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A Time-Spectral Algorithm for Fractional Wave Problems

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Abstract This paper develops a high-accuracy algorithm for time fractional wave problems, which employs a spectral method in the temporal discretization and a finite element method in the spatial discretization. Moreover, stability and convergence of this algorithm are derived, and numerical experiments are performed, demonstrating the exponential decay in the temporal discretization error provided the solution is sufficiently smooth.

Keywords Fractional wave problem · Spectral method · Finite element

1 Introduction

Let $1 < \gamma < 2$ and let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a polygon/polyhedron. This paper considers the fractional wave problem

$$\begin{cases} D_{0+}^{\gamma}(u - u_0 - tu_1) - \Delta u = f & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \\ u_t(\cdot, 0) = u_1 & \text{in } \Omega, \end{cases} \tag{1}$$

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where $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, and $f \in L^2(\Omega_T)$ with $\Omega_T := \Omega \times (0, T)$. Here u_t is the derivative of u with respect to the time variable t, and D_{0+}^{γ} is a Riemann–Liouville fractional differential operator.

The above problem is a particular case of time fractional diffusion-wave problems, which have attracted a considerable amount of research in the field of numerical analysis in the past twenty years. By now, most of the existing numerical algorithms employ the $L1$ scheme ([5, 11,17,27,28]), Grünwald-Letnikov discretization ([2,12,19,20,23,24]) or fractional linear multi-step method ([8,21,26]) to discrete the fractional derivatives. Generally, for those algorithms, the best temporal accuracy are $O(\tau^2)$ for the fractional diffusion problems and $O(\tau^{3-\gamma})$ for the fractional wave problems, where τ is the time step size.

Due to the nonlocal property of fractional differential operator, the memory and computing cost of an accuracy approximation to a fractional diffusion-wave problem is significantly more expensive than that to a corresponding normal diffusion-wave problem. To reduce the cost, high-accuracy algorithms are often preferred, especially those of high accuracy in the time direction. This motivates us to develop high-accuracy numerical algorithms for problem (1). The efforts in this aspect are summarized as follows. Li and Xu [10] proposed a space-time spectral algorithm for the fractional diffusion equation, and then Zheng et al. [29] constructed a high order space-time spectral method for the fractional Fokker–Planck equation. Gao et al. [7] proposed a new scheme to approximate Caputo fractional derivatives of order γ $(0 < \gamma < 1)$. Zayernouri and Karniadakis [25] developed an exponentially accurate fractional spectral collocation method. Yang et al. [22] developed a spectral Jacobi collocation method for the time fractional diffusion-wave equation. Recently, Ren et al. [14] investigated the superconvergence of finite element approximation to time fractional wave problems; however, the temporal accuracy order is only $O(\tau^{3-\gamma})$.

In this paper, using a spectral method in the temporal discretization and a finite element method in the spatial discretization, we design a high-accuracy algorithm for problem (1) and establish its stability and convergence. Our numerical experiments show the exponential decay in the temporal discretization errors, provided the underlying solution is sufficiently smooth.

The rest of this paper is organized as follows. Section 2 introduces some Sobolev spaces and the Riemann–Liouville fractional calculus operators. Section 3 describes a time-spectral algorithm and constructs the basis functions for the temporal discretization. Sections 4 and 5 establish the stability and convergence of the proposed algorithm, and Sect. 6 performs some numerical experiments to demonstrate its high accuracy. Finally, Sect. 7 provides some concluding remarks.

2 Notation

Let us first introduce some Sobolev spaces. For $0 < \alpha < \infty$, as usual, $H_0^\alpha(0, T)$, $H^\alpha(0, T)$, $H_0^\alpha(\Omega)$ and $H^\alpha(\Omega)$ are used to denote four standard Sobolev spaces; see [18]. Let X be a separable Hilbert space with an inner product $(\cdot, \cdot)_X$ and an orthonormal basis $\{e_k : k \in \mathbb{N}\}$. For $0 < \alpha < \infty$, define

$$H^\alpha(0, T; X) := \left\{ v \in L^2(0, T; X) : \sum_{k=0}^{\infty} \|(v, e_k)_X\|_{H^\alpha(0,T)}^2 < \infty \right\}$$

and endow this space with the norm

$$\|\cdot\|_{H^\alpha(0,T;X)} := \left(\sum_{k=0}^{\infty} \|(\cdot, e_k)_X\|_{H^\alpha(0,T)}^2\right)^{1/2},$$

where $L^2(0,T;X)$ is an X-valued Bochner L^2 space. For $v \in H^j(0,T;X)$ with $j \in \mathbb{N}_{\geqslant 1}$, the symbol $v^{(j)}$ denotes its jth weak derivative:

$$v^{(j)}(t) := \sum_{k=0}^{\infty} c_k^{(j)}(t) e_k, \quad 0 < t < T,$$

where $c_k(\cdot) := (v(\cdot), e_k)_X$ and $c_k^{(j)}$ is its jth weak derivative. Conventionally, $v^{(1)}$ and $v^{(2)}$ are also abbreviated to v' and v'', respectively.

Moreover, for $j \in \mathbb{N}$ we define

$$B^j(0,T;X) := \left\{ v \in L^2(0,T;X) : \sum_{k=0}^{\infty} \|(v, e_k)_X\|_{B^j(0,T)}^2 < \infty \right\}$$

and equip this space with the norm

$$\|\cdot\|_{B^j(0,T;X)} := \left(\sum_{k=0}^{\infty} \|(\cdot, e_k)_X\|_{B^j(0,T)}^2\right)^{1/2},$$

where the space $B^j(0,T)$ and its norm are respectively given by

$$B^j(0,T) := \left\{ v \in L^2(0,T) : \int_0^T t^i (T-t)^i \left| v^{(i)}(t) \right|^2 \mathrm{d}t < \infty, \ 0 \leqslant i \leqslant j \right\}$$

and

$$\|\cdot\|_{B^j(0,T)} := \left(\sum_{i=0}^{j} \int_0^T t^i (T-t)^i \left| (\cdot)^{(i)}(t) \right|^2 \mathrm{d}t \right)^{1/2}.$$

Then we introduce the Riemann–Liouville fractional operators. Let X be a Banach space and let $L^1(0,T;X)$ be an X-valued Bochner L^1 space.

Definition 2.1 For $0 < \alpha < \infty$, define $I_{0+}^{\alpha,X}, I_{T-}^{\alpha,X} : L^1(0,T;X) \to L^1(0,T;X)$, respectively, by

$$\left(I_{0+}^{\alpha,X} v\right)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) \, \mathrm{d}s, \quad 0 < t < T,$$
$$\left(I_{T-}^{\alpha,X} v\right)(t) := \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} v(s) \, \mathrm{d}s, \quad 0 < t < T,$$

for all $v \in L^1(0,T;X)$.

Definition 2.2 For $j-1 < \alpha < j$ with $j \in \mathbb{N}_{>0}$, define

$$D_{0+}^{\alpha,X} := D^j I_{0+}^{j-\alpha,X},$$
$$D_{T-}^{\alpha,X} := (-1)^j D^j I_{T-}^{j-\alpha,X},$$

where D is the first-order differential operator in the distribution sense.

Above $\Gamma(\cdot)$ is the Gamma function, and, for convenience, we shall simply use I_{0+}^{α}, I_{T-}^{α}, D_{0+}^{α} and D_{T-}^{α}, without indicating the underlying Banach space X. Each $v \in L^1(\Omega_T)$ also regarded as an element of $L^1(0, T; X)$ with $X = L^1(\Omega)$, and thus $D_{0+}^{\alpha} v$ and $D_{T-}^{\alpha} v$ mean $D_{0+}^{\alpha,X} v$ and $D_{T-}^{\alpha,X} v$, respectively, for all $0 < \alpha < \infty$.

