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Stormer-Numerov HDG Methods for Acoustic Waves

Bernardo Cockburn[1] · **Zhixing Fu**[2] · **Allan Hungria**[2] · **Liangyue Ji**[1] · **Manuel A. Sánchez**[1,3] · **Francisco-Javier Sayas**[2]

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Abstract We introduce and analyze the first energy-conservative hybridizable discontinuous Galerkin method for the semidiscretization in space of the acoustic wave equation. We prove optimal convergence and superconvergence estimates for the semidiscrete method. We then introduce a two-step fourth-order-in-time Stormer-Numerov discretization and prove energy conservation and convergence estimates for the fully discrete method. In particular, we show that by using polynomial approximations of degree two, convergence of order four is obtained. Numerical experiments verifying that our theoretical orders of convergence are sharp are presented. We also show experiments comparing the method with dissipative methods of the same order.

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✉ Manuel A. Sánchez
sanchez@umn.edu

Bernardo Cockburn
cockburn@math.umn.edu

Zhixing Fu
zhixingfu@icloud.com

Allan Hungria
allanh@udel.edu

Liangyue Ji
jil@umn.edu

Francisco-Javier Sayas
fjsayas@udel.edu

1 School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

2 Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

3 Institute for Mathematical and Computational Engineering, School of Engineering and Faculty of Mathematics, Pontificia Universidad Católica de Chile, Santiago, Chile

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1 Introduction

We introduce energy-conservative hybridizable discontinuous Galerkin (HDG) methods for the wave equation

$$\rho\ddot{u}(t) = \operatorname{div}(\kappa\nabla u)(t) + f(t) \qquad \text{in } \Omega, \quad \forall t \geq 0, \tag{1.1a}$$
$$u(t) = g(t) \qquad \text{on } \Gamma, \quad \forall t \geq 0, \tag{1.1b}$$
$$u(0) = u_0, \qquad \text{in } \Omega, \tag{1.1c}$$
$$\dot{u}(0) = v_0 \qquad \text{in } \Omega. \tag{1.1d}$$

We will use evolution equation notation for functions of space and time, where only time-dependence is shown explicitly. Differential operators act on the space variables and time-derivatives are shown in dot notation (i.e. $\dot{u}$ denotes the first derivative). Here Ω is a bounded open polygonal set in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ denoted as Γ. We assume that $\rho, \kappa \in L^\infty(\Omega)$, $\kappa \geq \kappa_0 > 0$ and $\rho \geq \rho_0 > 0$ almost everywhere, and $f : [0, \infty) \to L^2(\Omega)$, $g : [0, \infty) \to H^{1/2}(\Gamma)$ are continuous functions of the time variable. With the help of flux variable $\mathbf{q}(t) := -\kappa\nabla u(t)$, the system (1.1) can be rewritten in terms of u and $\mathbf{q}$ as follows:

$$\mathbf{q}(t) + \kappa\nabla u(t) = 0, \qquad \text{in } \Omega, \quad \forall t \geq 0, \tag{1.2a}$$
$$\rho\ddot{u}(t) + \nabla\cdot\mathbf{q}(t) = f(t) \qquad \text{in } \Omega, \quad \forall t \geq 0, \tag{1.2b}$$
$$u(t) = g(t) \qquad \text{on } \Gamma, \quad \forall t \geq 0, \tag{1.2c}$$
$$u(0) = u_0, \qquad \text{in } \Omega, \tag{1.2d}$$
$$\dot{u}(0) = v_0 \qquad \text{in } \Omega. \tag{1.2e}$$

The system of equations (1.2) have been subject of several numerical studies. Particularly, among finite element methods approximating its solution we find the following: continuous Galerkin methods [1,12], interior penalty methods [18], mixed methods [10,11,15,19], discontinuous Galerkin methods [3,13,14,21] and hybridizable discontinuous Galerkin methods [9,16,17,22]. For a more complete description and comparison of some of these methods see [22].

Let us describe our results. The first hybridizable discontinuous Galerkin (HDG) method was introduced by Cockburn, Gopalakrishnan and Lazarov in 2009 [7] in the framework of purely diffusion problems. The hybridization of finite element methods is a technique by which the method can be *statically condensed* and hence efficiently implemented. Discontinuous Galerkin methods to which this technique can be applied are called the HDG methods. Extensive numerical and theoretical results indicate that these new methods can also be more accurate and can be applied to a wide range of PDEs.

The first HDG method for wave propagation in acoustics and elastodynamics was introduced and numerically tested in 2011 [22]. The wave equation is rewritten as a first-order system in terms of the velocity $v := \dot{u}$, the flux $\mathbf{q}$ and the original variable u. The HDG method is then used to discretize in space and get an evolution equation for the approximations to the velocity and the flux; only the elementwise average of u is evolved in time. A theoretical a priori error of the semidiscrete HDG method was then provided by Cockburn and Quenneville-Bélair in 2014 [9]. For unstructured meshes of shape-regular simplexes, they showed that the approximations to the velocity converge with the optimal order of $k+1$ in the L^2-norm uniformly in time whenever polynomials of degree $k \geq 0$ are used. They also showed that a local postprocessing of the original scalar unknown u converges with order $k+2$ for $k \geq 1$. This HDG method, however, is dissipative.

In contrast, the staggered discontinuous Galerkin (SDG) method proposed by Chung and Engquist in 2009 [5] is not. The advantage of conservative methods, like the SDG method, is that they are known to provide better approximation for a long time. The SDG method discretizes in space the above-mentioned first-order system and achieves, for the approximations to the velocity and flux, the optimal order of $k+1$ in the L^2-norm uniformly in time whenever polynomials of degree $k \geq 0$ are used. The relation between the SDG and the HDG methods was uncovered by Chung, Cockburn and Fu in 2014 [4] in the framework of steady-state diffusion problems. They showed that the SDG method can be obtained as the limit of an HDG method when the stabilization function is set to zero or sent to infinity in a suitable manner which results in a non-dissipative method in the present setting. By using this fact, one can easily prove that the local postprocessing of the original scalar unknown u used for the HDG method also converges with order $k+2$ for $k \geq 1$ for the SDG method.

Other conservative methods are those that use mixed methods to discretize in space the equations of the first-order system; see, for example, the references in [9]. However, their mass matrix is not easily invertible since the $H(div)$-conformity of the space of fluxed forces precludes it from begin block-diagonal. This is the difficulty avoided by the SDG method and by any DG method, although most DG methods for first-order hyperbolic systems are actually dissipative. A conservative, local discontinuous Galerkin (LDG) method to discretize in space (on Cartesian meshes) the second-order equations (1.2) was proposed by Chou, Shu and Xing in 2014 [3]. When using polynomials of degree k, the approximation to u is proven to converge with the optimal order of $k+1$ in the L^2-norm uniformly in time.

In this paper, we construct the first energy-conservative HDG methods for wave propagation. Unlike the HDG methods considered in [9,22], to define the method, we use the second-order system (1.2), just as done in [3] and achieve the conservation of a discrete energy simply by using the *standard* HDG numerical traces. For the semidiscrete case, we show that, just as for the HDG methods considered in [9], the approximations to the velocity and flux converge with the optimal order of $k+1$ in the L^2-norm whenever piecewise-polynomial approximations of degree $k \geq 0$ are used. We also show that, an element-by-element post-processing the approximation to u *superconverges* with order $k+2$ for $k \geq 1$. As an example of a fully discretized scheme, we consider the method obtained by applying the Stormer-Numerov time-discretization to the above HDG semidiscrete scheme; see its application and analysis to other finite element method in [20]. We display the corresponding discrete energy, show that it is conserved and prove that the optimal convergence of the velocity and flux and the superconvergence of u can also be achieved. In particular, fourth order accuracy in the approximation of u holds when polynomials of degree two are used.

The paper is organized as follows. In Sect. 2, we introduce the semidiscrete HDG method, prove its energy-conserving property, and present and discuss the main results of its a priori error analysis. In Sect. 3, we display detailed proofs. In Sect. 4, we study the full discretization of the method by using the Stormer-Numerov method. The proofs of the corresponding results are provided in Sect. 5. We end in Sect. 6 with some concluding remarks.

2 The Semidiscrete Scheme

In this section, we introduce our HDG formulation of the equation (1.2), and state and briefly discuss its convergence properties.

2.1 Notation

Throughout the paper, we will use round brackets for 'volume' integrals on an open set $D \subset \mathbb{R}^d$, $(u, v)_D := \int_D u\, v$ and $(\mathbf{q}, \mathbf{r})_D := \int_D \mathbf{q} \cdot \mathbf{r}$, and angled brackets for integrals on flat $(d-1)$-manifolds or union thereof $\langle u, v\rangle_F := \int_F u\, v \mathrm{d}F$. On the surface of any given polygonal domain D, the unit normal vector field $\mathbf{n}_{\partial D} : \partial D \to \mathbb{R}^d$ will be taken pointing outwards. We use the standard notation of the norm and seminorm on Sobolev spaces. We also write $\|\mathbf{q}\|^2_{\kappa^{-1}} := (\kappa^{-1}\mathbf{q}, \mathbf{q})_{\mathcal{T}_h}$, $\|u\|^2_\rho := (\rho u, u)_{\partial \mathcal{T}_h}$ and $|u|^2_\tau := \langle \tau\, u, u\rangle_{\partial \mathcal{T}_h}$.

To describe the HDG method, we discretize our domain by a conforming triangulation $\mathcal{T}_h$ formed of triangles ($d = 2$) or tetrahedra ($d = 3$). The set of edges ($d = 2$) or faces ($d = 3$) of the elements of the triangulation is denoted $\mathcal{E}_h$. We will collect integrals over elements or their boundaries with the following notation: $(u, v)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} (u, v)_K$, $(\mathbf{q}, \mathbf{r})_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} (\mathbf{q}, \mathbf{r})_K$ and $\langle u, v\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \langle u, v\rangle_{\partial K}$. Also $\langle u, v\rangle_{\partial\mathcal{T}_h\setminus\Gamma} := \sum_{K\in\mathcal{T}_h} \langle u, v\rangle_{\partial K\setminus(\partial K\cap\Gamma)}$ and $\langle u, \mathbf{r}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \langle u, \mathbf{r}\cdot\mathbf{n}_{\partial K}\rangle_{\partial K}$.

The finite element spaces for the HDG semidiscretization are

$$
\begin{aligned}
\boldsymbol{V}_h &:= \left\{\mathbf{q} : \Omega \to \mathbb{R}^d \ : \ \mathbf{q}|_K \in \boldsymbol{\mathcal{P}}_p(K) \quad \forall K \in \mathcal{T}_h\right\},\\
W_h &:= \left\{u : \Omega \to \mathbb{R} \ : \ u|_K \in \mathcal{P}_p(K) \quad \forall K \in \mathcal{T}_h\right\},\\
M_h &:= \left\{\widehat{u} : \cup_{e\in\mathcal{E}_h} \to \mathbb{R} \ : \ \widehat{u}|_e \in \mathcal{P}_p(e) \quad \forall e \in \mathcal{E}_h\right\}.
\end{aligned}
$$

Here $\mathcal{P}_p(K)$ is the space of d-variate polynomials of degree less than or equal to p, $\boldsymbol{\mathcal{P}}_p(K) := \mathcal{P}_p(K)^d$ and $\mathcal{P}_p(e)$ is the space of $(d-1)$-variate polynomials of degree less than or equal to p on $e \in \mathcal{E}_h$. In this paper p is a fixed (but arbitrary) non-negative integer.

