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Unconditionally Optimal Error Estimates of a Linearized Galerkin Method for Nonlinear Time Fractional Reaction–Subdiffusion Equations

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Abstract This paper is concerned with unconditionally optimal error estimates of linearized Galerkin finite element methods to numerically solve some multi-dimensional fractional reaction–subdiffusion equations, while the classical analysis for numerical approximation of multi-dimensional nonlinear parabolic problems usually require a restriction on the time-step, which is dependent on the spatial grid size. To obtain the unconditionally optimal error estimates, the key point is to obtain the boundedness of numerical solutions in the L^∞-norm. For this, we introduce a time-discrete elliptic equation, construct an energy function for the nonlocal problem, and handle the error summation properly. Compared with integer-order nonlinear problems, the nonlocal convolution in the time fractional derivative causes much difficulties in developing and analyzing numerical schemes. Numerical examples are given to validate our theoretical results.

Keywords Unconditionally optimal error estimates · Linearized Galerkin method · Nonlinear fractional reaction–subdiffusion equations · High-dimensional nonlinear problems

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1 Introduction

We design stable numerical schemes and prove unconditionally optimal error estimates for multi-dimensional nonlinear fractional reaction–subdiffusion equations given by

$$u_t = {}_0^{RL}\mathcal{D}_t^{1-\gamma_1} \Delta u - \mu^2 \, {}_0^{RL}\mathcal{D}_t^{1-\gamma_2} u + f(u), \quad (x,t) \in \Omega \times [0,T] \tag{1.1}$$

with the boundary condition

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad \text{for } x \in \Omega. \tag{1.3}$$

Here $f \in C^2(\mathbb{R})$ represents the nonlinear source, the computational domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is a bounded, smooth and convex polygon/polyhedron, the given constant parameters satisfy $0 < \gamma_1, \gamma_2 < 1$ and $\mu > 0$, and ${}_0^{RL}\mathcal{D}_t^{1-\gamma} u$ stands for the Riemann-Liouville fractional derivative of order $1-\gamma$ defined by

$${}_0^{RL}\mathcal{D}_t^{1-\gamma} u(x,t) = \frac{1}{\Gamma(\gamma)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,s)}{(t-s)^{1-\gamma}} ds.$$

Fractional reaction–subdiffusion equation (1.1) is believed to provide a powerful tool for modeling plenty of nature phenomena in physics [2,4], biology [11,32,33] and chemistry [34,40,41]. Due to potential applications, the numerical simulation and analysis of fractional differential equations have received much attentions. For example, Lin et al. [26] presented a finite difference/Legendre spectral scheme for solving the linear time fractional cable equation. Yu and Jiang [39] studied stability and convergence of a fourth-order compact finite difference scheme. Langlands and Henry [18] considered accuracy and stability of an implicit solution method for the fractional diffusion equation. Jin et al. [14] presented error analysis of a Galerkin method for fractional diffusion equations. For more development of numerical methods and analysis for the fractional reaction–subdiffusion equations, we refer the readers to [6,10,15,16,23–25,30,31,44]. The works in above are interesting and instructive, but most of them focus on the analysis of numerical schemes for linear problems or one-dimensional problems.

The error estimate for high-dimensional nonlinear problems generally requires to prove the boundedness of numerical solutions in L^∞-norm. For this, the usual analysis may lead to certain stepsize restriction condition $\tau = O(h^c)$, where τ is the temporal stepsize, h is the spacial stepsize and c is a constant. One important reason is the application of induction methods with the following inverse inequality to bound the numerical solution, namely,

$$\begin{aligned} \|U_h^n\|_{L^\infty} &\le \|R_h u^n\|_{L^\infty} + \|R_h u^n - U_h^n\|_{L^\infty} \\ &\le \|R_h u^n\|_{L^\infty} + Ch^{-d/2} \|R_h u^n - U_h^n\|_{L^2} \\ &\le \|R_h u^n\|_{L^\infty} + Ch^{-d/2} \left(\tau^p + h^{r+1}\right). \end{aligned} \tag{1.4}$$

Here and below, R_h represents the projection operator, d is the dimension, p and $r+1$ are the convergence orders in the temporal and spacial directions, respectively, u^n and U_h^n are the exact and numerical solution, respectively. The above inequality results in the stepsize restriction condition $\tau = o(h^{\frac{d}{2p}})$. For more error estimates of high-dimensional nonlinear problems, various stepsize restriction conditions often appear in literatures, see [1,5,7,28, 29,42]. The time step restrictions (i.e., $\tau = o(h^c)$) may lead to the use of an unnecessarily small time step and make the computations much more time-consuming. The issue could become more serious when numerical methods are applied to simulate the nonlinear time fractional problems.

On the other hand, many works on error estimates of high-dimensional nonlinear time fractional differential equations [27,43,45,46] based on the assumption that the nonlinear term satisfies the lipschitz condition, i.e.,

$$\|f(u^n) - f(U_h^n)\| \le L\|u^n - U_h^n\|.$$

The above assumption is equivalent to

$$\|f(u^n) - f(U_h^n)\| = \|f'(\xi)(u^n - U_h^n)\| \le \|f'(\xi)\|\|u^n - U_h^n\|,$$

where the mean value theorem is used and ξ depends on exact solutions u^n and numerical solutions U_h^n in L^∞-norm. It implies that the stepsize restriction $\tau = O(h^c)$ is also required while the boundedness of $\|U_h^n\|_{L^\infty}$ is obtained by the inequality (1.4).

The goal of this paper is to advance the numerical analysis of a linearized Galkerin method for the multi-dimensional nonlinear fractional reaction–subdiffusion equation (1.1) to the same level as that obtaining for linear problems. Our analysis will show that the convergence order of the fully discrete numerical method is of $\mathcal{O}(\tau + h^{r+1})$, and the error estimate holds without any time-step restriction $\tau = \mathcal{O}(h^c)$. In order to estimate optimal error estimates, we adopt the idea for the integer order parabolic equations by first introducing a time-discrete fractional system, and prove suitable regularities of the solution U^n to the time discrete fractional system, with which, the boundedness of the finite element approximation U_h^n in L^∞-norm is obtained via

$$\begin{aligned}\|U_h^n\|_{L^\infty} &\le \|R_hU^n\|_{L^\infty} + \|R_hU^n - U_h^n\|_{L^\infty}\\ &\le \|R_hU^n\|_{L^\infty} + Ch^{-d/2}\|R_hU^n - U_h^n\|_{L^2}\\ &\le \|R_hU^n\|_{L^\infty} + Ch^{-d/2}h^2\\ &\le C,\end{aligned}$$

when τ and h are sufficiently small, respectively. After that, the unconditional optimal error estimates are obtained by using temporal-spatial error splitting argument proposed in [19] for the convergence analysis of the nonlinear integer-order parabolic problem.

While the temporal-spatial error splitting argument has very recently and successfully been applied to the integer-order PDEs, see [8,19–22,36,37], to avoid certain stepsize restriction condition $\tau = O(h^c)$, the extension of the spirit of this methodology to the numerical analysis for fractional nonlinear problems has so far received little attention. The main difficulty is due to the weakly singular kernel in the fractional derivative and the non-locality of the problems. The error estimates for fractional equations rely heavily on the some special constructed energy functions and rigorous study of error summation. The derivation is sharp contrast to that of the integer-order PDEs.

The outline of the paper is organized as follows. In Sect. 2, we present the linearized scheme and the main convergence result. In Sect. 3, we introduce the temporal discrete system and proved a priori estimate of the temporal and spatial errors. In Sect. 4, we present a detailed proof of the main result. In Sect. 5, numerical tests are given to verify the theoretical results of our method. Finally, conclusions and discussions are summarized in Sect. 6.

2 Fully Discrete Methods and Main Results

In this section, we present a fully discrete linearized Galerkin method for the fractional nonlinear equation (1.1) and, report the error estimate of the proposed scheme.

