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Linearized Galerkin FEMs for Nonlinear Time Fractional Parabolic Problems with Non-smooth Solutions in Time Direction

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Abstract

A Newton linearized Galerkin finite element method is proposed to solve nonlinear time fractional parabolic problems with non-smooth solutions in time direction. Iterative processes or corrected schemes become dispensable by the use of the Newton linearized method and graded meshes in the temporal direction. The optimal error estimate in the L^2-norm is obtained without any time step restrictions dependent on the spatial mesh size. Such unconditional convergence results are proved by including the initial time singularity into concern, while previous unconditional convergent results always require continuity and boundedness of the temporal derivative of the exact solution. Numerical experiments are conducted to confirm the theoretical results.

Keywords Time fractional parabolic problems · Unconditional convergence · Optimal error estimates · Linearized schemes

Mathematics Subject Classification 65N12 · 65N30 · 35K61

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1 Introduction

This paper is concerned with the construction and analysis of an effective fully discrete scheme for the following nonlinear time fractional parabolic equation (TFPE):

$$\partial_t^\alpha u = \Delta u + f(u), \quad x \in \Omega, \quad t \in (0, T] \tag{1.1}$$

with the following initial and boundary conditions

$$\begin{aligned} &u(x, 0) = u_0(x), && x \in \Omega, \\ &u(x, t) = 0 \ \text{ or } \nabla u(x, t) \cdot \mathbf{n} = 0, && x \in \partial\Omega, \quad t \in [0, T], \end{aligned} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is a bounded convex and smooth polygon/polyhedron and $\mathbf{n}$ denotes the outward directed boundary normal. The Caputo fractional derivative ∂_t^α is defined as [13]

$$\partial_t^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{1}{(t-s)^\alpha} ds, \quad 0 < \alpha < 1. \tag{1.3}$$

Here $\Gamma(\cdot)$ denotes the usual gamma function. The TFPE provides a powerful tool to model many anomalous slow diffusion in physics, biology and control systems, where u denotes the probability density of the diffusing particles that have a mean-square displacement proportional to t^α, see, e.g. [1,2,7,9,14,22,26,30,38,40]. Especially, when α tends to 1, the fractional derivative $\partial_t^\alpha u$ would converge to the first-order derivative $\frac{\partial u}{\partial t}$ [13, p. 70], and thus Eq. (1.1) reproduces the usual reaction–diffusion equation.

The typical solution of problem (1.1) is not so smooth regardless of the regularity of f. As shown in [12], if $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, problem (1.1) has a unique solution u such that $\|\partial_t u(t)\|_{L^2(\Omega)} \le Ct^{\alpha-1}$ and this estimate is sharp. It implies that u has an initial layer at the beginning and u_t blows up as $t \to 0^+$. To deal with such an initial layer, there have been some recent progress in developing effective numerical methods, such as the corrected methods [6,10,11,36,37], graded meshes for initial time steps [3,23,33,39], and so on. However, most efforts have been focused on linear problems.

Recently, Jin et al. [12] investigated the nonlinear TFPE (1.1) by applying the L1 scheme and convolution quadrature, respectively. The convergence order of α in temporal direction was only obtained by considering the initial layer. Cuesta et al. [5] and Mustapha et al. [28] considered several nonlinear integro-differential equations, where a Riemann–Liouville fractional integral operator appears in front of the Laplacian. As pointed out in [28], the integro-differential equations are closely related to the nonlinear Eq. (1.1) but own different smoothing properties. In fact, to the best of our knowledge, we have not seen any unconditionally optimal error analysis of effective numerical methods for 2D or 3D nonlinear TFPEs (1.1) with non-smooth solutions.

There are several difficulties in analysing numerical schemes for nonlinear time fractional problems. Firstly, the error relies heavily on the history part due to the nonlocality of the problem. The initial layer may lead to some possible loss of accuracy, see [12,18,33]. Secondly, the error grows non-monotonically because of the nonlinearity of the problem. Finally, it is difficult to obtain the unconditional convergence of the fully discrete scheme for the nonlinear fractional problems in term of the weak regularity of the solutions and nonlocality of the problem.

In this paper, we present an effective numerical method and a rigorous error analysis for the nonlinear TFPE (1.1) by considering all three difficulties. A fully discrete numerical scheme is constructed as follows. The Galerkin FEM is applied used for the spatial discretization; the L1-scheme on a graded mesh is applied to approximate the fractional derivative, and the Newton linearized method is adopted to handle the nonlinear term. The fully discrete scheme is linear and requires only one starting value. Iterative processes or the corrected schemes become dispensable. More importantly, using the recent discrete fractional Gronwall type inequality and the temporal-spatial error splitting argument, optimal error estimates are obtained without time-step restrictions on the spatial mesh size.

We remark that the key to the proof of unconditional convergence is the temporal-spatial error splitting argument, which has been successfully used in recent numerical analysis for nonlinear parabolic PDEs, e.g., [15–17,19–21,31,35]. However, the analyze of nonlinear time-fractional equations is very different and more difficult. On the one hand, the unconditional convergence is established under the weak regularity of the solution, while all previous results are proved under the assumption that the exact solutions and their derivatives in the temporal directions are continuous and bounded. On the other hand, due to the non-locality of the problem and the use of the graded time-step, we have to estimate errors and prove the boundedness of the non-local discrete operator involving all previous (non-uniform) time levels.

The rest of the article is organized as follows. In Sect. 2, we propose a Newton linearized FEM for solving problem (1.1) and present our main result. In Sect. 3, we prove the main result by using the temporal-spatial error splitting argument. In Sect. 4, we implement numerical experiments to verify the accuracy and the unconditional convergence. Finally, we give some conclusions in Sect. 5.

Through out the paper, we let C be a generic positive constant, which is independent of the mesh sizes and may be different under different circumstances.

2 Linearized FEMs and Main Results

For any integer $m \geq 0$ and $1 \leq p \leq \infty$, we denote by $W^{m,p}(\Omega)$ the Sobolev space of functions defined on Ω and by $\|\cdot\|_{W^{m,p}}$ the corresponding Sobolev norm. Especially, denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. Similar to the standard finite element discretization [20,34], we let $\mathcal{T}_h$ be a quasiuniform partition of Ω into triangles T_i $(i = 1, \dots, M)$ in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$, $h = \max_{1\leq i\leq M}\{\text{diam } T_i\}$ be the spatial mesh size and let V_h be the finite-dimensional subspace of $H_0^1(\Omega)$, which consists of continuous piecewise polynomials of degree r $(r \geq 1)$ on $\mathcal{T}_h$. Let $\mathcal{T}_\tau = \{t_n | t_n = T(n/N)^\delta; 0 \leq n \leq N, \delta \geq 1\}$.