3 Algorithm Definition

Let $\mathcal{K}_h$ be a triangulation of Ω consisting of d-simplexes, and let h be the maximum diameter of these simplexes in $\mathcal{K}_h$. Define

$$V_h := \left\{ v_h \in H^1(\Omega) : \ v_h|_K \in P_m(K) \ \text{ for all } K \in \mathcal{K}_h \right\},$$
$$\mathring{V}_h := \left\{ v_h \in H_0^1(\Omega) : \ v_h|_K \in P_m(K) \ \text{ for all } K \in \mathcal{K}_h \right\},$$

where m is a positive integer and $P_m(K)$ is the set of all polynomials defined on K of degree $\leqslant m$. For $j \in \mathbb{N}$, define

$$P_j[0, T] \otimes \mathring{V}_h := \operatorname{span}\left\{ q v_h : \ v_h \in \mathring{V}_h, \ q \in P_j[0, T] \right\},$$

where $P_j[0, T]$ is the set of all polynomials defined on $[0, T]$ of degree $\leqslant j$. Moreover, we introduce a projection operator $R_h : H_0^1(\Omega) \to \mathring{V}_h$ by

$$\left(\nabla (I - R_h) v, \nabla v_h \right)_{L^2(\Omega)} = 0, \quad \forall v \in H_0^1(\Omega), \ \forall v_h \in \mathring{V}_h.$$

Here and in the rest of this paper, I denotes the identity operator.

Now, let us describe a time-spectral algorithm for problem (1) as follows: seek $U \in P_M[0, T] \otimes \mathring{V}_h$ with $U(0) = R_h u_0$ such that

$$\left(D_{0+}^{\gamma_0} (U' - u_{h,1}), D_{T-}^{\gamma_0} V \right)_{L^2(\Omega_T)} + (\nabla U, \nabla V)_{L^2(\Omega_T)} = (f, V)_{L^2(\Omega_T)} \tag{2}$$

for all $V \in P_{M-1}[0, T] \otimes \mathring{V}_h$, where $M \geqslant 2$ is an integer, $\gamma_0 := (\gamma - 1)/2$, and $u_{h,1}$ is the $L^2(\Omega)$-projection of u_1 onto V_h.

Remark 3.1 In "Appendix A" we define the weak solution of problem (1). The numerical solution obtained by (2) is actually an approximation of the weak solution to problem (1).

Remark 3.2 It is well known that the solution to problem (1) generally has singularity in time, caused by the fractional derivative. However, in view of the basic properties of the operator D_{0+}^{γ}, it is anticipated that we can improve the performance of the above algorithm by enlarging $P_M[0, T]$ and $P_{M-1}[0, T]$ by some singular functions, such as t^{γ} for $P_M[0, T]$ and correspondingly $t^{\gamma-1}$ for $P_{M-1}[0, T]$.

The remainder of this section is devoted to the construction of the bases of $P_M[0, T]$ and $P_{M-1}[0, T]$, which is crucial in the implementation of the proposed algorithm. To this purpose, let us first introduce the well-known Jacobi polynomials; see [1,16] for more details. Given $-1 < \alpha, \beta < \infty$, the Jacobi polynomials $\{J_n^{(\alpha,\beta)} : \ n \in \mathbb{N}\}$ are defined by

$$J_n^{(\alpha,\beta)}(t) = w^{-\alpha,-\beta}(t) \frac{(-1)^n}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} w^{n+\alpha,n+\beta}(t), \quad -1 < t < 1, \ n \in \mathbb{N},$$

where

$$w^{r,s}(t) := (1-t)^r (1+t)^s,$$

for all $-1 < r, s < +\infty$. They form a complete orthogonal basis of $L^2_{w^{\alpha,\beta}}(-1,1)$, the weighted L^2 space with weight function $w^{\alpha,\beta}$.

Then we construct a basis $\{p_i\}_{i=0}^M$ of $P_M[0,T]$ and a basis $\{q_j\}_{j=0}^{M-1}$ of $P_{M-1}[0,T]$, respectively, by

$$\begin{cases} p_0(t) := 1, \\ p_i(t) := \dfrac{2t}{T} J_{i-1}^{(-\gamma_0,0)}(2t/T-1), \quad 1 \leqslant i \leqslant M, \end{cases}$$

and

$$q_j(t) = J_j^{(0,-\gamma_0)}(2t/T-1), \quad 0 \leqslant j \leqslant M-1.$$

By [3, Lemma 2.5] a straightforward computing yields

$$D_{0+}^{\gamma_0} p_i'(t) D_{T-}^{\gamma_0} q_j(t) = t^{-\gamma_0}(T-t)^{-\gamma_0}\zeta_{ij}(t) + t^{1-\gamma_0}(T-t)^{-\gamma_0}\varsigma_{ij}(t),$$

for all $0 \leqslant i \leqslant M$ and $0 \leqslant j < M$. Above $\zeta_{ij}(t)$ and $\varsigma_{ij}(t)$ are given respectively by

$$\zeta_{ij}(t) = C_{ij}\left(J_{i-1}^{(0,-\gamma_0)} J_j^{(-\gamma_0,0)}\right)(2t/T-1),$$
$$\varsigma_{ij}(t) = D_{ij}\left(J_{i-2}^{(1,1-\gamma_0)} J_j^{(-\gamma_0,0)}\right)(2t/T-1),$$

where

$$C_{ij} := \begin{cases} 0, & i = 0, \\ \frac{2}{T}\frac{\Gamma(i)\Gamma(j+1)}{\Gamma(j+1-\gamma_0)\Gamma(i-\gamma_0)}, & i \geqslant 1, \end{cases} \quad D_{ij} := \begin{cases} 0, & 0 \leqslant i \leqslant 1, \\ \frac{\Gamma(i+1-\gamma_0)}{\Gamma(i-\gamma_0)T} C_{ij}, & i \geqslant 2. \end{cases}$$

Then $\int_0^T D_{0+}^{\gamma_0} p_i' D_{T-}^{\gamma_0} q_j \, \mathrm{d}t$ is evaluated numerically by a suitable Jacobi-Gauss quadrature rule.

4 Main Results

Let us first introduce the following conventions: u and U are the solutions to problem (1) and (2), respectively; unless otherwise specified, C is a generic positive constant that is independent of any function and is bounded as $M \to \infty$; $a \lesssim b$ means that there exists a positive constant c, depending only on γ, T, Ω, m or the shape regular parameter of $\mathcal{K}_h$, such that $a \leqslant cb$; the symbol $a \sim b$ means $a \lesssim b \lesssim a$. The above shape regular parameter of $\mathcal{K}_h$ means

$$\max\{h_K/\rho_K : K \in \mathcal{K}_h\},$$

where h_K is the diameter of K, and ρ_K is the diameter of the circle ($d = 2$) or ball ($d = 3$) inscribed in K.

Then we introduce an interpolation operator. Let X be a separable Hilbert space and let $P_M[0,T;X]$ be the set of all X-valued polynomials defined on $[0,T]$ of degree $\leqslant M$. Define the interpolation operator

$$Q_M^X : H^{1+\gamma_0}(0,T;X) \to P_M[0,T;X]$$

as follows: for each $v \in H^{1+\gamma_0}(0,T;X)$, the interpolant $Q_M^X v$ fulfills

$$\begin{cases} \left(Q_M^X v\right)(0) = v(0), \\ \int_0^T D_{0+}^{\gamma_0}\left(v - Q_M^X v\right)' D_{T-}^{\gamma_0} q \,\mathrm{d}t = 0, \quad \forall q \in P_{M-1}[0,T]. \end{cases}$$

For convenience, we shall use Q_M instead of Q_M^X when no confusion will arise.

Remark 4.1 Let $\{e_k : k \in \mathbb{N}\}$ be an orthonormal basis of X. For any $v \in H^{\gamma_0}(0,T;X)$, the definition of $H^{\gamma_0}(0,T;X)$ implies that

$$(v, e_k)_X \in H^{\gamma_0}(0,T) \quad \text{for each } k \in \mathbb{N},$$

and hence, as Lemma 5.4 (in the next section) indicates

$$\left\| D_{0+}^{\gamma_0,\mathbb{R}} (v, e_k)_X \right\|_{L^2(0,T)} \sim \|(v, e_k)_X\|_{H^{\gamma_0}(0,T)},$$

it is evident that

$$\left\| D_{0+}^{\gamma_0,X} v \right\|_{L^2(0,T;X)} = \left(\sum_{k=0}^{\infty} \left\| D_{0+}^{\gamma_0,\mathbb{R}} (v, e_k)_X \right\|_{L^2(0,T)}^2 \right)^{\frac{1}{2}} \sim \|v\|_{H^{\gamma_0}(0,T;X)}.$$

Remark 4.2 Since $Q_M^{\mathbb{R}}$ is well-defined by Lemma 5.4, Q_M^X is evidently also well-defined and

$$Q_M^X v = \sum_{k=0}^{\infty} Q_M^{\mathbb{R}} (v, e_k)_X e_k, \quad \forall v \in H^{1+\gamma_0}(0,T;X).$$

Furthermore, we can redefine Q_M^X equivalently as follows: for each $v \in H^{1+\gamma_0}(0,T;X)$, the interpolant $Q_M^X v$ fulfills

$$\begin{cases} \left(Q_M^X v\right)(0) = v(0), \\ \int_0^T \left(D_{0+}^{\gamma_0}\left(v - Q_M^X v\right)', D_{T-}^{\gamma_0} q \right)_X \mathrm{d}t = 0, \quad \forall q \in P_{M-1}[0,T;X]. \end{cases}$$

Finally, we are ready to state the main results of this paper as follows.

