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Virtual Element Methods for Elliptic Variational Inequalities of the Second Kind

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Abstract
This paper is devoted to virtual element methods for solving elliptic variational inequalities (EVIs) of the second kind. First, a general framework is provided for the numerical solution of the EVIs and for its error analysis. Then virtual element methods are applied to solve two representative EVIs: a simplified friction problem and a frictional contact problem. Optimal order error estimates are derived for the virtual element solutions of the two representative EVIs, including the effects of numerical integration for the non-smooth term in the EVIs. A fast solver is introduced to solve the discrete problems. Several numerical examples are included to show the numerical performance of the proposed methods.

Keywords Virtual element method · Elliptic variational inequality of the second kind · Error estimate · ASSN algorithm

1 Introduction

Variational inequalities, as a class of important nonlinear problems, are frequently encountered in various applied sciences, including physical, engineering, financial, and management sciences. In the past four decades, their mathematical theories have been established extensively and thoroughly (cf. [7,25,27,34]). Moreover, many numerical methods have been developed for such nonlinear variational problems, for instance, the conforming element method (cf. [6,16,26,28,29]), the nonconforming element method (cf. [33]), the mixed ele-

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ment method (cf. [13,17,18,31]), the finite volume method (cf. [42]), and the discontinuous Galerkin method (cf. [38]).

A new family of numerical methods, known as virtual element methods (VEMs), is gaining popularity in the area of numerical solution of differential equations. VEMs are advantageous in handling complex geometries and in solving problems with high-order differential equations. The methods were introduced in [2,8,10]. More recent theoretical developments of VEMs can be found in [11,14,19]. VEMs have been applied to solve a variety of problems in solid mechanics (e.g., [4,5,9]), fluid mechanics (e.g., [3,12]), and so on. In [39], VEMs are applied to solve the obstacle problem, which is an elliptic variational inequality (EVI) of the first kind. In [40], VEMs are applied to solve a simplified friction problem, which is an EVI of the second kind. In this paper, we study VEMs to solve EVIs of the second kind. Unlike [40], the VEMs developed in this paper are directly implementable.

Motivated by the construction of VEMs [2,8,10], we first propose in this paper a general framework for the numerical solution of EVIs of the second kind. Under some reasonable assumptions, we provide error analysis for the numerical solution, including the influence of numerical integration of the nonsmooth term which is very important in actual computation. As examples of applications of this general framework, we construct and analyze VEMs for solving the simplified friction problem and the frictional contact problem.

Furthermore, we discuss how to apply a recently developed algorithm (cf. [41]), which is superlinearly convergent, to solve discrete problems arising from the VEM discretization of the previous two problems. A series of numerical results are also reported to illustrate computational performance of the VEMs proposed in this paper.

Finally, it should be emphasized that our general framework can be used for the numerical solution and error analysis of any EVI of the second kind, whenever the VEM related to the linear problem of the standard form

$$u \in V, \quad a(u, v) = f(v) \quad \forall\, v \in V$$

is available.

The rest of this paper is organized as follows. An abstract method and its error analysis are given in Sect. 2. The two virtual element methods and their error estimates are developed in Sect. 3 for the simplified friction problem and the frictional contact problem, respectively. In Sect. 4, a detailed discussion is presented on solving the discrete problems in Sect. 3, based on a regularized semi-smooth Newton method with projection steps mentioned in [41]. Finally, several numerical results are presented in Sect. 5 to illustrate the performance of the VEMs proposed in this paper.

2 A General Framework for Numerical Solution of EVIs of the Second Kind

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygon with the Lipschitz boundary $\partial\Omega$. Let V be a Hilbert space consisting of certain functions defined over Ω, which is equipped with the norm $\|\cdot\|_V$. We use V' to denote the dual space of V and write $\langle\cdot,\cdot\rangle$ for the duality pairing between V' and V. Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a continuous, coercive and symmetric bilinear form, $j(\cdot) : V \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ a proper, convex and lower semi-continuous functional, and $f(\cdot) : V \to \mathbb{R}$ a continuous functional. For $v \in V$, we also write $f(v) = \langle f, v\rangle$. An elliptic variational inequality (EVI) of the second kind can be described as follows ([28]).

Problem 2.1 *Find $u \in V$ such that*

$$a(u, v - u) + j(v) - j(u) \geq \langle f, v - u \rangle \quad \forall\, v \in V. \tag{2.1}$$

This problem has a unique solution under the stated conditions on the problem data (cf. [28]). Let us introduce a general framework on the numerical solution of Problem 2.1. Let $\mathcal{T}_h := \{K\}_{K \in \mathcal{T}_h}$ be a mesh of Ω into polygons, with each generic element denoted by K; $h := \max_{K \in \mathcal{T}_h} h_K$ and $h_K := \operatorname{diam}(K)$. With this mesh, we associate a finite dimensional subspace V_h of V. For a non-negative integer m and an element $K \in \mathcal{T}_h$, denote by $\mathbb{P}_m(K)$ the set of all polynomials on K with the total degree no more than m. Moreover, we assume that the bilinear form $a(\cdot, \cdot)$ can be decomposed as

$$a(v, w) = \sum_{K \in \mathcal{T}_h} a^K(v, w), \quad v, w \in V,$$

where $a^K(\cdot, \cdot)$ is a bilinear form over $V_K := V_{|K}$. For a function in V, we naturally view it as a function in V_K by its restriction to K. We equip the Hilbert space $V_{|K}$ with a norm or semi-norm $\| \cdot \|_{V,K}$ such that

$$\|v\|_V^2 = \sum_{K \in \mathcal{T}_h} \|v\|_{V,K}^2 \quad \forall\, v \in V, \tag{2.2}$$

and for all $K \in \mathcal{T}_h$ there holds

$$a^K(v, v) \lesssim \|v\|_{V,K}^2 \quad \forall\, v \in V_K, \tag{2.3}$$

where and in what follows, for any two quantities a and b, "$a \lesssim b$" stands for "$a \leq Cb$", with C being a generic constant independent of h_K or h, which may take different values at different occurrences. Assume f_h is an approximation of f, and the bilinear form $a_h(\cdot, \cdot)$ is obtained by

$$a_h(u, v) := \sum_{K \in \mathcal{T}_h} a_h^K(u, v),$$

where the symmetric bilinear form $a_h^K(\cdot, \cdot)$ has the following properties:

- k-Consistency: For all $p \in \mathbb{P}_k(K)$ and for all $v_h \in V_{h|K}$,

$$a_h^K(p, v_h) = a^K(p, v_h), \tag{2.4}$$

  where k is a given natural number.
- Stability: There exist two positive constants $\alpha_\star$ and $\alpha^\star$, independent of h_K and K, such that

$$\alpha_\star a^K(v_h, v_h) \leq a_h^K(v_h, v_h) \leq \alpha^\star a^K(v_h, v_h) \quad \forall\, v_h \in V_{h|K}. \tag{2.5}$$

Our general numerical method for Problem 2.1 is as follows.

Problem 2.2 *Find $u_h \in V_h$ such that*

$$a_h(u_h, v_h - u_h) + j(v_h) - j(u_h) \geq \langle f_h, v_h - u_h \rangle \quad \forall\, v_h \in V_h. \tag{2.6}$$

Remark 2.1 The construction of $a_h(\cdot, \cdot)$ is motivated by the ideas of the VEM introduced in [2,8,10]. Here, we first propose a general framework for the numerical solution of Problem 2.1. Based on this general framework, we can then easily devise and analyze the virtual element method for solving several concrete EVIs of the second kind in a unified way. For details, see Sect. 3.

Remark 2.2 Throughout this paper, we use the standard notation and symbols for Sobolev spaces and their norms and seminorms. For details, see [1]. As a typical example of the above general framework, let $V := H^1(\Omega)$ with the norm $\|\cdot\|_V := \|\cdot\|_{1,\Omega}$. Choose

$$a(v, w) := \int_\Omega (\nabla v \cdot \nabla w + vw)\, dx.$$

Then $V_K = H^1(K)$ equipped with the norm $\|\cdot\|_{V,K} = \|\cdot\|_{1,K}$, and

$$a^K(v, w) := \int_K (\nabla v \cdot \nabla w + vw)\, dx.$$

In order to derive error estimates for the method (2.6), we make two assumptions as follows.

Assumption A1. For each $K \in \mathcal{T}_h$, for every $v \in H^{k+1}(K)$, there exists a function $v_\pi \in \mathbb{P}_k(K)$ such that

$$\|v - v_\pi\|_{0,K} + h_K\|v - v_\pi\|_{V,K} \lesssim h_K^{k+1}|v|_{k+1,K}, \quad v \in H^{k+1}(K). \tag{2.7}$$

Assumption A2. For each $K \in \mathcal{T}_h$, there exists an interpolation operator $I_K : H^{k+1}(K) \to V_{h|K}$ satisfying the following estimate:

$$\|v - I_K v\|_{0,K} + h_K\|v - I_K v\|_{V,K} \lesssim h_K^{k+1}|v|_{k+1,K}, \quad v \in H^{k+1}(K). \tag{2.8}$$

Moreover, $v_I \in V_h$ if v is additionally in V, where v_I is defined piecewise by $v_I = I_K v$ on K, for each $K \in \mathcal{T}_h$.

For later uses, we always write v_I as the global interpolant of v whenever $v \in H^{k+1}(\Omega)$.

