



# Unconditional Convergence of a Fast Two-Level Linearized Algorithm for Semilinear Subdiffusion Equations

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## Abstract

A fast two-level linearized scheme with nonuniform time-steps is constructed and analyzed for an initial-boundary-value problem of semilinear subdiffusion equations. The two-level fast L1 formula of the Caputo derivative is derived based on the sum-of-exponentials technique. The resulting fast algorithm is computationally efficient in long-time simulations or small time-steps because it significantly reduces the computational cost  $O(MN^2)$  and storage  $O(MN)$  for the standard L1 formula to  $O(MN \log N)$  and  $O(M \log N)$ , respectively, for  $M$  grid points in space and  $N$  levels in time. The nonuniform time mesh would be graded to handle the typical singularity of the solution near the time  $t = 0$ , and Newton linearization is used to approximate the nonlinearity term. Our analysis relies on three tools: a recently developed discrete fractional Grönwall inequality, a global consistency analysis and a discrete  $H^2$  energy method. A sharp error estimate reflecting the regularity of solution is established without any restriction on the relative diameters of the temporal and spatial mesh sizes. Numerical examples are provided to demonstrate the effectiveness of our approach and the sharpness of error analysis.

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## 1 Introduction

A two-level linearized method is considered to numerically solve the following semilinear subdiffusion equation on a bounded domain

$$\mathcal{D}_t^\alpha u = \Delta u + f(u) \quad \text{for } \mathbf{x} \in \Omega \text{ and } 0 < t \leq T, \quad (1.1a)$$

$$u = u^0(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega \text{ and } t = 0, \quad (1.1b)$$

$$u = 0 \quad \text{for } \mathbf{x} \in \partial\Omega \text{ and } 0 < t \leq T, \quad (1.1c)$$

where  $\partial\Omega$  is the boundary of  $\Omega := (x_l, x_r) \times (y_l, y_r)$ , and the nonlinear function  $f(u)$  is smooth. In (1.1a),  $\mathcal{D}_t^\alpha = {}^C_0\mathcal{D}_t^\alpha$  denotes the Caputo fractional derivative of order  $\alpha$ :

$$(\mathcal{D}_t^\alpha v)(t) := \int_0^t \omega_{1-\alpha}(t-s)v'(s) ds, \quad 0 < \alpha < 1, \quad (1.2)$$

where the weakly singular kernel  $\omega_{1-\alpha}$  is defined by  $\omega_\mu(t) := t^{\mu-1}/\Gamma(\mu)$ . It is easy to verify that  $\omega'_\mu(t) = \omega_{\mu-1}(t)$  and  $\int_0^t \omega_\mu(s) ds = \omega_{\mu+1}(t)$  for  $t > 0$ .

In any numerical methods for solving semilinear fractional diffusion equations (1.1a), a key consideration is the singularity of the solution near the time  $t = 0$ , see [5,9,19,23]. For example, under the assumption that the nonlinear function  $f$  is Lipschitz continuous and the initial data  $u^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ , Jin et al. [5, Theorem 3.1] prove that problem (1.1) has a unique solution  $u$  for which  $u \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega))$ ,  $\mathcal{D}_t^\alpha u \in C([0, T]; L^2(\Omega))$  and  $\partial_t u \in L^2(\Omega)$  with

$$\|\partial_t u(t)\|_{L^2(\Omega)} \leq C_u t^{\alpha-1} \quad \text{for } 0 < t \leq T,$$

where  $C_u > 0$  is a constant independent of  $t$  but may depend on  $T$ . Their analysis of numerical methods for solving (1.1) is applicable to both the L1 scheme and backward Euler convolution quadrature on a uniform time grid of diameter  $\tau$ ; a lagging linearized technique is used to handle the nonlinearity  $f(u)$ , and [5, Theorem 4.5] shows that the discrete solution is  $O(\tau^\alpha)$  convergent in  $L^\infty(L^2)$ .

This work may be considered as a continuation of [14], in which a sharp error estimate for the L1 formula on nonuniform meshes was obtained for linear subdiffusion-reaction equations based on a discrete fractional Grönwall inequality and a global consistency analysis. In this paper, we combine the L1 formula and the sum-of-exponentials (SOEs) technique to develop a one-step fast difference algorithm for the semilinear subdiffusion problem (1.1) by using the Newton's linearization to approximate nonlinear term, and present the corresponding sharp error estimate of the proposed scheme without any restriction on the relative diameters of temporal and spatial mesh sizes.

It is known that the Caputo fractional derivative involves a convolution kernel. The total number of operations required to evaluate the sum of L1 formula is proportional to  $O(N^2)$ , and the active memory to  $O(N)$  with  $N$  representing the total time steps, which is prohibitively expensive for the practically large-scale and long-time simulations. Recently, a simple fast algorithm based on SOEs approximation is proposed to significantly reduce the computational complexity to  $O(N \log N)$  and  $O(\log N)$  when the final time  $T \gg 1$ , see [4,10]. For an evolution equation with memory, a fast summation algorithm was also proposed [17] by interval clustering. Very recently, another fast algorithm for the evaluation of

the fractional derivative has been proposed in [1], where the compression is carried out in the Laplace domain by solving the equivalent ODE with some one-step A-stable scheme. Here the technique of SOEs approximation is used to develop a two-level fast L1 formula by combining a nonuniform mesh suited to the initial singularity with a fast time-stepping algorithm for the historical memory in (1.2). As an interesting property, this scheme computes the current solution by only using the solution at previous time-level, so it would be useful to develop efficient parallel-in-time algorithms, cf. [21], for time-fractional differential equations.

On the other hand, the nonlinearity of the problem also results in the difficulty for the numerical analysis. To establish an error estimate of the two-level linearized scheme at time  $t_n$ , it always requires to prove the boundedness of the numerical solution at the previous time level, that is  $\|u^{n-1}\|_\infty \leq C_u$ . Traditionally it is gotten by using the mathematical induction and some inverse estimate, assuming the underlying scheme is accurate of order  $O(\tau^\beta + h^2)$  with  $\beta$  representing the temporal convergence order,

$$\begin{aligned} \|u^{n-1}\|_\infty &\leq \|U^{n-1}\|_\infty + \|U^{n-1} - u^{n-1}\|_\infty \\ &\leq \|U^{n-1}\|_\infty + h^{-1}\|U^{n-1} - u^{n-1}\| \\ &\leq \|U^{n-1}\|_\infty + C_u h^{-1}(\tau^\beta + h^2). \end{aligned}$$

This leads to a restriction of the time-space grid  $\tau = O(h^{1/\beta})$  in the theoretical analysis even though it is nonphysical and may be unnecessary in numerical simulations. In this paper, we will extend the discrete  $H^2$  energy method developed in [11–13] to prove unconditional convergence of our fully discrete solution without any time-space grid restrictions. The main idea of discrete  $H^2$  energy method is to separately treat the temporal and spatial truncation errors. This method avoids some nonphysical time-space grid restrictions in the error analysis. Another related approach in a finite element setting is discussed in [6–8].

The convergence rate of L1 formula for the Caputo derivative is limited by the smoothness of the solution. We approximate the Caputo fractional derivative (1.2) on a (possibly nonuniform) time mesh  $0 = t_0 < \dots < t_{k-1} < t_k < \dots < t_N = T$ , with the time-step sizes  $\tau_k := t_k - t_{k-1}$  for  $1 \leq k \leq N$ , the maximum time-step  $\tau = \max_{1 \leq k \leq N} \tau_k$  and the step size ratios

$$\rho_k := \tau_k / \tau_{k+1} \quad \text{for } 1 \leq k \leq N - 1.$$

The analysis here is based on the following assumptions on the continuous solution

$$\|u\|_{H^4(\Omega)} \leq C, \quad \|\partial_t u\|_{H^4(\Omega)} \leq C(1 + t^{\sigma-1}) \quad \text{and} \quad p \|\partial_{tt} u\|_{H^2(\Omega)} \leq C(1 + t^{\sigma-2}) \quad (1.3)$$

for  $0 < t \leq T$ , where  $\sigma \in (0, 1) \cup (1, 2)$  is a regularity parameter. To satisfy the regularity conditions in (1.3), appropriate regularity and compatibility assumptions should be imposed on the given data in problem (1.1). Investigating this is beyond the scope of the paper. Noting that, the first two regularity conditions in (1.3) can be relaxed by using the finite elements instead of finite differences [15]. Throughout the paper, any subscripted  $C$ , such as  $C_u, C_\gamma, C_\Omega, C_v, C_0$  and  $C_F$ , denotes a generic positive constant, not necessarily the same at different occurrences, which is always dependent on the given data and the continuous solution  $u$ , but independent of the time-space grid steps.

To resolve the singularity at  $t = 0$ , it is reasonable to use a nonuniform mesh that concentrates grid points near  $t = 0$ , see [2,3,14,16,20]. We make the following assumption on the time mesh:

**AssG** For a parameter  $\gamma \geq 1$ , there are positive constant  $C_\gamma$  and  $\tilde{C}_\gamma$ , independent of  $k$ , such that  $\tau_k \leq C_\gamma \tau \min\{1, t_k^{1-1/\gamma}\}$  for  $1 \leq k \leq N$ , and  $t_k \leq \tilde{C}_\gamma t_{k-1}$  for  $2 \leq k \leq N$ .

The assumption **AssG** implies that  $\tau_1 = O(\tau^\gamma)$ , and allows the time-step size  $\tau_k$  to increase as the time  $t_k$  increases, meanwhile, one may has  $\tau_k = O(\tau)$  as those bounded  $t_k$  away from  $t = 0$ . The parameter  $\gamma$  controls the distribution density of the grid points concentrated near  $t = 0$ : increasing  $\gamma$  will refine the time-step sizes near  $t = 0$  and so move mesh points closer to  $t = 0$ . A simple example of a family of meshes satisfying **AssG** is the graded grid  $t_k = T(k/N)^\gamma$ , see discussions in [2, 14, 16, 18–20]. Although nonuniform time meshes are flexible and reasonably convenient for practical implementations, they also significantly complicate the numerical analysis of schemes, both with respect to stability and consistency. In this paper, our analysis will rely on a generalized fractional Grönwall inequality [15], which would be applicable for any discrete fractional derivatives having the discrete convolution form. As the main result shown in Theorem 4.2, the proposed two-level linearized fast scheme is proved to be unconditionally convergent in the sense that ( $\epsilon$  is the SOE approximation error and  $h$  is the maximum spatial length)

$$\|U^k - u^k\|_\infty \leq \frac{C_u}{\sigma(1-\alpha)} \left( \tau^{\min\{2-\alpha, \gamma\sigma\}} + h^2 + \epsilon \right),$$

where  $C_u$  may depend on  $u$  and  $T$ , but is uniformly bounded with respect to  $\alpha$  and  $\sigma$ .

The paper is organized as follows. Section 2 presents the two-level fast L1 formula and the corresponding linearized fast scheme. The global consistency analysis of fast L1 formula and the Newton's linearization are presented in Sect. 3. A sharp error estimate for the linearized fast scheme is proved in Sect. 4. Two numerical examples in Sect. 5 are given to demonstrate the sharpness of our analysis.

