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The Temporal Second Order Difference Schemes Based on the Interpolation Approximation for the Time Multi-term Fractional Wave Equation

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Abstract
In this paper, two temporal second-order schemes are derived and analyzed for the time multi-term fractional diffusion-wave equation based on the order reduction technique. The weighted average at two time levels is applied to the discretization of the spatial derivative, in which the weight coefficient corresponds to the optimal point for the time discretization. The two difference schemes are proved to be uniquely solvable. The stability and convergence are rigorously investigated utilizing the energy method. In addition, a fast difference scheme is also presented. The applicability and the accuracy of the schemes are demonstrated by several numerical experiments.

Keywords Fractional diffusion-wave equation · Multi-term fractional derivatives · Difference scheme · Stability · Convergence · Fast algorithm

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1 Introduction

Recently, the fractional differential equations (FDES) have attracted more and more attention, which can simulate many physical and chemical processes more accuracy than the classical integer-order differential equations. FDES have been frequently used to solve many application problems [1–7]. The time fractional sub-diffusion and diffusion-wave equation are obtained from the classical diffusion or wave equation by replacing the first or second order time derivative by a fractional derivative of order α with $0 < \alpha < 1$ or $1 < \alpha < 2$, respectively. In practice, many processes can be described by the multi-term FDES, such as the underlying processes with loss [8], viscoelastic damping [9], oxygen delivery through a capillaryto tissues [10], the anomalous diffusion in highly heterogeneous aquifers and complex viscoelastic materials [11].

In particular, the multi-term time-fractional diffusion-wave equations can successfully describe the power-law frequency dependence in a continuous time random walk model [12].

For most fractional differential equations, it is very difficult to get the exact solutions. Many researchers have proposed various kinds of numerical methods for solving fractional differential equations [13–15]. Much work has been done numerically on the time diffusion-wave equations. For the approximation of the fractional derivative with order $\alpha \in (1, 2)$, Oldham and Spanier proposed the first-order GL formula based on the Grünwald-Letnikov derivative [16]. Sun and Wu [17] derived L1 formula using linear interpolation technique which keeps $(3-\alpha)$-order accuracy. Later, the L1 formula was used for solving the problem with diffusion-wave property [18–20], and the derived numerical schemes obtain $(3-\alpha)$-order accuracy in time. Zhao et al. [21] proposed a second-order formula using high-order interpolation for the variable-order fractional derivative with the order between 1 and 2, and applied the formula for solving wave propagation problem. Sun et al. [22] explored the $L2\text{-}1_\sigma$ formula for the fractional diffusion-wave problem and obtained the second-order scheme both in time and in space. Dehghan et al. [23] proposed a high-order numerical scheme to solve the space-time tempered fractional diffusion-wave equation. They employ the fourth-order technique to approximate the Riesz fractional derivative and a second-order approximation for the tempered fractional integral. The convergence order of the proposed method is $O(\tau^2+h^4)$. Ghazizadeh et al. [24] constructed a generalized MacCormack scheme and a fully implicit scheme for solving the fractional Cattaneo equation. The stability of the former scheme was analyzed using the Von Neumann stability criterion. The scheme keeps second-order spatial rate of convergence and $(1+\alpha)$-order temporal rate of convergence, where $\alpha \in (1, 2)$ is the order of fractional derivative. Li and Cao [25] presented an unconditional stable scheme with convergence order of O$(\tau^{3-\alpha} + h^2)$ for the 1D Cattaneo equation. Vong et al. [26] derived a fourth order finite difference scheme for the 1D generalized fractional Cattaneo equation combining L1 approximation for the time fractional derivative and compact difference scheme for the second-order space derivative. The stability and convergence were proved in the maximum norm by the energy method.

There are many numerical methods for multi-term fractional diffusion equation, such as Galerkin finite element [27], finite difference method [28,38], spectral method [29], and so on. Some research work on multi-term time fractional diffusion-wave equation has been made. In [30], Salehi applied a meshless collocation method to solve the multi-term time fractional diffusion-wave equation in two dimensions. The Caputo time fractional derivatives are approximated by a scheme of order $O(\tau^{3-\alpha})$, $\alpha \in (1, 2)$. Abdel-Rehim et al. [31] gave the simulations of the approximation solutions of time-fractional wave, forced wave (shear wave), and damped wave equations. The Von-Neumann stability conditions are also considered and discussed for these models. Liu [32] established a strong maximum principle for fractional diffusion equations with multiple Caputo derivatives in time, and investigate a related inverse problem of practical importance. Bhrawy and Zaky [33] proposed a shifted Jacobi tau method for both temporal and spatial discretizations for multi-term time-space fractional differential equation with Dirichlet boundary conditions. Dehghan et al. [34] constructed a high order difference scheme and Galerkin spectral technique for the numerical solution of multi-term time fractional partial differential equations. The proposed methods are based on a finite difference scheme in time, which have $(3-\alpha)$ order accuracy.

Ren and Sun [35] proposed some efficient numerical schemes to solve one-dimensional and two-dimensional multi-term time fractional diffusion-wave equation, by combining the compact difference approach for the spatial discretisation and an L1 approximation for the multi-term time Caputo fractional derivatives. Liu et al. [36] proposed a finite difference scheme for solving a two-term time-fractional wave-diffusion equation. Brunner et al. [37] introduced an artificial boundary and found the exact and approximate artificial boundary conditions for the time-fractional diffusion-wave equation on a two-dimensional unbounded spatial domain, which leads to a problem on a bounded computational domain.

It is noted that the above methods for multi-term fractional diffusion-wave equation are obtained mainly by applying directly the techniques which are used to handle the single-term fractional diffusion-wave equation, including L1 formula and GL formula. L1 formula can only achieve $3-\alpha$ order accuracy which is a little lower. Although GL formula can obtain 2 order accuracy, it requires the continuous zero-extension of the solution when $t<0$.

In [38], the authors proposed a numerical formula to approximate the multi-term Caputo fractional derivatives of order α_r $(0<\alpha_r\le 1)$ at the super-convergent point. The formula can achieve at least second-order accuracy at this point. And some effective difference schemes for solving the time multi-term fractional sub-diffusion equation and the time distributed-order sub-diffusion equation, respectively, are presented along with the theoretical analysis on the solvability, stability and convergence.

Motivated by the novel idea proposed in [38] and combining with the order reduction method, we will present two temporal second-order accuracy difference schemes based on the interpolation approximation for the time multi-term fractional wave equation. The unconditional stability and convergence of the proposed difference schemes in L_∞ norm are proved, and the convergence order of the two difference schemes is $O(\tau^2+h^2)$ and $O(\tau^2+h^4)$, respectively.

Most difference schemes for time fractional differential equations require storing the solution at all previous time steps for use and huge computational cost. Nowadays some efforts have been made to develop efficient fast numerical methods for the Caputo derivative. Jiang et al. [39] proposed a fast evaluation of Caputo fractional derivative based on the $L1$ formula which employed the sum-of-exponentials (SOE) approximation to the kernel function $t^{-1-\alpha}$. The fast algorithm keeps the accuracy of $O(\tau^{2-\alpha})$ and reduces the computational complexity significantly. Yan et al. [40] proposed a fast $\mathcal{F}L2\text{-}1_\sigma$ formula for the Caputo fractional derivative combining the $L2\text{-}1_\sigma$ formula with SOE approximation. The formula has high accuracy and reduces the storage and computational cost. We will develop a fast difference scheme by combining $\mathcal{F}L2\text{-}1_\sigma$ formula with the method of the order reduction for time fractional diffusion wave equation.

In this paper, consider the following time multi-term fractional wave equation

$$\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} u(x,t) = u_{xx}(x,t) + f(x,t), \quad 0 < x < L, \ 0 < t \le T, \tag{1.1}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad 0 < t \le T, \tag{1.2}$$

$$u(x,0) = w_1(x), \quad u_t(x,0) = w_2(x), \quad 0 \le x \le L, \tag{1.3}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_m$ are some positive constants, $1 < \alpha_m < \alpha_{m-1} < \cdots < \alpha_0 \le 2$ and at least one of α_i's belongs to $(1,2)$, ${}_0^C D_t^{\alpha} f(t)$ is the Caputo fractional derivative defined by

$${}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{f''(s)}{(t-s)^{\alpha-1}} ds.$$

This paper is arranged as follows. In Sect. 2, some useful notations and lemmas are introduced. A temporal and spatial second order difference scheme is presented for time multi-term fractional diffusion wave equation in Sect. 3. The stability and convergence of the difference scheme are discussed. Sect. 4 constructs a temporal second order and spatial fourth-order compact difference scheme. The stability and convergence of the compact difference scheme are also shown. In Sect. 5, a fast second-order difference scheme is presented for the time multi-term fractional diffusion wave equation. In Sect. 6, two numerical examples are demonstrated to verify the theoretical results. The paper ends with a brief conclusion in Sect. 7.

2 Preliminary

Denote

$$\gamma_r = \alpha_r - 1, \quad 0 \le r \le m$$

and

$$F(\sigma) = \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(3-\gamma_r)} \sigma^{1-\gamma_r} \left[\sigma - \left(1 - \frac{\gamma_r}{2}\right)\right] \tau^{2-\gamma_r}, \quad \sigma \ge 0.$$

It is easy to know that $0 < \gamma_m < \gamma_{m-1} < \cdots < \gamma_0 \le 1$.

Lemma 2.1 [38] *The equation* $F(\sigma) = 0$ *has a unique positive root* $\sigma^* \in [a,b]$, *where* $a = 1 - \frac{\gamma_0}{2}$, $b = 1 - \frac{\gamma_m}{2}$.

If $m = 0$, the root of $F(\sigma) = 0$ is $\sigma^* = 1 - \frac{\gamma_0}{2}$. If $m \ge 1$, the root σ^* of $F(\sigma) = 0$ can be obtained by the Newton iteration method.

Lemma 2.2 [38] *For* $m \ge 1$, *the Newton iteration sequence* $\{\sigma_k\}_{k=0}^{\infty}$, *generated by*

$$\begin{cases} \sigma_{k+1} = \sigma_k - \frac{F(\sigma_k)}{F'(\sigma_k)}, & k = 0, 1, 2, \ldots, \\ \sigma_0 = b, \end{cases} \tag{2.1}$$

is monotonically decreasing and convergent to σ^*.

For simplicity in writing hear and after, let $\sigma = \sigma^*$. For $0 < \gamma < 1$, a sequence $\{c_n^{(k+1,\gamma)}\}$ defined in [41] is introduced in the following.

$$a_0^{(\gamma)} = \sigma^{1-\gamma}, \ a_l^{(\gamma)} = (l+\sigma)^{1-\gamma} - (l-1+\sigma)^{1-\gamma}, \ l \ge 1,$$

$$b_l^{(\gamma)} = \frac{1}{2-\gamma}[(l+\sigma)^{2-\gamma} - (l-1+\sigma)^{2-\gamma}] - \frac{1}{2}[(l+\sigma)^{1-\gamma} + (l-1+\sigma)^{1-\gamma}], \ l \ge 1.$$

For $k = 0$

$$c_0^{(k+1,\gamma)} = a_0. \tag{2.2}$$

For $k \geq 1$

$$c_n^{(k+1,\gamma)} = \begin{cases} a_0^{(\gamma)} + b_1^{(\gamma)}, & n = 0, \\ a_n^{(\gamma)} + b_{n+1}^{(\gamma)} - b_n^{(\gamma)}, & 1 \leq n \leq k-1, \\ a_k^{(\gamma)} - b_k^{(\gamma)}, & n = k. \end{cases} \tag{2.3}$$

Denote

$$\hat{c}_n^{(k+1)} = \sum_{r=0}^{m} \lambda_r \frac{\tau^{-\gamma_r}}{\Gamma(2-\gamma_r)} c_n^{(k+1,\gamma_r)}, \quad n = 0, 1, \ldots, k$$

and

$$\hat{b}_n = \sum_{r=0}^{m} \lambda_r \frac{\tau^{-\gamma_r}}{\Gamma(2-\gamma_r)} b_n^{(\gamma_r)}, \quad n = 0, 1, \ldots, k.$$

The properties of the coefficients $\{\hat{c}_n^{(k)}\}$ and $\{\hat{b}_n\}$ will be stated in the following two lemmas.

Lemma 2.3 [38] *Given any non-negative integer m and positive constants $\lambda_0, \lambda_1, \ldots, \lambda_m$, for any $\gamma_i \in (0, 1]$, $i = 0, 1, \ldots, m$, it holds*

$$\hat{c}_1^{(k+1)} > \hat{c}_2^{(k+1)} > \cdots > \hat{c}_{k-2}^{(k+1)} > \hat{c}_{k-1}^{(k+1)} > \sum_{r=0}^{m} \lambda_r \frac{\tau^{-\gamma_r}}{\Gamma(2-\gamma_r)} \cdot \frac{1-\gamma_r}{2} (k-1+\sigma)^{-\gamma_r}.$$

In addition, there exists a $\tau_0 > 0$, such that

$$(2\sigma - 1)\hat{c}_0^{(k+1)} - \sigma \hat{c}_1^{(k+1)} > 0,$$

when $\tau \leq \tau_0$, $n = 2, 3, \ldots,$ and

$$\hat{c}_0^{(k+1)} > \hat{c}_1^{(k+1)}.$$

Lemma 2.4 [22] *The sequences $\{\hat{c}_n^{(k)}\}$ and $\{\hat{b}_n\}$ satisfy*

$$\hat{c}_n^{(k+1)} = \begin{cases} \hat{c}_n^{(k)}, & 0 \leq n \leq k-2, \\ \hat{c}_n^{(k)} + \hat{b}_{n+1}, & n = k-1. \end{cases}$$

In addition

$$\sum_{n=1}^{k} \hat{c}_n^{(k+1)} \leq \sum_{r=0}^{m} \lambda_r \frac{3\tau^{-\gamma_r}}{2\Gamma(2-\gamma_r)} (k+\sigma)^{1-\gamma_r}$$

and

$$\sum_{n=1}^{k} \hat{b}_n \leq \sum_{r=0}^{m} \lambda_r \frac{\gamma_r \tau^{-\gamma_r}}{2\Gamma(3-\gamma_r)} (k+\sigma)^{1-\gamma_r}.$$

Take two positive integers M, N and let $h = \frac{L}{M}, \tau = \frac{T}{N}$. Denote $x_i = ih, t_k = k\tau$, $\Omega_h = \{x_i \mid 0 \le i \le M\}$, $\Omega_\tau = \{t_k \mid 0 \le k \le N\}$ and $t_{k+\sigma} = t_k + \sigma\tau$.

