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WSGD-OSC Scheme for Two-Dimensional Distributed Order Fractional Reaction–Diffusion Equation

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Abstract In this paper, a new numerical approximation is discussed for the two-dimensional distributed-order time fractional reaction–diffusion equation. Combining with the idea of weighted and shifted Grünwald difference (WSGD) approximation (Tian et al. in Math Comput 84:1703–1727, 2015; Wang and Vong in J Comput Phys 277:1–15, 2014) in time, we establish orthogonal spline collocation (OSC) method in space. A detailed analysis shows that the proposed scheme is unconditionally stable and convergent with the convergence order $\mathscr{O}(\tau^2 + \Delta\alpha^2 + h^{r+1})$, where τ, $\Delta\alpha$, h and r are, respectively the time step size, step size in distributed-order variable, space step size, and polynomial degree of space. Interestingly, we prove that the proposed WSGD-OSC scheme converges with the second-order in time, where OSC schemes proposed previously (Fairweather et al. in J Sci Comput 65:1217–1239, 2015; Yang et al. in J Comput Phys 256:824–837, 2014) can at most achieve temporal accuracy of order which depends on the order of fractional derivatives in the equations and is usually less than two. Some numerical results are also given to confirm our theoretical prediction.

Keywords Distributed order fractional equation · WSGD operator · Orthogonal spline collocation scheme · Stability · Error estimate

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1 Introduction

In this article, we consider the following distributed order time fractional reaction diffusion equation:

$$\begin{aligned}\mathscr{D}_t^{\omega} u(\mathbf{x}, t) &= \kappa \Delta u(\mathbf{x}, t) - \nu u(\mathbf{x}, t) + f(\mathbf{x}, t),\\ &\quad \mathbf{x} = (x, y) \in \Omega = [0, L] \times [0, L] \subset \mathbb{R}^2, t \in (0, T],\end{aligned} \tag{1.1}$$

with initial condition

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1.2}$$

and boundary condition

$$u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T]. \tag{1.3}$$

where $\kappa > 0$ is the diffusion coefficient, $\nu > 0$ is the constant reaction rate, Δ is the Laplace differential operator, i.e., $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. The symbols Ω, $\partial\Omega$ denote the transport field and its boundary. $f(\mathbf{x}, t)$ and $\varphi(\mathbf{x})$ are given functions, and $\mathscr{D}_t^{\omega} u(\mathbf{x}, t)$ denotes the distributed order fractional derivative of u in time t, given by

$$\mathscr{D}_t^{\omega} u(\mathbf{x}, t) = \int_0^1 \omega(\alpha)\, {}_0^C D_t^{\alpha} u(\mathbf{x}, t) d\alpha. \tag{1.4}$$

Here $\omega(\alpha)$ is a continuous non-negative weight function, such that the conditions

$$\omega(\alpha) \geq 0; \quad \omega \neq 0, \ \alpha \in [0, 1]; \quad \int_0^1 \omega(\alpha) d\alpha = c_0 > 0$$

hold true, where c_0 is a positive constant. ${}_0^C D_t^{\alpha} u(\mathbf{x}, t)$ is the αth order time Caputo fractional derivative defined by

$${}_0^C D_t^{\alpha} u(\mathbf{x}, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(\mathbf{x}, s)}{\partial s} \frac{ds}{(t-s)^{\alpha}}, \quad 0 < \alpha < 1, \tag{1.5}$$

and $\Gamma(\cdot)$ is the Gamma function.

The sub-diffusion processes with the mean square displacement with a logarithmic growth have been introduced recently. One of the approaches for modelling of such processes is to employ time-fractional diffusion equations of distributed order. There are numerous references to the literature dealing with different methods and techniques for the analytical solutions and other properties of the distributed-order time fractional diffusion equations. We just mention a few part among them. Luchko [1,2] considered the maximum principle and some uniqueness and existence results, and subsequently the asymptotic behaviors of solutions is studied in [3]. Jia et al. [4] studied the well-posedness of the distributed-order fractional abstract Cauchy problem using functional calculus technique. Meerschaert et al. [5] established the explicit strong solutions and stochastic analogues. The fundamental solution of the distributed order time fractional diffusion equations in the unbounded domain was obtained by Mainardi et al. [6] in terms of the Fourier–Laplace representation.

In recent years, several numerical methods have been applied for the solution of fractional distributed order partial differential equations in one and two several space variables. Ye et al. [7,8] analysed the difference scheme and compact difference scheme for a distributed-order time-fractional diffusion-wave equations and Riesz space fractional diffusion equation, and proved unconditional stability and convergence by the discrete energy method and mathematical induction. Katsikadelis [9] developed a new numerical method for solving distributed order fractional differential equations of general form in the integration interval $[\alpha, \beta]$. Gao and Sun [10–12] proposed some different difference and alternating direction implicit (ADI) difference schemes for distributed order fractional equations. Du et al. [13] devoted a high-order difference schemes for the distributed-order time-fractional diffusion equations with smooth solutions both one and two space dimensions. Jin et al. [14] presented a rigorous numerical analysis of two fully discrete schemes for the distributed-order time fractional diffusion equation with non-smooth initial data. Chen et al. [15] developed a spectral and pseudospectral scheme for the distributed order time fractional reaction–diffusion equation on a semi-infinite spatial domain. Morgado and Rebelo [16] presented an implicit scheme for the numerical approximation of a distributed-order time-fractional reaction–diffusion equation with a nonlinear source term.

However, numerical approximation referring OSC method for distributed-order time fractional equations is still at an early stage of development. As pointed out in paper [17], in comparison with finite difference schemes, orthogonal spline collocation methods show continuous approximations to the solution and its spatial derivatives at all points of the domain of the problem and allow for arbitrarily high-order accuracy in the spatial approximation. Compared to finite element Galerkin methods the calculation of the coefficient of the mass and stiffness matrices in the system of linear equations determining the approximate solution is very fast and efficient, since no integrals need to be evaluated or approximated. Moreover, OSC scheme always lead to the almost block diagonal linear systems, which can be implemented by the software packages efficiently.

Therefore, OSC methods have evolved as a robust and valuable technique for the numerical solutions of a broad class of ordinary and partial differential equations; see [18] for a comprehensive survey. Bialecki et al. [19] formulated the extrapolated Crank–Nicolson OSC method with C^1 splines of degree ≥ 3, and established an optimal order error bound in the discrete maximum norm in time and the continuous maximum norm in space for a quasilinear parabolic problem with nonlocal boundary conditions. Fernandes and Fairweather [20] considered the ADI extrapolated Crank–Nicolson orthogonal spline collocation technique for the approximate solution of nonlinear reaction–diffusion systems in fixed domains, and Fernandes and Bialecki [21] subsequently generalized to evolving domains. Recently, we formulated OSC and ADI OSC methods to solve fractional partial differential equations [22–25], the convergence and stability were analyzed strictly.