The classical $L1$-approximation on the graded meshes to the Caputo fractional derivative is given by

$$
\begin{aligned}
\partial_t^\alpha u(x,t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{1}{(t-s)^\alpha} ds \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{n} \frac{u(x,t_j) - u(x,t_{j-1})}{\tau_j} \int_{t_{j-1}}^{t_j} \frac{1}{(t_n-s)^\alpha} ds + Q^n \\
&= \sum_{j=1}^{n} a_{n,j}(u(x,t_j) - u(x,t_{j-1})) + Q^n,
\end{aligned}
$$

where Q^n is the truncation error and $a_{n,j} = \frac{1}{\tau_j \Gamma(1-\alpha)} \int_{t_{j-1}}^{t_j} \frac{1}{(t_n-s)^\alpha} ds = \frac{1}{\Gamma(2-\alpha)} \frac{(t_n-t_{j-1})^{1-\alpha} - (t_n-t_j)^{1-\alpha}}{t_j - t_{j-1}}$.

Let $\{\omega^n\}_{n=0}^N$ be a sequence of functions. We denote

$$D_\tau \omega^n := \sum_{j=1}^{n} a_{n,j}(\omega^j - \omega^{j-1}), \quad n = 1, \dots, N.$$

With above notations, the Newton linearized $L1$-Galerkin FEM is to seek $U_h^n \in V_h$, such that

$$\left(D_\tau^\alpha U_h^n, v\right) + \left(\nabla U_h^n, \nabla v\right) = \left(f(U_h^{n-1}) + f_1(U_h^{n-1})(U_h^n - U_h^{n-1}),\ v\right), \tag{2.1}$$

for any $v \in V_h$, where $n = 1, 2, \dots, N$ and $f_1(U_h^{n-1}) = \frac{\partial}{\partial u} f\Big|_{u=U_h^{n-1}}$.

The unconditionally optimal error estimates of schemes (2.1) are presented in following theorem. The proof will be given in the next section.

Theorem 2.1 *Suppose that the system* (1.1)–(1.2) *has a unique solution u satisfying*

$$\|\partial_{tt} u(t)\|_{L^\infty(\Omega)} \le C(1 + t^{\alpha-2}) \tag{2.2}$$

and the exact solution u is sufficiently regular in spatial directions. Then there exist positive constants N_0 and h_0, such that when $N \ge N_0$ and $h < h_0$, the r-degree finite element system defined in (2.1) *admits a unique solution U_h^m, $m=1, 2, \dots, N$, satisfying*

$$\|u^m - U_h^m\|_{L^2} \le C_0(N^{-\delta\alpha} + h^{r+1}), \tag{2.3}$$

where $1 \le \delta \le \frac{2-\alpha}{\alpha}$ and C_0 is a positive constant independent of N and h.

Remark 2.1 The optimal error estimate holds unconditionally. It means that the optimal error estimate is obtained without certain time-step restrictions dependent on the spatial mesh size, while the classical error analysis for multi-dimensional nonlinear parabolic problems always required such temporal stepsize restrictions.

Remark 2.2 The typical solution of problem (1.1) has weak regularity in temporal direction. Therefore, we only assume (2.2) holds and that the exact solution u is sufficiently regular in spatial directions. The similar assumptions are widely used in analysing numerical scheme for time fractional problems. See e.g., [4,25,27,29,32].

Remark 2.3 If assumption (2.2) is replaced by the more general condition

$$\|\partial_{tt} u(t)\|_{L^\infty(\Omega)} \le C(1 + t^{\sigma-2}), \quad \sigma \in (0, 1),$$

one can have the following error estimates

$$\|u^m - U_h^m\|_{L^2} \le C_0(N^{-\delta\sigma} + h^{r+1}), \quad m = 1, 2, \dots, N, \tag{2.4}$$

where $1 \le \delta \le \frac{2-\alpha}{\sigma}$. The result can be proved similarly by using the method in the present paper and Remark 5 of reference [23].

3 Unconditional Optimal Error Estimates

In this section, we present the proof of our main result. Our analysis is focused on homogeneous Dirichlet boundary condition. The techniques can be further extended to the homogeneous Neumann boundary condition without any difficulties.

3.1 Preliminaries

According to the standard FEM theory [34], if we define the Ritz projection operator $R_h : H_0^1(\Omega) \to V_h$ by

$$(\nabla(v - R_h v), \nabla\omega) = 0, \quad \forall \omega \in V_h,$$

then it holds

$$\|v - R_h v\|_{L^2} + h\|\nabla(v - R_h v)\|_{L^2} \le Ch^s\|v\|_{H^s}, \quad \forall v \in H^s(\Omega) \cap H_0^1(\Omega) \tag{3.1}$$

for $1 \le s \le r+1$.

We also introduce the following coefficients [23]

$$P_{n,n-k} = \Gamma(2-\alpha)\begin{cases} \tau_n^\alpha, & 1 \le k = n, \\ \tau_k^\alpha \displaystyle\sum_{j=k+1}^{n} (a_{j,j-k-1} - a_{j,j-k})P_{n,n-j}, & 1 \le k \le n-1. \end{cases} \tag{3.2}$$

The coefficients are defined recursively and play an important role in the following lemmas. Besides, in the following lemmas, we always let $t_n = T(n/N)^\delta$, $\delta \ge 1$ and $\tau_n = t_n - t_{n-1}$ for $0 \le n \le N$.

Lemma 3.1 ([23]) *Suppose that the nonnegative sequences* $\{\omega^n, \xi^n\}_{n=0}^N$ *satisfy*

$$D_\tau^\alpha(\omega^n)^2 \le \lambda_1(\omega^n)^2 + \lambda_2(\omega^{n-1})^2 + \omega^n(\xi^n + \eta), \quad n \ge 1,$$

where η, λ_1 *and* λ_2 *are all positive constants independent of the time step* τ_n. *Then it holds*

$$\omega^n \le 2E_\alpha(2\lambda t_n^\alpha)\left(\omega^0 + \max_{1\le j\le n}\sum_{l=1}^{j} P_{j,j-l}\xi^l + \frac{t_n^\alpha}{\Gamma(1+\alpha)}\eta\right), \quad 1 \le n \le N, \tag{3.3}$$

whenever the maximum temporal stepsize $\tau_N \le (2\lambda\Gamma(2-\alpha))^{-\frac{\alpha}{2}}$, *where* $E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}$ *is the Mittag-Leffler function and* $\lambda = \lambda_1 + \lambda_2$.