2.2 The HDG Method

We look for $\mathbf{q}_h : [0, \infty) \to \boldsymbol{V}_h$, $u_h : [0, \infty) \to W_h$, and $\widehat{u}_h : [0, \infty) \to M_h$, satisfying

$$
\begin{aligned}
&\left(\kappa^{-1}\mathbf{q}_h(t), \mathbf{r}\right)_{\mathcal{T}_h} - (u_h(t), \nabla\cdot\mathbf{r})_{\mathcal{T}_h} + \langle\widehat{u}_h(t), \mathbf{r}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} = 0 && \forall \mathbf{r} \in \boldsymbol{V}_h, && (2.1a)\\
&(\rho\ddot{u}_h(t), w)_{\mathcal{T}_h} - (\mathbf{q}_h(t), \nabla w)_{\mathcal{T}_h} + \langle\widehat{\mathbf{q}}_h(t)\cdot\mathbf{n}, w\rangle_{\partial\mathcal{T}_h} = (f(t), w)_{\mathcal{T}_h} && \forall w \in W_h, && (2.1b)\\
&\widehat{\mathbf{q}}_h(t)\cdot\mathbf{n} := \mathbf{q}_h(t)\cdot\mathbf{n} + \tau(u_h(t) - \widehat{u}_h(t)) \quad \text{on } \partial\mathcal{T}_h, && && (2.1c)\\
&\langle\widehat{\mathbf{q}}_h(t)\cdot\mathbf{n}, \mu\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0 && \forall \mu \in M_h, && (2.1d)\\
&\langle\widehat{u}_h(t), \mu\rangle_\Gamma = \langle g(t), \mu\rangle_\Gamma && \forall \mu \in M_h, && (2.1e)
\end{aligned}
$$

for all $t \geq 0$, as well as the initial condition

$$
u_h(0) = u_{h,0}, \qquad \dot{u}_h(0) = v_{h,0}. \tag{2.1f}
$$

The initial data are suitably defined approximations in W_h of the initial data u_0 and v_0. The stabilization function τ is independent of t and is defined to be piecewise constant and non-negative on ∂K for all K. We assume that for each $K \in \mathcal{T}_h$, there exists at least one $e \in \mathcal{E}(K)$ (the set of edges/faces of K) where τ is strictly positive. Equation (2.1d) is then equivalent to demanding that the normal component of the numerical flux $\widehat{\mathbf{q}}_h(t)$ be single-valued on internal edges/faces of the triangulation. With some algebra it is possible to show that (2.1) can be expressed as a second order linearly implicit system of linear ordinary differential equations in the variable u_h with initial data (2.1f). Therefore, the initial value problem (2.1) has a unique solution $\mathbf{q}_h$, u_h and $\widehat{u}_h$ if $f : [0,\infty) \to L^2(\Omega)$ and $g : [0,\infty) \to L^2(\Gamma)$ are continuous.

We end the definition the HDG method by describing the element-by-element post-processing technique to compute the approximation $u_h^*(t)$ at any time $t \geq 0$. For any $K \in \mathcal{T}_h$ we define a new approximate displacement $u_h^*|_K \in \mathcal{P}_{p+1}(K)$ determined by

$$\begin{aligned} \left(\nabla u_h^*(t), \nabla w\right)_K &= (\mathbf{q}_h(t), \nabla w)_K\,, \quad \forall\, w \in \mathcal{P}_{p+1}(K),\\ \left(u_h^*(t), 1\right)_K &= (u_h(t), 1)_K\,. \end{aligned} \tag{2.2}$$

2.3 Energy Conservation

We begin by showing that the HDG method has the following two energy conservation properties.

Proposition 2.1 (Energy identities) *If* $(\mathbf{q}_h(t), u_h(t), \widehat{u}_h(t))$ *is a solution of (2.1) and*

$$\begin{aligned} E_h(t) &:= \tfrac{1}{2}\|\mathbf{q}_h(t)\|^2_{\kappa^{-1}} + \tfrac{1}{2}\|\dot{u}_h(t)\|^2_\rho + \tfrac{1}{2}|u_h(t) - \widehat{u}_h(t)|^2_\tau\\ F_h(t) &:= \tfrac{1}{2}\|\dot{\mathbf{q}}_h(t)\|^2_{\kappa^{-1}} + \tfrac{1}{2}\|\ddot{u}_h(t)\|^2_\rho + \tfrac{1}{2}|\dot{u}_h(t) - \dot{\widehat{u}}_h(t)|^2_\tau, \end{aligned}$$

then

$$\begin{aligned} \dot{E}_h(t) &= (f(t), \dot{u}_h(t))_{\mathcal{T}_h} - \langle \dot{g}(t), \mathbf{q}_h(t)\cdot\mathbf{n} + \tau(u_h(t) - \widehat{u}_h(t)\rangle_\Gamma,\\ \dot{F}_h(t) &= (\dot{f}(t), \ddot{u}_h(t))_{\mathcal{T}_h} - \langle \ddot{g}(t), \dot{\mathbf{q}}_h(t)\cdot\mathbf{n} + \tau(\dot{u}_h(t) - \dot{\widehat{u}}_h(t)\rangle_\Gamma. \end{aligned}$$

Note that, when $f \equiv 0$ and g is independent of time, the energies $E_h(t)$ and $F_h(t)$ are conserved.

Proof Differentiate (2.1a) with respect to time, and test the first equation with $\mathbf{q}_h(t)$, test the second equation with $\dot{u}_h(t)$, test (2.1d) with $-\dot{\widehat{u}}_h(t)$, and finally differentiate (2.1e) and test it with $-\widehat{\mathbf{q}}_h(t)\cdot\mathbf{n}$. Adding the results, we obtain the first energy identity.

To obtain the second, we first differentiate the whole set of equations in (2.1) with respect to time and then proceed as in the proof of the first identity. This completes the proof. □

2.4 Error Estimates

To obtain our a priori error estimates, we first obtain estimates of the projections of the errors $\boldsymbol{\varepsilon}_h^{\mathsf{q}}(t) := \boldsymbol{\Pi}\mathbf{q}(t) - \mathbf{q}_h(t)$, $\boldsymbol{\varepsilon}_h^{\dot{\mathsf{q}}}(t) := \boldsymbol{\Pi}\dot{\mathbf{q}}(t) - \dot{\mathbf{q}}_h(t)$, $\varepsilon_h^u(t) := \Pi u(t) - u_h(t)$, $\varepsilon_h^{\dot{u}}(t) := \Pi\dot{u}(t) - \dot{u}_h(t)$, $\varepsilon_h^{\ddot{u}}(t) := \Pi\ddot{u}(t) - \ddot{u}_h(t)$, $\widehat{\varepsilon}_h^u(t) := Pu(t) - \widehat{u}_h(t)$ and $\widehat{\varepsilon}_h^{\dot{u}}(t) := P\dot{u}(t) - \dot{\widehat{u}}_h(t)$. We let P be the standard L^2-projection onto M_h and $(\boldsymbol{\Pi}, \Pi)$ be the HDG projection we define next; see [8]. In addition, we denote by $P_{p-1} : \mathcal{T}_h \to \mathbb{R}$ the standard L^2-projection onto piecewise polynomials of degree at most $p-1$. From these estimates, we easily deduce the results of the corresponding, actual errors.

The HDG projection Given any function pair $(\mathbf{q}, u)$, we recall that the projection $(\boldsymbol{\Pi}\mathbf{q}, \Pi u) \in \boldsymbol{V}_h \times W_h$ is defined as the unique solution of the equations

$$(\boldsymbol{\Pi}\mathbf{q}, \mathbf{r})_K = (\mathbf{q}, \mathbf{r})_K \qquad \forall \mathbf{r} \in \boldsymbol{\mathcal{P}}_{p-1}(K), \tag{2.3a}$$

$$(\Pi u, w)_K = (u, w)_K \qquad \forall w \in \mathcal{P}_{p-1}(K), \tag{2.3b}$$

$$\langle \boldsymbol{\Pi}\mathbf{q}\cdot\mathbf{n} + \tau\,\Pi u, \mu\rangle_{\partial K} = \langle \mathbf{q}\cdot\mathbf{n} + \tau\,u, \mu\rangle_{\partial K} \qquad \forall \mu \in R_p(\partial K), \tag{2.3c}$$

where $R_p(\partial K)$ is the restriction of M_h to ∂K. Note that we follow the notation of [8], where the fact that both $\mathbf{\Pi q}$ and Πu depend simultaneously on $\mathbf{q}$ and u is not made explicit for the sake of simplicity. Let us recall the approximate result proven in the Appendix of [8].

Theorem 2.2 [8, Theorem 2.1] *Suppose $p \geq 0$, $\tau|_{\partial K}$ is nonnegative and $\tau_K^{\max} := \max \tau|_{\partial K} > 0$. Then the system (2.3) is uniquely solvable for $(\mathbf{\Pi q}, \Pi u)$. Furthermore, there is constant C independent of K and τ such that*

$$\|\Pi \mathbf{q} - \mathbf{q}\|_K \leq C h_K^{l_q+1} |\mathbf{q}|_{H^{l_q+1}(K)} + C h_K^{l_u+1}\, \tau_K^* |u|_{H^{l_u+1}(K)},$$

$$\|\Pi u - u\|_K \leq C h_K^{l_u+1}\, |u|_{H^{l_u+1}(K)} + C \frac{h_K^{l_q+1}}{\tau_K^{\max}} |\nabla \cdot \mathbf{q}|_{H^{l_q}(K)},$$

for l_u, l_q in $[0, p]$. Here $\tau_K^* := \max \tau|_{\partial K \setminus F^*}$, where F^* is a face of K at which $\tau|_{\partial K}$ is maximum.

It is not difficult to see that the projection converges with the optimal order $p+1$ provided the function $(\mathbf{q}, u)$ is smooth enough.

Estimates of the projection of the errors We now provide uniform-in-time estimates of the projection of the errors. We use the following notation

$$|||(\boldsymbol{r}, w, \mu)||| := \left[\|\boldsymbol{r}\|_{\kappa^{-1}}^2 + \|w\|_\rho^2 + |\mu|_\tau^2\right]^{1/2}.$$

Theorem 2.3 *For any $T > 0$ and $p \geq 0$, we have that*

$$|||(\boldsymbol{\varepsilon}_h^q, \varepsilon_h^{\dot{u}}, \varepsilon_h^u - \widehat{\varepsilon}_h^u)(T)||| \leq |||(\boldsymbol{\varepsilon}_h^q, \varepsilon_h^{\dot{u}}, \varepsilon_h^u - \widehat{\varepsilon}_h^u)(0)||| + \int_0^T \left(\|\mathbf{\Pi}\dot{\mathbf{q}} - \dot{\mathbf{q}}\|_{\kappa^{-1}} + \|\Pi \ddot{u} - \ddot{u}\|_\rho\right),$$

$$|||(\boldsymbol{\varepsilon}_h^{\dot{q}}, \varepsilon_h^{\ddot{u}}, \varepsilon_h^{\dot{u}} - \widehat{\varepsilon}_h^{\dot{u}})(T)||| \leq |||(\boldsymbol{\varepsilon}_h^{\dot{q}}, \varepsilon_h^{\ddot{u}}, \varepsilon_h^{\dot{u}} - \widehat{\varepsilon}_h^{\dot{u}})(0)||| + \int_0^T \left(\|\mathbf{\Pi}\ddot{\mathbf{q}} - \ddot{\mathbf{q}}\|_{\kappa^{-1}} + \|\Pi \dddot{u} - \dddot{u}\|_\rho\right).$$

Moreover, for $p \geq 1$ and if the following regularity hypothesis holds

$$\eta \in H_0^1(\Omega), \quad \nabla \cdot (\kappa \nabla \eta) \in L^2(\Omega) \qquad \Longrightarrow \qquad \eta \in H^2(\Omega), \tag{2.4}$$

then

$$\|\rho P_{p-1} \varepsilon_h^u(T)\|_\Omega \leq C \left(\|P_{p-1}\varepsilon_h^u(0)\|_\Omega + \|P_{p-1}\varepsilon_h^{\dot{u}}(0)\|_\Omega\right) + Ch \left(\|\mathbf{q}(0) - \mathbf{q}_h(0)\|_\Omega + \sup_{t \in (0,T)} \|\dot{\mathbf{q}}(t) - \dot{\mathbf{q}}_h(t)\|_\Omega\right) + Ch \left(\sup_{t\in(0,T)} \|\ddot{u}(t) - \ddot{u}_h(t)\|_\Omega\right).$$

Note that these estimates hold independently of the way we define the initial data $(u_h(0), \dot{u}_h(0))$. Next, we pick a particular choice which will give rise to optimal estimates and to superconvergence of the projection of the error in the approximation of the scalar variable u. If $\nabla\kappa \in L^\infty(\Omega)^d$, the regularity hypothesis (2.4) implies the existence of $C > 0$ such that

$$\|\eta\|_{H^2(\Omega)} \leq C\, \|\nabla \cdot (\kappa \nabla \eta)\|_\Omega, \qquad \forall \eta \in H_0^1(\Omega) \quad \text{s.t.} \quad \nabla \cdot (\kappa\nabla\eta) \in L^2(\Omega). \tag{2.5}$$

Note also that if κ is a constant and Ω is a convex polyhedron, the hypotheses (2.4)–(2.5) hold.