## 2 A Two-Level Fast Method

In space we use a standard finite difference method on a tensor product grid. Let  $M_1$  and  $M_2$  be two positive integers. Set  $h_1 = (x_r - x_l)/M_1$ ,  $h_2 = (y_r - y_l)/M_2$  and the maximum spatial length  $h = \max\{h_1, h_2\}$ . Then the fully discrete spatial grid

$$\bar{\Omega}_h := \{x_h = (x_l + ih_1, y_l + jh_2) \mid 0 \leq i \leq M_1, 0 \leq j \leq M_2\}.$$

Set  $\Omega_h = \bar{\Omega}_h \cap \Omega$  and the boundary  $\partial\Omega_h = \bar{\Omega}_h \cap \partial\Omega$ . Given a grid function  $v = \{v_{ij}\}$ , define

$$\begin{aligned} v_{i-\frac{1}{2},j} &= (v_{i,j} + v_{i-1,j})/2, \quad \delta_x v_{i-\frac{1}{2},j} = (v_{i,j} - v_{i-1,j})/h_1, \\ \delta_x^2 v_{ij} &= (\delta_x v_{i+\frac{1}{2},j} - \delta_x v_{i-\frac{1}{2},j})/h_1. \end{aligned}$$

Also, discrete operators  $v_{i,j-\frac{1}{2}}$ ,  $\delta_y v_{i,j-\frac{1}{2}}$ ,  $\delta_x \delta_y v_{i-\frac{1}{2},j-\frac{1}{2}}$  and  $\delta_y^2 v_{ij}$  can be defined analogously. The second-order approximation of  $\Delta v(x_h)$  for  $x_h \in \Omega_h$  is  $\Delta_h v_h := (\delta_x^2 + \delta_y^2)v_h$ . Let  $\mathcal{V}_h$  be the space of grid functions,

$$\mathcal{V}_h = \{v = (v_h)_{x_h \in \bar{\Omega}_h} \mid v_h = 0 \text{ for } x_h \in \partial\Omega_h\}.$$

For  $v, w \in \mathcal{V}_h$ , define the discrete inner product  $\langle v, w \rangle = h_1 h_2 \sum_{x_h \in \Omega_h} v_h w_h$ , the discrete  $L^2$  norm  $\|v\| := \sqrt{\langle v, v \rangle}$ , the discrete  $H^1$  seminorms

$$\|\delta_x v\| := \sqrt{h_1 h_2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2-1} (\delta_x v_{i-\frac{1}{2},j})^2}, \quad \|\delta_y v\| := \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2} (\delta_y v_{i,j-\frac{1}{2}})^2},$$

$\|\nabla_h v\| = \sqrt{\|\delta_x v\|^2 + \|\delta_y v\|^2}$  and the maximum norm  $\|v\|_\infty = \max_{x_h \in \Omega_h} |v_h|$ . For any  $v \in \mathcal{V}_h$ , by [13, Lemmas 2.1, 2.2 and 2.5] there exists a constant  $C_\Omega > 0$  such that

$$\|v\| \leq C_\Omega \|\nabla_h v\|, \quad \|\nabla_h v\| \leq C_\Omega \|\Delta_h v\|, \quad \|v\|_\infty \leq C_\Omega \|\Delta_h v\|. \tag{2.1}$$

### 2.1 A Fast Variant of the L1 Formula

On our nonuniform mesh, the standard L1 approximation of the Caputo derivative is

$$(D_\tau^\alpha v)^n := \sum_{k=1}^n \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} \omega_{1-\alpha}(t_n - s) \nabla_\tau v^k ds = \sum_{k=1}^n a_{n-k}^{(n)} \nabla_\tau v^k, \tag{2.2}$$

where  $\nabla_\tau v^k := v^k - v^{k-1}$  and the convolution kernel  $a_{n-k}^{(n)}$  is defined by

$$a_{n-k}^{(n)} := \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} \omega_{1-\alpha}(t_n - s) ds = \frac{1}{\tau_k} [\omega_{2-\alpha}(t_n - t_{k-1}) - \omega_{2-\alpha}(t_n - t_k)], \quad 1 \leq k \leq n. \tag{2.3}$$

**Lemma 2.1** *For any fixed integer  $n \geq 2$ , the convolution kernel  $a_{n-k}^{(n)}$  of (2.3) satisfies*

- (i)  $a_{n-k-1}^{(n)} > \omega_{1-\alpha}(t_n - t_k) > a_{n-k}^{(n)}, \quad 1 \leq k \leq n - 1;$
- (ii)  $a_{n-k-1}^{(n)} - a_{n-k}^{(n)} > \frac{1}{2} [\omega_{1-\alpha}(t_n - t_k) - \omega_{1-\alpha}(t_n - t_{k-1})], \quad 1 \leq k \leq n - 1.$

**Proof** The integral mean value theorem yields

$$a_{n-k}^{(n)} = \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} \omega_{1-\alpha}(t_n - s) ds = \omega_{1-\alpha}(t_n - s_k) \quad \text{for some } s_k \in (t_{k-1}, t_k),$$

which implies the result (i) directly since the kernel  $\omega_{1-\alpha}$  is decreasing, also see [14,23]. For any function  $q \in C^2[t_{k-1}, t_k]$ , let  $\Pi_{1,k} q$  be the linear interpolant of  $q(t)$  at  $t_{k-1}$  and  $t_k$ . Let

$$\widetilde{\Pi_{1,k} q} := q - \Pi_{1,k} q$$

be the error in this interpolant. For  $q(s) = \omega_{1-\alpha}(t_n - s)$  one has  $q''(s) = \omega_{-\alpha-1}(t_n - s) > 0$  for  $0 < s < t_n$ , so the Peano representation of the interpolation error [14, Lemma 3.1] shows that

$$\int_{t_{k-1}}^{t_k} (\widetilde{\Pi_{1,k} q})(s) ds < 0.$$

Thus the definition (2.3) of  $a_{n-k}^{(n)}$  yields

$$a_{n-k}^{(n)} - \frac{1}{2} \omega_{1-\alpha}(t_n - t_k) - \frac{1}{2} \omega_{1-\alpha}(t_n - t_{k-1})$$

$$= \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} (\widetilde{\Pi_{1,k}q})(s) \, ds < 0, \quad 1 \leq k \leq n - 1.$$

Subtract this inequality from (i) to obtain (ii) immediately. □

One can see that the direct evaluation of the L1 formula (2.2) is quite inefficient as it requires the information of solutions at all previous time levels while solving problem (1.1). This motivates us to develop a fast L1 formula based on the SOEs approach given in [4, 10, 22]. A basic result of SOE approximation (see [4, Theorem 2.5] or [22, Lemma 2.2]) is as follows:

**Lemma 2.2** *Given  $\alpha \in (0, 1)$ , an absolute tolerance error  $\epsilon \ll 1$ , a cut-off time  $\Delta t > 0$  and a final time  $T$ , there exists a positive integer  $N_q$ , positive quadrature nodes  $\theta^\ell$  and positive weights  $\varpi^\ell$  ( $1 \leq \ell \leq N_q$ ) such that*

$$\left| \omega_{1-\alpha}(t) - \sum_{\ell=1}^{N_q} \varpi^\ell e^{-\theta^\ell t} \right| \leq \epsilon \quad \forall t \in [\Delta t, T],$$

where the number  $N_q$  of quadrature nodes satisfies

$$N_q = O \left( \log \frac{1}{\epsilon} \left( \log \log \frac{1}{\epsilon} + \log \frac{T}{\Delta t} \right) + \log \frac{1}{\Delta t} \left( \log \log \frac{1}{\epsilon} + \log \frac{1}{\Delta t} \right) \right).$$

To design the fast L1 algorithm, we divide the Caputo derivative  $(\mathcal{D}_t^\alpha v)(t_n)$  of (1.2) into a sum of a local part (an integral over  $[t_{n-1}, t_n]$ ) and a history part (an integral over  $[0, t_{n-1}]$ ), and approximate  $v'$  by linear interpolation in the local part (as the same as the standard L1 method) and use the SOE technique of Lemma 2.2 to approximate the kernel  $\omega_{1-\alpha}(t - s)$  in the history part. It arrives at

$$\begin{aligned} (\mathcal{D}_t^\alpha u)(t_n) &\approx \int_{t_{n-1}}^{t_n} \omega_{1-\alpha}(t_n - s) \frac{\nabla_\tau u^n}{\tau_n} \, ds + \int_0^{t_{n-1}} \sum_{\ell=1}^{N_q} \varpi^\ell e^{-\theta^\ell(t_n-s)} u'(s) \, ds \\ &= a_0^{(n)} \nabla_\tau u^n + \sum_{\ell=1}^{N_q} \varpi^\ell e^{-\theta^\ell \tau_n} \mathcal{H}^\ell(t_{n-1}), \quad n \geq 1, \end{aligned}$$

where  $\mathcal{H}^\ell(t_k) := \int_0^{t_k} e^{-\theta^\ell(t_k-s)} u'(s) \, ds$  with  $\mathcal{H}^\ell(t_0) = 0$  for  $1 \leq \ell \leq N_q$ . To compute  $\mathcal{H}^\ell(t_k)$  efficiently we apply linear interpolation in each cell  $[t_{k-1}, t_k]$  to have

$$\mathcal{H}^\ell(t_k) = e^{-\theta^\ell \tau_k} \mathcal{H}^\ell(t_{k-1}) + \int_{t_{k-1}}^{t_k} e^{-\theta^\ell(t_k-s)} u'(s) \, ds \approx e^{-\theta^\ell \tau_k} \mathcal{H}^\ell(t_{k-1}) + b^{(k,\ell)} \nabla_\tau u^k,$$

where the positive coefficient is given by

$$b^{(k,\ell)} := \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} e^{-\theta^\ell(t_k-s)} \, ds, \quad k \geq 1, \quad 1 \leq \ell \leq N_q. \tag{2.4}$$

In summary, we now have the two-level fast L1 formula

$$(\mathcal{D}_f^\alpha u)^n := a_0^{(n)} \nabla_\tau u^n + \sum_{\ell=1}^{N_q} \varpi^\ell e^{-\theta^\ell \tau_n} H^\ell(t_{n-1}), \quad n \geq 1, \tag{2.5a}$$

where  $H^\ell(t_k)$  satisfies  $H^\ell(t_0) = 0$  and the recurrence relationship

$$H^\ell(t_k) = e^{-\theta^\ell \tau_k} H^\ell(t_{k-1}) + b^{(k,\ell)} \nabla_\tau u^k, \quad k \geq 1, \quad 1 \leq \ell \leq N_q. \tag{2.5b}$$

### 2.2 The Two-Level Linearized Scheme

Let  $U_h^n = u(\mathbf{x}_h, t_n)$  for  $\mathbf{x}_h \in \bar{\Omega}_h, 0 \leq n \leq N$ , and  $u_h^n$  be the discrete approximation of  $U_h^n$ . Using the fast L1 formula (2.5) and the Newton linearization, we obtain a linearized scheme for problem (1.1): find  $\{u_h^n\} \in \mathcal{V}_h$  such that

$$(D_f^\alpha u_h)^n = \Delta_h u_h^n + f(u_h^{n-1}) + f'(u_h^{n-1}) \nabla_\tau u_h^n, \quad \mathbf{x}_h \in \Omega_h, \quad 1 \leq n \leq N; \tag{2.6a}$$

$$u_h^0 = u^0(\mathbf{x}_h), \quad \mathbf{x}_h \in \bar{\Omega}_h. \tag{2.6b}$$

The Newton linearization of a general nonlinear function  $f = f(\mathbf{x}, t, u)$  at  $t = t_n$  is taken as the form

$$f(\mathbf{x}_h, t_n, u_h^n) \approx f(\mathbf{x}_h, t_n, u_h^{n-1}) + f'_u(\mathbf{x}_h, t_n, u_h^{n-1}) \nabla_\tau u_h^n.$$

The scheme (2.6) is a two-level procedure for computing  $\{u_h^n\}$ , because (2.6a) can be equivalently reformulated as

$$\left[ a_0^{(n)} - \Delta_h - f'(u_h^{n-1}) \right] \nabla_\tau u_h^n = \Delta_h u_h^{n-1} + f(u_h^{n-1}) - \sum_{\ell=1}^{N_q} \varpi^\ell e^{-\theta^\ell \tau_n} H_h^\ell(t_{n-1}), \tag{2.7}$$

$$H_h^\ell(t_n) = e^{-\theta^\ell \tau_n} H_h^\ell(t_{n-1}) + b^{(n,\ell)} \nabla_\tau u_h^n, \quad 1 \leq \ell \leq N_q. \tag{2.8}$$

Thus, once the values  $\{u_h^{n-1}, H_h^\ell(t_{n-1})\}$  at the previous time-level  $t_{n-1}$  are available, the current solution  $\{u_h^n\}$  can be found by (2.7) with a fast matrix solver and the historic term  $\{H_h^\ell(t_n)\}$  will be updated explicitly by the recurrence formula (2.8).