Lemma 3.2 ([23]) *Assume that* $v \in C^2(0, T]\bigcap C[0, T]$ *and there exists a constant* $C_v > 0$ *such that*

$$|v''(t)| \le C_v(1 + t^{\alpha-2}), \quad 0 < t \le T. \tag{3.4}$$

Then, it holds that,

$$\sum_{j=1}^{n} P_{n,n-j} \le \frac{t_n^\alpha}{\Gamma(1+\alpha)} \tag{3.5}$$

and

$$\sum_{j=1}^{n} P_{n,n-j}|\Upsilon^j| \le C_v T^\alpha N^{-\delta\alpha} + 4^{\delta-1}C_v T^\alpha \delta^2 N^{-\min\{\delta\alpha, 2-\alpha\}}, \quad n \ge 1,$$

where $\Upsilon^n = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}\frac{v'(s)}{(t-s)^\alpha}ds - D_\tau^\alpha v(t_n)$.

Lemma 3.3 *Suppose that* $v \in C[0,T] \cap C^2(0,T]$ *satisfies condition* (3.4)*, and the nonlinear function* $f \in C^2(\mathbb{R})$*. Let* $v^n = v(t_n)$ *and* $\mathcal{R}_v^n = f(v^n) - f(v^{n-1}) - f'(v^{n-1})(v^n - v^{n-1})$*. Then , it holds*

$$\sum_{j=1}^{n} P_{n,n-j} \left|\mathcal{R}_v^j\right| \le C N^{-\min\{3\delta\alpha,2\}}, \quad 1 \le n \le N. \tag{3.6}$$

Proof It follows from the Taylor expansion that

$$\mathcal{R}_v^i = \frac{1}{2}(v^i - v^{i-1})^2 \int_0^1 f''\big(v^{i-1} + s\nabla_\tau v^i\big)(1-s)\,\mathrm{d}s \le C\Big(\int_{t_{i-1}}^{t_i} \left|v'(t)\right| \mathrm{d}t\Big)^2.$$

By condition (3.4), we have $\left|\mathcal{R}_v^1\right| \le C\big(\int_{t_0}^{t_1} \left|v'(t)\right| \mathrm{d}t\big)^2 \le C\left(\tau_1^2 + \tau_1^{2\alpha}\right)$, and

$$\left|\mathcal{R}_v^j\right| \le C\Big(\int_{t_{j-1}}^{t_j} \left|v'(t)\right| \mathrm{d}t\Big)^2 \le C\big(\tau_j^2 + t_{j-1}^{2\alpha-2}\tau_j^2\big), \quad 2 \le j \le N.$$

Together with (3.5), we obtain

$$\begin{aligned}
\sum_{j=1}^{n} P_{n,n-j} \left|\mathcal{R}_v^j\right| \le\, & P_{n-1}^{(n)} \left|\mathcal{R}_v^1\right| + \sum_{j=2}^{n} P_{n-j}^{(n)} \left|\mathcal{R}_v^j\right| \\
\le\, & C\tau_1^\alpha \left|\mathcal{R}_v^1\right| + C t_n^\alpha \max_{2\le j\le n} \left|\mathcal{R}_v^j\right| \\
\le\, & C\tau_1^{3\alpha} + C t_n^\alpha \max_{2\le j\le n} \left(\tau_j^2 + t_{j-1}^{2\alpha-2}\tau_j^2\right) \\
\le\, & C\tau_1^{3\alpha} + C\tau_n^2 \\
\le\, & C N^{-\min\{3\delta\alpha,2\}},
\end{aligned}$$

which completes the proof. □

In order to prove unconditional error estimates of the proposed schemes (2.1), we introduce the following time-discrete system

$$D_\tau^\alpha U^n = \Delta U^n + f(U^{n-1}) + f_1(U^{n-1})(U^n - U^{n-1}), \quad n = 1, 2, \ldots, N, \tag{3.7}$$

where the initial and boundary conditions are given by

$$U^0(x) = u_0(x), \qquad \text{for } x \in \Omega. \tag{3.8}$$

$$U^n(x) = 0, \qquad \text{for } x \in \partial\Omega, \quad n = 1, 2, \ldots, N, \tag{3.9}$$

Then, the errors can be splitted into two parts

$$\|u^n - U_h^n\| \le \|u^n - U^n\| + \|U^n - U_h^n\|, \tag{3.10}$$

where $u^n = u(x, t_n)$, U^n and U_h^n are respectively the solutions of the time discrete system (3.7) and the fully discrete system (2.1). Our analysis relays heavily on the error estimates of $\|u^n - U^n\|$ and $\|U^n - U_h^n\|$, respectively.

3.2 Analyses of the Time Discrete System

We investigate the error analysis of the time discrete system and the boundedness of U^n in this subsection.

Firstly, the u^n satisfies the following equation

$$D_\tau^\alpha u^n = \Delta u^n + f(u^{n-1}) + f_1(u^{n-1})(u^n - u^{n-1}) + P^n, \quad n = 1, 2, \dots, N, \tag{3.11}$$

where

$$P^n = \left(D_\tau u^n - \partial_{t_n}^\alpha u\right) + \left(f(u^n) - f(u^{n-1}) - f_1(u^{n-1})(u^n - u^{n-1})\right). \tag{3.12}$$

Let $e^n = u^n - U^n$, $n = 0, 1, 2, \dots, N$. Subtracting (3.7) from (3.11), we have

$$D_\tau e^n = \triangle e^n + R^n + P^n, \tag{3.13}$$

where $R_1^n = f(u^{n-1}) + f_1(u^{n-1})(u^n - u^{n-1}) - f(U^{n-1}) - f_1(U^{n-1})(U^n - U^{n-1})$.

Let $K_1 = \max_{1\le n\le N} \|u^n\|_{L^\infty} + 1$.

Theorem 3.4 *Suppose the assumptions in Theorem 2.1 hold. Then the time-discrete system in (3.7)–(3.8) has a unique solution U^n. Moreover, there exists an $N_1^* > 0$ such that when $N \ge N_1^*$,*

$$\|e^n\|_{H^2} \le C_1^* N^{-\delta\alpha}, \tag{3.14}$$

$$\|U^n\|_{L^\infty} \le K_1, \tag{3.15}$$

and

$$\|D_\tau^\alpha U^n\|_{H^2} \le C_1^{**}, \tag{3.16}$$

where $1 \le \delta \le \frac{2-\alpha}{\alpha}$, C_1^* and C_1^{**} are positive constants independent of N, h and C_0.

