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Asymptotically Exact Posteriori Error Estimates for the Local Discontinuous Galerkin Method Applied to Nonlinear Convection–Diffusion Problems

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Abstract In this paper, we present and analyze implicit *a posteriori* error estimates for the local discontinuous Galerkin (LDG) method applied to nonlinear convection–diffusion problems in one space dimension. Optimal a priori error estimates for the solution and for the auxiliary variable that approximates the first-order derivative are derived in the L^2-norm for the semi-discrete formulation. More precisely, we identify special numerical fluxes and a suitable projection of the initial condition for the LDG scheme to achieve $p+1$ order of convergence for the solution and its spatial derivative in the L^2-norm, when piecewise polynomials of degree at most p are used. We further prove that the derivative of the LDG solution is superconvergent with order $p+1$ towards the derivative of a special projection of the exact solution. We use this result to prove that the LDG solution is superconvergent with order $p+3/2$ towards a special Gauss–Radau projection of the exact solution. Our superconvergence results allow us to show that the leading error term on each element is proportional to the $(p+1)$-degree right Radau polynomial. We use these results to construct asymptotically exact a posteriori error estimator. Furthermore, we prove that the a posteriori LDG error estimate converges at a fixed time to the true spatial error in the L^2-norm at $\mathcal{O}(h^{p+3/2})$ rate. Finally, we prove that the global effectivity index in the L^2-norm converge to unity at $\mathcal{O}(h^{1/2})$ rate. Our proofs are valid for arbitrary regular meshes using P^p polynomials with $p \geq 1$. Finally, several numerical examples are given to validate the theoretical results.

Keywords Local discontinuous Galerkin method · Nonlinear convection–diffusion problems · Superconvergence · *a posteriori* error estimation · Gauss–Radau projection

Mathematics Subject Classification 65M15 · 65M60 · 65M50 · 65N30 · 65N50

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1 Introduction

In this paper, we propose a residual-based a posteriori error estimator for the local discontinuous Galerkin (LDG) method for one-dimensional nonlinear convection–diffusion problems of the form

$$u_t + (f(u))_x = ku_{xx} + g(x,t), \quad x \in \Omega = [a,b],\ t \in [0,T], \tag{1.1a}$$

subject to the initial and periodic boundary conditions

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{1.1b}$$

$$u(a,t) = u(b,t), \quad u_x(a,t) = u_x(b,t), \quad t \in [0,T], \tag{1.1c}$$

where the diffusion coefficient $k > 0$ is a constant. Here, $g(x,t)$ and $u_0(x)$ are smooth functions possessing all the necessary derivatives. The function $f(u)$ is a nonlinear flux function. In our analysis, we assume that $f(u)$ is a differentiable function with respect to the variable u. For the sake of simplicity, we only consider the case of periodic boundary conditions. However, this assumption is not essential. We note that if other boundary conditions (e.g., Dirichlet or Neumann or mixed boundary conditions) are chosen, the LDG method can be easily designed; see [8,9,11,22,41] for some discussion. In our analysis, the initial condition, u_0, and the source term, $g(x,t)$, are assumed to be sufficiently smooth functions so that the exact solution, $u(x,t)$, is a smooth function on $[a,b] \times [0,T]$.

The LDG method we discuss in this paper is an extension of the discontinuous Galerkin (DG) method aimed at solving differential equations containing higher than first-order spatial derivatives. The LDG method for solving convection–diffusion problems was first introduced by Cockburn and Shu in [30]. LDG methods are robust and high-order accurate, can achieve stability without slope limiters, and are locally (element-wise) mass-conservative. This last property is very useful in the area of computational fluid dynamics, especially in situations where there are shocks, steep gradients or boundary layers. Moreover, LDG methods are extremely flexible in the mesh-design; they can easily handle meshes with hanging nodes, elements of various types and shapes, and local spaces of different orders. They further exhibit strong superconvergence that can be used to estimate the discretization errors. LDG schemes have been successfully applied to hyperbolic, elliptic, and parabolic partial differential equations [2,4–6,8,15,18,19,22,22,29–32,36,38,40], to mention a few. A review of the LDG methods is given in [8,10,16,18,22,26–28,41].

The LDG method for solving convection–diffusion problems was first introduced by Cockburn and Shu in [30]. They further studied the stability and error estimates for the LDG method. Castillo et al. [14,17] presented the first *a priori* error analysis for the LDG method for a model elliptic problem. They considered arbitrary meshes with hanging nodes and elements of various shapes and studied general numerical fluxes. They showed that, for smooth solutions, the L^2 errors in ∇u and in u are of order p and $p + 1/2$, respectively, when polynomials of total degree not exceeding p are used. Cockburn et al. [25] presented a superconvergence result for the LDG method for a model elliptic problem on Cartesian grids. They identified a special numerical flux for which the L^2-norms of the gradient and the potential are of orders $p + 1/2$ and $p + 1$, respectively, when tensor product polynomials of degree at most p are used.

Related theoretical results in the literature including superconvergence results and error estimates of the LDG methods for convection–diffusion problems are given in [4,5,13,18,19,21–23,33,38,39,41,43]. In particular, we mention the work of Castillo et al. [18,38] in which optimal a priori error estimates for the hp-version of the LDG method for linear convection–diffusion problems are investigated. Later Adjerid et al. [4,5] investigated the superconvergence of the LDG method applied to diffusion and transient convection–diffusion problems. Celiker and Cockburn [19] proved a new superconvergence property of a large class of finite element methods for one-dimensional steady state linear convection–diffusion problems. Cheng and Shu [21] studied the convergence behavior of the LDG methods when applied to one-dimensional time dependent linear convection–diffusion equations. They observed that the LDG solution is superconvergent towards a particular projection of the exact solution. The order of superconvergence is observed to be $p+2$ when polynomials of degree at most p are used. In [22], Cheng and Shu studied the superconvergence property for the DG and LDG methods for solving one-dimensional time-dependent linear convection and linear convection–diffusion equations. They proved superconvergence towards a particular projection of the exact solution. The order of superconvergence is proved to be $p + 3/2$, when p-degree piecewise polynomials with $p \geq 1$ are used. We also mention the recent work of Shu, Zhiming et al. [13,41] in which sharp superconvergence of the LDG method for linear convection–diffusion equations in one space dimension is analyzed. Finally, Cheng et al. [20] studied the LDG method based on the generalized alternating numerical fluxes for solving linear convection–diffusion equations in one dimension and two dimensions. They derived the optimal error estimate in the L^2-norm for the LDG method with generalized alternating fluxes. More precisely, they proved the $(p + 1)$th-order convergence rate in one-dimensional space and multi-dimensional space on Cartesian meshes with piecewise tensor product polynomials of degree at most $p \geq 0$.

In this paper, we study the superconvergence property for the LDG method for nonlinear convection–diffusion problems, extending the results in [22] for linear problems and the results in [7] for nonlinear scalar conservation laws in one space dimension. We also present and analyze a residual-based a posteriori error estimator for the spatial discretization error. We prove that the LDG solutions are $(p+1)$th order convergent in the L^2-norm, when the space of piecewise polynomials of degree p is used. Computational results indicate that the theoretical order of convergence is optimal. Moreover, we show that the derivative of the LDG solution is superconvergent with order $p + 1$ towards the derivative of a Gauss–Radau projection of the exact solution. We also prove that the LDG solution is superconvergent with order $p+3/2$ towards a Gauss–Radau projection of the exact solution, while computational results show higher $\mathcal{O}(h^{p+2})$ convergence rate. These results allow us to prove that the significant part of the spatial discretization error is proportional to a $(p + 1)$-degree right Radau polynomial. We use this result to develop a residual-based *a posteriori* error estimate of the spatial error. The leading term of the discretization error is estimated by solving a local steady problem with no boundary conditions on each element. We further prove that our LDG error estimate converges to the true spatial error at $\mathcal{O}(h^{p+3/2})$ rate, while computational results show higher $\mathcal{O}(h^{p+3})$ convergence rate. Finally, we prove that the global effectivity index in the L^2-norm converges to unity at $\mathcal{O}(h^{1/2})$ rate, while numerically it exhibits $\mathcal{O}(h^2)$ rate. In our analysis we proved these convergence results under mesh refinement and at a fixed time t and time discretization is assumed to be exact. Our proofs are valid for any regular meshes and using piecewise polynomials of degree $p \geq 1$.

This paper is organized as follows: In Sect. 2 we present the semi-discrete LDG method for solving nonlinear convection–diffusion problems. We also introduce some notation and definitions. In Sect. 3 we present the LDG error analysis and prove several optimal L^2 error estimates. In Sect. 4 we state and prove the main superconvergence results. In Sect. 5 we present and analyze our a posteriori error estimation procedure. In Sect. 6 we present numerical results to confirm the theoretical results. We conclude and discuss our results in Sect. 7.

2 The LDG Method for Convection–Diffusion Problems

Without loss of generality, we may assume that the diffusion coefficient $k = 1$. In order to construct the LDG scheme, we introduce an auxiliary variable $q = u_x$ and convert Eq. (1.1a) into the following first-order system

$$u_t + (f(u))_x - q_x = g(x,t), \qquad q - u_x = 0. \tag{2.1}$$

We divide the computational domain $\Omega = [a, b]$ into N intervals $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, N$, where $a = x_0 < x_1 < \cdots < x_N = b$. Let $h_i = x_i - x_{i-1}$ be the length of the interval I_i, and denote $h = \max\limits_{1\le i\le N} h_i$ and $h_{min} = \min\limits_{1\le i\le N} h_i$ to be the lengths of the largest and smallest intervals, respectively. In this paper, we consider regular meshes, that is $h \le K h_{min}$, where $K \ge 1$ is a constant during mesh refinement. For simplicity, we use $v\big|_i$ to denote the value of the continuous function $v = v(x,t)$ at $x = x_i$. We also use $v^-\big|_i$ and $v^+\big|_i$ to denote the left limit and the right limit of v at the discontinuity point x_i, i.e.,

$$v^-\big|_i = v(x_i^-, t) = \lim_{s\to 0^-} v(x_i + s, t), \qquad v^+\big|_i = v(x_i^+, t) = \lim_{s\to 0^+} v(x_i + s, t).$$

Multiplying the two equations in (2.1) by test functions v and w, respectively, integrating over an arbitrary element I_i, and using integration by parts, we get

$$\int_{I_i} u_t v dx + \int_{I_i} (q - f(u)) v_x dx + (f(u) - q)v\big|_i - (f(u) - q)v\big|_{i-1} = \int_{I_i} g v dx, \tag{2.2a}$$

$$\int_{I_i} q w dx + \int_{I_i} u w_x dx - uw\big|_i + uw\big|_{i-1} = 0. \tag{2.2b}$$

We introduce the following discontinuous finite element approximation space

$$V_h^p = \{v : v|_{I_i} \in P^p(I_i),\ i = 1, \ldots, N\},$$

where $P^p(I_i)$ denotes the space of polynomials of degree at most p on I_i with coefficients as functions of t. We would like to emphasize that polynomials in the finite element space V_h^p are allowed to be completely discontinuous at the mesh points.

Next, we replace the exact solutions u and q, at any fixed time t, by piecewise polynomials of degree at most p and denote them by $u_h \in V_h^p$ and $q_h \in V_h^p$, respectively. We also choose the test functions v and w to be piecewise polynomials of degree at most p. The LDG scheme can now be defined as: find approximations u_h and $q_h \in V_h^p$ such that $\forall\, i = 1, \ldots, N$,

$$\int_{I_i} (u_h)_t v dx + \int_{I_i} (q_h - f(u_h)) v_x dx + (\hat{f} - \hat{q}_h)v^-\big|_i - (\hat{f} - \hat{q}_h)v^+\big|_{i-1} = \int_{I_i} g v dx, \tag{2.3a}$$

$$\int_{I_i} q_h w dx + \int_{I_i} u_h w_x dx - \hat{u}_h w^-\big|_i + \hat{u}_h w^+\big|_{i-1} = 0, \tag{2.3b}$$

where $\hat{f}$, $\hat{u}_h$, and $\hat{q}_h$ are the so-called numerical fluxes. The numerical fluxes $\hat{u}_h$ and $\hat{q}_h$ are the discrete approximations to the traces of u and q at the nodes. The numerical flux $\hat{f}$ is a single-valued function defined at the nodes and in general depends on the values of u_h from both sides i.e., $\hat{f} = \hat{f}(u_h^-, u_h^+)$. Here, $\hat{f}$ is a monotone numerical flux, i.e., it satisfies the following three conditions: (i) it is locally Lipschitz continuous, (ii) it is consistent with the flux $f(u)$, and (iii) it is a nondecreasing function of its first argument and a nonincreasing function of its second argument. The popular monotone numerical fluxes are the Godunov flux, the Engquist–Osher flux, the Lax–Friedrichs flux, etc (see [35]). We would like to mention that the numerical fluxes have to be suitably chosen in order to ensure the stability of the method and also to improve the order of convergence. In this paper, we choose the following fluxes:

- The numerical flux $\hat{f}$ associated with the convection is taken as the upwind flux which depends on the sign of f' i.e.,

$$\hat{f}\big|_i = \begin{cases} f(u_h^-)\big|_i, & \text{if } f'(u_h) \geq 0, \\ f(u_h^+)\big|_i, & \text{if } f'(u_h) < 0, \end{cases} \quad i = 0, \ldots, N. \tag{2.3c}$$

- The numerical fluxes $\hat{u}_h$ and $\hat{q}_h$ associated with the diffusion terms are taken as the alternating fluxes (e.g., see [22]) i.e.,

$$\hat{u}_h\big|_i = u_h^-\big|_i, \qquad \hat{q}_h\big|_i = q_h^+\big|_i, \qquad i = 0, \ldots, N. \tag{2.3d}$$

  Even though the proofs of our results are given using the numerical fluxes (2.3d), the same results can be proved using the following numerical fluxes with only minor modifications

$$\hat{u}_h\big|_i = u_h^+\big|_i, \qquad \hat{q}_h\big|_i = q_h^-\big|_i, \qquad i = 0, \ldots, N.$$

  It is crucial that we take $\hat{u}_h$ and $\hat{q}_h$ from the opposite directions.

To complete the definition of the LDG scheme, we still need to define the discrete initial condition $u_h(x, 0) \in V_h^p$. In this paper we use a special projection of the exact initial condition $u_0(x)$. This particular projection will be defined later and is needed to achieve global superconvergence result towards the Gauss–Radau projection, which will be defined later.

Norms, projections, and properties of the finite element space: We define the inner product of two integrable functions, $u = u(x, t)$ and $v = v(x, t)$, on $I_i = [x_{i-1}, x_i]$ and at a fixed time t as $(u(\cdot, t), v(\cdot, t))_{I_i} = \int_{I_i} u(x, t)v(x, t)dx$. The standard L^2-norm of v over I_i is denoted by $\|u(\cdot, t)\|_{0,I_i} = (u(\cdot, t), u(\cdot, t))_{I_i}^{1/2}$. Moreover, the L^∞-norm of $u(\cdot, t)$ on I_i at time t is defined by $\|u(\cdot, t)\|_{\infty,I_i} = \sup_{x\in I_i} |u(x, t)|$. For any $s = 0, 1, \ldots$, we use $H^s(I_i)$ to denote the standard Sobolev space $H^s(I_i) = \left\{u : \int_{I_i} |\partial_x^k u(x, t)|^2 dx < \infty,\ 0 \leq k \leq s\right\}$. Moreover, the $H^s(I_i)$-norm is defined as $\|u(\cdot, t)\|_{s,I_i} = \left(\sum_{k=0}^s \left\|\partial_x^k u(\cdot, t)\right\|_{0,I_i}^2\right)^{1/2}$. The $H^s(I_i)$-seminorm of u on I_i is given by $|u(\cdot, t)|_{s,I_i} = \left\|\partial_x^s u(\cdot, t)\right\|_{0,I_i}$. We also define the norms on the whole computational domain Ω as follows:

$$\|u(\cdot, t)\|_{s,\Omega} = \left(\sum_{i=1}^N \|u(\cdot, t)\|_{s,I_i}^2\right)^{1/2}, \quad \|u(\cdot, t)\|_{\infty,\Omega} = \max_{1\leq i\leq N} \|u(\cdot, t)\|_{\infty,I_i}.$$

The seminorm on the whole computational domain Ω is defined as $|u(\cdot, t)|_{s,\Omega} = \left(\sum_{i=1}^N |u|_{s,I_i}^2\right)^{1/2}$. We note that if $u(\cdot, t) \in H^s(\Omega)$, the norm $\|u(\cdot, t)\|_{s,\Omega}$ on the whole computational domain is the standard Sobolev norm $\left(\sum_{k=0}^s \left\|\partial_x^k u\right\|_{0,\Omega}^2\right)^{1/2}$. For simplicity, if we consider the norm on the whole computational domain Ω, then the corresponding index will be omitted. Thus, we use $\|u\|$, $\|u\|_s$, and $\|u\|_\infty$ to denote $\|u\|_{0,\Omega}$, $\|u\|_{s,\Omega}$, and $\|u\|_{\infty,\Omega}$, respectively. We also use $\|u(t)\|$ to denote the value of $\|u(\cdot, t)\|$ at time t. In particular, we use $\|u(0)\|$ to denote $\|u(\cdot, 0)\|$. Throughout the paper, we omit the argument t and we use $\|u\|$ to denote $\|u(t)\|$ whenever confusion is unlikely.