**Remark 2.3** At each time level the scheme (2.6) requires  $O(MN_q)$  storage and  $O(MN_q)$  operations, where  $M = M_1 M_2$  is the total number of spatial grid points. Given a tolerance error  $\epsilon_0$ , by virtue of Lemma 2.2, the number of quadrature nodes  $N_q = O(\log N)$  if the final time  $T \gg 1$ . Hence our fast method is efficient for long time simulations since it computes the final solution using in total  $O(M \log N)$  storage and  $O(MN \log N)$  operations.

### 2.3 Discrete Fractional Grönwall Inequality

Our analysis relies on a generalized discrete fractional Grönwall inequality developed in [15], which is applicable for any discrete fractional derivative having the discrete convolution form

$$(D_t^\alpha v)^n \approx \sum_{k=1}^n A_{n-k}^{(n)} (v^k - v^{k-1}), \quad 1 \leq n \leq N, \tag{2.9}$$

provided that  $A_{n-k}^{(n)}$  and the time-steps  $\tau_n$  satisfy the following three assumptions:

**Ass1** The discrete kernel is monotone, that is,

$$A_{k-2}^{(n)} \geq A_{k-1}^{(n)} > 0 \quad \text{for } 2 \leq k \leq n \leq N.$$

**Ass2** There is a constant  $\pi_A > 0$  such that

$$A_{n-k}^{(n)} \geq \frac{1}{\pi_A} \int_{t_{k-1}}^{t_k} \frac{\omega_{1-\alpha}(t_n - s)}{\tau_k} ds \quad \text{for } 1 \leq k \leq n \leq N.$$

**Ass3** There is a constant  $\rho > 0$  such that the time-step ratios

$$\rho_k \leq \rho \quad \text{for } 1 \leq k \leq N - 1.$$

The complementary discrete kernel  $P_{n-k}^{(n)}$  was introduced by Liao et al. [14,15]; it satisfies the following identity

$$\sum_{j=k}^n P_{n-j}^{(n)} A_{j-k}^{(j)} \equiv 1 \quad \text{for } 1 \leq k \leq n \leq N. \tag{2.10}$$

Rearranging this identity yields a recursive formula that defines  $P_{n-k}^{(n)}$  :

$$P_0^{(n)} := 1/A_0^{(n)},$$

$$P_{n-j}^{(n)} := 1/A_0^{(j)} \sum_{k=j+1}^n \left( A_{k-j-1}^{(k)} - A_{k-j}^{(k)} \right) P_{n-k}^{(n)}, \quad 1 \leq j \leq n - 1. \tag{2.11}$$

From [15, Lemma 2.2] one can see that  $P_{n-k}^{(n)}$  is well-defined and non-negative if the assumption **Ass1** holds true. Furthermore, if **Ass2** holds true, then

$$\sum_{j=1}^n P_{n-j}^{(n)} \leq \pi_A \omega_{1+\alpha}(t_n) \quad \text{for } 1 \leq n \leq N. \tag{2.12}$$

Recall that the Mittag–Leffler function  $E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+k\alpha)}$ . We state the following (slightly simplified) version of [15, Theorem 3.2]. This result differs substantially from the fractional Grönwall inequality of Jin et al. [5, Theorem 4] since it is valid on very general nonuniform time meshes.

**Theorem 2.4** *Let Ass1–Ass3 hold true. Suppose that the sequences  $(\xi_1^n)_{n=1}^N, (\xi_2^n)_{n=1}^N$  are nonnegative. Assume that  $\lambda_0$  and  $\lambda_1$  are non-negative constants and the maximum step size  $\tau \leq 1/\sqrt[2]{2\pi_A \Gamma(2-\alpha)(\lambda_0 + \lambda_1)}$ . If the nonnegative sequence  $(v^k)_{k=0}^N$  satisfies*

$$\sum_{k=1}^n A_{n-k}^{(n)} \nabla_\tau v^k \leq \lambda_0 v^n + \lambda_1 v^{n-1} + \xi_1^n + \xi_2^n \quad \text{for } 1 \leq n \leq N,$$

then it holds that for  $1 \leq n \leq N$ ,

$$v^n \leq 2E_\alpha(2 \max\{1, \rho\} \pi_A (\lambda_0 + \lambda_1) t_n^\alpha) \left( v^0 + \max_{1 \leq k \leq n} \sum_{j=1}^k P_{k-j}^{(k)} \xi_1^j + \pi_A \omega_{1+\alpha}(t_n) \max_{1 \leq j \leq n} \xi_2^j \right).$$

To facilitate our analysis, we now eliminate the historic term  $H^\ell(t_n)$  from the fast L1 formula (2.5a) for  $(D_f^\alpha u)^n$ . From the recurrence relationship (2.5b), it is easy to see that

$$H^\ell(t_k) = \sum_{j=1}^k e^{-\theta^\ell(t_k-t_j)} b^{(j,\ell)} \nabla_\tau u^j, \quad k \geq 1, \quad 1 \leq \ell \leq N_q.$$

Inserting this in (2.5a) and using the definition (2.4), one obtains the alternative formula

$$(D_f^\alpha u)^n = a_0^{(n)} \nabla_\tau u^n + \sum_{k=1}^{n-1} \frac{\nabla_\tau u^k}{\tau_k} \int_{t_{k-1}}^{t_k} \sum_{\ell=1}^{N_q} \varpi^\ell e^{-\theta^\ell(t_n-s)} ds$$

$$= \sum_{k=1}^n A_{n-k}^{(n)} \nabla_\tau u^k, \quad n \geq 1, \tag{2.13}$$



where the discrete convolution kernel  $A_{n-k}^{(n)}$  is henceforth given as

$$A_0^{(n)} := a_0^{(n)}, \quad A_{n-k}^{(n)} := \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} \sum_{\ell=1}^{N_q} \varpi^\ell e^{-\theta^\ell (t_n-s)} ds, \quad 1 \leq k \leq n-1, \quad n \geq 1. \quad (2.14)$$

The formula (2.13) takes the form of (2.9), and we now verify that our  $A_{n-k}^{(n)}$  defined by (2.14) satisfy **Ass1** and **Ass2**, allowing us to apply Theorem 2.4 and establish the convergence of our computed solution. Part (I) of the next lemma ensures that **Ass1** is valid, while part (II) implies that **Ass2** holds true with  $\pi_A = \frac{3}{2}$ .

**Lemma 2.5** *If the tolerance error  $\epsilon$  of SOE satisfies  $\epsilon \leq \min \left\{ \frac{1}{3} \omega_{1-\alpha}(T), \alpha \omega_{2-\alpha}(T) \right\}$ , then the discrete convolutional kernel  $A_{n-k}^{(n)}$  of (2.14) satisfies*

- (I)  $A_{k-1}^{(n)} > A_k^{(n)} > 0, \quad 1 \leq k \leq n-1;$
- (II)  $A_0^{(n)} = a_0^{(n)}$  and  $A_{n-k}^{(n)} \geq \frac{2}{3} a_{n-k}^{(n)}, \quad 1 \leq k \leq n-1.$

**Proof** The definition (2.3) and Lemma 2.1 (i) yield

$$a_0^{(n)} - a_1^{(n)} > a_0^{(n)} - \omega_{1-\alpha}(\tau_n) = \frac{\alpha}{\tau_n} \omega_{2-\alpha}(\tau_n) \geq \alpha \omega_{2-\alpha}(T) \geq \epsilon.$$

The definition (2.14) and Lemma 2.2 imply that  $A_0^{(n)} = a_0^{(n)} > a_1^{(n)} + \epsilon > A_1^{(n)}$ . Lemma 2.2 also shows that  $\theta^\ell, \varpi^\ell > 0$  for  $\ell = 1, \dots, N_q$ ; the mean-value theorem now yields property (I). By Lemma 2.1 (i) and our hypothesis on  $\epsilon$  we have

$$\epsilon \leq \frac{1}{3} \omega_{-\alpha}(t_n) < \frac{1}{3} a_{n-1}^{(n)} \leq \frac{1}{3} a_{n-k}^{(n)}, \quad 1 \leq k \leq n-1.$$

Hence Lemma 2.2 gives

$$A_{n-k}^{(n)} \geq a_{n-k}^{(n)} - \epsilon \geq \frac{2}{3} a_{n-k}^{(n)} \quad \text{for } 1 \leq k \leq n-1.$$

The proof is complete. □

### 3 Global Consistency Error Analysis

We now proceed with the consistency error analysis of our fast linearized method, and begin with the consistency error of the standard L1 formula  $(D_\tau^\alpha u)^n$  of (2.2).

**Lemma 3.1** *For  $v \in C^2(0, T]$  with  $\int_0^T t |v''(t)| ds < \infty$ , one has*

$$\left| (D_t^\alpha v)(t_n) - (D_\tau^\alpha v)^n \right| \leq a_0^{(n)} G^n + \sum_{k=1}^{n-1} (a_{n-k-1}^{(n)} - a_{n-k}^{(n)}) G^k, \quad n \geq 1,$$

where the L1 kernel  $a_{n-k}^{(n)}$  is defined by (2.3) and  $G^k := 2 \int_{t_{k-1}}^{t_k} (t - t_{k-1}) |v''(t)| dt$ .

**Proof** From Taylor’s formula with integral remainder, the truncation error of the standard L1 formula at time  $t = t_n$  is (see [14, Lemma 3.3])

$$(D_t^\alpha v)(t_n) - (D_\tau^\alpha v)^n = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \omega_{1-\alpha}(t_n - s) \left( v'(s) - \nabla_\tau v^k / \tau_k \right) ds$$

$$= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} v''(t) (\widetilde{\Pi_{1,k}Q})(t) dt, \quad n \geq 1, \tag{3.1}$$

where  $Q(t) = \omega_{2-\alpha}(t_n - t)$  and we use the notation of the proof of Lemma 2.1. By the error formula for linear interpolation [14, Lemma 3.1], we have

$$(\widetilde{\Pi_{1,k}Q})(t) = \int_{t_{k-1}}^{t_k} \chi_k(t, y) Q''(y) dy, \quad t_{k-1} < t < t_k, \quad 1 \leq k \leq n,$$

where the Peano kernel  $\chi_k(t, y) = \max\{t - y, 0\} - \frac{t-t_{k-1}}{\tau_k}(t_k - y)$  satisfies

$$-\frac{t-t_{k-1}}{\tau_k}(t_k - t) \leq \chi_k(t, y) < 0 \quad \text{for any } t, y \in (t_{k-1}, t_k).$$

Observing that for each fixed  $n \geq 1$  the function  $Q$  is decreasing and  $Q''(t) = \omega_{-\alpha}(t_n - t) < 0$ , we arrive at the interpolation error  $(\widetilde{\Pi_{1,k}Q})(t) \geq 0$  for  $1 \leq k \leq n$ , with

$$\begin{aligned} (\widetilde{\Pi_{1,n}Q})(t) &\leq Q(t_{n-1}) - (\Pi_{1,n}Q)(t) = (t - t_{n-1})a_0^{(n)}, \\ (\widetilde{\Pi_{1,k}Q})(t) &\leq (t_{k-1} - t) \int_{t_{k-1}}^{t_k} Q''(t) dt \leq (t - t_{k-1})[\omega_{1-\alpha}(t_n - t_k) - \omega_{1-\alpha}(t_n - t_{k-1})] \\ &\leq 2(t - t_{k-1})(a_{n-k-1}^{(n)} - a_{n-k}^{(n)}), \quad t \in (t_{k-1}, t_k), \quad 1 \leq k \leq n - 1, \end{aligned}$$

where Lemma 2.1 (ii) is used in the last inequality. Thus, (3.1) yields

$$\begin{aligned} |(\mathcal{D}_t^\alpha v)(t_n) - (D_\tau^\alpha v)^n| &\leq \int_{t_{n-1}}^{t_n} |v''(t)| (\widetilde{\Pi_{1,n}Q})(t) dt + \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} |v''(t)| (\widetilde{\Pi_{1,k}Q})(t) dt \\ &\leq a_0^{(n)} \int_{t_{n-1}}^{t_n} (t - t_{n-1}) |v''(t)| dt + 2 \sum_{k=1}^{n-1} (a_{n-k-1}^{(n)} - a_{n-k}^{(n)}) \int_{t_{k-1}}^{t_k} (t - t_{k-1}) |v''(t)| dt, \end{aligned}$$

and the desired result follows from the definition of  $G^k$ . □

**Remark 3.2** Compared with the previous estimate in [14, Lemma 3.3], Lemma 3.1 removes the restriction of time-step ratios  $\rho_k \leq 1$ , which is an undesirable restriction on the mesh for problems that allow the rapid growth of the solution at the time far away from  $t = 0$ .