Proof Noting that (3.7) is a linear elliptic problem for every time level, we can obtain the existence and uniqueness of the solution U^n. We begin to prove (3.14) and (3.15) by mathematical induction. Firstly, the estimate holds for $n = 0$. Now, we assume that (3.14) and (3.15) hold for $0 \le n \le k-1$. Therefore, for $0 \le n \le k-1$, it holds that

$$\begin{aligned}
\|U^n\|_{L^\infty} &\le \|u^n\|_{L^\infty} + \|e^n\|_{L^\infty} \\
&\le \|u^n\|_{L^\infty} + C_\Omega \|e^n\|_{H^2} \\
&\le \|u^n\|_{L^\infty} + C_\Omega C_1^* N^{-\delta\alpha} \\
&\le K_1,
\end{aligned} \tag{3.17}$$

whenever $N > N_1 = (C_\Omega C_1^*)^{\frac{1}{\alpha}}$.

Due to the boundedness of $\|U^n\|_{L^\infty}$ and $\|u^n\|_{L^\infty}$ for $0 \le n \le k-1$, we have

$$\begin{aligned}
\|R_1^n\|_{H^2} &= \|f(u^{n-1}) + f_1(u^{n-1})(u^n - u^{n-1}) - f(U^{n-1}) - f_1(U^{n-1})(U^n - U^{n-1})\|_{H^2} \\
&\le \|f(u^{n-1}) - f(U^{n-1})\|_{H^2} + \|(f_1(u^{n-1}) - f_1(U^{n-1}))u^n\|_{H^2} \\
&\quad + \|f_1(U^{n-1})(u^n - U^n)\|_{H^2} + \|(f_1(u^{n-1}) - f_1(U^{n-1}))u^{n-1}\|_{H^2} \\
&\quad + \|f_1(U^{n-1})(u^{n-1} - U^{n-1})\|_{H^2} \\
&\le C_1 \|e^{n-1}\|_{H^2} + C_1 \|e^n\|_{H^2},
\end{aligned} \tag{3.18}$$

where C_1 is a constant dependent on u, K_1 and f.

Let $n=k$ in Eq. (3.13), multiplying both sides of Eq. (3.13) by e^k and integrating it over Ω, we obtain

$$\begin{aligned}(D_\tau e^k, e^k) &= (\triangle e^k, e^k) + (R_1^k, e^k) + (P^k, e^k)\\ &= -\|\nabla e^k\|_{L^2}^2 + (R_1^n, e^k) + (P^k, e^k)\\ &\le (R_1^k, e^k) + (P^k, e^k)\\ &\le \frac{3C_1}{2}\|e^k\|_{L^2}^2 + \frac{C_1}{2}\|e^{k-1}\|_{L^2}^2 + \|P^k\|_{L^2}\|e^n\|_{L^2}. \end{aligned} \tag{3.19}$$

By Lemma 3.1, there exists an N_2 such that when $N > N_2$, it holds

$$\|e^k\|_{L^2} \le 2E_\alpha(3C_1 t_k^\alpha)\Big(\max_{1\le j\le k}\sum_{l=1}^{j} P_{j,j-l}\|P^l\|_{L^2}\Big). \tag{3.20}$$

Similarly, multiplying (3.7) by $-\Delta e^k$ and $\Delta^2 e^k$, respectively, integrating it over Ω and using Lemma 3.1, we conclude that there exists an N_3 such that when $N > N_3$, it holds

$$\|\nabla e^k\|_{L^2} \le 2E_\alpha(3C_2 t_k^\alpha)\Big(\max_{1\le j\le k}\sum_{l=1}^{j} P_{j,j-l}\|\nabla P^l\|_{L^2}\Big), \tag{3.21}$$

and

$$\|\triangle e^k\|_{L^2} \le 2E_\alpha(3C_3 t_k^\alpha)\Big(\max_{1\le j\le k}\sum_{l=1}^{j} P_{j,j-l}\|\triangle P^l\|_{L^2}\Big), \tag{3.22}$$

where C_2 and C_3 are constants dependent on u, K_1 and f.

It follows from Lemma 3.2, Lemma 3.3 and (3.12) that

$$\max_{1\le j\le k}\sum_{l=1}^{j} P_{j,j-l}\|P^l\|_{H^2} \le \sum_{l=1}^{N} P_{N,N-l}\|P^l\|_{H^2} \le C_p N^{-\min\{\delta\alpha, 2-\alpha\}}, \tag{3.23}$$

where C_p is a constant, depending on u and f.

Now, following (3.20), (3.21) and (3.22), we obtain

$$\begin{aligned}\|e^k\|_{H^2} &\le 2\sqrt{3}\sqrt{E_\alpha^2(3C_1 t_k^\alpha) + E_\alpha^2(3C_2 t_k^\alpha) + E_\alpha^2(3C_3 t_k^\alpha)}C_p N^{-\min\{\delta\alpha, 2-\alpha\}}\\ &\le 2\sqrt{3}\sqrt{E_\alpha^2(3C_1 T^\alpha) + E_\alpha^2(3C_2 T^\alpha) + E_\alpha^2(3C_3 T^\alpha)}C_p N^{-\delta\alpha}\\ &= C_1^* N^{-\delta\alpha}, \end{aligned} \tag{3.24}$$

where $C_1^* = 2\sqrt{3}\sqrt{E_\alpha^2(3C_1 T^\alpha) + E_\alpha^2(3C_2 T^\alpha) + E_\alpha^2(3C_3 T^\alpha)}C_p$ and we have noted $\delta\alpha \le 2-\alpha$. The above formula further implies

$$\begin{aligned}\|U^k\|_{L^\infty} &\le \|u^k\|_{L^\infty} + \|e^k\|_{L^\infty}\\ &\le \|u^k\|_{L^\infty} + C_\Omega C_1^* N^{-\delta\alpha}\\ &\le K_1, \end{aligned} \tag{3.25}$$

whenever $N > N_1$.

Therefore, (3.14) and (3.15) hold for $n=k$ when we take $N_1^* = \max\{N_1, N_2, N_3\}$, and the mathematical induction is closed.

