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The Linear Barycentric Rational Quadrature Method for Auto-Convolution Volterra Integral Equations

Min Li[1] · **Chengming Huang**[1,2]

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Abstract
This paper is concerned with the numerical solution of auto-convolution Volterra integral equations. A composite quadrature method based on linear barycentric rational interpolation is introduced. The method is easy to be implemented because only a linear equation needs to be solved in each time step. Collocation method is used as the starting procedure. The boundedness and convergence of the numerical solution are studied in detail. Some numerical experiments are carried out to confirm the theoretical results.

Keywords Auto-convolution Volterra integral equation · Linear barycentric rational interpolation · Barycentric rational quadrature · Collocation method

1 Introduction

Volterra equations play an important role in describing many phenomena in physics. A wide range of numerical methods have been developed for solving this kind of equations. Popular methods include, but are not limited to, collocation methods, direct quadrature methods, Runge–Kutta methods, linear multistep methods, spectral methods and boundary value methods (see [1,3,4,6–11,14–24,26,27,29–31,33] and references therein). Many of them are based on polynomial interpolation, i.e., the approximation solution is a polynomial or piecewise polynomial. Recently, Berrut et al. [4] introduced two direct quadrature methods for second kind Volterra integral equations based on linear rational interpolation, and studied their convergence. Hosseini and Abdi [14] further investigated the stability of the composite barycentric rational quadrature method. Linear rational interpolation has polynomial inter-

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✉ Chengming Huang
chengming_huang@hotmail.com

Min Li
liminmaths@163.com

1 School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China

2 Hubei Key Laboratory of Engineering Modeling and Scientific Computing, Huazhong University of Science and Technology, Wuhan 430074, China

polation as a special case. In the case of equidistant nodes, however, it can avoid the Runge phenomenon of high degree polynomial interpolation if certain parameter value is selected. This is beneficial to construct high order methods with uniform meshes.

Nonstandard Volterra equations received attention only recently. Guan et al. [13] investigated collocation methods for the equation

$$u(t) = g(t) + \int_0^t K(t, s, u(t), u(s))ds, \quad t \in I := [0, T]. \tag{1}$$

Zhang et al. [32] considered collocation methods for auto-convolution Volterra integral equations (AVIEs) of the form

$$u(t) = g(t) + \int_0^t K(t, s)u(t - s)u(s)ds, \quad t \in I := [0, T], \tag{2}$$

where $u(t)$ is an unknown solution, and $g \in C(I)$, $K \in C(D)$ ($D = \{(t, s) : 0 \leq s \leq t \leq T\}$) are given functions. The existence, uniqueness, and regularity of the exact solution were studied, and the convergence order of the collocation solution was given.

In this paper, we also consider AVIEs. We will introduce direct quadrature methods based on linear barycentric rational interpolation [12] and the idea of composite quadrature [4], and we use the collocation method in [32] as the starting procedure. For the resulting composite barycentric rational quadrature method, we study the boundedness and convergence of the numerical solution in detail. Compared with the existent methods, this composite barycentric rational quadrature method is easier to be implemented because only one linear equation needs to be solved in each time step so that the method is essentially explicit. We also mention that the integrand in (2) does not satisfy a global Lipschitz condition because of the existence of the term $u(t-s)$, so that the analysis techniques in [4] cannot be applied to this case directly.

This paper is organized as follows. The barycentric rational interpolation and quadrature methods are discussed in Sect. 2. In Sect. 3, collocation starting procedure is presented. And then, the barycentric rational quadrature method and its convergence analysis are given in Sect. 4. In Sect. 5, some numerical experiments are given to validate the efficiency and accuracy of the methods. Finally, some conclusions are drawn in the last section.

2 The Composite Barycentric Rational Quadrature Rule

In this section, we first review the barycentric rational interpolation, and then consider the corresponding global quadrature and composite quadrature methods.

2.1 Barycentric Rational Interpolation

The polynomial barycentric interpolation was introduced in [5] systematically. For given function $g(t) : [0, T] \to \mathbb{R}$ and interpolating nodes $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n \leq T$, it has the following form

$$r_n(t) = \frac{\sum_{j=0}^{n} \frac{\tau_j}{t-t_j} g_j}{\sum_{j=0}^{n} \frac{\tau_j}{t-t_j}}, \quad \tau_j = \frac{1}{\prod_{i=0, i\neq j}^{n}(t_j - t_i)}.$$

After that, some scholars replaced the barycentric weights τ_j with other more suitable rational weights β_j that are independent of the interpolated functions. The merit lies in that this new kind of weights β_j may remove some bad properties of barycentric weights τ_j, such as unstable Runge phenomenon for equidistant nodes. Additionally, the convergence and stability properties are guaranteed. One of typical examples is the barycentric rational interpolation given by Floater and Hormann [12],

$$r_n(t) = \frac{\sum_{j=0}^{n} \frac{\beta_j}{t-t_j} g_j}{\sum_{j=0}^{n} \frac{\beta_j}{t-t_j}}, \tag{3}$$

with weights

$$\beta_j = \sum_{i\in J_j} (-1)^i \prod_{k=i,k\neq j}^{i+d} \frac{1}{t_j - t_k}, \quad J_j = \{i \in \{0,\dots,n-d\} : j-d \le i \le j\}, \tag{4}$$

where d is a nonnegative integer not exceeding n. In addition, if the interpolating nodes $\{t_j\}_{j=0}^n$ are equidistant, we can alternatively use

$$\beta_j = \frac{(-1)^{j-d}}{2^d} \sum_{i\in J_j} \binom{d}{j-i}, \quad J_j = \{i \in \{0,\dots,n-d\} : j-d \le i \le j\}. \tag{5}$$

Let $p_j(t)$ be the interpolant polynomial whose degree does not exceed d, and which is generated by interpolating values $g_j \cdots g_{j+d}$ at the $d+1$ nodes $t_j < \cdots < t_{j+d}$, then $r_n(t)$ satisfies

$$r_n(t) = \frac{\sum_{j=0}^{n-d} \lambda_j(t) p_j(t)}{\sum_{j=0}^{n-d} \lambda_j(t)}, \quad \text{and} \quad \lambda_j(t) = \frac{(-1)^j}{(t-t_j)\cdots(t-t_{j+d})}. \tag{6}$$

For more details, we can refer to [12]. In this paper, we use Floater and Hormann interpolation with barycentric form (3), which can also be represented as

$$r_n(t) = \sum_{j=0}^{n} g(t_j) l_j^{(\beta)}(t), \quad l_j^{(\beta)}(t) = \frac{\frac{\beta_j}{t-t_j}}{\sum_{j=0}^{n} \frac{\beta_j}{t-t_j}}. \tag{7}$$

And the interpolation error estimate is given by the following lemma.