The following result can be viewed as Céa's lemma for the numerical solution defined by Problem 2.2 for solving 2.1 (cf. [15,22]), which is useful in our further error estimates for the virtual element method of Problem 2.1.

Theorem 2.1 *If Assumptions* **A1–A2***,* (2.4) *and* (2.5) *hold and the true solution u of Problem* 2.1 *belongs to $H^{k+1}(\Omega)$ for some natural number k, then there holds*

$$\begin{aligned}\|u - u_h\|_V &\le \|u - u_I\|_V + \|u_I - u_h\|_V \\ &\lesssim h^k|u|_{k+1,\Omega} + \|f - f_h\|_{V_h'} + |R_1(u, u_I)|^{1/2},\end{aligned}$$

where

$$\|f - f_h\|_{V_h'} := \sup_{v_h \in V_h} \frac{\langle f - f_h, v_h\rangle}{\|v_h\|_V},$$

$$R_1(u, u_I) := a(u, u_I - u) + j(u_I) - j(u) - \langle f, u_I - u\rangle.$$

Proof Denote $w = u_I - u_h$. By the coerciveness of $a(\cdot,\cdot)$ and the stability condition for $a_h^K(\cdot,\cdot)$, there exists a positive constant α independent of h such that

$$\alpha\|w\|_V^2 \le a_h(u_I, w) - a_h(u_h, w). \tag{2.9}$$

Because of the k-consistency (2.4),

$$a_h^K(u_\pi, v_h) = a^K(u_\pi, v_h) \quad \forall\, v_h \in V_{h|K},$$

and we can write

$$a_h(u_I, w) = \sum_{K\in\mathcal{T}_h} \left(a_h^K(I_K u - u_\pi, w) + a^K(u_\pi - u, w)\right) + a(u, w). \tag{2.10}$$

Observe that

$$a(u,w) = a(u,u_I-u) + a(u,u-u_h) \\ \le a(u,u_I-u) + j(u_h) - j(u) - \langle f, u_h - u\rangle. \tag{2.11}$$

From the discrete variational inequality (2.6),

$$a_h(u_h,w) \ge j(u_h) - j(u_I) + \langle f_h, u_I - u_h\rangle. \tag{2.12}$$

Using (2.10), (2.11) and (2.12) in (2.9), we obtain

$$\alpha\|w\|_V^2 \le \sum_{K\in\mathcal{T}_h}\big(a_h^K(I_Ku-u_\pi,w)+a^K(u_\pi-u,w)\big) + \langle f-f_h,w\rangle + R_1(u,u_I). \tag{2.13}$$

It follows from the Cauchy–Schwarz inequality, (2.5) and (2.3) that

$$\sum_{K\in\mathcal{T}_h}\big(a_h^K(I_Ku-u_\pi,w)+a^K(u_\pi-u,w)\big) \\ \lesssim \left(\sum_{K\in\mathcal{T}_h}\|I_Ku-u_\pi\|_{V,K}^2 + \sum_{K\in\mathcal{T}_h}\|u-u_\pi\|_{V,K}^2\right)^{1/2}\|w\|_V.$$

So from (2.13), we have

$$\|w\|_V^2 \lesssim \left[\left(\sum_{K\in\mathcal{T}_h}\|I_Ku-u_\pi\|_{V,K}^2 + \sum_{K\in\mathcal{T}_h}\|u-u_\pi\|_{V,K}^2\right)^{1/2} + \|f-f_h\|_{V_h'}\right]\|w\|_V \\ +|R_1(u,u_I)|. \tag{2.14}$$

Applying the elementary result:

$$a,b,x\ge 0 \text{ and } x^2\le a\,x+b \quad\Longrightarrow\quad x^2\le a^2+2\,b,$$

we find from (2.14) that

$$\|w\|_V^2 \lesssim \sum_{K\in\mathcal{T}_h}\|I_Ku-u_\pi\|_{V,K}^2 + \sum_{K\in\mathcal{T}_h}\|u-u_\pi\|_{V,K}^2 \\ +\|f-f_h\|_{V_h'}^2 + |R_1(u,u_I)|. \tag{2.15}$$

Furthermore, we have by the estimates (2.7), (2.8) and the relation (2.2) that

$$\left(\sum_{K\in\mathcal{T}_h}\|u-u_\pi\|_{V,K}^2\right)^{1/2} \lesssim h^k|u|_{k+1,\Omega},$$

$$\|u-u_I\|_V = \left(\sum_{K\in\mathcal{T}_h}\|u-I_Ku\|_{V,K}^2\right)^{1/2} \lesssim h^k|u|_{k+1,\Omega}.$$

Noting that

$$\|u-u_h\|_V \le \|u-u_I\|_V + \|u_I-u_h\|_V,$$

we can obtain readily the desired result by using the last two inequalities and the estimate (2.15). □

In actual computation, we often require to carry out numerical integration for the quantity $j(v_h)$. In other words, $j(v_h)$ is replaced by another quantity $j_h(v_h)$. So instead of Problem 2.2, we actually solve the following problem.

Problem 2.3 *Find $u_h \in V_h$ such that*

$$a_h(u_h, v_h - u_h) + j_h(v_h) - j_h(u_h) \geq \langle f_h, v_h - u_h \rangle \quad \forall\, v_h \in V_h. \tag{2.16}$$

In this case, using the similar argument for deriving the above theorem, we can achieve the related error estimate under the following condition:

$$j_h(v_h) \geq j(v_h), \quad v_h \in V_h. \tag{2.17}$$

Theorem 2.2 *Assume* **A1**–**A2**, (2.4), (2.5) *and* (2.17) *hold true, and the true solution u of* (2.1) *belongs to $H^{k+1}(\Omega)$ for some natural number k. Let u_h be the solution of Problem* 2.3. *Then*

$$\|u - u_h\|_V \lesssim h^k |u|_{k+1,\Omega} + \|f - f_h\|_{V_h'} + |R(u, u_I)|^{1/2}, \tag{2.18}$$

where

$$R(u, u_I) := a(u, u_I - u) + j_h(u_I) - j(u) - \langle f, u_I - u \rangle. \tag{2.19}$$

Proof Take $v_h = u_I$ in (2.16),

$$a_h(u_h, u_I - u_h) + j_h(u_I) - j_h(u_h) \geq \langle f_h, u_I - u_h \rangle, \tag{2.20}$$

which implies

$$a_h(u_h, w) \geq j_h(u_h) - j_h(u_I) + \langle f_h, u_I - u_h \rangle,$$

where $w := u_I - u_h$. Arguing as in the derivation of (2.15), we have

$$\begin{aligned} \|w\|_V^2 \lesssim &\sum_{K \in \mathcal{T}_h} \|I_K u - u_\pi\|_{V,K}^2 + \sum_{K \in \mathcal{T}_h} \|u - u_\pi\|_{V,K}^2 \\ &+ \|f - f_h\|_{V_h'}^2 + |R(u, u_I)| + j(u_h) - j_h(u_h). \end{aligned} \tag{2.21}$$

Finally, noting the assumption (2.17) and following the same arguments after (2.15) in the proof of Theorem 2.1, we are able to derive the required result. □

3 Virtual Element Methods for EVIs of Second Kind

In this section, we intend to design and analyze some virtual element methods for solving two EVI problems, based on the framework in the last section. The partition $\mathcal{T}_h = \{K\}_{K \in \mathcal{T}_h}$ of the domain Ω has been introduced in Sect. 2. Following [19], we make the following assumption for the mesh $\mathcal{T}_h$.

Assumption A. For each $K \in \mathcal{T}_h$, there exists a "virtual triangulation" $\mathcal{T}_K$ of K such that $\mathcal{T}_K$ is uniformly shape regular and quasi-uniform. The corresponding mesh size of $\mathcal{T}_K$ is bounded below by a constant multiple of h_K. Each edge of K is a side of a triangle in $\mathcal{T}_K$.

It is evident to check that the above assumption covers the usual conditions satisfied by $K \in \mathcal{T}_h$, given as follows (cf. [2,8,10]).

C1 There exists a real number $\gamma > 0$ such that for each element $K \in \mathcal{T}_h$, it is star-shaped with respect to a disk of radius $\rho_K \geq \gamma h_K$.

C2 There exists a real number $\gamma_1 > 0$ such that for each element $K \in \mathcal{T}_h$, the distance between any two vertices of K is $\geq \gamma_1 h_K$.

We comment that under the Assumption **A**, it is easy to derive the estimate (2.7) using the classical Scott–Dupont theory in the case $V = H^1(\Omega)$ (cf. [15]), and if the virtual element space V_h is taken as those given in [2,8,10], the estimate (2.8) always holds by choosing I_K as the related nodal interpolation operator (cf. [14,19]).

Note that in general we can not achieve optimal error estimates for variational inequalities with high order elements due to the lack of sufficient solution regularity and the inequality feature of the problem. Hence, we restrict our attention to the lowest order virtual method introduced in [8]. From now on, we always assume the mesh $\mathcal{T}_h$ satisfies the Assumption **A**.