For $p \geq 1$, we define $P_h^{\pm}u$ as two special Gauss–Radau projections of u onto V_h^p as follows [22]: The restrictions of P_h^+u and P_h^-u to I_i are polynomials in $P^p(I_i)$ satisfying

$$\int_{I_i} (P_h^- u - u) v dx = 0, \ \forall\, v \in P^{p-1}(I_i), \quad \text{and} \quad (P_h^- u - u)^-\big|_i = 0, \tag{2.4a}$$

$$\int_{I_i} (P_h^+ u - u) v dx = 0, \ \forall\, v \in P^{p-1}(I_i), \quad \text{and} \quad (P_h^+ u - u)^+\big|_{i-1} = 0. \tag{2.4b}$$

These special projections are used in the error estimates of the DG methods to derive optimal L^2 error bounds in the literature, e.g., in [22]. They are mainly used to eliminate the jump terms at the element boundaries in the error estimates in order to prove the optimal L^2 error estimates. In our analysis, we need the following projection results [24]: If $u \in H^{p+1}(I_i)$, then there exists a positive constant C independent of the mesh size h, such that

$$\left\| u - P_h^{\pm} u \right\|_{0,I_i} + h_i \left\| (u - P_h^{\pm} u)_x \right\|_{0,I_i} \leq C h_i^{p+1} |u|_{p+1,I_i}. \tag{2.5}$$

In the rest of the paper, we will not differentiate between various constants, and instead will use a generic constant C (or accompanied by lower indices) to represent a positive constant independent of the mesh size h, but which may depend upon the exact smooth solution of the partial differential equation (1.1a) and its derivatives.

Next, we recall some inverse properties of the finite element space V_h^p that will be used in our error analysis: For any $v \in V_h^p$, there exists a positive constant C independent of v and h, such that

$$(i)\ \|v_x\| \leq C h^{-1} \|v\|, \quad (ii)\ \|v\|_\infty \leq C h^{-1/2} \|v\|,$$
$$(iii)\ \left(\sum_{i=1}^{N} v^2(x_i^+) + v^2(x_i^-) \right)^{1/2} \leq C h^{-1/2} \|v\|. \tag{2.6}$$

Finally, the Sobolev's inequality implies that there exists a positive constant C is independent of h such that

$$\left\| u - P_h^{\pm} u \right\|_\infty \leq C h^{p+1/2}. \tag{2.7}$$

The initial condition of the LDG method: To obtain a superconvergent LDG method, we carefully design a suitable projection of the initial condition of the LDG scheme. In our mathematical error analysis and numerical examples we approximate the initial condition of our numerical scheme on each interval as follows

$$u_h(x, 0) = P_h^1 u(x, 0), \quad x \in I_i, \quad i = 1, \ldots, N, \tag{2.8}$$

where P_h^1 is a special projection operator introduced by Cheng and Shu [22]. It is defined as follows: For any smooth function u, $P_h^1 u|_{I_i} \in P^p(I_i)$, and suppose $q_h \in V_h^p$ is the unique solution to

$$\int_{I_i} q_h w dx + \int_{I_i} P_h^1 u \, w_x dx - (P_h^1 u)^- w^-\big|_i + (P_h^1 u)^- w^+\big|_{i-1} = 0, \quad \forall\, w \in V_h^p. \tag{2.9}$$

Then, we require

$$(P_h^- u - P_h^1 u)^-\big|_{i-1} = (P_h^+ q - q_h)^+\big|_{i-1}, \tag{2.10a}$$

$$\int_{I_i} (P_h^- u - P_h^1 u) v dx = \int_{I_i} (P_h^+ q - q_h) v dx, \quad \forall\, v \in P^{p-1}(I_i). \tag{2.10b}$$

Proof for the existence and uniqueness of $P_h^1 u$ is provided in Cheng and Shu [22].

Lemma 2.1 *The operator P_h^1 exists and is unique. Moreover, we have the estimates*

$$\left\|(P_h^- u - P_h^1 u)(0)\right\| \le C\, h^{p+3/2}. \tag{2.11}$$

$$\left\|(P_h^+ u - q_h)(0)\right\| \le C h^{p+3/2}. \tag{2.12}$$

Proof Cf. Cheng and Shu [22]. More precisely, the estimate (2.11) can be found in its Lemma 3.1. The estimate (2.12) is proved in the proof of its Lemma 3.1 (see page 4064). □

3 A Priori Error Estimates

In this section, we will derive optimal L^2 error estimates for the LDG method. We assume that the flux function f in (1.1) is smooth enough, for example, $f \in C^2(\mathbb{R})$. In particular, we always assume that $f(u)$ satisfies the following conditions

Assumption 1 $f(u)$, $f'(u)$, and $f''(u)$ are continuous functions on $\mathbb{R}$.

Assumption 2 $f'(u) \ge 0$ so that $\hat{f}\big|_i = f(u_h^-)\big|_i$. The case $f'(u) < 0$ can be handled in a very similar manner. Let us emphasize that our conclusions actually holds true when general flux functions are used; see the numerical results in Sect. 6.

Assumption 3 There exists constants C_1 and C_2 such that $\left|f'(u)\right| \le C_1$ and $\left|f''(u)\right| \le C_2$ for all $u \in \mathbb{R}$. This assumption is reasonable for smooth solutions of (1.1); see [42] for more details.

By using the Mean Value Theorem, it can be shown that if f satisfies the above conditions, then f satisfies the following Lipschitz condition on $\mathbb{R}$ in the variable u with uniform Lipschitz constant $L = C_1$

$$\left|f(u) - f(v)\right| \le L\left|u - v\right|, \quad \text{for all } u \text{ and } v \in \mathbb{R}. \tag{3.1}$$

Throughout this paper, e_u and e_q, respectively, denote the errors between the exact solutions of (2.1) and the LDG solutions defined in (2.3), i.e., $e_u = u - u_h$ and $e_q = q - q_h$. Let the projection errors be defined as $\epsilon_u = u - P_h^- u$ and $\epsilon_q = q - P_h^+ q$ and the errors between the numerical solutions and the projection of the exact solutions be defined as $\bar{e}_u = P_h^- u - u_h$ and $\bar{e}_q = P_h^+ q - q_h$. We note that the true errors can be split as

$$e_u = \epsilon_u + \bar{e}_u, \quad e_q = \epsilon_q + \bar{e}_q. \tag{3.2}$$

We also note that, by the definitions of the projections $P_h^{\pm}$ (2.4), the following hold

$$\epsilon_u^-\big|_i = \epsilon_q^+\big|_{i-1} = 0 \quad \text{and} \quad \int_{I_i} \epsilon_u v_x dx = \int_{I_i} \epsilon_q v_x dx = 0,$$
$$\forall\, v \in \; P^p(I_i), \quad i = 1, \ldots, N, \tag{3.3}$$

since v is a polynomial of degree at most p and thus v_x is a polynomial of degree at most $p - 1$.

To deal with the nonlinearity of the flux $f(u)$, we would like to make an a priori assumption that, for small enough h and $p \ge 1$, there holds

$$\|\bar{e}_u\| \le Ch^2, \quad \forall\, t \in [0, T], \tag{3.4}$$

where C is a constant independent of h. This is obviously satisfied at time $t = 0$ since, initially, $||\bar{e}_u(0)|| \le h^{p+3/2} \le Ch^2,\ p \ge 1$, by (2.11). We will justify this a priori assumption for piecewise polynomials of degree $p \geq 1$ after Theorem 4.3. In remark 4.1, we will explain that this a priori assumption is unnecessary for the linear flux $f(u) = cu$, where c is a constant.

As a consequence of the a priori assumption (3.4), we have the following results.

Corollary 3.1 *Thea priori assumption* (3.4) *implies that,* $\forall\, t \in [0, T]$,

$$\left\|\bar{e}_u'\right\| \leq Ch. \tag{3.5}$$

$$\|\bar{e}_u\|_\infty \leq Ch^{3/2}. \tag{3.6}$$

$$\|e_u\|_\infty \leq Ch^{3/2}. \tag{3.7}$$

Proof Using the inverse property (2.6) and the *a priori* assumption (3.4), we obtain

$$\left\|\bar{e}_u'\right\| \leq C_1 h^{-1} \|\bar{e}_u\| \leq C_1 h^{-1}(C_2 h^2) \leq Ch,$$

which completes the proof of (3.5). In order to prove (3.6), we use the inverse property (2.6) and the a priori assumption (3.4), to obtain

$$\|\bar{e}_u\|_\infty \leq C_1 h^{-1/2} \|\bar{e}_u\| \leq C_1 h^{-1/2} C_2 h^2 \leq Ch^{3/2}.$$

Next, using (3.2), the triangle inequality, the estimate (2.7), and the estimate (3.6), we get,

$$\|e_u\|_\infty = \|\epsilon_u + \bar{e}_u\|_\infty \leq \|\epsilon_u\|_\infty + \|\bar{e}_u\|_\infty \leq C_1 h^{p+1/2} + C_2 h^{3/2} \leq Ch^{3/2}, \ \forall\, p \geq 1,$$

which completes the proof of (3.7). □

In the next theorem, we derive a priori error estimates for $\bar{e}_u$ and e_u in the L^2-norm.

Theorem 3.1 *Let $p \geq 1$ and (u, q) and (u_h, q_h) respectively, are solutions of* (2.1) *and* (2.3)*, where $u_h(x, 0) = P_h^1 u_0(x)$. Suppose that the a priori assumption* (3.4) *holds. Also, we assume that the flux function $f(u)$ is sufficiently smooth function with bounded derivatives. To be more precise, the condition $f(u) \in C_b^2(\mathbb{R})$ is enough, where $C_b^m(D)$ is the set of real m-times continuously differentiable functions which are bounded together with their derivatives up to the mth order. Then, for sufficiently small h, there exists a positive constant C independent of h such that,* $\forall\, t \in [0, T]$,

$$\|\bar{e}_u\| \leq Ch^{p+1}. \tag{3.8a}$$

$$\|e_u\| \leq Ch^{p+1}. \tag{3.8b}$$

Proof Subtracting (2.3) from (2.2) with $v,\ w \in V_h^p$ and using the numerical fluxes (2.3c) and (2.3d), we obtain the following error equations

$$\int_{I_i} (e_u)_t v dx - \int_{I_i} (f(u) - f(u_h) - e_q) v_x dx + (f(u) - f(u_h^-) - e_q^+) v^- \big|_i$$
$$-(f(u) - f(u_h^-) - e_q^+) v^+ \big|_{i-1} = 0, \tag{3.9a}$$

$$\int_{I_i} e_q w dx + \int_{I_i} e_u w_x dx - e_u^- w^- \big|_i + e_u^- w^+ \big|_{i-1} = 0. \tag{3.9b}$$

Using the classical Taylor's series with integral remainder in the variable u and using the relation $u - u_h = e_u$, we write the nonlinear term $f(u) - f(u_h)$ as

$$f(u) - f(u_h) = \theta(u - u_h) = \theta e_u = \theta(\bar{e}_u + \epsilon_u), \tag{3.10a}$$

where $\theta = \theta(x, t) = \int_0^1 f'(u + s(u_h - u))ds = \int_0^1 f'(u - se_u)ds$ is the mean value. Similarly, we write $f(u) - f(u_h^-)$ and $f(u_h^+) - f(u_h^-)$ as

$$f(u) - f(u_h^-) = \theta^-(u - u_h^-) = \theta^-(\bar{e}_u^- + \epsilon_u^-), \tag{3.10b}$$

$$f(u_h^+) - f(u_h^-) = f(u) - f(u_h^-) - (f(u) - f(u_h^+)) = \theta^-(\bar{e}_u^- + \epsilon_u^-) - \theta^+(\bar{e}_u^+ + \epsilon_u^+), \tag{3.10c}$$

where $\theta^\pm = \int_0^1 f'(u - se_u^\pm)ds$.

Substituting (3.10a) and (3.10b) into (3.9), using (3.2), and applying (3.3), we get

$$\int_{I_i} (\bar{e}_u)_t v dx - \int_{I_i} (\theta \bar{e}_u - \bar{e}_q) v_x dx + (\theta^- \bar{e}_u^- - \bar{e}_q^+) v^-\big|_i - (\theta^- \bar{e}_u^- - \bar{e}_q^+) v^+\big|_{i-1}$$
$$= \int_{I_i} \theta \epsilon_u v_x dx - \int_{I_i} (\epsilon_u)_t v dx, \tag{3.11a}$$
$$\int_{I_i} \bar{e}_q w dx + \int_{I_i} \bar{e}_u w_x dx - \bar{e}_u^- w^-\big|_i + \bar{e}_u^- w^+\big|_{i-1} = -\int_{I_i} \epsilon_q w dx, \tag{3.11b}$$

which, after integrating by parts, are equivalent to

$$\int_{I_i} (\bar{e}_u)_t v dx - \int_{I_i} \theta \bar{e}_u v_x dx + \theta^- \bar{e}_u^- v^-\big|_i - \theta^- \bar{e}_u^- v^+\big|_{i-1} - \int_{I_i} (\bar{e}_q)_x v dx - \left(\bar{e}_q^+ - \bar{e}_q^-\right) v^-\big|_i$$
$$= \int_{I_i} \theta \epsilon_u v_x dx - \int_{I_i} (\epsilon_u)_t v dx, \tag{3.12a}$$
$$\int_{I_i} \bar{e}_q w dx - \int_{I_i} (\bar{e}_u)_x w dx - \left(\bar{e}_u^+ - \bar{e}_u^-\right) w^+\big|_{i-1} = -\int_{I_i} \epsilon_q w dx. \tag{3.12b}$$

Taking $v = \bar{e}_u$ in (3.11a) and $w = \bar{e}_q$ in (3.12b) then adding the resulting equations, we get

$$\frac{1}{2}\frac{d}{dt}\int_{I_i} \bar{e}_u^2 dx + \int_{I_i} \bar{e}_q^2 dx - \int_{I_i} \theta \bar{e}_u (\bar{e}_u)_x dx + \theta^- (\bar{e}_u^-)^2\big|_i - \theta^- \bar{e}_u^- \bar{e}_u^+\big|_{i-1}$$
$$+ \bar{e}_u^- \bar{e}_q^+\big|_{i-1} - \bar{e}_u^- \bar{e}_q^+\big|_i = \int_{I_i} \theta \epsilon_u (\bar{e}_u)_x dx - \int_{I_i} (\epsilon_u)_t \bar{e}_u dx - \int_{I_i} \epsilon_q \bar{e}_q dx. \tag{3.13}$$

For each element $I_i = [x_{i-1}, x_i]$, we denote by u_i the value of the exact solution u at the point x_{i-1} and at time t. Adding and subtracting the constant $\theta_i = f'(u_i)$ to θ, we rewrite (3.13) as

$$\frac{1}{2}\frac{d}{dt}\int_{I_i} \bar{e}_u^2 dx + \int_{I_i} \bar{e}_q^2 dx = \sum_{k=1}^{5} T_{k,i}, \tag{3.14a}$$

where

$$T_{1,i} = \theta_i \left(\int_{I_i} \bar{e}_u (\bar{e}_u)_x dx - (\bar{e}_u^-)^2\big|_i + \bar{e}_u^- \bar{e}_u^+\big|_{i-1} \right), \tag{3.14b}$$
$$T_{2,i} = \int_{I_i} (\theta - \theta_i) \bar{e}_u (\bar{e}_u)_x dx - (\theta^- - \theta_i)(\bar{e}_u^-)^2\big|_i + (\theta^- - \theta_i) \bar{e}_u^- \bar{e}_u^+\big|_{i-1}, \tag{3.14c}$$
$$T_{3,i} = \int_{I_i} (\theta - \theta_i) \epsilon_u (\bar{e}_u)_x dx + \theta_i \int_{I_i} \epsilon_u (\bar{e}_u)_x dx, \tag{3.14d}$$
$$T_{4,i} = -\int_{I_i} (\epsilon_u)_t \bar{e}_u dx - \int_{I_i} \epsilon_q \bar{e}_q dx, \tag{3.14e}$$
$$T_{5,i} = -\bar{e}_u^- \bar{e}_q^+\big|_{i-1} + \bar{e}_u^- \bar{e}_q^+\big|_i. \tag{3.14f}$$

Summing the error equation (3.14a) over all elements, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\bar{e}_u\|^2 + \left\|\bar{e}_q\right\|^2 = \sum_{k=1}^{5} T_k, \tag{3.15}$$

where $T_k = \sum_{i=1}^{N} T_{k,i},\ k = 1,\ldots,5$. Next, we will estimate $T_k,\ k = 1,\ldots,5$ one by one.
Estimate of T_1. Taking $w = \bar{e}_u$ in (3.11b), we obtain

$$\int_{I_i} \bar{e}_q \bar{e}_u dx + \int_{I_i} \bar{e}_u (\bar{e}_u)_x dx - \bar{e}_u^- \bar{e}_u^- \big|_i + \bar{e}_u^- \bar{e}_u^+ \big|_{i-1} = -\int_{I_i} \epsilon_q \bar{e}_u dx.$$

Consequently, we have

$$\int_{I_i} \bar{e}_u (\bar{e}_u)_x dx - (\bar{e}_u^-)^2\big|_i + \bar{e}_u^- \bar{e}_u^+ \big|_{i-1} = -\int_{I_i} \bar{e}_q \bar{e}_u dx - \int_{I_i} \epsilon_q \bar{e}_u dx. \tag{3.16}$$