Lemma 1 [12] *Suppose* $d \ge 0$ *and* $g \in C^{d+2}[0,T]$, *and let* $h := \max_{0\le i\le n-1}(t_{i+1}-t_i)$. *If* $n-d$ *is odd, then*

$$\|r_n - g\| \le h^{d+1}(1+\mu)T\frac{\|g^{(d+2)}\|}{d+2}.$$

If $n-d$ *is even, then*

$$\|r_n - g\| \le h^{d+1}(1+\mu)\left(T\frac{\|g^{(d+2)}\|}{d+2} + \frac{\|g^{(d+1)}\|}{d+1}\right).$$

Here,

$$\|g\| := \max_{0\le x\le T} |g(x)|,$$

and

$$\mu = \begin{cases} \max\limits_{1\le i\le n-2} \min\{\frac{t_{i+1}-t_i}{t_i-t_{i-1}}, \frac{t_{i+1}-t_i}{t_{i+2}-t_{i+1}}\}, & \text{if } d = 0, \\ 0, & \text{if } d > 0. \end{cases}$$

For the case of $d=0$, if the nodes $\{t_i\}_{i=0}^n$ are equidistant, then we have $\mu = 1$.

2.2 Barycentric Rational Quadrature

For simplicity, we consider in the following sections a uniform mesh I_h of size h on $[0, T]$, i.e.,

$$t_i = ih, \quad h = T/N, \quad 0 \le i \le N.$$

Now, we consider $\int_{t_0}^{t_n} g(t)dt$, where the integrand is approximated by $r_n(t)$. Then, we have the barycentric rational quadrature formula,

$$Q_n(g) = \sum_{j=0}^{n} \int_{t_0}^{t_n} l_j^{(\beta)}(t)dt\, g(t_j) = h \sum_{j=0}^{n} w_j^{(n)} g(t_j), \tag{8}$$

with the rational quadrature weights

$$w_j^{(n)} = \int_0^n \frac{\frac{\beta_j}{v-j}}{\sum_{l=0}^{n} \frac{\beta_l}{v-l}} dv, \quad j = 0, 1, \ldots, n. \tag{9}$$

Remark 1 In general, $w_k^{(n)}$ cannot be calculated analytically. We need to use a numerical approximation method. As mentioned in [4], let $w_k^{(n)D}$ be the approximate weights. Thanks to the Chebfun system in Matlab [2,28], as well as Jacobi–Gauss type quadrature methods in [25], we can choose those methods whose orders are higher than the barycentric rational approximation. In this way, we can ignore the discretization error generated by using $w_k^{(n)D}$ to approximate $w_k^{(n)}$.

Remark 2 Since $w_k^{(n)}$ changes with n, we need to compute integral (9) for $n + 1$ times to obtain the numerical solution u_n in each time step when the global quadrature method is applied to integral equations (see Sect. 4). Thus, the computational cost will increase with n. When n is large enough, the calculation will cost a lot of time. So it is more suitable to use the composite quadrature method to reduce computation as well as to save time.

Before we introduce the composite quadrature formula, the parameters d and m should be given in advance. Let $d \ge 0, m \ge 1, d \le m \le n$ and $p := \lfloor n/m \rfloor - 1$, where $\lfloor x \rfloor$ is the largest integer that does not exceed x. In each subinterval $[t_{jm}, t_{(j+1)m}]\,(j = 0, 1, \ldots, p-1)$, we use the nodes $t_{jm}, t_{jm+1}, \ldots, t_{(j+1)m}$ to construct the rational interpolant $r_m^{(j)}(t)$. In the last subinterval $[t_{pm}, t_n]$, we use the nodes $t_{pm}, t_{pm+1}, \ldots, t_n$ to construct the rational interpolant $r_{n-pm}^{(p)}(t)$. Then we integrate them over their respective intervals to obtain the corresponding quadrature formulae. It is easy to observe that for all the subintervals of length mh, the quadrature weights $w_k^{(m)}$ remain unchanged. Hence, we have the following composite quadrature formula

$$\begin{aligned}\int_{t_0}^{t_n} g(t)dt &= \sum_{j=0}^{p-1} \int_{t_{jm}}^{t_{(j+1)m}} g(t)dt + \int_{t_{pm}}^{t_n} g(t)dt \\ &\approx h \sum_{j=0}^{p-1} \sum_{k=0}^{m} w_k^{(m)} g(t_{jm+k}) + h \sum_{k=0}^{n-pm} w_k^{(n-pm)} g(t_{pm+k}), \quad n \ge m+1, \end{aligned} \tag{10}$$

where

$$w_k^{(l)} = l \int_0^1 \frac{\frac{\beta_k}{ls-k}}{\sum_{j=0}^{l} \frac{\beta_j}{ls-j}} ds, \quad l = m, m+1, \ldots, 2m-1. \tag{11}$$

We remark that when $m < n < 2m$, (10) is a global quadrature formula which is the same as (8) since $p = 0$ and $\sum_{j=0}^{-1}$ means empty. And we do not use composite quadrature formula (10) until $n \geq 2m$.

Theorem 1 *Assume that n, $m \leq n$ are positive integers, parameter d $(0 \leq d \leq m)$ is nonnegative integer, and $g \in C^{d+2}[0, t_n]$. Then the absolute quadrature error corresponding to the composite quadrature method (10) with equidistant nodes is bounded by Ch^q, where*

$$q = \begin{cases} d+1, & m-d \text{ is even}, \\ d+2, & m-d \text{ is odd}, \end{cases} \tag{12}$$

and the constant C depends on d and m, on the derivatives of g, and on the interval length $t_n - t_0$.