The initial condition The initial data $(u_h(0), \dot{u}_h(0))$ is defined as follows. The function $u_h(0)$ is obtained by using the HDG method for the elliptic problem

$$\mathbf{q}(0) + \kappa \nabla u(0) = 0, \quad \nabla \cdot \mathbf{q}(0) = -\nabla \cdot (\kappa \nabla u_0), \quad \text{in } \Omega, \quad u(0) = g(0) \quad \text{on } \Gamma,$$

that is, we take $(\mathbf{q}_h(0), u_h(0), \widehat{u}_h(0)) \in \boldsymbol{V}_h \times W_h \times M_h$ as the solution of

$$\left(\kappa^{-1}\mathbf{q}_h(0), \mathbf{r}\right)_{\mathcal{T}_h} - (u_h(0), \nabla \cdot \mathbf{r})_{\mathcal{T}_h} + \langle \widehat{u}_h(0), \mathbf{r} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0, \tag{2.6a}$$

$$-(\mathbf{q}_h(0), \nabla w)_{\mathcal{T}_h} + \langle \widehat{\mathbf{q}}_h(0) \cdot \mathbf{n}, w \rangle_{\partial \mathcal{T}_h} = (-\nabla \cdot (\kappa \nabla u_0), w)_{\mathcal{T}_h}, \tag{2.6b}$$

$$\widehat{\mathbf{q}}_h(0) \cdot \mathbf{n} := \mathbf{q}_h(0) \cdot \mathbf{n} + \tau (u_h(0) - \widehat{u}_h(0)) \quad \text{on } \partial \mathcal{T}_h, \tag{2.6c}$$

$$\langle \widehat{\mathbf{q}}_h(0) \cdot \mathbf{n}, \mu \rangle_{\partial \mathcal{T}_h \backslash \Gamma} = 0, \tag{2.6d}$$

$$\langle \widehat{u}_h(0), \mu \rangle_{\Gamma} = \langle g(0), \mu \rangle_{\Gamma}, \tag{2.6e}$$

for all $(\mathbf{r}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$.

The function $\dot{u}_h(0)$ is obtained by using the auxiliary HDG projection $(\boldsymbol{\Pi}\mathbf{q}, \Pi u)$ of $(-\kappa \nabla v_0, v_0)$:

$$(\mathbf{s}_h(0), \dot{u}_h(0)) := (\boldsymbol{\Pi}(-\kappa \nabla v_0), \Pi v_0). \tag{2.6f}$$

Estimates of the errors It is now very easy to obtain the error estimates we were seeking by using the definition of the initial data and then applying the approximation properties of the projection $(\Pi \mathbf{q}, \Pi u)$ in the estimates of the projection the errors contained in Theorem 2.3.

Corollary 2.4 *Suppose that $(u_h(0), \dot{u}_h(0))$ is defined by (2.6). If $\ddot{u}(t) \in H^{p+1}(\Omega)$ for all t, then*

$$\begin{aligned} \|u(T) - u_h(T)\|_\rho &\leq C\, h^{p+1}, \\ \|\mathbf{q}(T) - \mathbf{q}_h(T)\|_{\kappa^{-1}} + \|\dot{u}(T) - \dot{u}_h(T)\|_\rho &\leq C\, h^{p+1}, \\ \|\dot{\mathbf{q}}(T) - \dot{\mathbf{q}}_h(T)\|_{\kappa^{-1}} + \|\ddot{u}(T) - \ddot{u}_h(T)\|_\rho &\leq C h^{p+1}. \end{aligned}$$

Moreover, for $p \geq 1$ and if the regularity hypotheses (2.4)–(2.5) hold, then

$$\|u(T) - u_h^*(T)\|_\Omega \leq C h^{p+2}.$$

The constant C depends on the time T, τ and the exact solution, but is independent of the mesh parameter h.

These results show that the convergence and supeconvergence properties of the dissipative HDG methods proposed in [9,22] do hold for the conservative HDG methods proposed here.

3 Proofs: The Semidiscrete HDG Method

In this section, we provide very brief proofs of Theorem 2.3 and Corollary 2.4. The analysis of this conservative HDG method runs parallel to that carried out in [9] for dissipative HDG methods. For this reason, we do not prove most lemmas and only provide brief sketches of proofs of the results which are significantly different. We proceed in several steps.

Step 1: The equations of the projection of the errors We begin by displaying the equation satisfied by the projection of errors.

Lemma 3.1 (Error equations). *We have*

$$\left(\kappa^{-1}\boldsymbol{\varepsilon}_h^q(t), \mathbf{r}\right)_{\mathcal{T}_h} - \left(\varepsilon_h^u(t), \nabla\cdot\mathbf{r}\right)_{\mathcal{T}_h} + \left\langle\widehat{\varepsilon}_h^u(t), \mathbf{r}\cdot\mathbf{n}\right\rangle_{\partial\mathcal{T}_h} = \left(\kappa^{-1}(\boldsymbol{\Pi}\mathbf{q}(t) - \mathbf{q}(t)), \mathbf{r}\right)_{\mathcal{T}_h}, \tag{3.1a}$$

$$\left(\rho\varepsilon_h^{\ddot{u}}(t), w\right)_{\mathcal{T}_h} - \left(\boldsymbol{\varepsilon}_h^q(t), \nabla w\right)_{\mathcal{T}_h} + \left\langle\widehat{\varepsilon}_h^q(t)\cdot\mathbf{n}, w\right\rangle_{\partial\mathcal{T}_h} = (\rho(\Pi\ddot{u}_h(t) - \ddot{u}(t)), w)_{\mathcal{T}_h}, \tag{3.1b}$$

$$\widehat{\varepsilon}_h^q(t)\cdot\mathbf{n} := \boldsymbol{\varepsilon}_h^q(t)\cdot\mathbf{n} + \tau\left(\varepsilon_h^u(t) - \widehat{\varepsilon}_h^{\dot{u}}(t)\right) \text{ on } \partial\mathcal{T}_h, \tag{3.1c}$$

$$\left\langle\widehat{\varepsilon}_h^q(t)\cdot\mathbf{n}, \mu\right\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0, \tag{3.1d}$$

$$\left\langle\widehat{\varepsilon}_h^u(t), \mu\right\rangle_{\Gamma} = 0, \tag{3.1e}$$

for all $(\mathbf{r}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$ *and* $t \geq 0$.

Step 2: Estimate of $\varepsilon_h^{\dot{u}}$ *and* $\varepsilon_h^{\ddot{u}}$ *by an energy argument* The same energy argument used in Proposition 2.1 yields the following two identities.

Lemma 3.2 *For the quantities*

$$\mathsf{E}_h(t) := \tfrac{1}{2}\|\boldsymbol{\varepsilon}_h^q(t)\|_{\kappa^{-1}}^2 + \tfrac{1}{2}\|\varepsilon_h^{\dot{u}}(t)\|_\rho^2 + \tfrac{1}{2}|\varepsilon_h^u(t) - \widehat{\varepsilon}_h^{\dot{u}}(t)|_\tau^2,$$
$$\mathsf{F}_h(t) := \tfrac{1}{2}\|\boldsymbol{\varepsilon}_h^{\dot{q}}(t)\|_{\kappa^{-1}}^2 + \tfrac{1}{2}\|\varepsilon_h^{\ddot{u}}(t)\|_\rho^2 + \tfrac{1}{2}|\varepsilon_h^{\dot{u}}(t) - \widehat{\varepsilon}_h^{\dot{u}}(t)|_\tau^2,$$

we have

$$\dot{\mathsf{E}}_h(t) = \left(\kappa^{-1}\left(\boldsymbol{\Pi}\dot{\mathbf{q}}(t) - \dot{\mathbf{q}}(t), \boldsymbol{\varepsilon}_h^q(t)\right)_{\mathcal{T}_h} + \left(\rho(\Pi\ddot{u}(t) - \ddot{u}(t)), \varepsilon_h^{\dot{u}}(t)\right)_{\mathcal{T}_h},\right.$$
$$\dot{\mathsf{F}}_h(t) = \left(\kappa^{-1}\left(\boldsymbol{\Pi}\ddot{\mathbf{q}}(t) - \ddot{\mathbf{q}}(t), \boldsymbol{\varepsilon}_h^{\dot{q}}(t)\right)_{\mathcal{T}_h} + \left(\rho(\Pi\dddot{u}(t) - \dddot{u}(t)), \varepsilon_h^{\ddot{u}}(t)\right)_{\mathcal{T}_h}.\right.$$

As an immediate consequence of this result, we obtain our first estimates.

Corollary 3.3 *For any time* $T > 0$*, we have*

$$\sqrt{2\mathsf{E}_h(T)} \leq \sqrt{2\mathsf{E}_h(0)} + \int_0^T \|\boldsymbol{\Pi}\dot{\mathbf{q}} - \dot{\mathbf{q}}\|_{\kappa^{-1}} + \int_0^T \|\Pi\ddot{u} - \ddot{u}\|_\rho,$$
$$\sqrt{2\mathsf{F}_h(T)} \leq \sqrt{2\mathsf{F}_h(0)} + \int_0^T \|\boldsymbol{\Pi}\ddot{\mathbf{q}} - \ddot{\mathbf{q}}\|_{\kappa^{-1}} + \int_0^T \|\Pi\dddot{u} - \dddot{u}\|_\rho.$$

The first two estimates of Theorem 2.3 are thus proved.

Step 3: Estimate of $P_{p-1}\varepsilon_h^u$ *by duality* In this step, we show that the projection of the error into a space of lower polynomial degree can superconvergence. We adapt the duality argument used in [9] to our setting. Let us start by introducing a terminal-time problem for any given function θ in $L^2(\Omega)$,

$$\rho\ddot{\Psi}(t) = \nabla\cdot(\kappa\nabla\Psi)(t) \quad \text{in } \Omega, \quad \forall t \in [0, T], \tag{3.2a}$$
$$\Psi(t) = 0 \quad \text{on } \Gamma, \quad \forall t \in [0, T], \tag{3.2b}$$
$$\Psi(T) = 0 \quad \text{on } \Omega, \tag{3.2c}$$
$$\dot{\Psi}(T) = \theta \quad \text{on } \Omega, \tag{3.2d}$$

as well as the accumulated field $\underline{\Psi}(t) := \int_t^T \Psi(s)ds$. Let us now recall the regularity inequalities proven in [9, Proposition 3.1].

Proposition 3.4 *There is a constant C' only depending on ρ and κ such that*

$$\sup_{t\in(0,T)} \|\Psi(t)\|_{H^1(\Omega)} + \sup_{t\in(0,T)} \|\dot{\Psi}(t)\|_\Omega \le C'\|\theta\|_\Omega.$$

Moreover, if (2.4)–(2.5) hold, then

$$\sup_{t\in(0,T)} \|\underline{\Psi}(t)\|_{H^2(\Omega)} \le C\|\theta\|_\Omega.$$

Since $1/\rho \in L^\infty(\Omega)$, we have

$$C\,\|P_{p-1}\varepsilon_h^u(T)\|_\Omega \le \|\rho P_{p-1}\varepsilon_h^u(T)\|_\Omega = \sup_{\theta\in C_0^\infty(\Omega)} \frac{\left(P_{p-1}\varepsilon_h^u(T), \rho\theta\right)_\Omega}{\|\theta\|_\Omega},$$

and we see that, to estimate $\|P_{p-1}\varepsilon_h^u(T)\|_\Omega$, we only need to obtain a suitable expression for the inner product $(P_{p-1}\varepsilon_h^u(T), \rho\theta)_{\mathcal{T}_h}$. Such an expression is contained in the following lemma.