Moreover, it follows from (3.14) that

$$\begin{aligned}
\|D_\tau e^n\|_{H^2} &= \|\sum_{j=1}^{n} a_{n,j}(e^j - e^{j-1})\|_{H^2}\\
&= \|a_{n,n}e^n - \sum_{j=1}^{n-1}(a_{n,j+1} - a_{n,j})e^j - a_{n,1}e^0\|_{H^2}\\
&= a_{n,n}\|e^n\|_{H^2} + \sum_{j=1}^{n-1}(a_{n,j+1} - a_{n,j})\|e^j\|_{H^2} + a_{n,1}\|e^0\|_{H^2}\\
&\le \Big(a_{n,n} + \sum_{j=1}^{n-1}(a_{n,j+1} - a_{n,j}) + a_{n,1}\Big)C_1^* N^{-\min\{\delta\alpha,2-\alpha\}}\\
&= 2a_{n,n}C_1^* N^{-\min\{\delta\alpha,2-\alpha\}}\\
&= \frac{2N^{\delta\alpha}}{\Gamma(2-\alpha)T^\alpha}C_1^* N^{-\min\{\delta\alpha,2-\alpha\}}\\
&\le \frac{2}{\Gamma(2-\alpha)T^\alpha}C_1^*,
\end{aligned} \tag{3.26}$$

where in the last inequality, we have used the assumption $\delta\alpha \le 2 - \alpha$. Therefore,

$$\|D_\tau U^n\|_{H^2} \le \|D_\tau u^n\|_{H^2} + \|D_\tau e^n\|_{H^2} \le C_1^{**}.$$

This completes the proof. □

3.3 Analyses of FEM Approximations

In this subsection, we show the boundedness of FEM approximations U_h^n based on the error estimates of $\|U^n - U_h^n\|_{L^2}$. It is known that $\|R_h v\|_{L^\infty} \le C\|v\|_{H^2}$ for any $v \in H^2(\Omega)$. By Lemmas in the previous subsection, we obtain the boundedness of $\|R_h U^n\|_{L^\infty}$ for $n = 1, 2, \ldots, N$. Then, we can define

$$K_2 := \max_{1\le n\le N} \|R_h U^n\|_{L^\infty} + 1. \tag{3.27}$$

The weak form of the time-discrete Eq. (3.7) can be defined by

$$\big(D_\tau U^n, v\big) = -\big(\nabla U^n, \nabla v\big) + \big(f(U^{n-1}) + f_1(U^{n-1})(U^n - U^{n-1}),\ v\big), \tag{3.28}$$

for all $v \in H_0^1(\Omega)$. Let

$$\theta_h^n = R_h U^n - U_h^n, \quad n = 0, 1, \ldots, N.$$

Subtracting (2.1) from (3.36), we get the error equation for θ_h^n, $n = 1, 2, \ldots, N$,

$$\big(D_\tau \theta_h^n, v\big) + \big(\nabla\theta_h^n, \nabla v\big) = \big(R_2^n, v\big) + \big(D_\tau(U^n - R_h U^n), v\big), \quad \forall v \in V_h, \tag{3.29}$$

where

$$R_2^n = f(U^{n-1}) + f_1(U^{n-1})(U^n - U^{n-1}) - f(U_h^{n-1}) - f_1(U_h^{n-1})(U_h^n - U_h^{n-1}).$$

Theorem 3.5 *Suppose that the system (1.1)–(1.2) has a unique solution u satisfying (2.2). Then the finite element system defined in (2.1) has a unique solution U_h^n, $n = 1, \ldots, N$, and there exist $N_2^* > 0$ and $h_2^* > 0$ such that when $N \ge N_2^*$, $h \le h_2^*$,*

$$\|\theta_h^n\|_{L^2} \le h^{\frac{7}{4}}, \tag{3.30}$$

and

$$\|U_h^n\|_{L^\infty} \le K_2. \tag{3.31}$$

Proof Noting that the coefficients matrices of system (2.1) are diagonal dominance, we obtain the existence and uniqueness of the FEM solution U_h^n. Now, we prove (3.30) and (3.31) by mathematical induction. Firstly, one can check that (3.30) holds for $n = 0$. Next, we assume that (3.30) holds for $n \le k-1$. Consequently, for $n = 1, 2, \ldots, k-1$, we have

$$\begin{aligned} \|U_h^n\|_{L^\infty} &\le \|R_hU^n\|_{L^\infty} + \|R_hU^n - U_h^n\|_{L^\infty} \\ &\le \|R_hU^n\|_{L^\infty} + C_\Omega h^{-\frac{d}{2}}\|R_hU^n - U_h^n\|_{L^2} \\ &\le \|R_hU^n\|_{L^\infty} + C_\Omega h^{-\frac{d}{2}} h^{\frac{7}{4}} \\ &\le \|R_hU^n\|_{L^\infty} + 1 \\ &\le K_2, \end{aligned} \tag{3.32}$$

when $h < h_1 = C_\Omega^{-\frac{4}{7-2d}}$.

It further implies

$$\begin{aligned} \|R_2^k\|_{L^2} &= \|f(U^{k-1}) + f_1(U^{k-1})(U^n - U^{k-1}) - f(U_h^{k-1}) - f_1(U_h^{k-1})(U_h^k - U_h^{k-1})\|_{L^2} \\ &\le \|f(U^{k-1}) - f(U_h^{k-1})\|_{L^2} + \|(f_1(U^{k-1}) - f_1(U_h^{k-1}))U^n\|_{L^2} \\ &\quad + \|f_1(U_h^{k-1})(U^k - U_h^k)\|_{L^2} + \|(f_1(U^{k-1}) - f_1(U_h^{k-1}))U^{k-1}\|_{L^2} \\ &\quad + \|f_1(U_h^{k-1})(U^{k-1} - U_h^{k-1})\|_{L^2} \\ &\le C_4\|U^{k-1} - U_h^{k-1}\|_{L^2} + C_4\|U^k - U_k^k\|_{L^2} \\ &\le C_4\|\theta_h^k\|_{L^2} + C_4\|\theta_h^{k-1}\|_{L^2} + 2C_4C_\Omega h^2, \end{aligned} \tag{3.33}$$

where C_4 is a constant dependent on K_2 and f.