Proof Since m is fixed and $n - pm < 2m$, we have

$$\begin{aligned} &\left| \sum_{j=0}^{p-1} \int_{t_{jm}}^{t_{(j+1)m}} (r_m^{(j)}(t) - g(t))\, dt + \int_{t_{pm}}^{t_n} (r_{n-pm}^{(p)}(t) - g(t))\, dt \right| \\ &\quad \leq \sum_{j=0}^{p-1} mh \max_{t_{jm} \leq t \leq t_{(j+1)m}} |r_m^{(j)}(t) - g(t)| + (n-pm)h \max_{t_{pm} \leq t \leq t_n} |r_{n-pm}^{(p)}(t) - g(t)| \\ &\quad \leq Ch^q, \end{aligned}$$

where we have used Lemma 1 and in particular, the quantity T in the estimate of Lemma 1 is replaced by the lengths of subintervals $[t_{jm}, t_{(j+1)m}]$ and $[t_{pm}, t_n]$, respectively. □

3 Collocation Starting Procedure

In this section, we consider how to solve AVIE (2) over the interval $[t_0, t_m]$. We regard $[t_0, t_m]$ as one collocation interval and apply the method in [32], i.e., we find a polynomial u_h of degree not exceeding $m - 1$ such that:

$$u_h(c_i mh) = g(c_i mh) + \int_0^{c_i mh} K(c_i mh, s) u_h(c_i mh - s) u_h(s) ds, \quad i = 1, 2, \ldots, m,$$

where $c_1, c_2, \ldots, c_m$ are collocation parameters. Let

$$l_j(s) = \prod_{l=1, l \neq j}^{m} \frac{s - c_l}{c_j - c_l},$$

and let $U_{0,i} := u_h(c_i mh)$. Then we have

$$\begin{aligned} U_{0,i} &= g(c_i mh) + mh \sum_{j,k=1}^{m} \int_0^{c_i} K(c_i mh, smh) l_j(c_i - s) l_k(s) ds\, U_{0,j} U_{0,k}, \\ & \quad i = 1, 2, \ldots, m. \end{aligned} \tag{13}$$

Denote

$$a_{j,k}^{(i)} = \int_0^{c_i} K(c_i mh, smh) l_j(c_i - s) l_k(s) ds,$$

then

$$U_{0,i}=g(c_imh)+mh(U_{0,1},U_{0,2},\dots U_{0,m})\begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} & \cdots & a_{1m}^{(i)} \\ a_{21}^{(i)} & a_{22}^{(i)} & \cdots & a_{2m}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^{(i)} & a_{m2}^{(i)} & \cdots & a_{mm}^{(i)} \end{pmatrix}\begin{pmatrix} U_{0,1} \\ U_{0,2} \\ \vdots \\ U_{0,m} \end{pmatrix}, \quad i=1,\dots,m. \tag{14}$$

The convergence of collocation method (14) has been studied in [32]. Here, we only consider the error of the collocation solution at collocation points on the first collocation interval $[t_0, t_m]$. The following result is implied in the proof of Theorem 4.3 in [32].

Lemma 2 [32] *Assume that the given functions g and K in AVIE (2) are m times continuously differentiable on their respective domains. Then there exists an $\bar{h}>0$, such that for any $h\in(0,\bar{h})$ and any choice of the collocation parameters $\{c_i\}_{i=1}^m$, we have*

$$\max_{1\le i\le m}\left|u(c_imh)-U_{0,i}\right|\le Ch^{m+1}, \tag{15}$$

where C depends on the collocation parameters $\{c_i\}_{i=1}^m$ and on $u^{(m)}$ but not on h.

Remark 3 We use the collocation solutions as the starting values of the quadrature methods introduced in the next section, i.e., we take collocation parameters $c_i=\frac{i}{m}$ and $u_j=U_{0j}$ $(j=1,2,\dots,m)$. From (15) it follows that $\{u_j\}_{j=1}^m$ are accurate of order $m+1$.

Remark 4 It is worth noting that (14) is an m dimensional nonlinear system, which can be solved by Newton iteration. But the iteration relies on the selection of initial values. Thus, Newton downhill method is often used in practical application.

4 The Barycentric Rational Quadrature Method

In this section, we first introduce a barycentric rational quadrature method for AVIE (2) based on the composite quadrature formula (17). Then we study the existence, uniqueness and uniform boundedness of the numerical solution. We finally give the corresponding convergence result. Before proceeding further, let us recall the regular properties of the exact solution of AVIE (2) in the following lemmas.

Lemma 3 [32] *Assume that the given functions in AVIE (2) satisfy $g\in C(I)$, $K\in C(D)$. Then AVIE (2) has a unique solution $u\in C(I)$.*

Lemma 4 [32] *Assume that the given functions in AVIE (2) satisfy $g\in C^m(I)$, $K\in C^m(D)$ for some integer $m\ge 1$. Then AVIE (2) has a unique solution $u\in C^m(I)$.*

Since the starting values u_j $(j=1,\dots,m)$ have been given by the collocation method in the previous section, we can give the numerical calculation scheme by apply the global rational quadrature method (8) to AVIE (2),

$$u_n=g(t_n)+h\sum_{k=0}^{n}w_k^{(n)}K(t_n,t_k)u_{n-k}u_k, \quad m+1\le n\le N. \tag{16}$$

However, as noted in Remark 2, for global quadrature method, we need to recalculate all the weights $w_k^{(n)}$ in each time step. If n is large enough, the cost of computation is huge. Hence, it is more reasonable to use the composite quadrature method based on (10) to reduce computation. For $n = m + 1, \ldots, N$, the composite barycentric rational quadrature method for AVIE (2) is as follows,

$$\begin{aligned} u_n &= g(t_n) + h \sum_{j=0}^{p-1} \sum_{k=0}^{m} w_k^{(m)} K(t_n, t_{jm+k}) u_{n-jm-k} u_{jm+k} \\ &\quad + h \sum_{k=0}^{n-pm} w_k^{(n-pm)} K(t_n, t_{pm+k}) u_{n-pm-k} u_{pm+k}, \end{aligned} \tag{17}$$

where $w_k^{(m)}$ and $w_k^{(n-pm)}$ are defined in (11).

Remark 5 It is worth pointing out that collocation method is used for the first m nodes. When $m < n < 2m$, (17) is actually a global method which is the same as (16) because $p = 0$ and $\sum_{j=0}^{-1}$ means empty. When $n \geq 2m$, (17) is different from (16), and we begin to use composite quadrature method. Besides, quadrature weights $w_k^{(m)}$ remain unchanged for all integrals with interval length mh.

4.1 The Uniform Boundedness of Numerical Solution

We mainly investigate the uniform boundedness of numerical solution in this subsection. The idea of the following lemma comes from [29].

