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Optimal Error Estimates of Penalty Based Iterative Methods for Steady Incompressible Magnetohydrodynamics Equations with Different Viscosities

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Abstract

In this paper, we consider the penalty based finite element methods for the 2D/3D stationary incompressible magnetohydrodynamics (MHD) equations with different Reynolds numbers. Penalty method is applied to address the incompressible constraint "$div\,\mathbf{u} = 0$" based on two different finite element pairs $P_1-P_0-P_1$ and $P_1b-P_1-P_1b$. Furthermore, the proposed methods are the interesting combination of three different iterations and two-level finite element algorithm such that the uniqueness condition holds. Besides, the rigorous analysis of stability and optimal error estimate with respect to the penalty parameter ϵ for the proposed methods are given. Extensive 2D/3D numerical tests demonstrated the competitive performance of penalty methods.

Keywords Magnetohydrodynamics equations · Penalty finite element method · Two-level method · Inf-sup condition · Error estimate

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1 Introduction

MHD mainly studies the behavior of the dynamics of electrically conducting fluids (such as liquid metals, plasmas, salt water, etc.) [1–3]. The corresponding incompressible MHD model is a system of PDEs, which are governed by the Navier–Stokes equations and coupled with the pre-Maxwell equations. Incompressible MHD has a number of technological and industrial applications such as metallurgical engineering, electromagnetic pumping, stirring of liquid metals, and measuring flow quantities based on induction [2]. More detail physical background knowledge refer to resources [4,5].

A considerable amount of finite element method research activity has been devoted to the analysis of the simulation of MHD flows in recent years. As far as we know that the basic research for the MHD equations can be traced back to Sermange et al. [6]. And Gunzburger et al. proposed the existence and uniqueness of the solution of a weak formulation of the MHD equations [4]. Then, Gerbeau et al. studied a stabilized method for the steady MHD equations in [7]. Recently, Wu et al. [8] given an efficient two-step algorithm for the stationary incompressible MHD equations. Zhang et al. [9] presented a streamline diffusion method for stationary incompressible MHD. Zhao et al. [10] proposed an anisotropic adaptive finite element method for MHD equations at high Hartmann numbers. And Hu et al. [11] given a stable finite element method preserving $\nabla \cdot \mathbf{B} = 0$ exactly for MHD models. More extensive investigations of the steady MHD equations can be referred to [12–15] and their references.

In this paper, we consider the following 2D/3D stationary incompressible MHD model:

$$\begin{cases} -R_e^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - S_c \mathrm{curl}\mathbf{B}\times\mathbf{B} = \mathbf{f}, & \text{in } \boldsymbol{\Omega}, \\ \mathrm{div}\mathbf{u} = 0, & \text{in } \Omega, \\ S_c R_m^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{B}) - S_c \mathrm{curl}(\mathbf{u}\times\mathbf{B}) = \mathbf{g}, & \text{in } \Omega, \\ \mathrm{div}\mathbf{B} = 0, & \text{in } \Omega, \end{cases} \tag{1}$$

under the boundary conditions:

$$\begin{cases} \mathbf{u}|_{\partial\Omega} = 0, & \text{(no-slip condition)}, \\ \mathbf{B}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \mathbf{n}\times\mathrm{curl}\mathbf{B}|_{\partial\Omega} = 0, & \text{(perfectly wall)}, \end{cases} \tag{2}$$

where Ω represents a convex polygonal/polyhedral domain in $\mathbb{R}^d$, d =2 or 3, with boundary $\partial\Omega$, $\mathbf{u}$ the velocity field, $\mathbf{B}$ the magnetic field, $\mathbf{f}$ and $\mathbf{g}$ the external force terms, p the pressure, R_e the hydrodynamic Reynolds number, R_m the magnetic Reynolds number, S_c the coupling number, and $\mathbf{n}$ is the outer unit normal of $\partial\Omega$.

It is observed that Eqs. (1) and (2) contain three nonlinear terms $(\mathbf{u}\cdot\nabla)\mathbf{u}$, $\mathrm{curl}\mathbf{B}\times\mathbf{B}$, $\mathrm{curl}(\mathbf{u}\times\mathbf{B})$ and velocity $\mathbf{u}$ and pressure p are coupled together by the incompressible constraint "$\mathrm{div}\mathbf{u} = 0$", which makes the coupled nonlinear system typically requires a very large number of degrees of freedom to resolve numerically. Hence, great attentions have been paid on iterative method to deal with the nonlinearity in recent years. The Stokes, Newton and Oseen iterative methods are considered for the stationary Navier–Stokes equations by He et al. [16] and it's references. Then, the iterative methods in finite element approximation for the incompressible MHD equations are investigated and analyzed in [17–20].

In order to handle the incompressible constrain, the general practice is to relax the incompressibility constraint in an approximate way, resulting in a class of pseudo-compressibility methods, among which are the penalty method, the pressure stabilization method, the artificial compressibility method and the projection method [21–25], etc. Besides, we also proposed some decoupling method with Uzawa-type idea for the incompressible MHD equations in [26,27]. In this study, we consider the penalty method to decouple the strong coupled stationary incompressible MHD equations.

The penalty method applied to (1) is to approximate the solution $(\mathbf{u}, p, \mathbf{B})$ by $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{B}_\epsilon)$ satisfying the following stationary MHD equations:

$$\begin{cases} -R_e^{-1}\Delta\mathbf{u}_\epsilon + (\mathbf{u}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon - S_c\text{curl}\mathbf{B}_\epsilon\times\mathbf{B}_\epsilon + \nabla p_\epsilon = \mathbf{f}, & \text{in } \Omega,\\ \text{div}\mathbf{u}_\epsilon + \frac{\epsilon}{\nu_e}p_\epsilon = 0, & \text{in } \Omega,\\ S_cR_m^{-1}\text{curl}(\text{curl}\mathbf{B}_\epsilon) - S_c\text{curl}(\mathbf{u}_\epsilon\times\mathbf{B}_\epsilon) = \mathbf{g}, & \text{in } \Omega,\\ \text{div}\mathbf{B}_\epsilon = 0, & \text{in } \Omega, \end{cases} \tag{3}$$

with the homogeneous boundary conditions:

$$\begin{cases} \mathbf{u}_\epsilon|_{\partial\Omega} = 0, & \text{(no-slip condition)},\\ \mathbf{B}_\epsilon\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \mathbf{n}\times\text{curl}\mathbf{B}_\epsilon|_{\partial\Omega} = 0, & \text{(perfectly wall)}, \end{cases} \tag{4}$$

where $0 < \epsilon < 1$ is a penalty parameter and $\nu_e = 1/R_e$.

Although, the iterative method and penalty method decoupled and lineralized the system, the final resulting system is still a large problem to solve. Two-level scheme is an efficient key to save a large amount of CPU time with reasonable results. This idea is put forward by Xu for the nonlinear elliptic boundary value problem in [28,29]. Recently, Layton et al. given a two-level method for the reduced MHD problem in [30,31] and Zhang al et. studied a two-level coupled correction and decoupled parallel correction finite element methods for solving the stationary MHD equations in [32].

To complete our previous work [19,20], we consider the two-level penalty finite element methods related to different Reynolds numbers for 2D/3D steady incompressible MHD equations in this article. In brief, we mainly consider the finite element space pair $\mathbf{X}_h\times\mathrm{M}_h\times\mathbf{W}_h$ which does not satisfy the discrete inf-sup condition ($P_1{-}P_0{-}P_1$) or satisfies the discrete inf-sup condition ($P_1b{-}P_1{-}P_1b$). For a small $\sigma := \frac{\sqrt{2}C_0^2\max\{1,\sqrt{2}S_c\}\|\mathbf{F}\|_{-1}}{(\min\{R_e^{-1},S_cC_1R_m^{-1}\})^2}$ satisfying the uniqueness condition $0 < \sigma \le 1 - (\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}}$, we propose three two-level penalty iterative finite element method by solving the iteration solution $((\mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H}^m), p_{\epsilon H}^m)$ on a coarse mesh and finding a correction solution $((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), p_{\epsilon mh})$ on a fine mesh. Specifically, in the case of $0 < \sigma \le \frac{2}{5}$, $((\mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H}^m), p_{\epsilon H}^m)$ is obtained by the Stokes, Newton or Oseen iteration; in the case of $\frac{2}{5} < \sigma \le \frac{5}{11}$, $((\mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H}^m), p_{\epsilon H}^m)$ is obtained by the Newton or Oseen iteration; in the case of $\frac{5}{11} < \sigma \le 1 - (\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}}$, $((\mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H}^m), p_{\epsilon H}^m)$ is obtained by the Oseen iteration. Furthermore, the rigorous analysis of the stability and optimal error estimate under the penalty parameter ϵ are given for the proposed schemes. Numerical tests verify the theoretical results.

The paper is organized as follows. In Sect. 2, some basic results are given. Penalty mixed finite element method is given in Sect. 3. Section 4 is devoted to uniform stability and convergence of the two-level penalty iterative methods. Numerical tests are given in Sect. 5. Finally we end with a short conclusion.

2 Functional Setting of the Stationary MHD Equations

In order to derive the appropriate variational form of problems (1) and (3), we introduce the following spaces

$$\mathbf{X} := H_0^1(\Omega)^d = \{\mathbf{u}\in H^1(\Omega)^d : \mathbf{u}|_{\partial\Omega} = 0\},$$
$$\mathbf{W} := H_n^1(\Omega)^d = \{\mathbf{v}\in H^1(\Omega)^d : \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\},$$
$$\mathbf{V}:= \{\mathbf{u}\in\mathbf{X} : \text{div}\mathbf{u} = 0 \text{ in } \Omega\},$$

$$\mathbf{V}_n := \{\mathbf{v} \in \mathbf{W} : \mathrm{div}\mathbf{v} = 0 \text{ in } \Omega\},$$
$$\mathrm{M} := L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q d\mathbf{x} = 0 \right\}.$$

Denote product space $\mathbf{W}_{0n} = H_0^1(\Omega)^d \times H_n^1(\Omega)^d$ equipped with the graph norm $\|(\mathbf{v},\mathbf{B})\|_1$, where $\|(\mathbf{v},\mathbf{B})\|_i = (\|\mathbf{v}\|_i^2 + \|\mathbf{B}\|_i^2)^{\frac{1}{2}}$ for all $\mathbf{v} \in H^i(\Omega)^d \cap \mathbf{X}$, $\mathbf{B} \in H^i(\Omega)^d \cap \mathbf{W}$ $(i = 0, 1, 2)$. And $H^{-1}(\Omega)^d$ denotes the dual of $H_0^1(\Omega)^d$ with norm $\|\mathbf{f}\|_{-1} = \sup\limits_{0 \neq \mathbf{w} \in H_0^1(\Omega)^d} \frac{\langle \mathbf{f}, \mathbf{w} \rangle}{\|\mathbf{w}\|_1}$, where $\langle \cdot, \cdot \rangle$ denotes duality product between the function space $H_0^1(\Omega)^d$ and its dual.

Besides, we set

$$\|\mathbf{F}\|_{-1} = \sup_{(0,0) \neq (\mathbf{v}, \boldsymbol{\Psi}) \in \mathbf{W}_{0n}} \frac{\langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi}) \rangle}{\|(\mathbf{v}, \boldsymbol{\Psi})\|_1}, \qquad \|\mathbf{F}\|_*^2 = \|\mathbf{f}\|_{-1}^2 + \|\mathbf{g}\|_0^2, \tag{5}$$

and we know that $\|\mathbf{F}\|_{-1} \leq \|\mathbf{F}\|_*$.

Then we define the following forms by

$$\begin{aligned}
A_0\left((\mathbf{v}, \boldsymbol{\Psi}), (\mathbf{w}, \boldsymbol{\Phi})\right) &= a_0(\mathbf{v}, \mathbf{w}) + b_0(\boldsymbol{\Psi}, \boldsymbol{\Phi}),\\
a_0(\mathbf{v}, \mathbf{w}) &= R_e^{-1}(\nabla \mathbf{v}, \nabla \mathbf{w}),\\
b_0(\boldsymbol{\Psi}, \boldsymbol{\Phi}) &= S_c R_m^{-1}(\mathrm{curl}\boldsymbol{\Psi}, \mathrm{curl}\boldsymbol{\Phi}) + S_c R_m^{-1}(\mathrm{div}\boldsymbol{\Psi}, \mathrm{div}\boldsymbol{\Phi}),\\
d\left((\mathbf{v}, \boldsymbol{\Phi}), q\right) &= (\mathrm{div}\mathbf{v}, q), \;\; \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi}) \rangle = \langle \mathbf{f}, \mathbf{v} \rangle + (\mathbf{g}, \boldsymbol{\Psi}),\\
A_1\left((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \boldsymbol{\Psi}), (\mathbf{w}, \boldsymbol{\Phi})\right) &= a_1(\mathbf{u}, \mathbf{v}, \mathbf{w}) + c(\boldsymbol{\Phi}, \mathbf{B}, \mathbf{v}) - c(\boldsymbol{\Psi}, \mathbf{B}, \mathbf{w}),\\
a_1(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \frac{1}{2}\left((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}\right) - \frac{1}{2}\left((\mathbf{u} \cdot \nabla)\mathbf{w}, \mathbf{v}\right),\\
c(\boldsymbol{\Phi}, \mathbf{B}, \mathbf{v}) &= S_c(\mathrm{curl}\boldsymbol{\Phi} \times \mathbf{B}, \mathbf{v}).
\end{aligned}$$

The variational formulation for (1) consists in finding $((\mathbf{u}, \mathbf{B}), p) \in \mathbf{W}_{0n} \times \mathrm{M}$ such that

$$\begin{aligned}
&A_0\left((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \boldsymbol{\Psi})\right) - d\left((\mathbf{v}, \boldsymbol{\Psi}), p\right) + d\left((\mathbf{u}, \mathbf{B}), q\right)\\
&\quad + A_1\left((\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}), (\mathbf{v}, \boldsymbol{\Psi})\right) = \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi}) \rangle,
\end{aligned} \tag{6}$$

for all $((\mathbf{v}, \boldsymbol{\Psi}), q) \in \mathbf{W}_{0n} \times \mathrm{M}$ and the variational formulation of (3) reads: find $((\mathbf{u}_\epsilon, \mathbf{B}_\epsilon), p_\epsilon) \in \mathbf{W}_{0n} \times \mathrm{M}$ such that for all $((\mathbf{v}, \boldsymbol{\Psi}), q) \in \mathbf{W}_{0n} \times \mathrm{M}$,

$$\begin{aligned}
&A_0\left((\mathbf{u}_\epsilon, \mathbf{B}_\epsilon), (\mathbf{v}, \boldsymbol{\Psi})\right) - d\left((\mathbf{v}, \boldsymbol{\Psi}), p_\epsilon\right) + d\left((\mathbf{u}_\epsilon, \mathbf{B}_\epsilon), q\right)\\
&\quad + A_1\left((\mathbf{u}_\epsilon, \mathbf{B}_\epsilon), (\mathbf{u}_\epsilon, \mathbf{B}_\epsilon), (\mathbf{v}, \boldsymbol{\Psi})\right) + \frac{\epsilon}{\nu_e}(p_\epsilon, q) = \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi}) \rangle.
\end{aligned} \tag{7}$$

Besides, $A_0(\cdot,\cdot)$ and $A_1(\cdot,\cdot,\cdot)$ possess the following properties in [4]: $\forall\ (\mathbf{u}, \mathbf{B}), (\mathbf{v}, \boldsymbol{\Psi}), (\mathbf{w}, \boldsymbol{\Phi}) \in \mathbf{W}_{0n}$, there holds

$$A_0\left((\mathbf{v}, \boldsymbol{\Psi}), (\mathbf{w}, \boldsymbol{\Phi})\right) \leq \overline{\nu}\|(\mathbf{v}, \boldsymbol{\Psi})\|_1 \|(\mathbf{w}, \boldsymbol{\Phi})\|_1, \tag{8}$$

$$A_0\left((\mathbf{v}, \boldsymbol{\Psi}), (\mathbf{v}, \boldsymbol{\Psi})\right) \geq \underline{\nu}\|(\mathbf{v}, \boldsymbol{\Psi})\|_1^2, \tag{9}$$

$$A_1\left((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \boldsymbol{\Psi}), (\mathbf{w}, \boldsymbol{\Phi})\right) \leq N\|(\mathbf{u}, \mathbf{B})\|_1 \|(\mathbf{v}, \boldsymbol{\Psi})\|_1 \|(\mathbf{w}, \boldsymbol{\Phi})\|_1, \tag{10}$$

$$A_1\left((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \boldsymbol{\Psi}), (\mathbf{v}, \boldsymbol{\Psi})\right) = 0, \tag{11}$$

$$A_1\left((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \boldsymbol{\Psi}), (\mathbf{w}, \boldsymbol{\Phi})\right) + A_1\left((\mathbf{v}, \boldsymbol{\Psi}), (\mathbf{u}, \mathbf{B}), (\mathbf{w}, \boldsymbol{\Phi})\right) \tag{12}$$

$$\quad + A_1\left((\mathbf{v}, \boldsymbol{\Psi}), (\mathbf{w}, \boldsymbol{\Phi}), (\mathbf{u}, \mathbf{B})\right) \tag{13}$$

$$\leq CN\|(\mathbf{u}, \mathbf{B})\|_1^{\frac{1}{2}} \|(\mathbf{u}, \mathbf{B})\|_0^{\frac{1}{2}} \|(\mathbf{w}, \boldsymbol{\Phi})\|_1 \|(\mathbf{v}, \boldsymbol{\Psi})\|_1, \tag{14}$$

where $\underline{\nu} := \min\{R_e^{-1}, S_c C_1 R_m^{-1}\}$, $\overline{\nu} := \max\{R_e^{-1}, (2+d) S_c R_m^{-1}\}$, $N := \sqrt{2} C_0^2 \max\{1, \sqrt{2} S_c\}$.

And we introduce two properties of trilinear form in [17]:

$$\begin{aligned} &|A_1 ((\mathbf{u}, \mathbf{B}), (\mathbf{w}, \boldsymbol{\Phi}), (\mathbf{v}, \boldsymbol{\Psi}))| \le CN \|(\mathbf{u}, \mathbf{B})\|_0 \|(\mathbf{w}, \boldsymbol{\Phi})\|_2 \|(\mathbf{v}, \boldsymbol{\Psi})\|_1, \\ &\quad \forall (\mathbf{u}, \mathbf{B}) \in L^2(\Omega)^d \times L^2(\Omega)^d, (\mathbf{w}, \boldsymbol{\Phi}) \in H^2(\Omega)^d \times H^2(\Omega)^d, (\mathbf{v}, \boldsymbol{\Psi}) \in \mathbf{W}_{0n}(\Omega), \\ &|A_1 ((\mathbf{u}, \mathbf{B}), (\mathbf{w}, \boldsymbol{\Phi}), (\mathbf{v}, \boldsymbol{\Psi}))| \le CN \|(\mathbf{u}, \mathbf{B})\|_2 \|(\mathbf{w}, \boldsymbol{\Phi})\|_1 \|(\mathbf{v}, \boldsymbol{\Psi})\|_0, \\ &\quad \forall (\mathbf{u}, \mathbf{B}) \in H^2(\Omega)^d \times H^2(\Omega)^d, (\mathbf{w}, \boldsymbol{\Phi}) \in \mathbf{W}_{0n}(\Omega), (\mathbf{v}, \boldsymbol{\Psi}) \in L^2(\Omega)^d \times L^2(\Omega)^d. \end{aligned} \tag{15}$$

For the sake of convenience, C or c (with or without a subscript) will denotes a generic positive constant throughout the paper and we set

$$|||(\mathbf{w}, \boldsymbol{\Phi})|||_i = \underline{\nu} \left(\|\mathbf{w}\|_i^2 + \|\boldsymbol{\Phi}\|_i^2\right)^{\frac{1}{2}}, \quad \forall \mathbf{w} \in H^i(\Omega)^d \cap \mathbf{X}, \ \boldsymbol{\Phi} \in H^i(\Omega)^d \cap \mathbf{W}, \ i = 0, 1, 2.$$

The following existence and uniqueness for (6) and (7) are classical results (see [19]).