3.1 The Simplified Friction Problem

Let Ω be a bounded polygonal domain of $\mathbb{R}^2$, $f \in L^2(\Omega)$ and $g > 0$ a constant. Decompose the boundary $\partial\Omega$ into two subsets Γ_D and Γ_C, which is aligned with the mesh $\mathcal{T}_h$. Then, the boundary value problem related to a simplified friction problem can be expressed as (cf. [28])

$$\begin{cases} -\Delta u + u = f \quad \text{in } \Omega, \\ |\partial_{\boldsymbol{\nu}} u| \leq g, \ u\partial_{\boldsymbol{\nu}} u + g|u| = 0 \quad \text{on } \Gamma_C, \\ u = 0 \quad \text{on } \partial\Omega \backslash \Gamma_C. \end{cases} \tag{3.1}$$

Define $V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\}$ equipped with the norm $\|\cdot\|_V = \|\cdot\|_{1,\Omega}$. Then, the variational inequality related to (3.1) reads

$$u \in V, \quad a(u, v-u) + j(v) - j(u) \geq \langle f, v-u \rangle \quad \forall\, v \in V, \tag{3.2}$$

where

$$a(u,v) := \int_\Omega (\nabla u \cdot \nabla v + uv)\, dx, \quad f(v) := \int_\Omega f v\, dx, \quad j(v) := \int_{\Gamma_C} g\, |v|\, ds.$$

Now, let us define a virtual element space of a lowest order ($k = 1$) as follows. For all $K \in \mathcal{T}_h$, as shown in [8],

$$V_1(K) := \left\{ v \in H^1(K) \mid \Delta v = 0 \text{ in } K,\ v|_{\partial K} \in C(\partial K),\ v|_e \in P_1(e) \text{ for each } e \in \partial K \right\}, \tag{3.3}$$

with the function values at the vertices of K as a set of degrees of freedom. Then the virtual element space V_h is defined by

$$V_h = \{v \in C(\overline{\Omega}) \mid v|_K \in V_1(K)\ \forall\, K \in \mathcal{T}_h\} \cap V.$$

Next, introduce a local H^1 projection $\Pi_1^\nabla : H^1(K) \to \mathbb{P}_1(K)$ as follows. For $v \in H^1(K)$,

$$\begin{cases} (\nabla \Pi_1^\nabla v, \nabla p)_K = (\nabla v, \nabla p)_K \quad \forall\, p \in \mathbb{P}_1(K), \\ \overline{\Pi_1^\nabla v} = \overline{v}, \end{cases}$$

where $(\cdot,\cdot)_K$ is the $L^2(K)$ inner product, and $\overline{v}$ is the integral average of v on the boundary ∂K of K. To simplify the presentation, we also use Π_1^∇ to represent the related element-wise defined global operator.

In addition, set

$$a_h(u,v) := (\nabla\Pi_1^\nabla u, \nabla\Pi_1^\nabla v) + (\Pi_1^\nabla u, \Pi_1^\nabla v) + \sum_{K\in\mathcal{T}_h} S_K(u,v), \tag{3.4}$$

$$S_K(u,v) := \sum_{i=1}^{n^K}(1+h_K^2)\chi_i^K((I-\Pi_1^\nabla)u)\cdot\chi_i^K((I-\Pi_1^\nabla)v), \tag{3.5}$$

where $(\cdot,\cdot)$ denotes the usual $L^2(\Omega)$ inner product, n^K denotes the number of vertices of K, while $\{\chi_i^K(\cdot)\}$ are the function values at the vertices of K. It is routine to examine that the third term in (3.4) can be assembled as a diagonal bilinear form in terms of the degrees of freedom $\{\chi_i(\cdot)\}$ of V_h.

On the other hand, we choose the approximation f_h of f such that (cf. [19])

$$\langle f_h, v_h\rangle := \sum_{K\in\mathcal{T}_h}\int_K f\cdot\Pi_1^\nabla v_h dx, \quad v_h\in V_h. \tag{3.6}$$

It is easy to see that this quantity can be expressed in terms of the degrees of freedom of V_h. Moreover, we have by the error estimate for the operator Π_1^∇ (cf. [19]) and the Cauchy–Schwarz inequality that

$$\begin{aligned}|\langle f-f_h, v_h\rangle| &= \Big|\sum_{K\in\mathcal{T}_h}\int_K f\cdot(v_h-\Pi_1^\nabla v_h)dx\Big| \le \sum_{K\in\mathcal{T}_h}\|f\|_{0,K}\|v_h-\Pi_1^\nabla v_h\|_{0,K}\\ &\lesssim \sum_{K\in\mathcal{T}_h}\|f\|_{0,K}h_K|v_h|_{1,K} \lesssim h\|f\|_{0,\Omega}\|v_h\|_V,\end{aligned}$$

i.e.,

$$\|f-f_h\|_{V_h'} \lesssim h. \tag{3.7}$$

As for the nonlinear functional $j(\cdot)$, we use the trapezoidal rule to carry out numerical integration at each edge of $\mathcal{T}_h$ on Γ_C, to get its approximation $j_h(\cdot)$. In detail, let $\{P_i\}_{i=1}^{m_0}$ be the nodes of $\mathcal{T}_h$ on Γ_C, which are numbered such that each line segment $e_i := \overline{P_iP_{i+1}}$, with length h_i, is an edge of a polygon $K\in\mathcal{T}_h$, lying on Γ_C. Then, for all $v_h\in V_h$,

$$j_h(v_h) := g\sum_{i=1}^{m_0-1}\frac{h_i}{2}(|v_h(P_i)|+|v_h(P_{i+1})|). \tag{3.8}$$

Now, we are ready to present a VEM for solving (3.2), described as follows.

$$u_h\in V_h, \quad a_h(u_h, v_h-u_h)+j_h(v_h)-j_h(u_h) \ge \langle f_h, v_h-u_h\rangle \quad \forall v_h\in V_h. \tag{3.9}$$

Theorem 3.1 *Assume that $f\in L^2(\Omega)$, and the exact solution u of problem (3.2) belong to $H^2(\Omega)$ and $u_{|\Gamma_i}\in H^2(\Gamma_i)$, with $\{\Gamma_i\}_{i=1}^{n_0}$ being the edges of $\partial\Omega$ on Γ_C. Moreover, u changes its sign only finite many times on Γ_C. Let $a_h(\cdot,\cdot)$, $f_h(\cdot)$ and $j_h(\cdot)$ be given by (3.4), (3.6) and (3.8), respectively. Then*

$$\|u-u_h\|_V \lesssim h,$$

where u_h is the solution of the method (3.9).

Proof It is shown in [2,8,19] that the above bilinear form $a_h(\cdot,\cdot)$ satisfies conditions (2.4) and (2.5). Since the mesh satisfies the Assumption **A**, Assumptions **A1** and **A2** naturally hold (cf. [14,15,19]). On the other hand, it is easy to check that

$$j_h(v_h) \geq j(v_h) \quad \forall\, v_h \in V_h.$$

Hence, all the assumptions in Theorem 2.2 hold, and we are able to bound the error of u_h by means of estimates (2.18)–(2.19).

On the other hand, recalling the construction of the nodal interpolation operator I_K (cf. [14, 19]), u_I is equal to the usual continuous $\mathbb{P}_1$ interpolant on Γ_C. Then, under the assumptions of u, following the similar argument for deriving [30, Theorem 3.2] and using the error estimate for the interpolation operator for Lagrange elements (cf. [15,22]), we find

$$|j_h(u_I) - j(u)| \lesssim h^2 \sum_{i=1}^{n_0} \|u\|_{W^{1,\infty}(\Gamma_i)} + \|u - u_I\|_{L^1(\Gamma_C)} \lesssim h^2 \sum_{i=1}^{n_0} \|u\|_{2,\Gamma_i}. \tag{3.10}$$

By integration by parts, using the first equation in (3.1), the property $u_I - u = 0$ on $\partial\Omega\backslash\Gamma_C$ and the Cauchy–Schwarz inequality, we have

$$|a(u, u_I - u) - \langle f, u_I - u\rangle| \leq \|\partial_{\boldsymbol{\nu}} u\|_{0,\Gamma_C} \|u - u_I\|_{0,\Gamma_C}.$$

Furthermore, invoking the error estimates for the nodal interpolation operator of Lagrange elements on Γ_C, we find

$$|a(u, u_I - u) - \langle f, u_I - u\rangle| \lesssim h^2. \tag{3.11}$$

Inserting the estimates (3.7), (3.10) and (3.11) into (2.18) and (2.19), we obtain the desired result. □

Remark 3.1 It deserves to point out some difference between our method (3.9) and the method studied in [40] to solve the simplified friction problem. For our method, the discrete right-hand side is computable with respect to the degrees of freedom of the VEM space. Moreover, the nonsmooth term $j(\cdot)$ is approximated through numerical integration so that the discrete problem can be solved efficiently by a fast algorithm developed in Sect. 4 while the proposed numerical method maintains the optimal convergence order.

3.2 The Frictional Contact Problem

In this subsection, we consider the deformation of an elastic body occupying a bounded domain Ω with a Lipschitz boundary Γ, with the physical setting depicted as in Fig. 1.

In this model, $\mathbb{C}\colon \mathbb{S}^2 \to \mathbb{S}^2$ is the fourth order elasticity tensor, which is bounded, symmetric and positive definite in Ω. Here, $\mathbb{S}^2$ denotes the set of all symmetric second order tensors in $\mathbb{R}^2$. The boundary is made up of Γ_D, Γ_N and Γ_C, where meas$(\Gamma_D) > 0$. We assume that the body is clamped on Γ_D. The surface traction of density $\boldsymbol{f}_2 \in (L^2(\Gamma_N))^2$ is applied to Γ_N. Γ_C is the contact surface with a rigid foundation. Volume forces of density $\boldsymbol{f}_1 \in (L^2(\Omega))^2$ act in Ω.