Theorem 4.1 *Problem* (2) *has a unique solution U. Moreover,*

$$\begin{aligned} &\|U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} + \|U(T)\|_{H_0^1(\Omega)} \\ &\lesssim \|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega_T)}. \end{aligned} \tag{3}$$

Theorem 4.2 *If $u \in H^2\left(0,T;H_0^1(\Omega) \cap H^2(\Omega)\right)$, then*

$$\|u - U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} \lesssim \eta_1 + \eta_2 + \eta_3 + \eta_4, \tag{4}$$

$$\|(u - U)(T)\|_{H_0^1(\Omega)} \lesssim \eta_1 + \eta_2 + \eta_3 + \eta_5, \tag{5}$$

where

$$\eta_1 := \left\| u_1 - u_{h,1} \right\|_{L^2(\Omega)},$$
$$\eta_2 := C M^{-1-2\gamma_0} \|(I - Q_M)\Delta u\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))},$$

$$\eta_3 := \|(I - R_h)u\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))},$$
$$\eta_4 := \|(I - Q_M R_h)u\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))},$$
$$\eta_5 := \|(u - Q_M R_h u)(T)\|_{H_0^1(\Omega)}.$$

Corollary 4.1 *If*

$$u \in H^2(0,T;H_0^1(\Omega) \cap H^2(\Omega)) \cap H^{1+\gamma_0}(0,T;H^{m+1}(\Omega)),$$
$$u'' \in B^r(0,T;H_0^1(\Omega) \cap H^2(\Omega)),$$

then

$$\|u - U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} \lesssim \xi_1 + \xi_2 + \xi_3 + \xi_4, \tag{6}$$
$$\|(u - U)(T)\|_{H_0^1(\Omega)} \lesssim \xi_1 + \xi_2 + \xi_3 + \xi_5, \tag{7}$$

where $r \in \mathbb{N}$ *and*

$$\xi_1 := h^{m+1}\|u_1\|_{H^{m+1}(\Omega)},$$
$$\xi_2 := CM^{-\gamma_0-2-r}\|u''\|_{B^r(0,T;H^2(\Omega))},$$
$$\xi_3 := h^{m+1}\|u\|_{H^{1+\gamma_0}(0,T;H^{m+1}(\Omega))},$$
$$\xi_4 := CM^{\gamma_0-1-r}\|u''\|_{B^r(0,T;L^2(\Omega))} + h^{m+1}\|u\|_{H^{1+\gamma_0}(0,T;H^{m+1}(\Omega))},$$
$$\xi_5 := CM^{-1.5-r}\|u''\|_{B^r(0,T;H_0^1(\Omega))} + h^m\|u(T)\|_{H^{m+1}(\Omega)}.$$

5 Proofs

5.1 Preliminaries

Lemma 5.1 *If* $v \in H_0^1(\Omega) \cap H^{m+1}(\Omega)$*, then*

$$\|(I - R_h)v\|_{L^2(\Omega)} + h\|(I - R_h)v\|_{H_0^1(\Omega)} \lesssim h^{m+1}\|v\|_{H^{m+1}(\Omega)}.$$

Lemma 5.2 *If* $v \in H^\alpha(0,T)$ *with* $\alpha > \gamma_0$*, then*

$$\inf_{q \in P_{M-1}[0,T]} \|v - q\|_{H^{\gamma_0}(0,T)} \leqslant CM^{\gamma_0-\alpha}\|v\|_{H^\alpha(0,T)}.$$

If $v \in H^2(0,T)$ *such that* $v'' \in B^j(0,T)$ *with* $j \in \mathbb{N}$*, then*

$$\inf_{q \in P_{M-1}[0,T]} \|v - q\|_{H^{1+\gamma_0}(0,T)} \leqslant CM^{\gamma_0-1-j}\|v''\|_{B^j(0,T)}.$$

Lemma 5.3 *The following properties hold:*

- *If* $0 < \alpha, \beta < \infty$*, then*

$$I_{0+}^\alpha I_{0+}^\beta = I_{0+}^{\alpha+\beta}, \quad I_{T-}^\alpha I_{T-}^\beta = I_{T-}^{\alpha+\beta}.$$

- *If* $0 < \alpha < \beta < \infty$*, then*

$$D_{0+}^\beta I_{0+}^\alpha = D_{0+}^{\beta-\alpha}, \quad D_{T-}^\beta I_{T-}^\alpha = D_{T-}^{\beta-\alpha}.$$

- *If* $0 < \alpha < \infty$, *then*

$$\left\| I_{0+}^{\alpha} v \right\|_{L^2(0,T)} \leqslant C \left\| v \right\|_{L^2(0,T)}, \quad \left\| I_{T-}^{\alpha} v \right\|_{L^2(0,T)} \leqslant C \left\| v \right\|_{L^2(0,T)},$$

where C *is a positive constant that only depends on* α *and* T.
- *If* $0 < \alpha < \infty$ *and* $u, v \in L^2(0,T)$, *then*

$$(I_{0+}^{\alpha} u, v)_{L^2(0,T)} = (u, I_{T-}^{\alpha} v)_{L^2(0,T)}.$$

Lemma 5.4 *We have the following properties.*

- *If* $v \in H^{\alpha}(0,T)$ *with* $0 < \alpha < 1/2$, *then*

$$\|v\|_{H^{\alpha}(0,T)} \sim \left\| D_{0+}^{\alpha} v \right\|_{L^2(0,T)} \sim \left\| D_{T-}^{\alpha} v \right\|_{L^2(0,T)} \sim \sqrt{\left(D_{0+}^{\alpha} v, D_{T-}^{\alpha} v\right)_{L^2(0,T)}}.$$

- *If* $v, w \in H^{\alpha}(0,T)$ *with* $0 < \alpha < 1/2$, *then*

$$\left(D_{0+}^{\alpha} v, D_{T-}^{\alpha} w\right)_{L^2(0,T)} \lesssim \|v\|_{H^{\alpha}(0,T)} \|w\|_{H^{\alpha}(0,T)}.$$

Above, the implicit constants are only depend on α *and* T.

Lemma 5.5 *If* $v \in H^2(0,T)$ *and* $w \in H^1(0,T)$, *then*

$$\left(D_{0+}^{\gamma}(v - v(0) - tv'(0)), w\right)_{L^2(0,T)} = \left(D_{0+}^{\gamma_0}(v' - v'(0)), D_{T-}^{\gamma_0} w\right)_{L^2(0,T)}.$$

Lemma 5.6 *Let* X *and* Y *be two separable Hilbert spaces, and let* $A : X \to Y$ *be a bounded linear operator. If* $v \in H^{1+\gamma_0}(0,T;X)$, *then*

$$A Q_M^X v = Q_M^Y A v.$$

Lemma 5.1 is standard (see [4]), and Lemma 5.3 follows from [16, Theorems 3.35–3.37] and the basic properties of the interpolation spaces. The proof of Lemma 5.3 is included in [13,15], and for convenience this lemma will be used implicitly in the forthcoming analysis. Lemma 5.4 is a direct consequence of [6, Lemma 2.4, Theorem 2.13 and Corollary 2.15], and Lemma 5.5 follows from [10, Lemma 2.6]. Finally, by Lemma 5.4 and the standard properties of the interpolation spaces and the Bochner integrals, a rigorous proof of Lemma 5.6 is tedious but straightforward, and so it is omitted here.