Theorem 2.1 *If R_e, R_m and S_c satisfy the uniqueness condition*

$$0 < \sigma < 1, \tag{16}$$

the problem (6) has a unique solution $((\boldsymbol{u}, \boldsymbol{B}), p) \in \boldsymbol{W}_{0n} \times \mathrm{M}$ which satisfies

$$|||(\boldsymbol{u}, \boldsymbol{B})|||_1 \le \|\boldsymbol{F}\|_{-1}. \tag{17}$$

Moreover, suppose that $\boldsymbol{f}$, $\boldsymbol{g} \in L^2(\Omega)^d$, then solution $((\boldsymbol{u}, \boldsymbol{B}), p)$ of the problem (6) satisfies the following regularity

$$|||(\boldsymbol{u}, \boldsymbol{B})|||_2 + \|p\|_1 \le C \|\boldsymbol{F}\|_0. \tag{18}$$

Theorem 2.2 *If R_e, R_m and S_c satisfy the uniqueness condition*

$$0 < \sigma < 1 \tag{19}$$

and $\epsilon c_0 \le 1$, then the problem (7) has a unique solution $((\boldsymbol{u}_\epsilon, \boldsymbol{B}_\epsilon), p_\epsilon) \in \boldsymbol{W}_{0n} \times \mathrm{M}$ which satisfies

$$|||(\boldsymbol{u}_\epsilon, \boldsymbol{B}_\epsilon)|||_1 \le \|\boldsymbol{F}\|_{-1}. \tag{20}$$

Moreover, suppose that $\boldsymbol{f}$, $\boldsymbol{g} \in L^2(\Omega)^d$, then solution $((\boldsymbol{u}_\epsilon, \boldsymbol{B}_\epsilon), p_\epsilon)$ of the problem (7) satisfies the following regularity

$$|||(\boldsymbol{u}_\epsilon, \boldsymbol{B}_\epsilon)|||_2 + \|p_\epsilon\|_1 \le C \|\boldsymbol{F}\|_0. \tag{21}$$

The bounds of the error $(\mathbf{u} - \mathbf{u}_\epsilon, \mathbf{B} - \mathbf{B}_\epsilon)$ and $p - p_\epsilon$ are stated in the following theorem (see [19] for detatil).

Theorem 2.3 *Under the assumptions of Theorem 2.2, we have*

$$|||(\boldsymbol{u} - \boldsymbol{u}_\epsilon, \boldsymbol{B} - \boldsymbol{B}_\epsilon)|||_1 + \|p - p_\epsilon\|_0 \le C\epsilon \|\boldsymbol{F}\|_0. \tag{22}$$

3 Penalty Finite Element Galerkin Discretization

For the finite element discretization, let $\{\tau_\mu\}$ be a family of triangulations or tetrahedrons of Ω into affine-equivalent finite elements K with $\bar{\Omega} = \bigcup_{K \in \tau_\mu} K$. Choose conforming finite element space $\mathbf{X}_H \subset \mathbf{X}$, $\mathrm{M}_H \subset \mathrm{M}$, $\mathbf{W}_H \subset \mathbf{W}$ and $(\mathbf{X}_H, \mathrm{M}_H, \mathbf{W}_H) \subset (\mathbf{X}_h, \mathrm{M}_h, \mathbf{W}_h)$.

Then we denote the set of all polynomials on K by $P_l(K)$, $l \geq 0$ and $\mathbf{W}_{0n}^{\mu} = \mathbf{X}_{\mu} \times \mathrm{M}_{\mu}$, $\mu = h$ *or* H.

We consider the following finite element pairs to investigate the relation of penalty parameter. In detail, $\mathbf{X}_{\mu} \times \mathrm{M}_{\mu} \times \mathbf{W}_{\mu}$ satisfies the following properties [13,16,17,24,33]:

Let ρ_{μ} denote the L^2-orthogonal projection which defined by

$$(\rho_{\mu} q, q_{\mu}) = (q, q_{\mu}), \quad \forall q \in \mathrm{M},\ q_{\mu} \in \mathrm{M}_{\mu}. \tag{23}$$

$(\mathcal{P}_1)$. Firstly, we consider the unstable finite element pair

$$\begin{aligned} \mathbf{X}_{\mu} &= \left\{ \mathbf{u} \in C^0(\bar{\Omega})^d \cap \mathbf{X} : \mathbf{u}|_K \in P_1(K)^d, \forall K \in \tau_{\mu} \right\}, \\ \mathrm{M}_{\mu} &= \left\{ q \in C^0(\bar{\Omega}) \cap \mathrm{M} : q|_K \in P_0(K), \forall K \in \tau_{\mu} \right\}, \\ \mathbf{W}_{\mu} &= \left\{ \mathbf{B} \in C^0(\bar{\Omega})^d \cap \mathbf{W} : \mathbf{B}|_K \in P_1(K)^d, \forall K \in \tau_{\mu} \right\}. \end{aligned}$$

And the pair $\mathbf{X}_{\mu} \times \mathrm{M}_{\mu}$ does not satisfy the inf-sup condition,

$$\sup_{(0,0) \neq (\mathbf{v}_{\mu}, \mathbf{B}_{\mu}) \in \mathbf{W}_{0n}^{\mu}} \frac{d\left(\left(\mathbf{v}_{\mu}, \mathbf{B}_{\mu}\right), q\right)}{\|(\mathbf{v}_{\mu}, \mathbf{B}_{\mu})\|_1} \geq \beta_0 \|q_{\mu}\|_0, \quad \forall q_{\mu} \in \mathrm{M}_{\mu}. \tag{24}$$

Then, there exists mappings $\pi_{\mu} : H^2(\Omega)^d \cap \mathbf{V} \to \mathbf{X}_{\mu}$ and $\rho_{\mu} : \mathrm{M} \to \mathrm{M}_{\mu}$ satisfy

$$\|\nabla(\mathbf{v} - \pi_{\mu}\mathbf{v})\|_0 \leq C\mu \|\mathbf{v}\|_2, \quad \|q - \rho_{\mu} q\|_0 \leq C\mu \|q\|_1, \tag{25}$$

for all $\mathbf{v} \in H^2(\Omega)^d \cap \mathbf{V}$, $q \in H^1(\Omega) \cap \mathrm{M}$, and a mapping $R_{\mu} : H^2(\Omega)^d \cap \mathbf{V}_n \to \mathbf{W}_{\mu}$ satisfy

$$\begin{aligned} (\nabla \times R_{\mu}\boldsymbol{\Phi}, \nabla \times \boldsymbol{\Psi}) + (\nabla \cdot R_{\mu}\boldsymbol{\Phi}, \nabla \cdot \boldsymbol{\Psi}) &= (\nabla \times \boldsymbol{\Phi}, \nabla \times \boldsymbol{\Psi}) + (\nabla \cdot \boldsymbol{\Phi}, \nabla \cdot \boldsymbol{\Psi}) \\ &= (\nabla \times \boldsymbol{\Phi}, \nabla \times \boldsymbol{\Psi}), \quad \forall \boldsymbol{\Psi} \in \mathbf{W}_{\mu}, \\ \|\boldsymbol{\Phi} - R_{\mu}\boldsymbol{\Phi}\|_0 + \mu \|\boldsymbol{\Phi} - R_{\mu}\boldsymbol{\Phi}\|_1 &\leq C\mu^2 \|\boldsymbol{\Phi}\|_2, \quad \forall \boldsymbol{\Phi} \in H^2(\Omega)^d \cap \mathbf{V}_n. \end{aligned} \tag{26}$$

It is important that this pair satisfy the relation

$$\mathrm{div}\mathbf{X}_{\mu} = \mathrm{M}_{\mu}. \tag{27}$$

$(\mathcal{P}_2)$. Next, we employ the following stable finite element pair

$$\begin{aligned} \mathbf{X}_{\mu} &= \left(P_{1,\mu}^b\right)^d \cap \mathbf{X}, \\ \mathrm{M}_{\mu} &= \left\{ q \in C^0(\bar{\Omega}) \cap \mathrm{M} : q|_K \in P_1(K),\ \forall K \in \tau_{\mu} \right\}, \\ \mathbf{W}_{\mu} &= \left(P_{1,\mu}^b\right)^d \cap \mathbf{W}, \end{aligned}$$

where $P_{1,\mu}^b = \{ v_{\mu} \in C^0(\bar{\Omega}) : v_{\mu}|_K \in P_1(K) \oplus span\{\hat{b}\},\ \forall K \in \tau_{\mu} \}$.

Here, $\mathbf{X}_{\mu} \times \mathrm{M}_{\mu}$ satisfies the discrete inf-sup condition (24). In addition, (27) does not hold. Besides, there exists mappings $\pi_{\mu} : H^2(\Omega)^d \cap \mathbf{X} \to \mathbf{X}_{\mu}$, $\rho_{\mu} : \mathrm{M} \to \mathrm{M}_{\mu}$ satisfy (25) and

$$\left(\nabla \cdot (\mathbf{v} - \pi_{\mu}\mathbf{v}), q\right) = 0, \quad \forall \mathbf{v} \in H^2(\Omega)^d \cap \mathbf{V}, \quad q \in \mathrm{M}_{\mu}. \tag{28}$$

Besides, mapping $R_{\mu} : H^2(\Omega)^d \cap \mathbf{V}_n \to \mathbf{W}_{\mu}$ satisfies (26).

Then, the penalty finite element discretization of (7) is: find $((\mathbf{u}_{\epsilon\mu}, \mathbf{B}_{\epsilon\mu}), p_{\epsilon\mu}) \in \mathbf{W}_{0n}^{\mu} \times \mathrm{M}_{\mu}$ such that

$$A_0\left(\left(\mathbf{u}_{\epsilon\mu}, \mathbf{B}_{\epsilon\mu}\right), (\mathbf{v}, \boldsymbol{\Psi})\right) + A_1\left(\left(\mathbf{u}_{\epsilon\mu}, \mathbf{B}_{\epsilon\mu}\right), \left(\mathbf{u}_{\epsilon\mu}, \mathbf{B}_{\epsilon\mu}\right), (\mathbf{v}, \boldsymbol{\Psi})\right) - d\left((\mathbf{v}, \boldsymbol{\Psi}), p_{\epsilon\mu}\right) + d\left(\left(\mathbf{u}_{\epsilon\mu}, \mathbf{B}_{\epsilon\mu}\right), q\right) + \frac{\epsilon}{\nu_e}(p_{\epsilon\mu}, q) = \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi})\rangle. \tag{29}$$

Next, we introduce the discrete analogue of space $\mathbf{V}$ as

$$\mathbf{V}_\mu = \left\{\mathbf{v} \in \mathbf{X}_\mu : d\left((\mathbf{v}, \boldsymbol{\Psi}), q\right) = 0, \forall q \in \mathrm{M}_\mu,\ \boldsymbol{\Psi} \in \mathbf{W}_\mu\right\}.$$

Here, we define discrete Stokes operator $\mathcal{A}_{1\mu} = -P_\mu \Delta_\mu$, and Δ_μ (see [34])

$$-(\Delta_\mu \mathbf{u}_\mu, \mathbf{v}_\mu) = (\nabla \mathbf{u}_\mu, \nabla \mathbf{v}_\mu), \quad \forall \mathbf{u}_\mu, \mathbf{v}_\mu \in \mathbf{X}_\mu,$$

where $P_\mu : L^2(\Omega)^d \to \mathbf{V}_\mu$ and define discrete operator $\mathcal{A}_{2\mu}\mathbf{B}_\mu = R_{0\mu}(\nabla_\mu \times \nabla \times \mathbf{B}_\mu + \nabla_\mu \nabla \cdot \mathbf{B}_\mu) \in \mathbf{W}_\mu$ as follows (see [33])

$$(\mathcal{A}_{2\mu}\mathbf{B}_\mu, \boldsymbol{\Psi}) = (\nabla \times \mathbf{B}_\mu, \nabla \times \boldsymbol{\Psi}) + (\nabla \cdot \mathbf{B}_\mu, \nabla \cdot \boldsymbol{\Psi}), \quad \forall \mathbf{B}_\mu, \boldsymbol{\Psi} \in \mathbf{W}_\mu,$$

where $R_{0\mu} : L^2(\Omega)^d \to \mathbf{W}_\mu$.

Recalling the following stability and optimal error estimate (see [19]).

Theorem 3.1 *Under the assumptions of Theorem 2.2 and if $\boldsymbol{X}_\mu \times M_\mu$ satisfies property $\mathcal{P}_k$, $k = 1, 2$, then (29) admits a unique solution $((\boldsymbol{u}_{\epsilon\mu}, \boldsymbol{B}_{\epsilon\mu}), p_{\epsilon\mu}) \in \mathbf{W}_{0n}^{\mu} \times M_\mu$ such that*

$$|||\left(\boldsymbol{u}_{\epsilon\mu}, \boldsymbol{B}_{\epsilon\mu}\right)|||_1 \le \|\boldsymbol{F}\|_{-1}, \quad |||\left(\mathcal{A}_{1\mu}\boldsymbol{u}_{\epsilon\mu}, \mathcal{A}_{2\mu}\boldsymbol{B}_{\epsilon\mu}\right)|||_0 \le C\|\boldsymbol{F}\|_0,$$

$$\|p_{\epsilon\mu}\|_0 \le \left(\frac{\nu_e}{\epsilon\underline{\nu}}\right)^{\frac{1}{2}} \|\boldsymbol{F}\|_{-1}, \quad for\ \mathcal{P}_1,$$

$$\|p_{\epsilon\mu}\|_0 \le C\|\boldsymbol{F}\|_{-1}, \quad for\ \mathcal{P}_2,$$

Theorem 3.2 *Under the assumptions of Theorem 2.2 and if $\boldsymbol{X}_\mu \times M_\mu$ satisfies property $\mathcal{P}_k$, $k = 1, 2$ and assume that $\mu \le \left(\frac{\|\boldsymbol{F}\|_{-1}}{\|\boldsymbol{F}\|_0}\right)^{\frac{1}{2}}(1-\sigma)$, then we have the following error estimate*

$$(1-\sigma)\epsilon^{\frac{1}{2}}|||\left(\boldsymbol{u}_\epsilon - \boldsymbol{u}_{\epsilon\mu}, \boldsymbol{B}_\epsilon - \boldsymbol{B}_{\epsilon\mu}\right)|||_0 + \mu\left(|||\left(\boldsymbol{u}_\epsilon - \boldsymbol{u}_{\epsilon\mu}, \boldsymbol{B}_\epsilon - \boldsymbol{B}_{\epsilon\mu}\right)|||_1 + \epsilon^{\frac{1}{2}}\|p_\epsilon - p_{\epsilon\mu}\|_0\right) \le C\epsilon^{-\frac{1}{2}}\mu^2\|\boldsymbol{F}\|_0,$$

$$(1-\sigma)|||\left(\boldsymbol{u}_\epsilon - \boldsymbol{u}_{\epsilon\mu}, \boldsymbol{B}_\epsilon - \boldsymbol{B}_{\epsilon\mu}\right)|||_0 + \mu\left(|||\left(\boldsymbol{u}_\epsilon - \boldsymbol{u}_{\epsilon\mu}, \boldsymbol{B}_\epsilon - \boldsymbol{B}_{\epsilon\mu}\right)|||_1 + \|p_\epsilon - p_{\epsilon\mu}\|_0\right) \le C\mu^2\|\boldsymbol{F}\|_0,$$

for $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively.

4 Penalty Iterative Methods for the 2D/3D Stationary MHD Equations

Three iterative methods in penalty method based on finite element pair $\mathcal{P}_1$ and $\mathcal{P}_2$ and several two-level schemes with different stability conditions are introduced as follows.

Method 1 (Stokes iterative method). Find $((\mathbf{u}_{\epsilon\mu}^n, \mathbf{B}_{\epsilon\mu}^n), p_{\epsilon\mu}^n) \in \mathbf{W}_{0n}^{\mu} \times \mathrm{M}_\mu$ such that for all $((\mathbf{v}, \boldsymbol{\Psi}), q) \in \mathbf{W}_{0n}^{\mu} \times \mathrm{M}_\mu$

$$A_0\left(\left(\mathbf{u}_{\epsilon\mu}^n, \mathbf{B}_{\epsilon\mu}^n\right), (\mathbf{v}, \boldsymbol{\Psi})\right) - d\left((\mathbf{v}, \boldsymbol{\Psi}), p_{\epsilon\mu}^n\right) + d\left(\left(\mathbf{u}_{\epsilon\mu}^n, \mathbf{B}_{\epsilon\mu}^n\right), q\right) + \frac{\epsilon}{\nu_e}(p_{\epsilon\mu}^n, q) + A_1\left(\left(\mathbf{u}_{\epsilon\mu}^{n-1}, \mathbf{B}_{\epsilon\mu}^{n-1}\right), \left(\mathbf{u}_{\epsilon\mu}^{n-1}, \mathbf{B}_{\epsilon\mu}^{n-1}\right), (\mathbf{v}, \boldsymbol{\Psi})\right) = \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi})\rangle. \tag{30}$$

Method 2 (Newton iterative method). Find $((\mathbf{u}^n_{\epsilon\mu}, \mathbf{B}^n_{\epsilon\mu}), p^n_{\epsilon\mu}) \in \mathbf{W}^{\mu}_{0n} \times \mathrm{M}_{\mu}$ such that for all $((\mathbf{v}, \boldsymbol{\Psi}), q) \in \mathbf{W}^{\mu}_{0n} \times \mathrm{M}_{\mu}$

$$\begin{aligned} &A_0\left(\left(\mathbf{u}^n_{\epsilon\mu}, \mathbf{B}^n_{\epsilon\mu}\right), (\mathbf{v}, \boldsymbol{\Psi})\right) - d\left((\mathbf{v}, \boldsymbol{\Psi}), p^n_{\epsilon\mu}\right) + d\left(\left(\mathbf{u}^n_{\epsilon\mu}, \mathbf{B}^n_{\epsilon\mu}\right), q\right) + \frac{\epsilon}{\nu_e}\left(p^n_{\epsilon\mu}, q\right) \\ &\quad + A_1\left(\left(\mathbf{u}^{n-1}_{\epsilon\mu}, \mathbf{B}^{n-1}_{\epsilon\mu}\right), \left(\mathbf{u}^n_{\epsilon\mu}, \mathbf{B}^n_{\epsilon\mu}\right), (\mathbf{v}, \boldsymbol{\Psi})\right) \\ &\quad + A_1\left(\left(\mathbf{u}^n_{\epsilon\mu}, \mathbf{B}^n_{\epsilon\mu}\right), \left(\mathbf{u}^{n-1}_{\epsilon\mu}, \mathbf{B}^{n-1}_{\epsilon\mu}\right), (\mathbf{v}, \boldsymbol{\Psi})\right) \\ &\quad = \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi})\rangle + A_1\left(\left(\mathbf{u}^{n-1}_{\epsilon\mu}, \mathbf{B}^{n-1}_{\epsilon\mu}\right), \left(\mathbf{u}^{n-1}_{\epsilon\mu}, \mathbf{B}^{n-1}_{\epsilon\mu}\right), (\mathbf{v}, \boldsymbol{\Psi})\right). \end{aligned} \tag{31}$$

Method 3 (Oseen iterative method). Find $((\mathbf{u}^n_{\epsilon\mu}, \mathbf{B}^n_{\epsilon\mu}), p^n_{\epsilon\mu}) \in \mathbf{W}^{\mu}_{0n} \times \mathrm{M}_{\mu}$ such that for all $((\mathbf{v}, \boldsymbol{\Psi}), q) \in \mathbf{W}^{\mu}_{0n} \times \mathrm{M}_{\mu}$

$$\begin{aligned} &A_0\left(\left(\mathbf{u}^n_{\epsilon\mu}, \mathbf{B}^n_{\epsilon\mu}\right), (\mathbf{v}, \boldsymbol{\Psi})\right) - d\left((\mathbf{v}, \boldsymbol{\Psi}), p^n_{\epsilon\mu}\right) + d\left(\left(\mathbf{u}^n_{\epsilon\mu}, \mathbf{B}^n_{\epsilon\mu}\right), q\right) + \frac{\epsilon}{\nu_e}\left(p^n_{\epsilon\mu}, q\right) \\ &\quad + A_1\left(\left(\mathbf{u}^{n-1}_{\epsilon\mu}, \mathbf{B}^{n-1}_{\epsilon\mu}\right), \left(\mathbf{u}^n_{\epsilon\mu}, \mathbf{B}^n_{\epsilon\mu}\right), (\mathbf{v}, \boldsymbol{\Psi})\right) = \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi})\rangle. \end{aligned} \tag{32}$$

Here, $((\mathbf{u}^0_{\epsilon\mu}, \mathbf{B}^0_{\epsilon\mu}), p^0_{\epsilon\mu})$ is defined by the discrete penalty equation:

$$\begin{aligned} &A_0\left(\left(\mathbf{u}^0_{\epsilon\mu}, \mathbf{B}^0_{\epsilon\mu}\right), (\mathbf{v}, \boldsymbol{\Psi})\right) - d\left((\mathbf{v}, \boldsymbol{\Psi}), p^0_{\epsilon\mu}\right) \\ &\quad + d\left(\left(\mathbf{u}^0_{\epsilon\mu}, \mathbf{B}^0_{\epsilon\mu}\right), q\right) + \tfrac{\epsilon}{\nu_e}\left(p^0_{\epsilon\mu}, q\right) = \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi})\rangle, \end{aligned} \tag{33}$$

for all $((\mathbf{v}, \boldsymbol{\Psi}), q) \in \mathbf{W}^{\mu}_{0n} \times \mathrm{M}_{\mu}$.