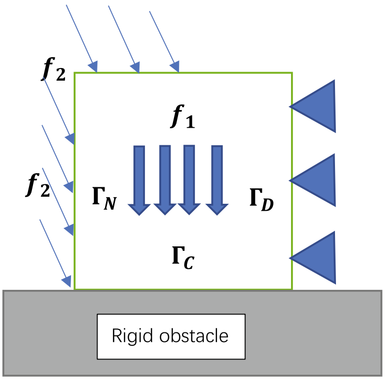

Fig. 1 Physical setting of the frictional contact problem

The mathematical model to this physical problem can be described as follows (cf. [25]).

$$\begin{cases} -\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}_1 \quad \text{in } \Omega, \\ \boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon} \quad \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D, \\ \boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_N, \\ u_\nu = 0 \quad \text{on } \Gamma_C, \\ |\boldsymbol{\sigma_\tau}| \le g \quad \text{on } \Gamma_C, \\ |\boldsymbol{\sigma_\tau}| < g \ \Rightarrow\ \boldsymbol{u_\tau} = \boldsymbol{0} \quad \text{on } \Gamma_C, \\ |\boldsymbol{\sigma_\tau}| = g \ \Rightarrow\ \boldsymbol{u_\tau} = -\lambda\boldsymbol{\sigma_\tau} \quad \text{on } \Gamma_C. \end{cases} \tag{3.12}$$

Here, for a vector $\boldsymbol{v}$, denote on the boundary $\partial\Omega$ by $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ its normal component and $\boldsymbol{v_\tau} = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$ the tangential component, respectively. Similarly, for a tensor $\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u})$, $\boldsymbol{\sigma} \in \mathbb{S}^2$, define its normal component as $\sigma_\nu = \boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu}$ and tangential component as $\boldsymbol{\sigma_\tau} = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$.

Let

$$\boldsymbol{V} := \{\boldsymbol{v} \in (H^1(\Omega))^2 \mid \boldsymbol{v}|_{\Gamma_D} = \boldsymbol{0},\ v_\nu|_{\Gamma_C} = 0\},$$

equipped with the inner product and the induced norm given by

$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}} := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) dx, \quad \|\boldsymbol{v}\|_{\boldsymbol{V}} := (\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{V}}^{1/2}.$$

Since meas$(\Gamma_D) > 0$, Korn's inequality holds ([35]) so that $\|\boldsymbol{v}\|_{\boldsymbol{V}}$ is a norm over $\boldsymbol{V}$, equivalent to the standard norm $\|\boldsymbol{v}\|_{1,\Omega}$. Over the space $\boldsymbol{V}$, let

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \tag{3.13}$$

$$f(\boldsymbol{v}) := \int_\Omega \boldsymbol{f}_1 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_N} \boldsymbol{f}_2 \cdot \boldsymbol{v}\, ds, \tag{3.14}$$

$$j(\boldsymbol{v}) := \int_{\Gamma_C} g\, |\boldsymbol{v_\tau}|\, ds. \tag{3.15}$$

Then the variational formulation of (3.12) is to find $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\boldsymbol{v}) - j(\boldsymbol{u}) \ge f(\boldsymbol{v} - \boldsymbol{u}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}. \tag{3.16}$$

Next, we choose

$$W_h := \{v \in C(\overline{\Omega}) \mid v|_K \in V_1(K)\ \forall K \in \mathcal{T}_h\},$$

and define

$$\boldsymbol{V}_h = (W_h)^2 \cap \boldsymbol{V}.$$

This is the virtual element space we want to use for numerically solving problem (3.16).

Furthermore, we briefly describe how to construct the bilinear form $a_h^K(\cdot,\cdot)$. Let $\boldsymbol{V}_h(K) := (V_1(K))^2$. Let $\boldsymbol{\Pi}_K$ be a projection operator from $\boldsymbol{V}_h(K)$ into $\mathbb{P}_0(K)_{sym}^{2\times 2}$ such that for any given $\boldsymbol{v}_h \in \boldsymbol{V}_h(K)$,

$$\int_K \boldsymbol{\Pi}_K(\boldsymbol{v}_h) : \boldsymbol{\varepsilon}^P dx = \int_K \boldsymbol{\varepsilon}(\boldsymbol{v}_h) : \boldsymbol{\varepsilon}^P dx \quad \forall \boldsymbol{\varepsilon}^P \in \mathbb{P}_0(K)_{sym}^{2\times 2},$$

where $\mathbb{P}_0(K)_{sym}^{2\times 2}$ stands for the set of all second order symmetric tensor fields with each entry being constant. Intuitively, $\boldsymbol{\Pi}_K(\boldsymbol{v}_h)$ is a constant projection of the strain field $\boldsymbol{\varepsilon}(\boldsymbol{v}_h)$ over K. Then, following [4, Eq. (12)], define

$$a_h^K(\boldsymbol{v}_h, \boldsymbol{w}_h) = \int_K \mathbb{C}\boldsymbol{\Pi}_K(\boldsymbol{v}_h) : \boldsymbol{\Pi}_K(\boldsymbol{w}_h)\, dx + b_h^K(\boldsymbol{v}_h, \boldsymbol{w}_h) \quad \forall \boldsymbol{v}_h,\ \boldsymbol{w}_h \in \boldsymbol{V}_h(K), \tag{3.17}$$

where the second term is a stabilization term. We mention that the first term on the right of (3.17) is essentially equivalent to the first term given in equation (4.1) of the paper [9]. However, the construction of $b_h^K(\cdot,\cdot)$ is rather involved, requiring that $b_h^K(\cdot,\cdot)$ is a symmetric and positive semidefinite bilinear form whose kernel is exactly $(\mathbb{P}_1(K))^2$ (cf. [9, pp. 808–809]). To simplify the presentation, we refer to [4,9] for details along this line.

Moreover, similar to the simplified friction problem, we also use the trapezoidal rule to produce its approximation $j_h(\cdot)$. In other words, for all $\boldsymbol{v} \in \boldsymbol{V}_h$,

$$j_h(\boldsymbol{v}) := g \sum_{i=1}^{m_0-1} \frac{h_i}{2}(|\boldsymbol{v}_{\boldsymbol{\tau}}(P_i)| + |\boldsymbol{v}_{\boldsymbol{\tau}}(P_{i+1})|). \tag{3.18}$$

For the right-hand side f, similar to the treatment for the above problem (cf. (3.6)), we define the approximation f_h such that

$$f_h(\boldsymbol{v}_h) := \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{f}_1 \cdot \boldsymbol{\Pi}_1^{\nabla} \boldsymbol{v}_h dx + \int_{\Gamma_N} \boldsymbol{f}_2 \cdot \boldsymbol{v}_h dx, \quad \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.19}$$

where $\boldsymbol{\Pi}_1^{\nabla}$ is the vectorized analog of Π_1^{∇}, i.e., for all $\boldsymbol{v} = (v_1, v_2)^T$,

$$\boldsymbol{\Pi}_1^{\nabla}\boldsymbol{v} := (\Pi_1^{\nabla} v_1, \Pi_1^{\nabla} v_2)^T.$$

We remark that the same notation applies to Sobolev spaces as well, e.g., $\boldsymbol{H}^2(\Omega) := (H^2(\Omega))^2$, etc.

It is easy to see that $f_h(\boldsymbol{v}_h)$ is computable because $\boldsymbol{\Pi}_1^{\nabla}$ is computable and $\boldsymbol{v}_h$ is piecewise linear function on Γ_N. Moreover, using the similar argument for getting (3.7) we know

$$\|f - f_h\|_{\boldsymbol{V}_h'} \lesssim h. \tag{3.20}$$

Now, we are ready to define a VEM for solving problem (3.16) as follows.

$$\boldsymbol{u}_h \in \boldsymbol{V}_h, \quad a_h(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + j_h(\boldsymbol{v}_h) - j_h(\boldsymbol{u}_h) \geq \langle \boldsymbol{f}_h, \boldsymbol{v}_h - \boldsymbol{u}_h \rangle \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.21}$$

In the error estimation for the above numerical method, we will assume the solution regularity $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$. Then for $\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}$, we have $\boldsymbol{\sigma} \in H^1(\Omega)^{2\times 2}$ and it follows that $\boldsymbol{\sigma}\boldsymbol{\nu} \in \boldsymbol{L}^2(\partial\Omega)$. Following the arguments in [32, Section 8.1], we know that the solution $\boldsymbol{u} \in \boldsymbol{V}$ of (3.16) satisfies the pointwise relations

$$-\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}_1 \quad \text{a.e. in } \Omega, \tag{3.22}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{a.e. on } \Gamma_N. \tag{3.23}$$

Theorem 3.2 *Assume that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, and the exact solution $\boldsymbol{u}$ of problem (3.16) belong to $\boldsymbol{H}^2(\Omega)$ and $\boldsymbol{u}_{|\Gamma_i} \in \boldsymbol{H}^2(\Gamma_i)$, with $\{\Gamma_i\}_{i=1}^{n_0}$ being the edges of $\partial\Omega$ on Γ_C. Moreover, the tangential component $\boldsymbol{u}_{\boldsymbol{\tau}}$ of $\boldsymbol{u}$ changes its sign only finite many times on Γ_C. Let $a_h(\cdot,\cdot)$ be given in [4], $f_h(\cdot)$ by (3.19) and $j_h(\cdot)$ by (3.18), respectively. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{V}} \lesssim h,$$

where $\boldsymbol{u}_h$ is the solution of the method (3.21).