Using (3.16), we rewrite $T_{1,i}$ as

$$T_{1,i} = \theta_i \left(-\int_{I_i} \bar{e}_q \bar{e}_u dx - \int_{I_i} \epsilon_q \bar{e}_u dx \right).$$

Summing over all elements, using the assumption $|f'| \le C$ on $\mathbb{R}$, applying the Cauchy–Schwarz inequality, the projection result (2.5), and the inequality $ab \le a^2 + \frac{1}{4}b^2$, we get

$$\begin{aligned} T_1 &\le \sum_{i=1}^{N} |\theta_i| \left(\int_{I_i} |\bar{e}_q||\bar{e}_u| dx + \int_{I_i} |\epsilon_q||\bar{e}_u| dx \right) \\ &\le C \sum_{i=1}^{N} \left(\left\|\bar{e}_q\right\|_{0,I_i} \|\bar{e}_u\|_{0,I_i} + \left\|\epsilon_q\right\|_{0,I_i} \|\bar{e}_u\|_{0,I_i} \right) \\ &\le C \left(\left\|\bar{e}_q\right\| \|\bar{e}_u\| + \left\|\epsilon_q\right\| \|\bar{e}_u\| \right) \\ &\le C_1 \left(h^{2p+2} + \|\bar{e}_u\|^2 \right) + \frac{1}{4}\left\|\bar{e}_q\right\|^2. \end{aligned} \tag{3.17}$$

Estimate of T_2. We first rewrite $\theta - \theta_i$ on each element I_i as follows

$$\theta - \theta_i = \int_0^1 f'(u - s e_u) ds - f'(u_i) = \int_0^1 \left(f'(u - s e_u) - f'(u_i) \right) ds,$$

since $f'(u_i)$ is a constant in s. Adding and subtracting $f'(u)$, we write

$$\theta - \theta_i = \int_0^1 \left(f'(u) - f'(u_i) \right) ds + \int_0^1 \left(f'(u - s e_u) - f'(u) \right) ds. \tag{3.18}$$

Using Taylor's theorem, we can bound the interpolation error $f'(u) - f'(u_i)$ on each element I_i as

$$\left| f'(u) - f'(u_i) \right| = \left| f''(\bar{u}) \right| \left| u - u_i \right| \le \left| f''(\bar{u}) \right| \left| u_x(\xi_i, t) \right| |x - x_{i-1}|,$$

where $\bar{u} = \lambda_1 u + (1 - \lambda_1) u_i$, $\xi_i = x_{i-1} + \lambda_2 (x - x_{i-1})$, and $\lambda_k \in [0, 1],\ k = 1, 2$. Using the smoothness of the exact solution u and f, we get

$$\left| f'(u) - f'(u_i) \right| \le C_1 C_2 |x - x_{i-1}| \le C_1 C_2 h_i \le C_3 h. \tag{3.19}$$

Similarly, we can bound the error $f'(u - s e_u) - f'(u)$ on each element I_i as

$$\left| f'(u - s e_u) - f'(u) \right| = \left| f''(\tilde{u}) \right| \left| -s e_u \right| \le \left| f''(\tilde{u}) \right| \left| e_u \right| \le \left| f''(\tilde{u}) \right| \|e_u\|_\infty,$$

since $s \in [0, 1]$, where $\tilde{u} = \lambda(u - se_u) + (1 - \lambda)u$, $\lambda \in [0, 1]$. Using the estimate (3.7) and the assumption $|f''| \le C_2$, we get

$$\left|f'(u - se_u) - f'(u)\right| \le C_2(C_1 h^{3/2}) \le C h^{3/2}. \tag{3.20}$$

Combining (3.18), (3.19), and (3.20), we conclude that

$$\begin{aligned} \left|\theta - \theta_i\right| &\le \int_0^1 \left|f'(u) - f'(u_i)\right| ds + \int_0^1 \left|f'(u - se_u) - f'(u)\right| ds \\ &\le \int_0^1 C_1 h ds + \int_0^1 C_2 h^{3/2} ds \le C h. \end{aligned} \tag{3.21}$$

We can use the same argument to show that the errors $(\theta^- - \theta_i)\big|_i$ and $(\theta^- - \theta_i)\big|_{i-1}$ are bounded by

$$\left|(\theta^- - \theta_i)\big|_i\right| \le C h, \quad \left|(\theta^- - \theta_i)\big|_{i-1}\right| \le C h. \tag{3.22}$$

Now, applying the Cauchy–Schwarz inequality, (3.21), (3.22), the inverse inequalities in (2.6), we obtain

$$\begin{aligned} T_2 &\le \sum_{i=1}^N \int_{I_i} \left|\theta - \theta_i\right| \left|\bar{e}_u\right| \left|(\bar{e}_u)_x\right| dx + \sum_{i=1}^N \left|\theta^- - \theta_i\right| \left|\bar{e}_u^-\right|^2\Big|_i + \sum_{i=1}^N \left|\theta^- - \theta_i\right| \left|\bar{e}_u^-\right| \left|\bar{e}_u^+\right|\Big|_{i-1} \\ &\le C_3 h \left(\int_\Omega \left|\bar{e}_u\right| \left|(\bar{e}_u)_x\right| dx + \sum_{i=1}^N \left|\bar{e}_u^-\right|^2\Big|_i + \left(\sum_{i=1}^N \left|\bar{e}_u^-\right|^2\Big|_{i-1} \right)^{1/2} \left(\sum_{i=1}^N \left|\bar{e}_u^+\right|^2\Big|_{i-1} \right)^{1/2} \right) \\ &\le C_3 h \left(\|\bar{e}_u\| \, \|(\bar{e}_u)_x\| + C_4 h^{-1} \|\bar{e}_u\|^2 + C_5 h^{-1/2} \|\bar{e}_u\| \, C_6 h^{-1/2} \|\bar{e}_u\| \right) \\ &\le C_3 h \left(C_7 h^{-1} \|\bar{e}_u\|^2 + C_4 h^{-1} \|\bar{e}_u\|^2 + C_7 h^{-1} \|\bar{e}_u\|^2 \right) \le C_2 \|\bar{e}_u\|^2. \end{aligned} \tag{3.23}$$

Estimate of T_3. Since ϵ_u is orthogonal to $(\bar{e}_u)_x \in P^{p-1}(I_i)$ (due to the properties in (2.4a)), $T_{3,i}$ simplifies to

$$T_{3,i} = \int_{I_i} (\theta - \theta_i) \epsilon_u (\bar{e}_u)_x dx.$$

Summing over all elements, using the estimate (3.21), applying the Cauchy–Schwarz inequality, invoking the inverse inequality (2.6), using the projection result (2.5), and applying the inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$, we get

$$T_3 \le \sum_{i=1}^N \int_{I_i} \left|\theta - \theta_i\right| \left|\epsilon_u\right| \left|(\bar{e}_u)_x\right| dx \le C_1 h \int_\Omega \left|\epsilon_u\right| \left|(\bar{e}_u)_x\right| dx \le C_1 h \|\epsilon_u\| \, \|(\bar{e}_u)_x\| \tag{3.24}$$

$$\le C_1 h C_2 h^{p+1} C_4 h^{-1} \|\bar{e}_u\| \le C_3 \left(h^{2p+2} + \|\bar{e}_u\|^2 \right). \tag{3.25}$$

Estimate of T_4. Applying the Cauchy–Schwarz inequality, the projection result (2.5), and the inequality $ab \le a^2 + \frac{1}{4}b^2$, we get

$$\begin{aligned} T_4 &\le \int_\Omega \left(\left|(\epsilon_u)_t\right| \left|\bar{e}_u\right| + \left|\epsilon_q\right| \left|\bar{e}_q\right| \right) dx \le \|(\epsilon_u)_t\| \, \|\bar{e}_u\| + \left\|\epsilon_q\right\| \left\|\bar{e}_q\right\| \\ &\le C_4 \left(h^{2p+2} + \|\bar{e}_u\|^2 \right) + \frac{1}{4} \left\|\bar{e}_q\right\|^2. \end{aligned} \tag{3.26}$$

Estimate of T_5. Using the periodic boundary conditions, we get

$$T_5 = \sum_{i=1}^{N} -\bar{e}_u^- \bar{e}_q^+\big|_{i-1} + \bar{e}_u^- \bar{e}_q^+\big|_i = -\bar{e}_u^- \bar{e}_q^+\big|_0 + \bar{e}_u^- \bar{e}_q^+\big|_N = 0. \tag{3.27}$$

Now, combining (3.15) with (3.17), (3.23), (3.25), (3.26), and (3.27), we arrive at

$$\frac{1}{2}\frac{d}{dt}\|\bar{e}_u\|^2 + \left\|\bar{e}_q\right\|^2 \le Ch^{2p+2} + C\|\bar{e}_u\|^2 + \frac{1}{2}\left\|\bar{e}_q\right\|^2. \tag{3.28}$$

Thus, we establish the estimate

$$\frac{1}{2}\frac{d}{dt}\|\bar{e}_u\|^2 \le C_1 h^{2p+2} + C_2\|\bar{e}_u\|^2.$$

Integrating this inequality over the interval $[0, t]$ and using the estimate (2.11) (note that initially $\bar{e}_u = P_h^- u - u_h = P_h^- u - P_h^1 u$), which is due to the special choice of the initial condition yields

$$\begin{aligned}\|\bar{e}_u\|^2 &\le \|\bar{e}_u(0)\|^2 + 2tC_1 t h^{2p+2} + 2C_2\int_0^t \|\bar{e}_u(s)\|^2 ds\\ &\le C_3 h^{2p+3} + C_4 h^{2p+2} + 2C_2\int_0^t \|\bar{e}_u(s)\|^2 ds.\end{aligned}$$

Invoking the classical Gronwall inequality (see, e.g., [37]), we get $\|\bar{e}_u\|^2 \le (C_3 h^{2p+3} + C_4 h^{2p+2})e^{2C_2 t} \le Ch^{2p+2}$, $\forall\, t \in [0, T]$, which completes the proof of (3.8a). Using $e_u = \bar{e}_u + \epsilon_u$ and applying the triangle inequality and the projection result (2.5), we obtain

$$\|e_u\| = \|\bar{e}_u + \epsilon_u\| \le \|\bar{e}_u\| + \|\epsilon_u\| \le C_1 h^{p+1} + C_2 h^{p+1} \le Ch^{p+1},$$

which establishes (3.8b).

Next, we state and prove optimal L^2 error estimates for $\|(e_u)_t\|$ and $\left\|e_q\right\|$.

Theorem 3.2 *Under the assumptions of Theorem 3.1, we have*

$$\|(\bar{e}_u)_t(0)\| \le Ch^{p+1}. \tag{3.29}$$
$$\|(\bar{e}_u)_t\| \le Ch^{p+1}. \tag{3.30}$$
$$\|(e_u)_t\| \le Ch^{p+1}. \tag{3.31}$$
$$\left\|\bar{e}_q\right\| \le Ch^{p+1}. \tag{3.32}$$
$$\left\|e_q\right\| \le Ch^{p+1}. \tag{3.33}$$

Proof At time $t = 0$, (2.11) gives

$$\|\bar{e}_u(0)\| \le Ch^{p+3/2}, \tag{3.34}$$
$$\left\|\bar{e}_q(0)\right\| \le Ch^{p+3/2}, \tag{3.35}$$

since initially $\bar{e}_u = P_h^- u - u_h = P_h^- u - P_h^1 u$.

Next, we will prove (3.29). Since $u_h(x, 0) = P_h^1 u(x, 0)$, (2.10) can be written as

$$\bar{e}_u^-\big|_{i-1} = \bar{e}_q^+\big|_{i-1}, \quad \int_{I_i} \bar{e}_u v_x dx = \int_{I_i} \bar{e}_q v_x dx, \ \forall\, v \in P^p(I_i), \tag{3.36}$$

since v is a polynomial of degree at most p and thus v_x is a polynomial of degree at most $p - 1$.

Combining (3.11a) with (3.36), we obtain, at $t = 0$,

$$\int_{I_i} (\bar{e}_u)_t v dx - \int_{I_i} \theta \bar{e}_u v_x dx + \theta^- \bar{e}_u^- v^- \big|_i - \theta^- \bar{e}_u^- v^+ \big|_{i-1} + \int_{I_i} \bar{e}_u v_x dx$$
$$- \bar{e}_u^- v^- \big|_i + \bar{e}_u^- v^+ \big|_{i-1} = \int_{I_i} \theta \epsilon_u v_x dx - \int_{I_i} (\epsilon_u)_t v dx.$$

Adding and subtracting the constant $\theta_i = f'(u_i)$ to θ and using the fact that $\int_{I_i} \theta_i \epsilon_u v_x dx = \theta_i \int_{I_i} \epsilon_u v_x dx = 0$, we get

$$\int_{I_i} (\bar{e}_u)_t v dx - \int_{I_i} (\theta - \theta_i) \bar{e}_u v_x dx + (\theta^- - \theta_i) \bar{e}_u^- v^- \big|_i - (\theta^- - \theta_i) \bar{e}_u^- v^+ \big|_{i-1}$$
$$- (\theta_i - 1) \left(\int_{I_i} \bar{e}_u v_x dx - \bar{e}_u^- v^- \big|_i + \bar{e}_u^- v^+ \big|_{i-1} \right) = \int_{I_i} (\theta - \theta_i) \epsilon_u v_x dx - \int_{I_i} (\epsilon_u)_t v dx. \tag{3.37}$$

We note that it follows from (3.11b) that, at time $t = 0$,

$$\int_{I_i} \bar{e}_q w dx + \int_{I_i} \bar{e}_u w_x dx - \bar{e}_u^- w^- \big|_i + \bar{e}_u^- w^+ \big|_{i-1} = - \int_{I_i} \epsilon_q w dx. \tag{3.38}$$

Combining (3.39) with (3.38) with $w = v$, we get, at $t = 0$,

$$\int_{I_i} (\bar{e}_u)_t v dx - \int_{I_i} (\theta - \theta_i) \bar{e}_u v_x dx + (\theta^- - \theta_i) \bar{e}_u^- v^- \big|_i - (\theta^- - \theta_i) \bar{e}_u^- v^+ \big|_{i-1}$$
$$+ (\theta_i - 1) \left(\int_{I_i} \bar{e}_q v dx + \int_{I_i} \epsilon_q v dx \right) = \int_{I_i} (\theta - \theta_i) \epsilon_u v_x dx - \int_{I_i} (\epsilon_u)_t v dx. \tag{3.39}$$

Taking $v = (\bar{e}_u)_t(x, 0)$, we obtain, at time $t = 0$,

$$\int_{I_i} (\bar{e}_u)_t^2 dx = \sum_{k=1}^{3} B_{k,i}, \tag{3.40a}$$

where

$$B_{1,i} = \int_{I_i} (\theta - \theta_i) \bar{e}_u (\bar{e}_u)_{xt} dx - (\theta^- - \theta_i) \bar{e}_u^- (\bar{e}_u)_t^- \big|_i + (\theta^- - \theta_i) \bar{e}_u^- (\bar{e}_u)_t^+ \big|_{i-1}, \tag{3.40b}$$

$$B_{2,i} = - (\theta_i - 1) \left(\int_{I_i} \bar{e}_q (\bar{e}_u)_t dx + \int_{I_i} \epsilon_q (\bar{e}_u)_t dx \right) - \int_{I_i} (\epsilon_u)_t (\bar{e}_u)_t dx, \tag{3.40c}$$

$$B_{3,i} = \int_{I_i} (\theta - \theta_i) \epsilon_u (\bar{e}_u)_{xt} dx. \tag{3.40d}$$

Summing over all elements, we obtain

$$\|(\bar{e}_u)_t(0)\|^2 = \sum_{k=1}^{3} B_k, \tag{3.41}$$

where $B_k = \sum_{i=1}^{N} B_{k,i}, \ k = 1, 2, 3$. Next, we will estimate $B_k, \ k = 1, 2, 3$ separately.