We now focus on the fast L1 method by taking the initial singularity into account. Here and hereafter, we denote  $\hat{T} = \max\{1, T\}$  and  $\hat{t}_n = \max\{1, t_n\}$  for  $1 \leq n \leq N$ . Next lemma presents the estimate of the global consistency error  $\sum_{j=1}^n P_{n-j}^{(n)} |\Upsilon^j|$  accumulating from  $t = t_1$  to  $t = t_n$  with the discrete convolution kernel  $P_{n-j}^{(n)}$ .

**Lemma 3.3** Assume that  $v \in C^2((0, T])$  and that there exists a constant  $C_v > 0$  such that

$$|v'(t)| \leq C_v(1 + t^{\sigma-1}), \quad |v''(t)| \leq C_v(1 + t^{\sigma-2}), \quad 0 < t \leq T, \tag{3.2}$$

where  $\sigma \in (0, 1) \cup (1, 2)$  is a parameter. Let

$$\Upsilon^j := (\mathcal{D}_t^\alpha v)(t_j) - (D_j^\alpha v)^j$$

denote the local consistency error of the fast L1 formula (2.13). Assume that the SOE tolerance error satisfies  $\epsilon \leq \frac{1}{3} \min\{\omega_{1-\alpha}(T), 3\alpha \omega_{2-\alpha}(T)\}$ . Then the global consistency error can be bounded by

$$\sum_{j=1}^n P_{n-j}^{(n)} |\Upsilon^j| \leq C_v \left( \frac{\tau_1^\sigma}{\sigma} + \frac{1}{1-\alpha} \max_{2 \leq k \leq n} (t_k - t_1)^\alpha t_{k-1}^{\sigma-2} \tau_k^{2-\alpha} + \frac{\epsilon}{\sigma} t_n^\alpha \hat{t}_{n-1}^2 \right) \tag{3.3}$$

for  $1 \leq n \leq N$ . Moreover, if the mesh satisfies **AssG**, then

$$\sum_{j=1}^n P_{n-j}^{(n)} |\Upsilon^j| \leq \frac{C_v}{\sigma(1-\alpha)} \tau^{\min\{2-\alpha, \gamma\sigma\}} + \frac{\epsilon}{\sigma} C_v t_n^\alpha \hat{t}_{n-1}^2, \quad 1 \leq n \leq N.$$

**Proof** The main difference between the fast L1 formula (2.13) and the standard L1 formula (2.2) is that the convolution kernel is approximated by SOEs with an absolute tolerance error  $\epsilon$ . Thus, comparing the standard L1 formula (2.2) with the corresponding fast L1 formula (2.13), by Lemma 2.2 and the regularity assumption (3.2) one has

$$\begin{aligned} |(D_f^\alpha v)^j - (D_\tau^\alpha v)^j| &\leq \sum_{k=1}^{j-1} \frac{|\nabla_\tau v^k|}{\tau_k} \int_{t_{k-1}}^{t_k} \left| \sum_{\ell=1}^{N_q} \varpi^\ell e^{-\theta^\ell(t_j-s)} - \omega_{1-\alpha}(t_j-s) \right| ds, \\ &\leq \epsilon \sum_{k=1}^{j-1} \int_{t_{k-1}}^{t_k} |v'(s)| ds \leq C_v(t_{j-1} + t_{j-1}^\sigma/\sigma)\epsilon \leq \frac{C_v}{\sigma} \hat{t}_{j-1}^2 \epsilon, \quad j \geq 1. \end{aligned}$$

Lemma 2.2 implies that  $|A_{n-k}^{(n)} - a_{n-k}^{(n)}| \leq \epsilon$  for  $1 \leq k \leq n - 1$ . Recalling that  $A_0^{(n)} = a_0^{(n)}$ , one has

$$a_{j-k-1}^{(j)} - a_{j-k}^{(j)} \leq A_{j-k-1}^{(j)} - A_{j-k}^{(j)} + 2\epsilon, \quad 1 \leq k \leq j - 1.$$

Then Lemma 3.1 and the regularity assumption (3.2) lead to

$$\begin{aligned} |(D_t^\alpha v)(t_j) - (D_\tau^\alpha v)^j| &\leq A_0^{(j)} G^j + \sum_{k=1}^{j-1} (A_{j-k-1}^{(j)} - A_{j-k}^{(j)}) G^k + 2\epsilon \sum_{k=1}^{j-1} G^k \\ &\leq A_0^{(j)} G^j + \sum_{k=1}^{j-1} (A_{j-k-1}^{(j)} - A_{j-k}^{(j)}) G^k + 4\epsilon \sum_{k=1}^{j-1} \int_{t_{k-1}}^{t_k} t |v''(t)| dt \\ &\leq A_0^{(j)} G^j + \sum_{k=1}^{j-1} (A_{j-k-1}^{(j)} - A_{j-k}^{(j)}) G^k + \frac{C_v}{\sigma} \hat{t}_{j-1}^2 \epsilon, \quad j \geq 1. \end{aligned}$$

Now a triangle inequality gives

$$|\Upsilon^j| \leq A_0^{(j)} G^j + \sum_{k=1}^{j-1} (A_{j-k-1}^{(j)} - A_{j-k}^{(j)}) G^k + \frac{C_v}{\sigma} \hat{t}_{j-1}^2 \epsilon, \quad j \geq 1. \tag{3.4}$$

Multiplying the above inequality (3.4) by  $P_{n-j}^{(n)}$  and summing the index  $j$  from 1 to  $n$ , one can exchange the order of summation and apply the definition (2.11) of  $P_{n-j}^{(n)}$  to obtain

$$\begin{aligned} \sum_{j=1}^n P_{n-j}^{(n)} |\Upsilon^j| &\leq \sum_{j=1}^n P_{n-j}^{(n)} A_0^{(j)} G^j + \sum_{j=2}^n P_{n-j}^{(n)} \sum_{k=1}^{j-1} (A_{j-k-1}^{(j)} - A_{j-k}^{(j)}) G^k \\ &\quad + C_v \frac{\epsilon}{\sigma} \sum_{j=2}^n P_{n-j}^{(n)} \hat{t}_{j-1}^2 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=1}^n G^j P_{n-j}^{(n)} A_0^{(j)} + \sum_{k=1}^{n-1} G^k \sum_{j=k+1}^n P_{n-j}^{(n)} (A_{j-k-1}^{(j)} - A_{j-k}^{(j)}) \\
 &\quad + C_v \hat{t}_{n-1}^2 \frac{\epsilon}{\sigma} \sum_{j=2}^n P_{n-j}^{(n)} \\
 &\leq \sum_{k=1}^n P_{n-k}^{(n)} A_0^{(k)} G^k + \sum_{k=1}^{n-1} P_{n-k}^{(n)} A_0^{(k)} G^k + \frac{C_v}{\sigma} t_n^\alpha \hat{t}_{n-1}^2 \epsilon, \tag{3.5}
 \end{aligned}$$

where the property (2.12) with  $\pi_A = 3/2$  is used in the last inequality. If the SOE approximation error  $\epsilon \leq \frac{1}{3} \min\{\omega_{1-\alpha}(T), 3\alpha \omega_{2-\alpha}(T)\}$ , Lemma 2.5 (II) and Lemma 2.1 (i) imply that

$$A_0^{(k)} = a_0^{(k)} = \omega_{2-\alpha}(\tau_k)/\tau_k, \quad A_{k-2}^{(k)} \geq \frac{2}{3} a_{k-2}^{(k)} \geq \frac{2}{3} \omega_{1-\alpha}(t_k - t_1),$$

and then

$$A_0^{(k)}/A_{k-2}^{(k)} \leq \frac{3}{2(1-\alpha)} (t_k - t_1)^\alpha \tau_k^{-\alpha}, \quad 2 \leq k \leq n \leq N.$$

Furthermore, the identical property (2.10) for the complementary kernel  $P_{n-j}^{(n)}$  gives

$$P_{n-1}^{(n)} A_0^{(1)} \leq 1 \quad \text{and} \quad \sum_{k=2}^{n-1} P_{n-k}^{(n)} A_{k-2}^{(k)} \leq \sum_{k=2}^n P_{n-k}^{(n)} A_{k-2}^{(k)} = 1.$$

The regularity assumption (3.2) gives

$$G^1 \leq C_v \tau_1^\sigma / \sigma \quad \text{and} \quad G^k \leq C_v t_{k-1}^{\sigma-2} \tau_k^2 \quad \text{for } 2 \leq k \leq n.$$

Thus it follows from (3.5) that

$$\begin{aligned}
 \sum_{j=1}^n P_{n-j}^{(n)} |\Upsilon^j| &\leq 2G^1 + 2 \sum_{k=2}^n P_{n-k}^{(n)} A_0^{(k)} G^k + \frac{C_v}{\sigma} t_n^\alpha \hat{t}_{n-1}^2 \epsilon \\
 &\leq C_v \frac{\tau_1^\sigma}{\sigma} + \frac{C_v}{1-\alpha} \sum_{k=2}^n P_{n-k}^{(n)} A_{k-2}^{(k)} (t_k - t_1)^\alpha t_{k-1}^{\sigma-2} \tau_k^{2-\alpha} + \frac{C_v}{\sigma} t_n^\alpha \hat{t}_{n-1}^2 \epsilon \\
 &\leq C_v \left( \frac{\tau_1^\sigma}{\sigma} + \frac{1}{1-\alpha} \max_{2 \leq k \leq n} (t_k - t_1)^\alpha t_{k-1}^{\sigma-2} \tau_k^{2-\alpha} + \frac{1}{\sigma} t_n^\alpha \hat{t}_{n-1}^2 \epsilon \right), \quad 1 \leq n \leq N.
 \end{aligned}$$

The claimed estimate (3.3) is verified. In particular, if AssG holds, one has

$$\begin{aligned}
 t_k^\alpha t_{k-1}^{\sigma-2} \tau_k^{2-\alpha} &\leq C_\gamma t_k^{\sigma-2+\alpha} \tau_k^{2-\alpha-\beta} \tau^\beta \min\{1, t_k^{\beta-\beta/\gamma}\} \\
 &\leq C_\gamma t_k^{\sigma-\beta/\gamma} (\tau_k/t_k)^{2-\alpha-\beta} \tau^\beta \leq C_\gamma t_k^{\max\{0, \sigma-(2-\alpha)\gamma\}} \tau^\beta, \quad 2 \leq k \leq N,
 \end{aligned}$$

where  $\beta = \min\{2 - \alpha, \gamma\sigma\}$ . The final estimate follows since  $\tau_1^\sigma \leq C_\gamma \tau^\gamma \sigma \leq C_\gamma \tau^\beta$ . □

Next lemma describes the global consistency error of Newton’s linearized approach, which is smaller than that generated by the above L1 approximation. In addition, there is no error in the linearized approximation if  $f = f(u)$  is a linear function.