Lemma 5 *Let*

$$I(\lambda) = \sup_{t \in [0,T]} \int_0^t e^{-2\lambda s(t-s)} ds, \quad \lambda \in (0, \infty)$$

then

$$\lim_{\lambda \to +\infty} I(\lambda) = 0.$$

Proof It is easy to see $I(\lambda) \geq 0$. For any $\epsilon > 0$, letting $\delta = \frac{\epsilon}{2}$ and $\Lambda = \frac{1}{\delta\epsilon}$, we prove that $I(\lambda) < \epsilon$ for all $\lambda \geq \Lambda$. In fact,

$$I(\lambda) = \max \left\{ \sup_{t \in [0,\delta]} \int_0^t e^{-2\lambda s(t-s)} ds, \sup_{t \in [\delta,T]} \int_0^t e^{-2\lambda s(t-s)} ds \right\}.$$

For the first term on the right-hand side, it is easy to find

$$\sup_{t \in [0,\delta]} \int_0^t e^{-2\lambda s(t-s)} ds < \delta < \epsilon.$$

For the second term on the right-hand side, we have

$$\begin{aligned} \sup_{t \in [\delta,T]} \int_0^t e^{-2\lambda s(t-s)} ds &= \sup_{t \in [\delta,T]} \Big(\int_0^\delta e^{-2\lambda s(t-s)} ds + \int_\delta^t e^{-2\lambda s(t-s)} ds \Big) \\ &\leq \frac{\epsilon}{2} + \sup_{t \in [\delta,T]} \int_\delta^t e^{-2\lambda\delta(t-s)} ds \end{aligned}$$

$$< \frac{\epsilon}{2} + \frac{1}{2\lambda\delta} \\ \le \epsilon.$$

A combination of two cases gives the desired conclusion. □

Theorem 2 *Assume that the given functions in AVIE (2) satisfy $g \in C(I)$, $K \in C(D)$. Then there exists an $\bar{h} > 0$, such that for all $h \in (0, \bar{h})$, the numerical solution $\{u_n\}_{n=0}^N$ is uniquely existent and uniformly bounded by*

$$\max_{0 \le n \le N} |u_n| \le (2\overline{G} + 1)e^{\lambda T^2},$$

where

$$\overline{G} := \max_{t \in I} |g(t)|, \quad \bar{K} := \max_{(t,s) \in D} |K(t,s)| \ \text{and} \ \lambda := \inf\left\{z > 0 : \ I(z) \le \frac{1}{16W\bar{K}(2\overline{G}+1)}\right\}.$$

Proof Some notations in the following come from the references [29,32]. Let

$$W := \max_{m \le l \le 2m-1} \max_{0 \le k \le l} |w_k^{(l)}|,$$

$U := (u_0, u_1, \ldots, u_N)^T$ and $G := (g(t_0), g(t_1), \ldots, g(t_N))^T$. Similarly to [29], we first introduce a new norm by

$$\|U\|_\lambda = \max\left\{\max_{0 \le n \le m} |u_n|, \max_{m+1 \le n \le N} e^{-\lambda t_n^2}|u_n|\right\}.$$

It is easy to verify that

$$e^{-\lambda T^2}\|U\|_\infty \le \|U\|_\lambda \le \|U\|_\infty.$$

And we define operators $T_N : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ and $\overline{T}_N : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ by

$$(T_N U)_n = \begin{cases} 0, & n = 0, \\ mh \sum_{j,k=1}^m a_{j,k}^{(n)} u_j u_k, & n = 1, 2, \ldots, m, \\ h \sum_{j=0}^{p-1} \sum_{k=0}^m w_k^{(m)} K(t_n, t_{jm+k}) u_{n-jm-k} u_{jm+k} & \\ \quad + h \sum_{k=0}^{n-pm} w_k^{(n-pm)} K(t_n, t_{pm+k}) u_{n-pm-k} u_{pm+k}, & n = m+1, \ldots, N. \end{cases}$$

and

$$(\overline{T}_N U)_n = g(t_n) + (T_N U)_n.$$

Consider the set $C(0, \overline{G}) := \{U \in \mathbb{R}^{N+1} : \|U\|_\lambda \le 2\overline{G} + 1\}$. For any $U \in C(0, \overline{G})$, we have

$$\|\overline{T}_N U\|_\lambda \le \|G\|_\lambda + \|T_N U\|_\lambda \\ \le \overline{G} + \max\left\{\max_{0 \le n \le m} |(T_N U)_n|, \max_{m+1 \le n \le N} e^{-\lambda t_n^2}|(T_N U)_n|\right\}. \tag{18}$$

It follows from Section 3 of [32] that there exists $h_1 > 0$, such that for $h \in (0, h_1)$,

$$\max_{1 \le n \le m} |(T_N U)_n| \le \frac{2\overline{G} + 1}{2}. \tag{19}$$

In addition, for $m + 1 \le n \le N$

$$
\begin{aligned}
&\max_{m+1\le n\le N} e^{-\lambda t_n^2}|(T_N U)_n| \\
&\le \max_{m+1\le n\le N} e^{-\lambda t_n^2}\Bigg(h\sum_{j=0}^{p-1}\sum_{k=0}^{m}|w_k^{(m)}||K(t_n,t_{jm+k})||u_{n-jm-k}||u_{jm+k}| \\
&\quad + h\sum_{k=0}^{n-pm}|w_k^{(n-pm)}||K(t_n,t_{pm+k})||u_{n-pm-k}||u_{pm+k}|\Bigg) \\
&\le \max_{m+1\le n\le N} 2hW\bar{K}e^{-\lambda t_n^2}\sum_{k=0}^{n}|u_{n-k}||u_k| \\
&= \max_{m+1\le n\le N} 2W\bar{K}h\sum_{k=0}^{n} e^{-\lambda t_n^2+\lambda t_{n-k}^2+\lambda t_k^2}\, e^{-\lambda t_{n-k}^2}|u_{n-k}|\, e^{-\lambda t_k^2}|u_k| \\
&\le 2W\bar{K}\|U\|_\lambda^2 \max_{m+1\le n\le N} h\sum_{k=0}^{n} e^{-\lambda t_n^2+\lambda t_{n-k}^2+\lambda t_k^2} \\
&= 2W\bar{K}\|U\|_\lambda^2 \max_{m+1\le n\le N} h\sum_{k=0}^{n} e^{-2\lambda t_k t_{n-k}}.
\end{aligned}
$$