Lemma 3.5 *Suppose that $p \ge 1$. Then, for any $\theta \in C_0^\infty(\Omega)$, we have*

$$\begin{aligned}(P_{p-1}\varepsilon_h^u(T), \rho\theta)_{\mathcal{T}_h} =\ & (\rho P_{p-1}\varepsilon_h^u, \dot{\Psi})_{\mathcal{T}_h}(0) - (\rho P_{p-1}\varepsilon_h^{\dot{u}}, \Psi)_{\mathcal{T}_h}(0)\\ &+ \left(\mathbf{q}(0) - \mathbf{q}_h(0), \kappa^{-1}\Pi_p^{BDM}(\kappa\nabla\underline{\Psi}(0)) - \nabla I_h\underline{\Psi}(0)\right)_{\mathcal{T}_h}\\ &+ \int_0^T \left(\dot{\mathbf{q}} - \dot{\mathbf{q}}_h, \kappa^{-1}\Pi_p^{BDM}(\kappa\nabla\underline{\Psi}) - \nabla I_h\underline{\Psi}\right)_{\mathcal{T}_h}\\ &+ \int_0^T \left(\ddot{u} - \ddot{u}_h, \rho I_h\Psi - P_{p-1}\rho\Psi\right)_{\mathcal{T}_h},\end{aligned}$$

where I_h is any h-uniformly bounded interpolant from $L^2(\Omega)$ into $W_h \cap H_0^1(\Omega)$ and Π_p^{BDM} is the BDM interpolation operator [2].

As a direct consequence of this result, we can obtain the last estimate of Theorem 2.3. Indeed, by the previous lemma, we have

$$\begin{aligned}|(P_{p-1}\varepsilon_h^u(T), \rho\theta)_{\mathcal{T}_h}| \le & H_1\|P_{p-1}\varepsilon_h^u(0)\|_\Omega + H_2\|P_{p-1}\varepsilon_h^{\dot{u}}(0)\|_\Omega + H_3\|\mathbf{q}(0) - \mathbf{q}_h(0)\|_\Omega\\ &+ H_4 \sup_{t\in(0,T)} \|\dot{\mathbf{q}}(t) - \dot{\mathbf{q}}_h(t)\|_\Omega + H_5 \sup_{t\in(0,T)} \|\ddot{u}(t) - \ddot{u}_h(t)\|_\Omega,\end{aligned}$$

where $H_1 = \|\rho\dot{\Psi}(0)\|_\Omega$, $H_2 = \|\rho\Psi(0)\|_\Omega$, $H_3 = \|\kappa^{-1}\Pi_p^{BDM}(\kappa\nabla\underline{\Psi})(0) - \nabla I_h\underline{\Psi}(0)\|_\Omega$, and

$$H_4 = \int_0^T \|\kappa^{-1}\Pi_p^{BDM}(\kappa\nabla\underline{\Psi}) - \nabla I_h\underline{\Psi}\|_\Omega, \qquad H_5 = \int_0^T \|\rho I_h\Psi - P_{p-1}\rho\Psi\|_\Omega.$$

Since $|H_3| \le C\,h\,\|\underline{\Psi}(0)\|_{H^2(\Omega)}$, $|H_4| \le C\,h \int_0^T \|\underline{\Psi}\|_{H^2(\Omega)}$, and $|H_5| \le Ch \int_0^T \|\nabla\Psi\|_\Omega$, by standard approximation estimates, the result now follows by using the regularity estimates of Proposition 3.4. This proves the third estimate of the Theorem 2.3 and completes the sketch of its proof.

Step 4: Error estimates at the starting time Here, we provide estimates of the errors in the approximation of the initial data.

Lemma 3.6 *Suppose that we take $u_h(0)$ as the solution of the elliptic problem (2.6). Then we have*

$$\|\boldsymbol{\varepsilon}_h^q(0)\|_{\kappa^{-1}}^2 + 2|\varepsilon_h^u(0) - \widehat{\varepsilon}_h^u(0)|_\tau^2 \le \|\mathbf{q}(0) - \boldsymbol{\Pi}\mathbf{q}(0)\|_{\kappa^{-1}}^2,$$
$$\|\varepsilon_h^{\ddot{u}}(0)\|_\rho \le \|\Pi\ddot{u}(0) - \ddot{u}(0)\|_\rho.$$

Moreover, if the elliptic regularity hypotheses (2.4)–(2.5) hold, we have

$$\|\varepsilon_h^u(0)\|_\Omega \le Ch^{\min\{p,1\}}\|\mathbf{q}(0) - \boldsymbol{\Pi}\mathbf{q}(0)\|_\Omega.$$

Proof The first and third estimates follow immediately from the results on HDG methods for steady-state diffusion problems in [7]. It remains to estimate $\varepsilon_h^{\ddot{u}}(0)$. Taking $w := \varepsilon_h^{\ddot{u}}(0)$ in the second error equation (3.1b) and recalling that $(\boldsymbol{\varepsilon}_h^q(0), \nabla w)_{\mathcal{T}_h} = \langle \widehat{\varepsilon}_h^q(0)\cdot\mathbf{n}, w\rangle_{\partial\mathcal{T}_h}$, by the second of the equations defining $u_h(0)$, (2.6), we get

$$\left(\rho\varepsilon_h^{\ddot{u}}(0), \varepsilon_h^{\ddot{u}}(0)\right)_{\mathcal{T}_h} = \left(\rho(\Pi\ddot{u}_h(0) - \ddot{u}(0)), \varepsilon_h^{\ddot{u}}(0)\right)_{\mathcal{T}_h},$$

which completes the proof. □

Let us now estimate the error in the initial data of the velocity.

Lemma 3.7 *If $\dot{u}_h(0)$ is computed using (2.6f), then $\varepsilon_h^{\dot{u}}(0) = 0$ and*

$$\|\boldsymbol{\varepsilon}_h^{\dot{q}}(0)\|_{\kappa^{-1}}^2 + 2|\varepsilon_h^{\dot{u}}(0) - \widehat{\varepsilon}_h^{\dot{u}}(0)|_\tau^2 \le \|\dot{\mathbf{q}}(0) - \boldsymbol{\Pi}\dot{\mathbf{q}}(0)\|_{\kappa^{-1}}^2.$$

Proof Since $\dot{u}_h(0) = \Pi v_0$ we have $\varepsilon_h^{\dot{u}}(0) = \Pi\dot{u}(0) - \dot{u}_h(0) = 0$. Differentiating the first error equation (3.1a) with respect to time and taking $\mathbf{r} := \boldsymbol{\varepsilon}_h^{\dot{q}}(0)$, we obtain

$$\left(\kappa^{-1}\boldsymbol{\varepsilon}_h^{\dot{q}}(0), \boldsymbol{\varepsilon}_h^{\dot{q}}(0)\right)_{\mathcal{T}_h} - \left(\varepsilon_h^{\dot{u}}(0), \nabla\cdot\boldsymbol{\varepsilon}_h^{\dot{q}}(0)\right)_{\mathcal{T}_h}$$
$$+\left\langle\widehat{\varepsilon}_h^{\dot{u}}(0), \boldsymbol{\varepsilon}_h^{\dot{q}}(0)\cdot\mathbf{n}\right\rangle_{\partial\mathcal{T}_h} = \left(\kappa^{-1}(\dot{\mathbf{q}}(0) - \boldsymbol{\Pi}\dot{\mathbf{q}}(0)), \boldsymbol{\varepsilon}_h^{\dot{q}}(0)\right)_{\mathcal{T}_h}.$$

Differentiating equations (3.1d) and (3.1e) with respect to time and using $\varepsilon_h^{\dot{u}}(0) = 0$, we can simplify the above identity to

$$\left(\kappa^{-1}\boldsymbol{\varepsilon}_h^{\dot{q}}(0), \boldsymbol{\varepsilon}_h^{\dot{q}}(0)\right)_{\mathcal{T}_h} + \langle\widehat{\varepsilon}_h^{\dot{u}}(0), \boldsymbol{\varepsilon}_h^{\dot{q}}(0)$$
$$-\widehat{\varepsilon}_h^{\dot{q}}(0)\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = \left(\kappa^{-1}(\dot{\mathbf{q}}(0) - \boldsymbol{\Pi}\dot{\mathbf{q}}(0)), \boldsymbol{\varepsilon}_h^q(0)\right)_{\mathcal{T}_h}.$$

Inserting the definition of $\widehat{\varepsilon}_h^{\dot{q}}(0)$ and using the fact that $\varepsilon_h^{\dot{u}}(0) = 0$, the result follows after simple manipulations. This concludes the proof. □

Step 5: *Conclusion* Applying the estimates obtained in the previous step, and using the approximation properties of the auxiliary HDG projection Theorem 2.2, we obtain the first three estimates of Corollary 2.4. The error estimate of $u - u_h^*$ can be proven in essentially the same way as in [9]. This concludes the proof of Corollary 2.4.

4 The Stormer-Numerov Time-Marching Scheme

In this section, we shall construct and analyze a two-step, fourth-order accurate scheme for the time discretization obtained by discretizing the semidiscrete HDG scheme by using the Stormer-Numerov method. The method was introduced by C. Stormer in [23].

4.1 The Fully Discrete HDG Method

Let k be the fixed time step, $t_n := n\,k, n = 0, 1 \ldots, N$, with $t_N = T$. Here $T > 0$ is a fixed but arbitrary final time. The design of our fully discrete scheme relies on the relation

$$\tfrac{1}{k^2}\left(y(t_{n+1}) - 2y(t_n) + y(t_{n-1})\right) = \tfrac{1}{12}\left(\ddot{y}(t_{n+1}) + 10\ddot{y}(t_n) + \ddot{y}(t_{n-1})\right) + O(k^4),$$

which holds for every sufficiently smooth function $y = y(t)$. This motivates the introduction of the central second difference operator and the average

$$\mathrm{D}_k^2 y^n = \tfrac{1}{k^2}\left(y^{n+1} - 2y^n + y^{n-1}\right), \qquad \mathrm{A}_k y^n := \tfrac{1}{12}\left(y^{n+1} + 10y^n + y^{n-1}\right),$$

applied to general sequences $\{y^n\}$. The data functions are sampled to provide the sequences $f^n := f(t_n)$ and $g^n := g(t_n)$.

Thus, for each $n \geq 1$, we look for $(\mathbf{q}_h^{n+1}, u_h^{n+1}, \widehat{u}_h^{n+1}) \in \boldsymbol{V}_h \times W_h \times M_h$ satisfying

$$\left(\kappa^{-1}\mathbf{q}_h^{n+1}, \mathbf{r}\right)_{\mathcal{T}_h} - \left(u_h^{n+1}, \nabla\cdot\mathbf{r}\right)_{\mathcal{T}_h} + \left\langle \widehat{u}_h^{n+1}, \mathbf{r}\cdot\mathbf{n}\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{4.1a}$$

$$\left(\rho \mathrm{D}_k^2 u_h^n, w\right)_{\mathcal{T}_h} - \left(\mathrm{A}_k \mathbf{q}_h^n, \nabla w\right)_{\mathcal{T}_h} + \left\langle \mathrm{A}_k \widehat{\mathbf{q}}_h^n\cdot\mathbf{n}, w\right\rangle_{\partial\mathcal{T}_h} = \left(\mathrm{A}_k f^n, w\right)_{\mathcal{T}_h}, \tag{4.1b}$$

$$\widehat{\mathbf{q}}_h^{n+1}\cdot\mathbf{n} := \mathbf{q}_h^{n+1}\cdot\mathbf{n} + \tau\left(u_h^{n+1} - \widehat{u}_h^{n+1}\right) \quad \text{on } \partial\mathcal{T}_h, \tag{4.1c}$$

$$\left\langle \widehat{\mathbf{q}}_h^{n+1}\cdot\mathbf{n}, \mu\right\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0, \tag{4.1d}$$

$$\left\langle \widehat{u}_h^{n+1}, \mu\right\rangle_\Gamma = \left\langle g^{n+1}, \mu\right\rangle_\Gamma, \tag{4.1e}$$

for all $(\mathbf{r}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$. This time-marching scheme is well defined under a very simple condition on the stabilization function τ as we see in the following result.