In this paper, our main purpose is to derive WSGD-OSC approximation for the problem (1.1)–(1.3). The main contribution of the paper consists of: (1) We construct discrete-time OSC methods for the numerical solution of the two dimensional distributed order time fractional reaction diffusion equation. Meanwhile we introduce the two-dimensional numerical example to demonstrate the effectiveness of proposed methods. (2) Based on the idea of weighted and shifted Grünwald difference operator, we establish discrete-time OSC schemes with temporal accuracy order equal to two, where OSC schemes proposed previously can at most achieve temporal accuracy of order which depends on the order of fractional derivatives in the equations and is usually less than two. (3) We carry out an analysis of stability and convergence of the proposed method. A convergence rate of order $\mathscr{O}(\tau^2 + \Delta\alpha^2 + h^{r+1})$ is rigourously proved. In particular, because the order of fractional derivatives in time distributed in the interval [0, 1], its stability and convergence analysis become more perplexing to handle. We first discretize the integral term in the time distributed-order using numerical approximation, thus the original problem is approximated by a multi-term time fractional reaction diffusion equation. We then use the weighted and shifted Grünwald difference operators [26–28] to approximate the fractional operators. Combined with OSC scheme in space, we introduce some new techniques for the convergence analysis, which is a generalization of the schemes proposed in [24] to the case of multi-term fractional derivatives.

The rest of this paper is organized as follows. In Sect. 2, we introduce some notations and auxiliary lemmas. Meanwhile, the discrete-time OSC method is derived. Section 3 presents the stability and convergence for WSGD-OSC scheme. Several examples are given in Sect. 4 and some conclusions are drawn in Sect. 5.

2 Discrete-Time OSC Scheme

2.1 Notations

In this subsection, we will introduce some notations, which will be frequently used in the subsequent of this article. For a positive integer N_x and N_y, a uniform partition of $\overline{I} = [0, 1]$ is defined as follows

$$\delta_x : 0 = x_0 < x_1 < \cdots < x_{N_x} = 1, \qquad \delta_y : 0 = y_0 < y_1 < \cdots < y_{N_y} = 1.$$

Let $\delta = \delta_x \times \delta_y$ of Ω be quasi-uniform [29], $h_k^x = x_k - x_{k-1}$, $h_l^y = y_l - y_{l-1}$ and $h = \max(\max\limits_{1\le k\le N_x} h_k^x, \max\limits_{1\le l\le N_y} h_l^y)$, $1 \le k \le N_x$, $1 \le l \le N_y$.

Set $\mathscr{M}(\delta)$ be the space of piecewise polynomials in x and y defined by

$$\mathscr{M}(\delta) = \mathscr{M}(r, \delta_x) \otimes \mathscr{M}(r, \delta_y),$$

where $\mathscr{M}(r, \delta_x) = \left\{v|v \in C^1(\bar{I}), v|_{[x_{k-1},x_k]} \in P_r, k = 1, 2, \ldots, N_x, v(0) = v(1) = 0\right\}$, and P_r denotes the set of polynomials of degree at most r, with $r \ge 3$. With $\mathscr{M}(r, \delta_y)$ defined similarly.

Define Gauss collocation points set in Ω: $\Lambda = \{\xi|\xi = (\xi^x, \xi^y), \xi^x \in \Lambda_x, \xi^y \in \Lambda_y\}$, where $\Lambda_x = \{\xi_{i,k}^x\}_{i,k=1}^{N_x,r-1}$, $\xi_{i,k}^x = x_{i-1} + \lambda_k h_i^x$, and $\{\lambda_k\}_{k=1}^{r-1}$ be the nodes of the $(r-1)$-point Gauss quadrature rule on $\overline{I}$. With Λ_y defined similarly.

Let $\{w_k\}_{k=1}^{r-1}$ be the weights of the $(r-1)$-point Gauss quadrature rule on $\overline{I}$. For any function u and v defined on Λ, the discrete inner product and norm are defined as follows

$$\langle u, v\rangle = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} h_i^x h_j^y \sum_{k=1}^{r-1}\sum_{l=1}^{r-1} w_k w_l (uv)(\xi_{i,k}^x, \xi_{j,l}^y), \qquad \|v\|_{\mathscr{M}_r}^2 = \langle v, v\rangle. \tag{2.1}$$

For m a nonnegative integer, let $H^m(\Omega)$ denote the usual Sobolev space with norm

$$\|v\|_{H^m} = \left(\sum_{\ell=0}^{m}\sum_{i+j=\ell}\left\|\frac{\partial^{i+j} v}{\partial x^i \partial y^j}\right\|^2\right)^{\frac{1}{2}},$$

where the norm $\|\cdot\|$ denotes the usual L^2 norm, sometimes written as $\|\cdot\|_{H^0}$ for convenience.

Lemma 1 [23] *The norms $\|\cdot\|_{\mathscr{M}_r}$ and $\|\cdot\|$ are equivalent on $\mathscr{M}(\delta)$.*

If X is a normed space with norm $\|\cdot\|_X$, then we denote $L^P(X)$ by

$$L^P(X) = \{v : v(\cdot, t) \in X, t \in [0, T]; \|v\|_{L^P(X)} < \infty\},$$

where

$$\|v\|_{L^P(X)} = \left(\int_0^T \|v\|_X^P dt\right)^{1/p}, \quad \|v\|_{L^\infty(X)} = \sup_{0\le t\le T} \|v\|_X.$$

Next, we introduce the composite trapezoid formula that will be frequently used in the discretization of the distributed order time derivative.

Divide the interval $[0, 1]$ into $2J$ subintervals with $\Delta\alpha = 1/2J$ and $\alpha_l = l\Delta\alpha$, $l = 0, 1, \cdots, 2J$.

Lemma 2 [11,12] *Let $s(\alpha) \in C^2[0, 1]$, then we have*

$$\int_0^1 s(\alpha)d\alpha = \Delta\alpha \sum_{l=0}^{2J} c_l s(\alpha_l) - \frac{\Delta\alpha^2}{12} s''(\varsigma), \quad \varsigma \in (0, 1), \tag{2.2}$$

where

$$c_l = \begin{cases} \frac{1}{2}, & l = 0,\ 2J; \\ 1, & 1 \le l \le 2J - 1. \end{cases}$$

2.2 Construction of the Fully Discrete Scheme

In this subsection, we will consider discrete-time OSC schemes for solving problem (1.1)–(1.3). Firstly, the continuous-time OSC scheme to the solution u of (1.1) is a differentiable map $u_h : (0, T] \to \mathscr{M}(\delta)$ such that

$$\begin{aligned} \mathscr{D}_t^\omega u_h(\xi_{i,k}^x, \xi_{j,l}^y, t) &= \kappa \Delta u_h(\xi_{i,k}^x, \xi_{j,l}^y, t) - \nu u_h(\xi_{i,k}^x, \xi_{j,l}^y, t) \\ &\quad + f_h(\xi_{i,k}^x, \xi_{j,l}^y, t), \ (\xi_{i,k}^x, \xi_{j,l}^y) \in \Lambda, \ t \in (0, T], \end{aligned} \tag{2.3}$$

where $f_h(\cdot, \cdot, t) \in \mathscr{M}(\delta)$ satisfying $\langle f_h, \vartheta\rangle = \langle f, \vartheta\rangle, \forall\vartheta \in \mathscr{M}(\delta)$.