Let $\epsilon = \frac{1}{16W\bar{K}(2\overline{G}+1)^2}$, the definition of definite integral reveals that there exists $h_2 > 0$, such that for $h \in (0, h_2)$,

$$\max_{m+1\le n\le N} h\sum_{k=0}^{n} e^{-2\lambda t_k t_{n-k}} < I(\lambda)+\epsilon.$$

Then, for any $U \in C(0, \overline{G})$,

$$
\begin{aligned}
\max_{m+1\le n\le N} e^{-\lambda t_n^2}|(T_N U)_n| &\le 2W\bar{K}\|U\|_\lambda^2\,(I(\lambda)+\epsilon) \\
&\le 2W\bar{K}\|U\|_\lambda^2\left(\frac{1}{16W\bar{K}(2\overline{G}+1)}+\frac{1}{16W\bar{K}(2\overline{G}+1)^2}\right) \\
&\le \frac{1}{4}(\overline{G}+1).
\end{aligned}
\tag{20}
$$

Combining (18), (19) and (20), we conclude that there exists $0 < \bar{h} < \min\{h_1, h_2\}$ such that for $h \in (0, \bar{h})$,

$$\overline{T}_N :\ \ C(0,\overline{G}) \longrightarrow C(0,\overline{G}).$$

Moreover, for all $\phi, \psi \in C(0, \overline{G})$,

$$\|\overline{T}_N\phi-\overline{T}_N\psi\|_\lambda = \max\left\{\max_{0\le n\le m}|(T_N\phi-T_N\psi)_n|,\ \max_{m+1\le n\le N} e^{-\lambda t_n^2}|(T_N\phi-T_N\psi)_n|\right\}.$$

On the one hand, from Section 3 of [32] it follows that there exists $h_1 > 0$, such that for $h \in (0, h_1)$,

$$\max_{1\le n\le m}|(T_N\phi-T_N\psi)_n| \le \frac{1}{2}\max_{1\le i\le m}|(\phi-\psi)_i| \le \frac{1}{2}\|\phi-\psi\|_\lambda.$$

On the other hand, when $h \in (0, \bar{h})$

$$\max_{m+1\le n\le N} e^{-\lambda t_n^2}|(T_N\phi-T_N\psi)_n|$$

$$\le \max_{m+1\le n\le N} e^{-\lambda t_n^2}\left| h\sum_{j=0}^{p-1}\sum_{k=0}^{m} w_k^{(m)} K(t_n, t_{jm+k})(\phi_{n-jm-k}\phi_{jm+k} - \psi_{n-jm-k}\psi_{jm+k}) \right.$$

$$\left. + h\sum_{k=0}^{n-pm} w_k^{(n-pm)} K(t_n, t_{pm+k})(\phi_{n-pm-k}\phi_{pm+k} - \psi_{n-pm-k}\psi_{pm+k}) \right|$$

$$\le h \max_{m+1\le n\le N} e^{-\lambda t_n^2}\left(\sum_{j=0}^{p-1}\sum_{k=0}^{m} |w_k^{(m)}||K(t_n, t_{jm+k})||\phi_{n-jm-k}\phi_{jm+k} - \psi_{n-jm-k}\psi_{jm+k}| \right.$$

$$\left. + \sum_{k=0}^{n-pm} |w_k^{(n-pm)}||K(t_n, t_{pm+k})||\phi_{n-pm-k}\phi_{pm+k} - \psi_{n-pm-k}\psi_{pm+k}| \right)$$

$$\le 2h \max_{m+1\le n\le N} W\bar{K}e^{-\lambda t_n^2}\sum_{k=0}^{n}\left|(\phi_{n-k}\phi_k - \phi_{n-k}\psi_k) + (\phi_{n-k}\psi_k - \psi_{n-k}\psi_k)\right|$$

$$\le 2h \max_{m+1\le n\le N} W\bar{K}e^{-\lambda t_n^2}\sum_{k=0}^{n}(|\ \phi_{n-k}\ | + |\ \psi_{n-k}\ |)\ |\ \phi_k - \psi_k\ |$$

$$\le \max_{m+1\le n\le N} 2W\bar{K}h\sum_{k=0}^{n} e^{-\lambda t_n^2+\lambda t_{n-k}^2+\lambda t_k^2}\ e^{-\lambda t_{n-k}^2}(|\ \phi_{n-k}\ | + |\ \psi_{n-k}\ |)\ e^{-\lambda t_k^2}|\phi_k - \psi_k|$$

$$\le 2W\bar{K}(\|\phi\|_\lambda + \|\psi\|_\lambda)\max_{m+1\le n\le N}\ h\sum_{k=0}^{n} e^{-2\lambda t_k t_{n-k}}\|\phi - \psi\|_\lambda \le \frac{1}{2}\|\phi - \psi\|_\lambda.$$

The last inequality above can be obtained by the same process as in (20). Thus, $\overline{T}_N : C(0,\overline{G}) \longrightarrow C(0,\overline{G})$ is a contraction mapping. By Banach contractive mapping theorem, we can conclude that there exists a unique uniformly bounded solution $U = (u_0, u_1, \ldots, u_N)^T$ in $C(0,\overline{G})$. Hence, the conclusion follows immediately. □

4.2 Consistency

As in [19], we first introduce some preliminary consistency definitions, which are necessary and convenient for our theoretical analysis.

Definition 1 Let u be the solution of Eq. (2). The function

$$\begin{aligned}
\delta(h, t_n) := &\sum_{j=0}^{p-1}\int_{t_{jm}}^{t_{(j+1)m}} K(t_n, s)u(t_n - s)u(s)ds \\
&- h\sum_{k=0}^{m} w_k^{(m)} K(t_n, t_{jm+k})u(t_n - t_{jm+k})u(t_{jm+k}) \\
&+ \int_{t_{pm}}^{t_n} K(t_n, s)u(t_n - s)u(s)ds \\
&- h\sum_{k=0}^{n-pm} w_k^{(n-pm)} K(t_n, t_{pm+k})u(t_n - t_{pm+k})u(t_{pm+k}),
\end{aligned} \tag{21}$$

is called the local error of barycentric quadrature method (17) for (2) at t_n.