Let Ω be a bounded and convex domain with the smooth boundary $\partial\Omega$ in C^2. We denote $\Im_h$ by a quasi-uniform partition of Ω into triangles $\mathcal{K}_j$, $j = 1, \ldots, M$ in $\mathbb{R}^2$ or tetrahedrons in $\mathbb{R}^3$, and let $h = \max_{1\le j\le M}\{\text{diam}\mathcal{K}_j\}$ be the mesh size. For a triangle $\mathcal{K}_j$ with two nodes (or a tetrahedron with three nodes) on the boundary, we denote by $\widetilde{\mathcal{K}}_j$ the triangle with one curved edge (or a tetrahedron with one curved face) with the same nodes as $\mathcal{K}_j$. For an interior element, $\widetilde{\mathcal{K}}_j = \mathcal{K}_j$. Let $\Omega_h = \cup_1^M \mathcal{K}_j$ and $x = G(\hat{x})$ be a map from Ω_h to Ω such that for each triangle $\mathcal{K}_j$, G maps $\mathcal{K}_j$ one-to-one onto $\widetilde{\mathcal{K}}_j$ [47]. For a given partition of Ω, we denote by

$$\widehat{V}_h^r = \left\{v_h \in H_0^1(\Omega_h) \cap C^0(\Omega_h);\, v_h|_{\mathcal{K}_j} \in P_r(\mathcal{K}_j)\right\},$$

the standard finite element space on Ω_h, where $P_r(\mathcal{K}_j)$ is the space of polynomials of degree r $(r \ge 1)$ on $\mathcal{K}_j$. Moreover, we define an operator $\mathcal{G}$ on $\widehat{V}_h^r$ by $\mathcal{G}v_h(x) := v_h(G^{-1}(x))$ for $x \in \Omega$. Then, the finite element space is defined by

$$V_h = \left\{\mathcal{G}v_h : v_h \in \widehat{V}_h^r\right\}.$$

For the time discretization, we divide the interval $[0, T]$ into N equally subintervals with a time step size $\tau = T/N$. Denote by $t_n = n\tau$ and $u^n = u(x, t_n)$, $0 \le n \le N$. The numerical approximation of the Riemann–Liouville fractional derivative is given by

$$\begin{aligned}
&\frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^{t_n}\frac{u(x,s)}{(t-s)^{1-\gamma}}ds\\
&= \frac{\tau^{-1}}{\Gamma(\gamma)}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\frac{u^j}{(t_n-s)^{1-\gamma}}ds - \frac{\tau^{-1}}{\Gamma(\gamma)}\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\frac{u^j}{(t_{n-1}-s)^{1-\gamma}}ds + Q^n\\
&= \frac{\tau^{\gamma-1}}{\Gamma(\gamma+1)}\left(u^n + \sum_{j=1}^{n-1}\left(b_j^{(\gamma)} - b_{j-1}^{(\gamma)}\right)u^{n-j}\right) + Q^n,
\end{aligned} \tag{2.1}$$

where $b_j^{(\gamma)} = (j+1)^\gamma - j^\gamma$, $j = 1, \ldots, n-1$, and the local truncation error Q^n satisfies [46]

$$\|Q^n\|_{L^2} \le C b_{n-1}^{(\gamma)}\tau^\gamma, \tag{2.2}$$

where C is a constant depends on u and t.

For a sequence of functions $\{\omega^n\}_{n=0}^N$, define

$$D_\tau\omega^n = \frac{1}{\tau}\left(\omega^n - \omega^{n-1}\right), \quad D_\tau^\gamma\omega^n = \frac{\tau^{\gamma-1}}{\Gamma(\gamma+1)}\left(\omega^n + \sum_{j=1}^{n-1}\left(b_j^{(\gamma)} - b_{j-1}^{(\gamma)}\right)\omega^{n-j}\right).$$

With these notations, the fully discrete linearized Galerkin method for the nonlinear problem (1.1) is to find $U_h^n \in V_h$, $n = 1, 2, \ldots, N$, such that

$$\left(D_\tau U_h^n, v\right) + \left(D_\tau^{\gamma_1}\nabla U_h^n, \nabla v\right) + \mu^2\left(D_\tau^{\gamma_2}U_h^n, v\right) = \left(f(U_h^{n-1}), v\right), \quad n = 1, \ldots, N, \tag{2.3}$$

for any $v \in V_h$. The initial value is calculated by $U_h^0 = \Pi_h u_0$, where Π_h denotes the interpolation operator.

In this paper, we assume that the solution of problem (1.1) satisfies

$$\|u_0\|_{H^{r+1}} + \|u\|_{L^\infty((0,T);H^{r+1})} + \|u_t\|_{L^\infty((0,T);H^{r+1})} + \|u_{tt}\|_{L^\infty((0,T);H^1)} \le K, \quad (2.4)$$

where K is a constant.

Remark The goal of this work is to prove the optimal error bound for our linearized scheme without the time step-restriction (unconditional) in terms of spatial steps. For simplify, we only make the assumption (2.4), which does not involve the regularity of the solution at the initial value at $t = 0$. It will be more complicate if the singularity at $t = 0$ is analyzed for the nonuniform mesh, such as graded mesh.

The main results are addressed as follows.

Theorem 1 *Suppose that the problem* (1.1)–(1.3) *has a unique solution u satisfying* (2.4) *and $\frac{1}{2} \le \gamma_2 < 1$. Then, there exist positive constants τ_0, h_0 such that when $\tau \le \tau_0$ and $h \le h_0$, the finite element system* (2.3) *admits a unique solution U_h^n, such that*

$$\max_{0 \le n \le N} \|u^n - U_h^n\|_{L^2} \le C_0 \left(\tau + h^{r+1}\right). \quad (2.5)$$

We point out that the error estimate (2.5) holds without time-step restrictions dependent on the spatial mesh size, i.e., $\tau = \mathcal{O}(h^c)$, and the detailed proof is presented in two sections below. In what follows, we denote by C a generic constant, independent of n, h, τ and C_0, and could be different in different places.

3 Boundedness of FEM Approximations

We now introduce a time-discrete system for (1.1) and prove the boundedness of its numerical solution in L^∞-norm.

3.1 Preliminaries

Define Ritz projection operator $R_h : H_0^1(\Omega) \to V_h$ by

$$(\nabla(v - R_h v), \nabla \omega) = 0, \quad \text{for all } \omega \in V_h. \quad (3.1)$$

According to the classical FEM theory [35], we have

$$\|v - R_h v\|_{L^2} + h\|\nabla(v - R_h v)\|_{L^2} \le Ch^s \|v\|_{H^s}, \quad \forall v \in V_h, \quad 1 \le s \le r+1, \quad (3.2)$$

and the inverse inequality

$$\|v_h\|_{L^\infty} \le Ch^{-\frac{d}{2}} \|v_h\|_{L^2}, \quad \forall v_h \in V_h, \quad d = 2, 3, \quad (3.3)$$

as well as the classical interpolation theory

$$\|\Pi_h v - v\|_{L^2} + h\|\nabla(\Pi_h v - v)\|_{L^2} \le Ch^{r+1} \|v\|_{H^{r+1}}. \quad (3.4)$$

The following lemmas will play an important role in proving our main results.

Lemma 1 ([9]) *Let* $b_n^{(\gamma)} = (1+n)^\gamma - n^\gamma$ *with* $0 < \gamma < 1$. *Then*

(1) $1 = b_0^{(\gamma)} > b_1^{(\gamma)} > \cdots > b_n^{(\gamma)} \to 0$,

(2) *there exist a positive constant C such that* $\tau \le C b_k^{(\gamma)} \tau^\gamma,\ k = 1, 2, \ldots$.

Lemma 2 ([12]) *Let* τ, B *and* a_k, b_k, c_k, γ_k, *for integers* $k > 0$, *be nonnegative numbers such that*

$$a_n + \tau \sum_{k=0}^{n} b_k \le \tau \sum_{k=0}^{n} \gamma_k a_k + \tau \sum_{k=0}^{n} c_k + B, \quad \text{for } n \ge 0.$$

Suppose that $\tau\gamma_k < 1$, *for all* k, *and set* $\sigma_k = (1 - \tau\gamma_k)^{-1}$. *Then,*

$$a_n + \tau \sum_{k=0}^{n} b_k \le \Big(\tau \sum_{k=0}^{n} c_k + B\Big) \exp\Big(\tau \sum_{k=0}^{n} \gamma_k \sigma_k\Big).$$

Lemma 3 *(Friedrichs' inequality) Let* Ω *be a bounded subset of Euclidean space* $\mathbb{R}^d$ *with diameter* $\varnothing$. *Suppose that* $v : \ \Omega \to R$ *lies in the sobolev space* $W_0^{k,p}(\Omega)$, *and the trace of* v *on the boundary* $\partial\Omega$ *is zero. Then*

$$\|v\|_{L^p} \le \varnothing^k \Big(\sum_{|a|=k} \|D^\alpha v\|_{L^p}^p \Big)^{1/p},$$

where D^α *is the mixed partial derivative* $D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}$.