Lemma 5.7 *If* $v \in L^2(0,T)$, *then*

$$\left\| I_{T-}^{2\gamma_0} v \right\|_{H^{2\gamma_0}(0,T)} \lesssim \|v\|_{L^2(0,T)}. \tag{8}$$

Moreover, if $v \in H^{\gamma_0}(0,T)$, *then*

$$\left\| I_{T-}^{2\gamma_0} v \right\|_{H^{3\gamma_0}(0,T)} \lesssim \|v\|_{H^{\gamma_0}(0,T)}. \tag{9}$$

Lemma 5.8 *If* $v \in H^2(0,T)$ *and* $w \in H^{\gamma_0}(0,T)$, *then*

$$\left((I - Q_M)v, w\right)_{L^2(0,T)} \lesssim CM^{-1-2\gamma_0} \|(I - Q_M)v\|_{H^{1+\gamma_0}(0,T)} \|w\|_{H^{\gamma_0}(0,T)}. \tag{10}$$

Lemma 5.9 *If* $v \in H^2(0,T)$ *and* $v'' \in B^j(0,T)$ *with* $j \in \mathbb{N}$, *then*

$$\|(I - Q_M)v\|_{H^{1+\gamma_0}(0,T)} \lesssim CM^{\gamma_0 - 1 - j} \left\| v'' \right\|_{B^j(0,T)}, \tag{11}$$

$$\|(I - Q_M)v\|_{L^2(0,T)} \lesssim CM^{-2-j} \left\| v'' \right\|_{B^j(0,T)}, \tag{12}$$

$$\|(I - Q_M)v\|_{C[0,T]} \lesssim CM^{-1.5-j} \left\| v'' \right\|_{B^j(0,T)}. \tag{13}$$

Proof of Lemma 5.7 Define

$$w(t) := \frac{1}{\Gamma(\gamma_0)} \int_t^\infty (s-t)^{\gamma_0 - 1} v(s)\,\mathrm{d}s, \quad -\infty < t < \infty,$$

where v is extended to $\mathbb{R}\backslash(0,T)$ by zero. Since $0 < \gamma_0 < 0.5$, a routine calculation yields $w \in L^2(\mathbb{R})$, and then [15, Theorem 7.1] implies

$$\mathcal{F}w(\xi) = (-\mathrm{i}\xi)^{-\gamma_0}\mathcal{F}v(\xi), \quad -\infty < \xi < \infty,$$

where $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Fourier transform operator, and i is the imaginary unit. Therefore, the well-known Plancherel Theorem yields

$$\|w\|_{H^{\gamma_0}(\mathbb{R})} \lesssim \|v\|_{L^2(0,T)},$$

and hence

$$\left\| I_{T-}^{\gamma_0} v \right\|_{H^{\gamma_0}(0,T)} \lesssim \|v\|_{L^2(0,T)}.$$

Furthermore, if $v \in H_0^1(0,T)$ then

$$\left\| I_{T-}^{\gamma_0} v \right\|_{H^{1+\gamma_0}(0,T)} \lesssim \|v\|_{H_0^1(0,T)},$$

by the evident equality $(I_{T-}^{\gamma_0} v)' = I_{T-}^{\gamma_0} v'$. Consequently, since $H_0^{\gamma_0}(0,T)$ coincides with $H^{\gamma_0}(0,T)$ with equivalent norms, applying [18, Lemma 22.3] gives

$$\left\| I_{T-}^{2\gamma_0} v \right\|_{H^{2\gamma_0}(0,T)} = \left\| I_{T-}^{\gamma_0} I_{T-}^{\gamma_0} v \right\|_{H^{2\gamma_0}(0,T)} \lesssim \left\| I_{T-}^{\gamma_0} v \right\|_{H_0^{\gamma_0}(0,T)} \lesssim \|v\|_{L^2(0,T)},$$

namely estimate (8). Analogously, we can obtain (9) and hence conclude the proof of the lemma. □

Proof of Lemma 5.8 Let $g := (I - Q_M)v$. Since a straightforward calculation yields

$$\left(I_{0+}^{1-\gamma_0} g' \right)(t) = \frac{g'(0)}{\Gamma(2-\gamma_0)} t^{1-\gamma_0} + \left(I_{0+}^{2-\gamma_0} g'' \right)(t), \quad 0 < t < T,$$

the fact $\gamma_0 < 0.5$ indicates that $I_{0+}^{1-\gamma_0} g' \in H^1(0,T)$ and $(I_{0+}^{1-\gamma_0} g')(0) = 0$. Then using integration by parts gives

$$\begin{aligned}
\left(D_{0+}^{\gamma_0} g', I_{T-}^{1+\gamma_0} w \right)_{L^2(0,T)} &= \left(\left(I_{0+}^{1-\gamma_0} g' \right)', I_{T-}^{1+\gamma_0} w \right)_{L^2(0,T)} \\
&= -\left(I_{0+}^{1-\gamma_0} g', \left(I_{T-}^{1+\gamma_0} w \right)' \right)_{L^2(0,T)} = \left(I_{0+}^{1-\gamma_0} g', I_{T-}^{\gamma_0} w \right)_{L^2(0,T)} \\
&= \left(g', I_{T-} w \right)_{L^2(0,T)}.
\end{aligned}$$

Hence, as the definition of Q_M implies $g(0) = 0$, we obtain

$$\left(D_{0+}^{\gamma_0} g', I_{T-}^{1+\gamma_0} w \right)_{L^2(0,T)} = \left(g', I_{T-} w \right)_{L^2(0,T)} = (g, w)_{L^2(0,T)},$$

which, combined with the evident equality

$$I_{T-}^{1+\gamma_0} w = D_{T-}^{\gamma_0} I_{T-}^{1+2\gamma_0} w,$$

yields

$$(g, w)_{L^2(0,T)} = \left(D_{0+}^{\gamma_0} g', D_{T-}^{\gamma_0} I_{T-}^{1+2\gamma_0} w \right)_{L^2(0,T)}.$$

Therefore, Lemma 5.4, the definition of Q_M and the Cauchy–Schwarz inequality imply

$$\left(g, w\right)_{L^2(0,T)} \lesssim \|g\|_{H^{1+\gamma_0}(0,T)} \inf_{q \in P_{M-1}[0,T]} \left\| I_{T-}^{1+2\gamma_0} w - q \right\|_{H^{\gamma_0}(0,T)}.$$

Clearly, to prove (10), by Lemma 5.2 it suffices to show

$$\left\| I_{T-}^{1+2\gamma_0} w \right\|_{H^{1+3\gamma_0}(0,T)} \lesssim \|w\|_{H^{\gamma_0}(0,T)}.$$

Therefore, since

$$\left\| I_{T-}^{1+2\gamma_0} w \right\|_{H^{1+3\gamma_0}(0,T)} \lesssim \left\| I_{T-}^{2\gamma_0} w \right\|_{H^{3\gamma_0}(0,T)},$$

using Lemma 5.7 completes the proof of Lemma 5.8. □

Proof of Lemma 5.9 Let us first consider (11). For each $p \in P_{M-1}[0, T]$, by Lemma 5.4, the definition of Q_M and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\left\|(Q_M v)' - p\right\|_{H^{\gamma_0}(0,T)}^2 \\
&\sim \left(D_{0+}^{\gamma_0}\left((Q_M v)' - p\right), D_{T-}^{\gamma_0}\left((Q_M v)' - p\right)\right)_{L^2(0,T)} \\
&= \left(D_{0+}^{\gamma_0}(v' - p), D_{T-}^{\gamma_0}\left((Q_M v)' - p\right)\right)_{L^2(0,T)} \\
&\lesssim \left\|v' - p\right\|_{H^{\gamma_0}(0,T)} \left\|(Q_M v)' - p\right\|_{H^{\gamma_0}(0,T)}.
\end{aligned}$$

It follows that

$$\left\|(Q_M v)' - p\right\|_{H^{\gamma_0}(0,T)} \lesssim \left\|v' - p\right\|_{H^{\gamma_0}(0,T)},$$

and so

$$\left\|(v - Q_M v)'\right\|_{H^{\gamma_0}(0,T)} \lesssim \left\|v' - p\right\|_{H^{\gamma_0}(0,T)}.$$

Therefore, as the fact $(v - Q_M v)(0) = 0$ implies

$$\|(I - Q_M)v\|_{H^{1+\gamma_0}(0,T)} \sim \left\|(v - Q_M v)'\right\|_{H^{\gamma_0}(0,T)},$$

using Lemma 5.2 proves (11).