Proof It is shown in [4,9,15,19] that the above bilinear form $a_h(\cdot,\cdot)$ satisfies conditions (2.4) and (2.5). Since the mesh satisfies the Assumption **A**, Assumption **A1** holds by the classical Scott–Dupont theory (cf. [14]). Moreover, there exists a nodal interpolation operator I_K: $H^2(K) \to V_1(K)$ such that Assumption **A2** holds (cf. [14,19]). We further write $\boldsymbol{I}_K$ as its vectorized analog, and for $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega)$ write its global interpolant as $\boldsymbol{v}_I$. It is easy to check that if $\boldsymbol{v} \in \boldsymbol{V}$, $\boldsymbol{v}_I \in \boldsymbol{V}_h$. Thus Assumption **A2** also holds. Moreover, it is easy to show that

$$j_h(\boldsymbol{v}) \geq j(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}_h.$$

Hence, we can bound the error of $\boldsymbol{u}_h$ using Theorem 2.2.

Using an argument similar to that in [30] (see also the proof of Theorem 3.1), we have

$$|j_h(\boldsymbol{u}_I) - j(\boldsymbol{u})| \lesssim h^2. \tag{3.24}$$

Using the definitions (3.13) and (3.14), the boundary conditions $\boldsymbol{u}_I = \boldsymbol{u} = \boldsymbol{0}$ on Γ_D, $u_{I\nu} = u_\nu = 0$ on Γ_C, and the pointwise relations (3.22)–(3.23), we have by integration by parts that

$$a(\boldsymbol{u}, \boldsymbol{u}_I - \boldsymbol{u}) - f(\boldsymbol{u}_I - \boldsymbol{u}) = \int_{\Gamma_C} \boldsymbol{\sigma}_{\boldsymbol{\tau}} \cdot ((\boldsymbol{u}_{\boldsymbol{\tau}})_I - \boldsymbol{u}_{\boldsymbol{\tau}})\, ds.$$

Hence, by the error estimates for the nodal interpolation operator of Lagrange elements, we immediately know

$$|a(\boldsymbol{u}, \boldsymbol{u}_I - \boldsymbol{u}) - f(\boldsymbol{u}_I - \boldsymbol{u})| \lesssim h^2. \tag{3.25}$$

Finally, inserting the estimates (3.20), (3.24) and (3.25) into (2.18) and (2.19), we obtain the desired estimate. □

4 Numerical Solution of the Discrete Problem

We now discuss efficient algorithms for solving the discrete Problem 2.3, in particular, the problems (3.9) and (3.21). As far as we know, there are two classes of algorithms in the literature for Problem 2.3. The first class of algorithms is based on the primal-dual (or the dual) formulation of Problem 2.3, with the dual problem solved by fixed-point algorithms (cf. [21,28,29] and the references therein). It is noted that such algorithms converge only linearly. The second class of algorithms depends on reformulating the original problem as a minimization problem which is solved by feasible optimization methods (cf. [20,24,28, 29,41]). Due to the rapid development of non-smooth convex programming in the past two decades, the latter one seems more attractive. In particular, a regularized semi-smooth Newton method with projection steps is proposed in [41] for solving a class of non-smooth convex problems. The method enjoys a remarkable advantage that it may be superlinearly convergent if a convex problem satisfies some reasonable conditions. Here, we will discuss how to use the method for solving the discrete problems (3.9) and (3.21).

Define

$$E(v_h) = \frac{1}{2} a_h(v_h, v_h) - f_h(v_h) + j_h(v_h).$$

It is well known that Problem 2.3 is equivalent to the following minimization problem.

Find $u_h \in V_h$ such that

$$E(u_h) = \inf \{E(v_h) \mid v_h \in V_h\}.$$

With the above reformulation in mind, after expressing the numerical solution in terms of shape basis functions of a given virtual element space, we can find after a direct manipulation that the minimization problem to either (3.9) or (3.21) can be expressed in the following form:

$$\min_{\boldsymbol{v} \in \mathbb{R}^N} \frac{1}{2} \boldsymbol{v}^T \boldsymbol{A} \boldsymbol{v} - \boldsymbol{b}^T \boldsymbol{v} + \boldsymbol{w}^T |\boldsymbol{v}|, \tag{4.1}$$

where $|\boldsymbol{v}|$ denotes a new vector formed by taking the absolute value for each entry of $\boldsymbol{v}$, $\boldsymbol{A} = (a_{ij})_{N \times N}$, with $a_{ij} = a_h(\phi_i, \phi_j)$, is the stiffness matrix, and $\boldsymbol{b} = (b_i)_{N \times 1}$, with $b_i = f_h(\phi_i)$, is the load vector. Here, $\{\phi_i\}_{i=1}^N$ are the shape basis functions of V_h or $\boldsymbol{V}_h$. Moreover, we number these functions such that the first N_1 ones correspond to the nodes on Γ_C. We remark that when dealing with the numerical solution of problem (3.21), we assume each edge of Ω on Γ_C is parallel to the coordinate axis.

Next, write the vector $\boldsymbol{v}$ in block form as $\boldsymbol{v} = ((\boldsymbol{v}_1)^T, (\boldsymbol{v}_2)^T)^T$ with $\boldsymbol{v}_1 \in \mathbb{R}^{N_1}$. Similarly,

$$\boldsymbol{A} := \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{12}^T & \boldsymbol{A}_{22} \end{pmatrix}, \quad \boldsymbol{b} := \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}, \quad \boldsymbol{w} := \begin{pmatrix} \boldsymbol{w}_1 \\ \boldsymbol{0} \end{pmatrix}, \tag{4.2}$$

Then the problem (4.1) can be recast as follows: Find $\boldsymbol{v}_1^* \in \mathbb{R}^{N_1}$ and $\boldsymbol{v}_2^* \in \mathbb{R}^{N_2}$ such that

$$\boldsymbol{F}(\boldsymbol{v}_1^*, \boldsymbol{v}_2^*) = \inf \left\{ \boldsymbol{F}(\boldsymbol{v}_1, \boldsymbol{v}_2) \mid \boldsymbol{v}_1 \in \mathbb{R}^{N_1}, \boldsymbol{v}_2 \in \mathbb{R}^{N_2} \right\}, \tag{4.3}$$

where $N_2 = N - N_1$, and

$$\boldsymbol{F}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \frac{1}{2} (\boldsymbol{v}_1^T \boldsymbol{A}_{11} \boldsymbol{v}_1 + 2 \boldsymbol{v}_1^T \boldsymbol{A}_{12} \boldsymbol{v}_2 + \boldsymbol{v}_2^T \boldsymbol{A}_{22} \boldsymbol{v}_2) - \boldsymbol{b}_1^T \boldsymbol{v}_1 - \boldsymbol{b}_2^T \boldsymbol{v}_2 + \boldsymbol{w}_1^T |\boldsymbol{v}_1|.$$

For the above optimization problem, from the first order condition with respect to $\boldsymbol{v}_2$ it follows that

$$\boldsymbol{0} = \boldsymbol{A}_{12}^T \boldsymbol{v}_1^* + \boldsymbol{A}_{22} \boldsymbol{v}_2^* - \boldsymbol{b}_2,$$

which implies

$$\boldsymbol{v}_2^* = \boldsymbol{A}_{22}^{-1} (\boldsymbol{b}_2 - \boldsymbol{A}_{12}^T \boldsymbol{v}_1^*). \tag{4.4}$$

Inserting this into (4.3), we find the above optimization problem can be expressed as follows.

Find $\boldsymbol{v}_1^* \in \mathbb{R}^{N_1}$ such that

$$\widetilde{\boldsymbol{F}}(\boldsymbol{v}_1^*) = \inf \left\{ \widetilde{\boldsymbol{F}}(\boldsymbol{v}_1) \mid \boldsymbol{v}_1 \in \mathbb{R}^{N_1} \right\}, \tag{4.5}$$

where

$$\widetilde{\boldsymbol{F}}(\boldsymbol{v}_1) = \frac{1}{2}\boldsymbol{v}_1^T \widetilde{\boldsymbol{A}_1} \boldsymbol{v}_1 - \widetilde{\boldsymbol{b}_1}^T \boldsymbol{v}_1 + \boldsymbol{w}_1^T |\boldsymbol{v}_1|,$$

with

$$\widetilde{\boldsymbol{A}_1} = \boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{12}^T, \quad \widetilde{\boldsymbol{b}_1} = \boldsymbol{b}_1 - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{b}_2.$$

The problem (4.5) can be described as the following separable convex optimization problem:

$$\min_{\boldsymbol{x}} f(\boldsymbol{x}) + g(\boldsymbol{x}), \tag{4.6}$$

where

$$f(\boldsymbol{x}) := \boldsymbol{w}_1^T |\boldsymbol{x}|, \quad g(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{x}^T \widetilde{\boldsymbol{A}_1}\boldsymbol{x} - \widetilde{\boldsymbol{b}_1}^T \boldsymbol{x}.$$

We next show how to use the algorithm in [41] for working out the solution of (4.6).