On the basis of our previous work [19,20], we have the following stability of the iterative method for $(\mathbf{e}^n, \mathbf{b}^n) = (\mathbf{u}_{\epsilon\mu} - \mathbf{u}^n_{\epsilon\mu}, \mathbf{B}_{\epsilon\mu} - \mathbf{B}^n_{\epsilon\mu})$ and $\eta^n = p_{\epsilon\mu} - p^n_{\epsilon\mu}$ for $n \geq 0$.

Theorem 4.1 *Under the assumptions of Theorem 3.2 and suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are valid, if $0 < \sigma \leq \frac{2}{5}$, then $(\boldsymbol{u}^m_{\epsilon\mu}, \boldsymbol{B}^m_{\epsilon\mu})$ and $p^m_{\epsilon\mu}$ defined by the Method 1 satisfy*

$$\begin{aligned} &|||\left(\boldsymbol{u}^m_{\epsilon\mu}, \boldsymbol{B}^m_{\epsilon\mu}\right)|||_1 \leq \tfrac{6}{5}\|\boldsymbol{F}\|_{-1}, \quad |||\left(\mathcal{A}_{1\mu}\boldsymbol{u}^m_{\epsilon\mu}, \mathcal{A}_{2\mu}\boldsymbol{B}^m_{\epsilon\mu}\right)|||_1 \leq C\|\boldsymbol{F}\|_0, \\ &\|p^m_{\epsilon\mu}\|_0 \leq \left(\tfrac{29\nu_e}{20\epsilon\underline{\nu}}\right)^{\frac{1}{2}}\|\boldsymbol{F}\|_{-1}, \ for\ \mathcal{P}_1, \ \|p^m_{\epsilon\mu}\|_0 \leq \beta_0^{-1}\left(\tfrac{6\overline{\nu}}{5\underline{\nu}} + \tfrac{8}{5}\right)\|\boldsymbol{F}\|_{-1}, \ for\ \mathcal{P}_2, \end{aligned} \tag{34}$$

and $(\boldsymbol{e}^m, \boldsymbol{b}^m)$, η^m satisfy the following bounds:

$$\begin{aligned} &|||\left(\boldsymbol{e}^m, \boldsymbol{b}^m\right)|||_1 \leq \left(\tfrac{11}{5}\sigma\right)^m \tfrac{2}{5}\|\boldsymbol{F}\|_{-1}, \\ &\|\eta^m\|_0 \leq \left(\tfrac{2\nu_e}{5\epsilon\underline{\nu}}\right)^{\frac{1}{2}}\left(\tfrac{11}{5}\sigma\right)^m\|\boldsymbol{F}\|_{-1}, \ for\ \mathcal{P}_1, \\ &\|\eta^m\|_0 \leq \beta_0^{-1}\left(\tfrac{\overline{\nu}}{\underline{\nu}} + 1\right)\tfrac{2}{5}\left(\tfrac{11}{5}\sigma\right)^m\|\boldsymbol{F}\|_{-1}, \ for\ \mathcal{P}_2, \end{aligned} \tag{35}$$

for all $m \geq 0$; if $0 < \sigma \leq \frac{5}{11}$, then $(\boldsymbol{u}^m_{\epsilon\mu}, \boldsymbol{B}^m_{\epsilon\mu})$ and $p^m_{\epsilon\mu}$ defined by the Method 2 satisfy

$$\begin{aligned} &|||\left(\boldsymbol{u}^m_{\epsilon\mu}, \boldsymbol{B}^m_{\epsilon\mu}\right)|||_1 \leq \tfrac{4}{3}\|\boldsymbol{F}\|_{-1}, \quad |||\left(\mathcal{A}_{1\mu}\boldsymbol{u}^m_{\epsilon\mu}, \mathcal{A}_{2\mu}\boldsymbol{B}^m_{\epsilon\mu}\right)|||_1 \leq C\|\boldsymbol{F}\|_0, \\ &\|p^m_{\epsilon\mu}\|_0 \leq \left(\tfrac{9\nu_e}{5\epsilon\underline{\nu}}\right)^{\frac{1}{2}}\|\boldsymbol{F}\|_{-1}, \ for\ \mathcal{P}_1, \ \ \|p^m_{\epsilon\mu}\|_0 \leq \left(\tfrac{4\overline{\nu}}{3\underline{\nu}} + \tfrac{17}{10}\right)\|\boldsymbol{F}\|_{-1}, \ for\ \mathcal{P}_2 \end{aligned} \tag{36}$$

and $(\boldsymbol{e}^m, \boldsymbol{b}^m)$, η^m satisfy the following bounds:

$$\begin{aligned} &|||\left(\boldsymbol{e}^m, \boldsymbol{b}^m\right)|||_1 \leq \left(\tfrac{33}{13}\sigma\right)^{2^m-1}\tfrac{5}{11}\|\boldsymbol{F}\|_{-1}, \\ &\|\eta^m\|_0 \leq \left(\tfrac{\nu_e}{\epsilon\overline{\nu}}\right)^{\frac{1}{2}}\sigma^2\left(\tfrac{33}{13}\sigma\right)^{2^m-\frac{3}{2}}\left(\tfrac{5}{11}\right)^{\frac{3}{2}}\|\boldsymbol{F}\|_{-1}, \ for\ \mathcal{P}_1, \\ &\|\eta^m\|_0 \leq \beta_0^{-1}\left(\tfrac{5\underline{\nu}}{11\overline{\nu}} + 3\sigma^2\right)\left(\tfrac{33}{13}\sigma\right)^{2^m-1}\|\boldsymbol{F}\|_{-1}, \ for\ \mathcal{P}_2, \end{aligned} \tag{37}$$

for all $m \geq 0$*; if* $0 < \sigma < 1$*, then* $(\boldsymbol{u}_{\epsilon\mu}^m, \boldsymbol{B}_{\epsilon\mu}^m)$ *and* $p_{\epsilon\mu}^m$ *defined by the Method 3 satisfy*

$$
\begin{aligned}
&||| \left(\boldsymbol{u}_{\epsilon\mu}^m, \boldsymbol{B}_{\epsilon\mu}^m\right) |||_1 \leq \|\boldsymbol{F}\|_{-1}, \quad ||| \left(\mathcal{A}_{1\mu}\boldsymbol{u}_{\epsilon\mu}^m, \mathcal{A}_{2\mu}\boldsymbol{B}_{\epsilon\mu}^m\right) |||_1 \leq C\|\boldsymbol{F}\|_0, \\
&\|p_{\epsilon\mu}^m\|_0 \leq \left(\frac{\nu_e}{\epsilon\underline{\nu}}\right)^{\frac{1}{2}} \|\boldsymbol{F}\|_{-1}, \ for\ \mathcal{P}_1, \ \|p_{\epsilon\mu}^m\|_0 \leq \beta_0^{-1}\left(\frac{\overline{\nu}}{\underline{\nu}}+2\right)\|\boldsymbol{F}\|_{-1}, \ for\ \mathcal{P}_2,
\end{aligned} \tag{38}
$$

and $(\boldsymbol{e}^m, \boldsymbol{b}^m)$ *and* η^m *satisfy the following bounds:*

$$
\begin{aligned}
&||| \left(\boldsymbol{e}^m, \boldsymbol{b}^m\right) |||_1 \leq \sigma^m \|\boldsymbol{F}\|_{-1}, \\
&\|\eta^m\|_0 \leq \left(\frac{\nu_e}{\epsilon\underline{\nu}}\right)^{\frac{1}{2}} \sigma^m \|\boldsymbol{F}\|_{-1}, \ for\ \mathcal{P}_1, \\
&\|\eta^m\|_0 \leq \left(\frac{\overline{\nu}}{\underline{\nu}}+2\right)\sigma^m \|\boldsymbol{F}\|_{-1}, \ for\ \mathcal{P}_2,
\end{aligned} \tag{39}
$$

for all $m \geq 0$.

Remark 4.1 From the formulation, Method 1 is the simplest and Method 2 is the most complicated one. Then, Methods 1, 2 and 3 are stable and convergent in terms of $0 < \sigma \leq 2/5$; Methods 2 and 3 are stable and convergent in terms of $2/5 < \sigma \leq 5/11$; Methods 3 is stable and convergent in terms of $5/11 < \sigma < 1$. Besides, Method 2 has the second order convergence rate and the best precision among them. The stability and error estimation of pressure p for finite element pair $P_1 - P_0 - P_1$ is related to the reciprocals of penalty parameter $\frac{1}{\epsilon}$. But, these estimations for finite element pair $P_1b - P_1 - P_1b$ are independent of ϵ.

4.1 Two-Level Penalty Iterative Methods with $0 < \sigma \leq \frac{2}{5}$

In this section, we consider the two-level penalty finite element methods with $0 < \sigma \leq \frac{2}{5}$. The methods includes two algorithms: m iteration steps by Stokes, Newton, Oseen technique on the coarse mesh H and once correction by the three corresponding iteration on the fine mesh h.

Step I. Find a coarse grid penalty iterative solution $((\mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H}^m), p_{\epsilon H}^m) \in \mathbf{W}_{0n}^H \times \mathrm{M}_H$ defined by Method 1, 2 and 3, respectively.

Step II. Find a fine grid solution $((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), p_{\epsilon mh}) \in \mathbf{W}_{0n}^h \times \mathrm{M}_h$ defined by the following Stokes, Newton and Oseen corrections, respectively.

Correction 1 Find a fine grid solution $((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), p_{\epsilon mh}) \in \mathbf{W}_{0n}^h \times \mathrm{M}_h$ defined by the Stokes correction problem

$$
\begin{aligned}
&A_0\left((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), (\mathbf{v}, \boldsymbol{\Psi})\right) - d\left((\mathbf{v}, \boldsymbol{\Psi}), p_{\epsilon mh}\right) + d\left((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), q\right) \\
&\quad + \frac{\epsilon}{\nu_e}(p_{\epsilon mh}, q) = \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi})\rangle - A_1\left(\left(\mathbf{u}_{\epsilon H}^m, \mathbf{u}_{\epsilon H}^m\right), \left(\mathbf{u}_{\epsilon H}^m, \mathbf{u}_{\epsilon H}^m\right), (\mathbf{v}, \boldsymbol{\Psi})\right),
\end{aligned} \tag{40}
$$

for all $((\mathbf{v}, \boldsymbol{\Psi}), q) \in \mathbf{W}_{0n}^h \times \mathrm{M}_h$.

Correction 2 Find a fine grid solution $((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), p_{\epsilon mh}) \in \mathbf{W}_{0n}^h \times \mathrm{M}_h$ defined by the Newton correction problem

$$
\begin{aligned}
&A_0\left((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), (\mathbf{v}, \boldsymbol{\Psi})\right) - d\left((\mathbf{v}, \boldsymbol{\Psi}), p_{\epsilon mh}\right) + d\left((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), q\right) \\
&+ \frac{\epsilon}{\nu_e}(p_{\epsilon mh}, q) + A_1\left((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), \left(\mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H}^m\right), (\mathbf{v}, \boldsymbol{\Psi})\right) \\
&\quad + A_1\left(\left(\mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H}^m\right), (\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), (\mathbf{v}, \boldsymbol{\Psi})\right) \\
&\quad = \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi})\rangle + A_1\left(\left(\mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H}^m\right), \left(\mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H}^m\right), (\mathbf{v}, \boldsymbol{\Psi})\right).
\end{aligned} \tag{41}
$$

for all $((\mathbf{v}, \boldsymbol{\Psi}), q) \in \mathbf{W}_{0n}^h \times \mathrm{M}_h$.

Correction 3 Find a fine grid solution $((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), p_{\epsilon mh}) \in \mathbf{W}_{0n}^h \times \mathrm{M}_h$ defined by the Oseen correction problem

$$\begin{aligned} &A_0\left((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), (\mathbf{v}, \boldsymbol{\Psi})\right) - d\left((\mathbf{v}, \boldsymbol{\Psi}), p_{\epsilon mh}\right) + d\left((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), q\right) \\ &\quad + \tfrac{\epsilon}{\nu_e}(p_{\epsilon mh}, q) + A_1\left(\left(\mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H}^m\right), (\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), (\mathbf{v}, \boldsymbol{\Psi})\right) = \langle \mathbf{F}, (\mathbf{v}, \boldsymbol{\Psi}) \rangle, \end{aligned} \tag{42}$$

for all $((\mathbf{v}, \boldsymbol{\Psi}), q) \in \mathbf{W}_{0n}^h \times \mathrm{M}_h$.

For the simplicity, we take $(\mathbf{e}_h, \mathbf{b}_h) = (\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon h} - \mathbf{B}_{\epsilon mh})$, $\eta_h = p_{\epsilon h} - p_{\epsilon mh}$. Then, we have the following theorem.

Lemma 4.1 *Under the assumptions of Theorem 4.1 and* $0 < \sigma \le \frac{2}{5}$, *then* $((\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh}), p_{\epsilon mh})$ *provided by Methods 1–3 with Corrections 1 and 3 satisfies the following stability and error estimates:*

$$\begin{aligned} &|||\,(\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh})\,|||_1 \le 2\|\boldsymbol{F}\|_{-1}, \\ &\|p_{\epsilon mh}\|_0 \le \left(\frac{10\nu_e}{\epsilon}\right)^{\frac{1}{2}} \|\boldsymbol{F}\|_{-1},\ for\ \mathcal{P}_1,\quad \|p_{\epsilon mh}\|_0 \le 2\beta_0^{-1}\left(\frac{\overline{\nu}}{\underline{\nu}} + 1\right)\|\boldsymbol{F}\|_{-1},\ for\ \mathcal{P}_2, \end{aligned} \tag{43}$$

and $(\boldsymbol{e}_h, \boldsymbol{b}_h)$, η_h *satisfy the following bounds:*

$$\begin{aligned} &|||\,(\boldsymbol{e}_h, \boldsymbol{b}_h)\,|||_1 \le C\left[\epsilon^{-1} H^2 \left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}} \|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}_{\epsilon H}^m, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}_{\epsilon H}^m\right)|||_1\right], \\ &\|\eta_h\|_0 \le C\left[\epsilon^{\frac{-3}{2}} H^2 \left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}} \|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}_{\epsilon H}^m, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}_{\epsilon H}^m\right)|||_1\right],\ for\ \mathcal{P}_1; \\ &|||\,(\boldsymbol{e}_h, \boldsymbol{b}_h)\,|||_1 \le C\left[H^2 \left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}} \|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}_{\epsilon H}^m, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}_{\epsilon H}^m\right)|||_1\right], \\ &\|\eta_h\|_0 \le C\left[H^2 \left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}} \|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}_{\epsilon H}^m, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}_{\epsilon H}^m\right)|||_1\right],\ for\ \mathcal{P}_2. \end{aligned} \tag{44}$$

Proof The stability estimate (43) and the error estimate obtained by Methods 1–3 with Correction 1 can be obtained by the similar technique used in [20].

Then, we give the error estimate for $((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), p_{\epsilon mh})$ obtained by Methods 1–3 with Correction 3.

And we have the error equation with (29) and (41)

$$\begin{aligned} &A_0\left((\mathbf{e}_h, \mathbf{b}_h), (\mathbf{v}, \boldsymbol{\Psi})\right) - d\left((\mathbf{v}, \boldsymbol{\Psi}), \eta_h\right) + d\left((\mathbf{e}_h, \mathbf{b}_h), q\right) + \tfrac{\epsilon}{\nu_e}(\eta_h, q) \\ &\quad + A_1\left(\left(\mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H}^m\right), (\mathbf{e}_h, \mathbf{b}_h), (\mathbf{v}, \boldsymbol{\Psi})\right) \\ &\quad + A_1\left((\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h} - \mathbf{B}_{\epsilon H}), (\mathbf{u}_{\epsilon h}, \mathbf{B}_{\epsilon h}), (\mathbf{v}, \boldsymbol{\Psi})\right) \\ &\quad + A_1\left(\left(\mathbf{u}_{\epsilon H} - \mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H} - \mathbf{B}_{\epsilon H}^m\right), (\mathbf{u}_{\epsilon h}, \mathbf{B}_{\epsilon h}), (\mathbf{v}, \boldsymbol{\Psi})\right) = 0. \end{aligned} \tag{45}$$

Taking $(\mathbf{v}, \boldsymbol{\Psi}) = (\mathbf{e}_h, \mathbf{b}_h)$, $q = \eta_h$ in (45), together with (9), (11), Lemma 4.1 and Theorem 3.2, we have

$$\begin{aligned} |||\,(\mathbf{e}_h, \mathbf{b}_h)\,|||_1 \le\ & \frac{CN}{\underline{\nu}^2} |||\,(\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h} - \mathbf{B}_{\epsilon H})\,|||_0 |||\,(\mathbf{u}_{\epsilon h}, \mathbf{B}_{\epsilon h})\,|||_0^{\frac{1}{2}} \\ & \times |||\,(\mathcal{A}_{1h}\mathbf{u}_{\epsilon h}, \mathcal{A}_{2h}\mathbf{B}_{\epsilon h})\,|||_0^{\frac{1}{2}} \\ & + \frac{N}{\underline{\nu}^2} |||\left(\mathbf{u}_{\epsilon H} - \mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H} - \mathbf{B}_{\epsilon H}^m\right)|||_1 \\ & \times |||\,(\mathbf{u}_{\epsilon h}, \mathbf{B}_{\epsilon h})\,|||_1 \\ \le\ & C\sigma \left(\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\right)^{\frac{1}{2}} |||\,(\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h} - \mathbf{B}_{\epsilon H})\,|||_0 \\ & + \sigma |||\left(\mathbf{u}_{\epsilon H} - \mathbf{u}_{\epsilon H}^m, \mathbf{B}_{\epsilon H} - \mathbf{B}_{\epsilon H}^m\right)|||_1. \end{aligned} \tag{46}$$

For $\mathcal{P}_1$, from Theorem 3.2 we have

$$\begin{aligned} \|| (\mathbf{e}_h, \mathbf{b}_h) \||_1 \leq C(\epsilon^{-1} H^2 \left(\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\right)^{\frac{1}{2}} \|\mathbf{F}\|_0 \\ + \|| \left(\mathbf{u}_{\epsilon H} - \mathbf{u}^m_{\epsilon H}, \mathbf{B}_{\epsilon H} - \mathbf{B}^m_{\epsilon H}\right) \||_1), \end{aligned} \tag{47}$$

and

$$\begin{aligned} \frac{\epsilon}{\nu_e} \|\eta_h\|_0^2 &\leq \frac{CN}{\underline{\nu}^3} \|| (\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h} - \mathbf{B}_{\epsilon H}) \||_0 \|| (\mathbf{u}_{\epsilon h}, \mathbf{B}_{\epsilon h}) \||_0^{\frac{1}{2}} \\ &\quad \times \|| (\mathcal{A}_{1h} \mathbf{u}_{\epsilon h}, \mathcal{A}_{2h} \mathbf{B}_{\epsilon h}) \||_0^{\frac{1}{2}} \|| (\mathbf{e}_h \mathbf{b}_h) \||_1 \\ &\quad + \frac{N}{\underline{\nu}^3} \|| \left(\mathbf{u}_{\epsilon H} - \mathbf{u}^m_{\epsilon H}, \mathbf{B}_{\epsilon H} - \mathbf{B}^m_{\epsilon H}\right) \||_1 \|| (\mathbf{u}_{\epsilon h}, \mathbf{B}_{\epsilon h}) \||_1 \|| (\mathbf{e}_h, \mathbf{b}_h) \||_1 \\ &\leq C\epsilon^{-2} H^4 \frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}} \|\mathbf{F}\|_0^2 + C \|| \left(\mathbf{u}_{\epsilon H} - \mathbf{u}^m_{\epsilon H}, \mathbf{B}_{\epsilon H} - \mathbf{B}^m_{\epsilon H}\right) \||_1^2, \end{aligned}$$

which implies that

$$\frac{\epsilon}{\nu_e} \|\eta_h\|_0 \leq C \left[\epsilon^{-\frac{3}{2}} H^2 \left(\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\right)^{\frac{1}{2}} \|\mathbf{F}\|_0 + \|| \left(\mathbf{u}_{\epsilon H} - \mathbf{u}^m_{\epsilon H}, \mathbf{B}_{\epsilon H} - \mathbf{B}^m_{\epsilon H}\right) \||_1 \right]. \tag{48}$$