If $w = \{w^k \mid 0 \le k \le N\}$ is a grid function defined on Ω_τ, denote

$$\delta_t w^{\frac{1}{2}} = \frac{1}{\tau}(w^1 - w^0),$$
$$D_{\hat{t}} w^k = \frac{1}{2\tau}\left[(2\sigma+1)w^{k+1} - 4\sigma w^k + (2\sigma-1)w^{k-1}\right],\ 1 \le k \le N-1$$

and

$$w^{k+\sigma} = \sigma w^{k+1} + (1-\sigma)w^k.$$

Let

$$\mathcal{U}_h = \{u \mid u = (u_0, \ldots, u_M), u_0 = u_M = 0\}.$$

For $u \in \mathcal{U}_h$, introduce the following notations

$$\delta_x u_{i+\frac{1}{2}} = \frac{1}{h}(u_{i+1} - u_i), \quad \delta_x^2 u_i = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}),$$
$$\mathcal{A}u_i = \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}).$$

For any $u, v \in \mathcal{U}_h$, the inner products and norms are defined by

$$(u, v) = h\sum_{i=1}^{M-1} u_i v_i, \quad (\delta_x u, \delta_x v) = h\sum_{i=0}^{M-1}\left(\delta_x u_{i+\frac{1}{2}}\right)\left(\delta_x v_{i+\frac{1}{2}}\right), \quad \langle u, v\rangle_{\mathcal{A}} = (u, \mathcal{A}v),$$
$$\|u\| = \sqrt{(u,u)}, \quad \|\delta_x u\| = \sqrt{(\delta_x u, \delta_x u)}, \quad \|u\|_{\mathcal{A}} = \sqrt{\langle u, u\rangle_{\mathcal{A}}}, \quad \|u\|_\infty = \max_{0\le i\le M}|u_i|.$$

Lemma 2.5 [38] *Suppose $f \in C^3([0,T])$, for any $\gamma_i \in (0,1], i = 0, 1, \ldots, m$ and $\gamma_0 > \gamma_1 > \cdots > \gamma_m$, then it holds*

$$\sum_{r=0}^{m}\lambda_r\, {}_0^C D_t^{\gamma_r} f(t_{k+\sigma}) = \sum_{n=0}^{k}\Big(\sum_{r=0}^{m}\lambda_r \frac{\tau^{-\gamma_r}}{\Gamma(2-\gamma_r)} c_n^{(k+1,\gamma_r)}\Big)\left[f(t_{k-n+1}) - f(t_{k-n})\right] + O(\tau^{3-\gamma_0})$$
$$= \sum_{n=0}^{k}\hat{c}_{k-n}^{(k+1)}\left[f(t_{n+1}) - f(t_n)\right] + O(\tau^{3-\gamma_0}).$$

Lemma 2.6 [22] *Suppose $f \in C^3([0,T])$. It holds*

$$D_{\hat{t}} f(t_k) \equiv \frac{1}{2\tau}\left[(2\sigma+1)f(t_{k+1}) - 4\sigma f(t_k) + (2\sigma-1)f(t_{k-1})\right]$$
$$= \frac{df}{dt}(t_{k+\sigma}) + O(\tau^2),\ k \ge 1.$$

Lemma 2.7 [38] *Suppose $\langle\cdot,\cdot\rangle_*$ is an inner product on $\mathcal{U}_h$, $\|\cdot\|_*$ is a norm deduced by the inner product. For any grid functions $v^0, v^1, \ldots, v^{k+1} \in \mathcal{U}_h$, we have the following inequality*

$$\Big\langle \sum_{n=0}^{k}\hat{c}_{k-n}^{(k+1)}(v^{n+1} - v^n), v^{k+\sigma}\Big\rangle_* \ge \frac{1}{2}\sum_{n=0}^{k}\hat{c}_{k-n}^{(k+1)}\left(\|v^{n+1}\|_*^2 - \|v^n\|_*^2\right).$$

Lemma 2.8 [22] *For any grid functions* $u^0, u^1, \ldots, u^N \in \mathcal{U}_h$*, we have the following inequality*

$$\left(D_{\hat{t}}u^k, u^{k+\sigma}\right) \geq \frac{1}{4\tau}(E^{k+1} - E^k),\ k \geq 1,$$

with

$$E^{k+1} = (2\sigma+1)\|u^{k+1}\|^2 - (2\sigma-1)\|u^k\|^2 + (2\sigma^2+\sigma-1)\|u^{k+1}-u^k\|^2,\ k \geq 0. \tag{2.4}$$

In addition, it holds

$$E^{k+1} \geq \frac{1}{\sigma}\|u^{k+1}\|^2,\ k \geq 0. \tag{2.5}$$

Lemma 2.9 [43] *For any* $u \in \mathcal{U}_h$*, we have*

$$\|u\|_\infty \leq \frac{\sqrt{L}}{2}\|\delta_x u\|, \quad \|u\| \leq \frac{L}{\sqrt{6}}\|\delta_x u\|,$$

and

$$\frac{2}{3}\|u\|^2 \leq \|u\|_{\mathcal{A}}^2 \leq \|u\|^2, \quad \|\mathcal{A}u\| \leq \|u\|.$$

Lemma 2.10 [42] *Assume the grid function* $\{w^k \mid 0 \leq k \leq N\}$ *is a nonnegative sequence and satisfies the inequality*

$$w^k \leq A + \tau B \sum_{p=1}^{k} w^p, \quad 0 \leq k \leq N,$$

where A, B *are nonnegative constants. Then, when* $\tau \leq \frac{1}{2B}$*, we have*

$$w^k \leq A\exp(2Bk\tau),\ 0 \leq k \leq N.$$

3 A Second-Order Difference Scheme in Time and Space

3.1 The Derivation of the Difference Scheme

Now, combining the super-convergence approximation [38] with the order reduction method, we construct the difference scheme for the problem (1.1)–(1.3).

Let $\gamma_r = \alpha_r - 1,\ 0 \leq r \leq m$ and

$$v(x,t) = u_t(x,t). \tag{3.1}$$

Then

$$\begin{aligned}\frac{\partial^{\alpha_r}u}{\partial t^{\alpha_r}}(x,t) =& \frac{1}{\Gamma(2-\alpha_r)}\int_0^t \frac{\partial^2 u}{\partial s^2}(x,s)\frac{1}{(t-s)^{\alpha_r-1}}ds \\ =& \frac{1}{\Gamma(1-\gamma_r)}\int_0^t \frac{\partial v}{\partial s}(x,s)\frac{1}{(t-s)^{\gamma_r}}ds \\ =& \frac{\partial^{\gamma_r}v}{\partial t^{\gamma_r}}(x,t).\end{aligned} \tag{3.2}$$

It follows from (3.1) that

$$(u_{xx})_t = v_{xx}.$$

Then, Eqs. (1.1)–(1.3) are equivalent to the following equation

$$\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\gamma_r} v(x,t) = u_{xx}(x,t) + f(x,t), \quad 0 < x < L, \ 0 < t \le T, \tag{3.3}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial^2 v}{\partial x^2}, \quad x \in (0,L), \ t \in (0,T], \tag{3.4}$$

$$u(x,0) = w_1(x), \ v(x,0) = w_2(x), \quad x \in [0,L], \tag{3.5}$$

$$u(0,t) = 0, \ u(L,t) = 0, \quad t \in (0,T], \tag{3.6}$$

$$v(0,t) = 0, \ v(L,t) = 0, \quad t \in (0,T]. \tag{3.7}$$

Suppose $u(x,t) \in C_{x,t}^{4,4}([0,L] \times [0,T])$. Define the grid functions

$$U_i^k = u(x_i, t_k), \quad V_i^k = v(x_i, t_k), \quad 0 \le i \le M, \quad 0 \le k \le N.$$

Considering (3.3) at the point $(x_i, t_{k+\sigma})$, we have

$$\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\gamma_r} v(x_i, t_{k+\sigma}) = u_{xx}(x_i, t_{k+\sigma}) + f(x_i, t_{k+\sigma}), \quad 1 \le i \le M-1, \ 0 \le k \le N-1. \tag{3.8}$$

Using Lemma 2.5, we obtain

$$\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\gamma_r} v(x_i, t_{k+\sigma}) = \sum_{n=0}^{k} \hat{c}_{k-n}^{(k+1)} \left(V_i^{n+1} - V_i^n\right) + O(\tau^{3-\gamma_0}),$$
$$1 \le i \le M-1, \ 0 \le k \le N-1. \tag{3.9}$$

By Taylor expansion, it yields

$$\begin{aligned} u_{xx}(x_i, t_{k+\sigma}) &= \sigma u_{xx}(x_i, t_{k+1}) + (1-\sigma) u_{xx}(x_i, t_k) + O(\tau^2) \\ &= \sigma \delta_x^2 U_i^{k+1} + (1-\sigma)\delta_x^2 U_i^k + O(\tau^2 + h^2), \\ &= \delta_x^2 U_i^{k+\sigma} + O(\tau^2 + h^2), \quad 1 \le i \le M-1, \ 0 \le k \le N-1. \end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we get

$$\sum_{n=0}^{k} \hat{c}_{k-n}^{(k+1)} \left(V_i^{n+1} - V_i^n\right) = \delta_x^2 U_i^{k+\sigma} + f_i^{k+\sigma} + R_i^{k+\sigma}, \quad 1 \le i \le M-1, \ 0 \le k \le N-1, \tag{3.11}$$

where $f_i^{k+\sigma} = f(x_i, t_{k+\sigma})$ and there exists a constant c_0 such that

$$|R_i^{k+\sigma}| \le c_0(\tau^2 + h^2), \quad 1 \le i \le M-1, \quad 0 \le k \le N-1. \tag{3.12}$$

Considering Eq. (3.4) at the points $(x_i, t_{\frac{1}{2}})$ and $(x_i, t_{k+\sigma})$, respectively, we have

$$(u_{xx})_t \left(x_i, t_{\frac{1}{2}}\right) = v_{xx}\left(x_i, t_{\frac{1}{2}}\right), \ 1 \le i \le M-1 \tag{3.13}$$

and

$$(u_{xx})_t(x_i, t_{k+\sigma}) = v_{xx}(x_i, t_{k+\sigma}),\ 1 \le i \le M-1,\ 1 \le k \le N-1. \quad (3.14)$$

By Taylor expansion, it follows from (3.13) that

$$\delta_t \delta_x^2 U_i^{\frac{1}{2}} = \delta_x^2 V_i^{\frac{1}{2}} + r_i^{\frac{1}{2}},\ 1 \le i \le M-1. \quad (3.15)$$

By Lemma 2.6 and Taylor expansion, it follows from (3.14) that

$$D_{\hat{t}} \delta_x^2 U_i^k = \delta_x^2 V_i^{k+\sigma} + r_i^{k+\sigma},\ 1 \le i \le M-1,\ 1 \le k \le N-1. \quad (3.16)$$

There exists a constant c_1 such that

$$|r_i^{\frac{1}{2}}| \le c_1(\tau^2 + h^2), \quad 1 \le i \le M-1, \quad (3.17)$$

$$|r_i^{k+\sigma}| \le c_1(\tau^2 + h^2), \quad 1 \le i \le M-1, \quad 1 \le k \le N-1. \quad (3.18)$$

In addition, noticing (3.4)–(3.6), we obtain

$$U_i^0 = w_1(x_i),\ V_i^0 = w_2(x_i),\ 1 \le i \le M-1, \quad (3.19)$$

$$U_0^k = 0,\ U_M^k = 0,\ 0 \le k \le N, \quad (3.20)$$

$$V_0^k = 0,\ V_M^k = 0,\ 0 \le k \le N. \quad (3.21)$$

Omitting the small terms in (3.11), (3.15) and (3.16) and noticing (3.19), (3.21) and we construct the difference scheme for the problem (1.1)–(1.3) as follows

$$\sum_{n=0}^{k} \hat{c}_{k-n}^{(k+1)}(v_i^{n+1} - v_i^n) = \delta_x^2 u_i^{k+\sigma} + f_i^{k+\sigma}, \quad 1 \le i \le M-1,\ 0 \le k \le N-1, \quad (3.22)$$

$$\delta_t \delta_x^2 u_i^{\frac{1}{2}} = \delta_x^2 v_i^{\frac{1}{2}},\ 1 \le i \le M-1, \quad (3.23)$$

$$D_{\hat{t}} \delta_x^2 u_i^k = \delta_x^2 v_i^{k+\sigma},\ 1 \le i \le M-1,\ 1 \le k \le N-1, \quad (3.24)$$

$$u_i^0 = w_1(x_i),\ v_i^0 = w_2(x_i),\ 1 \le i \le M-1, \quad (3.25)$$

$$u_0^k = 0,\ u_M^k = 0,\ 0 \le k \le N, \quad (3.26)$$

$$v_0^k = 0,\ v_M^k = 0,\ 0 \le k \le N. \quad (3.27)$$

3.2 The Unique Solvability of the Difference Scheme

Theorem 3.1 *The difference Scheme (3.22)–(3.27) is uniquely solvable.*

Proof Denote $u^k = (u_0^k, u_1^k, \ldots, u_M^k),\ v^k = (v_0^k, v_1^k, \ldots, u_M^k)$.