In order to obtain the unconditionally optimal error estimates of linearized fully discrete schemes (2.3), we now introduce a time-discrete system

$$D_\tau U^n = D_\tau^{\gamma_1} \Delta U^n - \mu^2 D_\tau^{\gamma_2} U^n + f(U^{n-1}), \quad n = 1, \ldots, N, \tag{3.5}$$

with the boundary and initial conditions given by

$$U^n(x) = 0, \quad \text{for } x \in \partial\Omega, \quad n = 1, 2, \ldots, N, \tag{3.6}$$

$$U^0(x) = u_0(x), \quad \text{for } x \in \Omega. \tag{3.7}$$

Applying solutions of the time discrete system, we split the errors into two parts

$$\|u^n - U_h^n\| \le \|u^n - U^n\| + \|U^n - U_h^n\|. \tag{3.8}$$

In the next two subsections, we shall respectively show the estimates $\|u^n - U^n\|_{H^2}$ and $\|U^n - U_h^n\|_{L^2}$.

3.2 A Primary Error Estimate of Time Discrete System

The exact solution of problem (1.1) satisfies

$$D_\tau u^n = D_\tau^{\gamma_1} \Delta u^n - \mu^2 D_\tau^{\gamma_2} u^n + f(u^{n-1}) + R_1^n, \quad n = 1, \ldots, N, \tag{3.9}$$

where the truncation error is given by

$$\begin{aligned} R_1^n = {} & D_\tau u^n - u_t^n + D_\tau^{\gamma_1} \Delta u^n - {}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_1} \Delta u \\ & - \mu^2 \Big(D_\tau^{\gamma_2} u^n - {}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_2} u \Big) + f(u^n) - f(u^{n-1}). \end{aligned}$$

By applying (2.2) and Taylor expansion, we have

$$\|R_1^n\|_{L^2} \leq C_R\tau + C_R b_{n-1}^{(\gamma_1)}\tau^{\gamma_1} + C_R b_{n-1}^{(\gamma_2)}\tau^{\gamma_2}, \tag{3.10}$$

where C_R is a constant.

Let $e^n = u^n - U^n$. Subtracting (3.5) from (3.9) gives

$$D_\tau e^n = D_\tau^{\gamma_1}\Delta e^n - \mu^2 D_\tau^{\gamma_2} e^n + f(u^{n-1}) - f(U^{n-1}) + R_1^n, \quad n = 1, \ldots, N. \tag{3.11}$$

Define

$$K_1 := \max_{1\leq n\leq N} \|u^n\|_{L^\infty} + 1.$$

We now present error estimates of $u^n - U^n$ in different norms.

Theorem 2 *Suppose that the problem* (1.1)–(1.3) *has a unique solution u satisfying* (2.4) *and* $\frac{1}{2} \leq \gamma_2 < 1$. *Then the time-discrete system* (3.5)–(3.7) *has a unique solution* U^n. *Moreover, there exists* $\tau_1^* > 0$ *such that when* $\tau \leq \tau_1^*$,

$$\|e^n\|_{H^1} + \tau^{\frac{1}{2}}\|e^n\|_{H^2} \leq C_1^*\tau, \tag{3.12}$$

$$\|U^n\|_{H^2} + \sum_{i=1}^{n} \tau\|D_\tau U^i\|_{H^2}^2 + \|D_\tau^{\gamma_2} U^n\|_{H^2} \leq C, \tag{3.13}$$

where C_1^* *is a positive constant independent of* τ, *h and* C_0 *(appeared in Theorem* 1*).*

Proof The existence and uniqueness of the solution U^n can be easily obtained since (3.5)–(3.7) is a linear elliptic problem. We begin to prove (3.12) by using mathematical induction. First, (3.12) holds obviously for $n = 0$. Then, we assume that (3.12) hold for $n = 0, 1, \ldots, k-1$. Therefore, for $n = 0, 1, \ldots, k-1$, we have

$$\|U^n\|_{L^\infty} \leq \|u^n\|_{L^\infty} + C\|e^n\|_{H^2} \leq \|u^n\|_{L^\infty} + CC_1^*\tau^{\frac{1}{2}} \leq K_1,$$

when $\tau \leq \tau_1 = \frac{1}{(CC_1^*)^2}$.

Therefore, by mean value theorem,

$$\|f(u^{n-1}) - f(U^{n-1})\|_{L^2} = \|(f'(\xi_1)e^{n-1}\|_{L^2} \leq C_{K_1}\|e^{n-1}\|_{L^2}, \tag{3.14}$$

where ξ_1 is determined by u^{n-1} and $\|U^{n-1}\|_{L^\infty}$ and C_{K_1} is a constant independent of n and τ.

Now, let $n = k$ in (3.11). Multiplying both sides of (3.11) by e^k and integrating the result over Ω yields

$$\begin{aligned}
\|e^k\|_{L^2}^2 &= \left(e^{k-1}, e^k\right) - \tau\left(D_\tau^{\gamma_1}\nabla e^k, \nabla e^k\right) - \tau\mu^2\left(D_\tau^{\gamma_2} e^k, e^k\right) \\
&\quad + \tau\left(f\left(u^{k-1}\right) - f\left(U^{k-1}\right), e^k\right) + \tau\left(R_1^k, e^k\right) \\
&\leq \frac{\|e^{k-1}\|_{L^2}^2 + \|e^k\|_{L^2}^2}{2} + \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)} \sum_{j=1}^{k-1} \left(b_{j-1}^{(\gamma_1)} - b_j^{(\gamma_1)}\right) \frac{\|\nabla e^{k-j}\|_{L^2}^2 + \|\nabla e^k\|_{L^2}^2}{2} \\
&\quad - \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)}\|\nabla e^k\|_{L^2}^2 + \frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)} \sum_{j=1}^{k-1} \left(b_{j-1}^{(\gamma_2)} - b_j^{(\gamma_2)}\right) \frac{\|e^{k-j}\|_{L^2}^2 + \|e^k\|_{L^2}^2}{2}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}\|e^k\|_{L^2}^2+\frac{C_{K_1}\tau}{2}\left(\|e^{k-1}\|_{L^2}^2+\|e^k\|_{L^2}^2\right)+\tau\left(R_1^k,e^k\right)\\
=&\frac{\|e^{k-1}\|_{L^2}^2+\|e^k\|_{L^2}^2}{2}+\frac{\tau^{\gamma_1}}{2\Gamma(\gamma_1+1)}\sum_{j=1}^{k-1}\left(b_{j-1}^{(\gamma_1)}-b_j^{(\gamma_1)}\right)\|\nabla e^{k-j}\|_{L^2}^2\\
&-\frac{\tau^{\gamma_1}}{2\Gamma(\gamma_1+1)}\left(1+b_{k-1}^{(\gamma_1)}\right)\|\nabla e^k\|_{L^2}^2+\frac{\mu^2\tau^{\gamma_2}}{2\Gamma(\gamma_2+1)}\sum_{j=1}^{k-1}\left(b_{j-1}^{(\gamma_2)}-b_j^{(\gamma_2)}\right)\|e^{k-j}\|_{L^2}^2\\
&-\frac{\mu^2\tau^{\gamma_2}}{2\Gamma(\gamma_2+1)}\left(1+b_{k-1}^{(\gamma_2)}\right)\|e^k\|_{L^2}^2+\frac{C_{K_1}\tau}{2}\|e^k\|_{L^2}^2\\
&+\frac{C_{K_1}\tau}{2}\|e^{k-1}\|_{L^2}^2+\tau\left(R_1^k,e^k\right)
\end{aligned}\tag{3.15}
$$

where we have used the fact $b_{j-1}^{(\gamma)}>b_j^{(\gamma)}$, $\gamma=\gamma_1$ or γ_2, $j=1,2,\ldots,k$.