For $\mathcal{P}_2$, it follows from (45)–(46) and (24), we observe that

$$\begin{aligned} \beta_0 \|\eta_h\|_0 &\leq \frac{\overline{\nu}}{\underline{\nu}} \|(\mathbf{e}_h, \mathbf{b}_h)\|_1 + \frac{CN}{\underline{\nu}^2} \|| \left(\mathbf{u}_{\epsilon H} - \mathbf{u}^m_{\epsilon H}, \mathbf{B}_{\epsilon H} - \mathbf{B}^m_{\epsilon H}\right) \||_1 \|| (\mathbf{u}_{\epsilon h}, \mathbf{B}_{\epsilon h}) \||_1 \\ &\quad + \frac{CN}{\underline{\nu}^2} \|| (\mathbf{u}_{\epsilon h} - \mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h} - \mathbf{B}_{\epsilon H}) \||_0 \|| (\mathbf{u}_{\epsilon h}, \mathbf{B}_{\epsilon h}) \||_0^{\frac{1}{2}} \\ &\quad \times \|| (\mathcal{A}_{1h} \mathbf{u}_{\epsilon h}, \mathcal{A}_{2h} \mathbf{B}_{\epsilon h}) \||_0^{\frac{1}{2}} \\ &\leq C \left[H^2 \left(\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\right)^{\frac{1}{2}} \|\mathbf{F}\|_0 + \|| \left(\mathbf{u}_{\epsilon H} - \mathbf{u}^m_{\epsilon H}, \mathbf{B}_{\epsilon H} - \mathbf{B}^m_{\epsilon H}\right) \||_1 \right]. \end{aligned} \tag{49}$$

Then, we complete the proof. □

Lemma 4.2 *Under the assumptions of Theorem 4.1 and $0 < \sigma \leq \frac{2}{5}$, then $((\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh}), p_{\epsilon mh})$ provided by Methods 1–3 with Correction 2 satisfies the following stability and error estimates:*

$$\begin{aligned} &\|| (\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh}) \||_1 \leq 2\|\boldsymbol{F}\|_{-1} + \frac{\sigma}{\|\boldsymbol{F}\|_{-1}} \|| \left(\boldsymbol{u}_{\epsilon mh} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon mh} - \boldsymbol{B}^m_{\epsilon H}\right) \||_1^2, \\ &\|p_{\epsilon mh}\|_0 \leq C \left(\frac{\nu_e}{\underline{\nu}\epsilon}\right)^{\frac{1}{2}} \left[\|\boldsymbol{F}\|_{-1} + \|| \left(\boldsymbol{u}_{\epsilon mh} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon mh} - \boldsymbol{B}^m_{\epsilon H}\right) \||_1^2\right], \ for\ \mathcal{P}_1, \\ &\|p_{\epsilon mh}\|_0 \leq C \left[\|\boldsymbol{F}\|_{-1} + \|| \left(\boldsymbol{u}_{\epsilon mh} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon mh} - \boldsymbol{B}^m_{\epsilon H}\right) \||_1^2\right], \ for\ \mathcal{P}_2, \end{aligned} \tag{50}$$

and $(\boldsymbol{e}_h, \boldsymbol{b}_h)$, η_h satisfy the following bounds:

$$\begin{aligned} &\|| (\boldsymbol{e}_h, \boldsymbol{b}_h) \||_1 \leq C \left[\epsilon^{-\frac{3}{2}} |\ln h| H^3 \frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}} \|\boldsymbol{F}\|_0 + \|| \left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right) \||_1^2\right], \\ &\|\eta_h\|_0 \leq C \left[\epsilon^{-2} |\ln h| H^3 \frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}} \|\boldsymbol{F}\|_0 + \|| \left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right) \||_1^2\right], \ for\ \mathcal{P}_1; \\ &\|| (\boldsymbol{e}_h, \boldsymbol{b}_h) \||_1 \leq C \left[|\ln h| H^3 \frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}} \|\boldsymbol{F}\|_0 + \|| \left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right) \||_1^2\right], \\ &\|\eta_h\|_0 \leq C \left[|\ln h| H^3 \frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}} \|\boldsymbol{F}\|_0 + \|| \left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right) \||_1^2\right], \ for\ \mathcal{P}_2, \end{aligned} \tag{51}$$

in the 2D case, and

$$
\begin{aligned}
&|||\,(\boldsymbol{e}_h,\boldsymbol{b}_h)\,|||_1 \le C\left[\epsilon^{-\frac{5}{4}}H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\\
&\|\eta_h\|_0 \le C\left[\epsilon^{-\frac{7}{4}}H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\ for\ \mathcal{P}_1;\\
&|||\,(\boldsymbol{e}_h,\boldsymbol{b}_h)\,|||_1 \le C\left[H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\\
&\|\eta_h\|_0 \le C\left[H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\ for\ \mathcal{P}_2,
\end{aligned}
\tag{52}
$$

in the 3D case.

Proof Let $(\mathbf{v},\boldsymbol{\Psi}) = (\mathbf{u}_{\epsilon mh},\mathbf{B}_{\epsilon mh})$ and $q = p_{\epsilon mh}$ in (41), we derive from (9) and (10) that

$$
\begin{aligned}
&\underline{\nu}\|(\mathbf{u}_{\epsilon mh},\mathbf{B}_{\epsilon mh})\|_1^2 + \frac{\epsilon}{\nu_e}\|p_{\epsilon mh}\|_0^2\\
&\quad \le N\|(\mathbf{u}^m_{\epsilon H},\mathbf{B}^m_{\epsilon H})\|_1\|(\mathbf{u}_{\epsilon mh}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon mh}-\mathbf{B}^m_{\epsilon H})\|_1\|(\mathbf{u}_{\epsilon mh},\mathbf{B}_{\epsilon mh})\|_1\\
&\quad + \|\mathbf{F}\|_{-1}\|(\mathbf{u}_{\epsilon mh},\mathbf{B}_{\epsilon mh})\|_1,
\end{aligned}
\tag{53}
$$

which implies that

$$
\begin{aligned}
|||\,(\mathbf{u}_{\epsilon mh},\mathbf{B}_{\epsilon mh})\,|||_1 &\le \frac{N}{\underline{\nu}^2}|||\left(\mathbf{u}^m_{\epsilon H},\mathbf{B}^m_{\epsilon H}\right)|||_1|||\left(\mathbf{u}_{\epsilon mh}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon mh}-\mathbf{B}^m_{\epsilon H}\right)\|_1\\
&\quad + \|\mathbf{F}\|_{-1}\\
&\le \frac{N}{4\underline{\nu}^2}|||\left(\mathbf{u}^m_{\epsilon H},\mathbf{B}^m_{\epsilon H}\right)|||_1^2\\
&\quad + \frac{N}{\underline{\nu}^2}|||\left(\mathbf{u}_{\epsilon mh}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon mh}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2 + \|\mathbf{F}\|_{-1}\\
&\le 2\|\mathbf{F}\|_{-1} + \frac{\sigma}{\|\mathbf{F}\|_{-1}}|||\left(\mathbf{u}_{\epsilon mh}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon mh}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2.
\end{aligned}
\tag{54}
$$

For $\mathcal{P}_1$, from (53), we have

$$
\begin{aligned}
\frac{\epsilon}{\nu_e}\|p_{\epsilon mh}\|_0^2 &\le \frac{N}{4\underline{\nu}^3}|||\left(\mathbf{u}^m_{\epsilon H},\mathbf{B}^m_{\epsilon H}\right)|||_1|||\left(\mathbf{u}_{\epsilon mh}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon mh}-\mathbf{B}^m_{\epsilon H}\right)\|_1\\
&\quad \times|||\,(\mathbf{u}_{\epsilon mh},\mathbf{B}_{\epsilon mh})\,|||_1 + \frac{1}{\underline{\nu}}\|\mathbf{F}\|_{-1}|||\,(\mathbf{u}_{\epsilon mh},\mathbf{B}_{\epsilon mh})\,|||_1\\
&\le C\frac{1}{\underline{\nu}}|||\left(\mathbf{u}_{\epsilon mh}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon mh}-\mathbf{B}^m_{\epsilon H}\right)\|_1^4 + C\frac{1}{\underline{\nu}}\|\mathbf{F}\|_{-1}^2,
\end{aligned}
\tag{55}
$$

which is that

$$
\|p_{\epsilon mh}\|_0 \le C\left(\frac{\nu_e}{\underline{\nu}\epsilon}\right)^{\frac{1}{2}}\left[\|\mathbf{F}\|_{-1} + |||\left(\mathbf{u}_{\epsilon mh}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon mh}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2\right].
\tag{56}
$$

Apply the similar technique used in Lemma 4.1, we have the following estimate for $\mathcal{P}_2$

$$
\begin{aligned}
\|p_{\epsilon mh}\|_0 &\le \frac{\overline{\nu}}{\underline{\nu}}|||\,(\mathbf{u}_{\epsilon mh},\mathbf{B}_{\epsilon mh})\,|||_1 + \frac{N}{\underline{\nu}^2}|||\left(\mathbf{u}^m_{\epsilon H},\mathbf{B}^m_{\epsilon H}\right)|||_1|||\,(\mathbf{u}_{\epsilon mh},\mathbf{B}_{\epsilon mh})\,|||_1\\
&\quad + \frac{N}{\underline{\nu}^2}|||\left(\mathbf{u}^m_{\epsilon H},\mathbf{B}^m_{\epsilon H}\right)|||_1|||\left(\mathbf{u}_{\epsilon mh}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon mh}-\mathbf{B}^m_{\epsilon H}\right)|||_1 + \|\mathbf{F}\|_{-1}\\
&\le C\left[\|\mathbf{F}\|_{-1} + |||\left(\mathbf{u}_{\epsilon mh}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon mh}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2\right].
\end{aligned}
\tag{57}
$$

Next, we will give the error estimate.

Subtracting (41) from (29) with $\mu = h$, we have

$$
\begin{aligned}
&A_0\left((\mathbf{e}_h,\mathbf{b}_h),(\mathbf{v},\boldsymbol{\Psi})\right) - d\left((\mathbf{v},\boldsymbol{\Psi}),\eta_h\right) + d\left((\mathbf{e}_h,\mathbf{b}_h),q\right) + \frac{\epsilon}{\nu_e}(\eta_h,q)\\
&\quad + A_1\left(\left(\mathbf{u}^m_{\epsilon H},\mathbf{B}^m_{\epsilon H}\right),(\mathbf{e}_h,\mathbf{b}_h),(\mathbf{v},\boldsymbol{\Psi})\right) + A_1\left((\mathbf{e}_h,\mathbf{b}_h),\left(\mathbf{u}^m_{\epsilon H},\mathbf{B}^m_{\epsilon H}\right),(\mathbf{v},\boldsymbol{\Psi})\right)\\
&\quad + A_1\left(\left(\mathbf{u}_{\epsilon h}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}^m_{\epsilon H}\right),\left(\mathbf{u}_{\epsilon h}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}^m_{\epsilon H}\right),(\mathbf{v},\boldsymbol{\Psi})\right) = 0.
\end{aligned}
\tag{58}
$$

Taking $(\mathbf{v}, \boldsymbol{\Psi}) = (\mathbf{e}_h, \mathbf{b}_h)$, $q = \eta_h$ in (58), with (9), (10), (15) and Theorem 3.1, we have

$$\begin{aligned}
&\tfrac{13}{25}\underline{\nu}\|(\mathbf{e}_h, \mathbf{b}_h)\|_1^2 + \tfrac{\epsilon}{\nu_e}\|\eta_h\|_0^2 \le \underline{\nu}(1-\tfrac{6}{5}\sigma)\|(\mathbf{e}_h, \mathbf{b}_h)\|_1^2 + \tfrac{\epsilon}{\nu_e}\|\eta_h\|_0^2 \\
&\le CN\|(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\|_1^{\frac{3}{2}}\|(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\|_0^{\frac{1}{2}}\|(\mathbf{e}_h, \mathbf{b}_h)\|_1 \\
&\quad +CN\|(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\|_1^{\frac{1}{2}}\|(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\|_0^{\frac{1}{2}} \\
&\quad \times\|(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\|_1\ \|(\mathbf{e}_h, \mathbf{b}_h)\|_1 \\
&\quad +N\|(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\|_1^2\|(\mathbf{e}_h, \mathbf{b}_h)\|_1,
\end{aligned} \tag{59}$$

which is that

$$\begin{aligned}
|||\,(\mathbf{e}_h, \mathbf{b}_h)\,|||_1 \le\ & \tfrac{C\sigma}{\|\mathbf{F}\|_{-1}}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1^{\frac{3}{2}} \\
&\times|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_0^{\frac{1}{2}} \\
&+\tfrac{C\sigma}{\|\mathbf{F}\|_{-1}}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1 \\
&\times|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_0 \\
&+\tfrac{5}{4}\tfrac{\sigma}{\|\mathbf{F}\|_{-1}}|||\,\left(\mathbf{u}_{\epsilon h}-\mathbf{u}^m_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2.
\end{aligned} \tag{60}$$

For $\mathcal{P}_1$, from (60), Theorems 3.2 and 4.1, we have

$$|||\,(\mathbf{e}_h, \mathbf{b}_h)\,|||_1 \le C\left[\epsilon^{-\frac{5}{4}}H^{\frac{5}{2}}\tfrac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\|\mathbf{F}\|_0 + |||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H}, \mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2\right], \tag{61}$$

and with (59) and (61), we have

$$\begin{aligned}
\tfrac{\epsilon}{\nu_e}\|\eta_h\|_0^2 \le\ & \tfrac{C\sigma}{\|\mathbf{F}\|_{-1}}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1^{\frac{3}{2}} \\
&\times|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_0^{\frac{1}{2}}|||\,(\mathbf{e}_h, \mathbf{b}_h)\,|||_1 \\
&+\tfrac{C\sigma}{\|\mathbf{F}\|_{-1}}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1 \\
&\times|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_0|||\,(\mathbf{e}_h, \mathbf{b}_h)\,|||_1 \\
&+\tfrac{5}{4}\tfrac{\sigma}{\|\mathbf{F}\|_{-1}}|||\,\left(\mathbf{u}_{\epsilon h}-\mathbf{u}^m_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2|||\,(\mathbf{e}_h, \mathbf{b}_h)\,|||_1 \\
\le\ & C\epsilon^{-\frac{5}{2}}H^5\left(\tfrac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\right)^2\|\mathbf{F}\|_0^2 + C|||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H}, \mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^4,
\end{aligned} \tag{62}$$

with some simple calculations, we complete the proof of the estimate for $\mathcal{P}_1$.

And for $\mathcal{P}_2$, from (60), Theorems 3.2 and 4.1, we have

$$|||\,(\mathbf{e}_h, \mathbf{b}_h)\,|||_1 \le C\left[H^{\frac{5}{2}}\tfrac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\|\mathbf{F}\|_0 + |||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H}, \mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2\right], \tag{63}$$

and with (24), (8), Theorems 3.2 and 4.1, we deduce that

$$\begin{aligned}
\beta_0\|\eta_h\|_0 \le\ & \tfrac{\overline{\nu}}{\underline{\nu}}|||\,(\mathbf{e}_h, \mathbf{b}_h)\,|||_1 + \tfrac{N}{\underline{\nu}^2}|||\,(\mathbf{e}_h, \mathbf{b}_h)\,|||_1|||\left(\mathbf{u}^m_{\epsilon h}, \mathbf{B}^m_{\epsilon h}\right)|||_1 \\
&+\tfrac{CN}{\underline{\nu}^2}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1^{\frac{3}{2}}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_0^{\frac{1}{2}} \\
&+\tfrac{CN}{\underline{\nu}^2}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_0 \\
&+\tfrac{5}{4}\tfrac{N}{\underline{\nu}^2}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H}, \mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1^2 \\
\le\ & C\left[H^{\frac{5}{2}}\tfrac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\|\mathbf{F}\|_0 + |||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H}, \mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2\right].
\end{aligned} \tag{64}$$

On the other hand, in the 2D case, applying the inverse inequality

$$\|v_\mu\|_{L^\infty} \le C|\ln\mu|^{\frac{1}{2}}\|v_\mu\|_1, \quad \|\Psi_\mu\|_{L^\infty} \le C|\ln\mu|^{\frac{1}{2}}\|\Psi_\mu\|_1, \tag{65}$$

$$\forall v_\mu \in \mathbf{X}_\mu, \Psi_\mu \in \mathbf{W}_\mu,\ \mu = h, H, \tag{66}$$

in the estimate of the trilinear term, from (64), Theorems 3.2 and 4.1, we derive that

$$\begin{aligned}\tfrac{13}{25}|||\,(\mathbf{e}_h,\mathbf{b}_h)\,|||_1 \le\ & \tfrac{C\sigma}{\|\mathbf{F}\|_{-1}}|\ln h|^{\frac{1}{2}}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1\\ &\times|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_0\\ &+\tfrac{C\sigma}{\|\mathbf{F}\|_{-1}}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1\\ &\times|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_0\\ &+\tfrac{5}{4}\tfrac{\sigma}{\|\mathbf{F}\|_{-1}}|||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2.\end{aligned}\tag{67}$$

For $\mathcal{P}_1$, with (67), Theorems 3.2 and 4.1, we have

$$|||\,(\mathbf{e}_h,\mathbf{b}_h)\,|||_1 \le C(\epsilon^{-\frac{3}{2}}|\ln h|H^3\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\|\mathbf{F}\|_0 \tag{68}$$

$$+|||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2), \tag{69}$$

and with (68), we have

$$\begin{aligned}\tfrac{\epsilon}{\nu_e}\|\eta_h\|_0^2 \le\ & \tfrac{C\sigma}{\|\mathbf{F}\|_{-1}}|\ln h|^{\frac{1}{2}}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1\\ &\times|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_0\|(\mathbf{e}_h,\mathbf{b}_h)\|_1\\ &+|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1\\ &\times|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_0\|(\mathbf{e}_h,\mathbf{b}_h)\|_1\\ &+\tfrac{5}{4}\tfrac{\sigma}{\|\mathbf{F}\|_{-1}}|\ln h|^{\frac{1}{2}}|||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2\|(\mathbf{e}_h,\mathbf{b}_h)\|_1\\ \le\ & C\tfrac{\|\mathbf{F}\|_0^2}{\|\mathbf{F}\|_{-1}^2}\|\mathbf{F}\|_0^2|\ln h|\epsilon^{-3}H^6+C|||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^4,\end{aligned}$$

which is that

$$\|\eta_h\|_0 \le C\left[|\ln h|\epsilon^{-2}H^3\tfrac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\|\mathbf{F}\|_0+|||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2\right].$$

And for $\mathcal{P}_2$, from Theorems 3.2, 4.1 and (67), we have

$$\begin{aligned}|||\,(\mathbf{e}_h,\mathbf{b}_h)\,|||_1 \le C\Big(&H^3|\ln h|\tfrac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\|\mathbf{F}\|_0\\ &+|||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2\Big),\end{aligned}\tag{70}$$

and with (24), (8), (70), Theorems 3.2 and 4.1, we deduce that

$$\begin{aligned}\beta_0\|\eta_h\|_0 \le\ & \overline{\nu}\|(\mathbf{e}_h,\mathbf{b}_h)\|_1+\tfrac{C\sigma}{\|\mathbf{F}\|_{-1}}|\ln h|^{\frac{1}{2}}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1\\ &\times|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_0\\ &+\tfrac{C\sigma}{\|\mathbf{F}\|_{-1}}|\ln h|^{\frac{1}{2}}|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_1\\ &\times|||\,(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})\,|||_0\\ &+\tfrac{5}{4}\tfrac{\sigma}{\|\mathbf{F}\|_{-1}}|\ln h|^{\frac{1}{2}}|||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2\\ \le\ & C\left[H^3|\ln h|\tfrac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\|\mathbf{F}\|_0+|||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2\right].\end{aligned}$$

Then, we complete the proof. □

Theorem 4.2 *Under the assumptions of Theorem 4.1, $((\boldsymbol{u}_{\epsilon mh},\boldsymbol{B}_{\epsilon mh}),p_{\epsilon mh})$ provided by Methods 1–3 with Corrections 1 and 3 satisfies the following error estimates for $\mathcal{P}_1$*

$$\begin{aligned}&|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh},\boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1 \le C\left[\epsilon+\epsilon^{-\frac{1}{2}}\left(h+\epsilon^{-\frac{1}{2}}H^2\left(\tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\right)\right]\|\boldsymbol{F}\|_0\\ &\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1,\\ &\|p-p_{\epsilon mh}\|_0 \le C\left[\epsilon+\epsilon^{-1}\left(h+\epsilon^{-\frac{1}{2}}H^2\left(\tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\right)\right]\|\boldsymbol{F}\|_0\\ &\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1,\end{aligned}$$

ϵ and H can be taken as $\epsilon = O(h^{\frac{1}{2}})$, $H^2 = O(\epsilon^{\frac{1}{2}} h)$ and the convergence rate is $O(h^{\frac{1}{2}})$; for $\mathcal{P}_2$, the optimal error estimates are

$$\begin{aligned}&|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh},\boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1+\|p-p_{\epsilon mh}\|_0\leq C\left(\epsilon+h+H^2\left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\right)\|\boldsymbol{F}\|_0\\&\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1,\end{aligned}$$

ϵ and H can be taken as $\epsilon = O(h)$, $H^2 = O(h)$ and the convergence rate is $O(h)$.