Next let us consider (12, 13). Proceeding as in the proof of Lemma 5.8 gives

$$\begin{aligned}
&\|(I - Q_M)v\|_{L^2(0,T)}^2 \\
&\lesssim \|(I - Q_M)v\|_{H^{1+\gamma_0}(0,T)} \inf_{q \in P_{M-1}[0,T]} \left\| I_{T-}^{1+2\gamma_0}(I - Q_M)v - q \right\|_{H^{\gamma_0}(0,T)} \\
&\lesssim C M^{-1-\gamma_0} \|(I - Q_M)v\|_{H^{1+\gamma_0}(0,T)} \|(I - Q_M)v\|_{L^2(0,T)},
\end{aligned}$$

which proves (12) by (11). Then, combining (11,12) and applying [18, Lemma 22.3] yield

$$\|(I - Q_M)v\|_{H^1(0,T)} \lesssim C M^{-1-j} \left\|v''\right\|_{B^j(0,T)},$$

so that (13) follows from (12) and the Gagliardo–Nirenberg interpolation inequality, namely,

$$\|w\|_{C[0,T]} \lesssim \|w\|_{L^2(0,T)}^{\frac{1}{2}} \|w\|_{H^1(0,T)}^{\frac{1}{2}}, \quad \forall w \in H^1(0, T).$$

This concludes the proof of Lemma 5.9. □

Remark 5.1 Assume that $P_M[0,T]$ and $P_{M-1}[0,T]$ are respectively replaced by

$$P_M[0,T] + \left\{cw^{1+2\gamma_0} : c \in \mathbb{R}\right\} \quad \text{and} \quad P_{M-1}[0,T] + \left\{cw^{2\gamma_0} : c \in \mathbb{R}\right\},$$

where $w(t) := T - t,\ 0 < t < T$. For each $v \in H^{1+\gamma_0}(0,T)$, the definition of Q_M implies

$$\int_0^T D_{0+}^{\gamma_0}(v - Q_M v)' D_{T-}^{\gamma_0} w^{2\gamma_0} \,\mathrm{d}t = 0,$$

and then, as in the previous remark, a straightforward computing yields

$$(v - Q_M v)(T) = 0.$$

Correspondingly, we can improve Corollary 4.1 by

$$\xi_5 := h^m \|u(T)\|_{H^{m+1}(\Omega)}.$$

5.2 Proofs of Theorems 4.1 and 4.2 and Corollary 4.1

Proof of Theorem 4.1 Since (3) contains the unique existence of U, it suffices to prove the former. Observe first that integration by parts yields

$$2(\nabla U, \nabla U')_{L^2(\Omega_T)} = \|U(T)\|_{H_0^1(\Omega)}^2 - \|U(0)\|_{H_0^1(\Omega)}^2$$

and that Lemma 5.4 implies

$$\begin{aligned} \left\|D_{0+}^{\gamma_0} u_{h,1}\right\|_{L^2(\Omega_T)} &\sim \left\|u_{h,1}\right\|_{H^{\gamma_0}(0,T;L^2(\Omega))} \sim \left\|u_{h,1}\right\|_{L^2(\Omega)}, \\ \left(D_{0+}^{\gamma_0} U', D_{T-}^{\gamma_0} U'\right)_{L^2(\Omega_T)} &\sim \left\|U'\right\|_{H^{\gamma_0}(0,T;L^2(\Omega))}^2 \sim \left\|D_{T-}^{\gamma_0} U'\right\|_{L^2(\Omega_T)}^2. \end{aligned}$$

Moreover, the fact that $u_{h,1}$ is the $L^2(\Omega)$-projection of u_1 onto V_h gives

$$\left\|u_{h,1}\right\|_{L^2(\Omega)} \leqslant \|u_1\|_{L^2(\Omega)}.$$

Consequently, by the Cauchy–Schwarz inequality and the Young's inequality with ϵ, inserting $V := U'$ into (2) yields

$$\begin{aligned} &\left\|U'\right\|_{H^{\gamma_0}(0,T;L^2(\Omega))} + \|U(T)\|_{H_0^1(\Omega)} \\ &\quad \lesssim \|U(0)\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega_T)}, \end{aligned}$$

which, combined with the estimate

$$\|U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} \sim \|U(0)\|_{L^2(\Omega)} + \left\|U'\right\|_{H^{\gamma_0}(0,T;L^2(\Omega))},$$

indicates

$$\begin{aligned} &\|U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} + \|U(T)\|_{H_0^1(\Omega)} \\ &\quad \lesssim \|U(0)\|_{H_0^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega_T)}. \end{aligned}$$

As the definition of R_h and the fact $U(0) = R_h u_0$ imply

$$\|U(0)\|_{H_0^1(\Omega)} \leqslant \|u_0\|_{H_0^1(\Omega)},$$

this proves (3) and thus concludes the proof of Theorem 4.1. □

Proof of Theorem 4.2 Set $\rho := (I - Q_M R_h)u$ and $\theta := U - Q_M R_h u$. By Lemma 5.5 and integration by parts, using (1) gives

$$\left(D_{0+}^{\gamma_0}(u' - u_1), D_{T-}^{\gamma_0}\theta'\right)_{L^2(\Omega_T)} + (\nabla u, \theta')_{L^2(\Omega_T)} = (f, \theta')_{L^2(\Omega_T)},$$

which, together with (2), yields

$$\left(D_{0+}^{\gamma_0}\theta', D_{T-}^{\gamma_0}\theta'\right)_{L^2(\Omega_T)} + (\nabla\theta, \nabla\theta')_{L^2(\Omega_T)} = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3,$$

where

$$\begin{aligned}
\mathbb{I}_1 &:= (\nabla\rho, \nabla\theta')_{L^2(\Omega_T)}, \\
\mathbb{I}_2 &:= \left(D_{0+}^{\gamma_0}\rho', D_{T-}^{\gamma_0}\theta'\right)_{L^2(\Omega_T)}, \\
\mathbb{I}_3 &:= -\left(D_{0+}^{\gamma_0}(u_1 - u_{h,1}), D_{T-}^{\gamma_0}\theta'\right)_{L^2(\Omega_T)}.
\end{aligned}$$

Moreover, the fact $\theta(0) = 0$ gives

$$(\nabla\theta, \nabla\theta')_{L^2(\Omega_T)} = \frac{1}{2}\|\theta(T)\|_{H_0^1(\Omega)}^2$$

by integration by parts, and Lemma 5.4 implies

$$\left(D_{0+}^{\gamma_0}\theta', D_{T-}^{\gamma_0}\theta'\right)_{L^2(\Omega_T)} \sim \left\|\theta'\right\|_{H^{\gamma_0}(0,T;L^2(\Omega))}^2.$$

Therefore, it follows

$$\left\|\theta'\right\|_{H^{\gamma_0}(0,T;L^2(\Omega))}^2 + \|\theta(T)\|_{H_0^1(\Omega)}^2 \lesssim \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3. \tag{14}$$

Let us first estimate $\mathbb{I}_1$. Since $R_h : H_0^1(\Omega) \to \mathring{V}_h$ and $-\Delta : H^2(\Omega) \to L^2(\Omega)$ are two bounded linear operators, Lemma 5.6 implies

$$Q_M R_h u = R_h Q_M u \quad \text{and} \quad Q_M(-\Delta u) = -\Delta Q_M u,$$

so that, by integration by parts and the definition of R_h, a straightforward calculation gives

$$\begin{aligned}
\mathbb{I}_1 &= \int_0^T \left(\nabla(I - R_h Q_M)u, \nabla\theta'\right)_{L^2(\Omega)} \mathrm{d}t \\
&= \int_0^T \left(\nabla(I - Q_M)u, \nabla\theta'\right)_{L^2(\Omega)} \mathrm{d}t \\
&= \int_0^T \left(-\Delta(I - Q_M)u, \theta'\right)_{L^2(\Omega)} \\
&= \int_0^T \left((I - Q_M)(-\Delta u), \theta'\right)_{L^2(\Omega)} \mathrm{d}t,
\end{aligned}$$

Therefore, Lemma 5.8 leads to

$$\mathbb{I}_1 \lesssim CM^{-1-2\gamma_0}\|(I - Q_M)\Delta u\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))}\left\|\theta'\right\|_{H^{\gamma_0}(0,T;L^2(\Omega))}. \tag{15}$$

