\frac{5}{\pi}\cos(2\pi x_2))$. We use the mesh shown in Fig. 1 and take $\Delta t = 0.01$. We set the stabilization parameter $\delta_0 = 10^{-3}$ for Scheme NSPstab.

Figures 6 and 7 show the stereographs of the solutions (u_h^n, p_h^n) at $t^n = 40$ by the both schemes. In Scheme NSTH, oscillation is clearly observed for both components of the velocity and they are far from the constant zero, while in Scheme NSPstab the velocity is almost zero although small ruggedness is observed. For the pressure, difference between the two schemes is small compared to the velocity but the solution by Scheme NSPstab is better. Figure 8 shows the cross-sections of the solutions. We cannot observe difference between the solution by Scheme NSPstab and the exact solution while that by Scheme NSTH takes different values.

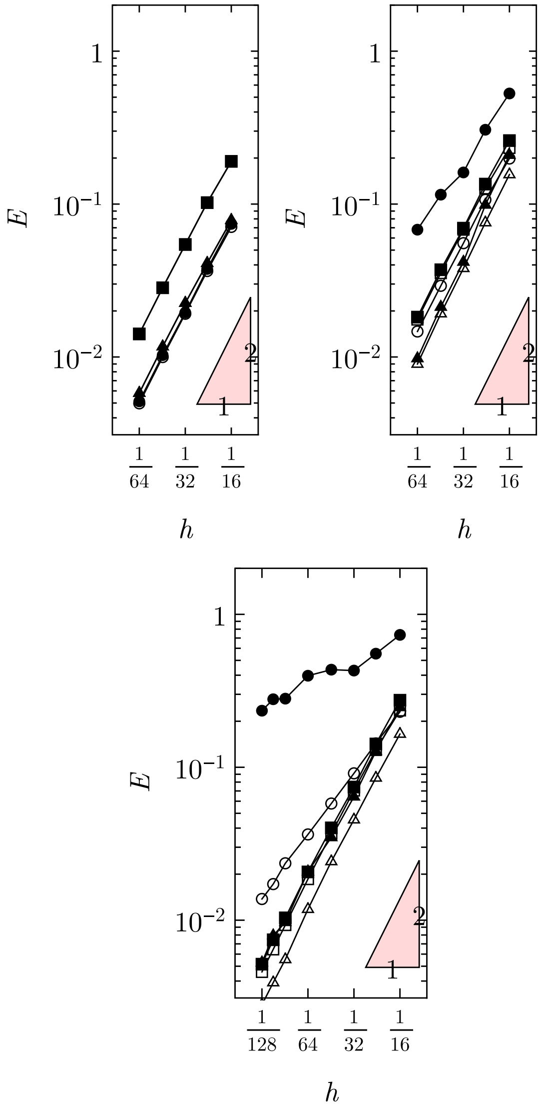

Fig. 4 Case (c). Relative errors versus h for $\nu = 10^{-2}$ (top left), $\nu = 10^{-4}$ (top right) and $\nu = 10^{-6}$ (bottom)

5 Concluding Remarks

We constructed a pressure-stabilized Lagrange–Galerkin scheme for the Oseen problem with high-order elements, and showed error estimates with the constants independent of the viscosity. The numerical examples showed the scheme has higher accuracy than the scheme with Taylor–Hood element especially for problems with small viscosity and large pressures. (i) Choice of the stabilization parameter in the pressure stabilization term, (ii) extension of the discussion to the Navier–Stokes problems, and (iii) numerical experiments of physically relevant problems will be future works. The error estimate of the pressure in the strong norm is a remaining issue. To achieve this estimate independent of the viscosity, we will need not only a special choice of the initial value of the velocity but also new arguments for an error