And $((\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh}), p_{\epsilon mh})$ provided by Methods 1–3 with Correction 2 satisfies the following error estimates, for $\mathcal{P}_1$

$$\begin{aligned}&|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh},\boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1\leq C\left[\epsilon+\epsilon^{-\frac{1}{2}}\left(h+\epsilon^{-1}|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)\right]\|\boldsymbol{F}\|_0\\&\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\\&\|p-p_{\epsilon mh}\|_0\leq C\left[\epsilon+\epsilon^{-1}\left(h+\epsilon^{-1}|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)\right]\|\boldsymbol{F}\|_0\\&\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\end{aligned}$$

ϵ and H can be taken as $\epsilon = O(h^{\frac{1}{2}})$, $H^3 = O(\epsilon h|\ln h|^{-1})$ and the convergence rate is $O(h^{\frac{1}{2}})$ in the 2D case; for $\mathcal{P}_2$, the optimal error estimates are

$$\begin{aligned}&|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh},\boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1+\|p-p_{\epsilon mh}\|_0\leq C\left[\epsilon+h+|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right]\|\boldsymbol{F}\|_0\\&\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\end{aligned}$$

ϵ and H can be taken as $\epsilon = O(h)$, $H^3 = O(h|\ln h|^{-1})$ and the convergence rate is $O(h)$ in the 2D case; and for $\mathcal{P}_1$

$$\begin{aligned}&|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh},\boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1\leq C\left[\epsilon+\epsilon^{-\frac{1}{2}}\left(h+\epsilon^{-\frac{3}{4}}H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)\right]\|\boldsymbol{F}\|_0\\&\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\\&\|p-p_{\epsilon mh}\|_0\leq C\left[\epsilon+\epsilon^{-1}\left(h+\epsilon^{-\frac{3}{4}}H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)\right]\|\boldsymbol{F}\|_0\\&\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\end{aligned}$$

ϵ and H can be taken as $\epsilon = O(h^{\frac{1}{2}})$, $H^{\frac{5}{2}} = O(\epsilon^{\frac{3}{4}} h)$ and the convergence rate is $O(h^{\frac{1}{2}})$ in the 3D case; for $\mathcal{P}_2$, the optimal error estimates are

$$\begin{aligned}&|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh},\boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1+\|p-p_{\epsilon mh}\|_0\leq C\left[\epsilon+h+H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right]\|\boldsymbol{F}\|_0\\&\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\end{aligned}$$

ϵ and H can be taken as $\epsilon = O(h)$, $H^{\frac{5}{2}} = O(h)$ and the convergence rate is $O(h)$ in the 3D case.

Proof We can finish the proof by Theorems 2.3, 3.2, Lemmas 4.1, 4.2, triangle inequality and some simple calculations. □

Furthermore, we can conclude the comparison for the presented methods from Theorem 4.2 as follows:

Remark 4.2 From the Theorem 4.2, we know that Methods 1, 2 and 3 are stable and convergent with $0 < \sigma \leq 2/5$. Accordingly, nine combinations are proposed with Methods i and Corrections j $(i, j = 1, 2, 3)$. And Method 2 with Correction 2 is the better choice for it's high precision and convergence rate in terms of $0 < \sigma \leq 2/5$.

4.2 Two-Level Iterative Penalty Methods with $\frac{2}{5} < \sigma \le \frac{5}{11}$

Here, we consider the two-level methods based on Newton and Oseen iterative solution $((\mathbf{u}^m_{\epsilon H}, \mathbf{B}^m_{\epsilon H}), p^m_{\epsilon H})$ on a coarse grid τ_H and the Stokes, Newton and Oseen correction solution $((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), p_{\epsilon mh})$ on a fine grid τ_h.

Step I. Find a coarse grid iterative solution $((\mathbf{u}^m_{\epsilon H}, \mathbf{B}^m_{\epsilon H}), p^m_{\epsilon H}) \in \mathbf{W}^H_{0n} \times \mathrm{M}_H$ defined by Method 2 and 3, respectively.

Step II. Find a fine grid solution $((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), p_{\epsilon mh}) \in \mathbf{W}^h_{0n} \times \mathrm{M}_h$ defined by Correction 1–3, respectively.

Lemma 4.3 *Under the assumptions of Theorem 4.1 and $\frac{2}{5} < \sigma \le \frac{5}{11}$, then $((\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh}), p_{\epsilon mh})$ provided by Methods 2–3 with Corrections 1 and 3 satisfies the following stability:*

$$
\begin{aligned}
&|||\,(\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh})\,|||_1 \le 2\|\boldsymbol{F}\|_{-1},\\
&\|p_{\epsilon mh}\|_0 \le \left(\frac{11\nu_e}{3\underline{\nu}\epsilon}\right)^{\frac{1}{2}} \|\boldsymbol{F}\|_{-1},\ for\ \mathcal{P}_1,\quad \|p_{\epsilon mh}\|_0 \le 2\beta_0^{-1}\left(\frac{\overline{\nu}}{\underline{\nu}}+1\right)\|\boldsymbol{F}\|_{-1},\ for\ \mathcal{P}_2,
\end{aligned}
$$

and $(\boldsymbol{e}_h, \boldsymbol{b}_h)$, η_h satisfy the following bounds:

$$
\begin{aligned}
&|||\,(\boldsymbol{e}_h, \boldsymbol{b}_h)\,|||_1 \le C\left[\epsilon^{-1}H^2\left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1\right],\\
&\|\eta_h\|_0 \le C\left[\epsilon^{\frac{-3}{2}}H^2\left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1\right],\ for\ \mathcal{P}_1;\\
&|||\,(\boldsymbol{e}_h, \boldsymbol{b}_h)\,|||_1 \le C\left[H^2\left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1\right],\\
&\|\eta_h\|_0 \le C\left[H^2\left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1\right],\ for\ \mathcal{P}_2.
\end{aligned}
$$

Proof The proof is completed by a similar procedure to that of Lemma 4.1. □

Lemma 4.4 *Under the assumptions of Theorem 4.1 and $\frac{2}{5} < \sigma \le \frac{5}{11}$, then $((\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh}), p_{\epsilon mh})$ provided by Methods 2–3 with Correction 2 satisfies the following stability and error estimates:*

$$
\begin{aligned}
&|||\,(\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh})\,|||_1 \le 2\|\boldsymbol{F}\|_{-1} + \frac{\sigma}{\|\boldsymbol{F}\|_{-1}}|||\left(\boldsymbol{u}_{\epsilon mh}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon mh}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\\
&\|p_{\epsilon mh}\|_0 \le C\left(\frac{\nu_e}{\underline{\nu}\epsilon}\right)^{\frac{1}{2}}\left[\|\boldsymbol{F}\|_{-1} + |||\left(\boldsymbol{u}_{\epsilon mh}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon mh}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\ for\ \mathcal{P}_1,\\
&\|p_{\epsilon mh}\|_0 \le C\left[\|\boldsymbol{F}\|_{-1} + |||\left(\boldsymbol{u}_{\epsilon mh}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon mh}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\ for\ \mathcal{P}_2,
\end{aligned}
$$

and $(\boldsymbol{e}_h, \boldsymbol{b}_h)$, η_h satisfy the following bounds:

$$
\begin{aligned}
&|||\,(\boldsymbol{e}_h, \boldsymbol{b}_h)\,|||_1 \le C\left[\epsilon^{-\frac{3}{2}}|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\\
&\|\eta_h\|_0 \le C\left[\epsilon^{-2}|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\ for\ \mathcal{P}_1;\\
&|||\,(\boldsymbol{e}_h, \boldsymbol{b}_h)\,|||_1 \le C\left[|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\\
&\|\eta_h\|_0 \le C\left[|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\ for\ \mathcal{P}_2,
\end{aligned}
$$

in the 2D case, and

$$\begin{aligned}
&|||\,(\boldsymbol{e}_h, \boldsymbol{b}_h)\,|||_1 \le C\left[\epsilon^{-\frac{5}{4}} H^{\frac{5}{2}} \tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}} \|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\\
&\|\eta_h\|_0 \le C\left[\epsilon^{-\frac{7}{4}} H^{\frac{5}{2}} \tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}} \|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right], \ for\ \mathcal{P}_1;\\
&|||\,(\boldsymbol{e}_h, \boldsymbol{b}_h)\,|||_1 \le C\left[H^{\frac{5}{2}} \tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}} \|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\\
&\|\eta_h\|_0 \le C\left[H^{\frac{5}{2}} \tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}} \|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right], \ for\ \mathcal{P}_2,
\end{aligned}$$

in the 3D case.

Proof Refer to the proof of Lemma 4.2, we can finish the proof. □

Theorem 4.3 *Under the assumptions of Theorem 4.3,* $((\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh}), p_{\epsilon mh})$ *provided by Methods 2–3 with Corrections 1 and 3 satisfies the following error estimates, for* $\mathcal{P}_1$

$$\begin{aligned}
&|||\,(\boldsymbol{u} - \boldsymbol{u}_{\epsilon mh}, \boldsymbol{B} - \boldsymbol{B}_{\epsilon mh})\,|||_1 \le C\left[\epsilon + \epsilon^{-\frac{1}{2}}\left(h + \epsilon^{-\frac{1}{2}} H^2 \left(\tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\right)\right]\|\boldsymbol{F}\|_0\\
&\quad + C|||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right)|||_1,\\
&\|p - p_{\epsilon mh}\|_0 \le C\left[\epsilon + \epsilon^{-1}\left(h + \epsilon^{-\frac{1}{2}} H^2 \left(\tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\right)\right]\|\boldsymbol{F}\|_0\\
&\quad + C|||(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H})|||_1,
\end{aligned}$$

ϵ *and* H *can be taken as* $\epsilon = O(h^{\frac{1}{2}})$, $H^2 = O(\epsilon^{\frac{1}{2}} h)$ *and the convergence rate is* $O(h^{\frac{1}{2}})$*; for* $\mathcal{P}_2$*, the optimal error estimates are*

$$\begin{aligned}
|||\,(\boldsymbol{u} - \boldsymbol{u}_{\epsilon mh}, \boldsymbol{B} - \boldsymbol{B}_{\epsilon mh})\,|||_1 + \|p - p_{\epsilon mh}\|_0 &\le C\left(\epsilon + h + H^2\left(\tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\right)\|\boldsymbol{F}\|_0\\
&\quad + C|||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right)|||_1,
\end{aligned}$$

ϵ *and* H *can be taken as* $\epsilon = O(h)$, $H^2 = O(h)$ *and the convergence rate is* $O(h)$*;*

And $((\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh}), p_{\epsilon mh})$ *provided by Methods 2–3 with Correction 2 satisfies the following error estimates, for* $\mathcal{P}_1$

$$\begin{aligned}
&|||\,(\boldsymbol{u} - \boldsymbol{u}_{\epsilon mh}, \boldsymbol{B} - \boldsymbol{B}_{\epsilon mh})\,|||_1 \le C\left[\epsilon + \epsilon^{-\frac{1}{2}}\left(h + \epsilon^{-1}|\ln h| H^3 \tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)\right]\|\boldsymbol{F}\|_0\\
&\quad + C|||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\\
&\|p - p_{\epsilon mh}\|_0 \le C\left[\epsilon + \epsilon^{-1}\left(h + \epsilon^{-1}|\ln h| H^3 \tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)\right]\|\boldsymbol{F}\|_0\\
&\quad + C|||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,
\end{aligned}$$

ϵ *and* H *can be taken as* $\epsilon = O(h^{\frac{1}{2}})$, $H^3 = O(\epsilon h |\ln h|^{-1})$ *and the convergence rate is* $O(h^{\frac{1}{2}})$ *in the 2D case; for* $\mathcal{P}_2$*, the optimal error estimates are*

$$\begin{aligned}
&|||\,(\boldsymbol{u} - \boldsymbol{u}_{\epsilon mh}, \boldsymbol{B} - \boldsymbol{B}_{\epsilon mh})\,|||_1 + \|p - p_{\epsilon mh}\|_0 \le C\left[\epsilon + h + |\ln h| H^3 \tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right]\|\boldsymbol{F}\|_0\\
&\quad + C|||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,
\end{aligned}$$

ϵ *and* H *can be taken as* $\epsilon = O(h)$, $H^3 = O(h|\ln h|^{-1})$ *and the convergence rate is* $O(h)$ *in the 2D case; and*

$$\begin{aligned}
&|||\,(\boldsymbol{u} - \boldsymbol{u}_{\epsilon mh}, \boldsymbol{B} - \boldsymbol{B}_{\epsilon mh})\,|||_1 \le C\left[\epsilon + \epsilon^{-\frac{1}{2}}\left(h + \epsilon^{-\frac{3}{4}} H^{\frac{5}{2}} \tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)\right]\|\boldsymbol{F}\|_0\\
&\quad + C|||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\\
&\|p - p_{\epsilon mh}\|_0 \le C\left[\epsilon + \epsilon^{-1}\left(h + \epsilon^{-\frac{3}{4}} H^{\frac{5}{2}} \tfrac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)\right]\|\boldsymbol{F}\|_0\\
&\quad + C|||\left(\boldsymbol{u}_{\epsilon H} - \boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H} - \boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,
\end{aligned}$$

ϵ and H can be taken as $\epsilon = O(h^{\frac{1}{2}})$, $H^{\frac{5}{2}} = O(\epsilon^{\frac{3}{4}}h)$ and the convergence rate is $O(h^{\frac{1}{2}})$ in the 3D case; for $\mathcal{P}_2$, the optimal error estimates are

$$\begin{aligned}|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1 + \|p-p_{\epsilon mh}\|_0 \le C\left[\epsilon + h + H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right]\|\boldsymbol{F}\|_0 \\ + C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1,\end{aligned}$$

ϵ and H can be taken as $\epsilon = O(h)$, $H^{\frac{5}{2}} = O(h)$ and the convergence rate is $O(h)$ in the 3D case.

Proof We can finish the proof by Theorems 2.3, 3.2, Lemmas 4.3, 4.4, triangle inequality and some simple calculations. □

Remark 4.3 From the Theorem 4.3, we know that Methods 2 and 3 are stable and convergent with $2/5 < \sigma \le 5/11$. Accordingly, six combinations are proposed with Methods i and Corrections j ($i = 2, 3$; $j = 1, 2, 3$). And Method 2 with Correction 2 is the better choice for it's high precision and convergence rate.

4.3 Two-Level Iterative Penalty Methods with $\frac{5}{11} < \sigma \le 1 - \left(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0}\right)^{\frac{1}{2}}$

Here, we consider the two-level methods based on the iterative solution $((\mathbf{u}^m_{\epsilon H}, \mathbf{B}^m_{\epsilon H}), p^m_{\epsilon H})$ on a coarse grid τ_H and the Stokes, Newton and Oseen correction solution $((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), p_{\epsilon mh})$ on a fine grid τ_h.

Step I. Find a coarse grid iterative solution $((\mathbf{u}^m_{\epsilon H}, \mathbf{B}^m_{\epsilon H}), p^m_{\epsilon H}) \in \mathbf{W}^H_{0n} \times \mathrm{M}_H$ defined by Method 3.

Step II. Find a fine grid solution $((\mathbf{u}_{\epsilon mh}, \mathbf{B}_{\epsilon mh}), p_{\epsilon mh}) \in \mathbf{W}^h_{0n} \times \mathrm{M}_h$ defined by the Correction 1–3, respectively.

Lemma 4.5 *Under the assumptions of Theorem 4.1 and $\frac{5}{11} < \sigma \le 1 - \left(\frac{\|\boldsymbol{F}\|_{-1}}{\|\boldsymbol{F}\|_0}\right)^{\frac{1}{2}}$, then $((\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh}), p_{\epsilon mh})$ provided by Method 3 with Corrections 1 and 3 satisfies the following stability and error estimates:*

$$\begin{aligned}&|||\,(\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh})\,|||_1 \le 2\|\boldsymbol{F}\|_{-1},\\ &\|p_{\epsilon mh}\|_0 \le \left(\frac{4\nu_e}{\underline{\nu}\epsilon}\right)^{\frac{1}{2}}\|\boldsymbol{F}\|_{-1},\ for\ \mathcal{P}_1,\quad \|p_{\epsilon mh}\|_0 \le 2\beta_0^{-1}\left(\frac{\overline{\nu}}{\underline{\nu}}+1\right)\|\boldsymbol{F}\|_{-1},\ for\ \mathcal{P}_2,\end{aligned} \tag{71}$$

and $(\boldsymbol{e}_h, \boldsymbol{b}_h)$, η_h, satisfy the following bounds:

$$\begin{aligned}&|||\,(\boldsymbol{e}_h, \boldsymbol{b}_h)\,|||_1 \le C\left[\epsilon^{-1}H^2\left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1\right],\\ &\|\eta_h\|_0 \le C\left[\epsilon^{\frac{-3}{2}}H^2\left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1\right],\ for\ \mathcal{P}_1;\\ &|||\,(\boldsymbol{e}_h, \boldsymbol{b}_h)\,|||_1 \le C\left[H^2\left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1\right],\\ &\|\eta_h\|_0 \le C\left[H^2\left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\|\boldsymbol{F}\|_0 + |||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1\right],\ for\ \mathcal{P}_2.\end{aligned} \tag{72}$$

Proof The stability can be derive by the same technique used in Lemma 4.1.