First, it is evident that the minimizer of (4.5) satisfies the following fixed-point formulation (cf. [24]):

$$\boldsymbol{x} = \boldsymbol{T}(\boldsymbol{x}) := \mathbf{prox}_{tf}(\boldsymbol{x} - t(\widetilde{\boldsymbol{A}_1}\boldsymbol{x} - \widetilde{\boldsymbol{b}_1})),$$

where $t > 0$ is a constant, and **prox** is the usual proximal operator (cf. [24]). The proximal operator corresponding to $f(\boldsymbol{x})$ is the shrinkage operator defined as

$$(\mathbf{prox}_{tf}(\boldsymbol{x}))_i = \text{sign}(x_i)\max(|x_i| - w_i t, 0).$$

Then we solve the equation

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{0} \tag{4.7}$$

by using the algorithms in [41]. The generalized Jacobian matrix of $\boldsymbol{F}(\boldsymbol{x})$ is

$$\boldsymbol{J}(\boldsymbol{x}) = \boldsymbol{I} - \boldsymbol{M}(\boldsymbol{x})(\boldsymbol{I} - t\widetilde{\boldsymbol{A}_1}),$$

where $\boldsymbol{M}(\boldsymbol{x}) \in \partial\, \mathbf{prox}_{tf}(\boldsymbol{x} - t\,(\widetilde{\boldsymbol{A}_1}\boldsymbol{x} - \widetilde{\boldsymbol{b}_1}))$ with ∂ denoting Clarke's generalized Jacobian (cf. [23,41]), and $\widetilde{\boldsymbol{A}_1}$ is the Hessian matrix of $g(\boldsymbol{x})$. Since $\widetilde{\boldsymbol{A}_1}$ is symmetric positive definite, we can choose $\boldsymbol{M}(\boldsymbol{x})$ as a diagonal matrix with the i-th diagonal entry given by

$$(\boldsymbol{M}(\boldsymbol{x}))_{ii} = \begin{cases} 1 & \text{if } |(\boldsymbol{x} - t(\widetilde{\boldsymbol{A}_1}\boldsymbol{x} - \widetilde{\boldsymbol{b}_1}))_i| > w_i t, \\ 0 & \text{otherwise.} \end{cases}$$

As in [41], introduce index sets

$$\mathcal{I} := \{1 \le i \le N_1 \mid |(\boldsymbol{x} - t(\widetilde{\boldsymbol{A}_1}\boldsymbol{x} - \widetilde{\boldsymbol{b}_1}))_i| > tw_i\} = \{1 \le i \le N_1 \mid (\boldsymbol{M}(\boldsymbol{x}))_{ii} = 1\},$$
$$\mathcal{J} := \{1 \le i \le N_1 \mid |(\boldsymbol{x} - t(\widetilde{\boldsymbol{A}_1}\boldsymbol{x} - \widetilde{\boldsymbol{b}_1}))_i| \le tw_i\} = \{1 \le i \le N_1 \mid (\boldsymbol{M}(\boldsymbol{x}))_{ii} = 0\}.$$

Then partition the matrix $\widetilde{\boldsymbol{A}_1}$ as a 2-by-2 block matrix in terms of the above partition of the index set $\{1, 2, \cdots, N_1\}$. Then we have

$$\boldsymbol{J}(\boldsymbol{x}) = \begin{pmatrix} t\,(\widetilde{\boldsymbol{A}_1})_{\mathcal{I}\mathcal{I}} & t\,(\widetilde{\boldsymbol{A}_1})_{\mathcal{I}\mathcal{J}} \\ \boldsymbol{0} & I \end{pmatrix}.$$

For an iterate $\boldsymbol{x}^k$, by choosing an element $\boldsymbol{J}_k$, a regularized Newton method for solving (4.7) is given as follows. Seek the incremental $\boldsymbol{d}_k$ by solving the regularized equations

$$(\boldsymbol{J}_k + \mu_k \boldsymbol{I})\boldsymbol{d}_k = -\boldsymbol{F}^k,$$

where $\boldsymbol{F}^k = \boldsymbol{F}(\boldsymbol{x}^k)$, $\mu_k = \lambda_k \|\boldsymbol{F}^k\|_2$ and $\lambda_k > 0$ is a regularization parameter. For our problem under discussion, the above equations are equivalent to

$$\begin{aligned} \boldsymbol{d}_{\mathcal{J}} &= -\frac{1}{1+\mu_k}\boldsymbol{F}_{\mathcal{J}}, \\ \left(t\,(\widetilde{\boldsymbol{A}}_1)_{\mathcal{I}\mathcal{I}} + \mu_k \boldsymbol{I}\right)\boldsymbol{d}_{\mathcal{I}} &= -\boldsymbol{F}_{\mathcal{I}} - t\,(\widetilde{\boldsymbol{A}}_1)_{\mathcal{I}\mathcal{J}}, \end{aligned} \tag{4.8}$$

from which we derive the incremental $\boldsymbol{d}_k = (\boldsymbol{d}_{\mathcal{I}}^T, \boldsymbol{d}_{\mathcal{J}}^T)^T$ and the update $\boldsymbol{u}^k = \boldsymbol{x}^k + \boldsymbol{d}_k$. Then $\boldsymbol{x}^{k+1}$ is obtained from (3.5)–(3.9) in [41].

Now, we can use the ASSN algorithm ([41, p. 372]) to solve (4.5). Since $\boldsymbol{F}$ is monotone and $\boldsymbol{T}$ is 1-averaged, the method is always convergent by Theorem 3.4 and the comment at the end of Section 3.2 in [41]. Moreover, since $\widetilde{\boldsymbol{A}}_1$ is symmetric and positive definite, $\boldsymbol{F}$ is semi-smooth and BD-regular, and the method may be superlinearly convergent ([41, Theorem 3.6]).

Remark 4.1 One main cost of the ASSN algorithm is to solve the linear system in (4.8). Since the coefficient matrix is symmetric and positive definite, we can use the usual conjugate gradient method, which involves the multiplication of $t\,(\widetilde{\boldsymbol{A}}_1)_{\mathcal{I}\mathcal{I}} + \mu_k \boldsymbol{I}$ by a vector. Hence, we do not need to form the matrix $\widetilde{\boldsymbol{A}}_1$ explicitly, and we only need to solve a linear system with the coefficient matrix $\boldsymbol{A}_{22}$.

5 Numerical Experiments

In this section, we report several numerical examples to illustrate the performance of the two methods (3.9) and (3.21). In our numerical simulation, the polygonal meshes are produced by an algorithm discussed in [37] and the codes are written based on the program described in [36]. The meshes with the element numbers $N = 64$ and $N = 256$ for the unit square are displayed in Fig. 2.

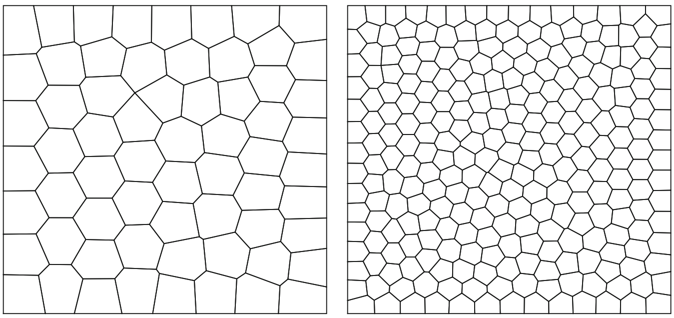

Fig. 2 The left is a polygon mesh for $N = 64$, and the right one corresponds to $N = 256$

5.1 Two numerical Examples on the Simplified Friction Problem

Example 1 The related data for the problem (3.2) is $\Omega = (0,1)\times(0,1)$, $\mu = 1$, $g = 1$, $\Gamma_C = \{1\}\times[0,1]$. We take the exact solution as

$$u(x,y) = (\sin x - x\sin 1)\sin(2\pi y), \quad (x,y)\in\Omega.$$

The corresponding source term is

$$f(x,y) = \left[(2+4\pi^2)\,\sin x - (1+4\pi^2)\,x\,\sin 1\right]\,\sin(2\pi y).$$

It is easy to check that

$$|u-u_h|_{1,\Omega} = \left(\sum_{K\in\mathcal{T}_h}|u-u_h|_{1,K}^2\right)^{1/2} \lesssim \left(\sum_{K\in\mathcal{T}_h}|u-\Pi_1^\nabla u|_{1,K}^2\right)^{1/2} + \left(\sum_{K\in\mathcal{T}_h}|\Pi_1^\nabla u-u_h|_{1,K}^2\right)^{1/2}.$$

Moreover, by the norm equivalence in [19], for all $u\in H^2(\Omega)$, $|\Pi_1^\nabla u-u_h|_{1,K}$ is equivalent to

$$S_K(\Pi_1^\nabla u-u_h) = \left(\sum_{i=1}^{n^K}(\chi_i^K(\Pi_1^\nabla u-u_h))^2\right)^{1/2},$$

where $\{\chi_i^K(\cdot)\}$ denote the function values of a function at the vertices of K, which is a computational quantity with respect to the degrees of freedom of V_h. Hence, we define a quantity to measure the error of u_h by

$$\|u-u_h\|_{1*,\Omega} = \left(\sum_{K\in\mathcal{T}_h}|u-\Pi_1^\nabla u|_{1,K}^2\right)^{1/2} + \left(\sum_{K\in\mathcal{T}_h}S_K(\Pi_1^\nabla u-u_h)^2\right)^{1/2}, \tag{5.1}$$

which is again a computable term with respect to the degrees of freedom of V_h. By the above argument, it is clear that

$$|u-u_h|_{1,\Omega} \lesssim \|u-u_h\|_{1*,\Omega}. \tag{5.2}$$

Moreover, we define the maximum error of u_h by

$$\|u-u_h\|_\infty = \max_i |u(P_i)-u_h(P_i)|,$$

where P_i is the i-th node of the mesh $\mathcal{T}_h$.