(1) For $k = 0$, we can obtain the system of linear algebraic equations about the unknowns u^1 and v^1 from (3.22), (3.23), (3.26) and (3.27). Considering its homogenous system, we have

$$\hat{c}_0^{(1)} v_i^1 = \sigma \delta_x^2 u_i^1, \quad 1 \le i \le M-1, \quad (3.28)$$

$$\frac{1}{\tau} \delta_x^2 u_i^1 = \frac{1}{2} \delta_x^2 v_i^1, \quad 1 \le i \le M-1, \quad (3.29)$$

$$u_0^1 = 0,\ u_M^1 = 0,\ v_0^1 = 0,\ v_M^1 = 0. \quad (3.30)$$

Solving $\delta_x^2 u_i^1$ from (3.29) and substituting the result into (3.28), we obtain

$$\hat{c}_0^{(1)} v_i^1 = \frac{\sigma\tau}{2}\delta_x^2 v_i^1, \quad 1 \le i \le M-1, \tag{3.31}$$

Taking the inner product of (3.31) with v^1 and using the summation by parts, we get

$$\hat{c}_0^{(1)} \|v^1\|^2 + \frac{\sigma\tau}{2}\|\delta_x v^1\|^2 = 0.$$

It implies that

$$v_i^1 = 0, \quad 1 \le i \le M-1.$$

Then, it follows from (3.28) that

$$\delta_x^2 u_i^1 = 0, \quad 1 \le i \le M-1. \tag{3.32}$$

Taking the inner product of (3.32) with u^1 and noticing (3.30), it yields

$$\|\delta_x u^1\| = 0, \quad \le k \le N-1.$$

Then we get

$$u_i^1 = 0, \quad 1 \le i \le M-1.$$

(2) For $k(1 \le k \le N-1)$, suppose that $\{u^{k-1},\ v^{k-1}, u^k, v^k\}$ have been determined, then we get a linear system of equations with respect to u^{k+1} and v^{k+1} from (3.22), (3.24), (3.26) and (3.27).

Consider the corresponding homogeneous system

$$\hat{c}_0^{(k+1)} v_i^{k+1} = \sigma \delta_x^2 u_i^{k+1}, \quad 1 \le i \le M-1, \tag{3.33}$$

$$\frac{2\sigma+1}{2\sigma\tau}\delta_x^2 u_i^{k+1} = \delta_x^2 v_i^{k+1}, \quad 1 \le i \le M-1, \tag{3.34}$$

$$u_0^{k+1} = 0,\ u_M^{k+1} = 0, \tag{3.35}$$

$$v_0^{k+1} = 0,\ v_M^{k+1} = 0. \tag{3.36}$$

Solving $\delta_x^2 u_i^{k+1}$ from (3.34) and substituting the result into (3.33), it yields

$$\hat{c}_0^{(k+1)} v_i^{k+1} = \frac{2\sigma^2\tau}{2\sigma+1}\delta_x^2 v_i^{k+1}, \quad 1 \le i \le M-1. \tag{3.37}$$

Taking the inner product of (3.37) with v^{k+1} and using the summation by parts, we obtain

$$\hat{c}_0^{(k+1)} \|v^{k+1}\|^2 + \frac{2\sigma^2\tau}{2\sigma+1}\|\delta_x v^{k+1}\|^2 = 0,$$

which yields that

$$v_i^{k+1} = 0, \quad 1 \le i \le M-1.$$

Consequently, it follows from (3.33) that

$$\delta_x^2 u_i^{k+1} = 0,\ 1 \le i \le M-1. \tag{3.38}$$

Taking the inner product of (3.38) with u^{k+1}, we have

$$\|\delta_x u^{k+1}\| = 0.$$

Then it yields

$$u_i^{k+1} = 0, \quad 1 \le i \le M-1.$$

According to the induction principle, this completes the proof. □

3.3 The Stability and Convergence of the Difference Scheme

Firstly, we present the priori estimate of the difference Scheme (3.22)–(3.27). The proof is divided into two steps, which correspond to the case $k = 0$ and $k \ge 1$.

Theorem 3.2 *Suppose* $\{p_i^k \mid 0 \le i \le M,\ 0 \le k \le N\}$ *and* $\{q_i^k \mid 0 \le i \le M,\ 0 \le k \le N\}$ *satisfy*

$$\sum_{n=0}^{k} \hat{c}_{k-n}^{(k+1)}(q_i^{n+1} - q_i^n) = \delta_x^2 p_i^{k+\sigma} + f_i^{k+\sigma}, \quad 1 \le i \le M-1,\ 0 \le k \le N-1, \tag{3.39}$$

$$\delta_t \delta_x^2 p_i^{\frac{1}{2}} = \delta_x^2 q_i^{\frac{1}{2}} + g_i^{\frac{1}{2}}, \quad 1 \le i \le M-1, \tag{3.40}$$

$$D_{\hat{t}} \delta_x^2 p_i^k = \delta_x^2 q_i^{k+\sigma} + g_i^{k+\sigma}, \quad 1 \le i \le M-1,\ 1 \le k \le N-1, \tag{3.41}$$

$$p_i^0 = w_1(x_i),\ q_i^0 = w_2(x_i), \quad 1 \le i \le M-1, \tag{3.42}$$

$$p_0^k = 0,\ p_M^k = 0,\ q_0^k = 0,\ q_M^k = 0,\ 0 \le k \le N, \tag{3.43}$$

where $w_1(x_i) = 0,\ w_2(x_i) = 0$ *for* $i = 0, M$. *Then there exists a constant* τ_0 *such that the following inequality holds when* $\tau \le \tau_0$,

$$\|\delta_x p^k\|^2 \le c_2 \exp\Big(\frac{4\sigma L^2}{3} T\Big) G_k, \quad \tau \sum_{n=1}^{k} \|q^n\|^2 \le c_3 G_k,\ 0 \le k \le N.$$

where c_2 *and* c_3 *are two constants and*

$$G_k = \|\delta_x p^0\|^2 + \|\delta_x^2 p^0\|^2 + \|q^0\|^2 + \|\delta_x q^0\|^2 + \|f^\sigma\|^2 + \|g^{\frac{1}{2}}\|^2 + \tau \sum_{l=1}^{k-1} \|f^{l+\sigma}\|^2 + \tau \sum_{l=1}^{k-1} \|g^{l+\sigma}\|^2.$$

Proof **Step1.** When $k = 0$, the system is as follows

$$\hat{c}_0^{(1)}(q_i^1 - q_i^0) = \sigma \delta_x^2 p_i^1 + (1-\sigma)\delta_x^2 p_i^0 + f_i^\sigma,\ 1 \le i \le M-1, \tag{3.44}$$

$$\delta_t \delta_x^2 p_i^{\frac{1}{2}} = \delta_x^2 q_i^{\frac{1}{2}} + g_i^{\frac{1}{2}},\ 1 \le i \le M-1, \tag{3.45}$$

$$p_i^0 = w_1(x_i),\ q_i^0 = w_2(x_i), \quad 1 \le i \le M-1, \tag{3.46}$$

$$p_0^1 = 0,\ p_M^1 = 0,\ q_0^1 = 0,\ q_M^1 = 0 \tag{3.47}$$

with $p_0^0 = 0,\ p_M^0 = 0,\ q_0^0 = 0,\ q_M^0 = 0$.

(I) Taking the inner product of (3.44) with q^1, we have

$$\hat{c}_0^{(1)}\|q^1\|^2 = \hat{c}_0^{(1)}(q^0, q^1) - \sigma\big(\delta_x p^1, \delta_x q^1\big) + (1-\sigma)(\delta_x^2 p^0, q^1) + (f^\sigma, q^1). \tag{3.48}$$

Taking the inner product of (3.45) with $-2\sigma p^1$ and by the summation by parts, it yields

$$\frac{2\sigma}{\tau}\|\delta_x p^1\|^2 = \frac{2\sigma}{\tau}(\delta_x p^0, \delta_x p^1) + \sigma(\delta_x q^1, \delta_x p^1) + \sigma(\delta_x q^0, \delta_x p^1) - 2\sigma(g^{\frac{1}{2}}, p^1). \tag{3.49}$$

Adding (3.48) with (3.49) and using Young inequality and Lemma 2.9, we obtain

$$\begin{aligned}
&\hat{c}_0^{(1)}\|q^1\|^2 + \frac{2\sigma}{\tau}\|\delta_x p^1\|^2 \\
&\quad \le \hat{c}_0^{(1)}(q^0, q^1) + (1-\sigma)(\delta_x^2 p^0, q^1) + (f^\sigma, q^1) \\
&\qquad + \frac{2\sigma}{\tau}(\delta_x p^0, \delta_x p^1) + \sigma(\delta_x q^0, \delta_x p^1) - 2\sigma(g^{\frac{1}{2}}, p^1) \\
&\quad \le \Big(\frac{\hat{c}_0^{(1)}}{3}\|q^1\|^2 + \frac{3\hat{c}_0^{(1)}}{4}\|q^0\|^2\Big) + \Big(\frac{\hat{c}_0^{(1)}}{3}\|q^1\|^2 + \frac{3(1-\sigma)^2}{4\hat{c}_0^{(1)}}\|\delta_x^2 p^0\|^2\Big) \\
&\qquad + \Big(\frac{\hat{c}_0^{(1)}}{3}\|q^1\|^2 + \frac{3}{4\hat{c}_0^{(1)}}\|f^\sigma\|^2\Big) + \Big(\frac{\sigma}{3\tau}\|\delta_x p^1\|^2 + \frac{3\sigma}{\tau}\|\delta_x p^0\|^2\Big) \\
&\qquad + \Big(\frac{\sigma}{3\tau}\|\delta_x p^1\|^2 + \frac{3\sigma\tau}{4}\|\delta_x q^0\|^2\Big) + \Big(\frac{\sigma}{3\tau}\|\delta_x p^1\|^2 + \frac{L^2\sigma}{2}\tau\|g^{\frac{1}{2}}\|^2\Big).
\end{aligned} \tag{3.50}$$

It follows that

$$\begin{aligned}
\|\delta_x p^1\|^2 &\le 3\|\delta_x p^0\|^2 + \frac{3\hat{c}_0^{(1)}\tau}{4\sigma}\|q^0\|^2 + \frac{3(1-\sigma)^2\tau}{4\hat{c}_0^{(1)}\sigma}\|\delta_x^2 p^0\|^2 + \frac{3\tau}{4\hat{c}_0^{(1)}\sigma}\|f^\sigma\|^2 \\
&\quad + \frac{3\tau^2}{4}\|\delta_x q^0\|^2 + \frac{L^2}{2}\tau^2\|g^{\frac{1}{2}}\|^2.
\end{aligned} \tag{3.51}$$

(II) It follows from (3.45) that

$$\delta_x^2 p_i^1 = \delta_x^2 p_i^0 + \tau\delta_x^2 q_i^{\frac{1}{2}} + \tau g_i^{\frac{1}{2}}, \quad 1 \le i \le M-1. \tag{3.52}$$

Substituting (3.52) into (3.44), we have

$$\hat{c}_0^{(1)}(q_i^1 - q_i^0) = \delta_x^2 p_i^0 + \sigma\tau\delta_x^2 q_i^{\frac{1}{2}} + f_i^\sigma + \sigma\tau g_i^{\frac{1}{2}}, \quad 1 \le i \le M-1. \tag{3.53}$$

Taking the inner product of (3.53) with $q^{\frac{1}{2}}$, we obtain

$$\hat{c}_0^{(1)}\left(q^1 - q^0, q^{\frac{1}{2}}\right) = \left(\delta_x^2 p^0, q^{\frac{1}{2}}\right) + \sigma\tau\left(\delta_x^2 q^{\frac{1}{2}}, q^{\frac{1}{2}}\right) + \left(f^\sigma, q^{\frac{1}{2}}\right) + \sigma\tau\left(g^{\frac{1}{2}}, q^{\frac{1}{2}}\right).$$

By the summation by parts and Young inequality $ab \le \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$ (taking $\varepsilon = \frac{3}{\hat{c}_0^{(1)}}$), it yields

$$\begin{aligned}
&\frac{\hat{c}_0^{(1)}}{2}(\|q^1\|^2 - \|q^0\|^2) \\
&= \left(\delta_x^2 p^0, q^{\frac{1}{2}}\right) - \sigma\tau\|\delta_x q^{\frac{1}{2}}\|^2 + \left(f^\sigma, q^{\frac{1}{2}}\right) + \sigma\tau\left(g^{\frac{1}{2}}, q^{\frac{1}{2}}\right) \\
&\le \left(\frac{1}{2\varepsilon}\|\delta_x^2 p^0\|^2 + \frac{\varepsilon}{2}\|q^{\frac{1}{2}}\|^2\right) + \left(\frac{1}{2\varepsilon}\|f^\sigma\|^2 + \frac{\varepsilon}{2}\|q^{\frac{1}{2}}\|^2\right) + \left(\frac{1}{2\varepsilon}\|\sigma\tau g^{\frac{1}{2}}\|^2 + \frac{\varepsilon}{2}\|q^{\frac{1}{2}}\|^2\right) \\
&\le \left(\frac{\hat{c}_0^{(1)}}{6}\|q^{\frac{1}{2}}\|^2 + \frac{3}{2\hat{c}_0^{(1)}}\|\delta_x^2 p^0\|^2\right) + \left(\frac{\hat{c}_0^{(1)}}{6}\|q^{\frac{1}{2}}\|^2 + \frac{3}{2\hat{c}_0^{(1)}}\|f^\sigma\|^2\right) \\
&\quad + \left(\frac{\hat{c}_0^{(1)}}{6}\|q^{\frac{1}{2}}\|^2 + \frac{3\sigma^2\tau^2}{2\hat{c}_0^{(1)}}\|g^{\frac{1}{2}}\|^2\right) \\
&\le \frac{\hat{c}_0^{(1)}}{4}\|q^1\|^2 + \frac{\hat{c}_0^{(1)}}{4}\|q^0\|^2 + \frac{3}{2\hat{c}_0^{(1)}}\|\delta_x^2 p^0\|^2 + \frac{3}{2\hat{c}_0^{(1)}}\|f^\sigma\|^2 + \frac{3\sigma^2\tau^2}{2\hat{c}_0^{(1)}}\|g^{\frac{1}{2}}\|^2. \qquad (3.54)
\end{aligned}$$