Estimate of B_1. Applying the Cauchy–Schwarz inequality, (3.21), (3.22), the inverse inequalities in (2.6), and the estimate (3.34), we obtain, at $t=0$,

$$\begin{aligned} B_1 &\le \sum_{i=1}^{N}\int_{I_i}|\theta-\theta_i||\bar{e}_u||(\bar{e}_u)_{xt}|dx+\sum_{i=1}^{N}|\theta^- -\theta_i||\bar{e}_u^-||(\bar{e}_u)_t^-||_i \\ &\quad +\sum_{i=1}^{N}|\theta^- -\theta_i||\bar{e}_u^-|(|\bar{e}_u)_t^+||_{i-1} \\ &\le C_2h\int_{\Omega}|\bar{e}_u||(\bar{e}_u)_{xt}|dx+C_2h\left(\sum_{i=1}^{N}|\bar{e}_u^-|^2|_i\right)^{1/2}\left(\sum_{i=1}^{N}|(\bar{e}_u)_t^+|^2|_i\right)^{1/2} \\ &\quad + C_2h\left(\sum_{i=1}^{N}|\bar{e}_u^-|^2|_{i-1}\right)^{1/2}\left(\sum_{i=1}^{N}|(\bar{e}_u)_t^+|^2|_{i-1}\right)^{1/2} \\ &\le C_3\|\bar{e}_u(0)\|\,\|(\bar{e}_u)_t(0)\|+C_3\|\bar{e}_u(0)\|\,\|(\bar{e}_u)_t(0)\|+C_3\|\bar{e}_u(0)\|\,\|(\bar{e}_u)_t(0)\| \\ &\le C_4\|\bar{e}_u(0)\|\,\|(\bar{e}_u)_t(0)\|\le C_1h^{p+3/2}\|(\bar{e}_u)_t(0)\|. \end{aligned} \tag{3.42}$$

Estimate of B_2. Using the assumption $|f'|\le C$ on $\mathbb{R}$, applying the Cauchy–Schwarz inequality, the projection result (2.5), and the estimate (3.35), we get, at time $t=0$,

$$\begin{aligned} B_2 &\le \sum_{i=1}^{N}(|\theta_i|+1)\left(\int_{I_i}|\bar{e}_q||(\bar{e}_u)_t|dx+\int_{I_i}|\epsilon_q||(\bar{e}_u)_t|dx\right)+\int_{I_i}|(\epsilon_u)_t||(\bar{e}_u)_t|dx \\ &\le C_3\sum_{i=1}^{N}\left(\|\bar{e}_q(0)\|_{0,I_i}+\|\epsilon_q(0)\|_{0,I_i}+\|(\epsilon_u)_t(0)\|_{0,I_i}\right)\|(\bar{e}_u)_t(0)\|_{0,I_i} \\ &\le C_3\left(\|\bar{e}_q(0)\|+\|\epsilon_q(0)\|+\|(\epsilon_u)_t(0)\|\right)\|(\bar{e}_u)_t(0)\| \\ &\le C_4\left(h^{p+3/2}+h^{p+1}\right)\|(\bar{e}_u)_t(0)\|\le C_2h^{p+1}\|(\bar{e}_u)_t(0)\|. \end{aligned} \tag{3.43}$$

Estimate of B_3. Applying the Cauchy–Schwarz inequality, (3.21), the projection result (2.5), and the inverse inequality, we get, at $t=0$,

$$\begin{aligned} B_3 &\le \sum_{i=1}^{N}\int_{I_i}|\theta-\theta_i||\epsilon_u||(\bar{e}_u)_{xt}|dx\le C_1h\int_{\Omega}|\epsilon_u||(\bar{e}_u)_{xt}|dx\le C_2\|\epsilon_u(0)\|\,\|(\bar{e}_u)_t(0)\| \\ &\le C_3h^{p+1}\|\bar{e}_u(0)\|\,\|(\bar{e}_u)_t(0)\|\le C_3h^{p+1}\|(\bar{e}_u)_t(0)\|. \end{aligned} \tag{3.44}$$

Now, combining (3.41) with (3.42), (3.43), and (3.44), we conclude that

$$\|(\bar{e}_u)_t(0)\|^2\le(C_1h^{p+3/2}+C_2h^{p+1}+C_3h^{p+1})\|(\bar{e}_u)_t(0)\|\le Ch^{p+1}\|(\bar{e}_u)_t(0)\|, \tag{3.45}$$

which completes the proof of the (3.29).

Next, we will show (3.30). Taking the first time derivation of (3.11a) and (3.12b), we get

$$\int_{I_i} (\bar{e}_u)_{tt} v dx - \int_{I_i} ((\theta \bar{e}_u)_t - (\bar{e}_q)_t) v_x dx + ((\theta^- \bar{e}_u^-)_t - (\bar{e}_q^+)_t) v^- \big|_i$$
$$- ((\theta^- \bar{e}_u^-)_t - (\bar{e}_q^+)_t) v^+ \big|_{i-1} = \int_{I_i} (\theta \epsilon_u)_t v_x dx - \int_{I_i} (\epsilon_u)_t v dx, \tag{3.46a}$$
$$\int_{I_i} (\bar{e}_q)_t w dx - \int_{I_i} (\bar{e}_u)_{xt} w dx - \left((\bar{e}_u^+)_t - (\bar{e}_u^-)_t\right) w^+ \big|_{i-1} = - \int_{I_i} (\epsilon_q)_t w dx, \tag{3.46b}$$

since (3.11a) and (3.12b) are satisfied when v and w are replaced with v_t and w_t.

Choosing $v = (\bar{e}_u)_t$ in (3.46a) and $w = (\bar{e}_q)_t$ in (3.46b) then adding the resulting equations, we obtain

$$\frac{1}{2}\frac{d}{dt} \int_{I_i} (\bar{e}_u)_t^2 dx + \int_{I_i} (\bar{e}_q)_t^2 dx - \int_{I_i} (\theta \bar{e}_u)_t (\bar{e}_u)_{xt} dx + \theta^- (\bar{e}_u^-))_t^2 \big|_i - \theta^- (\bar{e}_u)_t^- (\bar{e}_u)_t^+ \big|_{i-1}$$
$$+ \theta_t^- \bar{e}_u^- (\bar{e}_u^-)_t \big|_i - \theta_t^- \bar{e}_u^- (\bar{e}_u)_t^+ \big|_{i-1} + (\bar{e}_u)_t^- (\bar{e}_q)_t^+ \big|_{i-1} - (\bar{e}_u)_t^- (\bar{e}_q)_t^+ \big|_i$$
$$= \int_{I_i} (\theta \epsilon_u)_t (\bar{e}_u)_{xt} dx - \int_{I_i} (\epsilon_u)_{tt} (\bar{e}_u)_t dx - \int_{I_i} (\epsilon_q)_t (\bar{e}_q)_t dx. \tag{3.47}$$

Adding and subtracting the constant $\theta_i = f'(u_i)$ to θ, we rewrite (3.13) as

$$\frac{1}{2}\frac{d}{dt} \int_{I_i} (\bar{e}_u)_t^2 dx + \int_{I_i} (\bar{e}_q)_t^2 dx = \sum_{k=1}^{8} S_{k,i}, \tag{3.48a}$$

where

$$S_{1,i} = \theta_i \left(\int_{I_i} (\bar{e}_u)_t (\bar{e}_u)_{xt} dx - ((\bar{e}_u)_t^-)^2 \big|_i + (\bar{e}_u)_t^- (\bar{e}_u)_t^+ \big|_{i-1} \right), \tag{3.48b}$$
$$S_{2,i} = \int_{I_i} (\theta - \theta_i) \bar{e}_u (\bar{e}_u)_x dx - (\theta^- - \theta_i)((\bar{e}_u)_t^-)^2 \big|_i + (\theta^- - \theta_i)(\bar{e}_u)_t^- (\bar{e}_u)_t^+ \big|_{i-1}, \tag{3.48c}$$
$$S_{3,i} = \int_{I_i} (\theta - \theta_i)(\epsilon_u)_t (\bar{e}_u)_{xt} dx + \theta_i \int_{I_i} (\epsilon_u)_t (\bar{e}_u)_{xt} dx, \tag{3.48d}$$
$$S_{4,i} = - \int_{I_i} (\epsilon_u)_{tt} (\bar{e}_u)_t dx - \int_{I_i} (\epsilon_q)_t (\bar{e}_q)_t dx, \tag{3.48e}$$
$$S_{5,i} = - (\bar{e}_u)_t^- (\bar{e}_q)_t^+ \big|_{i-1} + (\bar{e}_u)_t^- (\bar{e}_q)_t^+ \big|_i, \tag{3.48f}$$
$$S_{6,i} = (\theta_i)_t \left(\int_{I_i} \bar{e}_u (\bar{e}_u)_{xt} dx - \bar{e}_u^- (\bar{e}_u^-)_t \big|_i + \bar{e}_u^- (\bar{e}_u)_t^+ \big|_{i-1} \right), \tag{3.48g}$$
$$S_{7,i} = \int_{I_i} (\theta - \theta_i)_t \bar{e}_u (\bar{e}_u)_{xt} dx - (\theta^- - \theta_i)_t \bar{e}_u^- (\bar{e}_u^-)_t \big|_i + (\theta^- - \theta_i)_t \bar{e}_u^- (\bar{e}_u)_t^+ \big|_{i-1}, \tag{3.48h}$$
$$S_{8,i} = \int_{I_i} (\theta - \theta_i)_t \epsilon_u (\bar{e}_u)_{xt} dx + (\theta_i)_t \int_{I_i} \epsilon_u (\bar{e}_u)_{xt} dx. \tag{3.48i}$$

Summing the error equation (3.48) over all elements, we obtain

$$\frac{1}{2}\frac{d}{dt} \|(\bar{e}_u)_t\|^2 + \left\|(\bar{e}_q)_t\right\|^2 = \sum_{k=1}^{8} S_k, \tag{3.49}$$

where $S_k = \sum_{i=1}^{N} S_{k,i}$, $k = 1, \ldots, 8$. From here, we can easily obtain the estimate (3.30) by following the same lines as in the proofs of (3.8a). In particular, estimates for S_k, $k = 1, \ldots, 5$ are similar to the estimates for T_k, $k = 1, \ldots, 5$. To estimate S_6, S_7, and S_8 we follow the steps used to estimate T_1, T_2, and T_3, respectively. Details are omitted for the sake of conciseness.

The proof of (3.31) follows from (3.2), the estimate (3.30), and the projection result (2.5)

$$\|(e_u)_t\| = \|(\bar{e}_u)_t + (\epsilon_u)_t\| \leq \|(\bar{e}_u)_t\| + \|(\epsilon_u)_t\| \leq C_1 h^{p+1} + C_2 h^{p+1} \leq C h^{p+1},$$

which establishes (3.31).

Next, we will show (3.32). We start with the estimate derived in (3.28). Since $\frac{1}{2}\frac{d}{dt}\|\bar{e}_u\|^2 = \int_\Omega \bar{e}_u (\bar{e}_u)_t dx$, we have

$$\left\|\bar{e}_q\right\|^2 \leq C\|\bar{e}_u\|^2 + Ch^{p+1}\|\bar{e}_u\| + Ch^{p+1}\left\|\bar{e}_q\right\| - \int_\Omega \bar{e}_u (\bar{e}_u)_t dx.$$

Applying the Cauchy–Schwarz inequality and using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ with $a = \left\|\bar{e}_q\right\|$ and $b = Ch^{p+1}$, we obtain

$$\left\|\bar{e}_q\right\|^2 \leq 2C\|\bar{e}_u\|^2 + 2Ch^{p+1}\|\bar{e}_u\| + C^2 h^{2p+2} + 2\|\bar{e}_u\|\,\|(\bar{e}_u)_t\|. \tag{3.50}$$

Using the estimates (3.8a) and (3.30), we establish (3.32). Since $e_q = \bar{e}_q + \epsilon_q$, the proof of estimate (3.33) follows straightforwardly from (3.32) and the projection result (2.5)

$$\left\|e_q\right\| = \left\|\bar{e}_q + \epsilon_q\right\| \leq \left\|\bar{e}_q\right\| + \left\|\epsilon_q\right\| \leq C_1 h^{p+1} + C_2 h^{p+1} \leq C h^{p+1}. \qquad \square$$

4 Superconvergence Error Analysis

In this section, we will prove that the derivative of the LDG solution $(u_h)_x$ is $\mathcal{O}(h^{p+1})$ superconvergent to $(P_h^- u)_x$. We will also prove that the LDG solution is $\mathcal{O}(h^{p+3/2})$ superconvergent to $P_h^- u$, when p-degree piecewise polynomials with $p \geq 1$ are used.

In our analysis, we need some properties of Radau polynomials. We denote by $\tilde{L}_p$ the Legendre polynomial of degree p on $[-1, 1]$, which can be defined by the Rodrigues formula [1]

$$\tilde{L}_p(\xi) = \frac{1}{2^p p!}\frac{d^p}{d\xi^p}\left((\xi^2 - 1)^p\right), \quad -1 \leq \xi \leq 1. \tag{4.1a}$$

The Legendre polynomial satisfies the properties $\tilde{L}_p(1) = 1$, $\tilde{L}_p(-1) = (-1)^p$, and the orthogonality relation

$$\int_{-1}^{1} \tilde{L}_p(\xi)\tilde{L}_q(\xi) d\xi = \frac{2}{2p+1}\delta_{pq}, \quad \text{where } \delta_{pq} \text{ is the Kronecker symbol.} \tag{4.1b}$$

We note that the $(p+1)$-degree Legendre polynomial on $[-1, 1]$ can be written as

$$\tilde{L}_{p+1}(\xi) = \frac{(2p+2)!}{2^{p+1}[(p+1)!]^2}\xi^{p+1} + \tilde{q}_p(\xi), \quad \text{where } \tilde{q}_p \in P^p([-1, 1]).$$

Next, we define the $(p+1)$-degree right Radau polynomial on $[-1, 1]$ as

$$\tilde{R}_{p+1}(\xi) = \tilde{L}_{p+1}(\xi) - \tilde{L}_p(\xi), \quad -1 \leq \xi \leq 1. \tag{4.1c}$$

The $(p+1)$-degree right Radau polynomial on $[-1, 1]$ has $p+1$ real distinct roots $-1 < \xi_0 < \cdots < \xi_p = 1$.

Mapping the physical element I_i into the reference element $[-1, 1]$ by the standard affine mapping

$$x(\xi, h_i) = \frac{x_i + x_{i-1}}{2} + \frac{h_i}{2}\xi, \tag{4.1d}$$

we obtain the p-degree shifted Legendre and right Radau polynomials on I_i

$$L_{p,i}(x) = \tilde{L}_p\left(\frac{2x - x_i - x_{i-1}}{h_i}\right), \quad R_{p,i}(x) = \tilde{R}_p\left(\frac{2x - x_i - x_{i-1}}{h_i}\right).$$

Using the mapping (4.1d) and the orthogonality relation (4.1b), we obtain

$$\|L_{p,i}\|^2_{0,I_i} = \int_{I_i} L^2_{p,i}(x)dx = \frac{h_i}{2}\int_{-1}^{1} \tilde{L}^2_p(\xi)d\xi = \frac{h_i}{2}\frac{2}{2p+1} = \frac{h_i}{2p+1} \le h_i. \tag{4.1e}$$

Throughout this paper the roots of $R_{p+1,i}(x),\ x \in I_i$ are denoted by

$$x_{i,j} = \frac{x_i + x_{i-1}}{2} + \frac{h_i}{2}\xi_j, \quad j = 0, 1, \ldots, p. \tag{4.1f}$$

Next, we define the $(p+1)$-degree monic right Radau polynomial on I_i as

$$\psi_{p+1,i}(x) = \prod_{j=0}^{p}(x - x_{i,j}) = \frac{((p+1)!)^2}{(2p+2)!}h_i^{p+1}R_{p+1,i}(x) = c_p h_i^{p+1} R_{p+1,i}(x),$$
$$\text{where } c_p = \frac{((p+1)!)^2}{(2p+2)!}. \tag{4.1g}$$

In the next lemma, we recall a result which will be needed in our *a posteriori* error analysis.

Lemma 4.1 *The* $(p+1)$*-degree monic Radau polynomials on* I_i*,* $\psi_{p+1,i}$*, satisfy the property*

$$\|\psi_{p+1,i}\|^2_{0,I_i} = \frac{2(2p+2)}{(2p+1)(2p+3)}c_p^2 h_i^{2p+3}, \quad \textit{where } c_p = \frac{[(p+1)!]^2}{(2p+2)!}. \tag{4.2}$$

Proof The proof of this lemma can be found in [8], more precisely in its Lemma 2.1. □

4.1 Superconvergence for the Derivative of the LDG Solution

Here, we prove that $(u_h)_x$ is $\mathcal{O}(h^{p+1})$ superconvergent to $(P_h^- u)_x$ in the L^2-norm.

Theorem 4.1 *Under the assumptions of Theorem 3.1, we have, at any fixed* $t \in [0, T]$*,*

$$\|(\bar{e}_u)_x\| \le Ch^{p+1}. \tag{4.3}$$

Proof Taking $w = (\bar{e}_u)_x - (-1)^p(\bar{e}_u)_x^+\big|_{i-1} L_{p,i}(x) \in P^p(I_i)$ in (3.12b), using the property $L_{p,i}(x_{i-1}) = (-1)^p$, and applying (4.1b), we get

$$\int_{I_i} (\bar{e}_u)^2_x dx = \int_{I_i} e_q \left((\bar{e}_u)_x - (-1)^p(\bar{e}_u)_x^+\big|_{i-1} L_{p,i}(x)\right) dx,$$

since for this choice $w^+\big|_{i-1} = 0$ and $\int_{I_i}(\bar{e}_u)_x L_{p,i}dx = 0$.

Applying the Cauchy–Schwarz inequality, the inverse inequality, and the estimate (4.1e) yields

$$\begin{aligned}\|(\bar{e}_u)_x\|^2_{0,I_i} &\le \int_{I_i} \left|e_q\right| \left(|(\bar{e}_u)_x| + \left|(\bar{e}_u)^+_x\big|_{i-1}\right| \left|L_{p,i}\right|\right) dx \\ &\le \left\|e_q\right\|_{0,I_i} \left(\|(\bar{e}_u)_x\|_{0,I_i} + \left|(\bar{e}_u)^+_x\big|_{i-1}\right| \left\|L_{p,i}\right\|_{0,I_i}\right) \\ &\le \left\|e_q\right\|_{0,I_i} \left(\|(\bar{e}_u)_x\|_{0,I_i} + C_3 h_i^{-1/2} \|(\bar{e}_u)_x\|_{0,I_i} h_i^{1/2}\right) \\ &\le (1+C_3) \left\|e_q\right\|_{0,I_i} \|(\bar{e}_u)_x\|_{0,I_i} \le C_4 \left\|e_q\right\|_{0,I_i} \|(\bar{e}_u)_x\|_{0,I_i}.\end{aligned}$$

Consequently, $\|(\bar{e}_u)_x\|_{0,I_i} \le C_4 \left\|e_q\right\|_{0,I_i}$. Squaring both sides, summing over all elements, and using the estimate (3.33), we obtain

$$\|(\bar{e}_u)_x\|^2 \le C_4 \left\|e_q\right\|^2 \le Ch^{2p+2},$$

which completes the proof of (4.3). □

4.2 Superconvergence for the LDG Solution Towards $P_h^- u$

Since $\bar{e}_u \in V_h^p$ and $\bar{e}_q \in V_h^p$ are piecewise polynomials, they can be written on each element I_i as

$$\bar{e}_u = a_i(t) + \frac{x - x_i}{h_i} r_i(x,t), \quad \bar{e}_q = b_i(t) + \frac{x - x_{i-1}}{h_i} s_i(x,t), \quad x \in I_i, \tag{4.4}$$

where $a_i = \bar{e}_u^-\big|_i$, $b_i = \bar{e}_q^+\big|_{i-1}$, and $r_i(\cdot,t),\ s_i(\cdot,t) \in P^{p-1}(I_i)$.