Let temporal domain $[0, T]$ to be divided by the uniform partition $\{t_k\}_{k=0}^K$ such that $t_k = k\tau$ and $\tau = T/K$, where K is a positive integer. In order to construct a second order finite difference scheme for the problem (1.1) in time, we first discretize the integral term in the distributed-order equation. Suppose $\omega(\alpha) \in C^2[0, 1]$, ${}_0^C D_t^\alpha(\cdot) \in C^2[0, 1]$. By using Lemma 2, we have

$$\begin{aligned} \mathscr{D}_t^\omega u(\xi_{i,k}^x, \xi_{j,l}^y, t) &= \int_0^1 \omega(\alpha) {}_0^C D_t^\alpha u(\xi_{i,k}^x, \xi_{j,l}^y, t) d\alpha \\ &= \Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l) {}_0^C D_t^{\alpha_l} u(\xi_{i,k}^x, \xi_{j,l}^y, t) + \mathscr{O}(\Delta\alpha^2). \end{aligned} \tag{2.4}$$

In the following parts of this paper, we assume $u(\mathbf{x}, 0) \equiv 0$, otherwise, we consider $u = u - \varphi$. In this case, for $0 < \alpha < 1$, if $g(0) = 0$, then we have ${}_0^C D_t^\alpha g(t) = {}_0^{RL} D_t^\alpha g(t)$, here ${}_0^{RL} D_t^\alpha g(t)$ denote the left-sided Riemann-Liouville fractional derivative of order α. Thus one can continuously extend the solution $u(\mathbf{x}, t)$ to be zero for $t < 0$. Now we consider the weighted and shifted Grünwald-Letnikov approximation [26] for ${}_0^C D_t^\alpha u(\mathbf{x}, t)$, that is

$${}_0^C D_t^\alpha u(\xi_{i,k}^x, \xi_{j,l}^y, t_{n+1}) = \tau^{-\alpha} \sum_{k=0}^{n+1} \lambda_k^{(\alpha)} u(\xi_{i,k}^x, \xi_{j,l}^y, t_{n+1-k}) + R_{n+1}^{(\alpha)}, \tag{2.5}$$

where

$$\lambda_0^{(\alpha)} = \frac{2+\alpha}{2} g_0^{(\alpha)}, \quad \lambda_k^{(\alpha)} = \frac{2+\alpha}{2} g_k^{(\alpha)} - \frac{\alpha}{2} g_{k-1}^{(\alpha)}, \ \ k \ge 1, \tag{2.6}$$

$$g_0^{(\alpha)} = 1, \quad g_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} = (1 - \frac{\alpha+1}{k}) g_{k-1}^{(\alpha)}, \quad k \geq 1.$$

Based on Theorem 1 in Tian et al. [26], we can derive the following estimate of the truncation error.

Lemma 3 *Suppose* $\forall \gamma > 0$ *and* ${}_0^{RL}D_t^{\gamma+2} u \in L(\mathbb{R})$. *We have*

$$\left| {}_0^{RL}D_t^{\gamma} u(t_n) - \tau^{-\gamma} \sum_{k=0}^{n} \lambda_k^{(\gamma)} u(t_{n-k}) \right| \leq C\tau^2 \left\| \mathfrak{F}[{}_0^{RL}D_t^{\gamma+2} u](\omega) \right\|_{L^1},$$

where $\mathfrak{F}$ denotes the fourier transform symbol.

Therefore, the truncation error $R_{n+1}^{(\alpha)}$ satisfies

$$\left| R_{n+1}^{(\alpha)} \right| \leq C_1 \tau^2 \left\| \mathfrak{F}[{}_0^{RL}D_t^{\alpha+2} u](\omega) \right\|_{L^1} = \mathscr{O}(\tau^2). \tag{2.7}$$

Then on using (2.3)–(2.5), the WSGD-OSC scheme for Eq. (1.1) consists in finding $\{u_h^n\}_{n=0}^{K-1} \subset \mathscr{M}(\delta)$ such that

$$\Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l) D_\tau^{\alpha_l} u_h^{n+1} - \kappa \Delta u_h^{n+1} + \nu u_h^{n+1} = f_h^{n+1}, n = 0, \cdots, K-1, \tag{2.8}$$

where, for convenience, we have used the symbol $D_\tau^{\alpha_l} u(t_{n+1}) = \tau^{-\alpha_l} \sum_{k=0}^{n+1} \lambda_k^{(\alpha_l)} u(t_{n-k+1})$ and omitted the dependence of $u^{n+1}(\xi_{i,k}^x, \xi_{j,l}^y)$ on $(\xi_{i,k}^x, \xi_{j,l}^y)$ in the above equation.

If the initial condition $u(\mathbf{x}, 0) = \varphi(\mathbf{x}) \neq 0$ the substitution $u = u - \varphi$ will be considered. Assume $\varphi \in H_0^2(\Omega)$, then we can write the homologous full-discrete form

$$\begin{aligned} &\Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l) D_\tau^{\alpha_l} u_h^{n+1} - \kappa \Delta u_h^{n+1} + \nu u_h^{n+1} \\ &= f_h^{n+1} + \kappa \Delta \varphi_h - \nu \varphi_h, \quad n = 0, \cdots, K-1, \end{aligned} \tag{2.9}$$

where the quantity $\varphi_h = u_h^0$ is a suitable approximation to the initial condition $\varphi(\mathbf{x})$, which we shall define later.

3 Analysis of the WSGD-OSC Scheme

In this section, we will derive and analyze the stability and convergence of fully-discrete scheme (2.8). We commence with the following Lemma which is critical for establishing the stability and convergence of the WSGD-OSC scheme.

Lemma 4 [28] *Let* $\{\lambda_n^{(\alpha)}\}_{n=0}^{\infty}$ *be defined as in* (2.6)*, then for any positive integer k and real vector* $(v_1, v_2, \cdots, v_k)^T \in \mathbb{R}^k$*, it holds that*

$$\sum_{n=0}^{k-1} \left(\sum_{p=0}^{n} \lambda_p^{(\alpha)} v_{n+1-p} \right) v_{n+1} \geq 0. \tag{3.1}$$

3.1 Stability

The stability of the WSGD-OSC scheme (2.8) is given in the following theorem.

Theorem 1 *The WSGD-OSC scheme* (2.8) *is unconditionally stable in the sense that for all* $\tau > 0$*, it holds*

$$\nu\tau \sum_{n=1}^{m} \|u_h^n\|_{\mathscr{M}_r}^2 \le \frac{\tau}{\nu} \sum_{n=1}^{m} \|f_h^n\|_{\mathscr{M}_r}^2, \;\; 1 \le m \le K. \tag{3.2}$$

Proof Making an inner product of (2.8) with u_h^{n+1}, we obtain

$$\Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l) \tau^{-\alpha_l} \left\langle \sum_{k=0}^{n+1} \lambda_k^{(\alpha_l)} u_h^{n-k+1}, u_h^{n+1} \right\rangle - \kappa \langle \Delta u_h^{n+1}, u_h^{n+1} \rangle$$
$$+ \nu \langle u_h^{n+1}, u_h^{n+1} \rangle = \langle f_h^{n+1}, u_h^{n+1} \rangle, \;\; n = 0, 1, \ldots, K-1. \tag{3.3}$$

Note that, from Fernandes and Fairweather [30], for $\vartheta \in \mathscr{M}(\delta)$, there exists a positive constant c such that

$$\langle -\Delta\vartheta, \vartheta \rangle \ge c\|\nabla\vartheta\|^2 \ge 0. \tag{3.4}$$