From (3.54), we obtain

$$\|q^1\|^2 \le 3\|q^0\|^2 + \frac{6}{(\hat{c}_0^{(1)})^2}\|\delta_x^2 p^0\|^2 + \frac{6}{(\hat{c}_0^{(1)})^2}\|f^\sigma\|^2 + \frac{6\sigma^2\tau^2}{(\hat{c}_0^{(1)})^2}\|g^{\frac{1}{2}}\|^2. \qquad (3.55)$$

Step 2. When $k \ge 1$, taking the inner product (3.39) with $q^{k+\sigma}$, we obtain

$$\left(\sum_{n=0}^{k}\hat{c}_{k-n}^{(k+1)}(q^{n+1} - q^n), q^{k+\sigma}\right) = \left(\delta_x^2 p^{k+\sigma}, q^{k+\sigma}\right) + \left(f^{k+\sigma}, q^{k+\sigma}\right),\ 1 \le k \le N-1. \qquad (3.56)$$

By Lemma 2.7 and Lemma 2.4, we have

$$\begin{aligned}
&\left(\sum_{n=0}^{k}\hat{c}_{k-n}^{(k+1)}(q^{n+1} - q^n), q^{k+\sigma}\right) \ge \frac{1}{2}\sum_{n=0}^{k}\hat{c}_{k-n}^{(k+1)}(\|q^{n+1}\|^2 - \|q^n\|^2) \\
&= \frac{1}{2}\left(\sum_{n=1}^{k+1}\hat{c}_{k-n+1}^{(k+1)}\|q^n\|^2 - \sum_{n=1}^{k}\hat{c}_{k-n}^{(k)}\|q^n\|^2 - \hat{b}_k\|q^1\|^2 - \hat{c}_k^{(k+1)}\|q^0\|^2\right), \\
&1 \le k \le N-1. \qquad (3.57)
\end{aligned}$$

Using Young inequality, for any $\varepsilon > 0$, it holds

$$\left|\left(f^{k+\sigma}, q^{k+\sigma}\right)\right| \le \varepsilon\|q^{k+\sigma}\|^2 + \frac{1}{4\varepsilon}\|f^{k+\sigma}\|^2. \qquad (3.58)$$

Substituting (3.57) and (3.58) into (3.56), it yields

$$\begin{aligned}
&\frac{1}{2}\left(\sum_{n=1}^{k+1}\hat{c}_{k-n+1}^{(k+1)}\|q^n\|^2 - \sum_{n=1}^{k}\hat{c}_{k-n}^{(k)}\|q^n\|^2 - \hat{b}_k\|q^1\|^2 - \hat{c}_k^{(k+1)}\|q^0\|^2\right) \\
&\le \left(\delta_x^2 p^{k+\sigma}, q^{k+\sigma}\right) + \varepsilon\|q^{k+\sigma}\|^2 + \frac{1}{4\varepsilon}\|f^{k+\sigma}\|^2, \quad 1 \le k \le N-1. \qquad (3.59)
\end{aligned}$$

Taking the inner product (3.41) with $-p^{k+\sigma}$, we get

$$-\left(D_{\hat{t}}\delta_x^2 p^k,\, p^{k+\sigma}\right) = -\left(\delta_x^2 q^{k+\sigma},\, p^{k+\sigma}\right) - \left(g^{k+\sigma},\, p^{k+\sigma}\right), \quad 1 \le k \le N-1. \tag{3.60}$$

Using Lemma 2.8, it yields

$$-\left(D_{\hat{t}}\delta_x^2 p^k,\, p^{k+\sigma}\right) = \left(D_{\hat{t}}\delta_x p^k,\, \delta_x p^{k+\sigma}\right) \ge \frac{1}{4\tau}(F^{k+1} - F^k), \tag{3.61}$$

where

$$F^{k+1} = (2\sigma+1)\|\delta_x p^{k+1}\|^2 - (2\sigma-1)\|\delta_x p^k\|^2 + (2\sigma^2+\sigma-1)\|\delta_x p^{k+1} - \delta_x p^k\|^2 \tag{3.62}$$

and

$$F^{k+1} \ge \frac{1}{\sigma}\|\delta_x p^{k+1}\|^2,\ k \ge 0. \tag{3.63}$$

By Cauchy-Schwarz inequality, we have

$$\left| -\left(g^{k+\sigma},\, p^{k+\sigma}\right)\right| \le \frac{1}{2}\|g^{k+\sigma}\|^2 + \frac{1}{2}\|p^{k+\sigma}\|^2, \quad 1 \le k \le N-1. \tag{3.64}$$

Substituting (3.61) and (3.64) into (3.60), it yields

$$\frac{1}{4\tau}(F^{k+1} - F^k) \le -\left(\delta_x^2 q^{k+\sigma},\, p^{k+\sigma}\right) + \frac{1}{2}\|g^{k+\sigma}\|^2 + \frac{1}{2}\|p^{k+\sigma}\|^2, \quad 1 \le k \le N-1. \tag{3.65}$$

Adding (3.59) with (3.65), we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\sum_{n=1}^{k+1}\hat{c}_{k-n+1}^{(k+1)}\|q^n\|^2 - \sum_{n=1}^{k}\hat{c}_{k-n}^{(k)}\|q^n\|^2 - \hat{b}_k\|q^1\|^2 - \hat{c}_k^{(k+1)}\|q^0\|^2\right) + \frac{1}{4\tau}(F^{k+1} - F^k)\\
&\quad \le \varepsilon\|q^{k+\sigma}\|^2 + \frac{1}{4\varepsilon}\|f^{k+\sigma}\|^2 + \frac{1}{2}\|g^{k+\sigma}\|^2 + \frac{1}{2}\|p^{k+\sigma}\|^2, \quad 1 \le k \le N-1.
\end{aligned} \tag{3.66}$$

Denote

$$H^{k+1} = 2\tau\sum_{n=1}^{k+1}\hat{c}_{k-n+1}^{(k+1)}\|q^n\|^2 + F^{k+1}.$$

Then, (3.66) can be rewritten as

$$\begin{aligned}
H^{k+1} &\le H^k + 2\tau\hat{b}_k\|q^1\|^2 + 2\tau\hat{c}_k^{(k+1)}\|q^0\|^2 + 4\tau\varepsilon\|q^{k+\sigma}\|^2 + \frac{\tau}{\varepsilon}\|f^{k+\sigma}\|^2\\
&\quad + 2\tau\|g^{k+\sigma}\|^2 + 2\tau\|p^{k+\sigma}\|^2\\
&\le H^1 + 2\tau\sum_{n=1}^{k}\hat{b}_n\|q^1\|^2 + 2\tau\sum_{n=1}^{k}\hat{c}_n^{(k+1)}\|q^0\|^2 + 8\tau\varepsilon\sum_{n=1}^{k+1}\|q^n\|^2 + \frac{\tau}{\varepsilon}\sum_{n=1}^{k}\|f^{n+\sigma}\|^2\\
&\quad + 2\tau\sum_{n=1}^{k}\|g^{n+\sigma}\|^2 + 4\tau\sum_{n=1}^{k+1}\|p^n\|^2, \quad 1 \le k \le N-1.
\end{aligned} \tag{3.67}$$

By Lemma 2.3, (3.62) and (3.63), when $\tau \le \tau_0$, we have

$$H^{k+1} \ge \left(\sum_{r=0}^{m}\lambda_r\frac{(1-\gamma_r)T^{-\gamma_r}}{\Gamma(2-\gamma_r)}\right)\tau\sum_{n=1}^{k+1}\|q^n\|^2 + \frac{1}{\sigma}\|\delta_x p^{k+1}\|^2, \quad 1 \le k \le N-1 \tag{3.68}$$

and

$$H^1 = 2\tau\hat{c}_0^{(1)}\|q^1\|^2 + F^1 \le 2\tau\hat{c}_0^{(1)}\|q^1\|^2 + (4\sigma^2 + 4\sigma - 1)\|\delta_x p^1\|^2 + (4\sigma^2 - 1)\|\delta_x p^0\|^2. \tag{3.69}$$

Substituting (3.68) and (3.69) into (3.67), it yields

$$\begin{aligned}
&\left(\sum_{r=0}^{m} \lambda_r \frac{(1-\gamma_r)T^{-\gamma_r}}{\Gamma(2-\gamma_r)}\right)\tau\sum_{n=1}^{k+1}\|q^n\|^2 + \frac{1}{\sigma}\|\delta_x p^{k+1}\|^2 \\
&\le 2\tau\hat{c}_0^{(1)}\|q^1\|^2 + (4\sigma^2 + 4\sigma - 1)\|\delta_x p^1\|^2 + (4\sigma^2 - 1)\|\delta_x p^0\|^2 + 2\tau\sum_{n=1}^{k}\hat{b}_n\|q^1\|^2 \\
&\quad + 2\tau\sum_{n=1}^{k}\hat{c}_n^{(k+1)}\|q^0\|^2 + 8\tau\varepsilon\sum_{n=1}^{k+1}\|q^n\|^2 + \frac{\tau}{\varepsilon}\sum_{n=1}^{k}\|f^{n+\sigma}\|^2 \\
&\quad + 2\tau\sum_{n=1}^{k}\|g^{n+\sigma}\|^2 + 4\tau\sum_{n=1}^{k+1}\|p^n\|^2.
\end{aligned} \tag{3.70}$$

Taking $\varepsilon = \frac{1}{16}\left(\sum\limits_{r=0}^{m}\lambda_r\frac{(1-\gamma_r)T^{-\gamma_r}}{\Gamma(2-\gamma_r)}\right)$ and using Lemma 2.4, (3.51) and (3.55), we have

$$\begin{aligned}
\|\delta_x p^{k+1}\|^2 &\le 4\sigma\tau\sum_{n=1}^{k+1}\|p^n\|^2 + c_2 G_{k+1} \\
&\le \frac{2L^2\sigma}{3}\tau\sum_{n=1}^{k+1}\|\delta_x p^n\|^2 + c_2 G_{k+1}, \quad 1 \le k \le N-1,
\end{aligned}$$

where c_2 is a constant.

By Lemma 2.10, it follows that

$$\|\delta_x p^{k+1}\|^2 \le c_2 \exp\left(\frac{4\sigma L^2}{3}T\right)G_{k+1}, \quad 1 \le k \le N-1. \tag{3.71}$$

Substituting (3.71) into (3.70), there exists a constant c_3 such that

$$\tau\sum_{n=1}^{k+1}\|q^n\|^2 \le c_3 G_{k+1}, \quad 1 \le k \le N-1.$$

This completes the proof. □

Theorem 3.2 implies the following theorem.

Theorem 3.3 *The solution of the difference scheme* (3.22)–(3.27) *is unconditionally stable with respect to the initial values* w_1, w_2 *and the right hand side function* f.

Proof Suppose $\{\theta_i^k \,|\, 0 \le i \le M,\ 0 \le k \le N\}$ and $\{z_i^k \,|\, 0 \le i \le M,\ 0 \le k \le N\}$ be the solution of

$$\sum_{n=0}^{k} \hat{c}_{k-n}^{(k+1)}(\theta_i^{n+1} - \theta_i^n) = \delta_x^2 z_i^{k+\sigma} + f_i^{k+\sigma} + \xi_i^k, \quad 1 \le i \le M-1,\ 0 \le k \le N-1, \tag{3.72}$$

$$\delta_t \delta_x^2 z_i^{\frac{1}{2}} = \delta_x^2 \theta_i^{\frac{1}{2}},\ 1 \le i \le M-1, \tag{3.73}$$

$$D_{\hat{t}} \delta_x^2 z_i^k = \delta_x^2 \theta_i^{k+\sigma},\ 1 \le i \le M-1,\ 1 \le k \le N-1, \tag{3.74}$$

$$z_i^0 = w_1(x_i) + \eta_{1i},\ \theta_i^0 = w_2(x_i) + \eta_{2i},\ 1 \le i \le M-1, \tag{3.75}$$

$$z_0^k = 0,\ z_M^k = 0,\ 0 \le k \le N, \tag{3.76}$$

$$\theta_0^k = 0, \theta_M^k = 0,\ 0 \le k \le N. \tag{3.77}$$

Denote

$$\nu_i^k = \theta_i^k - v_i^k, \quad \mu_i^k = z_i^k - u_i^k, \quad 0 \le i \le M, 0 \le k \le N.$$

Subtracting (3.72)–(3.77) from (3.21)–(3.27), we get the perturbation error equations

$$\sum_{n=0}^{k} \hat{c}_{k-n}^{(k+1)}(\nu_i^{n+1} - \nu_i^n) = \delta_x^2 \mu_i^{k+\sigma} + \xi_i^k, \quad 1 \le i \le M-1,\ 0 \le k \le N-1,$$
$$\delta_t \delta_x^2 \mu_i^{\frac{1}{2}} = \delta_x^2 \nu_i^{\frac{1}{2}}, \quad 1 \le i \le M-1,$$
$$D_{\hat{t}} \delta_x^2 \mu_i^k = \delta_x^2 \nu_i^{k+\sigma}, \quad 1 \le i \le M-1, \quad 1 \le k \le N-1,$$
$$\mu_i^0 = \eta_{1i}, \quad \nu_i^0 = \eta_{2i}, \quad 1 \le i \le M-1,$$
$$\mu_0^k = 0, \quad \mu_M^k = 0, \quad 0 \le k \le N,$$
$$\nu_0^k = 0, \nu_M^k = 0, \quad 0 \le k \le N.$$

By Theorem 3.2, we obtain

$$\|\delta_x \mu^k\|^2 \le \kappa_1 \exp\left(\frac{4\sigma L^2}{3} T\right) Q_k, \quad \tau \sum_{n=1}^{k} \|\nu^n\|^2 \le \kappa_2 Q_k,\ 0 \le k \le N,$$

where κ_1 and κ_2 are two constants and

$$Q_k = \|\delta_x \eta_1\|^2 + \|\delta_x^2 \eta_1\|^2 + \|\eta_2\|^2 + \|\delta_x \eta_2\|^2 + \|\xi^1\|^2 + \tau \sum_{l=2}^{k-1} \|\xi^l\|^2.$$

The proof ends. □

Next, we give the convergence of the scheme (3.22)–(3.27). We have the following theorem.