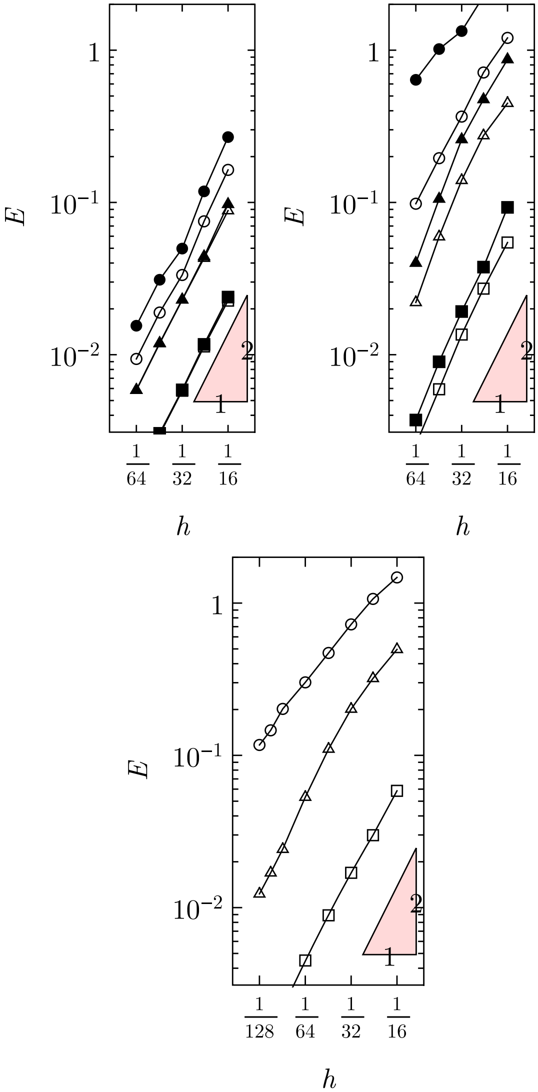

Fig. 5 Case (d). Relative errors versus h for $\nu = 10^{-2}$ (top left), $\nu = 10^{-4}$ (top right) and $\nu = 10^{-6}$ (bottom)

estimate of the time derivative of the velocity. Because of the term h^2 from the use of the locally linearized velocity, the order of convergence is optimal only when $k = 1$ or 2. A new technique for exact implementation of the LG scheme with higher order is also desired.

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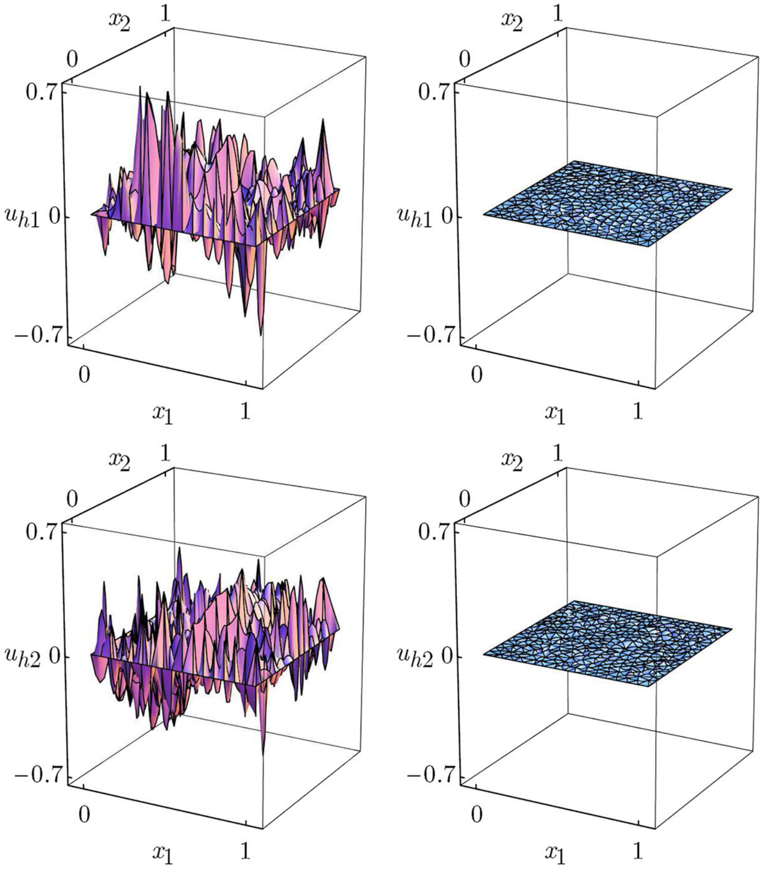

Fig. 6 Example 2. Stereographs of u^n_{h1} (top) and u^n_{h2} (bottom) at $t^n = 40$ by Scheme NSTH (left) and Scheme NSPstab (right)

A Estimates for LG Schemes

Lemma 6 is shown in [39, Lemma 5.7].

Lemma 6 *Let $w^* \in W^{1,\infty}(\Omega)^d$ and $X_1(w^*)$ be the mapping defined in (4). Under the condition $\Delta t|w^*|_{1,\infty} \le 1/4$, the estimate*

$$\frac{1}{2} \le \det\left(\frac{\partial X_1(w^*)}{\partial x}\right) \le \frac{3}{2}$$

holds, where $\det(\partial X_1(w^*)/\partial x)$ is the Jacobian of $X_1(w^*)$.

Lemma 7 is shown in [35, Lemma 1].