On using (3.4) with $\vartheta = u_h^{n+1}$ to the second term on LHS of (3.3), we have

$$-\kappa \langle \Delta u_h^{n+1}, u_h^{n+1} \rangle \ge 0. \tag{3.5}$$

Summing from $n = 0$ to $n = m - 1$ $(1 \le m \le K)$, and then multiplying the result equation by 2τ, noticing that $u_h^0 = 0$, we obtain

$$2\Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l) \tau^{1-\alpha_l} \sum_{n=0}^{m-1} \left\langle \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} u_h^{n-k+1}, u_h^{n+1} \right\rangle$$
$$-2\kappa\tau \sum_{n=0}^{m-1} \langle \Delta u_h^{n+1}, u_h^{n+1} \rangle + 2\nu\tau \sum_{n=0}^{m-1} \|u_h^{n+1}\|_{\mathscr{M}_r}^2 = 2\tau \sum_{n=0}^{m-1} \langle f_h^{n+1}, u_h^{n+1} \rangle. \tag{3.6}$$

Applying the Cauchy–Schwarz inequality to the term on the RHS of (3.6), also using (3.5) and Lemma 4, then removing the first and the second non-negative terms on the LHS of (3.6), we obtain

$$2\nu\tau \sum_{n=0}^{m-1} \|u_h^{n+1}\|_{\mathscr{M}_r}^2 \le \frac{\tau}{\nu} \sum_{n=0}^{m-1} \|f_h^{n+1}\|_{\mathscr{M}_r}^2 + \nu\tau \sum_{n=0}^{m-1} \|u_h^{n+1}\|_{\mathscr{M}_r}^2, \;\; 0 \le n \le K-1, \tag{3.7}$$

by a simple calculation, we complete the proof of Theorem 1.

3.2 Convergence

We now consider the convergence of our WSGD-OSC scheme. In the error analysis, we require the following elliptic projection W of the exact solution u. Define $W : [0, T] \to \mathscr{M}(\delta)$ by

$$\Delta\,(u - W) = 0 \;\; \text{on} \;\; \Lambda \times [0, T], \tag{3.8}$$

where u is the solution of (1.1)–(1.3). The next two lemmas provide estimates for $u - W$ and its time derivatives, which will be useful for the analysis in the after-mentioned part.

Lemma 5 [31] *If $\partial^l u/\partial t^l \in H^{r+3-j}$, $l = 0, 1$, $j = 0, 1$, and W is defined by (3.8), then there exists a constant C, independent of h, such that*

$$\left\| \frac{\partial^l (u - W)}{\partial t^l} \right\|_{H^j} \leq Ch^{r+1-j} \left\| \frac{\partial^l u}{\partial t^l} \right\|_{H^{r+3-j}}, \quad j = 0, 1, \quad l = 0, 1. \tag{3.9}$$

Lemma 6 [31] *If $\partial^i u/\partial t^i \in H^{r+3}$, for $t \in [0, T]$, $i = 0, 1$, then*

$$\left\| \frac{\partial^{l+i} (u - W)}{\partial x^{l_1} \partial y^{l_2} \partial t^i} \right\|_{\mathscr{M}_r} \leq Ch^{r+1-l} \left\| \frac{\partial^i u}{\partial t^i} \right\|_{H^{r+3}}, \tag{3.10}$$

where $0 \leq l = l_1 + l_2 \leq 4$.

For the error analysis of the fully discrete scheme (2.8), we have the following main convergence theorem.

Theorem 2 *Suppose u is the solution of (1.1)–(1.3), and u_h^n $(0 \leq n \leq K)$ is the solution of the problem (2.8) with $u_h^0 = W^0$. If the hypotheses of Lemma 3 are satisfied and if u, ${}_0^{RL}D_t^{\alpha_l} u \in L^\infty(H^{r+3})$, $0 \leq l \leq 2J$, then there exists a positive constant C, independent of h and τ, such that*

$$\begin{aligned} \frac{\nu}{4}\tau \sum_{n=1}^{m} \|u(t_n) - u_h^n\|^2 &\leq CTh^{2r+2} \left[\|u\|^2_{L^\infty(H^{r+3})} + \max_{0 \leq l \leq 2J} \|{}_0^{RL}D_t^{\alpha_l} u\|^2_{L^\infty(H^{r+3})} \right] \\ &+ C_u T \left(\tau^4 \max_{0 \leq l \leq 2J} \left\| \mathfrak{F}[{}_0^{RL}D_t^{\alpha_l+2} u](\omega) \right\|^2_{L^1(H^0)} + \Delta\alpha^4 \right), \ 1 \leq m \leq K. \end{aligned} \tag{3.11}$$

Proof With W defined in (3.8), we set

$$\eta^n = u^n - W^n, \quad \zeta^n = u_h^n - W^n, \qquad 0 \leq n \leq K, \tag{3.12}$$

so that

$$u^n - u_h^n = \eta^n - \zeta^n. \tag{3.13}$$

Since estimates of η^n are known from Lemmas 5 and 6, it is sufficient to bound ζ^n, then use the triangle inequality to bound $u^n - u_h^n$.

Firstly, from (1.1), (2.3)–(2.4), (2.8), (3.8) and (3.12), then for $v \in \mathscr{M}(\delta)$, we obtain

$$\begin{aligned} &\Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l) \langle D_\tau^{\alpha_l} \zeta^{n+1}, v \rangle - \kappa \langle \Delta \zeta^{n+1}, v \rangle + \nu \langle \zeta^{n+1}, v \rangle \\ &= \Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l) \langle D_\tau^{\alpha_l} \eta^{n+1}, v \rangle + \nu \langle \eta^{n+1}, v \rangle + \langle R^{n+1}, v \rangle, \ 0 \leq n \leq K - 1, \end{aligned} \tag{3.14}$$

here, by using (2.7)

$$|R^{n+1}| \leq c_u \max_{0 \leq l \leq 2J} \left\| \mathfrak{F}[{}_0^{RL}D_t^{\alpha_l+2} u](\omega) \right\|_{L^1} \tau^2 + c\Delta\alpha^2. \tag{3.15}$$

Taking $v = \zeta^{n+1}$ in (3.14), we have

$$\Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l) \tau^{-\alpha_l} \left\langle \sum_{k=0}^{n+1} \lambda_k^{(\alpha_l)} \zeta^{n-k+1}, \zeta^{n+1} \right\rangle - \kappa \langle \Delta \zeta^{n+1}, \zeta^{n+1} \rangle + \nu \|\zeta^{n+1}\|_{\mathscr{M}_r}^2$$
$$= \Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l) \langle D_\tau^{\alpha_l} \eta^{n+1}, \zeta^{n+1} \rangle$$
$$+ \nu \langle \eta^{n+1}, \zeta^{n+1} \rangle + \langle R^{n+1}, \zeta^{n+1} \rangle, \ 0 \le n \le K-1, \tag{3.16}$$

using Lemma 6, the second-to-last term in (3.16) can be bounded as

$$\nu \langle \eta^{n+1}, \zeta^{n+1} \rangle \le \nu \|\eta^{n+1}\|_{\mathscr{M}_r}^2 + \frac{\nu}{4} \|\zeta^{n+1}\|_{\mathscr{M}_r}^2$$
$$\le C\nu h^{2r+2} \|u\|_{L^\infty(H^{r+3})}^2 + \frac{\nu}{4} \|\zeta^{n+1}\|_{\mathscr{M}_r}^2. \tag{3.17}$$