Theorem 3.4 *Suppose the problem (3.3)–(3.7) has a unique smooth solution and $\{u_i^k, v_i^k \mid 0 \le i \le M,\ 0 \le k \le N\}$ is the solution of the difference scheme (3.22)–(3.27). Then when $\tau \le \tau_0$, there exists a constant C_1 such that*

$$\|e^k\|_\infty \le C_1(\tau^2 + h^2),\ \tau \sum_{n=1}^{k} \|\rho^n\| \le C_1(\tau^2 + h^2),\ 0 \le k \le N.$$

Proof Let

$$\rho_i^k = V_i^k - v_i^k, \quad e_i^k = U_i^k - u_i^k, \ 0 \le i \le M, \ 0 \le k \le N.$$

Subtracting (3.22)–(3.27) from (3.11), (3.15), (3.16), (3.19)–(3.21), respectively, we obtain the error equations as follows

$$\sum_{n=0}^{k} \hat{c}_{k-n}^{(k+1)}(\rho_i^{n+1} - \rho_i^n) = \delta_x^2 e_i^{k+\sigma} + R_i^{k+\sigma}, \quad 1 \le i \le M-1, \quad 0 \le k \le N-1, \quad (3.78)$$

$$\delta_t \delta_x^2 e_i^{\frac{1}{2}} = \delta_x^2 \rho_i^{\frac{1}{2}} + r_i^{\frac{1}{2}}, \quad 1 \le i \le M-1, \quad (3.79)$$

$$D_{\hat{t}} \delta_x^2 e_i^k = \delta_x^2 \rho_i^{k+\sigma} + r_i^{k+\sigma}, \quad 1 \le i \le M-1, \quad 1 \le k \le N-1, \quad (3.80)$$

$$e_i^0 = 0, \quad \rho_i^0 = 0, \quad 1 \le i \le M-1, \quad (3.81)$$

$$e_0^k = 0, \quad e_M^k = 0, \quad 0 \le k \le N, \quad (3.82)$$

$$\rho_0^k = 0, \quad \rho_M^k = 0, \quad 0 \le k \le N. \quad (3.83)$$

Using Theorem 3.2 and noticing (3.12), (3.17) and (3.18), we can obtain

$$\|\delta_x e^k\|^2 \le c_4(\tau^2 + h^2)^2, \quad \tau \sum_{n=1}^{k} \|\rho^n\|^2 \le c_4(\tau^2 + h^2)^2, \quad 0 \le k \le N,$$

where c_4 is a constant.

It follows from Lemma 2.9 and Cauchy-Schwarz inequality that

$$\|e^k\|_\infty \le C_1(\tau^2 + h^2), \quad \tau \sum_{n=1}^{k} \|\rho^n\| \le C_1(\tau^2 + h^2), \quad 0 \le k \le N,$$

where $C_1 = \max\{\sqrt{c_4 T}, \frac{\sqrt{c_4 L}}{2}\}$. The proof ends. □

4 A Fourth-Order Difference Scheme in Space

4.1 The Derivation of the Difference Scheme

Suppose $u(x,t) \in C_{x,t}^{6,4}([0, L] \times [0, T])$.

Considering (3.3) at the point $(x_i, t_{k+\sigma})$, we obtain

$$\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\gamma_r} v(x_i, t_{k+\sigma}) = u_{xx}(x_i, t_{k+\sigma}) + f(x_i, t_{k+\sigma}), \quad 1 \le i \le M-1, \quad 0 \le k \le N-1.$$

Using Lemma 2.5 and Taylor expansion, we obtain

$$\sum_{n=0}^{k} \hat{c}_{k-n}^{(k+1)} \left(V_i^{n+1} - V_i^n\right) = \sigma u_{xx}(x_i, t_{k+1}) + (1-\sigma) u_{xx}(x_i, t_k) + f_i^{k+\sigma} + O(\tau^2),$$
$$0 \le i \le M, \quad 0 \le k \le N-1. \quad (4.1)$$

Acting the averaging operator $\mathcal{A}$ on both sides of (4.1) and using Taylor expansion, we have

$$\begin{aligned}\sum_{n=0}^{k}\hat{c}_{k-n}^{(k+1)}\left(\mathcal{A}V_i^{n+1}-\mathcal{A}V_i^{n}\right)&=\sigma\mathcal{A}u_{xx}(x_i,t_{k+1})+(1-\sigma)\mathcal{A}u_{xx}(x_i,t_k)\\&\quad+\mathcal{A}f(x_i,t_{k+\sigma})+O(\tau^2)\\&=\delta_x^2U_i^{k+\sigma}+\mathcal{A}f_i^{k+\sigma}+S_i^{k+\sigma},\\&\quad 1\le i\le M-1,\quad 0\le k\le N-1,\end{aligned}\tag{4.2}$$

where there exists a constant c_5 such that

$$|S_i^{k+\sigma}|\le c_5(\tau^2+h^4),\quad 1\le i\le M-1,\quad 0\le k\le N-1,\tag{4.3}$$

Considering Eq. (3.4) at the points $(x_i,t_{\frac{1}{2}})$ and $(x_i,t_{k+\sigma})$, we have

$$u_{xxt}\left(x_i,t_{\frac{1}{2}}\right)=v_{xx}\left(x_i,t_{\frac{1}{2}}\right),\quad 0\le i\le M\tag{4.4}$$

and

$$u_{xxt}(x_i,t_{k+\sigma})=v_{xx}(x_i,t_{k+\sigma}),\quad 0\le i\le M,\quad 1\le k\le N-1.\tag{4.5}$$

Acting $\mathcal{A}$ on Eqs. (4.4) and (4.6), we get

$$\mathcal{A}u_{xxt}\left(x_i,t_{\frac{1}{2}}\right)=\mathcal{A}v_{xx}\left(x_i,t_{\frac{1}{2}}\right),\quad 1\le i\le M-1\tag{4.6}$$

and

$$\mathcal{A}u_{xxt}(x_i,t_{k+\sigma})=\mathcal{A}v_{xx}(x_i,t_{k+\sigma}),\quad 1\le i\le M-1,\quad 1\le k\le N-1.\tag{4.7}$$

Using Taylor expansion and Lemma 2.6, it yields

$$\delta_t\delta_x^2U_i^{\frac{1}{2}}=\delta_x^2V_i^{\frac{1}{2}}+s_i^{\frac{1}{2}},\quad 1\le i\le M-1\tag{4.8}$$

and

$$D_{\hat{t}}\delta_x^2U_i^k=\delta_x^2V_i^{k+\sigma}+s_i^{k+\sigma},\quad 1\le i\le M-1,\quad 1\le k\le N-1,\tag{4.9}$$

where there exists a constant c_6 such that

$$|s_i^{\frac{1}{2}}|\le c_6(\tau^2+h^4),\quad 1\le i\le M-1,\tag{4.10}$$

$$|s_i^{k+\sigma}|\le c_6(\tau^2+h^4),\quad 1\le i\le M-1.\tag{4.11}$$

Noticing the initial and boundary conditions, we get

$$U_i^0=w_1(x_i),\quad V_i^0=w_2(x_i),\quad 1\le i\le M-1,\tag{4.12}$$

$$U_0^k=0,\quad U_M^k=0,\quad 0\le k\le N,\tag{4.13}$$

$$V_0^k=0,\quad V_M^k=0,\quad 0\le k\le N.\tag{4.14}$$

Omitting the small terms in (4.2), (4.8) and (4.9) ans noticing (4.12)–(4.14), we construct the difference scheme for the problem (1.1)–(1.3) as follows

$$\sum_{n=0}^{k} \hat{c}_{k-n}^{(k+1)}(\mathcal{A}v_i^{n+1} - \mathcal{A}v_i^n) = \delta_x^2 u_i^{k+\sigma} + \mathcal{A}f_i^{k+\sigma}, \quad 1 \le i \le M-1, \quad 0 \le k \le N-1, \tag{4.15}$$

$$\delta_t \delta_x^2 u_i^{\frac{1}{2}} = \delta_x^2 v_i^{\frac{1}{2}}, \quad 1 \le i \le M-1, \tag{4.16}$$

$$D_{\hat{t}} \delta_x^2 u_i^k = \delta_x^2 v_i^{k+\sigma}, \quad 1 \le i \le M-1, \quad 1 \le k \le N-1, \tag{4.17}$$

$$u_i^0 = w_1(x_i), \quad v_i^0 = w_2(x_i), \quad 1 \le i \le M-1, \tag{4.18}$$

$$u_0^k = 0, \quad u_M^k = 0, \quad 0 \le k \le N, \tag{4.19}$$

$$v_0^k = 0, \quad v_M^k = 0, \quad 0 \le k \le N. \tag{4.20}$$

We know u^0 and v^0 from (4.18)–(4.20). Solving $\delta_x^2 u_i^1$ from (4.16) and then substituting the result into (4.15) with the superscript $k = 0$ yield a tri-diagonal system of linear algebraic equations about v^1. After v^1 is obtained, then u^1 can be got easily. Now suppose $\{u^l, v^l \mid 0 \le l \le k\}$ have been determined. Then, we solve $\delta_x^2 u_i^{k+1}$ from (4.16) and substitute it into (4.15) to obtain a tri-diagonal system of linear algebraic equations about v^{k+1}. When v^{k+1} is obtained, it is an easy work to get u^{k+1} by solving (4.16). We see that only two tri-diagonal systems of linear algebraic equations need be solved at each time level and the double weep method can be used.

4.2 The Unique Solvability of the Difference Scheme

Theorem 4.1 *The difference Scheme (4.15)–(4.20) is uniquely solvable.*

Proof (1) For $k = 0$, from (4.15), (4.16), (4.19) and (4.20), we can get the linear system of equations with respect to u^1 and v^1. Considering its homogenous system, we have

$$\hat{c}_0^{(1)} \mathcal{A}v_i^1 = \sigma \delta_x^2 u_i^1, \quad 1 \le i \le M-1, \tag{4.21}$$

$$\frac{1}{\tau}\delta_x^2 u_i^1 = \frac{1}{2}\delta_x^2 v_i^1, \quad 1 \le i \le M-1, \tag{4.22}$$

$$u_0^1 = 0, \quad u_M^1 = 0, \quad v_0^1 = 0, \quad v_M^1 = 0. \tag{4.23}$$

Solving $\delta_x^2 u_i^1$ from (4.22) and substituting the result into (4.21), then taking the inner product of the obtained equality with v^1, it yields

$$\hat{c}_0^{(1)} \|v^1\|_{\mathcal{A}}^2 = \frac{\sigma\tau}{2}(\delta_x^2 v^1, v^1) = -\frac{\sigma\tau}{2}\|\delta_x v\|^2.$$

It follows that

$$v_i^1 = 0, \quad 1 \le i \le M-1.$$

Then, from (4.21), it yields

$$\delta_x^2 u_i^1 = 0, \quad 1 \le i \le M-1. \tag{4.24}$$

Taking the inner product of (4.24) with u^1, we get

$$\|\delta_x u^1\| = 0.$$

Thus we have

$$u_i^1 = 0, \quad 1 \le i \le M-1.$$

(2) Suppose that $\{u^{k-1},\ v^{k-1}, u^k,\ v^k\}$ have been determined, then we get a linear system of equations with respect to u^{k+1} and v^{k+1} from (4.15), (4.17), (4.19) and (4.20). Consider the corresponding homogeneous system

$$\hat{c}_0^{(k+1)} \mathcal{A} v_i^{k+1} = \sigma \delta_x^2 u_i^{k+1}, \quad 1 \le i \le M-1, \tag{4.25}$$

$$\delta_x^2 u_i^{k+1} = \frac{2\sigma\tau}{2\sigma+1} \delta_x^2 v_i^{k+1}, \quad 1 \le i \le M-1, \tag{4.26}$$

$$u_0^{k+1} = 0, \quad u_M^{k+1} = 0, \tag{4.27}$$

$$v_0^{k+1} = 0, \quad v_M^{k+1} = 0. \tag{4.28}$$

Substituting (4.26) into (4.25) and then taking the inner product of the obtained equality with v^{k+1}, we obtain

$$\hat{c}_0^{(k+1)} \|v^{k+1}\|_{\mathcal{A}}^2 + \frac{2\sigma^2\tau}{2\sigma+1} \|\delta_x v^{k+1}\|^2 = 0.$$

It implies that

$$v_i^{k+1} = 0, \quad 1 \le i \le M-1.$$

Then, it follows from (4.25) that

$$\delta_x^2 u_i^{k+1} = 0, \quad 1 \le i \le M-1. \tag{4.29}$$

Taking the inner product of (4.29) with u^{k+1}, it yields

$$\|\delta_x u^{k+1}\| = 0.$$

Consequently, we get

$$u_i^{k+1} = 0, \quad 1 \le i \le M-1.$$

The proof ends. □

4.3 The Stability and Convergence of the Difference Scheme

Next, we investigate the stability and convergence of the difference scheme. The following theorem presents the prior estimate on the difference scheme (4.15)–(4.20).