Lemma 7 *Let $w^* \in W^{1,\infty}_0(\Omega)^d$ and $X_1(w^*)$ be the mapping defined in (4). Under the condition $\Delta t|w^*|_{1,\infty} \le 1/4$, there exists a positive constant c independent of Δt such that for $v \in L^2(\Omega)^d$*

$$\|v \circ X_1(w^*)\|_0^2 \le (1 + c|w^*|_{1,\infty}\Delta t)\|v\|_0^2.$$

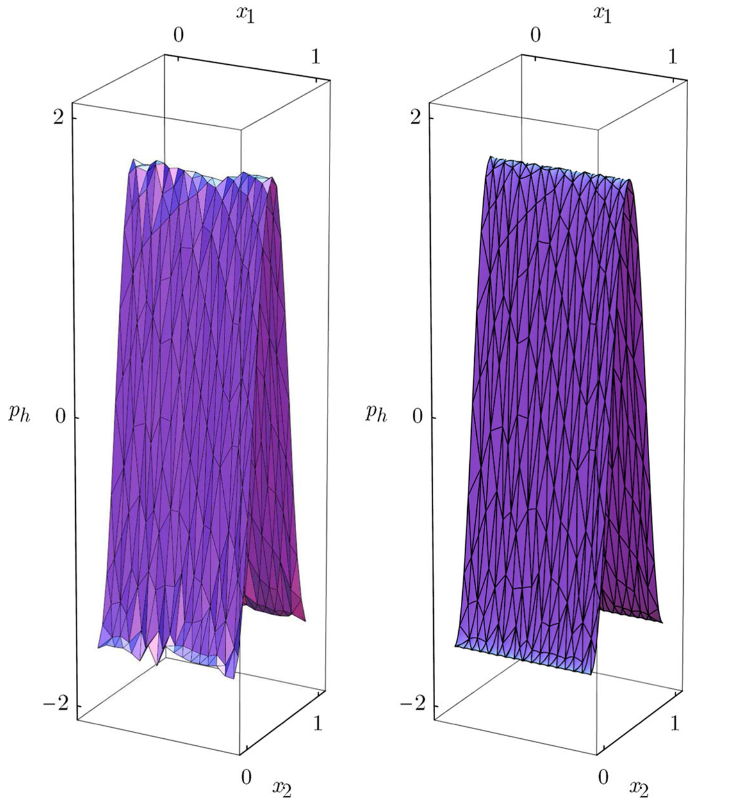

Fig. 7 Example 2. Stereographs of p_h^n at $t^n = 40$ by Scheme NSTH (left) and Scheme NSPstab (right)

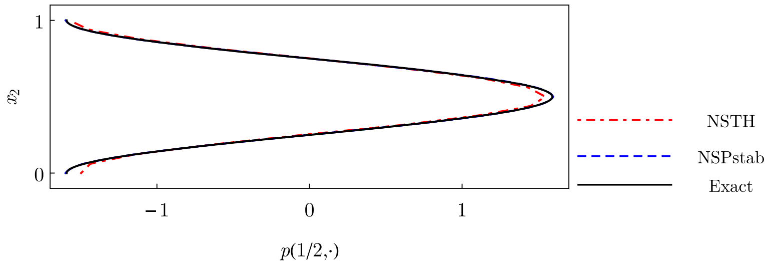

Fig. 8 Example 2. Cross-sections of $p_h^n(1/2, \cdot)$ and $p(1/2, \cdot)$

We now show an estimate for R_1^n in Lemma 8, or tools for estimating R_2^n and R_3^n in Lemmas 9 and 10, where R_i^n, $i = 1, 2, 3$, are defined in (16). Although these estimates are frequently used in the analysis of the LG method, e.g. [31,39], we show proofs of Lemmas 8 and 10 for completeness.

Lemma 8 *Suppose that* $u \in Z^2$, $w \in C(W_0^{1,\infty}) \cap H^1(L^\infty)$ *and* $\Delta t |w|_{C(W^{1,\infty})} \le 1/4$. *Then*

$$\|R_1^n\|_0 \le \sqrt{\Delta t}\left[\sqrt{\frac{2}{3}}(\|w^{n-1}\|_{0,\infty}^2+1)\|u\|_{Z^2(t^{n-1},t^n)} + \left\|\frac{\partial w}{\partial t}\right\|_{L^2(t^{n-1},t^n;L^\infty)}\|\nabla u^n\|_0\right].$$

Proof We estimate $\|R_1^n\|_0$ by dividing

$$\begin{aligned} R_1^n &= \left(\frac{\partial u^n}{\partial t} + (w^{n-1}\cdot\nabla)u^n - \frac{u^n - u^{n-1}\circ X_1(w^{n-1})}{\Delta t}\right) \\ &\quad + \big((w^n\cdot\nabla)u^n - (w^{n-1}\cdot\nabla)u^n\big) \\ &=: R_{11}^n + R_{12}^n. \end{aligned}$$