Applying (2.4), (2.7) and (3.15), the last term in (3.16) can estimated as

$$\langle R^{n+1}, \zeta^{n+1} \rangle$$
$$\le \frac{1}{\nu} c_u \left(\tau^4 \max_{0 \le l \le 2J} \left\| \mathfrak{F}[{}_0^{RL} D_t^{\alpha_l+2} u](\omega) \right\|_{L^1(H^0)}^2 + \Delta\alpha^4 \right) + \frac{\nu}{4} \|\zeta^{n+1}\|_{\mathscr{M}_r}^2. \tag{3.18}$$

Finally, in order to estimate the first term on RHS of (3.16), we first define a new elliptic projection $\widetilde{W}$ of the exact solution u by $\widetilde{W}$: $[0, T] \to \mathscr{M}(\delta)$ by

$$\Delta_0^{RL} D_t^\alpha u - \Delta \widetilde{W} = 0 \ \text{ on } \ \Lambda \times [0, T],$$

then from Theorem 3.4 in [29], it follows that

$$\|{}_0^{RL} D_t^\alpha u - \widetilde{W}\| \le C h^{r+1} \|{}_0^{RL} D_t^\alpha u\|_{H^{r+3}}, \tag{3.19}$$

by introducing ρ defined by

$$-\Delta\rho = {}_0^{RL} D_t^\alpha W - \widetilde{W}, \ \ in \ \ \Omega \times [0, T],$$
$$\rho = 0, \ \ on \ \ \partial\Omega \times [0, T].$$

According the proof of Lemma 3.5 in [31], and a straightforward modification of the argument given in the proof of Theorem 2.1 of [32], we can obtain

$$\|{}_0^{RL} D_t^\alpha W - \widetilde{W}\| \le C h^{r+1} \left(\|{}_0^{RL} D_t^\alpha u\|_{H^{r+2}} + \|u\|_{H^{r+2}} \right), \tag{3.20}$$

using the triangle inequality, (3.19), (3.20), we can obtain

$$\|{}_0^{RL} D_t^\alpha u - {}_0^{RL} D_t^\alpha W\| \le C h^{r+1} \left(\|{}_0^{RL} D_t^\alpha u\|_{H^{r+3}} + \|u\|_{H^{r+2}} \right). \tag{3.21}$$

Since $D_\tau^{\alpha_l} \eta^{n+1} = {}_0^{RL} D_t^{\alpha_l} \eta^{n+1} + \mathscr{O}(\tau^2)$, then using (3.21), we have

$$\Delta\alpha \sum_{l=0}^{2J} c_l \omega(\alpha_l) \langle D_\tau^{\alpha_l} \eta^{n+1}, \zeta^{n+1} \rangle$$
$$\le \frac{2}{\nu} \Delta\alpha \sum_{l=0}^{2J} c_l^2 (\omega(\alpha_l))^2 \|D_\tau^{\alpha_l} \eta^{n+1}\|_{\mathscr{M}_r}^2 + \Delta\alpha \sum_{l=0}^{2J} \frac{\nu}{8} \|\zeta^{n+1}\|_{\mathscr{M}_r}^2$$
$$\le C h^{2r+2} \left(\max_{0 \le l \le 2J} \|{}_0^{RL} D_t^{\alpha_l} u\|_{L^\infty(H^{r+3})}^2 + \|u\|_{L^\infty(H^{r+2})}^2 \right) + C_u \tau^4 + \frac{\nu}{4} \|\zeta^{n+1}\|_{\mathscr{M}_r}^2. \tag{3.22}$$

Substituting (3.17)–(3.18) and (3.22) in (3.16), then summing up (3.16) from $n = 0$ to $n = m - 1$ $(1 \leq m \leq K)$, and using (3.4) and Lemma 4, we attain

$$\begin{aligned}
&\frac{\nu}{4}\tau \sum_{n=1}^{m} \|\zeta^n\|^2_{\mathscr{M}_r} \\
&\leq Ch^{2r+2}\tau \sum_{n=1}^{m}\left[\|u\|^2_{L^\infty(H^{r+3})} + \max_{0\leq l\leq 2J} \|{}_0^{RL}D_t^{\alpha_l} u\|^2_{L^\infty(H^{r+3})}\right] \\
&\quad + C_u\tau \sum_{n=1}^{m}\left(\tau^4 \max_{0\leq l\leq 2J}\left\|\mathfrak{F}[{}_0^{RL}D_t^{\alpha_l+2}u](\omega)\right\|^2_{L^1(H^0)} + \Delta\alpha^4\right). \qquad (3.23)
\end{aligned}$$

Therefore, using the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_{\mathscr{M}_r}$ on $\mathscr{M}(\delta)$ in Lemmas 1, 5 and 6, and the triangle inequality, (3.23) then yield the desired result.

4 Numerical Experiments

In this section, we carry out numerical experiments using the new developed numerical algorithms to illustrate our theoretical statements. In our implementations, we used the space of piecewise Hermite bicubics ($r = 3$) with the standard value and scaled slope basis functions [17] on identical uniform partitions of $\overline{I}$. The initial condition is approximated by using $u_h^0 = W^0$, the OSC elliptic projection of φ, as specified in Theorem 2. The forcing function $f(\mathbf{x}, t)$ is approximated by using interpolant projection in the collocation point. For our method, we present L^∞ and L^2 errors and the corresponding rates of convergence, also the convergence rates determined by

$$Convergence\, rate \approx \frac{\log(e_m/e_{m+1})}{\log(h_m/h_{m+1})}, \qquad (4.1)$$

where $h = 1/N_m$ is the step size with $N = N_m$, and e_m is the norm of the corresponding error.

Example 1 We consider the following problem similar to [15]:

$$\begin{cases} \mathscr{D}_t^{\omega}u - \Delta u + u = f(x, y, t), \\ u(x, y, 0) = 0, \quad (x, y) \in \Omega, \\ u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T], \end{cases}$$

where $\omega(\alpha) = \Gamma(4 - \alpha)$, $\Omega = [0, 1] \times [0, 1]$, $T = 0.5$,

$$\begin{aligned}
f(x, y, t) = 8&\left[\frac{6(t^3 - t^2)}{\ln t} + (8\pi^2 + 1 - \frac{6}{(1+x)^2} - \frac{6}{(1+y)^2})t^3\right]\frac{\sin(2\pi x)\sin(2\pi y)}{(1+x)^2(1+y)^2} \\
&+ 8t^3[8\pi\frac{\cos(2\pi x)\sin(2\pi y)}{(1+x)^3(1+y)^2} + 8\pi\frac{\sin(2\pi x)\cos(2\pi y)}{(1+x)^2(1+y)^3}].
\end{aligned}$$

The exact solution of the example is given by $u(x, y, t) = 8t^3\frac{\sin(2\pi x)\sin(2\pi y)}{(1+x)^2(1+y)^2}$.