Theorem 4.2 *Suppose* $\{p_i^k | 0 \le i \le M, \quad 0 \le k \le N\}$ *and* $\{q_i^k | 0 \le i \le M, \quad 0 \le k \le N\}$ *satisfy*

$$\sum_{n=0}^{k} \hat{c}_{k-n}^{(k+1)} (\mathcal{A} q_i^{n+1} - \mathcal{A} q_i^n) = \delta_x^2 p_i^{k+\sigma} + \mathcal{A} f_i^{k+\sigma}, \quad 1 \le i \le M-1, \quad 0 \le k \le N-1, \tag{4.30}$$

$$\delta_t \delta_x^2 p_i^{\frac{1}{2}} = \delta_x^2 q_i^{\frac{1}{2}} + g_i^{\frac{1}{2}}, \quad 1 \le i \le M-1, \tag{4.31}$$

$$D_{\hat{t}} \delta_x^2 p_i^k = \delta_x^2 q_i^{k+\sigma} + g_i^{k+\sigma}, \quad 1 \le i \le M-1, \quad 1 \le k \le N-1, \tag{4.32}$$

$$p_i^0 = w_1(x_i), \quad q_i^0 = w_2(x_i), \quad 1 \le i \le M-1, \tag{4.33}$$

$$p_0^k = 0, \quad p_M^k = 0, \quad q_0^k = 0, \quad q_M^k = 0, \quad 0 \le k \le N, \tag{4.34}$$

where $w_1(x_i) = 0, \quad w_2(x_i) = 0$ *for* $i = 0, M$. *Then when* $\tau \le \tau_0$, *it holds that*

$$\|\delta_x p^k\|^2 \le c_7 \exp\Big(\frac{4\sigma L^2}{3} T\Big) L_k, \quad \tau \sum_{n=1}^{k} \|q^n\|^2 \le c_8 L_k, \quad 0 \le k \le N.$$

where c_7 *and* c_8 *are two constants and*

$$L_k = \|\delta_x p^0\|^2 + \|\delta_x^2 p^0\|^2 + \|q^0\|^2 + \|\delta_x q^0\|^2 + \|\mathcal{A} f^\sigma\|^2 + \|g^{\frac{1}{2}}\|^2 + \tau \sum_{l=1}^{k-1} \|f^{l+\sigma}\|_{\mathcal{A}}^2 + \tau \sum_{l=1}^{k-1} \|g^{l+\sigma}\|^2.$$

Proof **Step 1.** When $k = 0$, the system is as follows

$$\hat{c}_0^{(1)}(\mathcal{A}q_i^1 - \mathcal{A}q_i^0) = \sigma \delta_x^2 p_i^1 + (1-\sigma)\delta_x^2 p_i^0 + \mathcal{A} f_i^\sigma, \quad 1 \le i \le M-1, \tag{4.35}$$

$$\delta_t \delta_x^2 p_i^{\frac{1}{2}} = \delta_x^2 q_i^{\frac{1}{2}} + g_i^{\frac{1}{2}}, \quad 1 \le i \le M-1, \tag{4.36}$$

$$p_i^0 = w_1(x_i), \quad q_i^0 = w_2(x_i), \quad 1 \le i \le M-1, \tag{4.37}$$

$$p_0^1 = 0, \quad p_M^1 = 0, \quad q_0^1 = 0, \quad q_M^1 = 0 \tag{4.38}$$

with $p_0^0 = 0, \quad p_M^0 = 0, \quad q_0^0 = 0, \quad q_M^0 = 0.$

(I) Taking the inner product of (4.35) with q^1, we obtain

$$\hat{c}_0^{(1)} \|q^1\|_{\mathcal{A}}^2 = \hat{c}_0^{(1)}(\mathcal{A}q^0, q^1) - \sigma\big(\delta_x p^1, \delta_x q^1\big) + (1-\sigma)(\delta_x^2 p^0, q^1) + (\mathcal{A} f^\sigma, q^1).$$

Taking the inner product of (4.36) with $-2\sigma p^1$, we arrive at

$$\frac{2\sigma}{\tau}\|\delta_x p^1\|^2 = \frac{2\sigma}{\tau}(\delta_x p^0, \delta_x p^1) + \sigma(\delta_x q^1, \delta_x p^1) + \sigma(\delta_x q^0, \delta_x p^1) - 2\sigma\left(g^{\frac{1}{2}}, p^1\right).$$

Similar to the derivation of (3.51)and noticing $\|\mathcal{A}q^0\| \le \|q^0\|$, it yields

$$\|\delta_x p^1\|^2 \le 3\|\delta_x p^0\|^2 + \frac{3\tau}{4\hat{c}_0^{(1)}\sigma}\|q^0\|^2 + \frac{3(1-\sigma)^2\tau}{4\hat{c}_0^{(1)}\sigma}\|\delta_x^2 p^0\|^2 + \frac{3\tau}{4\hat{c}_0^{(1)}\sigma}\|\mathcal{A} f^\sigma\|^2 + \frac{3\tau^2}{4}\|\delta_x q^0\|^2 + \frac{L^2}{2}\tau^2\|g^{\frac{1}{2}}\|^2. \tag{4.39}$$

(II) It follows from (4.36) that

$$\delta_x^2 p_i^1 = \delta_x^2 p_i^0 + \tau \delta_x^2 q_i^{\frac{1}{2}} + \tau g_i^{\frac{1}{2}}, \quad 1 \le i \le M-1. \tag{4.40}$$

Substituting (4.40) into (4.35), we have

$$\hat{c}_0^{(1)}(\mathcal{A}q_i^1 - \mathcal{A}q_i^0) = \delta_x^2 p_i^0 + \sigma\tau\delta_x^2 q_i^{\frac{1}{2}} + \mathcal{A} f_i^\sigma + \sigma\tau g_i^{\frac{1}{2}}, \quad 1 \le i \le M-1. \tag{4.41}$$

Taking the inner product of (4.41) with $q^{\frac{1}{2}}$, we obtain

$$\hat{c}_0^{(1)}\left(\mathcal{A}q^1 - \mathcal{A}q^0, q^{\frac{1}{2}}\right) = \left(\delta_x^2 p^0, q^{\frac{1}{2}}\right) + \sigma\tau\left(\delta_x^2 q^{\frac{1}{2}}, q^{\frac{1}{2}}\right) + \left(\mathcal{A} f^\sigma, q^{\frac{1}{2}}\right) + \sigma\tau\left(g^{\frac{1}{2}}, q^{\frac{1}{2}}\right).$$

By Cauchy-Schwarz inequality and Lemma 2.9, it follows that

$$\begin{aligned}
&\frac{\hat{c}_0^{(1)}}{2}(\|q^1\|_{\mathcal{A}}^2 - \|q^0\|_{\mathcal{A}}^2) \\
&\le \frac{\hat{c}_0^{(1)}}{9}\|q^{\frac{1}{2}}\|^2 + \frac{9}{4\hat{c}_0^{(1)}}\|\delta_x^2 p^0\|^2 + \frac{\hat{c}_0^{(1)}}{9}\|q^{\frac{1}{2}}\|^2 + \frac{9}{4\hat{c}_0^{(1)}}\|\mathcal{A}f^\sigma\|^2 + \frac{\hat{c}_0^{(1)}}{9}\|q^{\frac{1}{2}}\|^2 \\
&\quad + \frac{9\sigma^2\tau^2}{4\hat{c}_0^{(1)}}\|g^{\frac{1}{2}}\|^2 \\
&\le \frac{\hat{c}_0^{(1)}}{4}\|q^1\|_{\mathcal{A}}^2 + \frac{\hat{c}_0^{(1)}}{4}\|q^0\|_{\mathcal{A}}^2 + \frac{9}{4\hat{c}_0^{(1)}}\|\delta_x^2 p^0\|^2 + \frac{9}{4\hat{c}_0^{(1)}}\|\mathcal{A}f^\sigma\|^2 + \frac{9\sigma^2\tau^2}{4\hat{c}_0^{(1)}}\|g^{\frac{1}{2}}\|^2.
\end{aligned}$$

Then, noticing $\|q^0\|_{\mathcal{A}} \le \|q^0\|$, we get

$$\|q^1\|_{\mathcal{A}}^2 \le 3\|q^0\|^2 + \frac{9}{(\hat{c}_0^{(1)})^2}\|\delta_x^2 p^0\|^2 + \frac{9}{(\hat{c}_0^{(1)})^2}\|\mathcal{A}f^\sigma\|^2 + \frac{9\sigma^2\tau^2}{(\hat{c}_0^{(1)})^2}\|g^{\frac{1}{2}}\|^2. \tag{4.42}$$

By $\|q^1\| \le \frac{3}{2}\|q^1\|_{\mathcal{A}}$, we obtain

$$\|q^1\| \le \frac{3}{2}\left[3\|q^0\|^2 + \frac{9}{(\hat{c}_0^{(1)})^2}\|\delta_x^2 p^0\|^2 + \frac{9}{(\hat{c}_0^{(1)})^2}\|\mathcal{A}f^\sigma\|^2 + \frac{9\sigma^2\tau^2}{(\hat{c}_0^{(1)})^2}\|g^{\frac{1}{2}}\|^2\right].$$

Step 2. When $k \ge 1$, taking the inner product (4.30) with $q^{k+\sigma}$, we obtain

$$\begin{aligned}
\left(\sum_{n=0}^{k}\hat{c}_{k-n}^{(k+1)}(\mathcal{A}q^{n+1} - \mathcal{A}q^n), q^{k+\sigma}\right) &= \left(\delta_x^2 p^{k+\sigma}, q^{k+\sigma}\right) \\
&\quad + \left(\mathcal{A}f^{k+\sigma}, q^{k+\sigma}\right), \quad 1 \le k \le N-1.
\end{aligned} \tag{4.43}$$

By Lemma 2.7 and Lemma 2.4, we have

$$\begin{aligned}
&\left(\sum_{n=0}^{k}\hat{c}_{k-n}^{(k+1)}(\mathcal{A}q^{n+1} - \mathcal{A}q^n), q^{k+\sigma}\right) \ge \frac{1}{2}\sum_{n=0}^{k}\hat{c}_{k-n}^{(k+1)}\left(\|q^{n+1}\|_{\mathcal{A}}^2 - \|q^n\|_{\mathcal{A}}^2\right) \\
&= \frac{1}{2}\left(\sum_{n=1}^{k+1}\hat{c}_{k-n+1}^{(k+1)}\|q^n\|_{\mathcal{A}}^2 - \sum_{n=1}^{k}\hat{c}_{k-n}^{(k)}\|q^n\|_{\mathcal{A}}^2 - \hat{b}_k\|q^1\|_{\mathcal{A}}^2 - \hat{c}_k^{(k+1)}\|q^0\|_{\mathcal{A}}^2\right), \\
&\quad 1 \le k \le N-1.
\end{aligned} \tag{4.44}$$

Using Young's inequality, for any $\varepsilon > 0$, it holds

$$\left|\left(\mathcal{A}f^{k+\sigma}, q^{k+\sigma}\right)\right| \le \|f^{k+\sigma}\|_{\mathcal{A}} \cdot \|q^{k+\sigma}\|_{\mathcal{A}} \le \varepsilon\|q^{k+\sigma}\|_{\mathcal{A}}^2 + \frac{1}{4\varepsilon}\|f^{k+\sigma}\|_{\mathcal{A}}^2. \tag{4.45}$$

Substituting (4.44) and (4.45) into (4.43), it yields

$$\begin{aligned}
&\frac{1}{2}\left(\sum_{n=1}^{k+1}\hat{c}_{k-n+1}^{(k+1)}\|q^n\|_{\mathcal{A}}^2 - \sum_{n=1}^{k}\hat{c}_{k-n}^{(k)}\|q^n\|_{\mathcal{A}}^2 - \hat{b}_k\|q^1\|_{\mathcal{A}}^2 - \hat{c}_k^{(k+1)}\|q^0\|_{\mathcal{A}}^2\right) \\
&\quad \le \left(\delta_x^2 p^{k+\sigma}, q^{k+\sigma}\right) + \varepsilon\|q^{k+\sigma}\|_{\mathcal{A}}^2 + \frac{1}{4\varepsilon}\|f^{k+\sigma}\|_{\mathcal{A}}^2, \quad 1 \le k \le N-1.
\end{aligned} \tag{4.46}$$

Taking the inner product (4.32) with $-p^{k+\sigma}$, we get

$$-\left(D_{\hat{t}}\delta_x^2 p^k, p^{k+\sigma}\right) = -\left(\delta_x^2 q^{k+\sigma}, p^{k+\sigma}\right) - \left(g^{k+\sigma}, p^{k+\sigma}\right), \quad 1 \le k \le N-1.$$

Similarly to the derivation of (3.65), it yields

$$\frac{1}{4\tau}(F^{k+1} - F^k) \le -\left(\delta_x^2 q^{k+\sigma}, p^{k+\sigma}\right) + \frac{1}{2}\|g^{k+\sigma}\|^2 + \frac{1}{2}\|p^{k+\sigma}\|^2, \quad 1 \le k \le N-1, \tag{4.47}$$

where F^{k+1} is defined by (3.62).