**Lemma 3.4** *Assume that  $v \in C([0, T]) \cap C^2((0, T])$  satisfies the regularity condition (3.2), and the nonlinear function  $f = f(u) \in C^2(\mathbb{R})$ . Denote  $v^n = v(t_n)$  and the local truncation error*

$$\mathcal{R}_f^n = f(v^n) - f(v^{n-1}) - f'(v^{n-1}) \nabla_\tau v^n$$

such that the global consistency error

$$\sum_{j=1}^n P_{n-j}^{(n)} |\mathcal{R}_f^j| \leq C_v \tau_1^\alpha (\tau_1^2 + \tau_1^{2\sigma} / \sigma^2) + C_v t_n^\alpha \max_{2 \leq j \leq n} (\tau_j^2 + t_{j-1}^{2\sigma-2} \tau_j^2), \quad 1 \leq n \leq N.$$

Moreover, if the assumption **AssG** holds, one has

$$\sum_{j=1}^n P_{n-j}^{(n)} |\mathcal{R}_f^j| \leq C_v \tau^{\min\{2, 2\gamma\sigma\}} \max\{1, \tau^{\gamma\alpha} / \sigma^2\}, \quad 1 \leq n \leq N.$$

**Proof** Applying the formula of Taylor expansion with integral remainder, one has

$$\mathcal{R}_f^j = (\nabla_\tau v^j)^2 \int_0^1 f''(v^{j-1} + s \nabla_\tau v^j) (1 - s) ds, \quad j \geq 1.$$

Under the regularity conditions, one has

$$|\mathcal{R}_f^1| \leq C_v \left( \int_{t_0}^{t_1} |v'(t)| dt \right)^2 \leq C_v (\tau_1^2 + \tau_1^{2\sigma} / \sigma^2),$$

and

$$|\mathcal{R}_f^j| \leq C_v \left( \int_{t_{j-1}}^{t_j} |v'(t)| dt \right)^2 \leq C_v (\tau_j^2 + t_{j-1}^{2\sigma-2} \tau_j^2), \quad 2 \leq j \leq N.$$

Note that, Lemma 2.5 (II) and the definition (2.3) give  $A_0^{(k)} = a_0^{(k)} = \omega_{2-\alpha}(\tau_k) / \tau_k$ , so the identical property (2.10) shows

$$P_{n-1}^{(n)} \leq 1/A_0^{(1)} \leq \Gamma(2 - \alpha) \tau_1^\alpha.$$

Moreover, the bounded estimate (2.12) with  $\pi_A = \frac{3}{2}$  gives

$$\sum_{j=2}^n P_{n-j}^{(n)} \leq \frac{3}{2} \omega_{1+\alpha}(t_n).$$

Thus, it follows that

$$\begin{aligned} \sum_{j=1}^n P_{n-j}^{(n)} |\mathcal{R}_f^j| &\leq P_{n-1}^{(n)} |\mathcal{R}_f^1| + \sum_{j=2}^n P_{n-j}^{(n)} |\mathcal{R}_f^j| \leq C_v \tau_1^\alpha |\mathcal{R}_f^1| + C_v t_n^\alpha \max_{2 \leq j \leq n} |\mathcal{R}_f^j| \\ &\leq C_v \tau_1^\alpha (\tau_1^2 + \tau_1^{2\sigma} / \sigma^2) + C_v t_n^\alpha \max_{2 \leq j \leq n} (\tau_j^2 + t_{j-1}^{2\sigma-2} \tau_j^2), \quad 1 \leq n \leq N. \end{aligned}$$

If **AssG** holds, one has

$$\tau_j^2 \leq C_\gamma \tau^2 \min\{1, t_j^{2-2/\gamma}\} \leq C_\gamma \tau^\beta \min\{1, t_j^{2-2/\gamma}\},$$

and

$$\begin{aligned} t_{j-1}^{2\sigma-2} \tau_j^2 &\leq C_\gamma t_j^{2\sigma-2} \tau_j^{2-\beta} \tau^\beta \min\{1, t_j^{\beta-\beta/\gamma}\} \\ &\leq C_\gamma t_j^{2\sigma-\min\{2, 2\gamma\sigma\}/\gamma} (\tau_k/t_k)^{2-\beta} \tau^\beta \leq C_\gamma t_k^{\max\{0, 2\sigma-2/\gamma\}} \tau^\beta, \quad 2 \leq j \leq N, \end{aligned}$$

where  $\beta = \min\{2, 2\gamma\sigma\}$ . The second estimate follows since  $\tau_1^{2\sigma} \leq C_\gamma \tau^{2\gamma\sigma} \leq C_\gamma \tau^\beta$ .  $\square$

### 4 Unconditional Convergence

Assume that the time mesh fulfills **Ass3** and **AssG** in the error analysis. We here extend the discrete  $H^2$  energy method in [11–13] to prove the unconditional convergence of discrete solutions to the two-level linearized scheme (2.6). In this section,  $K_0, \tau^*, \tau_0^*, \tau_1^*, h_0, \epsilon_0$  and any numeric subscripted  $c$ , such as  $c_0, c_1, c_2$  and so on, are fixed values, which are always dependent on the given data and the continuous solution, but independent of the time-space grid steps and the inductive index  $k$  in the mathematical induction as well. To make our ideas more clearly, four steps are listed to obtain unconditional error estimate as follows.

#### 4.1 STEP 1: Construction of Coupled Discrete System

We introduce a function  $w := \mathcal{D}_t^\alpha u - f(u)$  with the initial-boundary values  $w(\mathbf{x}, 0) := \Delta u^0(\mathbf{x})$  for  $\mathbf{x} \in \Omega$  and  $w(\mathbf{x}, t) := -f(0)$  for  $\mathbf{x} \in \partial\Omega$ . The problem (1.1a) can be formulated into

$$\begin{aligned} w &= \mathcal{D}_t^\alpha u - f(u), \quad \mathbf{x} \in \bar{\Omega}, \quad 0 < t \leq T; \\ w &= \Delta u, \quad \mathbf{x} \in \Omega, \quad 0 \leq t \leq T. \end{aligned}$$

Let  $w_h^n$  be the numerical approximation of function  $W_h^n = w(\mathbf{x}_h, t_n)$  for  $\mathbf{x}_h \in \bar{\Omega}_h$ . As done in subsection 2.2, one has an auxiliary discrete system: to seek  $\{u_h^n, w_h^n\}$  such that

$$w_h^n = (D_f^\alpha u_h)^n - f(u_h^{n-1}) - f'(u_h^{n-1}) \nabla_\tau u_h^n, \quad \mathbf{x}_h \in \bar{\Omega}_h, \quad 1 \leq n \leq N; \tag{4.1}$$

$$w_h^n = \Delta_h u_h^n, \quad \mathbf{x}_h \in \Omega_h, \quad 0 \leq n \leq N; \tag{4.2}$$

$$u_h^0 = u^0(\mathbf{x}_h), \quad \mathbf{x}_h \in \bar{\Omega}_h; \quad u_h^n = 0, \quad \mathbf{x}_h \in \partial\Omega_h, \quad 1 \leq n \leq N. \tag{4.3}$$

Obviously, by eliminating the auxiliary function  $w_h^n$  in above discrete system, one directly arrives at the computational scheme (2.6). Alternately, the solution properties of two-level linearized method (2.6) can be studied via the auxiliary discrete system (4.1)–(4.3).

#### 4.2 STEP 2: Reduction of Coupled Error System

Let  $\tilde{u}_h^n = U_h^n - u_h^n, \tilde{w}_h^n = W_h^n - w_h^n$  be the solution errors for  $\mathbf{x}_h \in \bar{\Omega}_h$ . The solution errors satisfy the governing equations as

$$\tilde{w}_h^n = (D_f^\alpha \tilde{u}_h)^n - \mathcal{N}_h^n + \xi_h^n, \quad \mathbf{x}_h \in \bar{\Omega}_h, \quad 1 \leq n \leq N; \tag{4.4}$$

$$\tilde{w}_h^n = \Delta_h \tilde{u}_h^n + \eta_h^n, \quad \mathbf{x}_h \in \Omega_h, \quad 0 \leq n \leq N; \tag{4.5}$$

$$\tilde{u}_h^0 = 0, \quad \mathbf{x}_h \in \bar{\Omega}_h; \quad \tilde{u}_h^n = 0, \quad \mathbf{x}_h \in \partial\Omega_h, \quad 1 \leq n \leq N, \tag{4.6}$$

where  $\xi_h^n$  and  $\eta_h^n$  denote temporal and spatial truncation errors, respectively, and

$$\begin{aligned} \mathcal{N}_h^n &:= f'(u_h^{n-1}) \nabla_\tau \tilde{u}_h^n + f(U_h^{n-1}) - f(u_h^{n-1}) + \left( f'(U_h^{n-1}) - f'(u_h^{n-1}) \right) \nabla_\tau U_h^n \\ &= f'(u_h^{n-1}) \nabla_\tau \tilde{u}_h^n + \tilde{u}_h^{n-1} \int_0^1 f'(sU_h^{n-1} + (1-s)u_h^{n-1}) ds \\ &\quad + \tilde{u}_h^{n-1} \nabla_\tau U_h^n \int_0^1 f''(sU_h^{n-1} + (1-s)u_h^{n-1}) ds. \end{aligned} \tag{4.7}$$

Acting the difference operators  $\Delta_h$  and  $D_f^\alpha$  on the Eqs. (4.4)–(4.5), respectively, gives

$$\begin{aligned} \Delta_h \tilde{w}_h^n &= (D_f^\alpha \Delta_h \tilde{u}_h)^n - \Delta_h \mathcal{N}_h^n + \Delta_h \xi_h^n, \quad \mathbf{x}_h \in \Omega_h, \quad 1 \leq n \leq N; \\ (D_f^\alpha \tilde{w}_h)^n &= (D_f^\alpha \Delta_h \tilde{u}_h)^n + (D_f^\alpha \eta_h)^n, \quad \mathbf{x}_h \in \Omega_h, \quad 1 \leq n \leq N. \end{aligned}$$

By eliminating the term  $(D_f^\alpha \Delta_h \tilde{u}_h)^n$  in the above two equations, one gets

$$(D_f^\alpha \tilde{w}_h)^n = \Delta_h \tilde{w}_h^n + \Delta_h \mathcal{N}_h^n + (D_f^\alpha \eta_h)^n - \Delta_h \xi_h^n \quad \mathbf{x}_h \in \Omega_h, \quad 1 \leq n \leq N; \tag{4.8}$$

$$\tilde{w}_h^0 = \eta_h^0, \quad \mathbf{x}_h \in \bar{\Omega}_h; \quad \tilde{w}_h^n = 0, \quad \mathbf{x}_h \in \partial\Omega_h, \quad 1 \leq n \leq N; \tag{4.9}$$

where the initial and boundary conditions are derived from the error system (4.4)–(4.6).