Throughout this section, $r \in V_h^{p-1}$ and $s \in V_h^{p-1}$ denote piecewise polynomials which are defined as follows:

$$r(x,t) = r_i(x,t), \quad s(x,t) = s_i(x,t), \quad \text{on } I_i. \tag{4.5}$$

In the next lemma, we recall the following results which will be needed in our analysis.

Lemma 4.2 *If $f(x) \in C^1(I_i)$, then*

$$\int_{I_i} \frac{x - x_{i-1}}{h_i} f(x) \frac{d}{dx}\left(\frac{x - x_i}{h_i} f(x)\right) dx = \frac{1}{2h_i} \int_{I_i} f^2(x) dx. \tag{4.6a}$$

$$\int_{I_i} \frac{x - x_i}{h_i} f(x) \frac{d}{dx}\left(\frac{x - x_{i-1}}{h_i} f(x)\right) dx = -\frac{1}{2h_i} \int_{I_i} f^2(x) dx. \tag{4.6b}$$

Proof The proof of this lemma can be found in [8], more precisely in its Lemma 2.3. □

Next, we prove the following theorem which will be needed to prove our main superconvergence result.

Theorem 4.2 *Suppose that the assumptions of Theorem 3.1 are satisfied. If $r \in V_h^{p-1}$ and $s \in V_h^{p-1}$ are given in (4.5) then there exists a positive constant C independent of h such that, at any fixed $t \in [0,T]$,*

$$\|r\| \le Ch^{p+2}, \quad \|s\| \le Ch^{p+2}. \tag{4.7}$$

Proof Substituting $\bar{e}_u = a_i(t) + \frac{x-x_i}{h_i} r_i(x,t)$ into (3.12b) and choosing $w = \frac{x-x_{i-1}}{h_i} r_i$, we get

$$\int_{I_i} \left(\frac{x-x_i}{h_i} r_i\right)_x \frac{x-x_{i-1}}{h_i} r_i dx = \int_{I_i} e_q \frac{x-x_{i-1}}{h_i} r_i dx,$$

since $(a_i(t))_x = 0$, $w^+\big|_{i-1} = 0$, and $e_q = \bar{e}_q + \epsilon_q$.

Using (4.6a) with $f = r_i$, we obtain

$$\frac{1}{2h_i}\int_{I_i} r_i^2 dx = \int_{I_i} e_q \frac{x-x_{i-1}}{h_i} r_i dx.$$

Since $x - x_{i-1} \le h_i \le h$ for all $x \in I_i$, we have

$$\int_{I_i} r_i^2 dx = 2\int_{I_i} e_q (x-x_{i-1}) r_i dx \le 2h \int_{I_i} \left|e_q\right| |r_i|\, dx.$$

Summing over all elements, applying the Cauchy–Schwarz inequality, and invoking the estimate in (3.33), we obtain

$$\|r\|^2 = \sum_{i=1}^{N} \int_{I_i} r_i^2 dx \le 2h \left\|e_q\right\| \|r\| \le C h^{p+2} \|r\|,$$

which completes the proof of the first estimate in (4.7).

Next, we show the second estimate in (4.7). Substituting $\bar{e}_q = b_i(t) + \frac{x-x_{i-1}}{h_i} s_i(x,t)$ into (3.12a) and taking $v = \frac{x-x_i}{h_i} s_i$, we obtain

$$\int_{I_i} \left(\frac{x-x_{i-1}}{h_i} s_i\right)_x \frac{x-x_i}{h_i} s_i dx = \int_{I_i} (e_u)_t \frac{x-x_i}{h_i} s_i dx - \int_{I_i} \theta e_u \left(\frac{x-x_i}{h_i} s_i\right)_x dx + \theta^- \bar{e}_u^- s_i^+\big|_{i-1}, \tag{4.8}$$

where we used the fact that $(b_i(t))_x = 0$, $v^-\big|_i = 0$, $v^+\big|_{i-1} = -s_i^+\big|_{i-1}$, and $e_u = \bar{e}_u + \epsilon_u$.

Applying (4.6b) with $f = s_i$ yields

$$-\frac{1}{2h_i}\int_{I_i} s_i^2 dx = \int_{I_i} (e_u)_t \frac{x-x_i}{h_i} s_i dx - \int_{I_i} \theta e_u \left(\frac{x-x_i}{h_i} s_i\right)_x dx + \theta^- \bar{e}_u^- s_i^+\big|_{i-1}. \tag{4.9}$$

Consequently, we have

$$\int_{I_i} s_i^2 dx = -2\int_{I_i} (e_u)_t (x-x_i) s_i dx + 2\int_{I_i} \theta e_u \left((x-x_i)s_i\right)_x dx - 2h_i \theta^- \bar{e}_u^- s_i^+\big|_{i-1}. \tag{4.10}$$

Adding and subtracting the constant $\theta_i = f'(u(x_{i-1},t))$ to θ then summing over all elements, we obtain

$$\|s\|^2 = \sum_{k=1}^{4} A_k, \tag{4.11a}$$

where $A_k = \sum_{i=1}^N A_{k,i},\ k = 1, \ldots, 4$ and

$$A_{1,i} = -2\int_{I_i} (e_u)_t (x-x_i) s_i dx, \tag{4.11b}$$

$$A_{2,i} = 2\int_{I_i} (\theta - \theta_i) e_u \left((x-x_i)s_i\right)_x dx, \tag{4.11c}$$

$$A_{3,i} = 2\theta_i \left(\int_{I_i} e_u \left((x - x_i)s_i\right)_x dx - h_i \bar{e}_u^- s_i^+ \big|_{i-1} \right), \tag{4.11d}$$

$$A_{4,i} = -2h_i(\theta^- - \theta_i)\bar{e}_u^- s_i^+ \big|_{i-1}. \tag{4.11e}$$

Next, we will estimate each of these terms separately.

Estimate of A_1. Using the fact that $x - x_i \le h$ for $x \in I_i$, applying the Cauchy–Schwarz inequality, and using the estimate (3.31), we get

$$A_1 \le 2\sum_{i=1}^{N} \int_{I_i} \left|(e_u)_t\right|\left|x - x_i\right|\left|s_i\right| dx \le 2h \sum_{i=1}^{N} \int_{I_i} \left|(e_u)_t\right|\left|s_i\right| dx$$
$$\le 2h \left\|(e_u)_t\right\| \|s\| \le C_1 h^{p+2} \|s\|. \tag{4.12}$$

Estimate of A_2. Applying (3.21), using the Cauchy–Schwarz inequality, the inverse inequality, the estimate $x - x_i \le h_i$ for $x \in I_i$, and invoking the estimate (3.8b), we obtain

$$A_2 \le 2\sum_{i=1}^{N} \int_{I_i} \left|\theta - \theta_i\right|\left|e_u\right| \left| ((x - x_i)s_i)_x \right| dx \le 2h \sum_{i=1}^{N} \int_{I_i} \left|e_u\right| \left| ((x - x_i)s_i)_x \right| dx$$
$$\le 2h \sum_{i=1}^{N} \|e_u\|_{0,I_i} \left\| ((x - x_i)s_i)_x \right\|_{0,I_i} \le 2hC_1 \sum_{i=1}^{N} \|e_u\|_{0,I_i} h_i^{-1} \|(x - x_i)s_i\|_{0,I_i}$$
$$\le 2hC_1 \sum_{i=1}^{N} \|e_u\|_{0,I_i} \|s_i\|_{0,I_i} \le 2hC_1 \|e_u\| \|s\| \le C_2 h^{p+2} \|s\|. \tag{4.13}$$

Estimate of A_3. Since $e_u = \bar{e}_u + \epsilon_u$ and ϵ_u is orthogonal to $((x - x_i)s_i)_x \in P^{p-1}(I_i)$ (due to the properties in (2.4a)), A_3 simplifies to

$$A_3 = 2\sum_{i=1}^{N} \theta_i \left(\int_{I_i} \bar{e}_u \left((x - x_i)s_i\right)_x dx - h_i \bar{e}_u^- s_i^+ \big|_{i-1} \right). \tag{4.14}$$

Taking $w = (x - x_i)s_i$ in (3.11b) yields

$$\int_{I_i} e_q (x - x_i)s_i dx + \int_{I_i} \bar{e}_u \left((x - x_i)s_i\right)_x dx - h_i \bar{e}_u^- s_i^+ \big|_{i-1} = 0,$$

since $w^-\big|_i = 0$ and $w^+\big|_{i-1} = -s_i^+\big|_{i-1}$. Multiplying by θ_i and summing over all elements, we get

$$\sum_{i=1}^{N} \theta_i \left(\int_{I_i} \bar{e}_u \left((x - x_i)s_i\right)_x dx - h_i \bar{e}_u^- s_i^+ \big|_{i-1} \right) = -\sum_{i=1}^{N} \theta_i \int_{I_i} e_q (x - x_i)s_i dx. \tag{4.15}$$

Combining (4.14) and (4.15), we obtain

$$A_3 = -2\sum_{i=1}^{N} \theta_i \int_{I_i} e_q (x - x_i)s_i dx.$$

Using the assumption $|f'| \le C_1$ on $\mathbb{R}$, the estimate $x - x_i \le h$ for $x \in I_i$, the Cauchy–Schwarz inequality, and the estimate (3.33) gives

$$A_3 \le 2\sum_{i=1}^{N} |\theta_i| \int_{I_i} |e_q||x-x_i||s_i|dx \le 2C_1 h \sum_{i=1}^{N}\int_{I_i} |e_q||s_i|dx$$
$$\le 2C_1 h \sum_{i=1}^{N} \left\|e_q\right\|_{0,I_i} \|s_i\|_{0,I_i} \le 2C_1 h \left\|e_q\right\| \|s\| \le C_3 h^{p+2}\|s\|. \tag{4.16}$$

Estimate of A_4. Using (3.22), we have

$$A_4 \le 2\sum_{i=1}^{N} h_i \left|(\theta^- - \theta_i)\right|_{i-1}\left|\left|\bar{e}_u^- s_i^+\right|_{i-1}\right| \le 2h \sum_{i=1}^{N} h_i \left|\bar{e}_u^- s_i^+\right|_{i-1}\right|. \tag{4.17}$$

On the other hand, we take $w = (x - x_i)s_i$ in (3.11b) to get

$$\int_{I_i} e_q (x-x_i)s_i dx + \int_{I_i} \bar{e}_u \left((x-x_i)s_i\right)_x dx - h_i \bar{e}_u^- s_i^+\big|_{i-1} = 0,$$

since $w^-\big|_i = 0$ and $w^+\big|_{i-1} = -s_i^+\big|_{i-1}$. Thus,

$$\sum_{i=1}^{N} h_i \left|\bar{e}_u^- s_i^+\big|_{i-1}\right| = \left|\sum_{i=1}^{N}\int_{I_i} e_q (x-x_i)s_i dx + \sum_{i=1}^{N}\int_{I_i} \bar{e}_u \left((x-x_i)s_i\right)_x dx\right|$$
$$\le \sum_{i=1}^{N}\int_{I_i} |e_q||x-x_i||s_i|dx + \sum_{i=1}^{N}\int_{I_i} |\bar{e}_u|\left|\left((x-x_i)s_i\right)_x\right|dx. \tag{4.18}$$

Combining (4.17) and (4.18), we obtain

$$A_4 \le 2h\sum_{i=1}^{N}\left(\int_{I_i} |e_q||x-x_i||s_i|dx + \int_{I_i} |\bar{e}_u|\left|\left((x-x_i)s_i\right)_x\right|dx\right).$$

Applying the Cauchy–Schwarz inequality, the inverse inequality, using the fact that $x - x_i \le h_i$ for $x \in I_i$, and invoking the estimates (3.33) and (3.8a) yields

$$A_4 \le 2h\sum_{i=1}^{N}\left(\left\|e_q\right\|_{0,I_i} \|(x-x_i)s_i\|_{0,I_i} + \|\bar{e}_u\|_{0,I_i}\left\|\left((x-x_i)s_i\right)_x\right\|_{0,I_i}\right)$$
$$\le 2h\sum_{i=1}^{N}\left(h_i\left\|e_q\right\|_{0,I_i}\|s_i\|_{0,I_i} + C_1 h_i^{-1}\|\bar{e}_u\|_{0,I_i}\|(x-x_i)s_i\|_{0,I_i}\right)$$
$$\le 2h^2\sum_{i=1}^{N}\left\|e_q\right\|_{0,I_i}\|s_i\|_{0,I_i} + 2hC_1\sum_{i=1}^{N}\|\bar{e}_u\|_{0,I_i}\|s_i\|_{0,I_i}$$
$$\le 2h^2\left\|e_q\right\|\|s\| + 2hC_1\|\bar{e}_u\|\|s\|$$
$$\le C_2 h^{p+3}\|s\| + C_3 h^{p+2}\|s\| \le C_4 h^{p+2}\|s\|. \tag{4.19}$$

Now, combining (4.11a) with the estimates (4.12), (4.13), (4.16), and (4.19), we deduce that

$$\|s\|^2 \le (C_1 + C_2 + C_3 + C_4)h^{p+2}\|s\| = Ch^{p+2}\|s\|,$$

which completes the proof of (4.7). □

Now, we are ready to prove that u_h is $\mathcal{O}(h^{p+3/2})$ superconvergent to $P_h^- u$ in the L^2-norm.

Theorem 4.3 *Suppose that the assumptions of Theorem 3.1 are satisfied. Then there exists a positive constant C independent of h such that, at any fixed $t \in [0, T]$,*

$$\|\bar{e}_u\| \leq Ch^{p+3/2}. \tag{4.20}$$

Proof We start from the error equation (3.15). We will be using the estimates (3.17), (3.23), and (3.27). However, we will use the superconvergence result (4.3) to obtain better estimates of (3.25) and (3.26).

Estimate of T_3. We start from (3.24). Using the estimate (4.3) and the projection result (2.5), we obtain

$$T_3 \leq C_1 h(C_2 h^{p+1})(C_4 h^{p+1}) \leq C_3 h^{2p+3}. \tag{4.21}$$

Estimate of T_4. Substituting the definitions of $\bar{e}_u$ and $\bar{e}_q$, given in (4.4), into (3.14e), and using the fact that $(\epsilon_u)_t$ and ϵ_q are orthogonal to any piecewise constant functions (due to the properties given in (2.4)), we get

$$T_4 = -\sum_{i=1}^{N} \int_{I_i} \left((\epsilon_u)_t \frac{x - x_i}{h_i} r_i + \epsilon_q \frac{x - x_{i-1}}{h_i} s_i \right) dx. \tag{4.22}$$

Using the fact that $|x - x_i| \leq h_i$ and $|x - x_{i-1}| \leq h_i$ for all $x \in I_i$, then applying the Cauchy–Schwarz inequality, the projection result (2.5), and (4.7), we get

$$T_4 \leq \sum_{i=1}^{N} \int_{I_i} \left(\left|(\epsilon_u)_t\right| \left|r_i\right| + \left|\epsilon_q\right| \left|s_i\right| \right) dx \leq \|(\epsilon_u)_t\| \, \|r\| + \left\|\epsilon_q\right\| \, \|s\| \leq C_4 h^{2p+3}. \tag{4.23}$$

Now, combining (3.15) with (3.17), (3.23), (4.21), (4.23), and (3.27), we obtain

$$\frac{1}{2}\frac{d}{dt} \|\bar{e}_u\|^2 + \left\|\bar{e}_q\right\|^2 \leq C_1 h^{2p+3} + C_2 \|\bar{e}_u\|^2 .$$

Integrating with respect to time and using (2.11) (note that initially $\bar{e}_u = P_h^- u - u_h = P_h^- u - P_h^1 u$), we get

$$\begin{aligned} \|\bar{e}_u\|^2 + \int_0^t \left\|\bar{e}_q(s)\right\|^2 ds &\leq \|\bar{e}_u(0)\|^2 + C_1 t h^{2p+3} + C_2 \int_0^t \|\bar{e}_u(s)\|^2 ds \\ &\leq C_3 h^{2p+3} + C_2 \int_0^t \|\bar{e}_u(s)\|^2 ds. \end{aligned}$$

By the continuous Gronwall inequality (see, e.g., [37]), we conclude that,

$$\|\bar{e}_u\|^2 + \int_0^t \left\|\bar{e}_q(s)\right\|^2 ds \leq C_3 h^{2p+3} e^{C_2 t} \leq C h^{2p+3}, \quad \forall\, t \in [0, T],$$

where C is independent of h. Thus, we completed the proof of the theorem. □

Justification of the a priori assumption: Now, to complete the proof of Theorem 4.3, let us justify the a priori assumption (3.4). We will follow the same arguments used in [34,39]. First of all, the a prior assumption is satisfied at $t = 0$ since, initially (due to (2.11)),

$$\|\bar{e}_u\| = \left\|P_h^- u - u_h\right\| = \left\|P_h^- u - P_h^1 u\right\| \leq Ch^{p+3/2} \leq Ch^{3/2}, \quad \text{for } p \geq 1.$$

Next, for $p \geq 1$, we can choose h small enough so that $Ch^{p+1} < \frac{1}{2}h^{3/2}$, where C is the constant in (3.8a) determined by the final time T. Define $t^* = \sup\{s \leq T : \|\bar{e}_u(t)\| \leq h^{3/2},\ t \in [0, s]\}$, then we have $\|\bar{e}_u(t^*)\| = h^{3/2}$ by continuity if $t^* < T$. On the other hand, our proof implies that (3.8a) holds for $t \le t^*$, in particular $\|\bar{e}_u(t^*)\| \le Ch^{p+1} < \frac{1}{2}h^{3/2}$. This is a contradiction if $t^* < T$. Hence $t^* \ge T$ and our a priori assumption (3.4) is justified.