From our theoretical estimates, the numerical convergence order of WSGD-OSC (2.8) is expected to be $\mathscr{O}(\tau^2 + \Delta\alpha^2 + h^4)$ when $r = 3$, so we can select the time steps $\tau = h^2$ (with J big enough) to verify the second order accuracy in t and fourth order accuracy in space concurrently for our proposed method. In Table 1, we take $\tau = h^2$, and fix $J = 100$ big

Table 1 Example 1: Numerical convergence orders in spatial and temporal direction with $\tau = h^2$, $J = 100$

N	L^2 error	Rate	L^∞ error	Rate
4	1.4199e−03		2.2491e−03	
8	9.1024e−05	3.9634	1.6154e−04	3.7994
16	5.7862e−06	3.9756	9.9709e−06	4.0180
32	3.6122e−07	4.0017	6.1927e−07	4.0091

Table 2 Example 1: Numerical convergence orders in distributed-order variable with fixing $\tau = h^2 = 1/32^2$

J	L^2 error	Rate	L^∞ error	Rate
2	2.0748e−04		3.8251e−04	
4	5.1790e−05	2.0022	9.5440e−05	2.0028
8	1.2900e−05	2.0053	2.3723e−05	2.0083
16	3.1938e−06	2.0140	5.8383e−06	2.0227

Table 3 Example 1: Numerical convergence orders in distributed-order variable and spatial direction with $\Delta\alpha = h^2$, $K = 2000$

N	L^2 error	Rate	L^∞ error	Rate
4	1.4199e−03		2.2491e−03	
8	9.1024e−05	3.9634	1.6154e−04	3.7994
16	5.7862e−06	3.9756	9.9709e−06	4.0180
32	3.6122e−07	4.0017	6.1927e−07	4.0091

enough to eliminate the errors caused by quadrature errors in the distributed-order variable. The data in Table 1 verify 2nd order accuracy in time and 4th order in space concurrently.

We can check the numerical accuracy in distributed-order variable by choosing $\tau = h^2$ small enough to eliminate the errors caused by temporal and spatial discretization. In Table 2 by fixing the time and space steps small enough ($\tau = h^2 = 1/32^2$), we also verify 2nd order accuracy in distributed-order variable.

Also we can select $\Delta\alpha = h^2$ (with K big enough to eliminate the contamination of the temporal error) to verify the second order accuracy in distributed-order variable and fourth order accuracy in space simultaneously. Table 3 shows the expected 2nd order accuracy in distributed-order variable and 4th order in space with $\Delta\alpha = h^2$ and $K = 2000$.

At the same time, in Table 4, we choose $\tau = \Delta\alpha = h$ so that the error stemming from the spatial approximation is negligible. In this case, we can verify 2nd order accuracy in time and distributed-order variable, simultaneously. The expected convergence rates can be seen in Table 4.

At last, with an optimal step ratio we can choose $\tau = \Delta\alpha = h^2$ to verify the second order accuracy in t, the second order accuracy in distributed-order variable, and fourth order accuracy in space, simultaneously. Table 5 lists the solution errors on the gradually refined grids, from which, one can read that, as the step sizes are reduced by a factor of 2, the errors are approximately decreased by a factor of 16. Hence, the convergence order in three directions matched that of the theoretical one.

Table 4 Example 1: Numerical convergence orders in temporal direction and distributed-order variable with $\tau = \Delta\alpha = h$

N	L^2 error	Rate	L^∞ error	Rate
10	1.4260e−03		2.6417e−03	
20	3.7593e−04	1.9234	7.0003e−04	1.9160
40	9.6403e−04	1.9633	1.7729e−04	1.9813
80	2.4402e−05	1.9821	4.4649e−05	1.9894
160	6.1381e−05	1.9911	1.1196e−05	1.9956
320	1.5393e−06	1.9955	2.8038e−06	1.9975

Table 5 Example 1: Numerical convergence orders in temporal, spatial direction and distributed-order variable with $\tau = \Delta\alpha = h^2$

N	L^2 error	Rate	L^∞ error	Rate
6	2.8462e−04		5.0224e−04	
12	1.8174e−05	3.9691	3.1017e−05	4.0172
24	1.1532e−06	3.9782	1.9661e−06	3.9797
48	7.2727e−08	3.9870	1.2272e−07	4.0019

Example 2 We consider the following problem similar to [11]:

$$\begin{cases} \mathscr{D}_t^{\omega} u - \Delta u + u = (\frac{t-1}{t \ln t}\Gamma(p+1) + 8\pi^2 + 1)2^p t^p \sin(2\pi x + 2\pi y), \\ u(x, y, 0) = 0, \quad (x, y) \in \Omega, \\ u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T], \end{cases}$$

where $\omega(\alpha) = \Gamma(p + 1 - \alpha)$, $\Omega = [0, 1] \times [0, 1]$, $T = 0.5$, $p > 0$.

The exact solution of the example is given by $u(x, y, t) = 2^p t^p \sin(2\pi x + 2\pi y)$.

In Table 6, we first test the temporal and spatial accuracy and convergence rates for our proposed method, and select $\tau = h^2$ with $J = 100$ big enough so that the error stemming from the quadrature approximation in the distributed-order variable is negligible. Table 6 verifies 2nd order accuracy in time and 4th order accuracy in space for all four different values of p ($p = 3, 2.5, 2, 1.5, 1$), which are in keeping with the theoretical predictions.

In Table 7, we choose $\tau = \Delta\alpha = h$ so that the error stemming from the spatial approximation is negligible. The data in Table 7 can tell us that the numerical convergence order in time and distributed-order variable are both two for $p = 2.5, 2, 1.5$. Whereas, the convergence order of both directions are polluted for $p = 1$. The reason might be that the exact solution behaves as $u(t) = t^1$ and therefore ${}_0^{RL}D_t^{\alpha+2}u \notin L(\mathbb{R})$ as required. Also, In Table 8, we take $\tau = h$ and $J = 100$ enough small such that the dominated errors come from the approximation for temporal direction. The second-order convergence in time can be suggested by the data in Table 8 for $p = 3.5, 3, 2.5, 2$. In the same way, the convergence order in time is polluted for $p = 1, 1.5$. It may be certain regularity conditions on the analytical solutions are required to ensure the expected numerical accuracy. How to establish an estimate with respect to the data regularity (instead of solution smoothness) which is one of the interesting problems relate to the fractional diffusion model [14,33–35].

Next, we can only check the numerical accuracy in distributed-order variable with fixed small enough temporal and spatial step sizes to eliminate the errors caused by temporal and

Table 6 Example 2: Numerical convergence orders in spatial and temporal direction with $\tau = h^2$, $J = 100$

p	N	L^2 error	Rate	L^∞ error	Rate
$p = 3$	4	4.2253e−03		6.6803e−03	
	8	2.6561e−04	3.9917	3.9842e−04	4.0667
	16	1.6752e−05	3.9869	2.4420e−05	4.0282
	32	9.2980e−07	4.1713	1.3353e−06	4.1928
$p = 2.5$	4	3.4919e−03		5.5212e−03	
	8	2.1393e−04	4.0288	3.2089e−04	4.1048
	16	1.3436e−05	3.9930	1.9586e−05	4.0342
	32	7.7292e−07	4.1196	1.1100e−06	4.1412
$p = 2$	4	3.2210e−03		5.0928e−03	
	8	1.9511e−04	4.0451	2.9267e−04	4.1211
	16	1.2243e−05	3.9943	1.7848e−05	4.0354
	32	7.2600e−07	4.0758	1.0426e−06	4.0975
$p = 1.5$	4	3.1335e−03		4.9546e−03	
	8	1.8897e−04	4.0515	2.8345e−04	4.1276
	16	1.1857e−05	3.9943	1.7285e−05	4.0355
	32	7.1629e−07	4.0491	1.0287e−06	4.0706
$p = 1$	4	3.1143e−03		4.9241e−03	
	8	1.8705e−04	4.0574	2.8058e−04	4.1334
	16	1.1664e−05	4.0033	1.7004e−05	4.0445
	32	6.9517e−07	4.0686	9.9836e−06	4.0902

spatial discretization. In Table 9, we take the fixed and sufficiently small temporal and spatial step sizes $\tau = h^2 = 1/50^2$, and the expected second-order convergence of WSGD-OSC (2.8) in distributed order can be observed for all four different values of $p = 2.5, 2, 1.5, 1$.