Let

$$
E^k=\|e^k\|_{L^2}^2+\frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)}\sum_{j=0}^{k-1}b_j^{(\gamma_1)}\|\nabla e^{k-j}\|_{L^2}^2+\frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}\sum_{j=0}^{k-1}b_j^{(\gamma_2)}\|e^{k-j}\|_{L^2}^2.\tag{3.16}
$$

Then, (3.15) can be rewritten as

$$
\begin{aligned}
E^k\le\ & E^{k-1}-\frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)}b_{k-1}^{(\gamma_1)}\|\nabla e^k\|_{L^2}^2-\frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}b_{k-1}^{(\gamma_2)}\|e^k\|_{L^2}^2+C_{K_1}\tau\|e^k\|_{L^2}^2\\
&+C_{K_1}\tau\|e^{k-1}\|_{L^2}^2+C_R\tau\left(\tau+b_{k-1}^{(\gamma_1)}\tau^{\gamma_1}+b_{k-1}^{(\gamma_2)}\tau^{\gamma_2},e^k\right)\\
\le\ & E^{k-1}-\frac{\tau^{\gamma_1}}{2\Gamma(\gamma_1+1)}b_{k-1}^{(\gamma_1)}\|\nabla e^k\|_{L^2}^2-\frac{C_1\tau^{\gamma_1}}{2\Gamma(\gamma_1+1)}b_{k-1}^{(\gamma_1)}\|e^k\|_{L^2}^2\\
&-\frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}b_{k-1}^{(\gamma_2)}\|e^k\|_{L^2}^2+C_{K_1}\tau E^k\\
&+C_{K_1}\tau E^{k-1}+C_R\left(\tau^2+b_{k-1}^{(\gamma_1)}\tau^{1+\gamma_1}+b_{k-1}^{(\gamma_2)}\tau^{1+\gamma_2},e^k\right),
\end{aligned}\tag{3.17}
$$

where we have used Lemma 3 in the second inequality (i.e., there exists a constant C_1 such that $-\|\nabla e^k\|_{L^2}^2\le -C_1\|e^k\|_{L^2}^2$).

Noting that

$$
\left(C_R\,b_{n-1}^{(\gamma_1)}\tau^{1+\gamma_1},e^k\right)\le\frac{C_1\tau^{\gamma_1}}{2\Gamma(\gamma_1+1)}b_{k-1}^{(\gamma_1)}\|e^k\|_{L^2}^2+\frac{C_R^2\Gamma(\gamma_1+1)}{2C_1}b_{k-1}^{(\gamma_1)}\tau^{\gamma_1+2},\tag{3.18}
$$

$$
\left(C_R\,b_{n-1}^{(\gamma_2)}\tau^{1+\gamma_2},e^k\right)\le\frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}b_{k-1}^{(\gamma_2)}\|e^k\|_{L^2}^2+\frac{C_R^2\Gamma(\gamma_2+1)}{4\mu^2}b_{k-1}^{(\gamma_2)}\tau^{\gamma_2+2},\tag{3.19}
$$

and by Lemma 1, there exists a constant C_2 such that

$$
\frac{\tau^{\gamma_1}}{2\Gamma(\gamma_1+1)}b_{k-1}^{(\gamma_1)}\ge C_2\tau.\tag{3.20}
$$

From (3.17), (3.18), (3.19) and (3.20), we arrive at

$$
E^k+C_2\tau\|\nabla e^k\|_{L^2}^2\le E^{k-1}+C_{K_1}\tau E^k+C_{K_1}\tau E^{k-1}+\frac{C_R^2\Gamma(\gamma_1+1)}{2C_1}b_{k-1}^{(\gamma_1)}\tau^{\gamma_1+2}
$$

$$+\frac{C_R^2\Gamma(\gamma_2+1)}{4\mu^2}b_{k-1}^{(\gamma_2)}\tau^{\gamma_2+2}+C_R^2\tau^3. \tag{3.21}$$

Summing over the above formula from 1 to k and using Lemma 2, we have that there exists a τ_2 such that when $\tau \le \tau_2$,

$$\begin{aligned}E^k+\sum_{j=1}^{k}C_2\tau\|\nabla e^j\|_{L^2}^2 &\le C_3\exp(2C_{K_1}T)\left(\sum_{j=1}^{k}b_{j-1}^{(\gamma_1)}\tau^{\gamma_1+2}+\sum_{j=1}^{k}b_{j-1}^{(\gamma_2)}\tau^{\gamma_2+2}+T\tau^2\right)\\ &= C_3\exp(2C_{K_1}T)((k\tau)^{\gamma_1}\tau^2+(k\tau)^{\gamma_2}\tau^2+T\tau^2)\\ &\le C_3\exp(2C_{K_1}T)(T^{\gamma_1}+T^{\gamma_2}+T)\tau^2\\ &\le 3TC_3\exp(2C_{K_1}T)\tau^2,\end{aligned} \tag{3.22}$$

where

$$C_3=\max\left\{\frac{C_R^2\Gamma(\gamma_1+1)}{2C_1},\ \frac{C_R^2\Gamma(\gamma_2+1)}{4\mu^2},\ C_R^2\right\}.$$

Therefore, we obtain

$$\|e^k\|_{L^2}\le\sqrt{3TC_3\exp(2C_{K_1}T)}\,\tau. \tag{3.23}$$

Next, let $n=k$ in (3.11). Multiplying both sides of (3.11) by $-\Delta e^k$ and integrating the result over Ω yields

$$\begin{aligned}\|\nabla e^k\|_{L^2}^2 &= \left(\nabla e^{k-1},\nabla e^k\right)-\tau\left(D_\tau^{\gamma_1}\Delta e^k,\Delta e^k\right)-\tau\left(D_\tau^{\gamma_2}\nabla e^k,\nabla e^k\right)\\ &\quad-\tau\left(f\left(u^{k-1}\right)-f\left(U^{k-1}\right),\Delta e^k\right)-\tau\left(\nabla R_1^k,\nabla e^k\right).\end{aligned}$$

Let

$$\begin{aligned}E_1^k &= \|\nabla e^k\|_{L^2}+\frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)}\sum_{j=0}^{k-1}b_j^{(\gamma_1)}\|\Delta e^{k-j}\|_{L^2}^2\\ &\quad+\frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}\sum_{j=0}^{k-1}b_j^{(\gamma_2)}\|\nabla e^{k-j}\|_{L^2}^2.\end{aligned} \tag{3.24}$$

Similar to the derivation of (3.21), there exists a τ_3 such that when $\tau\le\tau_3$,

$$\begin{aligned}E_1^k+C_2\tau\|\Delta e^k\|_{L^2}^2 &\le E_1^{k-1}+\overline{C}_{K_1}\tau E_1^k+\overline{C}_{K_1}\tau E_1^{k-1}+\frac{\overline{C}_R^2\Gamma(\gamma_1+1)}{2C_1}b_{k-1}^{(\gamma_1)}\tau^{\gamma_1+2}\\ &\quad+\frac{\overline{C}_R^2\Gamma(\gamma_2+1)}{4\mu^2}b_{k-1}^{(\gamma_2)}\tau^{\gamma_2+2}+\overline{C}_R^2\tau^3,\end{aligned} \tag{3.25}$$

where $\overline{C}_{K_1}$ and $\overline{C}_R$ are constants independent of τ and the induction variable k.