Proposition 4.1 *If $\tau > 0$ on $\partial\mathcal{T}_h$, then the solution of the equations (4.1) exists and is unique.*

To define the starting functions, namely, $(\mathbf{q}_h^n, u_h^n, \widehat{u}_h^n) \in \boldsymbol{V}_h \times W_h \times M_h$ for $n = 0, 1$, we proceed as follows. Given (u_h^n, g^n), we compute $(\mathbf{q}_h^n, \widehat{u}_h^n) \in \boldsymbol{V}_h \times M_h$ as the solution of

$$\left(\kappa^{-1}\mathbf{q}_h^n, \mathbf{r}\right)_{\mathcal{T}_h} - \left(u_h^n, \nabla\cdot\mathbf{r}\right)_{\mathcal{T}_h} + \left\langle \widehat{u}_h^n, \mathbf{r}\cdot\mathbf{n}\right\rangle_{\partial\mathcal{T}_h} = 0, \tag{4.2a}$$

$$\widehat{\mathbf{q}}_h^n\cdot\mathbf{n} := \mathbf{q}_h^n\cdot\mathbf{n} + \tau\left(u_h^n - \widehat{u}_h^n\right) \quad \text{on } \partial\mathcal{T}_h, \tag{4.2b}$$

$$\left\langle \widehat{\mathbf{q}}_h^n\cdot\mathbf{n}, \mu\right\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0, \tag{4.2c}$$

$$\left\langle \widehat{u}_h^n, \mu\right\rangle_\Gamma = \left\langle g^n, \mu\right\rangle_\Gamma, \tag{4.2d}$$

for all $(\mathbf{r}, \mu) \in \boldsymbol{V}_h \times M_h$ (see Proposition 4.2 below). The definition of $u_h^0 \in W_h$ and $u_h^1 \in W_h$ will be given later.

Computation of the approximate solution at t^{n+1}, $(\mathbf{q}_h^{n+1}, u_h^{n+1}, \widehat{u}_h^{n+1})$ in terms of the approximate solutions at t^n and t^{n-1} by using (4.1) is equivalent to solving a steady-state reaction-diffusion equation with the HDG method. This can be done using an equivalent hybridized formulation, where $\widehat{u}_h^{n+1}$ is computed by solving a system on the skeleton (only on the M_h degrees of freedom) and then $(\mathbf{q}_h^{n+1}, u_h^{n+1})$ are computed solving local problems. This is the gist of the hybridization techniques described in great detail in [7].

The computation of the starting value $(\mathbf{q}_h^n, \widehat{u}_h^n)$ in terms of u_h^n uses a method, see equations (4.2), which has not been considered elsewhere. We can see, however, that it is strongly related to the above-mentioned HDG method. In fact, the only difference is that its local problems are considerably simpler since they only involve the inversion of a mass matrix. The main properties of this method are captured in the following result.

Proposition 4.2 *The linear map* $\boldsymbol{\mathcal{S}} : \mathbf{L}^2(\Omega) \times W_h \times L^2(\Gamma) \to \boldsymbol{V}_h \times M_h$ *that associates* $(\mathbf{s}, v_h, z) \in \mathbf{L}^2(\Omega) \times W_h \times L^2(\Gamma)$ *to the solution* $(\mathbf{p}_h, \widehat{v}_h) \in \boldsymbol{V}_h \times M_h$ *of*

$$\begin{aligned}
&(\kappa^{-1}\mathbf{p}_h, \mathbf{r})_{\mathcal{T}_h} - (v_h, \nabla\cdot\mathbf{r})_{\partial\mathcal{T}_h} + \langle \widehat{v}_h, \mathbf{r}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} = (\kappa^{-1}\mathbf{s}, \mathbf{r})_{\mathcal{T}_h} && \forall \mathbf{r} \in \boldsymbol{V}_h,\\
&\widehat{\mathbf{p}}_h \cdot \mathbf{n} := \mathbf{p}_h\cdot\mathbf{n} + \tau(v_h - \widehat{v}_h) \quad \text{on } \partial\mathcal{T}_h, &&\\
&\langle \widehat{\mathbf{p}}_h\cdot\mathbf{n}, \mu\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0, && \forall \mu \in M_h,\\
&\langle \widehat{v}_h, \mu\rangle_\Gamma = \langle z, \mu\rangle_\Gamma && \forall \mu \in M_h,
\end{aligned}$$

is well defined. Moreover, if $z = 0$,

$$\left(\|\mathbf{p}_h\|^2_{\kappa^{-1}} + |v_h - \widehat{v}_h|^2_\tau\right)^{1/2} \le \sqrt{2}\, C_h\, \|v_h\|_\rho + \|\mathbf{s}\|_{\kappa^{-1}}, \tag{4.4}$$

where $C_h := \max_{K\in\mathcal{T}_h}\{C_{1,K}, C_{2,K}\}$, *and*

$$\begin{aligned}
C_{1,K} &:= \|\rho^{-1}\|^{1/2}_{L^\infty(K)}\, \|\kappa\|^{1/2}_{L^\infty(K)} \sup_{\mathbf{r}\in \boldsymbol{P}_p(K)\setminus\{\mathbf{0}\}} \frac{\|\nabla\cdot\mathbf{r}\|_K}{\|\mathbf{r}\|_K},\\
C_{2,K} &:= \|\tau\|^{1/2}_{L^\infty(\partial K)} \|\rho^{-1}\|^{1/2}_{L^\infty(K)} \sup_{w\in P_p(K)\setminus\{0\}} \frac{|w|_{\partial K}}{\|w\|_K}.
\end{aligned}$$

Note that, by construction of the method and of the initial conditions

$$\left(\mathbf{q}^n_h, \widehat{u}^n_h\right) = \boldsymbol{\mathcal{S}}\left(\mathbf{0}, u^n_h, g^n\right) \qquad \forall n \ge 0. \tag{4.5}$$

As we are going to see next, the constant C_h is strongly related to the CFL condition of the method which guarantees that the quantities conserved by the scheme are actually nonnegative.

4.2 Energy Conservation

From now on, we use the notation

$$\delta_k y^n := \left(y^n - y^{n-1}\right)/k, \qquad \underline{\delta}_k y^n := \left(y^{n+1} - y^{n-1}\right)/(2k) = \tfrac{1}{2}\left(\delta_k y^{n+1} + \delta_k y^n\right)$$

for the backwards and central discrete differentiation operators. We will also consider two functions that will serve to measure the evolution of discrete energy in our fully discrete scheme. They relate two elements of a sequence taking values in $\mathbf{V}_h \times W_h \times M_h$ and are given by

$$\begin{aligned}
\mathcal{E}_\star\left((\mathbf{p}, u, \widehat{u}), (\mathbf{p}^+, u^+, \widehat{u}^+)\right) := {}& \tfrac{1}{2}\|(u^+ - u)/k\|^2_\rho\\
&+\tfrac{1}{4}\|\mathbf{p}^+\|^2_{\kappa^{-1}} + \tfrac{1}{4}\|\mathbf{p}\|^2_{\kappa^{-1}} - \tfrac{5}{24}\|\mathbf{p}^+ - \mathbf{p}\|^2_{\kappa^{-1}}\\
&+\tfrac{1}{4}|u^+ - \widehat{u}^+|^2_\tau + \tfrac{1}{4}|u - \widehat{u}|^2_\tau\\
&-\tfrac{5}{24}|(u^+ - \widehat{u}^+) - (u - \widehat{u})|^2_\tau, \qquad (4.6)\\
\mathcal{E}\left((\mathbf{p}, u, \widehat{u}), (\mathbf{p}^+, u^+, \widehat{u}^+)\right) := {}& \tfrac{1}{2}c_0\|(u^+ - u)/k\|^2_\rho\\
&+\tfrac{1}{4}\|\mathbf{p}^+\|^2_{\kappa^{-1}} + \tfrac{1}{4}\|\mathbf{p}\|^2_{\kappa^{-1}} + \tfrac{1}{4}|u^+\\
&-\widehat{u}^+|^2_\tau + \tfrac{1}{4}|u - \widehat{u}|^2_\tau. \qquad (4.7)
\end{aligned}$$

Note that the time-step k appears in the discrete kinetic energy term in (4.6) and (4.7) and that a constant $c_0 \in [0, 1)$ scales the discrete kinetic energy term in (4.7). We have the following discrete version of the energy conservation properties of the semidiscrete case.

Proposition 4.3 (Energy identities) *Given the solution* $(\mathbf{q}_h^n, u_h^n, \widehat{u}_h^n)$ *of (4.1) we define the sequences*

$$E_{h,k}^{n+1/2} := \mathcal{E}_\star\left(\left(\mathbf{q}_h^n, u_h^n, \widehat{u}_h^n\right), \left(\mathbf{q}_h^{n+1}, u_h^{n+1}, \widehat{u}_h^{n+1}\right)\right), \qquad n \geq 0,$$
$$F_{h,k}^{m} := \mathcal{E}_\star\left(\delta_k\left(\mathbf{q}_h^m, u_h^m, \widehat{u}_h^m\right), \delta_k\left(\mathbf{q}_h^{m+1}, u_h^{m+1}, \widehat{u}_h^{m+1}\right)\right), \qquad m \geq 1.$$

Then, for $n \geq 1$ *and* $m \geq 2$,

$$\left(E_{h,k}^{n+1/2} - E_{h,k}^{n-1/2}\right)/k = \left(\mathrm{A}_k f^n, \underline{\delta}_k u_h^n\right)_{\mathcal{T}_h} - \left\langle \underline{\delta}_k g^n, \mathrm{A}_k(\widehat{\mathbf{q}}_h^n \cdot \boldsymbol{n})\right\rangle_\Gamma,$$
$$\left(F_{h,k}^{m} - F_{h,k}^{m-1}\right)/k = \left(\mathrm{A}_k \delta_k f^m, \underline{\delta}_k u_h^m\right)_{\mathcal{T}_h} - \left\langle \underline{\delta}_k \delta_k g^m, \mathrm{A}_k\left(\delta_k \widehat{\mathbf{q}}_h^m \cdot \boldsymbol{n}\right)\right\rangle_\Gamma.$$

Again, note that, when $f \equiv 0$ and g is independent of time, the quantities $E_{h,k}^{n+1/2}$ and $F_{h,k}^m$ are independent of n and m. Moreover, when g is independent of time, $E_{h,k}^{n+1/2}$ $F_{h,k}^m$ are actually nonnegative quantities provided the time step is not too big.

Proposition 4.4 *Suppose that* g *is independent of time, let* $c_0 \in [0, 1)$ *and assume the CFL condition* $k\, C_h \leq \sqrt{\frac{6}{5}(1 - c_0)}$ *is satisfied. Then, for* $n \geq 0$ *and* $m \geq 1$, *the quantities* $E_{h,k}^{n+1/2}$ *and* $F_{h,k}^m$ *are nonnegative and*

$$E_{h,k}^{n+1/2} \geq \mathcal{E}\left(\left(\mathbf{q}_h^n, u_h^n, \widehat{u}_h^n\right), \left(\mathbf{q}_h^{n+1}, u_h^{n+1}, \widehat{u}_h^{n+1}\right)\right),$$
$$F_{h,k}^{m} \geq \mathcal{E}\left(\delta_k\left(\mathbf{q}_h^m, u_h^m, \widehat{u}_h^m\right), \delta_k\left(\mathbf{q}_h^{m+1}, u_h^{m+1}, \widehat{u}_h^{m+1}\right)\right).$$

Note that the quantity $k\, C_h$ is dimensionless, as it is typical of CFL conditions. Indeed, we know that the sound speed of the medium is $\sqrt{\kappa/\rho}$. As a consequence, the constant $C_{1,h}$ (and $C_{2,h}$) has as dimension the inverse of the time, which proves our contention.