Table 10 lists the numerical errors and convergence orders under $\tau = \Delta\alpha = h^2$ with an optimal step ratio, and the expected convergence $\mathscr{O}(\tau^2 + \Delta\alpha^2 + h^4)$ can be seen by the Table 10 with different value of p ($p = 2.5, 2, 1.5, 1$). Hence, the convergence order in three directions matched that of the theoretical one.

In the following Example 3, we would like to test the efficiency of our WSGD-done globallyOSC method if the initial data $\varphi \neq 0$. The initial condition is approximated by using $u_h^0 = W^0$, the OSC elliptic projection of φ, as specified in Theorem 2. According to our analysis, the WSGD-OSC scheme (2.9) will be used under this conditions.

Example 3 In this example, the initial data are $u(x, y, 0) = xy(1-x)(1-y)e^{-x-y}$, $x, y \in [0, 1]$, and we choose the forcing function $f(x, y, t)$ so that

$$u(x, y, t) = (1 + t^{2.5})xy(1-x)(1-y)e^{-x-y}$$

is the exact solution. where we set $\omega(\alpha) = \Gamma(3.5 - \alpha)$, $\Omega = [0, 1] \times [0, 1]$, $T = 0.5$.

The numerical results for this example are presented in Tables 11, 12, 13 and 14. In Table 11, we take $\tau = h^2$ and fix $J = 1000$ big enough to eliminate the errors caused by quadrature errors in the distributed-order variable. The data in Table 1 verify 2nd order accuracy in time and 4th order in space concurrently. In Table 12 by fixing the time and space

Table 7 Example 2: Numerical convergence orders in temporal direction and distributed-order variable with $\tau = \Delta\alpha = h$

p	N	L^2 error	Rate	L^∞ error	Rate
$p = 2.5$	10	7.9206e−04		1.1173e−03	
	20	1.8505e−04	2.0977	2.6817e−04	2.0588
	40	4.5609e−05	2.0205	6.5301e−05	2.0380
	80	1.1381e−05	2.0027	1.6196e−05	2.0115
	160	2.8466e−06	1.9993	4.0383e−06	2.0038
	320	7.1191e−07	1.9995	1.0090e−06	2.0008
$p = 2$	10	1.9440e−04		2.7424e−04	
	20	3.3552e−05	2.5346	4.8621e−05	2.4958
	40	7.4099e−06	2.1789	1.0609e−05	2.1963
	80	1.7879e−06	2.0512	2.5443e−06	2.0599
	160	4.4261e−07	2.0142	6.2792e−07	2.0186
	320	1.1020e−07	2.0059	1.5677e−07	2.0019
$p = 1.5$	10	2.7677e−05		3.9043e−05	
	20	6.7720e−06	2.0310	9.8136e−06	1.9922
	40	2.5325e−06	1.4190	3.6260e−06	1.4364
	80	6.8623e−07	1.8838	9.7653e−07	1.8926
	160	1.7613e−07	1.9621	2.4988e−07	1.9664
	320	4.5077e−08	1.9662	6.4570e−08	1.9523
$p = 1$	10	1.8990e−05		2.6788e−05	
	20	1.0477e−05	0.8580	1.5182e−05	0.8192
	40	3.8977e−06	1.4265	5.5806e−06	1.4439
	80	1.2120e−06	1.6852	1.7247e−06	1.6941
	160	3.9064e−07	1.6335	5.5419e−07	1.6379
	320	1.3693e−07	1.5124	1.9470e−07	1.5091

steps small enough ($\tau = h^2 = 1/32^2$), we also verify 2nd order accuracy in distributed-order variable. In Table 13, we choose $\tau = \Delta\alpha = h$ so that the error stemming from the spatial approximation is negligible. We can verify 2nd order accuracy in time and distributed-order variable, simultaneously. At last, in Table 14 with an optimal step ratio we can choose $\tau = \Delta\alpha = h^2$ to verify the second order accuracy in time and distributed-order variable, and fourth order accuracy in space, simultaneously. Hence, the convergence order in three directions matched that of the theoretical one. Once again, we obtain the similar results as the Example 1 for the initial data $\varphi \neq 0$.

In the last Example 4, we would like to test the efficiency of WSGD-OSC method where the exact solution cannot be found readily.

Example 4 In this example, the initial data are $u(x, y, 0) = \sin(\pi x + \pi y)$, $x, y \in [0, 1]$, and the forcing function

$$f(x, y, t) = \left(\left(\frac{\Gamma(\frac{11}{3})(t^{\frac{8}{3}} - t^{\frac{5}{3}})}{lnt} - (1 - 2\pi^2)t^{\frac{8}{3}}\right)\sin(\pi(x + y)) + 4\pi\cos(\pi(x + y))t^{\frac{8}{3}}\right)e^{-x-y}.$$

where we set $\omega(\alpha) = \Gamma(11/3 - \alpha)$, $\Omega = [0, 1] \times [0, 1]$, $T = 0.5$.

Table 8 Example 2: Numerical convergence orders in temporal direction with $\tau = h$, $J = 100$

p	N	L^2 error	Rate	L^∞ error	Rate
$p = 3.5$	30	8.2848e−04		1.1845e−03	
	60	2.1248e−04	1.9631	3.0299e−04	1.9669
	120	5.3651e−05	1.9857	7.6190e−05	1.9916
	240	1.3320e−05	2.0100	1.8877e−05	2.0130
$p = 3$	30	2.9110e−04		4.1619e−04	
	60	7.3603e−05	1.9837	1.0495e−04	1.9875
	120	1.8426e−05	1.9980	2.6167e−05	2.0039
	240	4.5185e−06	2.0278	6.4034e−06	2.0308
$p = 2.5$	30	8.4863e−05		1.2133e−04	
	60	2.1060e−05	2.0106	3.0030e−05	2.0145
	120	5.2043e−06	2.0167	7.3906e−06	2.0226
	240	1.2407e−06	2.0685	1.7582e−06	2.0716
$p = 2$	30	1.5808e−05		2.2602e−05	
	60	3.7170e−06	2.0884	5.3002e−06	2.0923
	120	8.7849e−07	2.0810	1.2476e−06	2.0869
	240	1.8092e−07	2.2797	2.5645e−07	2.2824
$p = 1.5$	30	2.6497e−06		3.7884e−06	
	60	8.5093e−07	1.6387	1.2134e−06	1.6425
	120	2.4976e−07	1.7685	3.5471e−07	1.7743
	240	8.9595e−08	1.4791	1.2703e−07	1.4815
$p = 1$	30	5.0556e−06		7.2280e−06	
	60	1.7053e−06	1.5679	2.4317e−06	1.5716
	120	5.7270e−07	1.5742	8.1332e−07	1.5801
	240	2.1719e−07	1.3988	3.0785e−07	1.4016