Summing over the above formula from 1 to k and using Lemma 2, we have that there exists a τ_3 such that when $\tau\le\tau_3$,

$$E_1^k+\sum_{j=1}^{k}C_2\tau\|\Delta e^j\|_{L^2}^2\le 3T\overline{C}_3\exp(2\overline{C}_{K_1}T)\tau^2,$$

where

$$\overline{C}_3 = \max\left\{\frac{\overline{C}_R^2\Gamma(\gamma_1+1)}{2C_1},\ \frac{\overline{C}_R^2\Gamma(\gamma_2+1)}{4\mu^2},\ \overline{C}_R^2\right\}.$$

Hence, we arrive at

$$\|\nabla e^k\|_{L^2} \le \sqrt{3T\overline{C}_3\exp(2\overline{C}_{K_1}T)}\ \tau, \tag{3.26}$$

$$\sum_{j=1}^{k}\|\triangle e^j\|_{L^2}^2 \le \frac{3T\overline{C}_3\exp(2\overline{C}_{K_1}T)}{C_2}\ \tau, \tag{3.27}$$

and

$$\|\triangle e^k\|_{L^2} \le \sqrt{\sum_{j=1}^{k}\|\triangle e^j\|_{L^2}^2} \le \sqrt{\frac{3T\overline{C}_3\exp(2\overline{C}_{K_1}T)}{C_2}}\ \tau^{\frac{1}{2}}. \tag{3.28}$$

Together with (3.23), (3.26) and (3.28), we have

$$\|e^k\|_{H^1} \le \sqrt{3TC_3\exp(2C_{K_1}T) + 3T\overline{C}_3\exp(2\overline{C}_{K_1}T)}\ \tau, \tag{3.29}$$

and

$$\|e^k\|_{H^2} \le \sqrt{3TC_3\exp(2C_{K_1}T) + 3T\overline{C}_3\exp(2\overline{C}_{K_1}T) + \frac{3T\overline{C}_3\exp(2\overline{C}_{K_1}T)}{C_2}}\ \tau^{\frac{1}{2}}. \tag{3.30}$$

Now, taking $C_1^* = \sqrt{3TC_3\exp(2C_{K_1}T) + 3T\overline{C}_3\exp(2\overline{C}_{K_1}T) + \frac{3T\overline{C}_3\exp(2\overline{C}_{K_1}T)}{C_2}}$, we conclude that (3.12) holds for $n = k$. This completes the mathematical induction.

Further, we have

$$\|U^n\|_{L^\infty} \le \|u^n\|_{L^\infty} + C\|e^n\|_{H^2} \le \|u^n\|_{L^\infty} + CC_1^*\tau^{\frac{1}{2}} \le C, \tag{3.31}$$

and

$$\begin{aligned}
\|D_\tau^{\gamma_2}U^n\|_{H^2} &\le \|D_\tau^{\gamma_2}u^n\|_{H^2} + \|D_\tau^{\gamma_2}e^n\|_{H^2}\\
&\le \|D_\tau^{\gamma_2}u^n\|_{H^2} + \frac{\tau^{\gamma_2-1}}{\Gamma(\gamma_2+1)}\left(\|e^n\|_{H^2} + \sum_{j=1}^{n-1}(b_{j-1}^{(\gamma_2)} - b_j^{(\gamma_2)})\|e^{n-j}\|_{H^2}\right)\\
&\le \|D_\tau^{\gamma_2}u^n\|_{H^2} + \frac{\tau^{\gamma_2-1}}{\Gamma(\gamma_2+1)}\left(2 - b_{n-1}^{(\gamma_2)}\right)C_1^*\tau^{\frac{1}{2}}\\
&\le C.
\end{aligned} \tag{3.32}$$

Moreover, it follows from (3.27) that

$$\sum_{i=1}^{n}\tau\|\triangle D_\tau e^i\|_{L^2}^2 \le 4\tau^{-2}\sum_{i=1}^{n}\tau\|\triangle e^i\|_{L^2}^2 \le \frac{12T\overline{C}_3\exp(2\overline{C}_{K_1}T)}{C_2}.$$

By the theory of elliptic equation, $\|D_\tau e^n\|_{H^2} \le C\|\triangle D_\tau e^n\|_{L^2}$ for $n = 1, 2, \ldots, N$, we have

$$\sum_{i=1}^{n}\tau\|D_\tau e^i\|_{H^2}^2 \le C,$$

which further implies

$$\sum_{i=1}^{n} \tau \| D_\tau U^i \|_{H^2}^2 \le 4 \sum_{i=1}^{n} \tau \| D_\tau u^i \|_{H^2}^2 + 4 \sum_{i=1}^{n} \tau \| D_\tau e^i \|_{H^2}^2 \le C.$$

This completes the proof. □

3.3 A Primary Error Estimate of Fully Discrete System

The weak form of time-discrete system (3.5) satisfies

$$\left(D_\tau U^n, v\right) + \left(D_\tau^{\gamma_1} \nabla U^n, \nabla v\right) + \mu^2 \left(D_\tau^{\gamma_2} U^n, v\right) = \left(f(U^{n-1}), v\right) \tag{3.33}$$

for any $v \in V_h, n = 1, 2, \ldots, N$.

Let

$$\theta_h^n = R_h U^n - U_h^n, \quad n = 0, 1, \ldots, N.$$

Subtracting (2.3) from (3.33), we have the error equation for θ_h^n,

$$\begin{aligned} &\left(D_\tau \theta_h^n, v\right) + \left(D_\tau^{\gamma_1} \nabla \theta_h^n, \nabla v\right) + \mu^2 \left(D_\tau^{\gamma_2} \theta_h^n, v\right) \\ &= \left(f(U^{n-1}) - f(U_h^{n-1}), v\right) - \left(R_2^n, v\right), \end{aligned} \tag{3.34}$$

where

$$R_2^n = D_\tau (U^n - R_h U^n) + \mu^2 D_\tau^{\gamma_2} (U^n - R_h U^n).$$

Moreover, by (3.2), (3.12) and (3.13), we obtain

$$\sum_{i=1}^{n} \tau \| R_2^i \|_{L^2}^2 \le C \left(\sum_{i=1}^{n} \tau \| D_\tau U^n \|_{H^2}^2 + \sum_{i=1}^{n} \tau \| D_\tau^{\gamma_2} U^n \|_{H^2} \right) h^4 \le C h^4. \tag{3.35}$$

By Theorem 2, we have

$$\| R_h U^n \|_{L^\infty} \le C \| U^n \|_{H^2} \le C \| u^n \|_{H^2} + C \| e^n \|_{H^2} \le C, \quad n = 1, 2, \ldots, N.$$

Then, we can define

$$K_2 = \max_{1 \le n \le N} \| R_h U^n \|_{L^\infty} + 1.$$

Now, we are ready to present a primary error estimate of $U^n - U_h^n$ in L^2-norm.

Theorem 3 *Suppose that the problem* (1.1)–(1.3) *has a unique solution u satisfying* (2.4). *Then the finite element system* (2.3) *has a unique solution U_h^n, $n = 1, \ldots, N$, and there exist $\tau_2^* > 0$, $h_1^* > 0$ such that when $\tau \le \tau_2^*$, $h \le h_1^*$,*

$$\| U^n - U_h^n \|_{L^2} \le h^{\frac{7}{4}}, \tag{3.36}$$

$$\| U_h^n \|_{L^\infty} \le K_2. \tag{3.37}$$

Proof The existence and uniqueness of the FEM solution U_h^n hold because the coefficient matrices of system (2.3) are diagonally dominant. Next, we prove (3.36) by using mathematical induction. It is clear that (3.36) holds for $n = 0$. Next, we assume that (3.36) holds for $n \le k - 1$. Therefore, for $n = 1, 2, \ldots, k - 1$,

$$\begin{aligned} \| U_h^n \|_{L^\infty} &\le \| R_h U^n \|_{L^\infty} + \| R_h U^n - U_h^n \|_{L^\infty} \\ &\le \| R_h U^n \|_{L^\infty} + C h^{-\frac{d}{2}} \| R_h U^n - U_h^n \|_{L^2} \end{aligned}$$

$$\leq \|R_hU^n\|_{L^\infty} + Ch^{-\frac{d}{2}}h^{\frac{7}{4}}$$
$$\leq K_2,$$

for $d = 2, 3$, and $h \leq h_1 = C^{-\frac{4}{7-2d}}$. Thanks to the boundedness of $\|U^{k-1}\|_{L^\infty}$ and $\|U_h^{k-1}\|_{L^\infty}$, we have

$$\left\| f\left(U^{k-1}\right) - f\left(U_h^{k-1}\right)\right\|_{L^2} \leq C\|\left(U^{k-1} - U_h^{k-1}\right)\|_{L^2} \leq C\|\theta_h^{k-1}\|_{L^2} + Ch^2, \quad (3.38)$$

where we have used the mean value theorem and (3.4).