Next, we give the error estimate for $(\mathbf{e}_h, \mathbf{b}_h)$, η_h provided by Methods 3 with Correction 1. Refer to [20], we have

$$\begin{aligned}\frac{1}{\underline{\nu}}|||(\mathbf{e}_h,\mathbf{b}_h)|||_1^2+\frac{\epsilon}{\nu_e}\|\eta_h\|_0^2 &\le \frac{CN}{\underline{\nu}^3}|||(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})|||_0|||(\mathbf{e}_h,\mathbf{b}_h)|||_1\\ &\times(|||(\mathcal{A}_{1H}\mathbf{u}_{\epsilon H},\mathcal{A}_{2H}\mathbf{B}_{\epsilon H})|||_1+|||(\mathcal{A}_{1h}\mathbf{u}_{\epsilon h},\mathcal{A}_{2h}\mathbf{B}_{\epsilon h})|||_1)\\ &+\frac{CN}{\underline{\nu}^3}|||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1|||(\mathbf{e}_h,\mathbf{b}_h)|||_1\\ &\times\left(|||\left(\mathbf{u}^m_{\epsilon H},\mathbf{B}^m_{\epsilon H}\right)|||_1+|||(\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon H})|||_1\right),\end{aligned} \tag{73}$$

which we can get

$$\begin{aligned}(1-\sigma)|||(\mathbf{e}_h,\mathbf{b}_h)|||_1^2 \le \frac{C\sigma}{2}\left(\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\right)^2|||(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})|||_0^2\\ +\frac{C\sigma}{2}|||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1^2,\end{aligned} \tag{74}$$

that is

$$\begin{aligned}|||(\mathbf{e}_h,\mathbf{b}_h)|||_1 \le C\sigma\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}|||(\mathbf{u}_{\epsilon h}-\mathbf{u}_{\epsilon H},\mathbf{B}_{\epsilon h}-\mathbf{B}_{\epsilon H})|||_0\\ +C\sigma|||\left(\mathbf{u}_{\epsilon H}-\mathbf{u}^m_{\epsilon H},\mathbf{B}_{\epsilon H}-\mathbf{B}^m_{\epsilon H}\right)|||_1,\end{aligned} \tag{75}$$

which combines Theorem 3.2 and (24), we can finish the proof of the error estimates. □

Lemma 4.6 *Under the assumptions of Theorem 4.1 and* $\frac{5}{11}<\sigma\le 1-\left(\frac{\|\boldsymbol{F}\|_{-1}}{\|\boldsymbol{F}\|_0}\right)^{\frac{1}{2}}$, *then* $((\boldsymbol{u}_{\epsilon mh},\boldsymbol{B}_{\epsilon mh}),p_{\epsilon mh})$ *provided by Method 3 with Correction 2 satisfies the following stability and error estimates:*

$$\begin{aligned}&|||(\boldsymbol{u}_{\epsilon mh},\boldsymbol{B}_{\epsilon mh})|||_1 \le 2\|\boldsymbol{F}\|_{-1}+\frac{\sigma}{\|\boldsymbol{F}\|_{-1}}|||\left(\boldsymbol{u}_{\epsilon mh}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon mh}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\\ &\|p_{\epsilon mh}\|_0 \le C\left(\frac{\nu_e}{\underline{\nu}\epsilon}\right)^{\frac{1}{2}}\left[\|\boldsymbol{F}\|_{-1}+|||\left(\boldsymbol{u}_{\epsilon mh}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon mh}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\ for\ \mathcal{P}_1,\\ &\|p_{\epsilon mh}\|_0 \le C\left[\|\boldsymbol{F}\|_{-1}+|||\left(\boldsymbol{u}_{\epsilon mh}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon mh}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\ for\ \mathcal{P}_2,\end{aligned}$$

and $(\boldsymbol{e}_h,\boldsymbol{b}_h)$, η_h *satisfy the following bounds:*

$$\begin{aligned}&|||(\boldsymbol{e}_h,\boldsymbol{b}_h)|||_1 \le C\left[\epsilon^{-\frac{3}{2}}|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0+|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\\ &\|\eta_h\|_0 \le C\left[\epsilon^{-2}|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0+|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\ for\ \mathcal{P}_1;\\ &|||(\boldsymbol{e}_h,\boldsymbol{b}_h)|||_1 \le C\left[|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0+|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\\ &\|\eta_h\|_0 \le C\left[|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0+|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\ for\ \mathcal{P}_2,\end{aligned}$$

in the 2D case, and

$$\begin{aligned}&|||(\boldsymbol{e}_h,\boldsymbol{b}_h)|||_1 \le C\left[\epsilon^{-\frac{5}{4}}H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0+|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\\ &\|\eta_h\|_0 \le C\left[\epsilon^{-\frac{7}{4}}H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0+|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\ for\ \mathcal{P}_1;\\ &|||(\boldsymbol{e}_h,\boldsymbol{b}_h)|||_1 \le C\left[H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0+|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\\ &\|\eta_h\|_0 \le C\left[H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\|\boldsymbol{F}\|_0+|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H},\boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2\right],\ for\ \mathcal{P}_2,\end{aligned}$$

in the 3D case.

Proof We can complete the proof by the same fashion of the proof of lemmas 4.2 and 4.6. □

Theorem 4.4 *Under the assumptions of Theorem 4.3,* $((\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh}), p_{\epsilon mh})$ *provided by Method 2–3 with Correction 3 satisfies the following error estimates, for* $\mathcal{P}_1$

$$\begin{aligned}&|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1 \le C\left[\epsilon+\epsilon^{-\frac{1}{2}}\left(h+\epsilon^{-\frac{1}{2}}H^2\left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\right)\right]\|\boldsymbol{F}\|_0\\ &\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1,\\ &\|p-p_{\epsilon mh}\|_0 \le C\left[\epsilon+\epsilon^{-1}\left(h+\epsilon^{-\frac{1}{2}}H^2\left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\right)\right]\|\boldsymbol{F}\|_0\\ &\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1,\end{aligned}$$

ϵ *and* H *can be taken as* $\epsilon = O(h^{\frac{1}{2}})$, $H^2 = O(\epsilon^{\frac{1}{2}}h)$ *and the convergence rate is* $O(h^{\frac{1}{2}})$*; for* $\mathcal{P}_2$*, the optimal error estimates are*

$$\begin{aligned}&|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1+\|p-p_{\epsilon mh}\|_0 \le C\left(\epsilon+h+H^2\left(\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)^{\frac{1}{2}}\right)\|\boldsymbol{F}\|_0\\ &\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1,\end{aligned}$$

ϵ *and* H *can be taken as* $\epsilon = O(h)$, $H^2 = O(h)$ *and the convergence rate is* $O(h)$*;*

And $((\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}_{\epsilon mh}), p_{\epsilon mh})$ *provided by Method 3 with Correction 2 satisfies the following error estimates, for* $\mathcal{P}_1$

$$\begin{aligned}&|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1 \le C\left[\epsilon+\epsilon^{-\frac{1}{2}}\left(h+\epsilon^{-1}|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)\right]\|\boldsymbol{F}\|_0\\ &\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\\ &\|p-p_{\epsilon mh}\|_0 \le C\left[\epsilon+\epsilon^{-1}\left(h+\epsilon^{-1}|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)\right]\|\boldsymbol{F}\|_0\\ &\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\end{aligned}$$

ϵ *and* H *can be taken as* $\epsilon = O(h^{\frac{1}{2}})$, $H^3 = O(\epsilon h|\ln h|^{-1})$ *and the convergence rate is* $O(h^{\frac{1}{2}})$ *in the 2D case; for* $\mathcal{P}_2$*, the optimal error estimates are*

$$\begin{aligned}&|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1+\|p-p_{\epsilon mh}\|_0 \le C\left[\epsilon+h+|\ln h|H^3\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right]\|\boldsymbol{F}\|_0\\ &\qquad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\end{aligned}$$

ϵ *and* H *can be taken as* $\epsilon = O(h)$, $H^3 = O(h|\ln h|^{-1})$ *and the convergence rate is* $O(h)$ *in the 2D case; and for* $\mathcal{P}_1$

$$\begin{aligned}&|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1 \le C\left[\epsilon+\epsilon^{-\frac{1}{2}}\left(h+\epsilon^{-\frac{3}{4}}H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)\right]\|\boldsymbol{F}\|_0\\ &\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\\ &\|p-p_{\epsilon mh}\|_0 \le C\left[\epsilon+\epsilon^{-1}\left(h+\epsilon^{-\frac{3}{4}}H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right)\right]\|\boldsymbol{F}\|_0\\ &\quad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1^2,\end{aligned}$$

ϵ *and* H *can be taken as* $\epsilon = O(h^{\frac{1}{2}})$, $H^{\frac{5}{2}} = O(\epsilon^{\frac{3}{4}}h)$ *and the convergence rate is* $O(h^{\frac{1}{2}})$ *in the 3D case; for* $\mathcal{P}_2$*, the optimal error estimates are*

$$\begin{aligned}&|||\,(\boldsymbol{u}-\boldsymbol{u}_{\epsilon mh}, \boldsymbol{B}-\boldsymbol{B}_{\epsilon mh})\,|||_1+\|p-p_{\epsilon mh}\|_0 \le C\left[\epsilon+h+H^{\frac{5}{2}}\frac{\|\boldsymbol{F}\|_0}{\|\boldsymbol{F}\|_{-1}}\right]\|\boldsymbol{F}\|_0\\ &\qquad+C|||\left(\boldsymbol{u}_{\epsilon H}-\boldsymbol{u}^m_{\epsilon H}, \boldsymbol{B}_{\epsilon H}-\boldsymbol{B}^m_{\epsilon H}\right)|||_1,\end{aligned}$$

ϵ *and* H *can be taken as* $\epsilon = O(h)$, $H^{\frac{5}{2}} = O(h)$ *and the convergence rate is* $O(h)$ *in the 3D case.*

Table 1 Parameters of the investigations (2D)

Group	α	R_e	R_m	S_c	σ	C_0^2	$\|\mathbf{f}\|_{-1}$	$\|\mathbf{g}\|_*$	ε
1	0.1	1	1	1	2.00e–1 $\in (0, 2/5]$	3.18e–1	1.80e–2	3.14e–1	4.73e–1
2	0.22	1	1	1	4.41e–1 $\in (2/5, 5/11]$	3.18e–1	4.13e–2	6.91e–1	4.73e–1
3	0.28	1	1	1	5.61e–1 $\in (5/11, 1-\varepsilon]$	3.18e–1	5.42e–2	8.80e–1	4.73e–1
4	0.35	1	1	1	7.01e–1 $\in (1-\varepsilon, 1]$	3.18e–1	7.06e–2	1.10e–0	4.73e–1

Table 2 Parameters of the investigations (3D)

Group	α	R_e	R_m	S_c	σ	C_0^2	$\|\mathbf{f}\|_{-1}$	$\|\mathbf{g}\|_*$	ε
1	0.1	1	1	1	2.12e–1 $\in (0, 2/5]$	8.57e–1	9.21e–2	8.23e–2	4.03e–1
2	0.2	1	1	1	4.16e–1 $\in (2/5, 5/11]$	8.57e–1	1.77e–1	1.65e–1	4.06e–1
3	0.25	1	1	1	5.15e–1 $\in (5/11, 1-\varepsilon]$	8.57e–1	2.18e–1	2.07e–1	4.08e–1
4	0.35	1	1	1	7.08e–1 $\in (1-\varepsilon, 1]$	8.57e–1	2.93e–1	2.91e–1	4.11e–1

Table 3 $M_1 + C_i$ with $\epsilon = O(h^{1/2})$ and $\alpha = 0.1$ for $P_1 - P_0 - P_1$ element (2D)

Method	1/H	$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	Rate	CPU (s)
$M_1 + C_1$	12	108	2.13388e–1	/	1.35528e–1	/	1.45435e–2	/	1.50e1
$M_1 + C_1$	13	122	2.00809e–1	0.5	1.27411e–1	0.5	1.28747e–2	1.0	1.96e1
$M_1 + C_2$	12	108	2.13351e–1	/	1.35482e–1	/	1.45435e–2	/	2.45e1
$M_1 + C_2$	13	122	2.00770e–1	0.5	1.27387e–1	0.5	1.28747e–2	1.0	3.16e1
$M_1 + C_3$	12	108	2.13523e–1	/	1.35290e–1	/	1.45435e–2	/	2.00e1
$M_1 + C_3$	13	122	2.00910e–1	0.5	1.27220e–1	0.5	1.28747e–2	1.0	2.63e1

Table 4 $M_2 + C_i$ with $\epsilon = O(h^{1/2})$ and $\alpha = 0.1$ for $P_1 - P_0 - P_1$ element (2D)

Method	1/H	$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	Rate	CPU (s)
$M_2 + C_1$	12	108	2.13388e–1	/	1.35528e–1	/	1.45435e–2	/	1.48e1
$M_2 + C_1$	13	122	2.00809e–1	0.5	1.27411e–1	0.5	1.28747e–2	1.0	1.94e1
$M_2 + C_2$	12	108	2.13351e–1	/	1.35482e–1	/	1.45435e–2	/	2.47e1
$M_2 + C_2$	13	122	2.00770e–1	0.5	1.27387e–1	0.5	1.28747e–2	1.0	3.20e1
$M_2 + C_3$	12	108	2.13523e–1	/	1.35290e–1	/	1.45435e–2	/	1.93e1
$M_2 + C_3$	13	122	2.00910e–1	0.5	1.27220e–1	0.5	1.28747e–2	1.0	2.51e0

Proof We can finish the proof by Theorems 2.3, 3.2, Lemmas 4.5, 4.6, triangle inequality and some simple calculations. □

Remark 4.4 From the Theorem 4.4, we know that Method 3 is stable and convergent with $\frac{5}{11} < \sigma \leq 1 - \left(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0}\right)^{\frac{1}{2}}$. Hence, three combinations are proposed with Methods i and Corrections j ($i = 3$; $j = 1, 2, 3$). And Method 3 with Correction 2 is the better choice to solve the considered problem.

Table 5 $M_3 + C_i$ with $\epsilon = O(h^{1/2})$ and $\alpha = 0.1$ for $P_1 - P_0 - P_1$ element (2D)

Method	1/H	$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	Rate	CPU (s)
$M_3 + C_1$	12	108	2.13388e–1	/	1.35528e–1	/	1.45435e–2	/	1.52e1
$M_3 + C_1$	13	122	2.00809e–1	0.5	1.27411e–1	0.5	1.28747e–2	1.0	1.94e1
$M_3 + C_2$	12	108	2.13351e–1	/	1.35482e–1	/	1.45435e–2	/	2.50e1
$M_3 + C_2$	13	122	2.00770e–1	0.5	1.27387e–1	0.5	1.28747e–2	1.0	3.22e1
$M_3 + C_3$	12	108	2.13523e–1	/	1.35290e–1	/	1.45435e–2	/	1.97e1
$M_3 + C_3$	13	122	2.00910e–1	0.5	1.27220e–1	0.5	1.28747e–2	1.0	2.54e1

Table 6 $M_1 + C_i$ with $\epsilon = O(h)$ and $\alpha = 0.1$ for $P_1b - P_1 - P_1b$ element (2D)

Method	1/H	$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	Rate	CPU (s)
$M_1 + C_1$	10	100	3.53437e–2	/	5.48583e–3	/	1.49007e–2	/	2.09e1
$M_1 + C_1$	11	121	2.92195e–2	1.0	4.54141e–3	1.0	1.23149e–2	1.0	3.10e1
$M_1 + C_2$	10	100	3.50390e–2	/	3.96169e–3	/	1.50512e–2	/	3.52e1
$M_1 + C_2$	11	121	2.88992e–2	1.0	3.26110e–3	1.0	1.24175e–2	1.0	5.33e1
$M_1 + C_3$	10	100	3.51137e–2	/	3.41077e–3	/	1.50512e–2	/	2.79e1
$M_1 + C_3$	11	121	2.89587e–2	1.0	2.80065e–3	1.0	1.24175e–2	1.0	4.18e1

Table 7 $M_2 + C_i$ with $\epsilon = O(h)$ and $\alpha = 0.1$ for $P_1b - P_1 - P_1b$ element (2D)

Method	1/H	$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	Rate	CPU (s)
$M_2 + C_1$	10	100	3.56849e–2	/	5.48128e–3	/	1.50512e–2	/	2.04e1
$M_2 + C_1$	11	121	2.94549e–2	1.0	4.54701e–3	1.0	1.24175e–2	1.0	3.01e1
$M_2 + C_2$	10	100	3.50390e–2	/	3.96169e–3	/	1.50512e–2	/	3.61e1
$M_2 + C_2$	11	121	2.88992e–2	1.0	3.26110e–3	1.0	1.24175e–2	1.0	5.38e1
$M_2 + C_3$	10	100	3.51137e–2	/	3.41077e–3	/	1.50512e–2	/	2.81e1
$M_2 + C_3$	11	121	2.89587e–2	1.0	2.80065e–3	1.0	1.24175e–2	1.0	4.22e1

Table 8 $M_3 + C_i$ with $\epsilon = O(h)$ and $\alpha = 0.1$ for $P_1b - P_1 - P_1b$ element (2D)

Method	1/H	$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	Rate	CPU (s)
$M_3 + C_1$	10	100	3.56849e–2	/	5.48128e–3	/	1.50512e–2	/	2.05e1
$M_3 + C_1$	11	121	2.94549e–2	1.0	4.54701e–3	1.0	1.24175e–2	1.0	3.01e1
$M_3 + C_2$	10	100	3.50390e–2	/	3.96169e–3	/	1.50512e–2	/	3.57e1
$M_3 + C_2$	11	121	2.88992e–2	1.0	3.26110e–3	1.0	1.24175e–2	1.0	5.32e1
$M_3 + C_3$	10	100	3.51137e–2	/	3.41077e–3	/	1.50512e–2	/	2.79e1
$M_3 + C_3$	11	121	2.89587e–2	1.0	2.80065e–3	1.0	1.24175e–2	1.0	4.15e1

Table 9 $M_1 + C_i$ with $\epsilon = O(h^{1/2})$ and $\alpha = 0.1$ for $P_1 - P_0 - P_1$ element (3D)

Method	1/H	$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	Rate	CPU (s)
$M_1 + C_1$	5	14	4.46088e–2	/	5.03246e–1	/	4.91089e–2	/	1.77e1
$M_1 + C_1$	6	18	3.49609e–2	1.0	4.44626e–1	0.5	3.81997e–2	1.0	4.51e1
$M_1 + C_2$	5	14	4.46087e–2	/	5.03180e–1	/	4.91069e–2	/	1.97e1
$M_1 + C_2$	6	18	3.49611e–2	1.0	4.44508e–1	0.5	3.81986e–2	1.0	5.63e1
$M_1 + C_3$	5	14	4.46087e–2	/	5.03180e–1	/	4.91069e–2	/	2.00 e1
$M_1 + C_3$	6	18	3.49611e–2	1.0	4.44508e–1	0.5	3.81986e–2	1.0	2.63e1

Table 10 $M_2 + C_i$ with $\epsilon = O(h^{1/2})$ and $\alpha = 0.1$ for $P_1 - P_0 - P_1$ element (3D)

Method	1/H	$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	Rate	CPU (s)
$M_2 + C_1$	5	14	4.46088e–2	/	5.00325e–1	/	4.91089e–2	/	1.78e1
$M_2 + C_1$	6	18	3.49609e–2	1.0	4.44626e–1	0.5	3.81997e–2	1.0	4.58e1
$M_2 + C_2$	5	14	4.46099e–2	/	5.02854e–1	/	4.91066e–2	/	2.50e1
$M_2 + C_2$	6	18	3.49619e–2	1.0	4.44264e–1	0.5	3.81983e–2	1.0	8.98e1
$M_2 + C_3$	5	14	4.46087e–2	/	5.03180e–1	/	4.91069e–2	/	1.93e1
$M_2 + C_3$	6	18	3.49611e–2	1.0	4.44508e–1	0.5	3.81986e–2	1.0	5.05e1

Table 11 $M_3 + C_i$ with $\epsilon = O(h^{1/2})$ and $\alpha = 0.1$ for $P_1 - P_0 - P_1$ element (3D)

Method	1/H	$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	Rate	CPU (s)
$M_3 + C_1$	5	14	4.46088e–2	/	5.03246e–1	/	4.91089e–2	/	1.83e1
$M_3 + C_1$	6	18	3.49609e–2	1.0	4.44626e–1	0.5	3.81997e–2	1.0	4.59e1
$M_3 + C_2$	5	14	4.46099e–2	/	5.02854e–1	/	4.91066e–2	/	2.53e1
$M_3 + C_2$	6	18	3.49619e–2	1.0	4.44264e–1	0.5	3.81983e–2	1.0	5.91e1
$M_3 + C_3$	5	14	4.46087e–2	/	5.03180e–1	/	4.91069e–2	/	1.96e1
$M_3 + C_3$	6	18	3.49611e–2	1.0	4.44508e–1	0.5	3.81986e–2	1.0	4.86e1

Table 12 $M_1 + C_i$ with $\epsilon = O(h)$ and $\alpha = 0.1$ for $P_1b - P_1 - P_1b$ element (3D)