4.3 Error Estimates

To obtain our a priori error estimates, we mimic the procedure done for the semidiscrete case.
Estimates of the projection of the errors We start by obtaining estimates of the projection of the errors $\boldsymbol{\varepsilon}_{h,k}^{q,n} := \boldsymbol{\Pi}\mathbf{q}^n - \mathbf{q}_h^n$, $\varepsilon_{h,k}^{u,n} := \Pi u^n - u_h^n$ and $\widehat{\varepsilon}_{h,k}^{u,n} := P u^n - \widehat{u}_h^n$, where $\mathbf{q}^n = \mathbf{q}(t_n)$, $u^n := u(t_n)$, and $\widehat{u}^n := u(t_n)|_{\partial\mathcal{T}_h}$. The approximation error $\mathbf{a}_h^n := \boldsymbol{\Pi}\mathbf{q}^n - \mathbf{q}^n$ as well as the sequence,

$$\Theta_h^n := \mathrm{D}_k^2 \Pi u^n - \mathrm{A}_k \ddot{u}^n = \mathrm{D}_k^2(\Pi u^n - u^n) + \mathrm{D}_k^2 u^n - \mathrm{A}_k \ddot{u}^n,$$

which collects HDG projection error and the consistency error for the Stormer-Numerov scheme, will also be relevant in our estimates.

Theorem 4.5 *Let*

$$\mathbb{E}_{h,k}^{n+1/2} := \mathcal{E}\left(\left(\boldsymbol{\varepsilon}_{h,k}^{q,n}, \varepsilon_{h,k}^{u,n}, \widehat{\varepsilon}_{h,k}^{u,n}\right), \left(\boldsymbol{\varepsilon}_{h,k}^{q,n+1}, \varepsilon_{h,k}^{u,n+1}, \widehat{\varepsilon}_{h,k}^{u,n+1}\right)\right)$$
$$\mathbb{F}_{h,k}^{m} := \mathcal{E}\left(\delta_k\left(\boldsymbol{\varepsilon}_{h,k}^{q,m}, \varepsilon_{h,k}^{u,m}, \widehat{\varepsilon}_{h,k}^{u,m}\right), \delta_k\left(\boldsymbol{\varepsilon}_{h,k}^{q,m+1}, \varepsilon_{h,k}^{u,m+1}, \widehat{\varepsilon}_{h,k}^{u,m+1}\right)\right).$$

If k satisfies the CFL condition $k\,C_h \le \sqrt{\frac{3}{5}(1-c_0)}$, *then,*

$$\sqrt{\mathbb{E}_{h,k}^{n+1/2}} \le \sqrt{\tfrac{1}{c_0}\mathbb{E}_{h,k}^{1/2}} + 3\,k\sum_{\ell=0}^{n}\|\delta_k \mathbf{a}_h^{\ell+1}\|_{\kappa^{-1}} + \tfrac{1}{\sqrt{2\,c_0}}\,k\sum_{\ell=1}^{n}\|\Theta_h^{\ell}\|_{\rho}, \qquad n\ge 1,$$

$$\sqrt{\mathbb{F}_{h,k}^{m}} \le \sqrt{\tfrac{1}{c_0}\mathbb{F}_{h,k}^{1}} + 3k\sum_{\ell=0}^{m}\|\delta_k^2 \mathbf{a}_h^{\ell+1}\|_{\kappa^{-1}} + \tfrac{1}{\sqrt{2\,c_0}}\,k\sum_{\ell=1}^{m}\|\delta_k\Theta_h^{\ell}\|_{\rho}, \qquad m\ge 2.$$

Moreover, for $p\ge 1$ *and if the regularity hypotheses (2.4)–(2.5) hold, then*

$$\begin{aligned}\|\rho P_{p-1}\varepsilon_h^{u,N}\|_{\Omega} \le C\left(\|P_{p-1}\varepsilon_h^{u,1}\|_{\Omega} + \|P_{p-1}\delta_k\varepsilon_h^{u,1}\|_{\Omega} + h\|\mathbf{q}^0-\mathbf{q}_h^0\|_{\Omega} + h\|\boldsymbol{\varepsilon}_{h,k}^{q,0}\|_{\Omega}\right)\\ + Ch\left(\max_{n=1,\dots,N}\|\delta_k\left(\mathbf{q}^n-\mathbf{q}_h^n\right)\|_{\Omega} + \max_{n=2,\dots,N-1}\|\mathrm{D}_k^2\left(u^n-u_h^n\right)\|_{\Omega}\right)\\ + C\max_{n=2,\dots,N-1}\|\mathrm{D}_k^2 u^n - \mathrm{A}_k\ddot{u}^n\|_{\Omega}.\end{aligned}$$

Note that these estimates hold independently of the way we define the initial data (u_h^0, u_h^1). However, in order to guarantee optimal estimates and superconvergence of the projection of the error in the approximation of the scalar variable $u(T)$, we pick the starting functions (u_h^0, u_h^1) as follows.
The starting functions u_h^1 *and* u_h^0 The starting values of our scheme are obtained by using a discrete version of what was done for the semidiscrete case. The idea is as follows. First, we compute $\mathrm{A}_k u_h^1$ by solving an elliptic problem by the HDG method and then $\mathrm{D}_k^2 u_h^1$ by using the second equation defining our Stormer-Numerov HDG method. This would provide u_h^1 since

$$u_h^1 := \mathrm{A}_k u_h^1 - \tfrac{k^2}{12}\mathrm{D}_k^2 u_h^1.$$

Then we compute u_h^0 in such a way that $\delta_k u_h^1 = \Pi\delta_k u^1$.

Thus, we take

$$u_h^1 := u_{h,A} - \tfrac{k^2}{12}u_{h,D^2}, \tag{4.8a}$$
$$u_h^0 := \Pi\mathrm{T}u(0) + u_h^1 - \Pi\mathrm{T}u(k), \tag{4.8b}$$

where

$$\mathrm{T}u(t) := u(0) + t\,\dot{u}(0) + \tfrac{1}{2}t^2\ddot{u}(0) + \tfrac{1}{6}t^3\dddot{u}(0) + \tfrac{1}{24}t^4\ddddot{u}(0) \tag{4.8c}$$

and where the functions $u_{h,A}$ and u_{h,D^2} are the elements of W_h defined as follows. The function $(\mathbf{q}_{h,A}, u_{h,A}, \widehat{u}_{h,A})$ is the approximation provided by the HDG method for the elliptic problem

$$\mathbf{q}_A + \kappa\nabla u_A = 0, \quad \nabla\cdot\mathbf{q}_A = -\nabla\cdot(\kappa\mathrm{A}_k\nabla\mathrm{T}u^1), \quad \text{in }\Omega, \quad u_A = \mathrm{A}_k g^1, \quad \text{on }\Gamma,$$

that is, it is the solution of

$$(\kappa^{-1}\mathbf{q}_{h,A}, \mathbf{r})_{\mathcal{T}_h} - (u_{h,A}, \nabla\cdot\mathbf{r})_{\mathcal{T}_h} + \langle\widehat{u}_{h,A}, \mathbf{r}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{4.9a}$$
$$-(\mathbf{q}_{h,A}, \nabla w)_{\mathcal{T}_h} + \langle\widehat{\mathbf{q}}_{h,A}\cdot\mathbf{n}, w\rangle_{\partial\mathcal{T}_h} = \left(-\nabla\cdot(\kappa\mathrm{A}_k\nabla\mathrm{T}u^1), w\right)_{\mathcal{T}_h}, \tag{4.9b}$$
$$\widehat{\mathbf{q}}_{h,A} := \mathbf{q}_{h,A} + \tau(u_{h,A} - \widehat{u}_{h,A})\mathbf{n} \quad \text{on }\partial\mathcal{T}_h, \tag{4.9c}$$
$$\langle\widehat{\mathbf{q}}_{h,A}\cdot\mathbf{n}, \mu\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0, \tag{4.9d}$$
$$\langle\widehat{u}_{h,A}, \mu\rangle_{\Gamma} = \langle\mathrm{A}_k g^1, \mu\rangle_{\Gamma}, \tag{4.9e}$$

for all $(\mathbf{r}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$. Note that $\mathrm{A}_k \nabla \mathrm{T}u^1 = \nabla \mathrm{A}_k \mathrm{T}u^1$ involves the computation of $\mathrm{T}u^1 := \mathrm{T}u(k)$ and $\mathrm{T}u^2 := \mathrm{T}u(2k)$. The function $u_{D^2,h}$ is the solution of

$$(\rho\, u_{D^2,h}, w)_{\mathcal{T}_h} - (\mathbf{q}_{h,A}, \nabla w)_{\mathcal{T}_h} + \langle \widehat{\mathbf{q}}_{h,A} \cdot \mathbf{n}, w\rangle_{\partial \mathcal{T}_h} = (\mathrm{A}_k f^1, w)_{\mathcal{T}_h} \qquad \forall\, w \in W_h. \quad (4.10)$$

Estimates of the errors

Theorem 4.6 *Assume that* k *satisfies the CFL condition* $k\, C_h \le \sqrt{\frac{3}{5}(1-c_0)}$. *Then, for* $p \ge 1$, *we have*

$$\max_{0\le n\le N} \|u^n - u_h^n\|_\rho \le C(h^{p+1} + k^4),$$

$$\max_{0\le n\le N} \|\mathbf{q}^n - \mathbf{q}_h^n\|_{\kappa^{-1}} + \max_{1\le n\le N} \|\delta_k u^n - \delta_k u_h^n\|_\rho \le C(h^{p+1} + k^4),$$

$$\max_{1\le m\le N} \|\delta_k \mathbf{q}^m - \delta_k \mathbf{q}_h^m\|_{\kappa^{-1}} + \max_{2\le m\le N-1} \|\mathrm{D}_k^2 u^m - \mathrm{D}_k^2 u_h^m\|_\rho \le C(h^{p+1} + k^3).$$

Moreover, if (2.4)–(2.5) hold, then

$$\|u(T) - u_h^*(T)\|_\Omega \le C(h^{p+2} + k^4).$$

The constant C *depends on the time* T, *the stabilization parameter* τ, *the CFL condition and on derivatives of the exact solution, but it is independent of the mesh parameters* h *and* k.

This result states, in particular, that, if we use piecewise quadratic approximations, we can easily achieve fourth-order accuracy for smooth enough solutions. Moreover, we can obtain higher-order accuracy using polynomials of degree $p > 2$ and time step k of order $h^{(p+2)/4}$.

5 Proofs: The Stormer-Numerov HDG Method

5.1 Properties of the Mapping $\boldsymbol{\mathcal{S}}$

Let us prove Proposition 4.2 on the mapping $\boldsymbol{\mathcal{S}}$. Since the system defining $\boldsymbol{\mathcal{S}}$ is square, it is well defined if the inequality holds. It remains to prove the inequality (4.4). Taking $(\mathbf{r}, \mu) := (\mathbf{p}_h, -\widehat{u}_h)$, remembering that $z = 0$, and adding the equations, we obtain, after simple manipulations, that

$$\begin{aligned} N_h^2 &:= (\kappa^{-1}\mathbf{p}_h, \mathbf{p}_h)_{\mathcal{T}_h} + \langle \tau(v_h - \widehat{v}_h), v_h - \widehat{v}_h\rangle_{\partial\mathcal{T}_h} \\ &= (v_h, \nabla\cdot\mathbf{p}_h)_{\mathcal{T}_h} + \langle \tau\, v_h, v_h - \widehat{v}_h\rangle_{\partial\mathcal{T}_h} + (\kappa^{-1}\mathbf{s}, \mathbf{p}_h)_{\mathcal{T}_h} \\ &= \sum_{K\in\mathcal{T}_h} \big((v_h, \nabla\cdot\mathbf{p}_h)_K + \langle \tau\, v_h, v_h - \widehat{v}_h\rangle_{\partial K}\big) + (\kappa^{-1}\mathbf{s}, \mathbf{p}_h)_{\mathcal{T}_h}. \end{aligned}$$

Therefore,

$$\begin{aligned} N_h^2 &\le \sum_{K\in\mathcal{T}_h} \big(\|v_h\|_K \|\nabla\cdot\mathbf{p_h}\|_K + |v_h|_{\tau,\partial K} |v_h - \widehat{v}_h|_{\tau,\partial K}\big) + \|\mathbf{s}\|_{\kappa^{-1}} \|\mathbf{p}_h\|_{\kappa^{-1}} \\ &\le \sum_{K\in\mathcal{T}_h} \big(C_{1,K} \|v_h\|_{\rho,K} \|\mathbf{p}_h\|_{\kappa^{-1}} + C_{2,K} \|v_h\|_{\rho,K} |v_h - \widehat{v}_h|_{\tau,\partial K}\big) + \|\mathbf{s}\|_{\kappa^{-1}} \|\mathbf{p}_h\|_{\kappa^{-1}} \\ &\le C_h \sum_{K\in\mathcal{T}_h} \|v_h\|_{\rho,K} \big(\|\mathbf{p}_h\|_{\kappa^{-1}} + |v_h - \widehat{v}_h|_{\tau,\partial K}\big) + \|\mathbf{s}\|_{\kappa^{-1}} \|\mathbf{p}_h\|_{\kappa^{-1}} \\ &\le \big(\sqrt{2}\, C_h \|v_h\|_\rho + \|\mathbf{s}\|_{\kappa^{-1}}\big) N_h, \end{aligned}$$

and the result follows. This completes the proof of Proposition 4.2.