Now, let $n = k$ and set $v = \theta_h^k$ in (4.2). Using (3.38), we have

$$\begin{aligned}
\|\theta_h^k\|_{L^2}^2 &= \left(\theta_h^{k-1}, \theta_h^k\right) - \tau\left(D_\tau^{\gamma_1}\nabla\theta_h^k, \theta_h^k\right) \\
&\quad - \tau\mu^2\left(D_\tau^{\gamma_2}\theta_h^k, \theta_h^k\right) + \tau\left(f\left(U^{k-1}\right) - f\left(U_h^{k-1}\right), \theta_h^k\right) + \tau\left(R_2^k, \theta_h^k\right) \\
&\leq \frac{\|\theta_h^{k-1}\|_{L^2}^2 + \|\theta_h^k\|_{L^2}^2}{2} + \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)}\sum_{j=1}^{k-1}\left(b_{j-1}^{(\gamma_1)} - b_j^{(\gamma_1)}\right)\frac{\|\nabla\theta_h^{k-j}\|_{L^2}^2 + \|\nabla\theta_h^k\|_{L^2}^2}{2} \\
&\quad - \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)}\|\nabla\theta_h^k\|_{L^2}^2 + \frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}\sum_{j=1}^{k-1}\left(b_{j-1}^{(\gamma_2)} - b_j^{(\gamma_2)}\right)\frac{\|\theta_h^{k-j}\|_{L^2}^2 + \|\theta_h^k\|_{L^2}^2}{2} \\
&\quad - \frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}\|\theta_h^k\|_{L^2}^2 + C\tau\|\theta_h^{k-1}\|_{L^2}^2 + C\tau\|\theta_h^k\|_{L^2}^2 + C\tau h^4 + \tau\|R_2^k\|^2 \\
&= \frac{\|\theta_h^{k-1}\|_{L^2}^2 + \|\theta_h^k\|_{L^2}^2}{2} + \frac{\tau^{\gamma_1}}{2\Gamma(\gamma_1+1)}\sum_{j=1}^{k-1}\left(b_{j-1}^{(\gamma_1)} - b_j^{(\gamma_1)}\right)\|\nabla\theta_h^{k-j}\|_{L^2}^2 \\
&\quad - \frac{\tau^{\gamma_1}}{2\Gamma(\gamma_1+1)}\left(1 + b_{k-1}^{(\gamma_1)}\right)\|\nabla\theta_h^k\|_{L^2}^2 + \frac{\mu^2\tau^{\gamma_2}}{2\Gamma(\gamma_2+1)}\sum_{j=1}^{k-1}\left(b_{j-1}^{(\gamma_2)} - b_j^{(\gamma_2)}\right)\|\theta_h^{k-j}\|_{L^2}^2 \\
&\quad - \frac{\mu^2\tau^{\gamma_2}}{2\Gamma(\gamma_2+1)}\left(1 + b_{k-1}^{(\gamma_2)}\right)\|e^k\|_{L^2}^2 + C\tau\|\theta_h^{k-1}\|_{L^2}^2 + C\tau\|\theta_h^k\|_{L^2}^2 \\
&\quad + C\tau h^4 + \tau\|R_2^k\|^2. \qquad (3.39)
\end{aligned}$$

Let

$$\Theta^k = \|\theta_h^k\|_{L^2}^2 + \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)}\sum_{j=0}^{k-1} b_j^{(\gamma_1)}\|\nabla\theta_h^{k-j}\|_{L^2}^2 + \frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}\sum_{j=0}^{k-1} b_j^{(\gamma_2)}\|\theta_h^{k-j}\|_{L^2}^2. \quad (3.40)$$

Then, (3.39) can be rewritten as

$$\begin{aligned}
\Theta^k &\leq \Theta^{k-1} - \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)} b_{k-1}^{(\gamma_1)}\|\nabla\theta^k\|_{L^2}^2 - \frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)} b_{k-1}^{(\gamma_2)}\|\theta^k\|_{L^2}^2 + C\tau\|\theta^k\|_{L^2}^2 \\
&\quad + C\tau\|\theta^{k-1}\|_{L^2}^2 + C\tau h^4 + \tau\|R_2^n\|^2 \\
&\leq \Theta^{k-1} + C\tau\Theta^k + C\tau\Theta^{k-1} + C\tau h^4 + \tau\|R_2^k\|^2. \qquad (3.41)
\end{aligned}$$

□

Summing over (3.41) from 1 to k and then using (3.35) and Lemma 2, there exists a τ_4 such that when $\tau \leq \tau_4$,

$$\Theta^k \leq Ch^4,$$

which further implies

$$\|\theta^k\|_{L^2} \le Ch^2 \le h^{\frac{7}{4}},$$

when $h \le h_2 = C^{-4}$.

Therefore, (3.36) holds for $n = k$ by using the triangle inequality. Moreover, we have the following estimate,

$$\begin{aligned}\|U_h^n\|_{L^\infty} &\le \|U^n\|_{L^\infty} + \|U^n - U_h^n\|_{L^\infty}\\ &\le \|u^n\|_{L^\infty} + \|u^n - U^n\|_{L^\infty} + \|U^n - U_h^n\|_{L^\infty}\\ &\le \|u^n\|_{L^\infty} + C\|u^n - U^n\|_{H^2} + Ch^{-\frac{d}{2}}\|U^n - U_h^n\|_{L^2}\\ &\le \|u^n\|_{L^\infty} + CC_1^*\tau^{\frac{1}{2}} + Ch^{\frac{7}{4}-\frac{d}{2}}\\ &\le K_2,\end{aligned} \tag{3.42}$$

when $\tau \le \tau_5 = \frac{1}{(2CC_1^*)^2}$ and $h \le h_3 = (2C)^{-\frac{4}{7-2d}}$. Taking $\tau_2^* = \min\{\tau_1^*, \tau_4, \tau_5\}$ and $h_1^* = \min\{h_1, h_2, h_3\}$ we conclude that (3.36) and (3.37) hold. The proof is complete.

4 Proof of Theorem 1

We now prove the unconditionally optimal error estimate (2.5) in Theorem 1.

Proof The weak form of problem (1.1) satisfies: for any $v \in V_h$, $n = 1, 2, \ldots, N$

$$\left(u_t^n, v\right) + \left({}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_1}\nabla u, \nabla v\right) + \mu^2\left({}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_2}u, v\right) = \left(f(u^n), v\right). \tag{4.1}$$

Let

$$\eta_h^n = R_h u^n - U_h^n, \quad n = 0, 1, \ldots, N.$$

Subtracting (2.3) from (4.1), the error equation for η_h^n satisfies

$$\left(D_\tau \eta_h^n, v\right) + \left(D_\tau^{\gamma_1}\nabla\eta_h^n, \nabla v\right) + \mu^2\left(D_\tau^{\gamma_2}\eta_h^n, v\right) = (R_4^n, v), \tag{4.2}$$

where

$$\begin{aligned}R_4^n = &\left(f(u^n) - f\left(U_h^{n-1}\right)\right) + \left(D_\tau R_h u^n - u_t^n\right) - \left(D_\tau^{\gamma_1}\triangle R_h u^n - {}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_1}\triangle u\right)\\ &+ \mu^2\left(D_\tau^{\gamma_2}R_h u^n - {}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_2}u\right).\end{aligned} \tag{4.3}$$