### 4.3 STEP 3: Continuous Analysis of Truncation Error

According to the first regularity condition in (1.3), one has

$$\|\eta^n\| \leq c_1 h^2, \quad 0 \leq n \leq N. \tag{4.10}$$

Since the spatial error  $\eta_h^n$  is defined uniformly at the time  $t = t_n$  [there is no temporal error in the Eq. (4.2)], we can define a continuous function  $\eta_h(t)$  for  $\mathbf{x}_h = (x_i, y_j) \in \Omega_h$ ,

$$\begin{aligned} \eta_h(t) &= \frac{h_1^2}{6} \int_0^1 [\partial_x^{(4)} u(x_i - sh_1, y_j, t) + \partial_x^{(4)} u(x_i + sh_1, y_j, t)](1-s)^3 ds \\ &\quad + \frac{h_2^2}{6} \int_0^1 [\partial_y^{(4)} u(x_i, y_j - sh_2, t) + \partial_y^{(4)} u(x_i, y_j + sh_2, t)](1-s)^3 ds, \end{aligned}$$

such that  $\eta_h^n = \eta_h(t_n)$ . The second condition in (1.3) implies

$$\|\eta'(t)\| \leq C_u h^2 (1 + t^{\sigma-1}).$$

Hence, applying the fast L1 formula (2.13) and the equality (2.10), one has

$$\sum_{j=1}^n P_{n-j}^{(n)} \|(D_f^\alpha \eta)^j\| \leq \sum_{j=1}^n P_{n-j}^{(n)} \sum_{k=1}^j A_{j-k}^{(j)} \|\nabla_\tau \eta^k\| = \sum_{k=1}^n \|\nabla_\tau \eta^k\| \leq \frac{c_2}{\sigma} \hat{t}_n^2 h^2. \tag{4.11}$$

Since the time truncation error  $\xi_h^n$  in (4.4) is defined uniformly with respect to grid point  $\mathbf{x}_h \in \bar{\Omega}_h$ , we can define a continuous function  $\xi^n(\mathbf{x}) = \xi_1^n(\mathbf{x}) + \xi_2^n(\mathbf{x})$ , where  $\xi_1^n, \xi_2^n$  denote the truncation errors of fast L1 formula and Newton’s linearized approach, namely,

$$\xi_1^n = (D_t^\alpha u)(t_n) - (D_f^\alpha u)^n, \quad \xi_2^n = (\nabla_\tau u(t_n))^2 \int_0^1 f''(u(t_{n-1}) + s \nabla_\tau u(t_n))(1-s) ds,$$

such that  $\xi_h^n = \xi^n(x_i, y_j)$  for  $\mathbf{x}_h \in \bar{\Omega}_h$ . By the Taylor expansion formula, one has

$$\begin{aligned} \Delta_h(\xi_1^n)_{ij} &= \int_0^1 [\partial_{xx} \xi_1^n(x_i - sh_1, y_j) + \partial_{xx} \xi_1^n(x_i + sh_1, y_j)](1-s) ds \\ &\quad + \int_0^1 [\partial_{yy} \xi_1^n(x_i, y_j - sh_2) + \partial_{yy} \xi_1^n(x_i, y_j + sh_2)](1-s) ds, \quad 1 \leq n \leq N. \end{aligned}$$

Applying Lemma 3.3 with the second and third regularity conditions in (1.3), we have

$$\sum_{j=1}^n P_{n-j}^{(n)} \|\Delta_h \xi_1^j\| \leq \frac{C_u}{\sigma(1-\alpha)} \tau^{\min\{2-\alpha, \gamma\sigma\}} + \frac{C_u}{\sigma} t_n^\alpha \hat{t}_{n-1}^2 \epsilon, \quad 1 \leq n \leq N.$$

Similarly, one may have an integral expression of  $\Delta_h(\xi_2^n)_{ij}$  by using the Taylor expansion. Assuming  $f \in C^4(\mathbb{R})$  and taking the maximum time-step size

$$\tau \leq \tau_1^* := \sqrt[\gamma]{\sigma} \quad \text{such that} \quad \tau^{\gamma\alpha} \leq (\tau_1^*)^{\gamma\alpha} = \sigma,$$

we apply Lemma 3.4 with the second regularity condition in (1.3) to find,

$$\sum_{j=1}^n P_{n-j}^{(n)} \|\Delta_h \xi_2^j\| \leq C_u \tau^{\min\{2, 2\gamma\sigma\}} \max\{1, \tau^{\gamma\alpha}/\sigma^2\} \leq \frac{C_u}{\sigma} \tau^{\min\{2, 2\gamma\sigma\}}, \quad 1 \leq n \leq N.$$

Thus, the triangle inequality leads to

$$\sum_{j=1}^n P_{n-j}^{(n)} \|\Delta_h \xi^j\| \leq \frac{C_3}{\sigma(1-\alpha)} \tau^{\min\{2-\alpha, \gamma\sigma\}} + \frac{C_4}{\sigma} t_n^\alpha \hat{t}_{n-1}^2 \epsilon, \quad 1 \leq n \leq N. \tag{4.12}$$

#### 4.4 STEP 4: Error Estimate by Mathematical Induction

For a positive constant  $C_0$ , let  $\mathcal{B}(0, C_0)$  be a ball in the space of grid functions on  $\bar{\Omega}_h$  such that

$$\max \{ \|\psi\|_\infty, \|\nabla_h \psi\|, \|\Delta_h \psi\| \} \leq C_0$$

for any grid function  $\{\psi_h\} \in \mathcal{B}(0, C_0)$ . Always, we need the following result to treat the nonlinear terms but leave the proof to Appendix A.

**Lemma 4.1** *Let  $F \in C^2(\mathbb{R})$  and a grid function  $\{\psi_h\} \in \mathcal{B}(0, C_0)$ . Thus there is a constant  $C_F > 0$  dependent on  $C_0$  and  $C_\Omega$  such that,*

$$\|\Delta_h [F(\psi)v]\| \leq C_F \|\Delta_h v\|.$$

Under the regularity assumption (1.3) with  $U_h^k = u(x_h, t_k)$ , we define a constant

$$K_0 = \frac{1}{3} \max_{0 \leq k \leq N} \{ \|U^k\|_\infty, \|\nabla_h U^k\|, \|\Delta_h U^k\| \}.$$

For a smooth function  $F \in C^2(\mathbb{R})$  and any grid function  $\{v_h\} \in \mathcal{V}_h$ , we denote the maximum value of  $C_F$  in Lemma 4.1 as  $c_0$  such that

$$\|\Delta_h [F(w)v]\| \leq c_0 \|\Delta_h v\| \quad \text{for any grid function } \{w_h\} \in \mathcal{B}(0, K_0 + 1). \tag{4.13}$$

Let  $c_5$  be the maximum value of  $C_\Omega$  to verify the embedding inequalities in (2.1), and

$$c_6 = \max\{1, c_5\} E_\alpha(3 \max\{1, \rho\}(2K_0 + 3)c_0 T^\alpha), \quad c_7 = 3c_1 + \frac{2c_2}{\sigma} \hat{T}^2 + 3(2K_0 + 3)c_0 c_1 T^\alpha.$$

Also let

$$\tau_0^* := \frac{1}{\sqrt[2]{3\Gamma(2-\alpha)(2K_0 + 3)c_0}}, \quad \tau^* := \sqrt[\gamma]{\frac{\sigma(1-\alpha)}{6c_3c_6}} \leq \tau_1^*, \tag{4.14}$$



and

$$h_0 := \frac{1}{\sqrt{3c_6c_7}}, \quad \epsilon_0 := \min \left\{ \frac{\sigma}{6c_4c_6\hat{T}^2T^\alpha}, \frac{1}{3}\omega_{1-\alpha}(T), \alpha\omega_{2-\alpha}(T) \right\}. \tag{4.15}$$

For the simplicity of presentation, we define the following notations for  $1 \leq k \leq N$ ,

$$E_k := E_\alpha(3 \max\{1, \rho\}(2K_0 + 3)c_0t_k^\alpha),$$

$$\mathcal{T}^k := \frac{2c_3}{\sigma(1-\alpha)}\tau^{\min\{2-\alpha, \gamma\sigma\}} + \left(2c_1 + \frac{2c_2}{\sigma}\hat{t}_k^2 + 3(2K_0 + 3)c_0c_1t_k^\alpha\right)h^2 + \frac{2c_4}{\sigma}t_k^\alpha\hat{t}_{k-1}^2\epsilon.$$

We now apply the mathematical induction to prove that

$$\|\Delta_h \tilde{u}^k\| \leq E_k \mathcal{T}^k + c_1 h^2 \quad \text{for } 1 \leq k \leq N, \tag{4.16}$$

if the time-space grids and the SOE approximation satisfy

$$\tau \leq \min\{\tau_0^*, \tau^*\}, \quad h \leq h_0, \quad \epsilon \leq \epsilon_0. \tag{4.17}$$

Here  $\tau_0^*$ ,  $\tau^*$ ,  $h_0$  and  $\epsilon_0$  are fixed constants defined by (4.14)–(4.15). Note that, the restrictions in (4.17) ensures the error function  $\{\tilde{u}_h^k\} \in \mathcal{B}(0, 1)$  for  $1 \leq k \leq N$ .

Consider  $k = 1$  firstly. Since  $\tilde{u}_h^0 = 0$ ,  $\{u_h^0\} \in \mathcal{B}(0, K_0) \subset \mathcal{B}(0, K_0 + 1)$  and the nonlinear term (4.7) gives  $\mathcal{N}_h^1 = f'(u_h^0)\tilde{u}_h^1$ . For the function  $f \in C^3(\mathbb{R})$ , the inequality (4.13) implies

$$\|\Delta_h \mathcal{N}^1\| = \|\Delta_h (f'(u^0)\tilde{u}^1)\| \leq c_0 \|\Delta_h \tilde{u}^1\| \leq c_0 \|\tilde{w}^1\| + c_0 c_1 h^2, \tag{4.18}$$

where the Eq. (4.5) and the estimate (4.10) are used. Taking the inner product of the Eq. (4.8) (for  $n = 1$ ) by  $\tilde{w}_h^1$ , one gets

$$A_0^{(1)} \langle \nabla_\tau \tilde{w}^1, \tilde{w}^1 \rangle \leq \langle \Delta_h \mathcal{N}^1, \tilde{w}^1 \rangle + \langle (D_f^\alpha \eta)^1 - \Delta_h \xi^1, \tilde{w}^1 \rangle,$$

because the zero-valued boundary condition in (4.9) leads to  $\langle \Delta_h \tilde{w}^1, \tilde{w}^1 \rangle \leq 0$ . With the view of Cauchy–Schwarz inequality and (4.18), one has

$$\langle \nabla_\tau \tilde{w}^1, \tilde{w}^1 \rangle \geq \|\tilde{w}^1\| \|\nabla_\tau (\|\tilde{w}^1\|)\|$$

and then

$$A_0^{(1)} \|\nabla_\tau (\|\tilde{w}^1\|)\| \leq \|\Delta_h \mathcal{N}^1\| + \|(D_f^\alpha \eta)^1 - \Delta_h \xi^1\| \leq c_0 \|\tilde{w}^1\| + \|(D_f^\alpha \eta)^1 - \Delta_h \xi^1\| + c_0 c_1 h^2.$$

Setting  $\tau_1 \leq \tau_0^* \leq 1/\sqrt[3]{3\Gamma(2-\alpha)}$ , we apply Theorem 2.4 (discrete fractional Grönwall inequality) with  $\xi_1^1 = \|(D_f^\alpha \eta)^1 - \Delta_h \xi^1\|$  and  $\xi_2^1 = c_0 c_1 h^2$  to get

$$\begin{aligned} \|\tilde{w}^1\| &\leq E_\alpha(3 \max\{1, \rho\}c_0t_1^\alpha) \left(2\|\eta^0\| + 2P_0^{(1)}\|(D_f^\alpha \eta)^1 - \Delta_h \xi^1\| + 3c_0c_1\omega_{1+\alpha}(t_1)h^2\right) \\ &\leq E_1 \left(\frac{2c_3}{\sigma(1-\alpha)}\tau^{\min\{2-\alpha, \gamma\sigma\}} + 2c_1h^2 + \frac{2c_2}{\sigma}\hat{t}_1^2h^2 + 3c_0c_1\omega_{1+\alpha}(t_1)h^2\right) \leq E_1 \mathcal{T}^1, \end{aligned}$$

where the initial condition (4.9) and the error estimates (4.10)–(4.12) are used. Thus, the Eq. (4.5) and the inequality (4.10) yield the estimate (4.16) for  $k = 1$ ,

$$\|\Delta_h \tilde{u}^1\| \leq \|\tilde{w}^1\| + \|\eta^1\| \leq E_1 \mathcal{T}^1 + c_1 h^2.$$

Assume that the error estimate (4.16) holds for  $1 \leq k \leq n - 1$  ( $n \geq 2$ ). Thus we apply the embedding inequalities in (2.1) to get

$$\max \{ \| \tilde{u}^k \|_\infty, \| \nabla_h \tilde{u}^k \|, \| \Delta_h \tilde{u}^k \| \} \leq \max \{ 1, c_5 \} (E_k \mathcal{T}^k + c_1 h^2), \quad 1 \leq k \leq n - 1.$$

Under the priori settings in (4.17), we have the error function  $\{ \tilde{u}_h^k \} \in \mathcal{B}(0, 1)$ , the discrete solution  $\{ u_h^k \} \in \mathcal{B}(0, K_0 + 1)$  for  $1 \leq k \leq n - 1$ , and the continuous solution

$$\{ U_h^k \} \in \mathcal{B}(0, K_0) \subset \mathcal{B}(0, K_0 + 1).$$

Then, for the function  $f \in C^4(\mathbb{R})$ , one applies the inequality (4.13) to find that

$$\begin{aligned} \| \Delta_h [ f'(u^{n-1}) \nabla_\tau \tilde{u}^n ] \| &\leq c_0 \| \Delta_h \nabla_\tau \tilde{u}^n \| \leq c_0 \| \Delta_h \tilde{u}^n \| + c_0 \| \Delta_h \tilde{u}^{n-1} \|, \\ \| \Delta_h [ \tilde{u}^{n-1} f'(sU^{n-1} + (1-s)u^{n-1}) ] \| &\leq c_0 \| \Delta_h \tilde{u}^{n-1} \|, \\ \| \Delta_h [ \tilde{u}^{n-1} \nabla_\tau U^n f''(sU^{n-1} + (1-s)u^{n-1}) ] \| &\leq c_0 \| \Delta_h (\tilde{u}^{n-1} \nabla_\tau U^n) \| \leq 2c_0 K_0 \| \Delta_h \tilde{u}^{n-1} \|, \end{aligned}$$

where  $0 \leq s \leq 1$ . From the expression (4.7) of  $\mathcal{N}^n$  and the triangle inequality, one has

$$\begin{aligned} \| \Delta_h \mathcal{N}^n \| &\leq c_0 \| \Delta_h \tilde{u}^n \| + 2(K_0 + 1)c_0 \| \Delta_h \tilde{u}^{n-1} \| \\ &\leq c_0 \| \tilde{w}^n \| + 2(K_0 + 1)c_0 \| \tilde{w}^{n-1} \| + (2K_0 + 3)c_0 c_1 h^2, \end{aligned} \tag{4.19}$$

where the Eq.(4.5) and the estimate (4.10) are used.