Letting $n = k$ in (3.29), setting $v = \theta_h^k$ and using (3.33), we have, there exists an N_4 such that when $N > N_4$,

$$\begin{aligned} \left(D_\tau\theta_h^k, \theta_h^k\right) &\le -\|\nabla\theta_h^k\|_{L^2}^2 + \frac{C_4}{2}\|\theta_h^{k-1}\|_{L^2}^2 + \frac{3C_4}{2}\|\theta_h^k\|_{L^2}^2 + \left(\|D_\tau(U^k - R_hU^k)\|_{L^2} + 2C_4C_\Omega h^2\right)\|\theta_h^k\|_{L^2} \\ &\le \frac{C_4}{2}\|\theta_h^{k-1}\|_{L^2}^2 + \frac{3C_4}{2}\|\theta_h^k\|_{L^2}^2 + \left(C_\Omega\|D_\tau U^k\|_{H^2}h^2 + 2C_4C_\Omega h^2\right)\|\theta_h^k\|_{L^2}. \end{aligned} \tag{3.34}$$

Applying Lemma 3.1, we obtain

$$\begin{aligned} \|\theta_h^k\|_{L^2} &\le 2E_\alpha(4C_4t_k^\alpha)\left(\|\theta^0\|_{L^2} + \frac{t_k^\alpha}{\Gamma(1+\alpha)}(C_\Omega\|D_\tau U^k\|_{H^2}h^2 + 2C_4C_\Omega h^2)\right) \\ &\le 2E_\alpha(4C_4T^\alpha)\left(C_\Omega + \frac{T^\alpha}{\Gamma(1+\alpha)}(C_\Omega\|D_\tau U^k\|_{H^2} + 2C_4C_\Omega)\right)h^2 \\ &\le h^{\frac{7}{4}}, \end{aligned} \tag{3.35}$$

when $h < h_2 = \left(2E_\alpha(4C_4T^\alpha)(C_\Omega + \frac{T^\alpha}{\Gamma(1+\alpha)}(C_\Omega\|D_\tau U^k\|_{H^2} + 2C_4C_\Omega))\right)^{-1/4}$.

By inverse inequality and (3.35), one can further verify that

$$\|U_h^k\|_{L^\infty} \le \|R_hU^k\|_{L^\infty} + \|\theta_h^k\|_{L^\infty} \le \|R_hU^k\|_{L^\infty} + C_\Omega h^{-d/2}\|\theta_h^k\|_{L^2} \le K_2,$$

when $h < h_1$.

Therefore, the estimate (3.30) and (3.31) hold for $n = k$. The mathematical induction is closed and the proof is complete. □

3.4 Optimal Error Estimates

In this subsection, we consider optimal error estimates of the fully discrete systems.

The exact solution satisfies, for $\forall v \in H_0^1(\Omega)$,

$$\left(D_\tau u^n, v\right) = -\left(\nabla u^n, \nabla v\right) + \left(f(u^{n-1}) + f_1(u^{n-1})(u^n - u^{n-1}), v\right) + \left(P^n, v\right). \quad (3.36)$$

Let

$$\eta_h^n = R_hu^n - u_h^n, \quad n = 0, 1, \ldots, N.$$

Subtracting (2.1) from (3.36), we get the error equation for η_h^n, $n = 1, 2, \ldots, N$,

$$\left(D_\tau\eta_h^n, v\right) + \left(\nabla\eta_h^n, \nabla v\right) = \left(R_3^n, v\right) + \left(D_\tau(R_hu^n - u^n), v\right) + \left(P^n, v\right), \quad \forall v \in V_h, \quad (3.37)$$

where $R_3^n = f(u^{n-1}) + f_1(u^{n-1})(u^n - u^{n-1}) - f(U_h^{n-1}) - f_1(U_h^{n-1})(U_h^n - U_h^{n-1})$.

In the previous subsection, we obtain the boundedness of $\|U_h^n\|_{L^\infty}$ without certain time-step restrictions dependent on the spatial mesh size, which together with the boundedness of $\|u^n\|_{L^\infty}$ for $1 \le n \le N$ leads to

$$\begin{aligned}\|R_3^n\|_{L^2} &= \|f(u^{n-1}) + f_1(u^{n-1})(u^n - u^{n-1}) - f(U_h^{n-1}) - f_1(U_h^{n-1})(U_h^n - U_h^{n-1})\|_{L^2}\\ &\le \|f(u^{n-1}) - f(U_h^{n-1})\|_{L^2} + \|(f_1(u^{n-1}) - f_1(U_h^{n-1}))u^n\|_{L^2} + \|f_1(U_h^{n-1})(u^n - U_h^n)\|_{L^2}\\ &\quad + \|(f_1(u^{n-1}) - f_1(U_h^{n-1}))u^{n-1}\|_{L^2} + \|f_1(U_h^{n-1})(u^{n-1} - U_h^{n-1})\|_{L^2}\\ &\le C_5\|u^{n-1} - U_h^{n-1}\|_{L^2} + C_5\|u^n - U_h^n\|_{L^2}\\ &\le C_5\|\eta_h^{n-1}\|_{L^2} + C_5\|\eta_h^n\|_{L^2} + C_5C_\Omega\|u\|_{r+1}h^{r+1},\end{aligned} \quad (3.38)$$

where C_5 is a constant dependent on u and f.

Now, setting $v = \eta_h^n$ in (3.37) and using (3.38), we have

$$\begin{aligned}\left(D_\tau\eta_h^n, \eta_h^n\right) &\le -\|\nabla\eta_h^n\|^2 + \frac{C_5}{2}\|\eta_h^{n-1}\|_{L^2}^2 + \frac{3C_5}{2}\|\eta_h^n\|_{L^2}^2\\ &\quad + \left(\|D_\tau(u^n - R_hu^n)\|_{L^2} + \|P^n\|_{L^2}\right)\|\eta_h^n\|_{L^2}\\ &\le \frac{C_5}{2}\|\eta_h^{n-1}\|_{L^2}^2 + \frac{3C_5}{2}\|\eta_h^n\|_{L^2}^2 + \left(C_\Omega\|D_\tau u^n\|_{H^{r+1}}h^{r+1} + \|P^n\|_{L^2}\right)\|\eta_h^n\|_{L^2}.\end{aligned} \quad (3.39)$$

Applying Lemma 3.1, we conclude that there exists an N_5 such that when $N > N_5$, it holds $\|\eta_h^n\|_{L^2} \le 2E_2(4C_5t_n^\alpha)\left(\|\eta_h^0\|_{L^2} + C_\Omega\|D_\tau u^n\|_{H^{r+1}}h^{r+1} + \max_{1\le j\le k}\sum_{l=1}^{j} P_{j,j-l}\|P^l\|_{L^2}\right)$. By the initial error together with Lemmas 3.2 and 3.3, we have $\|\eta_h^n\|_{L^2} \le 2E_2(4C_5t_n^\alpha)\left(C_\Omega h^{r+1} + C_\Omega\|D_\tau u^n\|_{H^{r+1}}h^{r+1} + C_pN^{-\delta\alpha}\right)$, where we have noted the fact $\delta\alpha \le 2 - \alpha$.