5.2 The Identities for the Discrete Energies

Proof of the energy identities of Proposition 4.3 We mimic the proof of the energy identities of the continuous case. Thus, we begin by noting that, the equations defining the method, (4.1), give, for $n \geq 1$, that

$$\begin{aligned} \left(\kappa^{-1}\underline{\delta}_k \mathbf{q}_h^n, \mathbf{r}\right)_{\mathcal{T}_h} - \left(\underline{\delta}_k u_h^n, \nabla\cdot\mathbf{r}\right)_{\mathcal{T}_h} + \left\langle(\underline{\delta}_k \widehat{u}_h^n, \mathbf{r}\cdot\mathbf{n}\right\rangle_{\partial\mathcal{T}_h} &= 0,\\ \left(\rho \mathrm{D}_k^2 u_h^n, w\right)_{\mathcal{T}_h} - \left(\mathrm{A}_k \mathbf{q}_h^n, \nabla w\right)_{\mathcal{T}_h} + \left\langle \mathrm{A}_k \widehat{\mathbf{q}}_h^n\cdot\mathbf{n}, w\right\rangle_{\partial\mathcal{T}_h} &= (\mathrm{A}_k f^n, w)_{\mathcal{T}_h},\\ \left\langle \mathrm{A}_k \widehat{\mathbf{q}}_h^n\cdot\mathbf{n}, \mu\right\rangle_{\partial\mathcal{T}_h\setminus\Gamma} &= 0,\\ \left\langle \underline{\delta}_k \widehat{u}_h^n, \mu\right\rangle_{\Gamma} - \left\langle \underline{\delta}_k g^n, \mu\right\rangle_{\Gamma} &= 0, \end{aligned}$$

for all $(\mathbf{r}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h$. Taking $\mathbf{r} := \mathrm{A}_k \mathbf{q}_h^n$ in the first equation, $w := \underline{\delta}_k u_h^n$ in the second, $\mu := -\underline{\delta}_k \widehat{u}_h^n$ in the third, $\mu := -\mathrm{A}_k \widehat{\mathbf{q}}_h^n \cdot \mathbf{n}$, in the fourth and adding them, we get, after simple algebraic manipulations, that

$$DE_{h,k}^n = \left(\mathrm{A}_k f^n, \underline{\delta}_k u_h^n\right)_{\mathcal{T}_h} - \left\langle \underline{\delta}_k g^n, \mathrm{A}_k(\widehat{\mathbf{q}}_h^n\cdot\boldsymbol{n})\right\rangle_{\Gamma},$$

where

$$DE_{h,k}^n := \left(\rho \mathrm{D}_k^2 u_h^n, \underline{\delta}_k u_h^n\right)_{\mathcal{T}_h} + \left(\kappa^{-1}\underline{\delta}_k \mathbf{q}_h^n, \mathrm{A}_k \mathbf{q}_h^n\right)_{\mathcal{T}_h} + \left\langle \underline{\delta}_k \left(u_h^n - \widehat{u}_h^n\right), \mathrm{A}_k\left(\widehat{\mathbf{q}}_h^n - \mathbf{q}_h^n\right)\cdot\mathbf{n}\right\rangle_{\partial\mathcal{T}_h}.$$

It remains to show that $DE_{h,k}^n = (E_{h,k}^{n+1/2} - E_{h,k}^{n-1/2})/k$. However, this easily follows by using the identity $(a-2b+c)(a-c) = (a-b)^2 - (b-c)^2$ on the first term, and then inserting in the third term the definition of the numerical trace $\widehat{\mathbf{q}}_h^n$ and applying the identity

$$(a+10b+c)(a-c) = \left[6a^2 + 6b^2 - 5(a-b)^2\right] - \left[6b^2 + 6c^2 - 5(b-c)^2\right],$$

to the second and third terms.

After applying the finite difference operator δ_k to the equations (4.1), the second identity is proven in a similar manner. This completes the proof of Proposition 4.3.

The discrete energies We can now prove Proposition 4.4. We begin by using Proposition 4.2. Recalling (4.5), and assuming that g is independent of time, we have $(\mathbf{q}_h^{n+1} - \mathbf{q}_h^n, \widehat{u}_h^{n+1}, \widehat{u}_h^n) = \mathcal{S}(\mathbf{0}, u_h^{n+1} - u_h^n, g^{n+1} - g^n)) = \mathcal{S}(\mathbf{0}, u_h^{n+1} - u_h^n, 0)$, and thus

$$\left\|\mathbf{q}_h^{n+1} - \mathbf{q}_h^n\right\|_{\kappa^{-1}}^2 + \left|\left(u_h^{n+1} - \widehat{u}_h^{n+1}\right) - \left(u_h^n - \widehat{u}_h^n\right)\right|_{\tau}^2 \leq 2\, C_h^2 \left\|u_h^{n+1} - u_h^n\right\|_{\rho}^2.$$

We then obtain

$$\begin{aligned} E_{h,k}^{n+1/2} \geq{}& \tfrac{1}{2}\left(1 - \tfrac{5}{6}C_h^2 k^2\right)\left\|\left(u_h^{n+1} - u_h^n\right)/k\right\|_{\rho}^2\\ &+ \tfrac{1}{4}\|\mathbf{q}_h^{n+1}\|_{\kappa^{-1}}^2 + \tfrac{1}{4}\|\mathbf{q}_h^n\|_{\kappa^{-1}}^2 + \tfrac{1}{4}|u_h^{n+1} - \widehat{u}_h^{n+1}|_{\tau}^2 + \tfrac{1}{4}|u_h^n - \widehat{u}_h^n|_{\tau}^2. \end{aligned}$$

The first estimate follows after noting that, by the CFL condition of Proposition 4.4, we have that $\frac{1}{2}(1 - \frac{5}{6}C_h^2 k^2) \geq \frac{1}{2}\, c_0$. The second is obtained in the same fashion. This completes the proof of Proposition 4.4.

5.3 Existence and Uniqueness of the Stormer-Numerov Scheme

Next we show that by using the first energy identity of the Proposition 4.3, we can guarantee the existence and uniqueness of the solution of the time-marching scheme given by (4.1). Indeed, since the system of equations (4.1) is square, to prove the existence and uniqueness of the solution $(\mathbf{q}_h^{n+1}, u_h^{n+1}, \widehat{u}_h^{n+1})$, for $n > 1$, we only have to show that if we set the data equal to zero, the only solution is the trivial one. So, we set $(\mathbf{q}_h^m, u_h^m, \widehat{u}_h^m)$ to zero for $m = n, n-1$, which implies that $E_{h,k}^{n-1/2} = 0$. We also set g^{n+1} and $A_k f^n$ to zero, which implies, by the first energy identity of Proposition 4.3, that $E_h^{n+1/2} = E_h^{n-1/2}$. As a consequence, $E_{h,k}^{n+1/2} = 0$ and, by definition, we have

$$0 = \mathcal{E}_\star\left((\mathbf{0},0,0),\left(\mathbf{q}_h^{n+1}, u_h^{n+1}, \widehat{u}_h^{n+1}\right)\right), = \tfrac{1}{2}\|u_h^{n+1}/k\|_\rho^2 + \tfrac{1}{24}\,\|\mathbf{q}_h^{n+1}\|_{\kappa^{-1}}^2 + \tfrac{1}{24}\,|u_h^{n+1} - \widehat{u}_h^{n+1}|_\tau^2,$$

and we see that $(\mathbf{q}_h^{n+1}, u_h^{n+1}, \widehat{u}_h^{n+1}) = (\mathbf{0}, 0, 0)$ since $\tau > 0$. This completes the proof.

5.4 The Error Estimates

Step 1: The equations of the projection of the errors The projection of errors satisfy the following equations.

Lemma 5.1 (Error equations) *If we denote* $\boldsymbol{\varepsilon}_{h,k}^{,q,n}\cdot\mathbf{n} := \boldsymbol{\varepsilon}_{h,k}^{q,n}\cdot\mathbf{n} + \tau(\varepsilon_{h,k}^{u,n} - \widehat{\varepsilon}_{h,k}^{u,n})$, *then, for all* $n \geq 0$,

$$\left(\kappa^{-1}\boldsymbol{\varepsilon}_{h,k}^{q,n}, \mathbf{r}\right)_{\mathcal{T}_h} - \left(\varepsilon_{h,k}^{u,n}, \nabla\cdot\mathbf{r}\right)_{\mathcal{T}_h} + \left\langle\widehat{\varepsilon}_{h,k}^{u,n}, \mathbf{r}\cdot\mathbf{n}\right\rangle_{\partial\mathcal{T}_h} = \left(\kappa^{-1}\mathbf{a}_h^n, \mathbf{r}\right)_{\mathcal{T}_h}, \tag{5.1a}$$

$$\left\langle\widehat{\boldsymbol{\varepsilon}}_{h,k}^{q,n}\cdot\mathbf{n}, \mu\right\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0, \tag{5.1b}$$

$$\left\langle\widehat{\varepsilon}_{h,k}^{u,n}, \mu\right\rangle_{\Gamma} = 0, \tag{5.1c}$$

for all $(\mathbf{r},\mu) \in \boldsymbol{V}_h \times M_h$. *If* $n \geq 1$, *then*

$$\left(\rho \mathrm{D}_k^2\varepsilon_{h,k}^{u,n}, w\right)_{\mathcal{T}_h} - \left(\mathrm{A}_k\boldsymbol{\varepsilon}_{h,k}^{q,n}, \nabla w\right)_{\mathcal{T}_h} + \left\langle\mathrm{A}_k\widehat{\boldsymbol{\varepsilon}}_{h,k}^{q,n}, w\right\rangle_{\partial\mathcal{T}_h} = \left(\rho\,\Theta_h^n, w\right)_{\mathcal{T}_h}, \tag{5.1d}$$

for all $w \in W_h$.

Step 2: Estimate of $\delta_k\varepsilon_{h,k}^{u,n}$ *and* $\mathrm{D}_k^2\varepsilon_{h,k}^{u,n}$ The same energy argument used to obtain Proposition 4.1 yields the following discrete energy identities for the projection of the errors.