Now, taking the inner product of (4.8) by  $\tilde{w}_h^n$ , one gets

$$\langle (D_f^\alpha \tilde{w})^n, \tilde{w}^n \rangle \leq \langle \Delta_h \mathcal{N}^n, \tilde{w}^n \rangle + \langle (D_f^\alpha \eta)^n - \Delta_h \xi^n, \tilde{w}^n \rangle, \tag{4.20}$$

because the zero-valued boundary condition in (4.9) leads to  $\langle \Delta_h \tilde{w}^n, \tilde{w}^n \rangle \leq 0$ . Lemma 2.5 (I) says that the kernels  $A_{n-k}^{(n)}$  are decreasing, so the Cauchy–Schwarz inequality gives

$$\begin{aligned} \langle (D_f^\alpha \tilde{w})^n, \tilde{w}^n \rangle &\geq A_0^{(n)} \| \tilde{w}^n \|^2 - \sum_{k=1}^{n-1} (A_{n-k-1}^{(n)} - A_{n-k}^{(n)}) \| \tilde{w}^k \| \| \tilde{w}^n \| - A_{n-1}^{(n)} \| \tilde{w}^0 \| \| \tilde{w}^n \| \\ &= \| \tilde{w}^n \| \left[ A_0^{(n)} \| \tilde{w}^n \| - \sum_{k=1}^{n-1} (A_{n-k-1}^{(n)} - A_{n-k}^{(n)}) \| \tilde{w}^k \| - A_{n-1}^{(n)} \| \tilde{w}^0 \| \right] \\ &= \| \tilde{w}^n \| \sum_{k=1}^n A_{n-k}^{(n)} \nabla_\tau (\| \tilde{w}^k \|). \end{aligned}$$

Thus with the help of Cauchy–Schwarz inequality and (4.19), it follows from (4.20) that

$$\begin{aligned} \sum_{k=1}^n A_{n-k}^{(n)} \nabla_\tau (\| \tilde{w}^k \|) &\leq \| \Delta_h \mathcal{N}^n \| + \| (D_f^\alpha \eta)^n - \Delta_h \xi^n \| \\ &\leq c_0 \| \tilde{w}^n \| + 2(K_0 + 1)c_0 \| \tilde{w}^{n-1} \| + \| (D_f^\alpha \eta)^n - \Delta_h \xi^n \| + (2K_0 + 3)c_0 c_1 h^2. \end{aligned}$$

Setting the maximum time-step size

$$\tau \leq \tau_0^* = 1 / \sqrt[3]{3\Gamma(2 - \alpha)(2K_0 + 3)c_0},$$

we apply Theorem 2.4 with  $\xi_1^n = \|(D_f^\alpha \eta)^n - \Delta_h \xi^n\|$  and  $\xi_2^n = (2K_0 + 3)c_0 c_1 h^2$  to get

$$\begin{aligned} \|\tilde{w}^n\| &\leq E_n \left( 2\|\eta^0\| + 2 \max_{1 \leq j \leq n} \sum_{k=1}^j P_{j-k}^{(j)} \|(D_f^\alpha \eta)^k - \Delta_h \xi^k\| + 3(2K_0 + 3)c_0 c_1 \omega_{1+\alpha}(t_n) h^2 \right) \\ &\leq E_n \left( \frac{2c_3}{\sigma(1-\alpha)} \tau^{\min\{2-\alpha, \gamma\sigma\}} + \frac{2c_4}{\sigma} t_n^\alpha \hat{t}_{n-1}^2 \epsilon \right) \\ &\quad + E_n \left( 2c_1 + \frac{2c_2}{\sigma} \hat{t}_n^2 + 3(2K_0 + 3)c_0 c_1 \omega_{1+\alpha}(t_n) \right) h^2 \leq E_n T^n, \end{aligned}$$

where the initial data (4.9) and the three estimates (4.10)–(4.12) are used. Then the error equation (4.5) with (4.10) implies that the claimed error estimate (4.16) holds for  $k = n$ ,

$$\|\Delta_h \tilde{u}^n\| \leq E_n T^n + c_1 h^2.$$

The principle of induction and the third inequality in (2.1) give the following result.

**Theorem 4.2** *Assume that the nonlinear function  $f \in C^4(\mathbb{R})$  and the solution of nonlinear subdiffusion problem (1.1) fulfills the regularity assumption (1.3) with a regularity parameter  $\sigma \in (0, 1) \cup (1, 2)$ . Suppose that the SOE approximation error  $\epsilon$ , the maximum time-step size  $\tau$ , and the maximum spatial length  $h$  satisfy*

$$\epsilon \leq \epsilon_0, \quad \tau \leq \min\{\tau_0^*, \tau^*\}, \quad h \leq h_0,$$

where  $\epsilon_0, \tau_0^*, \tau^*$  and  $h_0$  are fixed constants defined by (4.14)–(4.15). Then the discrete solution of two-level linearized fast scheme (2.6), on the nonuniform time mesh satisfying Ass3 and AssG, is unconditionally convergent in the maximum norm, that is,

$$\|U^k - u^k\|_\infty \leq \frac{c_8}{\sigma(1-\alpha)} E_\alpha(3 \max\{1, \rho\}(2K_0 + 3)c_0 t_k^\alpha) \left( \tau^{\min\{2-\alpha, \gamma\sigma\}} + h^2 + \epsilon \right),$$

for  $1 \leq k \leq N$ , where

$$c_8 = \max\{1, c_5\} \max\{2c_3, 4c_1 + 2c_2 \hat{T}^2 + 3(2K_0 + 3)c_0 c_1 T^\alpha, 2c_4 T^\alpha \hat{T}^2\}.$$

The numerical solution achieves an optimal time accuracy of order  $O(\tau^{2-\alpha})$  if the grading parameter is taken by  $\gamma \geq \max\{1, (2-\alpha)/\sigma\}$ .

### 5 Numerical Experiments

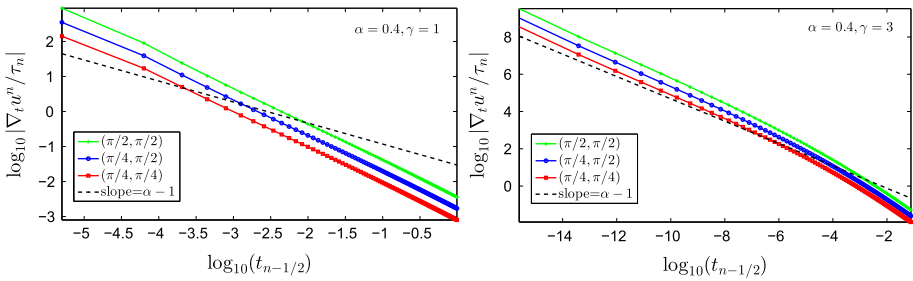
Two numerical examples are reported here to support our theoretical analysis. The two-level linearized scheme (2.6) runs for solving the fractional Fisher equation

$$\mathcal{D}_t^\alpha u = \Delta u + u(1-u) + g(\mathbf{x}, t), \quad (\mathbf{x}, t) \in (0, \pi)^2 \times (0, T],$$

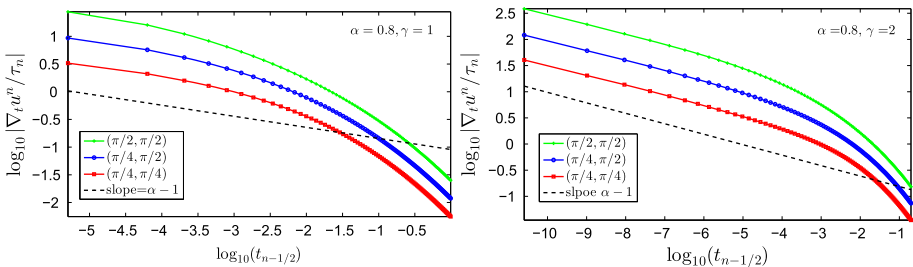
subject to zero-valued boundary data, with two different initial data and exterior forces:

- (Example 1)  $u^0(\mathbf{x}) = \sin x \sin y$  and  $g(\mathbf{x}, t) = 0$  such that no exact solution is available;
- (Example 2)  $g(\mathbf{x}, t)$  is specified such that  $u(\mathbf{x}, t) = \omega_\sigma(t) \sin x \sin y, 0 < \sigma < 2$ .

Note that, Example 2 with the regularity parameter  $\sigma$  is set to examine the sharpness of predicted time accuracy on nonuniform meshes. Actually, our present theory also fits for the semilinear problem with a nonzero force  $g(\mathbf{x}, t) \in C(\bar{\Omega} \times [0, T])$ .



**Fig. 1** The log–log plot of difference quotient  $\nabla_\tau u_h^n/\tau_n$  versus the time for Example 1 ( $\alpha = 0.4$ ) with two grading parameters  $\gamma = 1$  (left) and  $\gamma = 3$  (right)



**Fig. 2** The log–log plot of difference quotient  $\nabla_\tau u_h^n/\tau_n$  versus the time for Example 1 ( $\alpha = 0.8$ ) with two grading parameters  $\gamma = 1$  (left) and  $\gamma = 2$  (right)

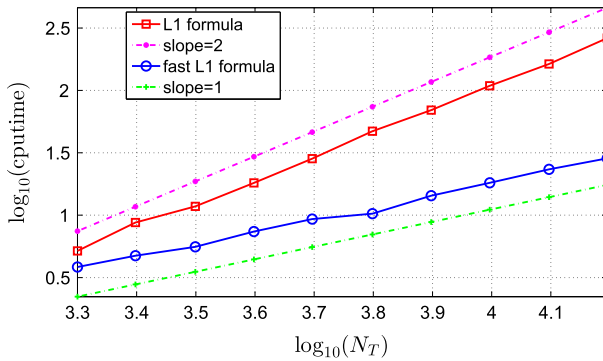
In our simulations, the spatial domain  $\Omega$  is divided uniformly into  $M$  parts in each direction ( $M_1 = M_2 = M$ ) and the time interval  $[0, T]$  is divided into two parts  $[0, T_0]$  and  $[T_0, T]$  with total  $N_T$  subintervals. According to the suggestion in [14], the graded mesh  $t_k = T_0(k/N)^\gamma$  is applied in the cell  $[0, T_0]$  and the uniform mesh with time step size  $\tau \geq \tau_N$  is used over the remainder interval. Given certain final time  $T$  and a proper number  $N_T$ , here we would take  $T_0 = \min\{1/\gamma, T\}$ ,  $N = \lceil \frac{N_T}{T+1-\gamma^{-1}} \rceil$  such that

$$\tau = \frac{T - T_0}{N_T - N} \geq \frac{T + 1 - \gamma^{-1}}{N_T} \geq N^{-1} \geq \tau_N.$$

Always, the absolute tolerance error of SOE approximation is set to  $\epsilon = 10^{-12}$  such that the two-level L1 formula (2.5a) is comparable with the L1 formula (2.2) in time accuracy.