Remark 4.1 We remark that when $f(u)$ is linear, we have $f'(u)$ is a constant. Therefore, $T_{2,i} = 0$. Consequently, the a priori assumption (3.4) is unnecessary.

4.3 Superconvergence Towards the Right Radau Interpolating Polynomial

Here, we prove an important superconvergence result towards the p-degree right Radau interpolating polynomial. This result will be used to show that the actual error can be split into a significant part (proportional to the $(p+1)$-degree right Radau polynomial) and a less significant part (converges to zero in the L^2-norm at $\mathcal{O}(h^{p+3/2})$ rate).

We first define two interpolation operators π and $\hat{\pi}$ [8]. The projection π is defined as follows: For any function u, $\pi u\big|_{I_i} \in P^p(I_i)$ and interpolates u at the roots of the $(p+1)$-degree right Radau polynomial shifted to I_i, $x_{i,j}$, $j = 0, 1, \ldots, p$. Next, the interpolation operator $\hat{\pi}$ is such that $\hat{\pi} u\big|_{I_i} \in P^{p+1}(I_i)$ and is defined as follows: $\hat{\pi} u\big|_{I_i}$ interpolates u at $x_{i,j}$, $j = 0, 1, \ldots, p$, and at an additional point $\bar{x}_i$ in I_i with $\bar{x}_i \ne x_{i,j}$, $j = 0, 1, \ldots, p$.

Remark 4.2 The operator $\hat{\pi}$ is only needed for technical reasons in the proof of the error estimates. We also would like to emphasize that the polynomial $\hat{\pi} u$ depends on the additional point $\bar{x}_i$. For clarity of presentation, we may choose $\bar{x}_i = x_{i-1}$ (left-end point of I_i). We note that $\bar{x}_i \ne x_{i,j}$, $j = 0, 1, \ldots, p$. Moreover, we can easily verify that $\hat{\pi} u$ is given by

$$\hat{\pi} u = \pi u + c_i(t)\psi_{p+1,i}(x), \tag{4.24}$$

since both $\psi_{p+1,i}(x)$ vanish at the Radau points $x_{i,k}$, $k = 0, 1, \ldots, p$. Using (4.24) and the fact that $\hat{\pi} u(x_{i-1}, t) = u(x_{i-1}, t)$, we find $c_i(t) = \frac{u(x_{i-1},t) - \pi u(x_{i-1},t)}{\psi_{p+1,i}(x_{i-1})}$. We note that $\psi_{p+1,i}(x_{i-1}) \ne 0$.

In the next lemma, we recall some properties of P_h^- and π [8], which play important roles in our *a posteriori* error analysis. In particular, we show that the interpolation error can be divided into a significant and a less significant parts.

Lemma 4.3 *Let $u \in H^{p+2}(I_i)$, $t \in [0, T]$ fixed, and P_h^- and π as defined above. Then the interpolation error can be split as:*

$$u - \pi u = \phi_i + \gamma_i, \quad \phi_i = \alpha_i(t)\psi_{p+1,i}(x), \quad \gamma_i = u - \hat{\pi} u, \quad \text{on } I_i, \tag{4.25a}$$

where $\alpha_i(t)$ is the coefficient of x^{p+1} in the $(p+1)$-degree polynomial $\hat{\pi} u$ and

$$\begin{aligned} \|\phi_i\|_{k,I_i} &\le C h_i^{p+1-k} \|u\|_{p+1,I_i}, \quad 0 \le k \le p, \\ \|\gamma_i\|_{k,I_i} &\le C h_i^{p+2-k} \|u\|_{p+2,I_i}, \quad 0 \le k \le p+1. \end{aligned} \tag{4.25b}$$

Moreover,

$$\left\|\pi u - P_h^- u\right\|_{0,I_i} \le C h_i^{p+2} \|u\|_{p+2,I_i}. \tag{4.26}$$

Proof The proof of this lemma can be found in [8], more precisely in its Lemma 2.4.

Now, we are ready to prove our main superconvergence result towards the right Radau interpolating polynomial. Furthermore, we show that the significant part of the discretization error for the LDG solution is proportional to the $(p+1)$-degree right Radau polynomial.

Theorem 4.4 *Under the assumptions of Theorem 3.1, there exists a constant C such that*

$$\|u_h - \pi u\| \leq C h^{p+3/2}, \tag{4.27}$$

and

$$e_u(x,t) = \alpha_i(t)\psi_{p+1,i}(x) + \omega_i(x,t), \quad \text{on } I_i, \tag{4.28a}$$

where $\omega_i = \gamma_i + \pi u - u_h$, and

$$\sum_{i=1}^{N} \left\| \partial_x^k \omega_i \right\|_{0,I_i}^2 \leq C h^{2(p-k)+3}, \quad k = 0, 1, \quad \forall\, t \in [0, T]. \tag{4.28b}$$

Proof Adding and subtracting $P_h^- u$ to $u_h - \pi u$, we write

$$u_h - \pi u = (u_h - P_h^- u) + (P_h^- u - \pi u) = -\bar{e}_u + P_h^- u - \pi u.$$

Taking the L^2-norm and applying the triangle inequality, we get

$$\|u_h - \pi u\| \leq \|\bar{e}_u\| + \left\| P_h^- u - \pi u \right\|.$$

Using the estimates (4.20) and (4.26), we establish (4.27).

Adding and subtracting πu to e_u , we get

$$e_u = u - \pi u + \pi u - u_h.$$

Furthermore, one can split the interpolation error $u - \pi u$ on I_i as in (4.25a) to obtain

$$e_u = \phi_i + \gamma_i + \pi u - u_h = \phi_i + \omega_i, \quad \text{where } \omega_i = \gamma_i + \pi u - u_h. \tag{4.29}$$

Next, we use the Cauchy–Schwarz inequality and the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ to write

$$\|\omega_i\|_{0,I_i}^2 = \|\gamma_i\|_{0,I_i}^2 + 2\int_{I_i} \gamma_i(\pi u - u_h)dx + \|\pi u - u_h\|_{0,I_i}^2 \leq 2\left(\|\gamma_i\|_{0,I_i}^2 + \|\pi u - u_h\|_{0,I_i}^2\right).$$

Summing over all elements and applying (4.25b) and (4.27) yields

$$\sum_{i=1}^{N} \|\omega_i\|_{0,I_i}^2 \leq C_1 h^{2p+4} + C_2 h^{2p+3} \leq C h^{2p+3},$$

which complete the proof of (4.28b) for $k = 0$. Next, we use the Cauchy–Schwarz inequality and the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ to get

$$\|(\omega_i)_x\|_{0,I_i}^2 = \int_{I_i} \left((\gamma_i + \pi u - u_h)_x\right)^2 dx \leq 2\left(\|(\gamma_i)_x\|_{0,I_i}^2 + \|(\pi u - u_h)_x\|_{0,I_i}^2\right).$$

Using the inverse inequality $\|(\pi u - u_h)_x\|_{0,I_i} \leq c h^{-1} \|\pi u - u_h\|_{0,I_i}$, we obtain the estimates

$$\|(\omega_i)_x\|_{0,I_i}^2 \leq C\left(\|(\gamma_i)_x\|_{0,I_i}^2 + h^{-2} \|\pi u - u_h\|_{0,I_i}^2\right).$$

Summing over all elements and applying (4.27) and the standard error estimate (4.25b), we establish (4.28b) for $k = 1$. □

5 Global *a Posteriori* Error Estimation

In this section, we present a residual-based a posteriori error estimator for the LDG method for nonlinear convection–diffusion problems. It is computed by solving a local steady problem with no boundary conditions on each element. The proposed LDG error estimate is shown to converge to the true spatial error as $h \to 0$.

In order to find a procedure for computing the *a posteriori* error estimate for nonlinear convection–diffusion problems, we replace u by $u_h + e_u$ and q by $q_h + e_q$ in the second equation of (2.1) i.e., $q - u_x = 0$ to obtain

$$(e_u)_x = q_h - (u_h)_x + e_q, \quad x \in I_i. \tag{5.1}$$

Multiplying (5.1) by a smooth test function v and integrating over I_i, we get

$$\int_{I_i} (e_u)_x v dx = \int_{I_i} \left(q_h - (u_h)_x + e_q\right) v dx. \tag{5.2}$$

Substituting (4.28a) into the left-hand side of (5.2) and choosing $v = L_{p,i}(x)$ yields

$$\left(\int_{I_i} \left(\psi_{p+1,i}\right)' L_{p,i} dx\right) \alpha_i = \int_{I_i} \left(q_h - (u_h)_x + e_q - (\omega_i)_x\right) L_{p,i} dx. \tag{5.3}$$

Next, we compte $\int_{I_i} \left(\psi_{p+1,i}\right)' L_{p,i} dx$. Using the definition of $\psi_{p+1,i}$ given by (4.1g) and the orthogonality relation (4.1b), we get

$$\int_{I_i} \left(\psi_{p+1,i}\right)' L_{p,i} dx = c_p h_i^{p+1} \int_{I_i} \left(L'_{p+1,i} - L'_{p,i}\right) L_{p,i} dx = c_p h_i^{p+1} \int_{I_i} L'_{p+1,i} L_{p,i} dx,$$

since $L'_{p,i}$ is a polynomial of degree $p-1$ on I_i.

Using a simple integration by parts and the orthogonality relation (4.1b), we obtain

$$\begin{aligned}
\int_{I_i} \left(\psi_{p+1,i}\right)' L_{p,i} dx &= c_p h_i^{p+1} \Big(L_{p+1,i}(x_i) L_{p,i}(x_i) - L_{p+1,i}(x_{i-1}) L_{p,i}(x_{i-1}) \\
&\quad - \int_{I_i} L_{p+1,i} L'_{p,i} dx \Big) \\
&= c_p h_i^{p+1} \left(L_{p+1,i}(x_i) L_{p,i}(x_i) - L_{p+1,i}(x_{i-1}) L_{p,i}(x_{i-1}) \right).
\end{aligned}$$

By the definition of the Legendre polynomial, we have $\tilde{L}_p(1) = 1$ and $\tilde{L}_p(-1) = (-1)^p$. Therefore, the shifted Legendre polynomials on I_i satisfy $L_{p+1,i}(x_i) = L_{p,i}(x_i) = 1$, $L_{p,i}(x_{i-1}) = (-1)^p$, and $L_{p+1,i}(x_{i-1}) = (-1)^{p+1}$. Thus,

$$\int_{I_i} \left(\psi_{p+1,i}\right)' L_{p,i} dx = c_p h_i^{p+1} \left((1)(1) - (-1)^{p+1}(-1)^p\right) = 2 c_p h_i^{p+1}. \tag{5.4}$$

Using (5.4), we obtain from (5.3)

$$\alpha_i(t) = \frac{1}{2 c_p h_i^{p+1}} \int_{I_i} \left(q_h - (u_h)_x + e_q - (\omega_i)_x\right) L_{p,i} dx. \tag{5.5}$$

Our error estimate procedure consists of approximating the true error on each element I_i by the leading term as

$$e_u(x,t) \approx E_u(x,t) = a_i(t) \psi_{p+1,i}(x), \quad x \in I_i, \tag{5.6}$$

where the coefficient of the leading term of the error at fixed time t, $a_i(t)$, is obtained from the coefficient $\alpha_i(t)$ defined in (5.5) by neglecting the terms ω_i and e_q, i.e.,

$$a_i(t) = \frac{1}{2c_p h_i^{p+1}} \int_{I_i} (q_h - (u_h)_x)\, L_{p,i} dx. \tag{5.7}$$

We note that E_u is a computable quantity since it only depends on the LDG solutions u_h and q_h. Thus, our LDG error estimate is computationally simple and is obtained by solving a local steady problem with no boundary conditions on each element.

An accepted efficiency measure of *a posteriori* error estimate is the effectivity index. In this paper, we use the global effectivity index $\Theta_u(t) = \frac{\|E_u\|}{\|e_u\|}$. Ideally, the global effectivity index should stay close to one and should converge to one under mesh refinement.

The main results of this section are stated in the following theorem. In particular, we will show that the error estimate E_u converges to the true error e_u in the L^2-norm as $h \to 0$. Furthermore, we will prove the convergence to unity of the global effectivity index $\Theta_u(t)$ under mesh refinement.

Theorem 5.1 *Suppose that the assumptions of Theorem 3.1 are satisfied. If α_i and a_i are given by (5.5) and (5.7), respectively, and $E_u(x,t) = a_i(t)\psi_{p+1,i}(x)$, then there exists a constant C independent of h such that*

$$\|e_u - E_u\| \le C h^{p+3/2}. \tag{5.8}$$

Consequently, we have

$$\big|\, \|e_u\| - \|E_u\| \,\big| \le C_1 h^{p+3/2}. \tag{5.9}$$

Finally, if there exists a constant $c = c(u) > 0$ independent of h with

$$\|e_u\| \ge c h^{p+1}, \tag{5.10}$$

then, at a fixed time t, the global effectivity index in the L^2 converges to unity at $\mathcal{O}(h^{1/2})$ rate i.e.,

$$\Theta_u(t) = \frac{\|E_u\|}{\|e_u\|} = 1 + \mathcal{O}(h^{1/2}). \tag{5.11}$$

Proof First, we will prove (5.8). Since $e_u = \alpha_i \psi_{p+1,i} + \omega_i$, and $E_u = a_i \psi_{p+1,i}$, we have

$$\|e_u - E_u\|_{0,I_i}^2 = \left\|(\alpha_i - a_i)\psi_{p+1,i} + \omega_i\right\|_{0,I_i}^2 \le 2(\alpha_i - a_i)^2 \left\|\psi_{p+1,i}\right\|_{0,I_i}^2 + 2\left\|\omega_i\right\|_{0,I_i}^2,$$

where we used the inequality $(a+b)^2 \le 2a^2 + 2b^2$. Summing over all elements and applying the estimate (4.28b) with $k = 0$ yields

$$\begin{aligned} \|e_u - E_u\|^2 = \sum_{i=1}^N \|e_u - E_u\|_{0,I_i}^2 &\le 2\sum_{i=1}^N (\alpha_i - a_i)^2 \left\|\psi_{p+1,i}\right\|_{0,I_i}^2 + 2\sum_{i=1}^N \|\omega_i\|_{0,I_i}^2 \\ &\le 2\sum_{i=1}^N (\alpha_i - a_i)^2 \left\|\psi_{p+1,i}\right\|_{0,I_i}^2 + C_1 h^{2p+3}. \end{aligned} \tag{5.12}$$

Next, we will estimate $\sum_{i=1}^N (\alpha_i - a_i)^2 \left\|\psi_{p+1,i}\right\|_{0,I_i}^2$. Subtracting (5.7) from (5.5) and applying the triangle inequality, we get

$$|\alpha_i - a_i| = \left|\frac{1}{2c_p h_i^{p+1}} \int_{I_i} \left(e_q - (\omega_i)_x\right) L_{p,i} dx\right| \le \frac{1}{2c_p h_i^{p+1}} \int_{I_i} \left(|e_q| + |(\omega_i)_x|\right) |L_{p,i}| dx.$$

Using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ yields

$$(\alpha_i - a_i)^2 \leq \frac{1}{2c_p^2 h_i^{2p+2}} \left[\left(\int_{I_i} |e_q||L_{p,i}|dx \right)^2 + \left(\int_{I_i} |(\omega_i)_x||L_{p,i}|dx \right)^2 \right].$$

Applying the Cauchy–Schwarz inequality and the estimate (4.1e), we obtain

$$(\alpha_i - a_i)^2 \leq \frac{\|L_{p,i}\|_{0,I_i}^2}{2c_p^2 h_i^{2p+2}} \left(\|e_q\|_{0,I_i}^2 + \|(\omega_i)_x\|_{0,I_i}^2 \right) \leq \frac{1}{2c_p^2 h_i^{2p+1}} \left(\|e_q\|_{0,I_i}^2 + \|(\omega_i)_x\|_{0,I_i}^2 \right). \tag{5.13}$$

Multiplying by $\|\psi_{p+1,i}\|_{0,I_i}^2$ and using (4.2) yields

$$\begin{aligned}(\alpha_i - a_i)^2 \|\psi_{p+1,i}\|_{0,I_i}^2 &\leq \frac{\|\psi_{p+1,i}\|_{0,I_i}^2}{2c_p^2 h_i^{2p+1}} \left(\|e_q\|_{0,I_i}^2 + \|(\omega_i^-)_x\|_{0,I_i}^2 \right) \\ &= C_p h_i^2 \left(\|e_q\|_{0,I_i}^2 + \|(\omega_i)_x\|_{0,I_i}^2 \right),\end{aligned}$$

where $C_p = \frac{(2p+2)}{(2p+1)(2p+3)}$.