Table 1 Errors and convergence rates in temporal direction (Example 4.1)

N	$\alpha = 0.8$		$\alpha = 0.6$		$\alpha = 0.4$		$\alpha = 0.2$	
	Error	Order	Error	Order	Error	Order	Error	Order
8	1.08e−04	–	7.58e−05	–	9.96e−05	–	3.10e−04	–
16	3.79e−05	1.51	2.34e−05	1.70	3.41e−05	1.55	1.08e−04	1.51
32	1.42e−05	1.42	8.06e−06	1.54	1.23e−05	1.47	3.75e−05	1.53
64	5.51e−06	1.36	2.99e−06	1.43	4.44e−06	1.47	1.27e−05	1.56
128	2.19e−06	1.33	1.15e−06	1.39	1.57e−06	1.50	4.17e−06	1.61

The above formula further implies that $\|u^n - U_h^n\|_{L^2} \le C_0(N^{-\delta\alpha} + h^{r+1})$. This completes the proof.

4 Numerical Examples

We present several numerical experiments to confirm our theoretical results in this section. As the convergence orders of the L1-scheme on the graded meshes with different δ were tested in [23,33] for linear problems, in this section we only set $\delta = \frac{2-\alpha}{\alpha}$ and test unconditionally optimal error estimates of the fully discrete schemes. In all of the following numerical experiments, the errors are under $L^\infty([0, T]; L^2(\Omega))$ without special instruction. All the computations are implemented by using the software FEniCS [24].

Example 1 Consider the following two-dimensional fractional fisher equation

$$\partial_t^\alpha u = \Delta u + u(1-u) + g(x,t), \quad x \in \Omega = (0,1)^2, \ 0 < t \le 1, \tag{4.1}$$

$$\nabla u(x,t) \cdot \mathbf{n} = 0, \quad x \in \partial\Omega, \ 0 < t \le 1, \tag{4.2}$$

where initial condition and the function g are given from the exact solution $u = (t^\alpha + t^2)x_1^2(1-x_1)^2x_2^2(1-x_2)^2$. The convergence rates in temporal direction are tested by using linear FEMs with $h = 0.5/N$ and different grades meshes. The numerical errors and convergence orders with different α are presented in Table 1. Then the spatial stepsize is refined with $\alpha = 0.6$ and $N = \lfloor h^{(r+1)/(\alpha-2)} \rfloor (\lfloor x \rfloor := \max\{m \in \mathbb{Z} \mid m \le x\})$. The numerical errors and convergence orders in spatial direction are listed in Table 2. These numerical results verify the convergence of the proposed method.

To confirm the unconditional convergence, problem (4.2) is solved by using linear and quadratic FEMs with different stepsizes. The errors are shown in Fig. 1. It can be seen that for a fixed N, the errors asymptotically tend to a constant, which implies that there is no time-step restrictions dependent on the spatial mesh size.

Example 2 In this example, we consider the following two-dimensional fractional Huxley equation

$$\partial_t^\alpha u = \Delta u + u(1-u)(u-1) + g(x,t), \qquad x \in \Omega = (0,1)^2, \ 0 < t \le 1, \tag{4.3}$$

$$\nabla u(x,t) \cdot \mathbf{n} = 0, \qquad x \in \partial\Omega, \ 0 < t \le 1. \tag{4.4}$$

where initial condition and the function g are determined by the exact solution $u = (t^\alpha + t^3)\sin^2 x_1(1-x_1)^2 \sin^2 x_2(1-x_2)^2$.

Table 2 Errors and convergence rates in spatial direction (Example 4.1)

N	$r=1$		$r=2$	
	Error	Order	Error	Order
8	2.03e−04	–	1.15e−05	–
16	5.41e−05	1.91	1.48e−06	2.97
32	1.37e−05	1.98	1.86e−07	2.99
64	3.45e−06	1.99	2.33e−08	3.00
128	8.63e−07	2.00	2.91e−09	3.00

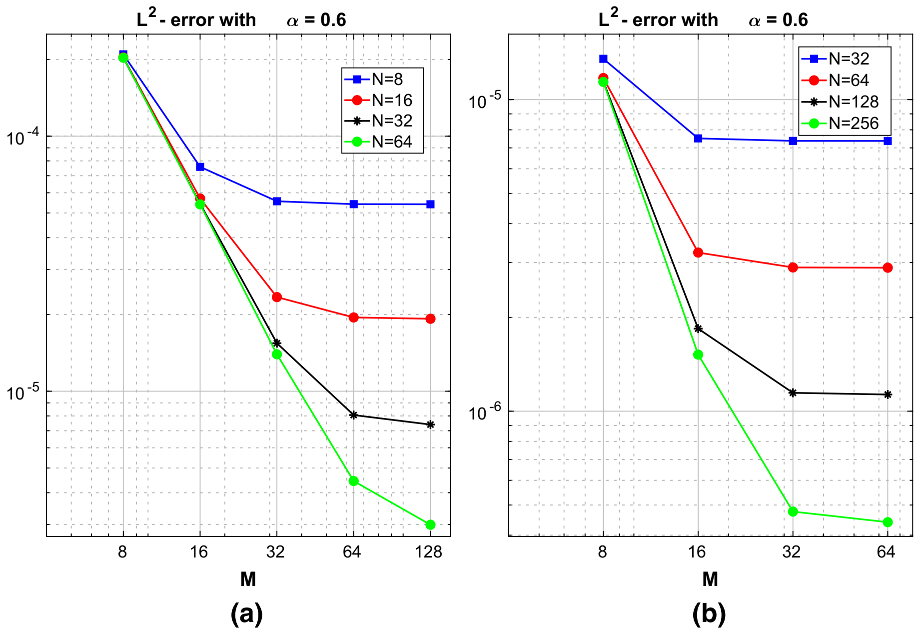

Fig. 1 L^2 errors on gradually refined meshes with fixed N (Example 4.1). **a** $r=1$; **b** $r=2$

Problem (4.3) is solved by using linear FEM with $h=1/N$ and different graded meshes. The numerical errors and the convergence orders in temporal directions are listed in Table 3. Then, we set $\alpha=0.6$ and $N=\lfloor h^{(r+1)/(\alpha-2)}\rfloor(\lfloor x\rfloor := \max\{m\in\mathbb{Z} \mid m\le x\})$ and refine the spatial stepsize. The numerical errors and convergence orders in spatial direction are given in Table 4. These results further verify the convergence of our linearized method.

To further confirm the unconditional convergence, problem (4.3) is solved by using linear and quadratic FEMs with different stepsizes, respectively. The errors are shown in Fig. 2. The figures imply that for a given N, the errors in L_2−norm asymptotically tend to a constant. The results further confirm the theoretical findings.