Adding (4.46) with (4.47), we obtain

$$\frac{1}{2}\left[\sum_{n=1}^{k+1}\hat{c}_{k-n+1}^{(k+1)}\|q^n\|_{\mathcal{A}}^2 - \sum_{n=1}^{k}\hat{c}_{k-n}^{(k)}\|q^n\|_{\mathcal{A}}^2 - \hat{b}_k\|q^1\|_{\mathcal{A}}^2 - \hat{c}_k^{(k+1)}\|q^0\|_{\mathcal{A}}^2\right] + \frac{1}{4\tau}(F^{k+1} - F^k)$$
$$\le \varepsilon\|q^{k+\sigma}\|_{\mathcal{A}}^2 + \frac{1}{4\varepsilon}\|f^{k+\sigma}\|_{\mathcal{A}}^2 + \frac{1}{2}\|g^{k+\sigma}\|^2 + \frac{1}{2}\|p^{k+\sigma}\|^2, \quad 1 \le k \le N-1. \tag{4.48}$$

Denote

$$J^{k+1} = 2\tau\sum_{n=1}^{k+1}\hat{c}_{k-n+1}^{(k+1)}\|q^n\|_{\mathcal{A}}^2 + F^{k+1}.$$

Then (4.48) can be rewritten as

$$\begin{aligned} J^{k+1} &\le J^k + 2\tau\hat{b}_k\|q^1\|_{\mathcal{A}}^2 + 2\tau\hat{c}_k^{(k+1)}\|q^0\|_{\mathcal{A}}^2 + 4\tau\varepsilon\|q^{k+\sigma}\|_{\mathcal{A}}^2 + \frac{\tau}{\varepsilon}\|\mathcal{A}f^{k+\sigma}\|^2 \\ &\quad + 2\tau\|g^{k+\sigma}\|^2 + 2\tau\|p^{k+\sigma}\|^2 \\ &\le J^1 + 2\tau\sum_{n=1}^{k}\hat{b}_n\|q^1\|_{\mathcal{A}}^2 + 2\tau\sum_{n=1}^{k}\hat{c}_n^{(k+1)}\|q^0\|_{\mathcal{A}}^2 + 8\tau\varepsilon\sum_{n=1}^{k+1}\|q^n\|_{\mathcal{A}}^2 + \frac{\tau}{\varepsilon}\sum_{n=1}^{k}\|f^{n+\sigma}\|_{\mathcal{A}}^2 \\ &\quad + 2\tau\sum_{n=1}^{k}\|g^{n+\sigma}\|^2 + 4\tau\sum_{n=1}^{k+1}\|p^n\|^2, \quad 1 \le k \le N-1. \end{aligned} \tag{4.49}$$

Using Lemma 2.3, (3.62) and (3.63), when $\tau \le \tau_0$, we obtain

$$J^{k+1} \ge \left(\sum_{r=0}^{m}\lambda_r\frac{(1-\gamma_r)T^{-\gamma_r}}{\Gamma(2-\gamma_r)}\right)\tau\sum_{n=1}^{k+1}\|q^n\|_{\mathcal{A}}^2 + \frac{1}{\sigma}\|\delta_x p^{k+1}\|^2, \quad 1 \le k \le N-1 \tag{4.50}$$

and

$$J^1 = 2\tau\hat{c}_0^{(1)}\|q^1\|_{\mathcal{A}}^2 + F^1 \le 2\tau\hat{c}_0^{(1)}\|q^1\|_{\mathcal{A}}^2 + (4\sigma^2 + 4\sigma - 1)\|\delta_x p^1\|^2 + (4\sigma^2 - 1)\|\delta_x p^0\|^2. \tag{4.51}$$

Substituting (4.50) and (4.51) into (4.49), we have

$$\begin{aligned}&\left(\sum_{r=0}^{m}\lambda_r\frac{(1-\gamma_r)T^{-\gamma_r}}{\Gamma(2-\gamma_r)}\right)\tau\sum_{n=1}^{k+1}\|q^n\|_{\mathcal{A}}^2+\frac{1}{\sigma}\|\delta_x p^{k+1}\|^2\\&\le 2\tau\hat{c}_0^{(1)}\|q^1\|_{\mathcal{A}}^2+(4\sigma^2+4\sigma-1)\|\delta_x p^1\|^2+(4\sigma^2-1)\|\delta_x p^0\|^2+2\tau\sum_{n=1}^{k}\hat{b}_n\|q^1\|_{\mathcal{A}}^2\\&+2\tau\sum_{n=1}^{k}\hat{c}_n^{(k+1)}\|q^0\|^2+8\tau\varepsilon\sum_{n=1}^{k+1}\|q^n\|_{\mathcal{A}}^2+\frac{\tau}{\varepsilon}\sum_{n=1}^{k}\|f^{n+\sigma}\|_{\mathcal{A}}^2\\&+2\tau\sum_{n=1}^{k}\|g^{n+\sigma}\|^2+4\tau\sum_{n=1}^{k+1}\|p^n\|^2,\\&1\le k\le N-1.\end{aligned}\tag{4.52}$$

Taking $\varepsilon=\frac{1}{16}\left(\sum\limits_{r=0}^{m}\lambda_r\frac{(1-\gamma_r)T^{-\gamma_r}}{\Gamma(2-\gamma_r)}\right)$ and using Lemma 2.4, (4.39) and (4.42), we have

$$\|\delta_x p^{k+1}\|^2\le 4\sigma\tau\sum_{n=1}^{k+1}\|p^n\|^2+c_7L_{k+1}\le\frac{2L^2\sigma}{3}\tau\sum_{n=1}^{k+1}\|\delta_x p^n\|^2+c_7L_{k+1},\quad 1\le k\le N-1,$$

where c_7 is a constant. By Lemma 2.10, it follows that

$$\|\delta_x p^{k+1}\|^2\le c_7\exp\left(\frac{4\sigma L^2}{3}T\right)L_{k+1},\quad 1\le k\le N-1.\tag{4.53}$$

Substituting (4.53) into (4.52) and using Lemma 2.9, we get

$$\tau\sum_{n=1}^{k+1}\|q^n\|^2\le\frac{3}{2}\tau\sum_{n=1}^{k+1}\|q^n\|_{\mathcal{A}}^2\le c_8L_{k+1},\quad 1\le k\le N-1,\quad 1\le k\le N-1,$$

where c_8 is a constant.

This completes the proof. □

From the theorem above, we can obtain the stability of the difference scheme.

Theorem 4.3 *The solution of the difference Scheme (4.15)–(4.20) is unconditionally stable with respect to the initial values w_1, w_2 and the right hand side function f.*

Next, we prove the convergence of the difference Scheme (4.15)–(4.20).

Let

$$\rho_i^k=V_i^k-v_i^k,\quad e_i^k=U_i^k-u_i^k,\quad 0\le i\le M,\quad 0\le k\le N.$$

Subtracting (4.15)–(4.20) from (4.2), (4.8), (4.9), (4.12)–(4.14), respectively, we get the error equations as follows

$$\sum_{n=0}^{k} \hat{c}_{k-n}^{(k+1)}(\mathcal{A}\rho_i^{n+1} - \mathcal{A}\rho_i^n) = \delta_x^2 e_i^{k+\sigma} + S_i^{k+\sigma}, \quad 1 \le i \le M-1, \quad 0 \le k \le N-1, \tag{4.54}$$

$$\delta_t \delta_x^2 e_i^{\frac{1}{2}} = \delta_x^2 \rho_i^{\frac{1}{2}} + s_i^{\frac{1}{2}}, \quad 1 \le i \le M-1, \tag{4.55}$$

$$D_{\hat{t}} \delta_x^2 e_i^k = \delta_x^2 \rho_i^{k+\sigma} + s_i^{k+\sigma}, \quad 1 \le i \le M-1, \quad 1 \le k \le N-1, \tag{4.56}$$

$$e_i^0 = 0, \quad \rho_i^0 = 0, \quad 1 \le i \le M-1, \tag{4.57}$$

$$e_0^k = 0, \quad e_M^k = 0, \quad 0 \le k \le N, \tag{4.58}$$

$$\rho_0^k = 0, \quad \rho_M^k = 0, \quad 0 \le k \le N. \tag{4.59}$$

Theorem 4.4 *Suppose the problem (3.3)–(3.7) has a unique smooth solution and $\{u_i^k, v_i^k \mid 0 \le i \le M, \ 0 \le k \le N\}$ is the solution of the difference Scheme (4.15)–(4.20). Then when $\tau \le \tau_0$, there exists a constant C_2 such that*

$$\|e^k\|_\infty \le C_2(\tau^2 + h^4), \quad \tau \sum_{n=1}^{k} \|\rho^n\| \le C_2(\tau^2 + h^4), \quad 0 \le k \le N.$$

Proof By Theorem 4.2 and noticing (4.3), (4.10) and (4.11), it yields

$$\|\delta_x e^k\|^2 \le c_9(\tau^2 + h^4)^2, \quad \tau \sum_{n=1}^{k} \|\rho^n\|^2 \le c_9(\tau^2 + h^4)^2, \quad 0 \le k \le N,$$

where c_9 is a constant.

Using Lemma 2.3 and Cauchy-Schwarz inequality, we have

$$\|e^k\|_\infty \le C_2(\tau^2 + h^4), \quad \tau \sum_{n=1}^{k} \|\rho^n\| \le C_2(\tau^2 + h^4), \quad 0 \le k \le N,$$

where $C_2 = \max\{\sqrt{c_9 T}, \frac{\sqrt{c_9 L}}{2}\}$. The proof ends. □

5 A Fast Second-Order Difference Scheme

In this section, we present a fast difference scheme for multi-term fractional diffusion wave equation based on the ${}^{\mathcal{F}}L2\text{-}1_\sigma$ formula [40], which can reduce the computational complexity significantly.

In [39,40], the kernel function $t^{-\alpha}$ in Caputo derivative is approximated by the sum-of-exponentials. For the given $\alpha \in (0, 1)$, tolerance error ε, cut-off time step size $\hat{\tau}$ and final time T, there is one positive integer $N_{\exp}$, exponential coefficients s_l and corresponding positive weights ω_l, $(l = 1, 2, \ldots, N_{\exp})$ satisfying

$$\left| t^{-\alpha} - \sum_{l=1}^{N_{\exp}} \omega_l e^{-s_l t} \right| \le \varepsilon, \quad \forall t \in [\hat{\tau}, T].$$

In addition, the number of exponentials has the following order

$$N_{exp} = O\Big(\log\frac{1}{\varepsilon}\Big(\log\log\frac{1}{\varepsilon}+\log\frac{T}{\hat{\tau}}\Big)+\log\frac{1}{\hat{\tau}}\Big(\log\log\frac{1}{\varepsilon}+\log\frac{T}{\hat{\tau}}\Big)\Big).$$

The fast evaluation of Caputo derivative, ${}^{\mathcal{F}}L2\text{-}1_\sigma$ formula, is given as follows

$$\begin{aligned}{}_0^CD_t^{\gamma_r}v(t_{k+\sigma}) &= {}^{\mathcal{FH}}\mathcal{D}_t^{\gamma_r}v^{k+\sigma}+O(\varepsilon+\tau^2)\\ &=\sum_{l=1}^{N_{\exp}}\hat{w}_l\hat{V}_l^k+\frac{\sigma^{1-\gamma_r}}{\tau^{\gamma_r}\Gamma(2-\gamma_r)}(v^{k+1}-v^k)+O(\varepsilon+\tau^2),\end{aligned}$$

where $\hat{w}_l=\frac{1}{\Gamma(1-\gamma_r)}w_l$ and $\hat{V}_l^k$ is obtained by the following recurrence relation

$$\hat{V}_l^k=e^{-s_l\tau}\hat{V}_l^{k-1}+A_l(v^k-v^{k-1})+B_l(v^{k+1}-v^k),$$

with $\hat{V}_l^0=0,\ (l=1,\ldots,N_{\exp})$ and

$$A_l=\int_0^1\Big(\frac{3}{2}-s\Big)e^{-s_l\tau(\sigma+1-s)}\quad ds,\quad B_l=\int_0^1\Big(s-\frac{1}{2}\Big)e^{-s_l\tau(\sigma+1-s)}\quad ds.$$

Thus, we obtain

$$\sum_{r=0}^m\lambda_r\,{}_0^CD_t^{\gamma_r}v(t_{k+\sigma})=\sum_{r=0}^m\lambda_r\left(\sum_{l=1}^{N_{\exp}}\hat{w}_l\hat{V}_l^k+\frac{\sigma^{1-\gamma_r}}{\tau^{\gamma_r}\Gamma(2-\gamma_r)}(v^{k+1}-v^k)\right)+O(\varepsilon+\tau^2),$$
$$1\le i\le M-1,\quad 0\le k\le N-1.$$

Then, we can construct a fast second-order recursion difference scheme for Eqs. (3.3)–(3.7) as follows

$$\sum_{r=0}^m\lambda_r\left(\sum_{l=1}^{N_{\exp}}\hat{w}_l\hat{V}_l^k+\frac{\sigma^{1-\gamma_r}}{\tau^{\gamma_r}\Gamma(2-\gamma_r)}(v_i^{k+1}-v_i^k)\right)=\delta_x^2u_i^{k+\sigma}+f_i^{k+\sigma},$$
$$1\le i\le M-1,\quad 0\le k\le N-1, \tag{5.1}$$
$$\delta_t\delta_x^2u_i^{\frac{1}{2}}=\delta_x^2v_i^{\frac{1}{2}},\quad 1\le i\le M-1, \tag{5.2}$$
$$D_{\hat{t}}\delta_x^2u_i^k=\delta_x^2v_i^{k+\sigma},\quad 1\le i\le M-1,\quad 1\le k\le N-1, \tag{5.3}$$
$$u_i^0=w_1(x_i),\quad v_i^0=w_2(x_i),\quad 1\le i\le M-1, \tag{5.4}$$
$$u_0^k=0,\quad u_M^k=0,\quad 0\le k\le N, \tag{5.5}$$
$$v_0^k=0,\quad v_M^k=0,\quad 0\le k\le N, \tag{5.6}$$
$$\hat{V}_l^0=0,\quad 1\le l\le N_{\exp}, \tag{5.7}$$
$$\hat{V}_l^k=e^{-s_l\tau}\hat{V}_l^{k-1}+A_l(v_i^k-v_i^{k-1})+B_l(v_i^{k+1}-v_i^k),$$
$$1\le l\le N_{\exp},\quad 1\le i\le M-1,\quad 1\le k\le N-1,. \tag{5.8}$$

From (5.4)–(5.6), we know u^0 and v^0. Solving $\delta_x^2u^1$ from (5.2) and substituting the result into (5.1) with the superscript $k=0$ then noting (5.7) achieve a tri-diagonal system of linear algebraic equations about v^1. After v^1 is obtained, then u^1 can be got easily from (5.2). Now suppose $\{u^{k-1},v^{k-1},u^k,v^k\}$ and $\{\hat{V}_l^{k-1}\,|\,1\le l\le N_{\exp}\}$ have been determined. Then, we solve $\delta_x^2u^{k+1}$ from (5.3) and substitute the result and (5.8) into (5.1) to obtain a tri-diagonal system of linear algebraic equations about v^{k+1}. When v^{k+1} is obtained, by solving (5.3) to get u^{k+1}. Simultaneously, we get $\{\hat{V}_l^k \mid 1 \le l \le N_{\exp}\}$ from (5.8). We find that only two tri-diagonal systems of linear algebraic equations need be solved at each time level and the double weep method can be used.