We apply the virtual element method (3.9) for discretization, and obtain the discrete solution by solving the related optimization problem using the ASSN algorithm in [41], with the related parameters taken by

$$\tau=\eta_1=1/2,\ \nu=\eta_2=3/4,\ \gamma_1=2,\ \gamma_2=3,\ t=1/\lambda_{\max}(\text{eigs}(\widetilde{\boldsymbol{A}_1})),\ \alpha=1,\ \beta=1/2, \tag{5.3}$$

respectively.

Let

$$Err := \|u - u_h\|_{1*,\Omega}, \quad err := \frac{\|u - u_h\|_{1*,\Omega}}{h}.$$

Recall that h is the mesh size of a mesh $\mathcal{T}_h$. Generally speaking, h is proportional to $N^{-1/2}$, where N is the total number of elements in the mesh. The errors of the numerical solutions for different meshes in the sense of (5.1) and L^∞-norm are given in Tables 1 and 2, respectively.

From these numerical results, we may conclude that this virtual element method performs well and there holds $|u - u_I|_{1,\Omega} \lesssim h$, matching with the theoretical result in Theorem 3.1.

Table 1 The errors for different meshes in the sense of (5.1)

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
Err	0.0508	0.0217	0.0098	0.0047	0.0023
err	0.4064	0.3472	0.3008	0.2990	0.2944

Table 2 L^∞-norm error for different meshes

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\|u - u_h\|_\infty$	2.70e−3	4.22e−4	1.47e−4	4.66e−5	1.20e−5

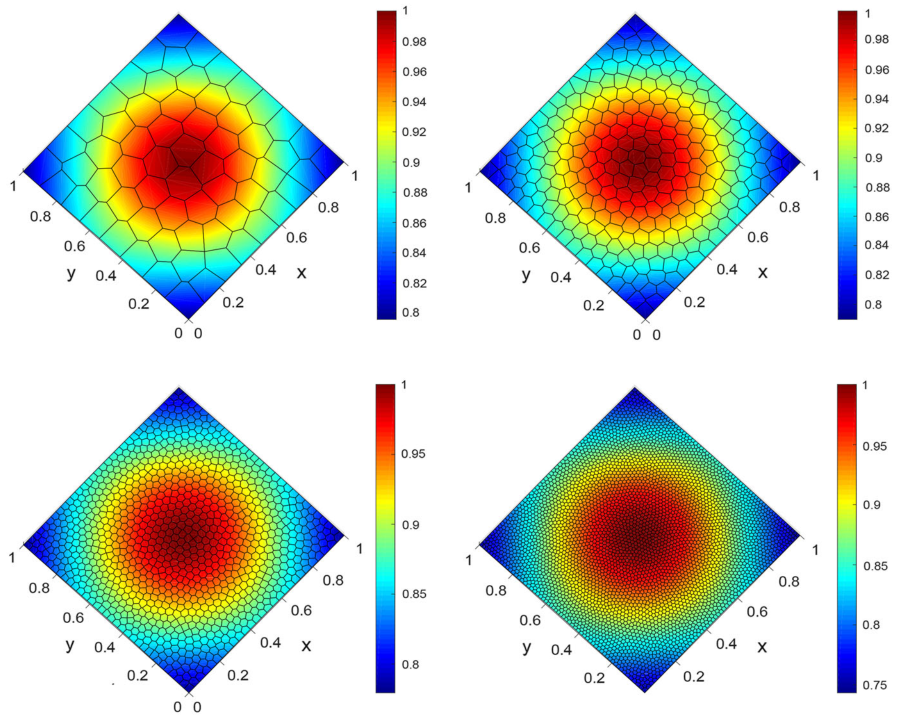

Fig. 3 The numerical solution related to different numbers of elements: $N = 64$ (upper left), $N = 256$ (upper right), $N = 1024$ (bottom left) and $N = 4096$ (bottom right)

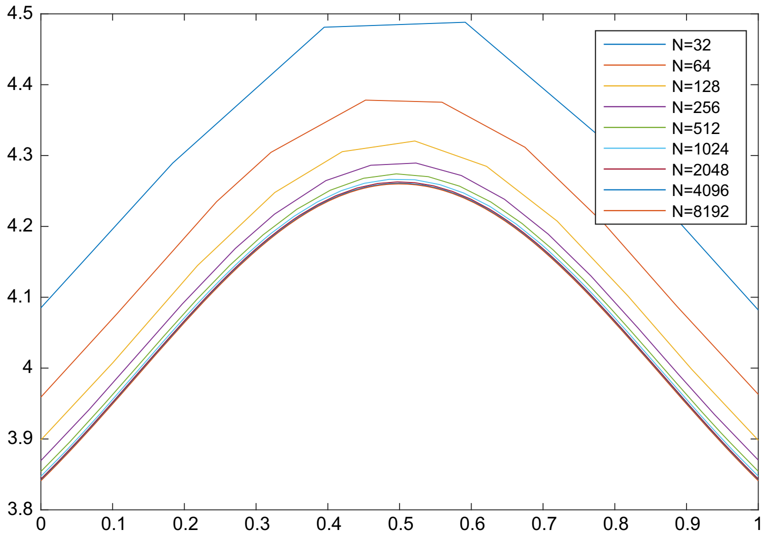

Fig. 4 Numerical solution on $\{1\} \times [0, 1]$ for several meshes

Example 2 In this example, the problem is to find $u \in H^1(\Omega)$ such that

$$\int_\Omega [\nabla u \cdot \nabla(v-u) + u(v-u)]\,dx + g\int_\Gamma |v|\,ds - g\int_\Gamma |u|\,ds \\ \geq \int_\Omega f\,(v-u)\,dx \quad \forall\, v \in H^1(\Omega).$$

Here, $\Omega = (0, 1) \times (0, 1)$, $\Gamma_C = \partial\Omega = \Gamma$, $g = 1$ and $f = -\Delta w + w$ with

$$w(x, y) = \sin(\pi x)\,\sin(\pi y).$$

Similar to the previous example, we solve the discrete problem by using the ASSN algorithm with the parameters given in (5.3).

The numerical solutions corresponding to several meshes with $N = 64$, $N = 256$, $N = 1024$, $N = 4096$ are displayed in Fig. 3, respectively. A convergence trend is evident for the numerical solutions as N increases.

Since we do not have a closed-form solution for this problem, we show in Fig. 4 the numerical solutions on the boundary $\{1\} \times [0, 1]$ for further numerical evidence of convergence.

5.2 An Example of Frictional Contact Problem

Now, we show the performance of the numerical method (3.21) for solving the frictional contact problem (3.12) through an example taken from [13].

Example 3 The domain $\Omega = (0, 4) \times (0, 4)$ is the cross section of a three-dimensional linearly elastic body and the plane stress condition is imposed. The boundary $\partial\Omega$ is decomposed into three parts: $\Gamma_D = \{4\} \times (0, 4)$ where the body is clamped, $\Gamma_C = (0, 4) \times \{0\}$ where frictional contact takes place, and the remaining part $\Gamma_N = (\{0\} \times (0, 4)) \cup ((0, 4) \times \{4\})$ for traction

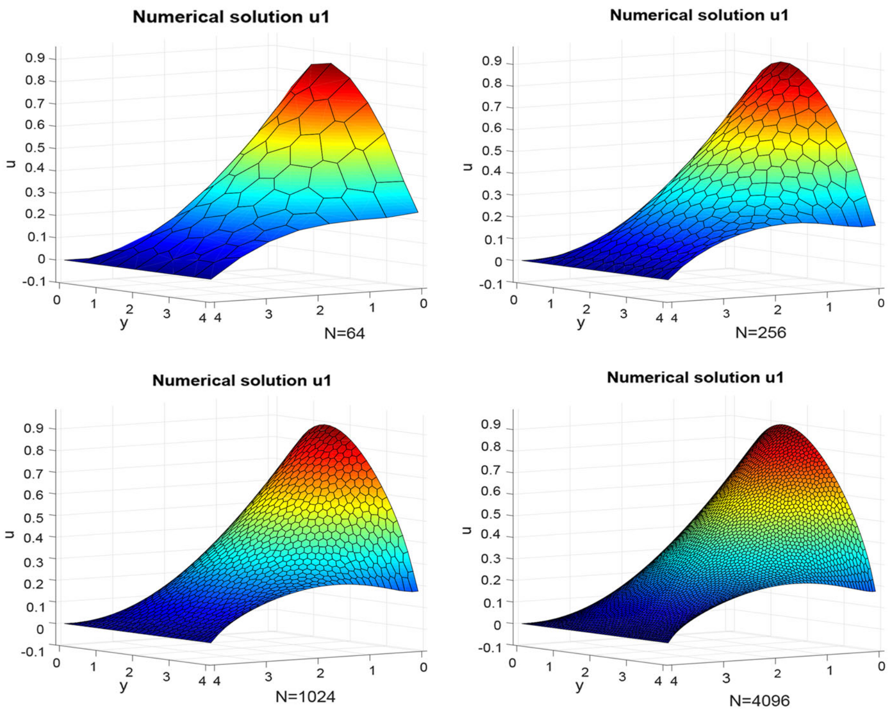

Fig. 5 The numerical solution for several values of N

boundary condition. The elasticity tensor $\mathbb{C}$ is given by

$$(\mathbb{C}\boldsymbol{\varepsilon})_{ij} = \frac{E\nu}{1-\nu^2}(\varepsilon_{11}+\varepsilon_{22})\delta_{ij} + \frac{E}{1+\nu}\varepsilon_{ij}, \quad 1 \le i, j \le 2,$$

where E is the Young modulus, ν is the Poisson ratio of the material and δ_{ij} is the Kronecker delta. We use the following data:

$$E = 2000\,\text{daN/mm}^2, \quad \nu = 0.4, \quad g = 450\ \text{daN/mm}^2,$$

$$\boldsymbol{f}_1 = (0,0)^T\ \text{daN/mm}^2, \quad \boldsymbol{f}_2(x,y) = \begin{cases} (200(5-y), -200)^T\ \text{daN/mm}^2 & \text{on } \{0\}\times(0,4), \\ \mathbf{0} \quad \text{on } (0,4)\times\{4\}. \end{cases}$$

We apply the ASSN algorithm to solve the discrete problem (3.21) with the parameters specified in (5.3). The numerical solution with several values of N is shown in Fig. 5; a convergence trend is evident as N increases.