For R_{11}^n, we set

$$y(x,s) := x - s w^{n-1}(x)\Delta t,\ \ t(s) := t^n - s\Delta t,$$

and use Taylor's theorem to get

$$\begin{aligned} (u^{n-1}\circ X_1(w^{n-1}))(x) &= u^n(x) - \Delta t\left(\frac{\partial u^n}{\partial t} + (w^{n-1}\cdot\nabla)u^n\right)(x) \\ &\quad + \Delta t^2\int_0^1 (1-s)\left(\frac{\partial}{\partial t} + w^{n-1}(x)\cdot\nabla\right)^2 u(y(x,s),t(s))ds. \end{aligned}$$

Using the property of the Bochner integral, we then have

$$\begin{aligned} \|R_{11}^n\|_0 &\le \Delta t\int_0^1 \left\|(1-s)\left(\frac{\partial}{\partial t} + w^{n-1}(\cdot)\cdot\nabla\right)^2 u(y(\cdot,s),t(s))\right\|_0 ds \\ &\le \Delta t\left(\int_0^1 (1-s)^2 dx\right)^{1/2}\left(\int_0^1 \left\|\left(\frac{\partial}{\partial t} + w^{n-1}(\cdot)\cdot\nabla\right)^2 u(y(\cdot,s),t(s))\right\|_0^2 ds\right)^{1/2} \\ &\le \sqrt{2/3}\sqrt{\Delta t}(\|w^{n-1}\|_{0,\infty}^2+1)\|u\|_{Z^2(t^{n-1},t^n)}. \end{aligned}$$

where we have used the transformation of independent variables from x to y and s to t, and the estimate $|\det(\partial x/\partial y)| \le 2$ by virtue of Lemma 6. It is easy to show

$$\|R_{12}^n\|_0 \le \sqrt{\Delta t}\left\|\frac{\partial w}{\partial t}\right\|_{L^2(t^{n-1},t^n;L^\infty)}\|\nabla u^n\|_0.$$

Combining the two estimate, we have the conclusion. □

Lemma 9 is a direct consequence of [1, Lemma 4.5] and Lemma 6.

Lemma 9 *Let $1 \le q < \infty$, $1 \le p \le \infty$, $1/p + 1/p' = 1$ and $w_i \in W_0^{1,\infty}(\Omega)^d$, $i = 1,2$. Under the condition $\Delta t|w_i|_{1,\infty} \le 1/4$, it holds that, for $v \in W^{1,qp'}(\Omega)^d$,*

$$\|v\circ X_1(w_1) - v\circ X_1(w_2)\|_{0,q} \le 2^{1/(qp')}\Delta t\|w_1 - w_2\|_{0,pq}\|\nabla v\|_{0,qp'},$$

where $X_1(\cdot)$ is defined in (4).

Lemma 10 *Suppose that $v \in H^1(H^1)$, $w^* \in W_0^{1,\infty}(\Omega)^d$, and $\Delta t|w^*|_{1,\infty} \le 1/4$. Then*

$$\left\|v^n - v^{n-1}\circ X_1(w^*)\right\|_0 \le \sqrt{2\Delta t}\left(\left\|\frac{\partial v}{\partial t}\right\|_{L^2(t^{n-1},t^n;L^2)} + \|w^*\|_{0,\infty}\|\nabla v\|_{L^2(t^{n-1},t^n;L^2)}\right),$$

where $X_1(\cdot)$ is defined in (4).

Proof Similar to the proof of Lemma 8, by defining

$$y(x, s) := x - sw^*(x)\Delta t, \ t(s) := t^n - s\Delta t,$$

and by using the property of the Bochner integral, we have the estimate

$$\left\| v^n - v^{n-1} \circ X_1(w^*) \right\|_0 \le \Delta t \int_0^1 \left\| \left(\frac{\partial}{\partial t} + (w^*(\cdot) \cdot \nabla) \right) v(y(\cdot, s), t(s)) \right\|_0 ds.$$

The conclusion follows from the transformation of the independent variables from x to y and s to t, and the estimate $|\det(\partial x / \partial y)| \le 2$ by virtue of Lemma 6. □

Lemma 11 is an extension of [17, Lemma 1] obtained in the 1D case.

Lemma 11 *Let $w^* \in W_0^{1,\infty}(\Omega)^d$. Under the condition $\Delta t |w^*|_{1,\infty} \le 1/4$, there exists a positive constant c independent of Δt such that, for $v \in L^2(\Omega)^d$,*

$$\|v - v \circ X_1(w)\|_{-1} \le c\Delta t \|w\|_{1,\infty} \|v\|_0,$$

where $X_1(\cdot)$ is defined in (4) and $\|\cdot\|_{-1}$ is the norm in H^{-1}.

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