Finally, summing over all elements and using the fact that $h = \max_{1\leq i\leq N} h_i$, we arrive at

$$\sum_{i=1}^{N} (\alpha_i - a_i)^2 \|\psi_{p+1,i}\|_{0,I_i}^2 \leq C_p h^2 \left(\|e_q\|^2 + \sum_{i=1}^{N} \|(\omega_i)_x\|_{0,I_i}^2 \right).$$

Applying the estimates (3.33) and (4.28b), we establish

$$\sum_{i=1}^{N} (\alpha_i - a_i)^2 \|\psi_{p+1,i}\|_{0,I_i}^2 \leq C h^{2p+3}. \tag{5.15}$$

Now, combining (5.12) with (5.15) yields

$$\|e_u - E_u\|^2 \leq C_1 h^{2p+3} + C_2 h^{2p+3} \leq C h^{2p+3},$$

which completes the proof of (5.8).

In order to prove (5.9), we use the reverse triangle inequality to have

$$\left| \|E_u\| - \|e_u\| \right| \leq \|E_u - e_u\|. \tag{5.16}$$

Combining (5.16) and (5.8) completes the proof of (5.9).

In order to prove (5.11), we divide the inequality in (5.16) by $\|e_u\|$ and we use the estimate (5.8) and the assumption (5.10) to obtain

$$\left| \frac{\|E_u\|}{\|e_u\|} - 1 \right| \leq \frac{\|E_u - e_u\|}{\|e_u\|} \leq \frac{C_1 h^{p+3/2}}{c h^{p+1}} \leq C h^{1/2}.$$

Therefore, $\frac{\|E_u\|}{\|e_u\|} = 1 + \mathcal{O}(h^{1/2})$, which completes the proof of (5.11). □

In the previous theorem, we proved that the residual-based *a posteriori* error estimate converges to the true spatial error at $\mathcal{O}(h^{p+3/2})$ rate. We also proved that the global effectivity index in the L^2-norm converges to unity at $\mathcal{O}(h^{1/2})$ rate. We note that $||E_u||$ is computationally efficient because our LDG error estimate is obtained by solving a local steady problem with no boundary conditions on each element. Additionally, (5.11) indicates that the computable quantity $\|E_u\|$ provides an asymptotically exact a posteriori estimator on the actual error $\|e_u\|$. Finally, we would like to mention that the computable quantity $u_h + E_u$ converge to the exact solution u at $\mathcal{O}(h^{p+3/2})$ rate. We emphasize that this accuracy enhancement is achieved by adding the error estimate to the approximate solution only once at the end of the computation i.e., at $t = T$.

Remark 5.1 The assumption in (5.10) imply that the term of order $\mathcal{O}(h^{p+1})$ is present in the error. Even though the proof of (5.11) is valid under the assumption (5.10), our computational results given in the next section suggest that the global effectivity index Θ_u in the L^2-norm converges to unity with at least $\mathcal{O}(h)$ rate. Thus, the proposed a posteriori error estimation technique is an excellent measure of the error and (5.11) indicates that our a posteriori error estimator is asymptotically exact.

6 Numerical Examples

The purpose of this section is to numerically validate our superconvergence results and the global convergence of the proposed residual-based *a posteriori* error estimates. The initial condition is determined by $u_h(x,0) = P_h^1 u(x,0)$. Temporal integration is performed by the fourth-order classical implicit Runge–Kutta method. A time step Δt is chosen so that temporal errors are small relative to spatial errors. We do not discuss the influence of the time discretization error in this paper.

Example 6.1 In this example, we consider the following nonlinear convection–diffusion problem subject to the periodic boundary condition

$$\begin{cases} u_t + (u^3+u)_x = u_{xx} + \big((2+3e^{\sin(x+t)})\cos(x+t) - \sin(x+t) + \cos^2(x+t)\big) \\ e^{\sin(x+t)}, \ x \in [0,2\pi], \ t \in [0,1], \\ u(x,0) = e^{\sin(x)}, \quad x \in [0,2\pi]. \end{cases} \tag{6.1}$$

The exact solution is given by $u(x,t) = e^{\sin(x+t)}$. Since $f'(u) = 3u^2 + 1 \geq 0$, we use the upwind flux $\hat{f}\big|_i = f(u_h^-)\big|_i$. We solve this problem using the LDG method on uniform meshes obtained by partitioning the computational domain $[0, 2\pi]$ into N subintervals with $N = 10, 20, 30, 40, 50, 60, 70, 80$ and using the spaces P^p with $p = 1-4$. Figure 1 shows the L^2 errors $||e_u||$ and $||e_q||$ at $t = 1$ with log-log scale as well as their orders of convergence. The errors are plotted in log scale just for easy visualization. For each P^p space, we fit, in a least-squares sense, the data sets with a linear function and then calculate from the fitting result the slopes of the fitting lines. The slopes of the fitting lines are shown on the graph. These results indicate that $||e_u||$ and $||e_q||$ are both $\mathcal{O}(h^{p+1})$. Thus, the error estimates proved in this paper are optimal in the exponent of the parameter h. In Fig. 2, we report the L^2-norm of the errors $||\bar{e}_u||$ and $||u_h - \pi u||$ as well as their orders of convergence. We observe that $||\bar{e}_u|| = \mathcal{O}(h^{p+2})$ and $||u_h - \pi u|| = \mathcal{O}(h^{p+2})$. Thus, the LDG solution u_h is superconvergent with order $p+2$ to the particular projection $P_h^- u$ and to the interpolating right Radau polynomial πu. Although the superconvergence rate is proved to be of order $p+3/2$, our computational results indicate that the observed numerical convergence rate is higher than the theoretical rate. In Fig. 3 we present the global errors $\big|||e_u|| - ||E_u||\big|$ and $||e_u - E_u||$ at $t = 1$. These results indicate that $\big|||e_u|| - ||E_u||\big| = \mathcal{O}(h^{p+2})$ and $||e_u - E_u|| = \mathcal{O}(h^{p+2})$. We note that the observed numerical convergence rates are higher than the theoretical rates established in Theorem 5.1. The results shown in Table 1 indicate that the global effectivity

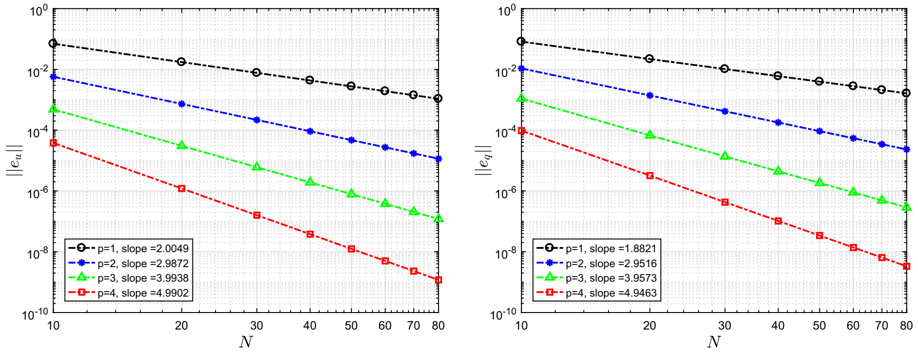

Fig. 1 Log–log plots of $||e_u||$ (left) and $||e_q||$ (right) versus mesh sizes h for Example 6.1 on uniform meshes having $N = 10, 20, 30, 40, 50, 60, 70, 80$ elements using P^p, $p = 1$ to 4

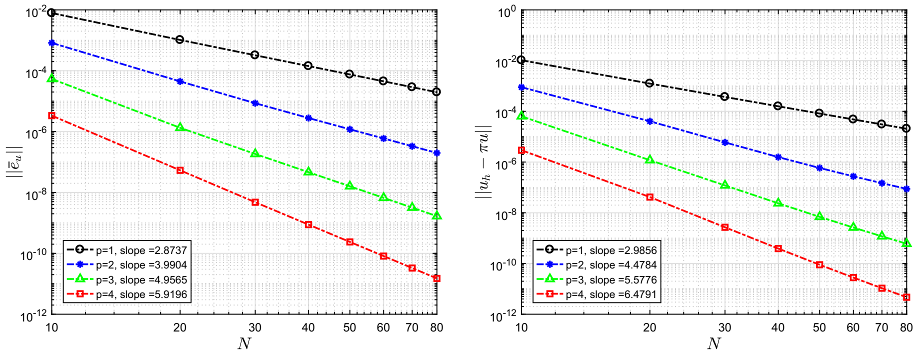

Fig. 2 Log–log plots of $||\bar{e}_u||$ (left) and $||u_h - \pi u||$ (right) versus mesh sizes h for Example 6.1 on uniform meshes having $N = 10, 20, 30, 40, 50, 60, 70, 80$ elements using P^p, $p = 1$ to 4

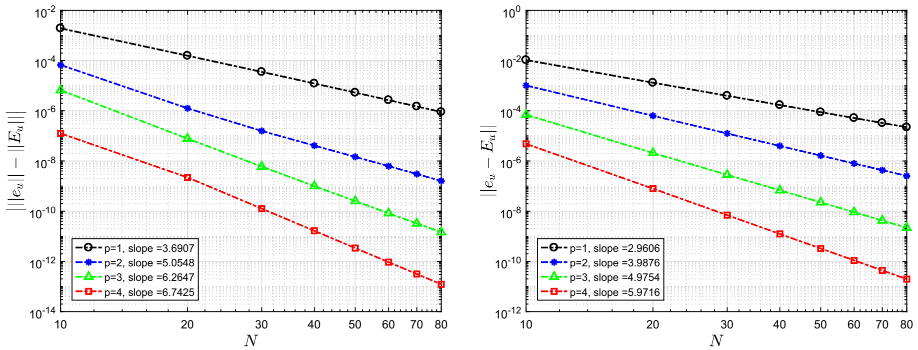

Fig. 3 Log–log plots of $\big|||e_u|| - ||E_u||\big|$ (left) and $||e_u - E_u||$ (right) versus mesh sizes h for Example 6.1 on uniform meshes having $N = 10, 20, 30, 40, 50, 60, 70, 80$ elements using P^p, $p = 1$ to 4

indices converge to unity under h-refinement. The numerical convergence rates at $t = 1$ for $\big|\Theta_u - 1\big|$ are also shown in Table 1, which suggest that the convergence rates is higher than the theoretical rate established in Theorem 5.1.

Table 1 Θ and the errors $\left|\Theta_u - 1\right|$ with their orders of convergence at $t = 1$ for Example 6.1 on uniform meshes having $N = 20, 30, 40, 50, 60, 70, 80$ elements using P^p, $p = 1$ to 4

N	$p = 1$			$p = 2$		
	Θ_u	$\left\|\Theta_u - 1\right\|$	Order	Θ_u	$\left\|\Theta_u - 1\right\|$	Order
20	0.9910	9.0336e−3		0.9983	1.7255e−3	
30	0.9954	4.6129e−3	1.6576	0.9993	7.2502e−4	2.1385
40	0.9972	2.8304e−3	1.6978	0.9996	4.4504e−4	1.6964
50	0.9981	1.9228e−3	1.7327	0.9997	3.0996e−4	1.6210
60	0.9986	1.3947e−3	1.7612	0.9998	2.2925e−4	1.6544
70	0.9989	1.0593e−3	1.7844	0.9998	1.7624e−4	1.7059
80	0.9992	8.3249e−4	1.8044	0.9999	1.3946e−4	1.7529
N	$p = 3$			$p = 4$		
	Θ_u	$\left\|\Theta_u - 1\right\|$	Order	Θ_u	$\left\|\Theta_u - 1\right\|$	Order
20	0.9974	2.5616e−3		0.9982	1.8370e−3	
30	0.9990	9.9372e−4	2.3354	0.9992	7.9506e−4	2.0655
40	0.9995	5.2023e−4	2.2497	0.9996	4.3487e−4	2.0974
50	0.9997	3.2143e−4	2.1578	0.9997	2.7282e−4	2.0894
60	0.9998	2.1914e−4	2.1011	0.9998	1.8687e−4	2.0754
70	0.9998	1.5929e−4	2.0693	0.9999	1.3592e−4	2.0652
80	0.9999	1.2114e−4	2.0503	0.9999	1.0332e−4	2.0537

Example 6.2 In this example, we solve the following viscous Burgers' equation with a source term

$$\begin{cases} u_t + (u^2/2)_x = u_{xx} + \cos(x+t) - \sin(x+t) + \frac{1}{2}\sin(2x+2t), \ x \in [0, 2\pi], \ t \in [0, 1], \\ u(x, 0) = \sin(x), \quad x \in [0, 2\pi], \end{cases} \tag{6.2}$$

subject to the periodic boundary conditions (1.1c). The exact solution is given by $u(x,t) = \sin(x+t)$. We note that $f'(u) = u$ changes sign in the computational domain. In this case, we use the Godunov flux which is an upwind flux. We solve this problem using the LDG method on uniform meshes having $N = 10, 20, 30, 40, 50, 60, 70, 80$ elements and using P^p polynomials with $p = 1-4$. The L^2 LDG errors $||e_u||$ and $||e_q||$ at time $t = 1$ shown in Fig. 4 suggest optimal $\mathcal{O}(h^{p+1})$ convergence rate. Figure 5 shows that $||\bar{e}_u||$ and $||u_h - \pi u||$ at $t = 1$ are $\mathcal{O}(h^{p+2})$ convergent. Consequently, the LDG solution u_h is superconvergent with order $p+2$ to the Gauss–Radau projection $P_h^- u$ and to the p-degree interpolating right Radau polynomial πu. Again the computational results indicate that the numerical convergence rate is higher than the theoretical rate, which is proved to be of order $p + 3/2$. In Fig. 6 we present the errors $\left|||e_u|| - ||E_u||\right|$, $||e_u - E_u||$, and their order of convergence at $t = 1$. Clearly both errors converge with order $p+2$ under mesh refinement. Table 2 lists the global effectivity indices and the errors $\left|\Theta_u - 1\right|$ with their order of convergence at $t = 1$. These results indicate that the proposed *a posteriori* LDG error estimate is asymptotically exact under mesh refinement. The convergence rate at $t = 1$ for $\left|\Theta_u - 1\right|$ is $\mathcal{O}(h)$. Even though the assumption $f'(u) \geq 0$ does not always hold true, the same results are observed.