Next let us estimate $\mathbb{I}_2$ and $\mathbb{I}_3$. The definition of Q_M gives

$$\mathbb{I}_2 = \left(D_{0+}^{\gamma_0}(u - Q_M R_h u)', D_{T-}^{\gamma_0}\theta'\right)_{L^2(\Omega_T)} = \left(D_{0+}^{\gamma_0}(u - R_h u)', D_{T-}^{\gamma_0}\theta'\right)_{L^2(\Omega_T)},$$

so that the Cauchy–Schwarz inequality and Lemma 5.4 indicate

$$\mathbb{I}_2 \lesssim \|(I - R_h)u\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))}\left\|\theta'\right\|_{H^{\gamma_0}(0,T;L^2(\Omega))}. \tag{16}$$

By the evident estimate

$$\left\| u_1 - u_{h,1} \right\|_{H^{\gamma_0}(0,T;\Omega_T)} \sim \left\| u_1 - u_{h,1} \right\|_{L^2(\Omega)},$$

the Cauchy–Schwarz inequality and Lemma 5.4 also yield

$$\mathbb{I}_3 \lesssim \left\| u_1 - u_{h,1} \right\|_{L^2(\Omega)} \left\| \theta' \right\|_{H^{\gamma_0}(0,T;L^2(\Omega))}. \tag{17}$$

Finally, by the Young's inequality with ϵ, combining (14), (15), (16), (17) gives

$$\left\| \theta' \right\|_{H^{\gamma_0}(0,T;L^2(\Omega))} + \left\| \theta(T) \right\|_{H_0^1(\Omega)} \lesssim \eta_1 + \eta_2 + \eta_3.$$

Since $\theta(0) = 0$ implies

$$\left\| \theta \right\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} \sim \left\| \theta' \right\|_{H^{\gamma_0}0,T;L^2(\Omega))},$$

it follows

$$\left\| \theta \right\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} + \left\| \theta(T) \right\|_{H_0^1(\Omega)} \lesssim \eta_1 + \eta_2 + \eta_3.$$

As (4), (5) are evident from the above estimate, this concludes the proof of Theorem 4.2. □

Proof of Corollary 4.1 It suffices to prove $\eta_i \lesssim \xi_i$ for all $1 \leqslant i \leqslant 5$, where $\{\eta_i\}_{i=1}^5$ are defined in Theorem 4.2. Observing that $\eta_1 \lesssim \xi_1$ is a standard result [4], that $\eta_2 \lesssim \xi_2$ follows from Lemma 5.9, and that $\eta_3 \lesssim \xi_3$ follows from Lemma 5.1, we only need to prove $\eta_4 \lesssim \xi_4$ and $\eta_5 \lesssim \xi_5$.

Let us first consider $\eta_4 \lesssim \xi_4$. By Lemma 5.4, the definition of Q_M implies

$$\left\| Q_M(I - R_h)u \right\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} \lesssim \left\| (I - R_h)u \right\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))},$$

so that Lemma 5.1 and [18, Lemma 22.3] yield

$$\left\| Q_M(I - R_h)u \right\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} \lesssim h^{m+1} \left\| u \right\|_{H^{1+\gamma_0}(0,T;H^{m+1}(\Omega))}.$$

Moreover, Lemma 5.9 gives

$$\left\| (I - Q_M)u \right\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} \lesssim C M^{\gamma_0 - 1 - r} \left\| u'' \right\|_{B^r(0,T;L^2(\Omega))}.$$

Consequently, $\eta_4 \lesssim \xi_4$ is a direct consequence of the inequality

$$\begin{aligned} &\left\| (I - Q_M R_h)u \right\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} \\ &\quad \leqslant \left\| (I - Q_M)u \right\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} + \left\| Q_M(I - R_h)u \right\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))}. \end{aligned}$$

Then let us consider $\eta_5 \lesssim \xi_5$. Since Lemma 5.6 gives $R_h Q_M u = Q_M R_h u$, the definition of R_h yields

$$\left\| (R_h u - Q_M R_h u)(T) \right\|_{H_0^1(\Omega)} \leqslant \left\| (u - Q_M u)(T) \right\|_{H_0^1(\Omega)},$$

and hence Lemma 5.9 indicates

$$\left\| (R_h u - Q_M R_h u)(T) \right\|_{H_0^1(\Omega)} \lesssim C M^{-1.5 - r} \left\| u'' \right\|_{B^r(0,T;H_0^1(\Omega))}.$$

Therefore, as Lemma 5.1 implies

$$\left\| (I - R_h)u(T) \right\|_{H_0^1(\Omega)} \lesssim h^m \left\| u(T) \right\|_{H^{m+1}(\Omega)},$$

the estimate $\eta_5 \lesssim \xi_5$ follows from the inequality

$$\left\| (u - Q_M R_h u)(T) \right\|_{H_0^1(\Omega)}$$

Table 1 The errors for Example 1 with $M = 20$

m	$1/h$	$\|(u-U)(T)\|_{H_0^1(\Omega)}$		$\|u-U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))}$	
		Error	Order	Error	Order
1	2	1.19e−01	–	8.68e−02	–
	4	6.12e−02	0.95	1.94e−02	2.17
	8	3.06e−02	1.01	4.52e−03	2.10
	16	1.52e−02	1.01	1.10e−03	2.03
	32	7.61e−03	1.00	2.74e−04	2.01
2	2	3.12e−02	–	1.18e−02	–
	4	8.28e−03	1.91	1.63e−03	2.86
	8	2.11e−03	1.97	2.12e−04	2.95
	16	5.31e−04	1.99	2.67e−05	2.98
	32	1.33e−04	2.00	3.35e−06	3.00
3	2	4.92e−03	–	1.50e−03	–
	4	5.94e−04	3.05	9.13e−05	4.04
	8	7.28e−05	3.03	5.51e−06	4.05
	16	9.01e−06	3.02	3.36e−07	4.04
	32	1.12e−06	3.01	2.07e−08	4.02

$$\leqslant \|(I-R_h)u(T)\|_{H_0^1(\Omega)} + \|(R_h u - Q_M R_h u)(T)\|_{H_0^1(\Omega)}.$$

This concludes the proof of Corollary 4.1. □

6 Numerical Experiments

This section performs some numerical experiments to demonstrate the high order accuracy of the proposed algorithm in two dimensional case. Throughout this section we set $\gamma := 1.5$, $T := 1$ and $\Omega := (0, 1)^2$.

Example 1 In this example the solution to problem (1) is

$$u(x,t) := t^{20} x_1 x_2 (1-x_1)(1-x_2), \quad (x,t) \in \Omega_T,$$

where $x = (x_1, x_2)$. Let us first consider the spatial discretization errors of the proposed algorithm, and, to this end, we set $M := 20$ to ensure that the temporal discretization errors are negligible compared with the former. The corresponding numerical results, presented in Table 1, illustrate that the convergence orders of

$$\|(u-U)(T)\|_{H_0^1(\Omega)} \quad \text{and} \quad \|u-U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))}$$

are m and $m+1$ respectively, which agrees well with Corollary 4.1. Then let us consider the temporal discretization errors and hence set $m := 4$ and $h := 1/32$ to ensure that the temporal discretization error is dominant. We plot the log-linear relationship between the errors and the polynomial degree M in Fig. 1. As indicated by Corollary 4.1, these numerical results demonstrate that the errors reduce exponentially as M increases.

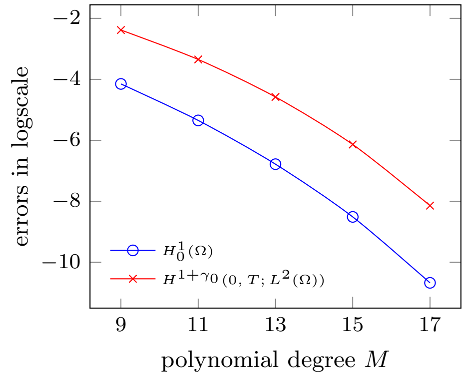

Fig. 1 The log-linear relationship between the errors and the polynomial degree M for Example 1 with $m = 4$ and $h = 1/32$

Table 2 The errors for Example 2 with $\beta = 2.5$

M	$\|(u-U)(T)\|_{H_0^1(\Omega)}$		$\|u-U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))}$	
	Error	Order	Error	Order
7	3.80e−5	–	3.00e−03	–
9	1.60e−5	3.44	1.94e−03	1.73
11	6.32e−6	4.63	1.35e−03	1.81
13	2.77e−6	4.93	9.94e−04	1.84
15	1.38e−6	4.86	7.64e−04	1.85
17	7.40e−7	4.99	6.06e−04	1.84

Example 2 This example adopts

$$u(x,t) := t^2\,|1-2t|^\beta\, x_1(1-x_1)\sin(\pi x_2), \quad (x,t) \in \Omega_T$$

as the solution to problem (1), where β is a positive constant. Here we only consider the temporal discretization errors and hence set $m := 6$ and $h := 2^{-4}$ to ensure that the temporal discretization errors are dominant. The corresponding numerical results are presented in Tables 2 and 3. Observing that

$$|1-2t|^\beta \in H^{\beta+0.5-\epsilon}(0,T) \quad \text{for all } \epsilon > 0,$$

by Corollary 4.1 and [18, Lemma 22.3] we have

$$\|(u-U)(T)\|_{H_0^1(\Omega)} \lesssim C(\epsilon)M^{-\beta+\epsilon},$$
$$\|u-U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))} \lesssim C(\epsilon)M^{0.75-\beta+\epsilon},$$

where $C(\epsilon)$ is a constant that depends on ϵ. Evidently, for the convergence order of $\|u-U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))}$, the numerical results are in agreement with Corollary 4.1. However, in this case, $\|(u-U)(T)\|_{H_0^1(\Omega)}$ reduces significantly faster than that predicted by Corollary 4.1.