Now, setting $v = \eta_h^n$ in (4.2), we have

$$\begin{aligned}\|\eta_h^n\|_{L^2} = &\left(\eta_h^{n-1}, \eta_h^n\right) - \tau\left(D_\tau^{\gamma_1}\nabla\eta_h^n, \nabla\eta_h^n\right) - \tau\mu^2\left(D_\tau^{\gamma_2}\eta_h^n, \eta_h^n\right) + \tau\left(R_4^n, \eta_h^n\right)\\ \le &\frac{\|\eta_h^{n-1}\|_{L^2}^2 + \|\eta_h^n\|_{L^2}^2}{2} + \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)}\sum_{j=1}^{n-1}\left(b_{j-1}^{(\gamma_1)} - b_j^{(\gamma_1)}\right)\frac{\|\nabla\eta_h^{n-j}\|_{L^2}^2 + \|\nabla\eta_h^n\|_{L^2}^2}{2}\\ &- \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)}\|\nabla\eta_h^n\|_{L^2}^2 + \frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}\sum_{j=1}^{n-1}\left(b_{j-1}^{(\gamma_2)} - b_j^{(\gamma_2)}\right)\frac{\|\eta_h^{n-j}\|_{L^2}^2 + \|\eta_h^n\|_{L^2}^2}{2}\\ &- \frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}\|\eta_h^n\|_{L^2}^2 + \tau\left(R_4^n, \eta_h^n\right)\end{aligned}$$

$$= \frac{\|\eta_h^{n-1}\|_{L^2}^2 + \|\eta_h^n\|_{L^2}^2}{2} - \frac{\tau^{\gamma_1}}{2\Gamma(\gamma_1+1)}\left(1+b_{n-1}^{(\gamma_1)}\right)\|\nabla\eta_h^n\|_{L^2}^2$$
$$+ \frac{\tau^{\gamma_1}}{2\Gamma(\gamma_1+1)}\sum_{j=1}^{n-1}\left(b_{j-1}^{(\gamma_1)} - b_j^{(\gamma_1)}\right)\|\nabla\eta_h^{n-j}\|_{L^2}^2$$
$$- \frac{\mu^2\tau^{\gamma_2}}{2\Gamma(\gamma_2+1)}\left(1+b_{n-1}^{(\gamma_2)}\right)\|\eta_h^n\|_{L^2}^2$$
$$+ \frac{\mu^2\tau^{\gamma_2}}{2\Gamma(\gamma_2+1)}\sum_{j=1}^{n-1}\left(b_{j-1}^{(\gamma_2)} - b_j^{(\gamma_2)}\right)\|\eta_h^{n-j}\|_{L^2}^2 + \tau\left(R_4^n, \eta_h^n\right). \tag{4.4}$$

Let

$$E_h^n = \|\eta_h^n\|_{L^2}^2 + \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)}\sum_{j=0}^{n-1} b_j^{(\gamma_1)}\|\nabla\eta_h^{n-j}\|_{L^2}^2 + \frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}\sum_{j=0}^{n-1} b_j^{(\gamma_2)}\|\eta_h^{n-j}\|_{L^2}^2.$$

Then, (4.4) can be rewritten as

$$E_h^n \le E_h^{n-1} - \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)} b_{n-1}^{(\gamma_1)}\|\nabla\eta_h^n\|_{L^2}^2 - \frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)} b_{n-1}^{(\gamma_2)}\|\eta_h^n\|_{L^2}^2 + 2\tau\left(R_4^n, \eta_h^n\right)$$
$$\le E_h^{n-1} - \frac{C_4\tau^{\gamma_1}}{\Gamma(\gamma_1+1)} b_{n-1}^{(\gamma_1)}\|\eta_h^n\|_{L^2}^2 - \frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)} b_{n-1}^{(\gamma_2)}\|\eta_h^n\|_{L^2}^2 + 2\tau\left(R_4^n, \eta_h^n\right), \tag{4.5}$$

where, again, we used the result in Lemma 3 (i.e., there exists a constant C_4 such that $-\|\nabla\eta_h^n\|_{L^2}^2 \le -C_4\|\eta_h^n\|_{L^2}^2$).

We now turn to estimate $\tau(R_4^n, \eta_h^n)$, where R_4^n is defined in (4.3).

By using the boundedness of $\|U_h^n\|_{L^\infty}$ in Theorem 3 and (3.2), we have

$$\tau\left(f\left(u^n\right) - f\left(U_h^{n-1}\right), \eta_h^n\right)$$
$$= \tau\left(f\left(u^n\right) - f\left(u^{n-1}\right), \eta_h^n\right) + \tau\left(f\left(u^{n-1}\right) - f\left(U_h^{n-1}\right), \eta_h^n\right)$$
$$\le \tau\|f(u^n) - f(u^{n-1})\|_{L^2}\|\eta_h^n\|_{L^2} + C\tau\|u^{n-1} - R_h u^{n-1} + \eta_h^{n-1}\|_{L^2}\|\eta_h^n\|_{L^2}$$
$$\le C\tau\left(\tau^2 + h^{2r+2}\right) + C\tau\|\eta_h^n\|_{L^2}^2 + C\tau\|\eta_h^{n-1}\|_{L^2}^2. \tag{4.6}$$

By (2.2) and (3.2), we have

$$\tau\left(D_\tau R_h u^n - u_t^n, \eta_h^n\right) = \tau\left(D_\tau R_h u^n - R_h u_t^n, \eta_h^n\right) + \tau\left(R_h u_t^n - u_t^n, \eta_h^n\right)$$
$$\le C\tau\left(\tau^2 + h^{2r+2}\right) + C\tau\|\eta_h^n\|_{L^2}^2, \tag{4.7}$$

$$-\tau\left(D_\tau^{\gamma_1}\triangle R_h u^n - {}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_1}\triangle u, \eta_h^n\right)$$
$$= -\tau\left(D_\tau^{\gamma_1}\triangle R_h u^n - {}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_1}\triangle R_h u, \eta_h^n\right) - \tau\left({}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_1}\triangle R_h u - {}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_1}\triangle u, \eta_h^n\right)$$
$$= -\tau\left(D_\tau^{\gamma_1}\triangle R_h u^n - {}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_1}\triangle R_h u, \eta_h^n\right)$$
$$\le \left(C b_{n-1}^{(\gamma_1)}\tau^{1+\gamma_1}, |\eta_h^n|\right)$$
$$\le \frac{C_4\tau^{\gamma_1}}{\Gamma(\gamma_1+1)} b_{n-1}^{(\gamma_1)}\|\eta_h^n\|_{L^2}^2 + \frac{C^2\Gamma(\gamma_1+1) b_{n-1}^{(\gamma_1)}\tau^{\gamma_1}}{4C_4}\tau^2, \tag{4.8}$$

and

$$\begin{aligned}
&\mu^2\tau\Big(D_\tau^{\gamma_2}R_hu^n-{}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_2}u,\eta_h^n\Big)\\
&\quad=\mu^2\tau\Big(D_\tau^{\gamma_2}R_hu^n-{}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_2}R_hu,\ \eta_h^n\Big)+\mu^2\tau\Big({}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_2}R_hu-{}_0^{RL}\mathcal{D}_{t_n}^{1-\gamma_2}u,\ \eta_h^n\Big)\\
&\quad\le C\mu^2\Big(b_{n-1}^{(\gamma_2)}\tau^{1+\gamma_2},|\eta_h^n|\Big)+\mu^2\tau\Big(Ch^{r+1},\ |\eta_h^n|\Big)\\
&\quad\le\frac{\mu^2\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}b_{k-1}^{(\gamma_2)}\|\eta_h^n\|_{L^2}^2+\frac{C^2\Gamma(\gamma_2+1)}{4\mu^2}b_{k-1}^{(\gamma_2)}\tau^{\gamma_2+2}+C\tau h^{2r+2}+C\tau\|\eta_h^n\|_{L^2}^2. \quad (4.9)
\end{aligned}$$

Substituting the estimates (4.6)–(4.9) into (4.5) yields

$$\begin{aligned}
E_h^n&\le E_h^{n-1}+C\tau\|\eta_h^n\|_{L^2}^2+C\tau\|\eta_h^{n-1}\|_{L^2}^2+Cb_{n-1}^{(\gamma_1)}\tau^{\gamma_1+2}+Cb_{n-1}^{(\gamma_2)}\tau^{\gamma_2+2}+C\tau^3+C\tau h^{2r+2}\\
&\le E_h^{n-1}+C\tau E_h^n+C\tau E_h^{n-1}+Cb_{n-1}^{(\gamma_1)}\tau^{\gamma_1+2}+Cb_{n-1}^{(\gamma_2)}\tau^{\gamma_2+2}+C\tau^3+C\tau h^{2r+2}
\end{aligned} \quad (4.10)$$

Summing over the above formula from 1 to n and using Lemma 2, there exists a τ_6 such that when $\tau\le\tau_6$,

$$\begin{aligned}
E_h^n&\le C\left(\sum_{j=1}^n b_{j-1}^{(\gamma_1)}\tau^{\gamma_1}\tau^2+\sum_{j=1}^n b_{j-1}^{(\gamma_2)}\tau^{\gamma_2+2}+n\tau^3+n\tau h^{2r+2}\right)\\
&\le C\left((n\tau)^{\gamma_1}\tau^2+(n\tau)^{\gamma_2}\tau^2+n\tau^3+n\tau h^{2r+2}\right)\\
&\le 3CT\Big(\tau^2+h^{2r+2}\Big)\\
&\le 3CT\Big(\tau+h^{r+1}\Big)^2,
\end{aligned} \quad (4.11)$$

which further implies

$$\|\theta^n\|_{L^2}\le C\Big(\tau+h^{r+1}\Big).$$

Therefore, taking $\tau_0=\min\{\tau_2^*,\tau_6\}$ and $h_0=h_1^*$, the finite element system (2.3) admits a unique solution U_h^n, such that

$$\|u^n-U_h^n\|_{L^2}\le C_0\Big(\tau+h^{r+1}\Big). \quad (4.12)$$

This completes the proof. □

5 Numerical Examples

In this section, several numerical experiments are carried out to illustrate the theoretical results. All the computations are performed by using the software FreeFEM++.