Definition 2 [19] Let Θ be a class of equations of the form (2). If for every equation in Θ,

$$\lim_{h\to 0} \max_{m+1\le n\le N} \|\delta(h, t_n)\| = 0,$$

then the approximation method (17) is said to be consistent with (2) for the class of equations in Θ. If for every equation in Θ, there exists a constant C which is independent of h, such that

$$\max_{m+1\le n\le N} \|\delta(h, t_n)\| \le Ch^p,$$

then the method is said to be consistent of order p in Θ.

4.3 Convergence

Lemma 6 [19] *Let the sequence* $\xi_0, \xi_1, \ldots$ *satisfy*

$$|\xi_n| \le A\sum_{i=0}^{n-1} |\xi_i| + B_n, \quad n = r,\ r+1, \ldots,$$

where

$$A > 0, \quad |B_n| \le B, \quad \sum_{i=0}^{r-1} |\xi_i| \le \eta.$$

Then

$$|\xi_n| \le (1 + A)^{n-r}(B + A\eta), \quad n = r, r+1, \ldots.$$

Moreover, if $A = h\bar{K}$, $t_n = nh$, *then*

$$|\xi_n| \le (B + h\bar{K}\eta)e^{\bar{K}t_n}.$$

With the prior preliminary results, we can give the following convergence theorem.

Theorem 3 *Assume that*

(i) The given functions in the AVIE (2) *satisfy* $g \in C^{m+2}(I)$, $K \in C^{m+2}(D)$.
(ii) Collocation method (14) *with* m *collocation parameters* $c_i = \frac{i}{m}$ *is used in the starting procedure.*
(iii) After the starting procedure, AVIE (2) *is approximated by barycentric rational quadrature method* (17) *with integer parameter* d $(0 \le d \le m)$.

Then the convergence result is described by,

$$\max_{i=m+1,\ldots,N} \left|u(t_i) - u_i\right| \le Ch^q, \tag{22}$$

where

$$q = \begin{cases} d+1, & m-d \ is\ even, \\ d+2, & m-d\ is\ odd. \end{cases} \tag{23}$$

Proof Let $\varepsilon_n = u(t_n) - u_n$. Then, for $n = m+1, \ldots, N$, we have

$$\varepsilon_n = u(t_n) - u_n$$
$$= \sum_{j=0}^{p-1}\left(\int_{t_{jm}}^{t_{(j+1)m}} K(t_n, s)u(t_n - s)u(s)ds - h\sum_{k=0}^{m} w_k^{(m)} K(t_n, t_{jm+k})u_{n-jm-k}u_{jm+k}\right)$$

$$+\int_{t_{pm}}^{t_n} K(t_n,s)u(t_n-s)u(s)ds - h\sum_{k=0}^{n-pm} w_k^{(n-pm)} K(t_n,t_{pm+k})u_{n-pm-k}u_{pm+k}.$$

One can see from (21) that,

$$\begin{aligned}
\varepsilon_n &= \delta(h,t_n) + h\sum_{j=0}^{p-1}\sum_{k=0}^{m} w_k^{(m)} K(t_n,t_{jm+k})\big[u(t_n-t_{jm+k})u(t_{jm+k}) \\
&\quad - u_{n-jm-k}u(t_{jm+k}) + u_{n-jm-k}u(t_{jm+k}) - u_{n-jm-k}u_{jm+k}\big] \\
&\quad + h\sum_{k=0}^{n-pm} w_k^{(n-pm)} K(t_n,t_{pm+k})\big[u(t_n-t_{pm+k})u(t_{pm+k}) \\
&\quad - u_{n-pm-k}u(t_{pm+k}) + u_{n-pm-k}u(t_{pm+k}) - u_{n-pm-k}u_{pm+k}\big] \\
&= \delta(h,t_n) + h\sum_{j=0}^{p-1}\sum_{k=0}^{m} w_k^{(m)} K(t_n,t_{jm+k})\big[\varepsilon_{n-jm-k}u(t_{jm+k}) + u_{n-jm-k}\varepsilon_{jm+k}\big] \\
&\quad + h\sum_{k=0}^{n-pm} w_k^{(n-pm)} K(t_n,t_{pm+k})\big[\varepsilon_{n-pm-k}u(t_{pm+k}) + u_{n-pm-k}\varepsilon_{pm+k}\big].
\end{aligned}$$

According to Lemma 3 and Theorem 2, when h is small enough, the exact solution and the numerical solution of AVIE (2) are uniformly bounded. Without loss of generality, let C be their common upper bound, $\bar{K}$ be upper bound of $K(t,s)$ on D and $W = \max_{m\le l\le 2m-1}\max_{0\le k\le l}|w_k^{(l)}|$. Hence,

$$|\varepsilon_n| \le |\delta(h,t_n)| + 2hW\bar{K}C\sum_{i=0}^{n}|\varepsilon_i| + 2hW\bar{K}C\sum_{i=0}^{n}|\varepsilon_{n-i}|,$$

which gives,

$$|\varepsilon_n| \le \frac{4hW\bar{K}C}{1-4hW\bar{K}C}\sum_{i=0}^{n-1}|\varepsilon_i| + \frac{|\delta(h,t_n)|}{1-4hW\bar{K}C}.$$

Applying Lemma 6 to the above inequality yields

$$|\varepsilon_n| \le \frac{1}{1-4hW\bar{K}C}\left\{\max_{m+1\le i\le n}|\delta(h,t_i)| + 4hW\bar{K}C\sum_{i=0}^{m}|\varepsilon_i|\right\} e^{\frac{4W\bar{K}C}{1-4hW\bar{K}C}t_n},\ n=m+1,\ldots,N. \tag{24}$$

Since $u_0 = u(t_0) = g(t_0)$, we have $\varepsilon_0 = 0$. And it follows from Lemma 2 that

$$|\varepsilon_i| = |u_i - u(t_i)| = O(h^{m+1}),\quad i=1,\ldots,m.$$

Theorem 1 implies that the barycentric rational quadrature method (17) is consistent of order q, i.e.,

$$\max_{m+1\le i\le N}|\delta(h,t_i)| \le Ch^q,$$

where q is stated in (23). Then, the convergence result stated in (22) and (23) follows from (24) immediately. □