Method	1/H	$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	Rate	CPU (s)
$M_1 + C_1$	4	16	3.75461e–2	/	9.09759e–2	/	4.27608e–2	/	7.50e1
$M_1 + C_1$	5	20	3.00399e–2	1.0	6.48166e–2	1.5	3.42093e–2	1.0	1.07e2
$M_1 + C_2$	4	16	3.75451e–2	/	8.87178e–2	/	4.27533e–2	/	8.92e1
$M_1 + C_2$	5	20	3.00394e–2	1.0	6.32902e–2	1.5	3.42054e–2	1.0	1.34e2
$M_1 + C_3$	4	16	3.7545e–2	/	9.0736e–2	/	4.27548e–2	/	5.23e1
$M_1 + C_3$	5	20	3.00394e–2	1.0	6.45071e–2	1.5	3.42062e–2	1.0	1.05e2

Table 13 $M_2 + C_i$ with $\epsilon = O(h)$ and $\alpha = 0.1$ for $P_1b-P_1-P_1b$ element (3D)

Method	1/H	$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	Rate	CPU (s)
$M_2 + C_1$	4	16	3.75461e–2	/	9.09759e–2	/	4.27608e–2	/	1.78e1
$M_2 + C_1$	5	20	3.00399e–2	1.0	6.48166e–2	1.5	3.42093e–2	1.0	4.58e1
$M_2 + C_2$	4	16	3.75451e–2	/	8.87178e–2	/	4.27533e–2	/	2.50e1
$M_2 + C_2$	5	20	3.00394e–2	1.0	6.32902e–2	1.5	3.42054e–2	1.0	8.98e1
$M_2 + C_3$	4	16	3.7545e–2	/	9.0736 e–2	/	4.27548e–2	/	1.93e1
$M_2 + C_3$	5	20	3.00394e–2	1.0	6.45071e–2	1.5	3.42062e–2	1.0	5.05e1

Table 14 $M_3 + C_i$ with $\epsilon = O(h)$ and $\alpha = 0.1$ for $P_1b-P_1-P_1b$ element (3D)

Method	1/H	$1/h$	$\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$	Rate	CPU (s)
$M_3 + C_1$	4	16	3.75461e–2	/	9.09759e–2	/	4.27608e–2	/	1.83e1
$M_3 + C_1$	5	20	3.00399e–2	1.0	6.48166e–2	1.5	3.42093e–2	1.0	4.59e1
$M_3 + C_2$	4	16	3.75451e–2	/	8.87178e–2	/	4.27533e–2	/	2.53e1
$M_3 + C_2$	5	20	3.00394e–2	1.0	6.32902e–2	1.5	3.42054e–2	1.0	5.91e1
$M_3 + C_3$	4	16	3.75450e–2	/	9.07360e–2	/	4.27548e–2	/	1.96e1
$M_3 + C_3$	5	20	3.00394e–2	1.0	6.45071e–2	1.5	3.42062e–2	1.0	4.86e1

5 Numerical Results

In this section, we investigate the numerical performance of the proposed methods via a flow problem with a smooth solution and a driven cavity flow problem for the 2D/3D MHD problem. Penalty parameter ϵ is selected as $\epsilon = O(h^{\frac{1}{2}})$ for $\mathcal{P}_1$ and $\epsilon = O(h)$ for $\mathcal{P}_2$ based on Theorems 4.2, 4.3 and 4.4 in all the following numerical tests. Then, by defining $M_i + C_j$ denote the combination of the Method i and Correction j $(i,\ j = 1, 2, 3)$. The stopping criterion is based on a relative reduction in the energy norm with a tolerance of 10^{-10}.

5.1 Problems with Smooth Solutions

We test the accuracy performance of our proposed methods with 2D/3D smooth solution in this case. On the square domain $\Omega = [0, 1]^d$, $d = 2, 3$ and the exact solutions be given by

$$\begin{cases} u_1 = \alpha x^2(x-1)^2 y(y-1)(2y-1), & u_2 = \alpha y^2(y-1)^2 x(x-1)(2x-1), \\ B_1 = \alpha \sin(\pi x)\cos(\pi y), & B_2 = -\alpha \sin(\pi y)\cos(\pi x), \\ p = \alpha(2x-1)(2y-1), \end{cases}$$

for $d = 2$ and

$$\begin{cases} u_1 = \alpha(y^4 + z), \quad u_2 = \alpha(x + z^3), \quad u_3 = \alpha(x^2 + y^2), \\ B_1 = \alpha \sin(yz), \quad B_2 = -\alpha \sin(x+z), \quad B_3 = -\alpha y \sin(x^2), \\ p = \alpha(2x-1)(2y-1)(2z-1), \end{cases}$$

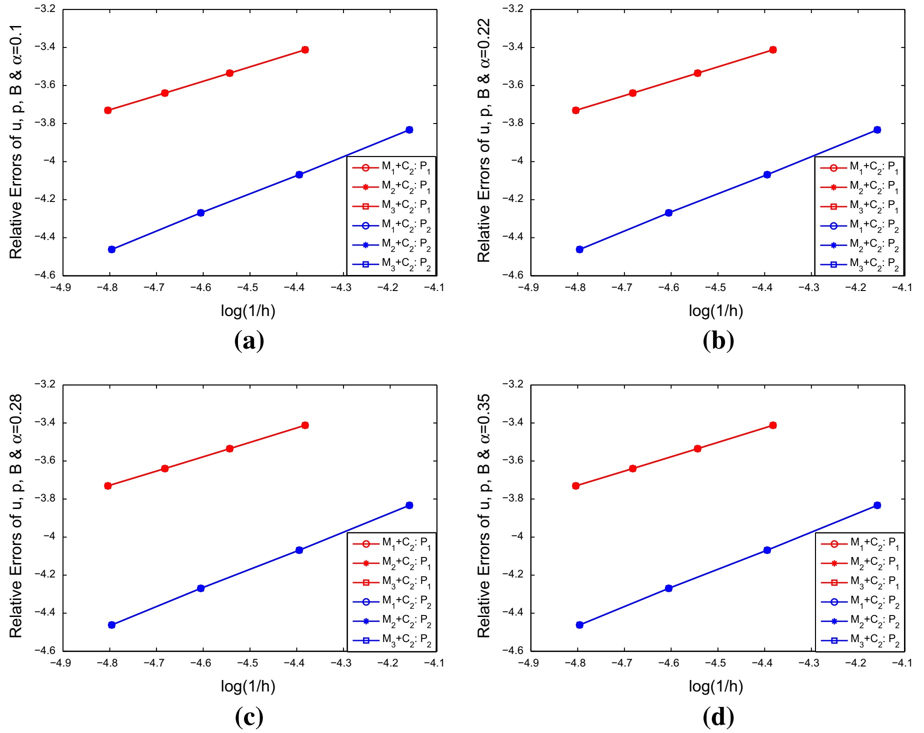

Fig. 1 (2D) Relative error for $\alpha = 0.1$ (**a**); $\alpha = 0.22$ (**b**); $\alpha = 0.28$ (**c**); $\alpha = 0.35$ (**d**)

$d = 3$. Here, α is chosen such that $0 < \sigma \leq \frac{2}{5}, \frac{5}{11}$ and $1 - \varepsilon$, where $\varepsilon = \left(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0}\right)^{\frac{1}{2}}$. And the body forces $\mathbf{f}$, $\mathbf{g}$ are determined accordingly for any R_e, R_m and S_c.

Tables 1 and 2 present several groups of the parameter (Refer to the Remark 6.1 in our previous work [20] for the detail computational formulas). Numerical results of the presented methods with the first group parameters of Tables 1 and 2 are shown in Tables 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14. It is observed that the results are in very good agreement with the convergence rates that are predicted by the analysis in Theorems 4.2, 4.3 and 4.4. Specifically, the magnetic field $\mathbf{B}$ and the velocity field $\mathbf{u}$ for $\mathcal{P}_1$ has an improved convergence rate, it is even higher than the theoretical result $O(h^{1/2})$ on count of the chosen smooth solutions for the 3D case. Since, the degrees of freedom for $P_1b{-}P_1{-}P_1b$ is greater, the relative errors of $P_1b{-}P_1{-}P_1b$ is smaller and the operating speed is slower than the $P_1{-}P_0{-}P_1$ one.

Then, the comparison results indicate that the relative errors are almost the same for the same finite element pairs with three different iterative methods. Moreover, $M_1 + C_j$ is the quickest and $M_2 + C_j$ is the slowest one for both $\mathcal{P}_1$ and $\mathcal{P}_2$. The complexity of the discretization of the nonlinear terms and the order of the finite element pair are the main reasons for it. For fixed iterative methods $M_i (i = 1, 2, 3)$, $M_i + C_2$ is more accurate than $M_i + C_1$ and $M_i + C_3$; $M_i + C_1$ is the fastest one and $M_i + C_2$ is the slowest one.

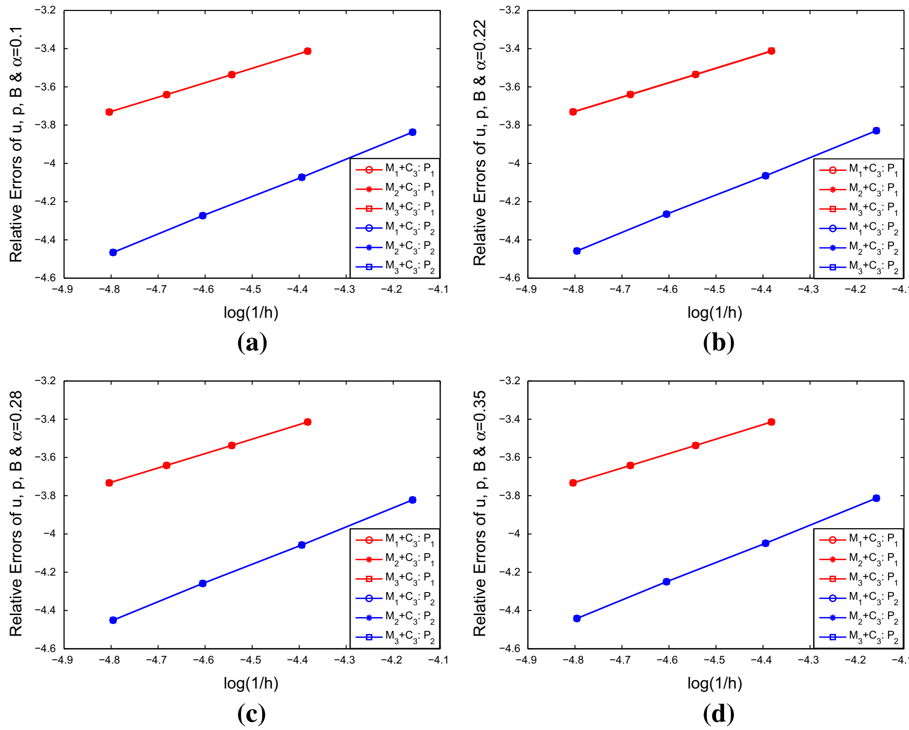

Fig. 2 (2D) Relative error for $\alpha = 0.1$ (**a**); $\alpha = 0.22$ (**b**); $\alpha = 0.28$ (**c**); $\alpha = 0.35$ (**d**)

To study the effective combinative methods, the two-level combinations with respect to M_2 and M_3 are only considered in this paper. The relative error of $M_i + C_2$ and $M_i + C_3$ for $\mathcal{P}_1$ and $\mathcal{P}_2$ with fixed α given in Tables 1 and 2 are shown in Figs. 1, 2, 3 and 4. The relative error is defined by $\frac{|||((\mathbf{u}-\mathbf{u}_{\epsilon mh},\mathbf{B}-\mathbf{B}_{\epsilon mh}),p-p_{\epsilon mh})|||}{|||((\mathbf{u},\mathbf{B}),p)|||}$, where $|||((\mathbf{u},\mathbf{B}),\ p)||| = |||(\mathbf{u},\mathbf{B})|||_1 + \|p\|_0$. The numerical results indicate that the Correction C_j, $(j = 2, 3)$ is fixed, the nearly same convergence rates and relative errors can be obtained for current parameter configurations. Figures 5 and 6 present the comparison results of $M_i + C_2$ and $M_i + C_3$, $(i = 2, 3)$ for larger α to investigate which combination is more effective. But it is difficult to separate the major difference between them for the estimation of C_0 is not exactly sharp (refer to our previous work [20]). Then as a consequence, we can only conclude that $M_i + C_2$ is more effective than $M_i + C_3$ in accuracy comparing and we can not figure out $M_2 + C_2$ and $M_3 + C_2$ which one is the superior one from this experiment.

But in our previous work [19,20], we closely compared M_2 and M_3 in numerical computation aspects. Based on these experiences, we can conclude that $M_2 + C_2$ has higher convergence speed than $M_3 + C_2$. And $M_3 + C_2$ has the most adaptability for σ. Therefore, we only consider the scheme $M_2 + C_2$ in the next driven cavity flow problem.

5.2 Driven Cavity Flow

Let us consider a 2D/3D driven cavity flow which used in fluid dynamics. In this example, we consider the implementation domain $\Omega = (-1, 1)^d$, $d = 2$ and $\Omega = (0, 1)^3$, $d = 3$ with

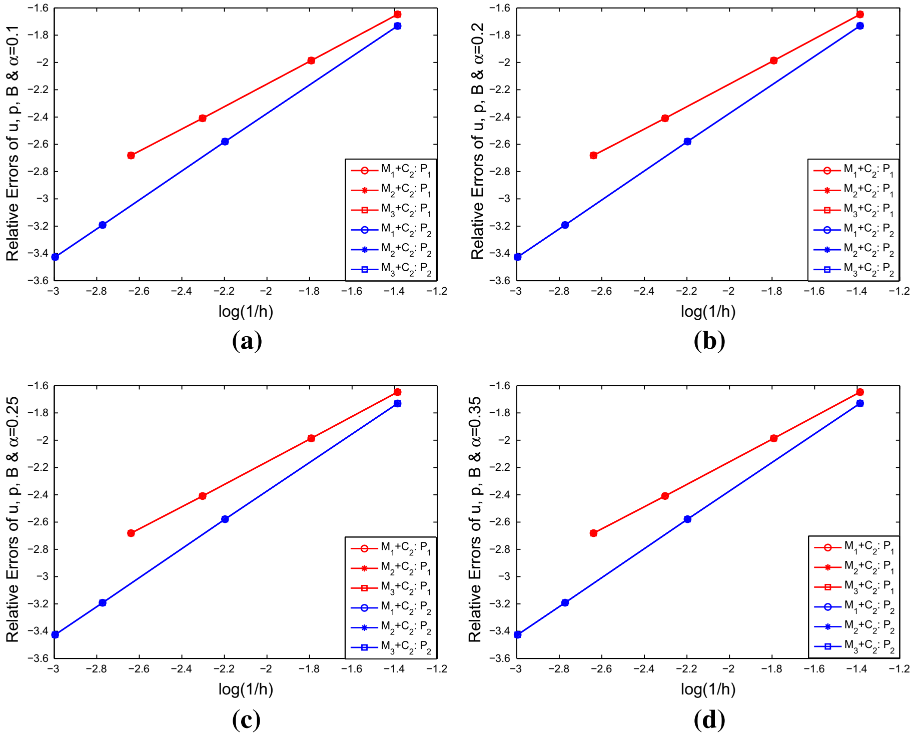

Fig. 3 (3D) Relative error for $\alpha = 0.1$ (**a**); $\alpha = 0.2$ (**b**); $\alpha = 0.25$ (**c**); $\alpha = 0.35$ (**d**)

$\Gamma_D = \partial\Omega$, and set the source terms to be zero. The boundary conditions are prescribed as follows:

$$\begin{cases} \mathbf{u} = 0, \quad \text{on } x = \pm 1 \text{ and } \; y = -1, \\ \mathbf{u} = (1, 0), \quad \text{on } \; y = 1, \\ \mathbf{n} \times \mathbf{B} = \mathbf{n} \times \mathbf{B}_D, \quad \text{on } \; \partial\Omega, \end{cases}$$

where $\mathbf{B}_D = (1, 0)$ for $d = 2$;

$$\begin{cases} \mathbf{u} = 0, \quad \text{on } x = 0, x = 1, y = 0, y = 1 \text{ and } \; z = 0, \\ \mathbf{u} = (1, 0, 0), \quad \text{on } \; z = 1, \\ \mathbf{n} \times \mathbf{B} = \mathbf{n} \times \mathbf{B}_D, \quad \text{on } \; \partial\Omega, \end{cases}$$

where $\mathbf{B}_D = (1, 0, 0)$ for $d = 3$.

According to the numerical analysis above, we only relate to the scheme $M_2 + C_2$ with $P_1b{-}P_1{-}P_1b$ finite element pair for the 2D/3D driven cavity flow problem in this example. Here, we develop the further research for the hydrodynamic number R_e. Numerical results of $M_2 + C_2$ are compared with the standard two-level iterative method to show the performance of the proposed scheme.

The horizontal velocity and magnetic field distribution at the mid-width with different R_e for 2D/3D case are presented in Figs. 7 and 8. It is observed that our results show an excellent agreement with the standard two-level iterative method. Figures 9, 10, 11, 12, 13

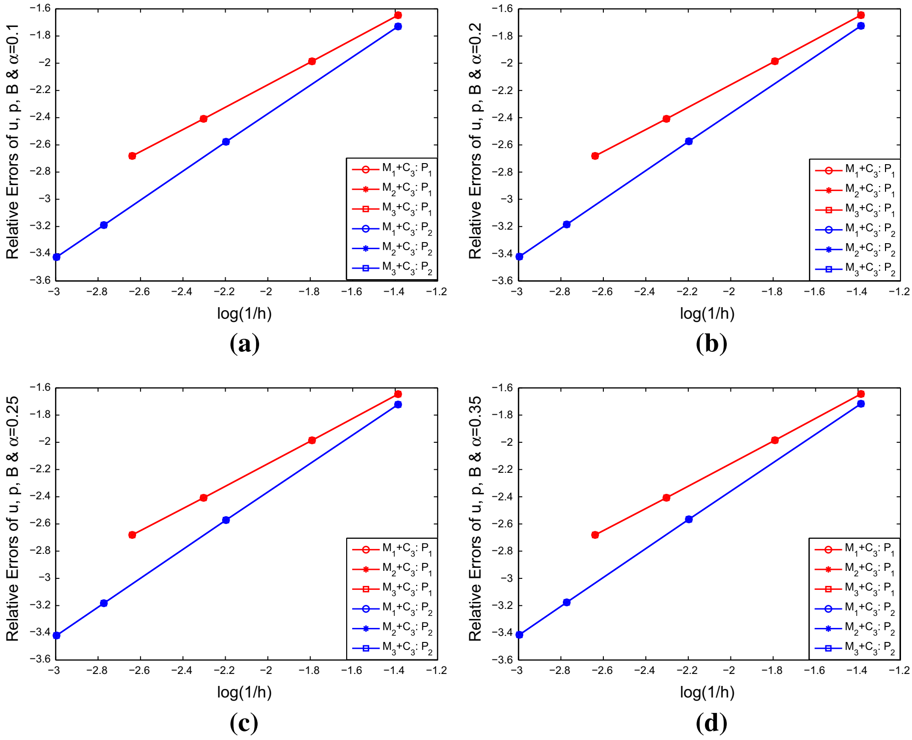

Fig. 4 (3D) Relative error for $\alpha = 0.1$ (**a**); $\alpha = 0.2$ (**b**); $\alpha = 0.25$ (**c**); $\alpha = 0.35$ (**d**)

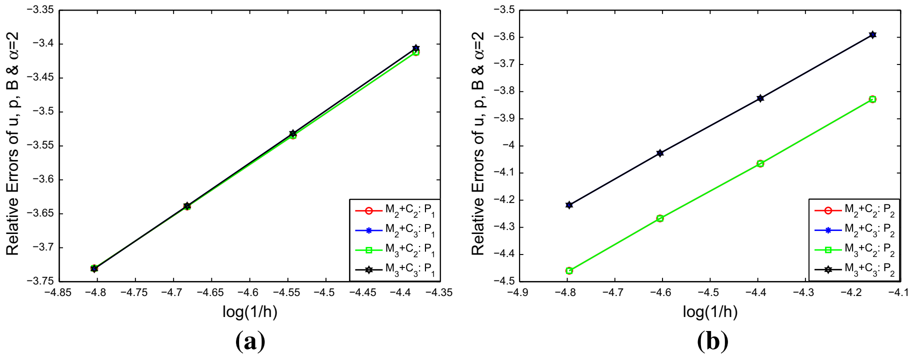

Fig. 5 (2D) Relative error for $\alpha = 2$

and 14 illustrate the numerical results of $M_2 + C_2$ for $R_e = 1, 5 \times 10^2, 5 \times 10^3$ and $R_e = 10^{-1}, 10, 10^2$ with $R_m = 1$, $S_c = 1$ in 2D case and 3D case, respectively. As can be seen that the velocity main vortex grows into several small ones and become more complex with the increase of R_e. It can be inferred that more resolved vortexes may captured with the increase of R_e.