Table 9 Example 2: Numerical convergence orders in distributed-order variable with fixing $\tau = h^2 = 1/50^2$

p	J	L^2 error	Rate	p	J	L^2 error	Rate
$p = 2.5$	2	2.0250e−04		$p = 2$	2	1.2500e−04	
	4	5.0514e−05	2.0032		4	3.1156e−05	2.0043
	8	1.2520e−05	2.0124		8	7.6911e−06	2.0182
	16	3.0222e−06	2.0506		16	1.8247e−06	2.0755
$p = 1.5$	2	8.4587e−05		$p = 1$	2	6.5337e−05	
	4	2.1057e−05	2.0061		4	1.6257e−05	2.0068
	8	5.1700e−06	2.0261		8	3.9773e−06	2.0312
	16	1.1978e−06	2.1098		16	9.0678e−07	2.1330

In order to confirm the expected convergence rates in space, we take the numerical solution with $N = 48$, $M = 2304$, $J = 1000$ as the "true" solution, and select $\tau = h^2$, $J = 1000$. Just as we hope, the results in Table 15 affirm the expected convergence rates of 4th (when $r = 3$) order in space and 2nd order accuracy in time.

Table 10 Example 2: Numerical convergence orders in temporal, spatial direction and distributed-order variable with $\tau = \Delta\alpha = h^2$

p	N	L^2 error	Rate	L^∞ error	Rate
$p = 2.5$	6	6.7442e−04		8.9218e−04	
	12	4.2350e−05	3.9932	6.2350e−05	3.8392
	24	2.6775e−06	3.9834	3.8647e−06	4.0117
	48	1.6870e−07	3.9884	2.4106e−07	4.0029
$p = 2$	6	6.1749e−04		8.1686e−04	
	12	3.8589e−05	4.0002	5.6802e−05	3.8461
	24	2.4369e−06	3.9851	3.5174e−06	4.0134
	48	1.5350e−07	3.9887	2.1933e−06	4.0033
$p = 1.5$	6	5.9910e−04		7.9253e−04	
	12	3.7368e−05	4.0029	5.5004e−05	3.8489
	24	2.3582e−06	3.9860	3.4037e−06	4.0144
	48	1.4845e−07	3.9896	2.1212e−07	4.0042
$p = 1$	6	5.9415e−04		7.8599e−04	
	12	3.6886e−05	4.0097	5.4295e−05	3.8556
	24	2.2972e−06	4.0051	3.3157e−06	4.0334
	48	1.3798e−07	4.0573	1.9716e−07	4.0719

Table 11 Example 3: Numerical convergence orders in spatial and temporal direction with $\tau = h^2$, $J = 1000$

N	L^2 error	Rate	L^∞ error	Rate
4	2.4496e−05		3.8508e−05	
8	1.5973e−06	3.9388	2.3824e−06	4.0147
16	1.0243e−07	3.9629	1.4848e−07	4.0041
32	6.4224e−09	3.9954	9.2154e−09	4.0101

Table 12 Example 3: Numerical convergence orders in distributed-order variable with fixing $\tau = h^2 = 1/32^2$

J	L^2 error	Rate	L^∞ error	Rate
2	1.9305e−05		2.7639e−05	
4	4.8160e−06	2.0031	6.8950e−06	2.0031
8	1.1988e−06	2.0062	1.7163e−06	2.0062
16	2.9480e−07	2.0238	4.2204e−07	2.0239

We take the numerical solution with $N = M = J = 320$ as the "true" solution when verifying the temporal and distributed-order accuracy and convergence rates for our proposed method, and select $\tau = h = \Delta\alpha$ so that the error stemming from the spatial approximation is negligible. Table 16 verifies 2nd order accuracy in time and in distributed-order variable, which are in keeping with the theoretical predictions.

In Table 17, we take the numerical solution with $N = 48$, $M = 2304$, $J = 2304$ as the "true" solution and select $\tau = \Delta\alpha = h^2$. Table 17 verifies the second order accuracy in

Table 13 Example 3: Numerical convergence orders in temporal direction and distributed-order variable with $\tau = \Delta\alpha = h$

N	L^2 error	Rate	L^∞ error	Rate
10	7.0609e−05		1.0383e−04	
20	1.7826e−05	1.9859	2.5722e−05	2.0131
40	4.4884e−06	1.9897	6.4040e−06	2.0060
80	1.1264e−06	1.9945	1.5979e−06	2.0028
160	2.8216e−07	1.9971	3.9904e−07	2.0016
320	7.0605e−08	1.9987	9.9700e−08	2.0009

Table 14 Example 3: Numerical convergence orders in temporal, spatial direction and distributed-order variable with $\tau = \Delta\alpha = h^2$

N	L^2 error	Rate	L^∞ error	Rate
4	2.3382e−05		3.6762e−05	
8	1.5252e−06	3.9383	2.2752e−06	4.0141
16	9.7866e−08	3.9620	1.4188e−07	4.0033
32	6.2053e−09	3.9792	8.9046e−09	3.9940

Table 15 Example 4: Numerical convergence orders in spatial and temporal direction with $\tau = h^2$, $J = 1000$

N	L^2 error	Rate	L^∞ error	Rate
4	1.5430e−04		2.1506e−04	
8	9.6805e−06	3.9945	1.3487e−05	3.9951
16	1.9133e−06	3.9986	2.7137e−06	3.9545
32	6.0533e−07	4.0003	8.6029e−07	3.9933

Table 16 Example 4: Numerical convergence orders in temporal direction and distributed-order variable with $\tau = \Delta\alpha = h$

N	L^2 error	Rate	L^∞ error	Rate
10	3.7715e−04		5.3117e−04	
20	9.4218e−05	2.0011	1.3328e−04	1.9947
40	2.3576e−05	1.9987	3.3381e−05	1.9974
80	5.8973e−06	1.9992	8.3524e−06	1.9988

time and in distributed-order variable, and fourth order accuracy in space, simultaneously. So, from the three Tables 15, 16 and 17, we can see that the scheme still works properly in this situation.

Table 17 Example 4: Numerical convergence orders in temporal, spatial direction and distributed-order variable with $\tau = \Delta\alpha = h^2$

N	L^2 error	Rate	L^∞ error	Rate
4	1.4941e−04		2.0815e−04	
8	9.3777e−06	3.9939	1.3075e−05	3.9927
12	1.8538e−06	3.9981	2.6303e−06	3.9550
16	5.8671e−07	3.9991	8.3399e−07	3.9927

5 Conclusion

In the present work, we have developed an effective WSGD-OSC scheme for the two-dimensional distributed order time fractional reaction diffusion equation (1.1)–(1.3). So far, we have not found any reports on the OSC method for solving the problem. Based on WSGD approximation in time, we establish discrete-time OSC method. The unconditional stability and convergence are strictly analyzed. Numerical experiments illustrate the efficiency and numerical accuracy of the proposed WSGD-OSC scheme. In the future, we will prove the stability and convergence in the H^2-norm. Future work of OSC methods will involve extending to problem involving space fractional diffusions. Extension of the OSC method to evolving domain in R^3 is also of interest for our future research.

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