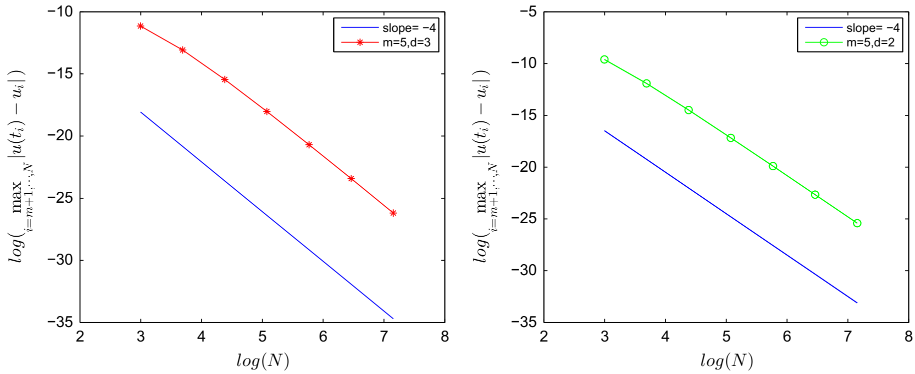

Fig. 1 The convergence order of composite barycentric rational quadrature method (17) with $(m, d) = (5, 2)$ and $(m, d) = (5, 3)$ for Example 1

5 Numerical Experiments

In this section, we apply composite quadrature method (17) to some auto-convolution Volterra integral equations, to demonstrate the efficiency and accuracy of the proposed scheme.

Example 1 Consider the equation

$$u(t) = g(t) + \int_0^t cos(t-s)u(t-s)u(s)ds, \quad t \in [0, 1],$$

where

$$g(t) = t^2 + 3 - (21sin(t) - 6t + t^2 sin(t) - 6t\, cos(t)),$$

such that the equation has the exact solution $u(t) = t^2 + 3$.

Collocation method with 5 collocation points is used in the starting procedure. Figure 1 shows the maximal absolute errors of the composite barycentric rational quadrature method (17) for the rest grid points with $(m, d) = (5, 2)$ and $(m, d) = (5, 3)$. For $(m, d) = (5, 2)$, because $m - d$ is odd, the theoretical order of convergence should be $d + 2 = 4$. And for $(m, d) = (5, 3)$, since $m - d$ is even, the corresponding convergence order should be $d + 1 = 4$. One may observe that all the numerical convergence orders are approximately $p = 4$ from Fig. 1, which matches the theoretical results in Theorem 3.

Example 2 Secondly, we consider the auto-convolution Volterra integral equation

$$u(t) = g(t) + \int_0^t (t+s)^2 u(t-s)u(s)ds, \quad t \in [0, 1],$$

where

$$g(t) = t^3 - \frac{41}{2520}t^9,$$

such that the exact solution is $u(t) = t^3$.

Collocation method with 3 collocation parameters is implemented in the starting procedure. And the composite barycentric rational quadrature method (17) with $(m, d) = (3, 2)$ and $(m, d) = (3, 3)$ are used for the rest grid points. For $(m, d) = (3, 2)$, because $m - d$

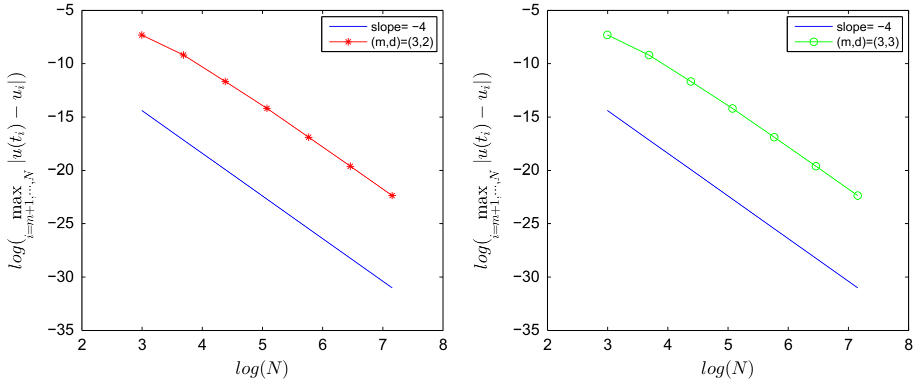

Fig. 2 The convergence order of composite barycentric rational quadrature method (17) with $(m, d) = (3, 2)$ and $(m, d) = (3, 3)$ for Example 2

Table 1 Values of $\max \{|\varepsilon_i|,\ i = m+1, \ldots, N\}$ corresponding to (17) for Example 3

N	$(m, d) = (5, 2)$		$(m, d) = (5, 4)$		$(m, d) = (5, 5)$	
	Error	Rate	Error	Rate	Error	Rate
20	4.4134e−07	–	5.1319e−10	–	5.2328e−10	–
40	2.8316e−08	3.9622	1.1123e−11	5.5279	8.2395e−12	5.9889
80	1.7903e−09	3.9833	2.1677e−13	5.6812	1.9101e−13	5.4308
160	1.1243e−10	3.9931	3.3862e−15	6.0004	3.1641e−15	5.9157
320	7.0417e−12	3.9970	1.6653e−16	4.3458	1.6653e−16	4.2479

is odd, the theoretical order of convergence should be $d + 2 = 4$. And for $(m, d) = (3, 3)$, since $m - d$ is even, the corresponding convergence order should be $d + 1 = 4$. We display the convergence results in Fig. 2. It can be seen that all the numerical convergence orders are approximately $p = 4$, which confirms the theoretical results in Theorem 3 well.

Example 3 Finally, we consider the example in [32]

$$u(t) = \beta t e^{-\gamma t} - \beta^2 e^{-\gamma t}[2sin(t) - t - tcos(t)] + \int_0^t K(t, s)u(t - s)u(s)ds, \quad t \in [0, 1],$$

with $K(t, s) = cos(t - s)$. And the exact solution is $u(t) = \beta t e^{-\gamma t}$.

We also choose $\gamma = 1$, $\beta = 1$ in this numerical test. 5 collocation points are chosen in the starting procedure, and the composite rational quadrature methods with parameter $d = 2, 4, 5$ are used for the rest nodes respectively. Convergence results are shown in Table 1. For $(m, d) = (5, 5)$, since $m - d$ is even, the theoretical order of convergence is $d + 1 = 6$. And for the cases of $(m, d) = (5, 2)$ and $(m, d) = (5, 4)$, since $m - d$ are odd, theoretical convergence order should be $d + 2 = 4$ and 6 respectively. And from Table 1, one can observe that the numerical results coincide with the theoretical ones. Note that the case of $(m, d) = (5, 4)$ and $(m, d) = (5, 5)$, they reach machine precision when $N = 160$, which explains why the experimental orders of the last row in Table 1 are smaller than 6.