In Fig. 6, we plot the numerical solution along tangential direction on the boundary $[0,4]\times\{0\}$; a similar convergence trend is clearly observed.

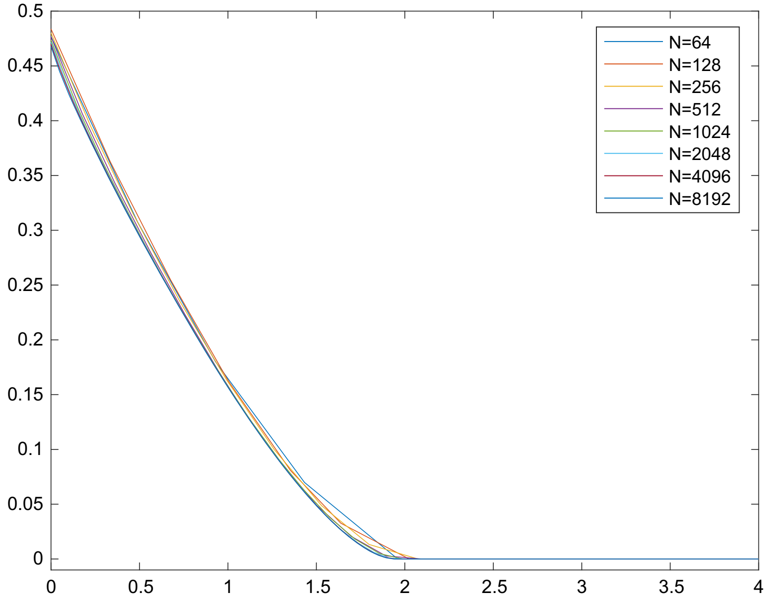

Fig. 6 The numerical solution along tangential direction on $[0, 4] \times \{0\}$ for several values of N

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References

1. Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
2. Alsaedi, A., Brezzi, F., Marini, L., Russo, A.: Equivalent projectors for virtual element methods. Comput. Math. Appl. **66**(3), 376–391 (2013)
3. Antonietti, P.F., Beirão Da Veiga, L., Mora, D., Verani, M.: A stream function formulation of the Stokes problem for the virtual element method. SIAM J. Numer. Anal. **52**, 386–404 (2014)
4. Artioli, E., Beirão Da Veiga, L., Lovadina, C., et al.: Arbitrary order 2D virtual elements for polygonal meshes: part I, elastic problem. Comput. Mech. **60**(3), 355–377 (2017)
5. Artioli, E., Beirão Da Veiga, L., Lovadina, C., Sacco, E.: Arbitrary order 2D virtual elements for polygonal meshes: part II, inelastic problem. Comput. Mech. **60**, 643–657 (2017)
6. Badea, L., Krause, R.: One- and two-level Schwarz methods for variational inequalities of the second kind and their application to frictional contact. Numer. Math. **120**(4), 573–599 (2012)
7. Baiocchi, C., Capelo, A.: Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems. Wiley, Chichester (1984)
8. Beirão Da Veiga, L., Brezzi, F., Cangiani, A., Manzini, G., et al.: Basic principles of virtual element methods. Math. Models Methods Appl. Sci. **23**(1), 1–16 (2013)
9. Beirão Da Veiga, L., Brezzi, F., Marini, L.D.: Virtual elements for linear elasticity problems. SIAM J Numer. Anal. **51**(2), 794–812 (2013)
10. Beirão Da Veiga, L., Brezzi, F., Marini, L.D., Russo, A.: The Hitchhiker's guide to the virtual element method. Math. Models Methods Appl. Sci. **24**(8), 1541–1573 (2014)
11. Beirão Da Veiga, L., Lovadina, C., Russo, A.: Stability analysis for the virtual element method. Math. Models Methods Appl. Sci. **27**(13), 2557–2594 (2017)
12. Beirão Da Veiga, L., Lovadina, C., Vacca, G.: Virtual elements for the Navier–Stokes problem on polygonal meshes. SIAM J. Numer. Anal. **56**, 1210–1242 (2018)
13. Bostan, V., Han, W.: A posteriori error analysis for finite element solutions of a frictional contact problem. Comput. Methods Appl. Mech. Eng. **195**(9–12), 1252–1274 (2006)

14. Brenner, S.C., Guan, Q., Sung, L.: Some estimates for virtual element methods. Comput. Methods Appl. Math. **17**(4), 553–574 (2017)
15. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods, 3rd edn. Springer, New York (2008)
16. Brezzi, F., Hager, W., Raviart, P.A.: Error estimates for the finite element solution of variational inequalities I. Primal Theory. Numer. Math. **28**, 431–443 (1977)
17. Brezzi, F., Hager, W., Raviart, P.A.: Error estimates for the finite element solution of variational inequalities II. Mixed methods. Numer. Math. **31**, 1–16 (1978)
18. Bürg, M., Schröder, A.: A posteriori error control of hp-finite elements for variational inequalities of the first and second kind. Comput. Math. Appl. **70**, 2783–2802 (2015)
19. Chen, L., Huang, J.: Some error analysis on virtual element methods. Calcolo **55**(1), 5 (2018)
20. Chen, P., Huang, J., Zhang, X.: A primal-dual fixed point algorithm for convex separable minimization with applications to image restoration. Inverse Probl. **29**(2), 025011 (33 pp) (2013)
21. Cheng, X., Han, W.: Inexact Uzawa algorithms for variational inequalities of the second kind. Comput. Methods Appl. Mech. Eng. **192**(11–12), 1451–1462 (2003)
22. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
23. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
24. Combettes, P.L., Wajs, V.R.: Signal recovery by proximal forward-backward splitting. Multiscale Model. Simul. **4**(4), 1168–1200 (2005)
25. Duvaut, G., Lions, J.L.: Inequalities in Mechanics and Physics. Springer, Berlin (1976)
26. Falk, R.S.: Error estimates for the approximation of a class of variational inequalities. Math. Comput. **28**(128), 963–971 (1974)
27. Friedman, A.: Variational Principles and Free-Boundary Problems. Wiley, New York (1982)
28. Glowinski, R.: Numerical Methods for Nonlinear Variational Problems. Springer, New York (1984)
29. Glowinski, R., Lions, J.L., Tremolieres, I.: Numerical Analysis of Variational Inequalities. North-Holland, Amsterdam (1981)
30. Han, W., Jensen, S., Reddy, B.D.: Numerical approximations of internal variable problems in plasticity: error analysis and solution algorithm. Numer. Linear Algebra Appl. **4**(3), 191–204 (1997)
31. Han, W., Reddy, B.D.: On the finite element method for mixed variational inequalities arising in elastoplasticity. SIAM J. Numer. Anal. **32**(6), 1778–1807 (1995)
32. Han, W., Sofonea, M.: Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity. American Mathematical Society and International Press, London (2002)
33. Han, W., Wang, L.: Nonconforming finite element analysis for a plate contact problem. SIAM J. Numer. Anal. **40**(5), 1683–1697 (2002)
34. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York (1980)
35. Nečas, J., Hlaváček, I.: Mathematical Theory of Elastic and Elastico-Plastic Bodies: An Introduction. Elsevier, Amsterdam (1981)
36. Sutton, O.J.: The virtual element method in 50 lines of MATLAB. Numer. Algorithms. **75**, 1–19 (2016)
37. Talischi, C., Paulino, G.H., Pereira, A., Menezes, I.F.M.: PolyMesher: a general-purpose mesh generator for polygonal elements written in Matlab. Struct. Multidiscip. Optim. **45**(3), 309–320 (2012)
38. Wang, F., Han, W., Cheng, X.: Discontinuous Galerkin methods for solving elliptic variational inequalities. SIAM J. Numer. Anal. **48**(2), 708–733 (2010)
39. Wang, F., Wei, H.: Virtual element methods for the obstacle problem. IMA J. Numer. Anal. (2018). https://doi.org/10.1093/imanum/dry055
40. Wang, F., Wei, H.: Virtual element method for simplified friction problem. Appl. Math. Lett. **85**, 125–131 (2018)
41. Xiao, X., Li, Y., Wen, Z., Zhang, L.: A regularized semi-smooth Newton method with projection steps for composite convex programs. J. Sci. Comput. **76**, 364–389 (2018)
42. Zhang, T., Tang, L.: Finite volume method for the variational inequalities of first and second kinds. Math. Methods Appl. Sci. **38**(17), 3980–3989 (2016)