Example 3 This example investigates the temporal accuracy of the algorithm in the case that the underlying solution has singularity at $t = 0$. The solution to problem (1) is

$$u(x,t) = t^\beta x_1 x_2(1-x_1)(1-x_2), \quad (x,t) \in \Omega_T,$$

Table 3 The errors for Example 2 with $\beta = 2.1$

M	$\|(u-U)(T)\|_{H_0^1(\Omega)}$		$\|u-U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))}$	
	Error	Order	Error	Order
7	1.24e−5	–	1.05e−03	–
9	5.48e−6	3.24	7.49e−03	1.36
11	2.32e−6	4.28	5.64e−04	1.41
13	1.08e−6	4.56	4.45e−04	1.42
15	5.72e−7	4.46	3.63e−04	1.43
17	3.22e−7	4.59	3.03e−04	1.42

Table 4 The errors for Example 3 with $\beta = 1.2$

M	$\|(u-U)(T)\|_{H_0^1(\Omega)}$		$\|u-U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))}$	
	Error	Order	Error	Order
10	1.82e−06	–	1.20e−03	–
11	1.27e−06	3.78	1.10e−03	0.90
12	8.80e−07	4.20	1.02e−03	0.91
13	6.13e−07	4.53	9.47e−04	0.91
14	4.48e−07	4.23	8.85e−04	0.91

Table 5 The errors for Example 3 with $\beta = 1.5$

M	$\|(u-U)(T)\|_{H_0^1(\Omega)}$		$\|u-U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))}$	
	Error	Order	Error	Order
10	8.96e−07	–	4.18e−04	–
11	5.90e−07	4.37	3.62e−04	1.51
12	3.89e−07	4.80	3.18e−04	1.51
13	2.58e−07	5.11	2.81e−04	1.51
14	1.81e−07	4.82	2.51e−04	1.51

Table 6 The errors for Example 3 with $\beta = 1.8$

M	$\|(u-U)(T)\|_{H_0^1(\Omega)}$		$\|u-U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))}$	
	Error	Order	Error	Order
10	1.74e−07	–	6.48e−05	–
11	1.08e−07	4.97	5.29e−05	2.12
12	6.77e−08	5.39	4.40e−05	2.12
13	4.29e−08	5.70	3.71e−05	2.12
14	2.88e−08	5.40	3.17e−05	2.12

where $\beta = 1.2$, 1.5 or 1.8. We set $m := 4$ and $h := 2^{-5}$, and display the corresponding numerical results in Tables 4, 5 and 6. These numerical results illustrate that both $\|(u-U)(T)\|_{H_0^1(\Omega)}$ and $\|u-U\|_{H^{1+\gamma_0}(0,T;L^2(\Omega))}$ converge significantly faster than that implied by Corollary 4.1.

7 Conclusions

In this paper, a high accuracy algorithm for time fractional wave problems is developed, which adopts a spectral method to approximate the fractional derivative and uses a finite element method in the spatial discretization. Stability and a priori error estimates of this algorithm are derived, and numerical experiments are also performed to verify its high accuracy.

In future work, we shall consider the following issues. Firstly, the optimal error estimates of $\|(u-U)(T)\|_{L^\infty(\Omega)}$ and $\|(u-U)(T)\|_{L^2(\Omega)}$ are not established. Secondly, it is worth applying the idea of approximating fractional differential operators of order γ $(1<\gamma<2)$ by spectral methods to other fractional differential equations, such as nonlinear fractional ordinary differential equations and nonlinear time fractional wave equations.

Appendix A: Weak Solution

We call

$$u \in H^{(\gamma+1)/2}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$$

a weak solution to problem (1) if $u(0)=u_0$ and

$$\left(D_{0+}^{(\gamma+1)/2}(u-u_0-tu_1),\, D_{T-}^{(\gamma-1)/2}v\right)_{L^2(\Omega_T)} + (\nabla u,\nabla v)_{L^2(\Omega_T)} = (f,v)_{L^2(\Omega_T)} \tag{18}$$

for all $v \in H^{(\gamma-1)/2}(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.

To prove that problem (1) admits a unique weak solution, we first consider the following problem: given $c_0, c_1 \in \mathbb{R}$ and $g \in L^2(0,T)$, seek $y \in H^\gamma(0,T)$ such that

$$D_{0+}^\gamma(y-c_0-c_1t)+\lambda y = g, \tag{19}$$

where λ is a positive constant such that $\lambda \geqslant 1$.

Lemma A.1 *Suppose that* $v \in H^{(\gamma+1)/2}(0,T)$ *and* $D_{0+}^\gamma v \in L^2(0,T)$*, then*

$$\|v\|_{H^\gamma(0,T)} \lesssim \left\|D_{0+}^\gamma v\right\|_{L^2(0,T)}. \tag{20}$$

Proof Since $D_{0+}^\gamma v \in L^2(0,T)$, by [9, Lemmas A.4] we conclude that $I_{0+}^\gamma D_{0+}^\gamma v \in H^\gamma(0,T)$ with

$$\left\|I_{0+}^\gamma D_{0+}^\gamma v\right\|_{H^\gamma(0,T)} \lesssim \left\|D_{0+}^\gamma v\right\|_{L^2(0,T)}. \tag{21}$$

A simple calculation yields

$$v = c_0 t^{\gamma-2} + c_1 t^{\gamma-1} + I_{0+}^\gamma D_{0+}^\gamma v,$$

which indicates that $c_0=c_1=0$ by the fact $v \in H^{(\gamma+1)/2}(0,T)$. Then (20) follows from (21). This completes the proof. □

Lemma A.2 *Suppose that* $v \in H^{(\gamma+1)/2}(0,T)$ *with* $v(0)=0$*, then we have the following properties.*

(a) It holds that

$$\left(D_{0+}^{\frac{\gamma+1}{2}} v,\, D_{T-}^{\frac{\gamma-1}{2}} v'\right)_{L^2(0,T)} \sim \|v\|^2_{H^{(\gamma+1)/2}(0,T)}. \tag{22}$$

(b) For any $w \in H^{(\gamma-1)/2}(0,T)$*, it holds that*

$$\left(D_{0+}^{\frac{\gamma+1}{2}} v, D_{T-}^{\frac{\gamma-1}{2}} w\right)_{L^2(0,T)} \lesssim \|v\|_{H^{(\gamma+1)/2}(0,T)} \|w\|_{H^{(\gamma-1)/2}(0,T)}. \tag{23}$$

(c) For any $\varphi \in C_0^\infty(0,T)$*, it holds that*

$$\left\langle D_{0+}^{\gamma} v, \varphi \right\rangle = \left(D_{0+}^{\frac{\gamma+1}{2}} v, D_{T-}^{\frac{\gamma-1}{2}} \varphi\right)_{L^2(0,T)}. \tag{24}$$

Proof Let us first prove (a). Since $v \in H^{(\gamma+1)/2}(0,T)$ and $v(0)=0$, we have

$$\left\|v'\right\|_{H^{(\gamma-1)/2}(0,T)} \sim \|v\|_{H^{(\gamma+1)/2}(0,T)}. \tag{25}$$

In addition, a straightforward calculation gives

$$D_{0+}^{\frac{\gamma+1}{2}} v = D^2 I_{0+}^{\frac{3-\gamma}{2}} I_{0+} v' = D^2 I_{0+}^{\frac{5-\gamma}{2}} v' = D_{0+}^{\frac{\gamma-1}{2}} v'. \tag{26}$$

So (22) follows from (25), (26) and Lemma 5.4.