6 Conclusion

In this paper, the barycentric rational quadrature method is applied to AVIEs, and the convergence analysis is established. Since the cost of the global quadrature method is huge, we choose the composite barycentric rational quadrature method to reduce computation cost. This method only needs to solve a linear equation in each time step and can achieve a high order of convergence if parameter d is appropriately large. Both theoretical analysis and numerical tests show that the composite barycentric rational quadrature method is efficient for the numerical solution of AVIEs.

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References

1. Baker, C.T.H.: The Numerical Treatment of Integral Equations. Monographs on Numerical Analysis. Clarendon Press, Oxford (1977)
2. Battles, Z., Trefethen, L.N.: An extension of matlab to continuous functions and operators. SIAM J. Sci. Comput. **25**(5), 1743–1770 (2004)
3. Bellen, A., Jackiewicz, Z., Vermiglio, R., Zennaro, M.: Stability analysis of Runge–Kutta methods for Volterra integral equations of the second kind. IMA J. Numer. Anal. **10**(1), 103–118 (1990)
4. Berrut, J.P., Hosseini, S., Klein, G.: The linear barycentric rational quadrature method for Volterra integral equations. SIAM J. Sci. Comput. **36**(1), A105–A123 (2014)
5. Berrut, J.P., Trefethen, L.N.: Barycentric Lagrange interpolation. SIAM Rev. **46**(3), 501–517 (2004)
6. Brunner, H.: Collocation Methods for Volterra Integral and Related Functional Differential Equations. Cambridge University Press, Cambridge (2004)
7. Brunner, H., Hairer, E., Nørsett, S.: Runge–Kutta theory for Volterra integral equations of the second kind. Math. Comput. **39**(159), 147–163 (1982)
8. Brunner, H., van der Houwen, P.J.: The Numerical Solution of Volterra Equations. CWI Monographs, vol. 3. North-Holland Publishing Co., Amsterdam (1986)
9. Brunner, H., Yan, N.: On global superconvergence of iterated collocation solutions to linear second-kind Volterra integral equations. J. Comput. Appl. Math. **67**(1), 185–189 (1996)
10. Chen, H., Zhang, C.: Block boundary value methods for solving Volterra integral and integro-differential equations. J. Comput. Appl. Math. **236**(11), 2822–2837 (2012)
11. Chen, Y., Tang, T.: Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equations with a weakly singular kernel. Math. Comput. **79**(269), 147–167 (2010)
12. Floater, M.S., Hormann, K.: Barycentric rational interpolation with no poles and high rates of approximation. Numer. Math. **107**(2), 315–331 (2007)
13. Guan, Q., Zhang, R., Zou, Y.: Analysis of collocation solutions for nonstandard Volterra integral equations. IMA J. Numer. Anal. **32**(4), 1755–1785 (2011)
14. Hosseini, S.A., Abdi, A.: On the numerical stability of the linear barycentric rational quadrature method for Volterra integral equations. Appl. Numer. Math. **100**, 1–13 (2016)
15. Hu, Q.: Superconvergence of numerical solutions to Volterra integral equations with singularities. SIAM J. Numer. Anal. **34**(5), 1698–1707 (1997)
16. Huang, C.: Stability of linear multistep methods for delay integro-differential equations. Comput. Math. Appl. **55**(12), 2830–2838 (2008)
17. Huang, C., Vandewalle, S.: Stability of Runge–Kutta–Pouzet methods for Volterra integro-differential equations with delays. Front. Math. China **4**(1), 63–87 (2009)
18. Liang, H., Brunner, H.: On the convergence of collocation solutions in continuous piecewise polynomial spaces for Volterra integral equations. BIT **56**(4), 1339–1367 (2016)
19. Linz, P.: Analytical and Numerical Methods for Volterra Equations. SIAM, Philadelphia (1985)
20. Lubich, C.: On the stability of linear multistep methods for Volterra convolution equations. IMA J. Numer. Anal. **3**(4), 439–465 (1983)
21. Lubich, C.: Runge–Kutta theory for Volterra and Abel integral equations of the second kind. Math. Comput. **41**(163), 87–102 (1983)

22. Ma, J., Xiang, S.: A collocation boundary value method for linear Volterra integral equations. J. Sci. Comput. **71**(1), 1–20 (2017)
23. Ming, W., Huang, C.: Collocation methods for Volterra functional integral equations with non-vanishing delays. Appl. Math. Comput. **296**, 198–214 (2017)
24. Pouzet, P.: étude en vue de leur traitement numérique des équations intégrales de type Volterra. Rev. Franç. Trait. Inf. Chiffres **6**, 79–112 (1963)
25. Shen, J., Tang, T., Wang, L.L.: Spectral Methods: Algorithms, Analysis and Applications, vol. 41. Springer, Berlin (2011)
26. Tang, T.: Superconvergence of numerical solutions to weakly singular Volterra integro-differential equations. Numer. Math. **61**(3), 373–382 (1992)
27. Tang, T., Xu, X., Cheng, J.: On spectral methods for Volterra integral equations and the convergence analysis. J. Comput. Math. **26**, 825–837 (2008)
28. Trefethen, L.N., et al.: Chebfun version 4.2, the Chebfun development team (2011)
29. Wang, Z., Yang, Z., Zhao, C.: Theoretical and numerical analysis for third-kind auto-convolution Volterra integral equations. Commun. Nonlinear Sci. Numer. Simul. (under review)
30. Xie, H., Zhang, R., Brunner, H.: Collocation methods for general Volterra functional integral equations with vanishing delays. SIAM J. Sci. Comput. **33**(6), 3303–3332 (2011)
31. Xie, Z., Li, X., Tang, T.: Convergence analysis of spectral galerkin methods for Volterra type integral equations. J. Sci. Comput. **53**(2), 414–434 (2012)
32. Zhang, R., Liang, H., Brunner, H.: Analysis of collocation methods for generalized auto-convolution Volterra integral equations. SIAM J. Numer. Anal. **54**(2), 899–920 (2016)
33. Zhao, J., Cao, Y., Xu, Y.: Legendre spectral collocation methods for Volterra delay-integro-differential equations. J. Sci. Comput. **67**(3), 1110–1133 (2016)