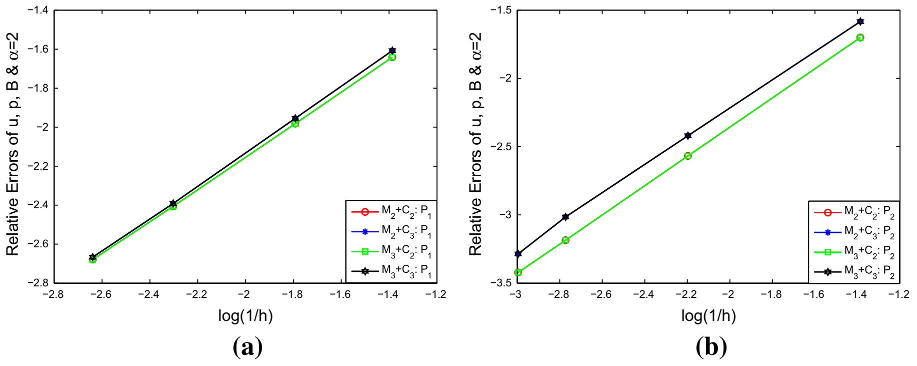

Fig. 6 (3D) Relative error for $\alpha = 2$

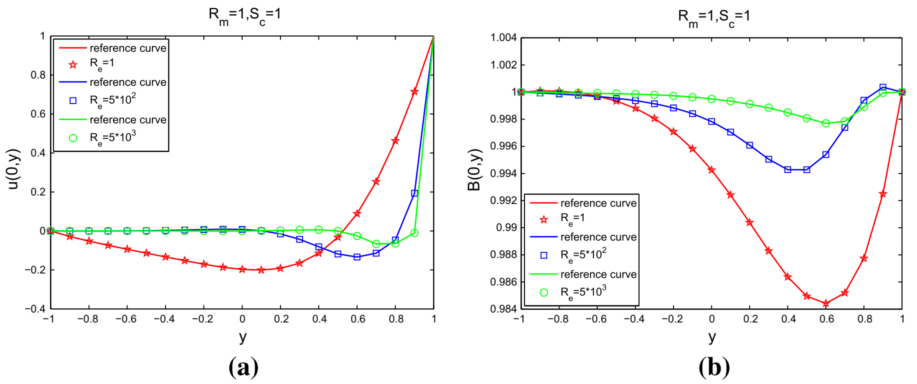

Fig. 7 (2D) Comparison results versus R_e

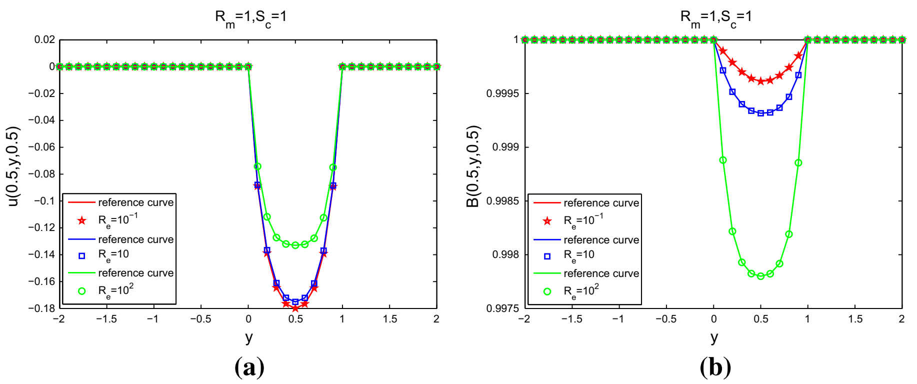

Fig. 8 (3D) Comparison results versus R_e

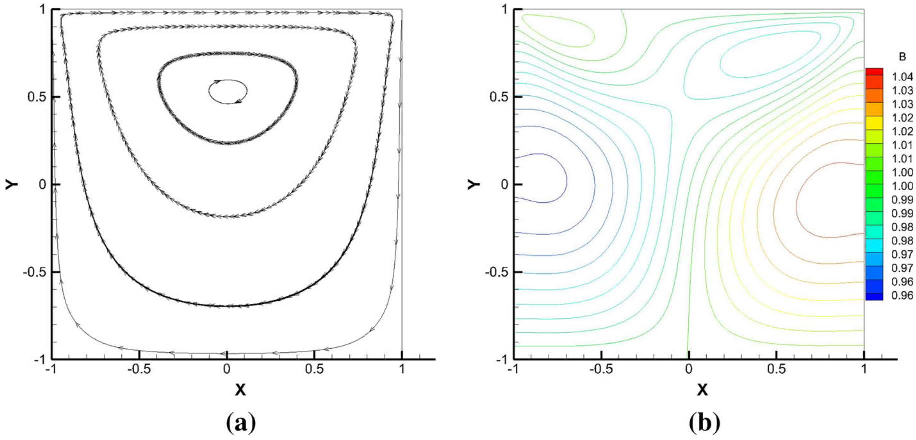

Fig. 9 (2D) Numerical streamlines (**a**); isodynamic (**b**) with $R_e = 1$

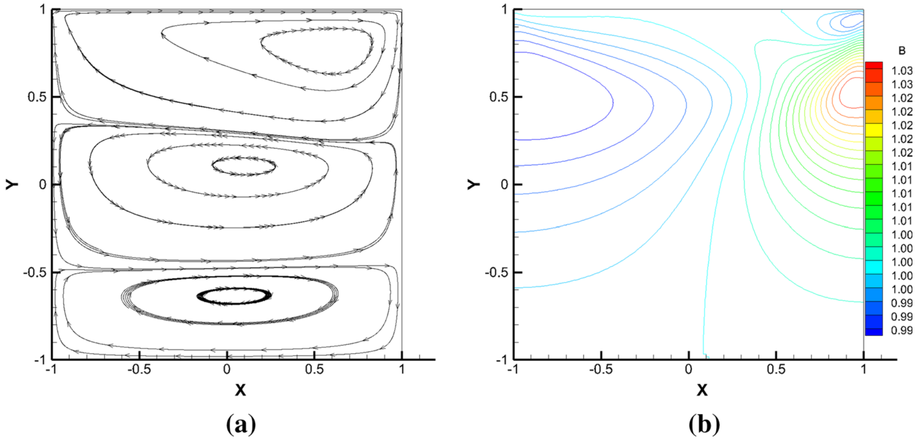

Fig. 10 (2D) Numerical streamlines (**a**); isodynamic (**b**) with $R_e = 5 \times 10^2$

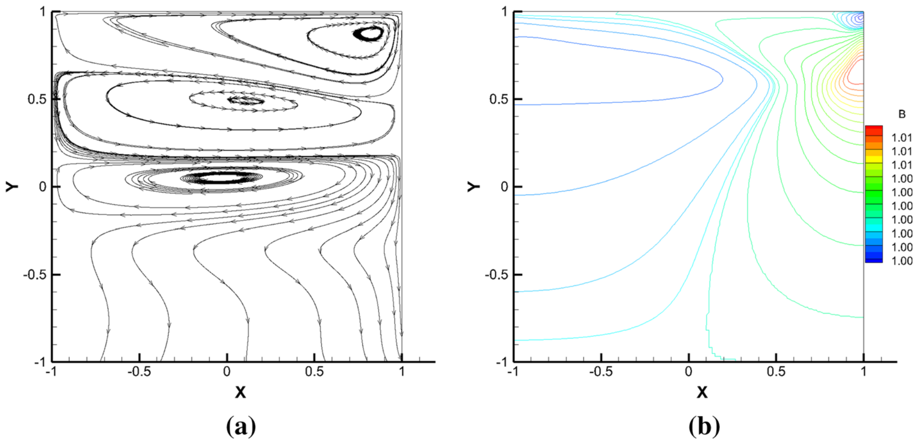

Fig. 11 (2D) Numerical streamlines (**a**); isodynamic (**b**) with $R_e = 5 \times 10^3$

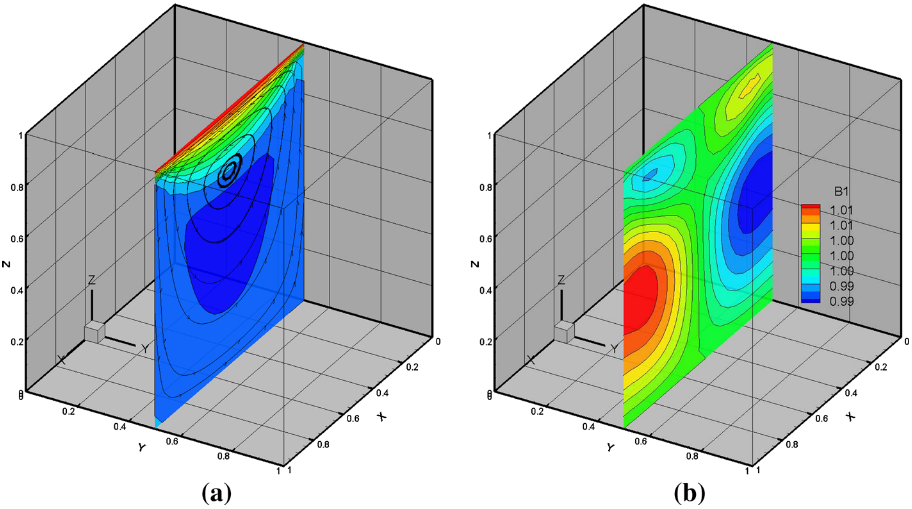

Fig. 12 (3D) Numerical streamlines (**a**); isodynamic (**b**) with $R_e = 10^{-1}$

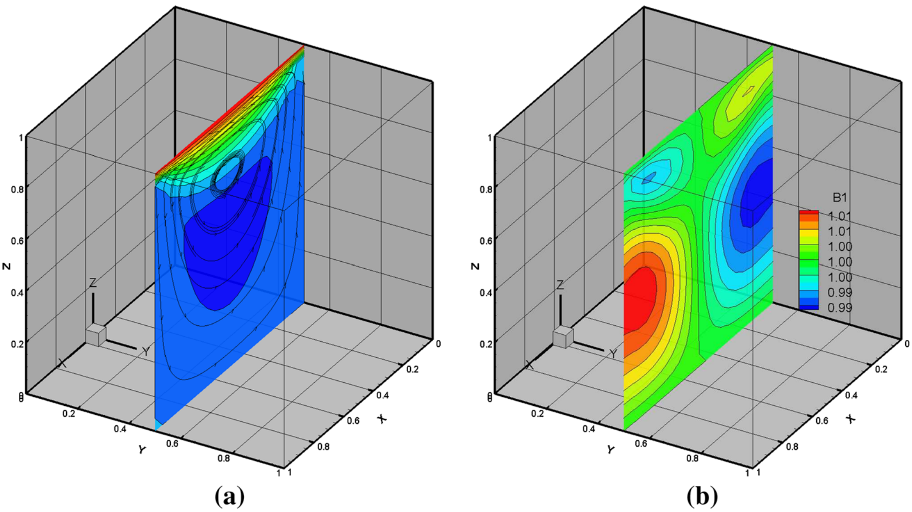

Fig. 13 (3D) Numerical streamlines (**a**); isodynamic (**b**) with $R_e = 10$

6 Conclusions

In this article, we have presented some two-level penalty finite element methods for solving the 2D/3D steady incompressible MHD equations. And we give the rigorous proof of the stability and optimal error estimate for different finite element pair $\mathcal{P}_1$ and $\mathcal{P}_2$ under the penalty parameter ϵ. Based on the theoretical analysis, we know that $M_i + C_2$ $(i = 1, 2, 3)$ is the best choice with the convergence speed and accurancy perspective than $M_i + C_1$ $(i =$

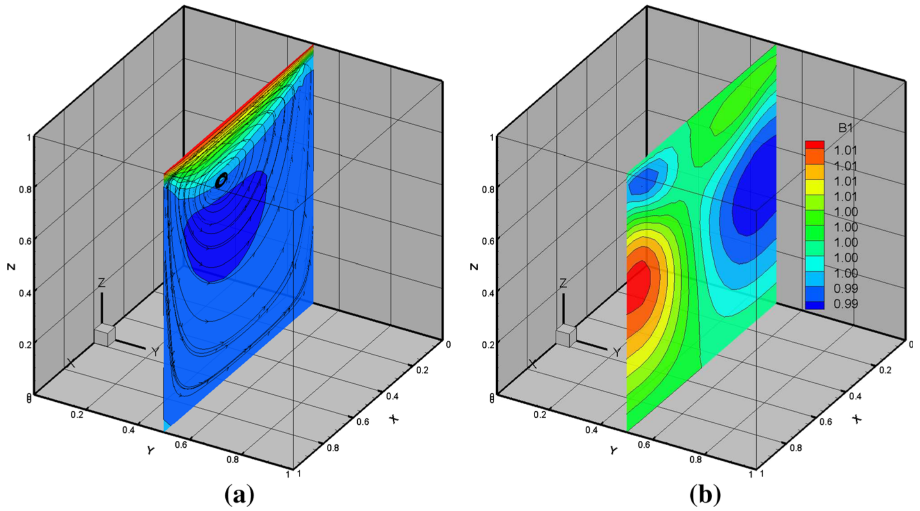

Fig. 14 (3D) Numerical streamlines (**a**); isodynamic (**b**) with $R_e = 10^2$

1, 2, 3) and $M_i + C_3$ ($i = 1, 2, 3$). Moreover, $M_3 + C_j$ ($j = 1, 2, 3$) has the most adaptability for σ. Numerical simulation results verified the theoretical results. Furthermore, $M_3 + C_2$ is the best choice to deal with large Reynolds number flows from the numerical simulations.

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References

1. Davidson, P.: An Introduction to Magnetohydrodynamics. Cambridge University Press, Cambridge (2001)
2. Gerbeau, J., Bris, C., Lelièvre, T.: Mathematical Methods for the Magnetohydrodynamics of Liquid Metals, Numerical Mathematics and Scientific Computation. Oxford University Press, New York (2006)
3. Müller, U., Bühler, L.: Magnetofluiddynamics in Channels and Containers. Springer, Berlin (2001)
4. Gunzburger, M., Meir, A., Peterson, J.: On the existence, uniquess and finite element approximation of solutions of the equations of sationary, incompressible magnetohydrodynamic. Math. Comput. **56**, 523–563 (1991)
5. Moreau, R.: Magneto-hydrodynamics. Kluwer Academic Publishers, Dordrecht (1990)
6. Sermange, M., Temam, R.: Some mathematical questions related to the MHD equations. Commun. Pure Appl. Math. **36**, 635–664 (1983)
7. Gerbeau, J.: A stabilized finite element method for the incompressible magnetohydrodynamic equations. Numer. Math. **87**, 83–111 (2000)
8. Wu, J., Liu, D., Feng, X., Huang, P.: An efficient two-step algorithm for the stationary incompressible magnetohydrodynamic equations. Appl. Math. Comput. **302**, 21–33 (2017)
9. Zhang, G., He, Y., Zhang, Y.: Streamline diffusion finite element method for stationary incompressible magnetohydrodynamics. Numer. Methods Partial Differ. Equ. **30**, 1877–1901 (2014)
10. Zhao, J., Mao, S., Zheng, W.: Anisotropic adaptive finite element method for magnetohydrodynamic flow at high Hartmann numbers. Appl. Math. Mech. **37**, 1479–1500 (2016)
11. Hu, K., Ma, Y., Xu, J.: Stable finite element methods preserving $\nabla \cdot B = 0$ exactly for MHD models. Numer. Math. **135**, 371–396 (2017)
12. Li, L., Zheng, W.: A robust solver for the finite element approximation of stationary incompressible MHD equations in 3D. J. Comput. Phys. **351**, 254–270 (2017)
13. Ravindran, S.: Linear feedback control and approximation for a system governed by unsteady MHD equations. Comput. Methods Appl. Mech. Eng. **198**, 524–541 (2008)

14. Salah, N., Soulaimani, A., Habashi, W.: A finite element method for magnetohydrodynamics. Comput. Methods Appl. Mech. Eng. **190**, 5867–5892 (2001)
15. Meir, A., Schmidt, P.: Analysis and numerical approximation of a stationary MHD flow problem with nonideal boundary. SIAM J. Numer. Anal. **36**, 1304–1332 (1999)
16. He, Y., Li, J.: Convergence of three iterative methods based on the finite element discretization for the stationary Navier-Stokes equations. Comput. Methods Appl. Mech. Eng. **198**, 1351–1359 (2009)
17. Dong, X., He, Y., Zhang, Y.: Convergence analysis of three finite element iterative methods for the 2D/3D stationary incompressible magnetohydrodynamics. Comput. Methods Appl. Mech. Eng. **276**, 287–311 (2014)
18. Dong, X., He, Y.: Two-level Newton iterative method for the 2D/3D stationary incompressible magnetohydrodynamics. J. Sci. Comput. **63**, 426–451 (2015)
19. Su, H., Feng, X., Huang, P.: Iterative methods in penalty finite element discretization for the steady MHD equations. Comput. Methods Appl. Mech. Eng. **304**, 521–545 (2016)
20. Su, H., Feng, X., Zhao, J.: Two-level penalty Newton iterative method for the 2D/3D stationary incompressible magnetohydrodynamics equations. J. Sci. Comput. **70**(3), 1144–1179 (2017)
21. Shen, J.: On error estimates of some higher order projection and penalty-projection methods for Navier–Stokes equations. Numer. Math. **62**, 49–73 (1992)
22. Shen, J.: On error estimates of the penalty method for unsteady Navier–Stokes equations. SIAM J. Numer. Anal. **32**, 386–403 (1995)
23. He, Y.: Optimal error estimate of the penalty finite element method for the time-dependent Navier–Stokes equations. Math. Comput. **74**, 1201–1216 (2005)
24. He, Y., Li, J., Yang, X.: Two-level penalized finite element methods for the stationary Navier–Stoke equations. Int. J. Inf. Syst. Sci. **2**, 131–143 (2006)
25. An, R., Li, Y.: Error analysis of first-order projection method for time-dependent magnetohydrodynamics equations. Appl. Numer. Math. **112**, 167–181 (2017)
26. Zhu, T., Su, H., Feng, X.: Some Uzawa-type finite element iterative methods for the steady incompressible magnetohydrodynamic equations. Appl. Math. Comput. **302**, 34–47 (2017)
27. Zhang, Q., Su, H., Feng, X.: A partitioned finite element scheme based on Gauge–Uzawa method for time-dependent MHD equations. Numer. Algorithms **78**(1), 277–295 (2018)
28. Xu, J.: A novel two two-grid method for semilinear elliptic equations. SIAM J. Sci. Comput. **15**, 231–237 (1994)
29. Xu, J.: Two-grid discretization techniques for linear and nonlinear PDEs. SIAM J. Numer. Anal. **33**, 1759–1777 (1996)
30. Layton, W., Lenferink, H., Peterson, J.: A two-level Newton finite element algorithm for approximating electrically conducting incompressible fluid flows. Comput. Math. Appl. **28**, 21–31 (1994)
31. Layton, W., Meir, A., Schmidtz, P.: A two-level discretization method for the stationary MHD equations. Electron. Trans. Numer. Anal. **6**, 198–210 (1997)
32. Zhang, G., Zhang, Y., He, Y.: Two-level coupled and decoupled parallel correction methods for stationary incompressible magnetohydrodynamics. J. Sci. Comput. **65**, 920–939 (2015)
33. He, Y.: Unconditional convergence of the Euler semi-implicit scheme for the 3D incompressible MHD equations. IMA J. Numer. Anal. **35**, 767–801 (2014)
34. He, Y.: Two-level method based on fnite element and Crank–Nicolson extrapolation for the time-dependent Navier–Stokes equations. SIAM J. Numer. Anal. **41**, 1263–1285 (2003)