Example 3 Finally, we test the unconditional convergence results by using the system (4.3)–(4.4) in three dimension domain $\Omega=(0,1)^3$. The boundary, initial conditions and g are given by using the following exact solution $u=(t^\alpha+t^3)\sin^2 x_1(1-x_1)^2\sin^2 x_2(1-x_2)^2\sin^2 x_3(1-x_3)^2$. The 3D problem is solved by using linear and quadratic FEMs with different stepsizes, respectively. Numerical results are presented in Fig. 3, Again, for a fixed

Table 3 Errors e_u^N and convergence rates in temporal direction (Example 4.2)

N	$\alpha = 0.8$		$\alpha = 0.6$		$\alpha = 0.4$		$\alpha = 0.2$	
	Error	Order	Error	Order	Error	Order	Error	Order
8	5.61e−05	–	5.12e−05	–	5.01e−05	–	5.03e−05	–
16	1.94e−05	1.53	1.57e−05	1.71	1.46e−05	1.78	1.50e−05	1.74
32	7.65e−06	1.35	5.26e−06	1.58	4.50e−06	1.70	4.63e−06	1.70
64	3.22e−06	1.25	1.89e−06	1.47	1.45e−06	1.64	1.44e−06	1.68
128	1.39e−06	1.21	7.07e−07	1.42	4.77e−07	1.60	4.49e−07	1.69

Table 4 Errors e_u^N and convergence rates in spatial direction (Example 4.2)

N	$r = 1$		$r = 2$	
	Error	Order	Error	Order
8	1.68e−04	–	9.86e−06	–
16	4.44e−05	1.92	1.27e−06	2.96
32	1.13e−05	1.98	1.60e−07	2.99
64	2.83e−06	1.99	2.00e−08	3.00
128	7.07e−07	2.00	2.50e−09	3.00

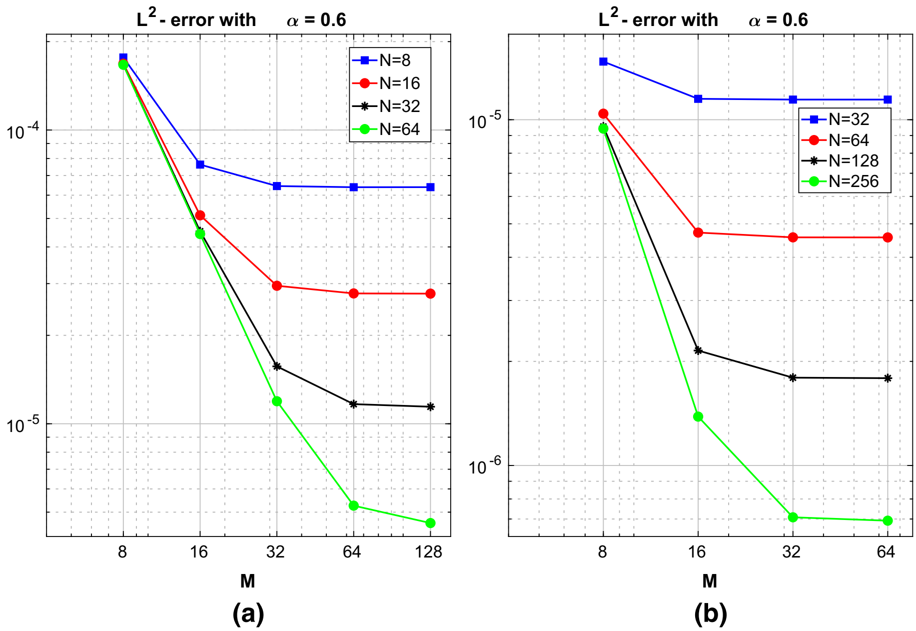

Fig. 2 L^2 errors on gradually refined meshes with fixed N (Example 4.2). **a** $r = 1$; **b** $r = 2$

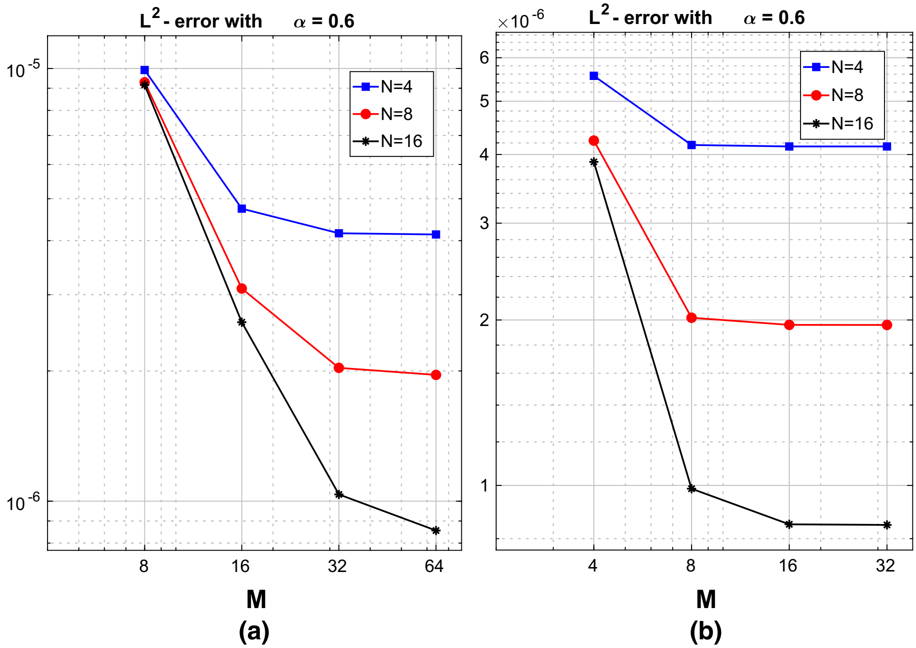

Fig. 3 L^2 errors on gradually refined meshes with fixed N (Example 4.3). **a** $r = 1$; **b** $r = 2$

N, the errors asymptotically tend to a constant, which implies that there is no certain time-step restrictions dependent on the spatial mesh size.

5 Conclusions

In order to effectively solve the problems, we apply the L1-scheme with graded meshes to approximate the time fractional derivative, the Newton linearized method to approximate the nonlinear term and the finite element method to discrete the spatial variable. By the use of the discrete fractional Gronwall type inequality on the non-uniform meshes and the temporal-spatial error splitting argument, the optimal error estimates of the Newton linearized scheme are obtained without the time-step restrictions dependent on the spatial mesh size. Such unconditional convergence results are proved by including the initial time singularity into concern.

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