Then let us prove (b). In view of (25), (26), using Lemma 5.4 yields (23).

Finally we prove (c). Observe that (26) implies $I_{0+}^{\frac{3-\gamma}{2}} v' \in H^1(0,T)$, and a simple computation implies

$$\left(I_{0+}^{\frac{3-\gamma}{2}} v'\right)(t) \leqslant \frac{t^{\frac{1-\gamma}{2}} \left\|v'\right\|_{L^2(0,t)}}{\Gamma(\frac{3-\gamma}{2})\sqrt{2-\gamma}}, \quad 0 < t \leqslant T.$$

Thus,

$$\left(I_{0+}^{\frac{3-\gamma}{2}} v'\right)(0) = 0.$$

Using integration by parts gives

$$\begin{aligned}
\left\langle D_{0+}^{\gamma} v, \varphi \right\rangle &= \left\langle D^2 I_{0+}^{2-\gamma} v, \varphi \right\rangle \\
&= (I_{0+}^{2-\gamma} v, \varphi'')_{L^2(0,T)} = (I_{0+}^{3-\gamma} v', \varphi'')_{L^2(0,T)} \\
&= (I_{0+}^{\frac{3-\gamma}{2}} v', I_{T-}^{\frac{3-\gamma}{2}} \varphi'')_{L^2(0,T)} = -(D I_{0+}^{\frac{3-\gamma}{2}} v', I_{T-}^{\frac{3-\gamma}{2}} \varphi')_{L^2(0,T)} \\
&= (D_{0+}^{\frac{\gamma-1}{2}} v', D_{T-}^{\frac{\gamma-1}{2}} \varphi)_{L^2(0,T)} = (D_{0+}^{\frac{\gamma+1}{2}} v, D_{T-}^{\frac{\gamma-1}{2}} \varphi)_{L^2(0,T)}
\end{aligned}$$

for all $\varphi \in C_0^\infty(0,T)$. This shows (24) and completes the proof of this lemma. □

Lemma A.3 *Problem* (19) *has a unique solution* $y \in H^\gamma(0,T)$*, and* y *satisfies that* $y(0) = c_0$ *and*

$$\left(D_{0+}^{\frac{\gamma+1}{2}} (y - c_0 - c_1 t),\ D_{T-}^{\frac{\gamma-1}{2}} z\right)_{L^2(0,T)} + \lambda (y,z)_{L^2(0,T)} = (g,z)_{L^2(0,T)} \tag{27}$$

for all $z \in H^{\frac{\gamma-1}{2}}(0,T)$*. Moreover,*

$$\|y\|_{H^{\frac{\gamma+1}{2}}(0,T)} + \lambda^{\frac{1}{2}} \|y\|_{L^2(0,T)} \lesssim \|g\|_{L^2(0,T)} + \lambda^{\frac{1}{2}} |c_0| + |c_1|. \tag{28}$$

Proof Set

$$b(z) := (g, z)_{L^2(0,T)} + \left(D_{0+}^{\frac{\gamma+1}{2}} (c_1 t), D_{T-}^{\frac{\gamma-1}{2}} z \right)_{L^2(0,T)} - \lambda (c_0, z)_{L^2(0,T)}$$

for all $z \in H^{\frac{\gamma-1}{2}}(0, T)$. Since Lemma 5.4 implies $b \in H^{\frac{1-\gamma}{2}}(0, T)$, Lemma A.2 and the well-known Lax-Milgram Theorem guarantee the unique existence of $w \in H^{\frac{\gamma+1}{2}}(0, T)$ with $w(0) = 0$ such that

$$\left(D_{0+}^{\frac{\gamma+1}{2}} w, D_{T-}^{\frac{\gamma-1}{2}} z \right)_{L^2(0,T)} + \lambda(w, z)_{L^2(0,T)} = b(z) \tag{29}$$

for all $z \in H^{\frac{\gamma-1}{2}}(0, T)$. Using Lemma A.2 gives

$$\left\langle D_{0+}^{\gamma} w, \varphi \right\rangle = \left(D_{0+}^{\frac{\gamma+1}{2}} w, D_{T-}^{\frac{\gamma-1}{2}} \varphi \right)_{L^2(0,T)},$$
$$\left\langle D_{0+}^{\gamma} (c_1 t), \varphi \right\rangle = \left(D_{0+}^{\frac{\gamma+1}{2}} (c_1 t), D_{T-}^{\frac{\gamma-1}{2}} \varphi \right)_{L^2(0,T)}$$

for all $\varphi \in C_0^\infty(0, T)$, so that from (29) it follows that

$$D_{0+}^{\gamma}(w - c_1 t) = g - \lambda(w + c_0).$$

Putting $y := w + c_0$ gives

$$D_{0+}^{\gamma}(y - c_0 - c_1 t) + \lambda y = g,$$

and then by Lemma A.1 and A.2 it is evident that y is the unique $H^\gamma(0, T)$-solution to problem (19). Also, $y(0) = c_0$ is obvious, and (27) follows directly from (29).

Now let us prove (28). Firstly, substituting $z := y'$ into (27) and using integration by parts yield

$$\left(D_{0+}^{\frac{\gamma+1}{2}} (y - c_0 - c_1 t),\ D_{T-}^{\frac{\gamma-1}{2}} y' \right)_{L^2(0,T)} + \frac{\lambda}{2} y^2(T) = \frac{\lambda}{2} c_0^2 + (g, y')_{L^2(0,T)}.$$

Therefore, Lemma A.2, the Cauchy–Schwarz inequality and the Young's inequality with ϵ imply

$$\|y - c_0\|^2_{H^{\frac{\gamma+1}{2}}(0,T)} + \lambda y^2(T) \lesssim \|g\|^2_{L^2(0,T)} + \lambda c_0^2 + c_1^2,$$

and so

$$\|y\|_{H^{\frac{\gamma+1}{2}}(0,T)} \lesssim \|g\|_{L^2(0,T)} + \lambda^{\frac{1}{2}} |c_0| + |c_1|. \tag{30}$$

Secondly, substituting $z := y$ into (27) yields

$$\lambda \|y\|^2_{L^2(0,T)} = (g, y)_{L^2(0,T)} - \left(D_{0+}^{\frac{\gamma+1}{2}} (y - c_0 - c_1 t),\ D_{T-}^{\frac{\gamma-1}{2}} y \right)_{L^2(0,T)},$$

so that using Lemmas 5.4 and A.2, the Cauchy–Schwarz inequality and the Young's inequality with ϵ gives

$$\lambda \|y\|^2_{L^2(0,T)} \lesssim \|y - c_0 - c_1 t\|_{H^{\frac{\gamma+1}{2}}(0,T)} \|y\|_{H^{\frac{\gamma-1}{2}}(0,T)} + \lambda^{-1} \|g\|^2_{L^2(0,T)},$$

which, together with (30), yields

$$\lambda^{\frac{1}{2}} \|y\|_{L^2(0,T)} \lesssim \|g\|_{L^2(0,T)} + \lambda^{\frac{1}{2}} |c_0| + |c_1| . \tag{31}$$

Finally, collecting (30), (31) proves (28), and thus proves this lemma. □

Finally, by the above lemma and the Galerkin method, we readily conclude that problem (1) admits a unique weak solution indeed. We summarize the result as follows.

Theorem A.1 *The weak solution u of problem (1) satisfies that $u(0) = u_0$ and that*

$$\left(D_{0+}^{\frac{\gamma+1}{2}} (u - u_0 - tu_1),\ D_{T-}^{\frac{\gamma-1}{2}} v \right)_{L^2(\Omega_T)} + (\nabla u, \nabla v)_{L^2(\Omega_T)} = (f, v)_{L^2(\Omega_T)} \tag{32}$$

for all $v \in H^{\frac{\gamma-1}{2}}(0, T; H_0^1(\Omega))$. Furthermore, we have

$$\|u\|_{H^{\frac{\gamma+1}{2}}(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H_0^1(\Omega))} \lesssim \left(\|f\|_{L^2(\Omega_T)} + \|u_1\|_{L^2(\Omega)} + \|u_0\|_{H_0^1(\Omega)} \right).$$

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