Example 1 Consider two dimensional Michaelis–Menten reaction equation [40,45]

$$\frac{\partial u}{\partial t}={}_0^{RL}\mathcal{D}_t^{\gamma}\Delta u-{}_0^{RL}\mathcal{D}_t^{\gamma}u+u-u^2+g,\quad x\in[0,1]^2,\quad 0<t\le 1, \quad (5.1)$$

where the function g, the initial and boundary conditions are determined by the exact solution

$$u=t^3\sin(\pi x)\sin(\pi y).$$

Table 1 L^2 errors and convergence rates

M	L-FEM		Q-FEM	
	Error	Orders	Error	Orders
2D				
4	6.72E−2	–	3.39E−3	–
8	1.79E−2	1.91	4.63E−4	2.87
16	4.57E−3	1.97	5.96E−5	2.96
32	1.15E−3	1.99	7.48E−6	2.99

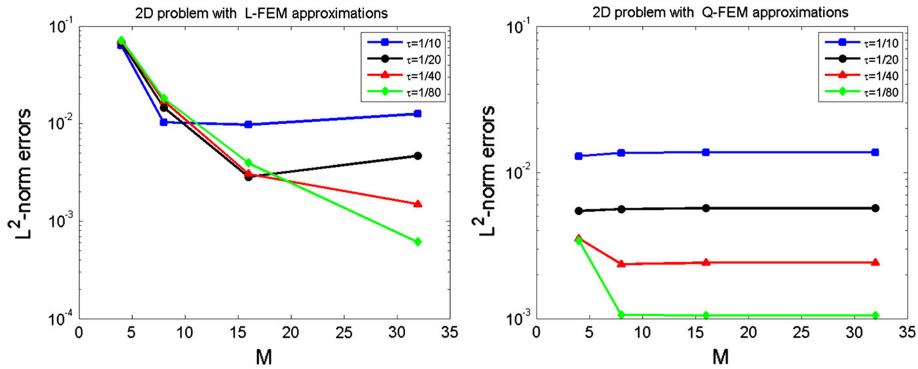

Fig. 1 2D problem: L^2-errors with fixed τ by changing spatial mesh sizes

We present the accuracy tests by using a uniform triangular partition with $M+1$ nodes in each spatial direction here and below. In the simulations, we set $\gamma = 0.6$, let $N = M^2$ and $N = M^3$ when the linear and quadratic element approximation are applied, respectively. The L^2-errors at time $T = 1$ and convergence rates for the two dimensional problem are shown in Table 1. These results imply that the numerical solutions converge to the exact solution in the order of $O(\tau + h^{r+1})$ for the r-th degree finite element methods.

To verify unconditional convergence, we solve the problem by fixing τ and changing spatial mesh size h. The L^2-norm errors at time $T = 1$ for two and three dimensional problems are shown in Figs. 1 and 2, respectively. One can learn from these two figures that for a fixed τ, the errors in L^2-norm asymptotically convergence to a constant, which implies that the time step restriction $\tau = h^c$ is unnecessary.

Example 2 Consider three dimensional nonlinear fractional cable equation [3,26,46], given by

$$\frac{\partial u}{\partial t} = {}_0^{RL}\mathcal{D}_t^{\gamma_1} \Delta u - {}_0^{RL}\mathcal{D}_t^{\gamma_2} u + u - u^3 + g, \quad x \in [0,1]^3, \quad 0 < t \leq 1, \tag{5.2}$$

where the function g, initial and boundary conditions are determined by

$$u = t^2\Big(x^2 - x^5\Big)\Big(y^2 - y^5\Big)\Big(z^2 - z^5\Big).$$

In this numerical experiment, we set the parameter $\gamma_1 = 0.2$, $\gamma_2 = 0.8$, $N = M^2$ and $N = M^3$ when the linear and quadratic element approximation are applied, respectively. The L^2-errors at time $T = 1$ and convergence rates for the three dimensional problems are listed in Table 2. These results further illustrate convergence of the proposed methods

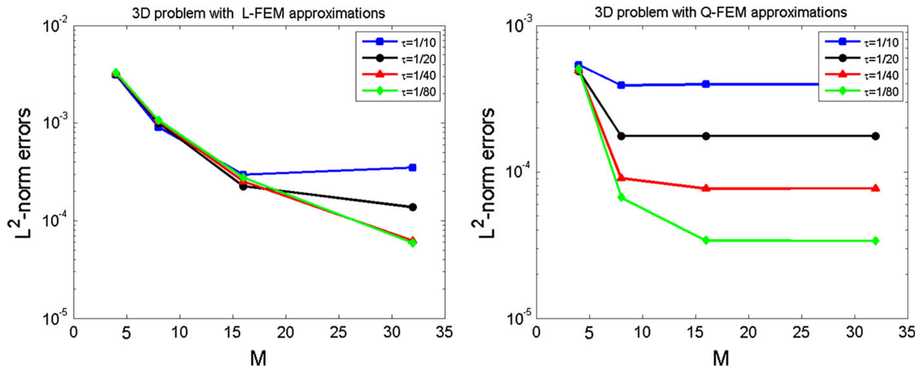

Fig. 2 3D problem: L^2-errors with fixed τ by changing spatial mesh sizes

Table 2 L^2 errors and convergence rates

M	L-FEM		Q-FEM	
	Error	Orders	Error	Orders
3D				
4	2.33E−3	–	2.64E−4	–
8	7.12E−4	1.71	3.26E−5	3.02
16	1.89E−4	1.91	4.03E−6	3.01
32	4.76E−5	1.98	5.03E−7	3.00

Again, to verify unconditional convergence, we solve the problems with fixed τ by changing spatial mesh sizes. The L^2-norm errors at time $T = 1$ for three dimensional problems are shown in Fig. 2. For a fixed τ, the errors in L^2-norm asymptotically convergence to a constant. It implies that the time step restriction is unnecessary.

6 Conclusions

In this paper, a fully discrete linearized Galerkin finite element method is proposed to solve the multi-dimensional fractional reaction–subdiffusion equations. By introducing the time-discrete elliptic equations and constructing the energy functions, we obtain the H_2 boundedness of the solution of the time-discrete system and L^∞ boundedness of the corresponding finite element solution. Then, the optimal error estimates are proved without any stepsize restriction, i.e., $\tau = O(h^c)$. Numerical examples in both two and three dimensional cases are presented to confirm our theoretical results.

Several issues are deserving further investigation. First, due to the non-locality of the problem, it is difficult to get the boundedness of $\|D_\tau^{\gamma_2} U^n\|_{H^2}$ when $0 < \gamma_2 < \frac{1}{2}$. Therefore, it is still an open problem to prove the unconditional error estimate under the assumption $0 < \gamma_2 < \frac{1}{2}$. Second, in view of the regularity of the solution to the problem (1.1), it is interesting to develop nonuniform time-step schemes to approximate the time-fractional derivative and provide rigorous error analysis. Third, the computational storage and cost are huge for long time simulations of high-dimensional problems, this motivates us to consider high-order accuracy scheme with fast evaluation of the time-fractional derivative. The resulting fast algorithm should not only keep the same accuracy as the direct evaluation of the fractional derivative, but also reduce significantly the computational storage and cost, see relative works [13,17,38]. In addition, it is interesting to investigate the global existence of the solution for the nonlinear problem (1.1).

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