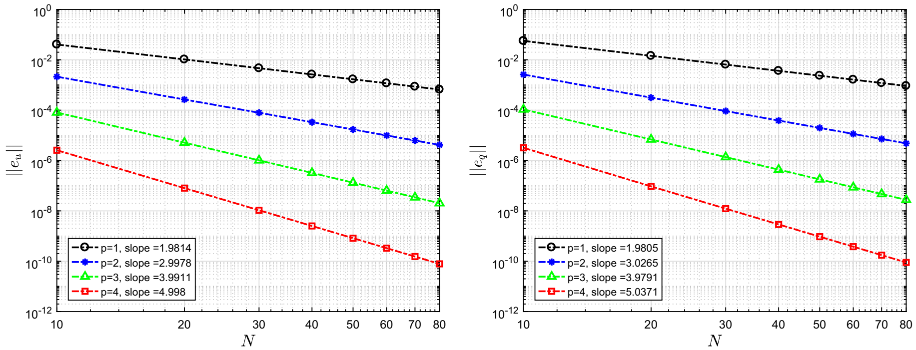

Fig. 4 Log–log plots of $||e_u||$ (left) and $||e_q||$ (right) versus mesh sizes h for Example 6.2 on uniform meshes having $N = 10, 20, 30, 40, 50, 60, 70, 80$ elements using P^p, $p = 1$ to 4

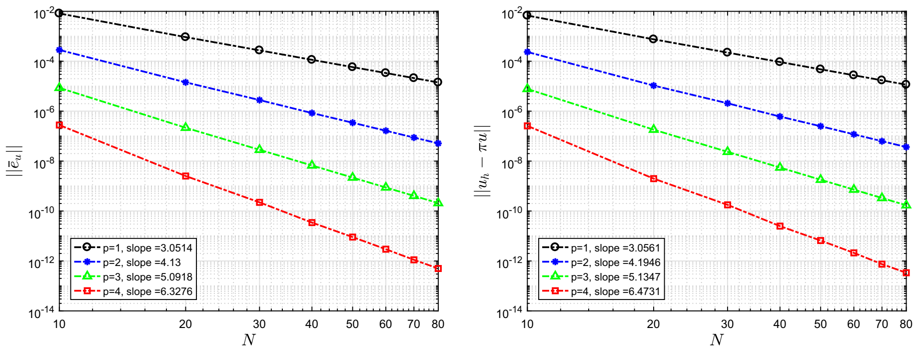

Fig. 5 Log–log plots of $||\bar{e}_u||$ (left) and $||u_h - \pi u||$ (right) versus mesh sizes h for Example 6.2 on uniform meshes having $N = 10, 20, 30, 40, 50, 60, 70, 80$ elements using P^p, $p = 1$ to 4

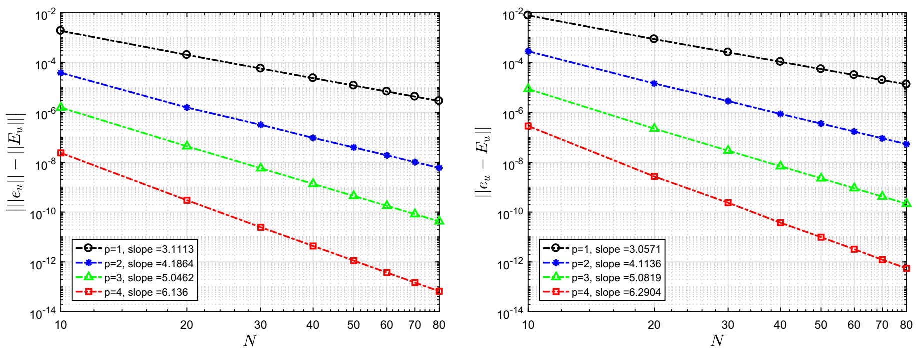

Fig. 6 Log–log plots of $\big|||e_u|| - ||E_u||\big|$ (left) and $||e_u - E_u||$ (right) versus mesh sizes h for Example 6.2 on uniform meshes having $N = 10, 20, 30, 40, 50, 60, 70, 80$ elements using P^p, $p = 1$ to 4

Table 2 Θ and the errors $\left|\Theta_u - 1\right|$ with their orders of convergence at $t = 1$ for Example 6.2 on uniform meshes having $N = 20, 30, 40, 50, 60, 70, 80$ elements using P^p, $p = 1$ to 4

N	$p = 1$			$p = 2$		
	Θ_u	$\left\|\Theta_u - 1\right\|$	Order	Θ_u	$\left\|\Theta_u - 1\right\|$	Order
20	0.9804	1.9605e−2		0.9941	5.9039e−3	
30	0.9876	1.2397e−2	1.1304	0.9960	4.0231e−3	0.9460
40	0.9910	9.0547e−3	1.0921	0.9972	2.8389e−3	1.2119
50	0.9929	7.1284e−3	1.0719	0.9977	2.3052e−3	0.9333
60	0.9941	5.8754e−3	1.0603	0.9981	1.9182e−3	1.0080
70	0.9950	4.9973e−3	1.0501	0.9984	1.6213e−3	1.0909
80	0.9957	4.3466e−3	1.0447	0.9986	1.4218e−3	0.9833
N	$p = 3$			$p = 4$		
	Θ_u	$\left\|\Theta_u - 1\right\|$	Order	Θ_u	$\left\|\Theta_u - 1\right\|$	Order
20	0.9915	8.4945e−3		0.9963	3.7201e−3	
30	0.9944	5.6280e−3	1.0153	0.9977	2.3280e−3	1.1561
40	0.9958	4.2057e−3	1.0126	0.9983	1.7355e−3	1.0210
50	0.9966	3.3581e−3	1.0086	0.9986	1.3676e−3	1.0676
60	0.9972	2.7953e−3	1.0061	0.9989	1.1226e−3	1.0828
70	0.9976	2.3937e−3	1.0062	0.9990	9.6768e−4	0.9634
80	0.9979	2.0932e−3	1.0046	0.9992	8.4147e−4	1.0466

Example 6.3 We consider the viscous Burgers' equation with mixed Dirichlet–Neumann boundary conditions

$$\begin{cases} u_t + (u^2/2)_x = u_{xx} + \frac{1}{2} e^{-2t} \sin(2x), & x \in [0, 2\pi],\ t \in [0, 1], \\ u(x, 0) = \sin(x), & x \in [0, 2\pi] \\ u(0, t) = 0, \quad u_x(2\pi, t) = e^{-t}, & t \in [0, 1]. \end{cases} \tag{6.3}$$

The exact solution is given by $u(x, t) = e^{-t} \sin(x)$. In this example, we test our superconvergence results and the global convergence of our error estimates using mixed Dirichlet–Neumann boundary conditions. The numerical flux $\hat{f}$ associated with the convection is taken as the Godunov flux. The numerical fluxes $\hat{u}_h$ and $\hat{q}_h$ associated with the diffusion terms are taken as

$$\hat{u}_h\big|_i = \begin{cases} u(0, t), & i = 0, \\ u_h^-\big|_i, & i = 1, \ldots, N, \end{cases}, \quad \hat{q}_h\big|_i = \begin{cases} q_h^+\big|_i, & i = 0, \ldots, N-1, \\ u_x(2\pi, t), & i = N. \end{cases} \tag{6.4}$$

We test this example using P^p polynomials with $p = 1 - 4$ on a uniform mesh. Figure 7 shows the errors $||e_u||$ and $||e_q||$ at time $t = 1$. We observe that the order of convergence of the errors e_u and e_q is $p + 1$. This is in full agreement with our theoretical results. In Fig. 8 we present the errors $||\bar{e}_u||$ and $||u_h - \pi u||$ at $t = 1$. We observe that both errors converge at an $\mathcal{O}(h^{p+2})$ rate. Again these rates are higher than the theoretical rate, which is proved to be $\mathcal{O}(h^{p+3/2})$. In Fig. 9 we present the errors $\left|||e_u|| - ||E_u||\right|$, $||e_u - E_u||$, and their order of convergence at $t = 1$. We observe that these errors achieve at least $(p + 2)$th order of convergence. Although Theorem 5.1 indicates that the convergence rate is $\mathcal{O}(h^{p+3/2})$, we observe higher convergence rate. Next, we present the global effectivity indices and the

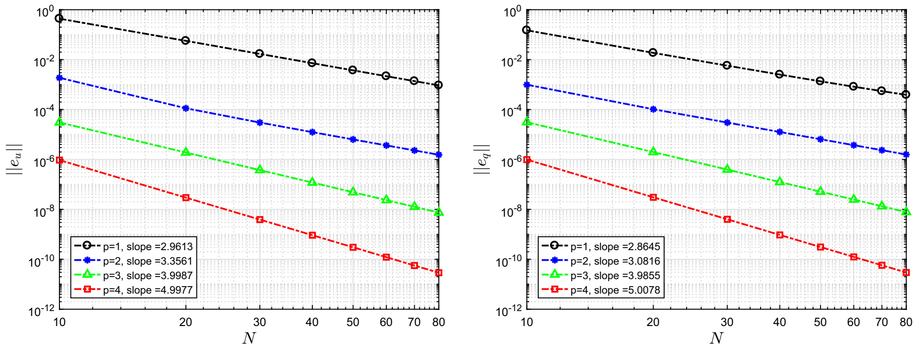

Fig. 7 Log–log plots of $||e_u||$ (left) and $||e_q||$ (right) versus mesh sizes h for Example 6.3 on uniform meshes having $N = 10, 20, 30, 40, 50, 60, 70, 80$ elements using P^p, $p = 1$ to 4

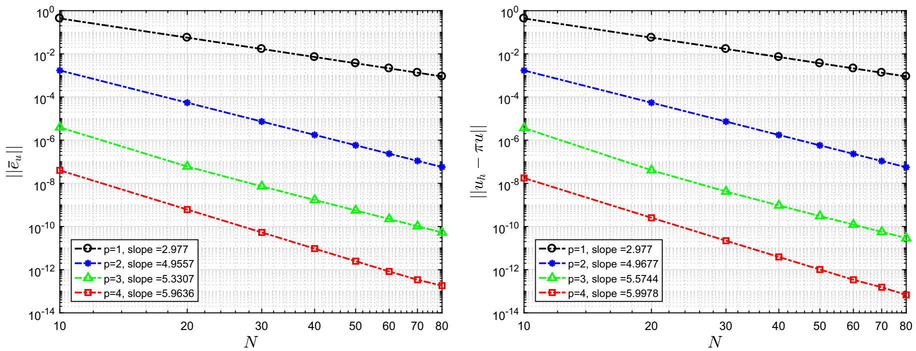

Fig. 8 Log–log plots of $||\bar{e}_u||$ (left) and $||u_h - \pi u||$ (right) versus mesh sizes h for Example 6.3 on uniform meshes having $N = 10, 20, 30, 40, 50, 60, 70, 80$ elements using P^p, $p = 1$ to 4

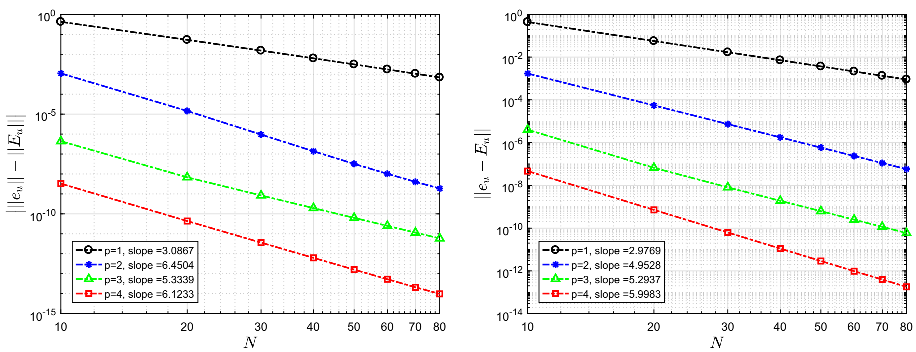

Fig. 9 Log–log plots of $\big| ||e_u|| - ||E_u|| \big|$ (left) and $||e_u - E_u||$ (right) versus mesh sizes h for Example 6.3 on uniform meshes having $N = 10, 20, 30, 40, 50, 60, 70, 80$ elements using P^p, $p = 1$ to 4

errors $\left|\Theta_u - 1\right|$ with their order of convergence at $t = 1$ in Table 3. We observe that the Θ_u converges to unity under h-refinement. The convergence rate at $t = 1$ for $\left|\Theta_u - 1\right|$ is $\mathcal{O}(h)$. This example demonstrates that all conclusions hold true for convection–diffusion problems subject to mixed boundary conditions.

Table 3 Θ and the errors $|\Theta_u - 1|$ with their orders of convergence at $t = 1$ for Example 6.3 on uniform meshes having $N = 20, 30, 40, 50, 60, 70, 80$ elements using P^p, $p = 2$ to 4

N	$p = 2$			$p = 3$		
	Θ_u	$\|\Theta_u - 1\|$	Order	Θ_u	$\|\Theta_u - 1\|$	Order
20	0.8721	1.2794e−1		0.9964	3.5949e−3	
30	0.9686	3.1438e−2	3.4616	0.9978	2.2439e−3	1.1624
40	0.9889	1.1082e−2	3.6245	0.9984	1.6429e−3	1.0837
50	0.9949	5.0715e−3	3.5031	0.9987	1.2965e−3	1.0612
60	0.9972	2.7878e−3	3.2820	0.9989	1.0707e−3	1.0496
70	0.9982	1.7507e−3	3.0181	0.9991	9.1186e−4	1.0417
80	0.9988	1.2133e−3	2.7460	0.9992	7.9400e−4	1.0365

N	$p = 4$		
	Θ_u	$\|\Theta_u - 1\|$	Order
20	0.9985	1.4824e−3	
30	0.9991	9.3201e−4	1.1445
40	0.9993	6.7764e−4	1.1079
50	0.9995	5.3184e−4	1.0857
60	0.9996	4.3790e−4	1.0660
70	0.9996	3.7297e−4	1.0411
80	0.9997	3.2546e−4	1.0204

Remark 6.1 We also solved (6.3) but subject to the Dirichlet boundary conditions $u(0, t) = u(2\pi, t) = 0$ and observed similar results. Moreover, we repeated the previous experiments with all parameters kept unchanged except for meshes where we used nonuniform meshes. We also observed similar conclusions. These results are not included to save space.

Example 6.4 In this example, we solve a convection–diffusion problem, where the exact solution has a steep front. We consider the viscous Burgers' equation with mixed Dirichlet–Neumann boundary conditions

$$\begin{cases} u_t + (u^2/2)_x = u_{xx} + f(x, t), \quad x \in [0, 1], \ t \in [0, T], \\ u(x, 0) = 1 - 2\tanh(100x), \quad u(0, t) = 1 - 2\tanh(-100t), \\ u_x(1, t) = -200\,\text{sech}^2(100(1 - t)). \end{cases} \tag{6.5}$$

The source term $f(x, t)$ is chosen so that the exact solution is the smooth function

$$u(x, t) = 1 - 2\tanh(100(x - t)),$$

which has a steep front along the line $x = t$. In particular, at time $t = 0.5$, the exact solution is smooth function but it has a steep front at the neighborhood of the point $x = 0.5$. In Fig. 10 we present the exact solution, the LDG solution, and the exact error using $N = 100$, $\Delta t = 10^{-5}$, $T = 0.5$, and $p = 1$. The results indicate that the scheme converges towards the analytic solution and the steep front is well captured. The global effectivity index $\Theta_u(t)$ is shown in Fig. 11 using $p = 1$, $T = 1$, and $N = 40, \ 60$. We observe that the effectivity index remains constant as we refine the mesh. Moreover, Θ_u approaches 1 as we refine the mesh These computational results indicate that our estimator is effective.

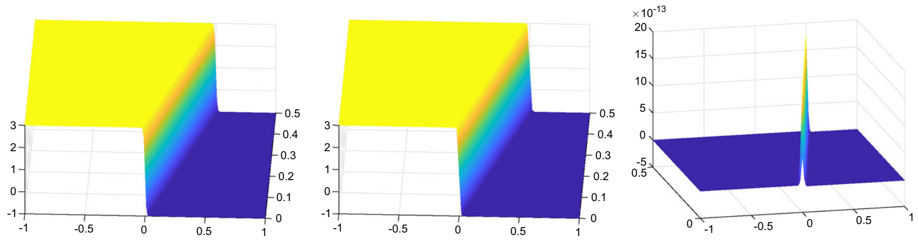

Fig. 10 Graph of the exact solution u (left), the LDG solution u_h (middle, and the exact error $u - u_h$ (right) for Example 6.4 using $N = 100$, $\Delta t = 10^{-5}$, and $p = 1$

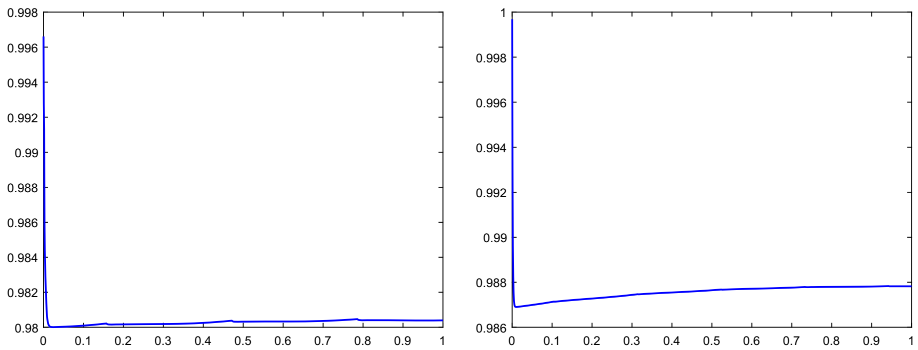

Fig. 11 Global effectivity index $\Theta_u(t)$, $t \in [0, 1]$ versus time for Example 6.4 using $p = 1$ on uniform meshes having $N = 40$ elements (left) and $N = 60$ (right)

7 Concluding Remarks

In this paper, we proposed and analyzed a posteriori error estimate for local discontinuous Galerkin (LDG) method applied to nonlinear convection–diffusion problems in one space dimension. We proved several L^2 error estimates and superconvergence results towards a special projection, when the upwind flux is used for the convection term and the alternating flux is used for the diffusion term. More precisely, we showed that the LDG solutions converge to the exact solutions with order $p + 1$, when the space of piecewise polynomials of degree $p \geq 1$ is used. We further proved that the derivative of the LDG solution is superconvergent with order $p + 1$ towards the derivative of a Gauss–Radau projection of the exact solution. Moreover, we proved that the LDG solution is $\mathcal{O}(h^{p+3/2})$ superconvergent towards Gauss–Radau projection of the exact solution. We used the superconvergence results to construct asymptotically exact *a posteriori* error estimates by solving a local steady problem with no boundary conditions on each element. We further proved that the proposed *a posteriori* error estimates converge to the true spatial errors at $\mathcal{O}(h^{p+3/2})$ rate. Finally, we proved that the global effectivity indices in the L^2-norm converge to unity at $\mathcal{O}(h^{1/2})$ rate. We are currently investigating the superconvergence properties and the asymptotic exactness of *a posteriori* error estimates for LDG methods applied to two-dimensional convection–diffusion and wave equations on rectangular and triangular meshes. Our future work will focus on extending our a posteriori error analysis to problems on tetrahedral meshes. We are also planning to use the *a posteriori* error estimators to construct efficient adaptive LDG methods and reach similar conclusions as in our previous work [3], where we tested similar *a posteriori* error estimates of DG discretization errors [12] for hyperbolic problems on adaptively refined unstructured triangular meshes. We expect that our error estimates will converge to the true error under adaptive mesh refinement.

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Compliance with Ethical Standards

Conflict of interest The author declares that he has no conflict of interest.

Ethical Statement The author agrees that this manuscript has followed the rules of ethics presented in the journal's Ethical Guidelines for Journal Publication.

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