In Example 1, we investigate the asymptotic behavior of solution near  $t = 0$  and the computational efficiency of the linearized method (2.6). Setting  $M = 100$ ,  $T = 1/\gamma$  and  $N_T = 100$ , Figs. 1 and 2 depict, in log–log plot, the numerical behaviors of first-order difference quotient  $\nabla_\tau u_h^n/\tau_n$  at three spatial points near the initial time for different fractional orders and grading parameters. Observations suggest that  $\log |u_t(\mathbf{x}, t)| \approx C_u(\mathbf{x}) + (\alpha - 1) \log t$  as  $t \rightarrow 0$ , and the solution is weakly singular near the initial time. Compared with the uniform grid, the graded mesh always concentrates much more points in the initial time layer and provides better resolution for the initial singularity.

To see the effectiveness of our linearized method (2.6), we also consider another linearized method by replacing the two-level fast L1 formula  $(D_f^\alpha u_h)^n$  with the nonuniform L1 formula  $(D_\tau^\alpha u_h)^n$  defined in (2.2). Setting  $\alpha = 0.5$ ,  $\gamma = 2$ , and  $M = 50$ , the two schemes are run for Example 1 to the final time  $T = 50$  with different total numbers  $N_T$ . Figure 3 shows the CPU time in seconds for both linearized procedures versus the total number  $N_T$  of subintervals.



**Fig. 3** The log–log plot of CPU time versus the total number  $N_T$  of time levels for the linearized method in solving Example 1 with two different formulas of Caputo derivative

**Table 1** Numerical temporal accuracy for  $\sigma = 2 - \alpha$  and  $\gamma = 1$

$N$	$\alpha = 0.4, \sigma = 1.6$		$\alpha = 0.6, \sigma = 1.4$		$\alpha = 0.8, \sigma = 1.2$	
	$e(N)$	Order	$e(N)$	Order	$e(N)$	Order
50	5.69e-04	–	1.14e-03	–	2.57e-03	–
100	1.57e-04	1.86	4.65e-04	1.30	1.23e-03	1.07
200	4.40e-05	1.84	1.88e-04	1.31	5.80e-04	1.08
400	1.45e-05	1.60	7.51e-05	1.32	2.71e-04	1.10
800	5.02e-06	1.53	2.98e-05	1.34	1.25e-04	1.12
$\min\{\gamma\sigma, 2 - \alpha\}$		1.60		1.40		1.20

We observe that the proposed method has almost linear complexity in  $N_T$  and is much faster than the direct scheme using traditional L1 formula.

Since the spatial error  $O(h^2)$  is standard, the time accuracy due to the numerical approximations of Caputo derivative and nonlinear reaction is examined in Example 2 with  $T = 1$ . The maximum norm error  $e(N, M) = \max_{1 \leq l \leq N} \|U(t_l) - u^l\|_\infty$ . To test the sharpness of our error estimate, we consider three different scenarios, respectively, in Tables 1, 2, and 3:

Table 1:  $\sigma = 2 - \alpha$  and  $\gamma = 1$  with fractional orders  $\alpha = 0.4, 0.6$  and  $0.8$ .

Table 2:  $\alpha = 0.4$  and  $\sigma = 0.4$  with grid parameters  $\gamma = 1, \frac{3}{4}\gamma_{\text{opt}}, \gamma_{\text{opt}}$  and  $\frac{5}{4}\gamma_{\text{opt}}$ .

Table 3:  $\alpha = 0.4$  and  $\sigma = 0.8$  with grid parameters  $\gamma = 1, \frac{3}{4}\gamma_{\text{opt}}, \gamma_{\text{opt}}$  and  $\frac{5}{4}\gamma_{\text{opt}}$ .

Table 1 lists the solution errors, for  $\sigma = 2 - \alpha$ , on the gradually refined grids with the coarsest grid of  $N = 50$ . Numerical data indicates that the optimal time order is of about  $O(\tau^{2-\alpha})$ , which dominates the spatial error  $O(h^2)$ . Always, we take  $M = N$  in Tables 1, 2, and 3 such that  $e(N, M) \approx e(N)$ . The experimental rate (listed as Order in tables) of convergence is estimated by observing that  $e(N) \approx C_u \tau^\beta$  and then  $\beta \approx \log_2 [e(N)/e(2N)]$ .

Numerical results in Tables 2 and 3 (with  $\alpha = 0.4$  and  $\sigma < 2 - \alpha$ ) support the predicted time accuracy in Theorem 4.2 on the smoothly graded mesh  $t_k = T(k/N)^\gamma$ . In the case of a uniform mesh ( $\gamma = 1$ ), the solution is accurate of order  $O(\tau^\sigma)$ , and the nonuniform meshes improve the numerical precision and convergence rate of solution evidently. The optimal time accuracy  $O(\tau^{2-\alpha})$  is observed when the grid parameter  $\gamma \geq (2 - \alpha)/\sigma$ .

**Table 2** Numerical temporal accuracy for  $\alpha = 0.4$ ,  $\sigma = 0.4$  and  $\gamma_{\text{opt}} = 4$

$N$	$\gamma = 1$		$\gamma = 3$		$\gamma = 4$		$\gamma = 5$	
	$e(N)$	Order	$e(N)$	Order	$e(N)$	Order	$e(N)$	Order
50	5.47e-02	–	3.82e-03	–	1.65e-03	–	1.32e-03	–
100	4.64e-02	0.24	1.68e-03	1.18	5.78e-04	1.52	4.60e-04	1.52
200	3.78e-02	0.30	7.36e-04	1.19	1.99e-04	1.54	1.58e-04	1.54
400	3.00e-02	0.33	3.21e-04	1.20	6.78e-05	1.55	5.37e-05	1.56
800	2.34e-02	0.36	1.40e-04	1.20	2.30e-05	1.56	1.81e-05	1.57
$\min\{\gamma\sigma, 2 - \alpha\}$		0.40		1.20		1.60		1.60

**Table 3** Numerical temporal accuracy for  $\alpha = 0.4$ ,  $\sigma = 0.8$  and  $\gamma_{\text{opt}} = 2$

$N$	$\gamma = 1$		$\gamma = 3/2$		$\gamma = 2$		$\gamma = 5/2$	
	$e(N)$	Order	$e(N)$	Order	$e(N)$	Order	$e(N)$	Order
50	3.46e-03	–	8.72e-04	–	5.80e-04	–	7.52e-04	–
100	2.20e-03	0.65	3.93e-04	1.15	1.39e-04	2.08	1.77e-04	2.08
200	1.34e-03	0.72	1.75e-04	1.17	3.80e-05	1.87	4.06e-05	2.13
400	7.95e-04	0.75	7.70e-05	1.18	1.32e-05	1.53	8.88e-06	2.19
600	5.83e-04	0.77	4.76e-05	1.19	7.06e-06	1.54	4.22e-06	1.55
800	4.67e-04	0.77	3.38e-05	1.19	4.52e-06	1.55	2.70e-06	1.55
$\min\{\gamma\sigma, 2 - \alpha\}$		0.80		1.20		1.60		1.60

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### A Proof of Lemma 4.1

**Proof** Consider  $F(\psi) = \psi$  firstly. It is easy to check that, at point  $x_h = (x_i, y_j) \in \Omega_h$ ,

$$\delta_x^2(\psi_{ij}v_{ij}) = \psi_{ij}(\delta_x^2v_{ij}) + \delta_x\psi_{i-\frac{1}{2},j}(\delta_xv_{i-\frac{1}{2},j}) + \delta_x\psi_{i+\frac{1}{2},j}(\delta_xv_{i+\frac{1}{2},j}) + v_{ij}(\delta_x^2\psi_{ij}),$$

so that

$$\|\delta_x^2(\psi v)\| \leq C_0 (\|v\| + \|\delta_x v\| + \|\delta_x^2 v\|).$$

Similarly,

$$\|\delta_y^2(\psi v)\| \leq C_0 (\|v\| + \|\delta_y v\| + \|\delta_y^2 v\|).$$

Moreover, one has

$$\|\delta_y\delta_x(\psi v)\| \leq C_0 (\|v\| + \|\delta_x v\| + \|\delta_y v\| + \|\delta_y\delta_x v\|),$$

due to the fact

$$\begin{aligned} \delta_y\delta_x(\psi_{i-\frac{1}{2},j-\frac{1}{2}}v_{i-\frac{1}{2},j-\frac{1}{2}}) &= \psi_{i-\frac{1}{2},j-\frac{1}{2}}(\delta_y\delta_xv_{i-\frac{1}{2},j-\frac{1}{2}}) + \delta_y\psi_{i-\frac{1}{2},j-\frac{1}{2}}(\delta_xv_{i-\frac{1}{2},j-\frac{1}{2}}) \\ &\quad + \delta_x\psi_{i-\frac{1}{2},j-\frac{1}{2}}(\delta_yv_{i-\frac{1}{2},j-\frac{1}{2}}) + (\delta_y\delta_x\psi_{i-\frac{1}{2},j-\frac{1}{2}})v_{i-\frac{1}{2},j-\frac{1}{2}}. \end{aligned}$$

Noticing that  $\|\Delta_h v\|^2 = \|\delta_x^2 v\|^2 + 2\|\delta_x\delta_y v\|^2 + \|\delta_y^2 v\|^2$ , we apply the embedding inequalities in (2.1) to obtain, also see [11, Lemma 2.2],

$$\|\Delta_h(\psi v)\| \leq C_u (\|v\| + \|\Delta_h v\|) \leq C_F \|\Delta_h v\|,$$

where the constant  $C_F$  is dependent on  $C_0$  and  $C_\Omega$ . For the general case  $F \in C^2(\mathbb{R})$ , one has

$$\begin{aligned} \delta_x^2[F(\psi_{ij})v_{ij}] &= F(\psi_{ij})(\delta_x^2v_{ij}) + \delta_x F(\psi_{i-\frac{1}{2},j})(\delta_xv_{i-\frac{1}{2},j}) \\ &\quad + \delta_x F(\psi_{i+\frac{1}{2},j})(\delta_xv_{i+\frac{1}{2},j}) + v_{ij}[\delta_x^2 F(\psi_{ij})]. \end{aligned}$$

The formula of Taylor expansion with integral remainder gives

$$\begin{aligned} \delta_x F(\psi_{i-\frac{1}{2},j}) &= (F(\psi_{ij}) - F(\psi_{i-1,j}))/h_1 = \delta_x\psi_{i-\frac{1}{2},j} \int_0^1 F'(s\psi_{ij} + (1-s)\psi_{i-1,j}) ds, \\ \delta_x^2 F(\psi_{ij}) &= (\delta_x^2\psi_{ij})F'(\psi_{ij}) + (\delta_x\psi_{i-\frac{1}{2},j})^2 \int_0^1 F''(s\psi_{ij} + (1-s)\psi_{i-1,j})(1-s) ds \\ &\quad + (\delta_x\psi_{i+\frac{1}{2},j})^2 \int_0^1 F''(s\psi_{ij} + (1-s)\psi_{i+1,j})(1-s) ds, \end{aligned}$$

such that  $\|\delta_x F(\psi)\| \leq C_F$  and  $\|\delta_x^2 F(\psi)\| \leq C_F$ . Therefore, simple calculations arrive at

$$\|\delta_x^2[F(\psi)v]\| \leq C_F (\|v\| + \|\delta_x v\| + \|\delta_x^2 v\|).$$

By presenting similar arguments as those in the above simple case, it is straightforward to get claimed estimate and complete the proof. □



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