The determination of $\{u^{k+1}, v^{k+1}\}$ and $\{\hat{V}_l^k \mid 1 \le l \le N_{\exp}\}$ is only dependent on $\{u^{k-1}, v^{k-1}, u^k, v^k\}$ and $\{\hat{V}_l^{k-1} \mid 1 \le l \le N_{\exp}\}$. We only need store the values at two time levels. This reduces the storage and computational cost significantly.

The analysis of the stability and convergence of the difference scheme (5.1)–(5.8) is too long, which we omit here.

6 Numerical Experiments

In this section, we provide two numerical examples. The first example is to demonstrate the accuracy of the difference scheme (3.22)–(3.27) and the scheme (4.15)–(4.20). A comparison with the difference scheme based on $L1$ formula is also presented. The second example is to compare the difference scheme (3.22)–(3.27) with the fast difference scheme (6.1)–(6.3), which shows that the fast difference scheme can reduce the CPU time greatly.

Denote

$$E(h,\tau) = \max_{0\le k\le N} \|U^k - u^k\|_\infty,$$
$$\text{Order}_\tau = \log_2(E(h,2\tau)/E(h,\tau)), \quad \text{Order}_h = \log_2(E(2h,\tau)/E(h,\tau)).$$

Example 6.1 In (1.1)–(1.3), take $T = 1$, $[0, L] = [0, \pi]$. Consider the problem (1.1)–(1.3) with the source term

$$f(x,t) = \left(\sum_{r=0}^{2} \frac{24\lambda_r t^{4-\alpha_r}}{\Gamma(5-\alpha_r)} + t^4\right)\sin x$$

and the initial the boundary values

$$u(0,t) = 0,\ u(1,t) = 0, \quad u(x,0) = 0,\ u_t(x,0) = 0.$$

The problem has an exact solution

$$u(x,t) = t^4 \sin x.$$

With different values of $\lambda_0, \lambda_1, \lambda_2$ and $\alpha_0, \alpha_1, \alpha_2$, the difference scheme (3.22)–(3.27) and the scheme (4.15)–(4.20) will be used to numerically solve this problem, respectively.

Firstly, we examine the numerical accuracy in time. Taking the fixed and sufficiently small h, the maximum errors and convergence orders are shown in Table 1. From Table 1, one can see that both difference schemes can achieve the second-order accuracy in time. The computational results are in a good agreement with theoretical results.

Secondly, the numerical accuracy of the difference scheme (3.22)–(3.27) and the scheme (4.15)–(4.20) in space is tested. We fix the temporal step size $\tau = \frac{1}{5000}$, Table 2 presents the maximum errors and convergence orders for the different space step sizes. From Table 2, we can find that, the second-order convergence of the difference schemes (3.22)–(3.27) and the fourth-order convergence of the scheme (4.15)–(4.20) in space are verified, respectively.

Next, we show the efficiency of proposed difference scheme comparing with the difference scheme based on $L1$ formula. The difference scheme for the problem (1.1)–(1.3) based on $L1$ formula is as follows [27]:

Table 1 Maximum errors and convergence orders of the two difference schemes in time

$(\lambda_0, \lambda_1, \lambda_2)$	$(\alpha_0, \alpha_1, \alpha_2)$	τ	scheme (3.22)–(3.27) ($M = 2000$)		scheme (4.15)–(4.20) ($M = 100$)	
			$E(h,\tau)$	order$_\tau$	$E(h,\tau)$	order$_\tau$
(3, 2, 1)	(4/3, 5/4, 6/5)	1/20	6.8898e−3	–	6.8904e − 3	–
		1/40	1.7801e−3	1.953	1.7802e−3	1.953
		1/80	4.5256e−4	1.976	4.5260e−4	1.976
		1/160	1.1410e−4	1.988	1.1411e−4	1.988
	(5/3, 3/2, 4/3)	1/20	4.6355e−3	–	4.6377e−3	–
		1/40	1.1896e−3	1.962	1.1902e−3	1.962
		1/80	3.0177e−4	1.979	3.0192e−4	1.979
		1/160	7.6062e−5	1.988	7.6099e−5	1.988
(1, 2, 3)	(4/3, 5/4, 6/5)	1/20	7.1942e−3	–	7.1945e−3	–
		1/40	1.8587e−3	1.953	1.8587e−3	1.953
		1/80	4.7232e−4	1.976	4.7234e−4	1.976
		1/160	1.1900e−4	1.989	1.1900e−4	1.989
	(5/3, 3/2, 4/3)	1/20	5.3556e−3	–	5.3570e−3	–
		1/40	1.3640e−3	1.973	1.3644e−3	1.973
		1/80	3.4270e−4	1.993	3.4280e−4	1.993
		1/160	8.5480e−5	2.003	8.5505e−5	2.003

Table 2 Maximum errors and convergence orders of the two difference schemes in space ($N = 5000$)

$(\lambda_0, \lambda_1, \lambda_2)$	$(\alpha_0, \alpha_1, \alpha_2)$	scheme (3.22)–(3.27)			scheme (4.15)–(4.20)		
		h	$E(h,\tau)$	order$_h$	h	$E(h,\tau)$	order$_h$
(3, 2, 1)	(4/3, 5/4, 6/5)	$\pi/4$	1.7719e−4	–	$\pi/4$	5.7628e−7	–
		$\pi/8$	4.6079e−5	1.944	$\pi/8$	4.0833e−8	3.819
		$\pi/16$	1.1591e − 5	1.989	$\pi/16$	2.6492e−9	3.947

$$\sum_{r=0}^{m} \frac{\tau^{1-\alpha_r}}{\Gamma(3-\alpha_r)} \left[a_0^{(\alpha_r)} \delta_t u_i^{k+\frac{1}{2}} - \sum_{n=1}^{k-1} \left(a_{k-n-1}^{(\alpha_r)} - a_{k-n}^{(\alpha_r)} \right) \delta_t u_i^{n+\frac{1}{2}} - a_{k-1}^{(\alpha_r)} w_2(x_i) \right]$$
$$= \delta_x^2 u_i^{k+\frac{1}{2}} + f_i^{k+\frac{1}{2}}, \quad 1 \le i \le M-1,\ 1 \le k \le N-1, \tag{6.1}$$
$$u_i^0 = w_1(x_i), \quad 1 \le i \le M-1, \tag{6.2}$$
$$u_0^k = 0,\ u_M^k = 0,\ 0 \le k \le N, \tag{6.3}$$

where $\delta_t u^{k+\frac{1}{2}} = \frac{u^{k+1}-u^k}{\tau}$, $a_0^{(\alpha_r)} = 1$, $a_l^{(\alpha_r)} = (l+1)^{2-\alpha_r} - l^{2-\alpha_r}$.

Table 3 lists the errors and the orders of the scheme (3.22)–(3.27) and the scheme (6.1)–(6.3). For the different temporal step sizes $\frac{1}{40}$, $\frac{1}{80}$, $\frac{1}{160}$ and $\frac{1}{320}$, we choose the spatial step sizes by $h = \tau^{\frac{1}{2}\min\{2-\gamma_r\}}$ for the scheme (6.1)–(6.3) and $h = \tau$ for the scheme (3.22)–(3.27). From Table 3, we can see the scheme (6.1)–(6.3) has $\min\{2-\gamma_r\}$ order accuracy, while the scheme (3.22)–(3.27) can achieve 2-order accuracy. It shows that the scheme (3.22)–(3.27) is more efficient than the scheme (6.1)–(6.3).

Table 3 Maximum errors and convergence orders of the two difference schemes

$(\lambda_0, \lambda_1, \lambda_2)$	$(\alpha_0, \alpha_1, \alpha_2)$	N	scheme (3.22)–(3.27)			scheme (6.1)–(6.3)		
			M	$E(h,\tau)$	order$_\tau$	M	$E(h,\tau)$	order$_\tau$
(3, 2, 1)	(5/3, 3/2, 4/3)	40	40	5.6545e−5	–	137	8.9636e−3	–
		80	80	1.2950e−5	2.126	345	3.5804e−3	1.324
		160	160	2.9663e−6	2.126	869	1.4154e−3	1.339
		320	320	6.8341e−7	2.118	2189	5.5638e−4	1.347
	(9/5, 7/4, 6/5)	40	40	7.1120e−5	–	84	1.9274e−2	–
		80	80	1.6401e−5	2.116	192	8.3824e−3	1.201
		160	160	3.7659e−6	2.123	442	3.6221e−3	1.211
		320	320	8.6452e−7	2.123	1014	1.5593e−3	1.216
(1, 2, 3)	(5/3, 3/2, 4/3)	40	40	5.1573e−5	–	137	5.5068e−3	–
		80	80	1.1904e−5	2.115	345	2.1182e−3	1.378
		160	160	2.7499e−6	2.114	869	8.0600e−4	1.394
		320	320	6.3915e−7	2.105	2189	3.0521e−4	1.401
	(9/5, 7/4, 6/5)	40	40	6.6939e−5	–	84	1.6068e−2	–
		80	80	1.5347e−5	2.125	192	6.8266e−3	1.235
		160	160	3.5063e−6	2.130	442	2.8790e−3	1.246
		320	320	8.0210e−7	2.128	1014	1.2092e−3	1.251

Table 4 Numerical errors and convergence orders of the difference scheme (5.1)–(5.6) in time with $M = 1000$ and $(\lambda_0, \lambda_1, \lambda_2) = (3, 2, 1)$, $(\alpha_0, \alpha_1, \alpha_2) = (4/3, 5/4, 6/5)$ for Example 6.2

τ	difference scheme (5.1)–(5.6)		difference scheme (3.22)–(3.27)	
	E(h,τ)	order$_\tau$	E(h,τ)	order$_\tau$
1/20	4.4867e−3	–	4.4867e−3	–
1/40	1.1482e−3	1.966	1.1482e−3	1.966
1/80	2.9008e−4	1.985	2.9031e−4	1.984
1/160	7.3043e−5	1.990	7.2975e−5	1.992

Example 6.2 In (1.1)–(1.3), take $T = 1$, $[0, L] = [0, \pi]$. Consider the problem (1.1)–(1.3) with the source term

$$f(x,t) = \left(\sum_{r=0}^{2} \lambda_r \frac{\Gamma(3+\alpha_0)}{\Gamma(3+\alpha_0-\alpha_r)} t^{2+\alpha_0-\alpha_r} + t^{2+\alpha_0}\right) \sin x$$

and the initial and boundary values

$$u(0,t) = 0, \;\; u(1,t) = 0, \quad u(x,0) = 0, \;\; u_t(x,0) = 0.$$

The problem has an exact solution

$$u(x,t) = t^{2+\alpha_0} \sin x.$$

From Table 4 and Table 5, we can see that the difference scheme (5.1)–(5.6) can achieve second order accuracy both in time and in space. We take $\varepsilon = 10^{-10}$ and $\hat{\tau} = \sigma\tau$ in the simulation. The CPU time for both schemes are also shown in Table 5 which verifies the

Table 5 Numerical errors and convergence orders of two difference schemes in space with $N = 5000$ and $(\lambda_0, \lambda_1, \lambda_2) = (3, 2, 1)$, $(\alpha_0, \alpha_1, \alpha_2) = (4/3, 5/4, 6/5)$ for Example 6.2

h	difference scheme (5.1)–(5.6)			difference scheme (3.22)–(3.27)		
	$E(h, \tau)$	order$_h$	CPU	$E(h, \tau)$	order$_h$	CPU
$\pi/4$	1.1701e−3	–	49s	1.2046e−3	–	18.77h
$\pi/8$	2.7123e−4	2.109	84s	3.0567e−4	1.979	19.15h
$\pi/16$	4.2327e−5	2.680	116s	7.6757e−5	1.994	19.48h

efficiency of the scheme (5.1)–(5.6). From Table 5, we find the difference scheme (5.1)–(5.6) can reduce the computational cost significantly.

7 Conclusion

Motivated by the idea in [38], we propose two temporal second-order accuracy difference schemes at the super-convergence point by the order reduction technique for time multi-term fractional diffusion wave equation. The schemes based on the interpolation approximation can achieve higher-order accuracy than L1 formula and more efficient than GL formula which requires continuous zero-extension of the solution when $t < 0$. The unconditional stability and convergence of the two schemes are proved rigorously by the energy method. We also present a fast difference scheme which can reduce the computational cost significantly. The numerical examples are presented to verify the theoretical results.

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