Lemma 5.2 *If*

$$\mathsf{E}_{h,k}^{n+1/2} := \mathcal{E}_\star\left(\left(\boldsymbol{\varepsilon}_{h,k}^{q,n}, \varepsilon_{h,k}^{u,n}, \widehat{\varepsilon}_{h,k}^{u,n}\right), \left(\boldsymbol{\varepsilon}_{h,k}^{q,n+1}, \varepsilon_{h,k}^{u,n+1}, \widehat{\varepsilon}_{h,k}^{u,n+1}\right)\right), \qquad n \geq 0,$$

$$\mathsf{F}_{h,k}^{m} := \mathcal{E}_\star\left(\delta_k\left(\boldsymbol{\varepsilon}_{h,k}^{q,m}, \varepsilon_{h,k}^{u,m}, \widehat{\varepsilon}_{h,k}^{u,m}\right), \delta_k\left(\boldsymbol{\varepsilon}_{h,k}^{q,m+1}, \varepsilon_{h,k}^{u,m+1}, \widehat{\varepsilon}_{h,k}^{u,m+1}\right)\right), \qquad n \geq 1,$$

then

$$\left(\mathsf{E}_{h,k}^{n+1/2} - \mathsf{E}_{h,k}^{n-1/2}\right)/k = \left(\kappa^{-1}\underline{\delta}_k \mathbf{a}_h^n, \mathrm{A}_k \boldsymbol{\varepsilon}_{h,k}^{q,n}\right)_{\mathcal{T}_h} + \left(\rho\Theta_h^n, \underline{\delta}_k \varepsilon_{h,k}^{u,n}\right)_{\mathcal{T}_h}, \qquad n \geq 1,$$

$$\left(\mathsf{F}_{h,k}^{m} - \mathsf{F}_{h,k}^{m-1}\right)/k = \left(\kappa^{-1}\underline{\delta}_k \delta_k \mathbf{a}_h^m, \mathrm{A}_k \delta_k \boldsymbol{\varepsilon}_{h,k}^{q,m}\right)_{\mathcal{T}_h} + \left(\rho\delta_k\Theta_h^m, \underline{\delta}_k \delta_k \varepsilon_{h,k}^{u,m}\right)_{\mathcal{T}_h}, \qquad m \geq 2.$$

In order to prove first two estimates in Theorem 4.5, we are going to use the following discrete integral inequality.

Lemma 5.3 *If* $\{\alpha^{n-1/2}\}$, $\{\zeta^{n-1/2}\}$ *and* $\{\eta^n\}$ *are sequences of nonnegative numbers satisfying*

$$\left(\alpha^{n+1/2}\right)^2 \leq \left(\zeta^{n+1/2}\right)^2 + \left(\alpha^{1/2}\right)^2 + k\sum_{\ell=1}^{n} \eta^\ell \left(\alpha^{\ell+1/2} + \alpha^{\ell-1/2}\right) \qquad \forall n \geq 1,$$

then

$$\alpha^{n+1/2} \leq \alpha^{1/2} + 2\sum_{\ell=0}^{n} \zeta^{\ell+1/2} + k\sum_{\ell=1}^{n} \eta^\ell.$$

Proof Set $\chi^{n+1/2}$ equal to the right-hand side of the first inequality when $n \geq 1$ and equal to $(\alpha_{1/2})^2$ (we are setting $\zeta^{1/2} := 0$) when $n = 0$. Then, for $n \geq 1$, we have

$$\begin{aligned}
\chi^{n+1/2} - \chi^{n-1/2} &= \left(\zeta^{n+1/2}\right)^2 - \left(\zeta^{n-1/2}\right)^2 + \eta^n \left(\alpha^{n+1/2} + \alpha^{n-1/2}\right) k \\
&\leq \max\left\{0, \left(\zeta^{\ell+1/2} - \zeta^{\ell-1/2}\right)/k\right\} \left(\zeta^{n+1/2} + \zeta^{n-1/2}\right) k \\
&\quad + \eta^n \left(\alpha^{n+1/2} + \alpha^{n-1/2}\right) k \\
&\leq \left(\max\left\{0, \left(\zeta^{\ell+1/2} - \zeta^{\ell-1/2}\right)/k\right\} + \eta^n\right) k \left(\sqrt{\chi^{n+1/2}}\right. \\
&\quad \left. + \sqrt{\chi^{n-1/2}}\right),
\end{aligned}$$

since, by definition, $\max\{\alpha^{\ell+1/2}, \zeta^{\ell+1/2}\} \leq \sqrt{\chi^{\ell+1/2}}$. This implies that

$$\begin{aligned}
\sqrt{\chi^{n+1/2}} - \sqrt{\chi^{n-1/2}} &\leq \left(\max\left\{0, (\zeta^{\ell+1/2} - \zeta^{\ell-1/2})/k\right\} + \eta^n\right) k \\
&\leq \zeta^{\ell+1/2} + \zeta^{\ell-1/2} + \eta^n\, k.
\end{aligned}$$

This last estimate is quite crude, but is enough for our purposes. Finally, the result follows by summing on n and noting that $\sqrt{\chi^{1/2}} = \alpha^{1/2}$. This completes the proof. □

We are now ready to prove Theorem 4.5.

Proof Let us prove the first inequality. From Lemma 5.2, we get that

$$\mathsf{E}_{h,k}^{n+1/2} = \mathsf{E}_{h,k}^{1/2} + \sum_{\ell=1}^{n}\left(\kappa^{-1}\underline{\delta}_k \mathbf{a}_h^\ell, \mathrm{A}_k \boldsymbol{\varepsilon}_{\ell,k}^{q,\ell}\right)_{\mathcal{T}_h} + \sum_{\ell=1}^{n}\left(\rho\Theta_h^\ell, \underline{\delta}_k \varepsilon_{h,k}^{u,\ell}\right)_{\mathcal{T}_h}.$$

We claim that under the CFL condition of Theorem 4.5, it follows that

$$\mathbb{E}_{h,k}^{n+1/2} - \tfrac{5}{12}\left\|\mathbf{a}_h^{n+1} - \mathbf{a}_h^n\right\|_{\kappa^{-1}}^2 \le \mathsf{E}_{h,k}^{n+1/2} \le \tfrac{1}{c_0}\mathbb{E}_{h,k}^{n+1/2}.$$

Then, since, by definition, we have

$$\max\left\{\|\mathrm{A}_k \boldsymbol{\varepsilon}_{h,k}^{q,n}\|_{\kappa^{-1}}, \sqrt{2\,c_0}\|\underline{\delta}_k \varepsilon_{h,k}^{u,n}\|_\rho\right\} \le \sqrt{\mathbb{E}_{h,k}^{n+1/2}} + \sqrt{\mathbb{E}_{h,k}^{n-1/2}},$$

we easily obtain that

$$\begin{aligned}\mathbb{E}_{h,k}^{n+1/2} &\le \tfrac{5}{12}\,\|\mathbf{a}_h^{n+1} - \mathbf{a}_h^n\|_{\kappa^{-1}}^2 + \tfrac{1}{c_0}\mathbb{E}_{h,k}^{1/2} \\ &\quad + \sum_{\ell=1}^{n}\left(\|\underline{\delta}_k \mathbf{a}_h^\ell\|_{\kappa^{-1}} + \tfrac{1}{\sqrt{2\,c_0}}\|\Theta_h^\ell\|_\rho\right)\left(\sqrt{\mathbb{E}_{h,k}^{\ell+1/2}} + \sqrt{\mathbb{E}_{h,k}^{\ell-1/2}}\right) k.\end{aligned}$$

Now, a direct application of Lemma 5.3 with $\alpha^{\ell+1/2} := \sqrt{\mathbb{E}_{h,k}^{\ell+1/2}}$ when $\ell \ge 1$ and $\alpha^{1/2} := \sqrt{\mathbb{E}_{h,k}^{1/2}/c_0}$, $\zeta^{\ell+1/2} := \sqrt{\tfrac{5}{12}}\,\|\mathbf{a}_h^{\ell+1} - \mathbf{a}_h^\ell\|_{\kappa^{-1}}$ and $\eta^\ell := \|\underline{\delta}_k \mathbf{a}_h^\ell\|_{\kappa^{-1}} + \tfrac{1}{c_0}\|\Theta_h^\ell\|_\rho$ gives

$$\sqrt{\mathbb{E}_{h,k}^{n+1/2}} \le \sqrt{\tfrac{1}{c_0}\mathbb{E}_{h,k}^{1/2}} + \sqrt{\tfrac{5}{3}}\,k\sum_{\ell=0}^{n}\|\delta_k \mathbf{a}_h^{\ell+1}\|_{\kappa^{-1}} + k\sum_{\ell=1}^{n}\left(\|\underline{\delta}_k \mathbf{a}_h^\ell\|_{\kappa^{-1}} + \tfrac{1}{\sqrt{2\,c_0}}\|\Theta_h^\ell\|_\rho\right),$$

and the result follows.

It remains to prove the claim. To do that, we use Proposition 4.2. First, note that by Lemma 5.1, $(\boldsymbol{\varepsilon}_{h,k}^{q,n+1} - \boldsymbol{\varepsilon}_{h,k}^{q,n}, \widehat{\varepsilon}_{h,k}^{u,n+1} - \widehat{\varepsilon}_{h,k}^{u,n}) = \boldsymbol{\mathcal{S}}(\mathbf{a}_h^{n+1} - \mathbf{a}_h^n, \varepsilon_{h,k}^{u,n+1} - \varepsilon_{h,k}^{u,n}, 0)$, for $n \ge 0$. Therefore

$$\begin{aligned}&\left\|\boldsymbol{\varepsilon}_{h,k}^{q,n+1} - \boldsymbol{\varepsilon}_{h,k}^{q,n}\right\|_{\kappa^{-1}}^2 + \left|\left(\varepsilon_{h,k}^{u,n+1} - \widehat{\varepsilon}_{h,k}^{u,n+1}\right) - \left(\varepsilon_{h,k}^{u,n} - \widehat{\varepsilon}_{h,k}^{u,n}\right)\right|_\tau^2 \le 4\,C_h^2\,\left\|\varepsilon_{h,k}^{u,n+1}\right. \\ &\left.-\varepsilon_{h,k}^{u,n}\right\|_\rho^2 + 2\,\left\|\mathbf{a}_h^{n+1} - \mathbf{a}_h^n\right\|_{\kappa^{-1}}^2,\end{aligned}$$

and the claim follows after applying the CFL condition of Theorem 4.5.

The second inequality can be proven in exactly the same manner. This completes the proof of the first two estimates in Theorem 4.5. □

Step 3: Estimate of $P_{p-1}\varepsilon_{h,k}^{u,N}$ *by duality* As in the semidiscrete case, we only need to obtain a suitable expression for the term $(P_{p-1}\varepsilon_{h,k}^{u,N}, \rho\theta)_{\mathcal{T}_h}$. Such an expression is contained in the following lemma.

To state it, we use the following notation. For any function $\mu : [0, T] \to \mathbb{R}$, we define $\mathrm{I}_k\mu$ as the continuous piecewise-linear interpolation of the values $\mathrm{I}_k\mu(t^n)$ given by

$$\mathrm{I}_k\mu(t_n) := \begin{cases} \frac{2}{k}\int_0^k \frac{1}{k}(t-k)\mu(t)dt, & n = 0 \\ \frac{1}{k}\int_{t_{n-1}}^{t_n}\frac{1}{k}(t - t_{n-1})\mu(t)dt + \frac{1}{k}\int_{t_n}^{t_{n+1}}\frac{1}{k}(t_{n+1} - t)\mu(t)dt & 0 < n < N \\ \frac{2}{k}\int_{T-k}^{T}\frac{1}{k}(t - (T-k))\mu(t)dt, & n = N. \end{cases}$$

Also, given an set of real numbers $\{\eta^n\}_{n=0}^N$, we denote by $\eta : [0, T] \to \mathbb{R}$ the piecewise-linear function such that $\eta(t^n) := \eta^n$, $n = 0, \ldots, N$. Note that, for $t \in (t_{n-1}, t_n)$, we have that $\dot{\eta}(t) = (\eta^n - \eta^{n-1})/k$.

Lemma 5.4 *Suppose that $p \ge 1$. Then, for any $\theta \in C_0^\infty(\Omega)$, we have*

$$\begin{aligned}\left(P_{p-1}\varepsilon_{h,k}^{u,N}, \rho\theta\right)_{\mathcal{T}_h} &= \left(\rho P_{p-1}\varepsilon_{h,k}^{u,0}, \dot{\Psi}(0)\right)_{\mathcal{T}_h} - \left(\rho P_{p-1}\delta_k\varepsilon_{h,k}^{u,1}, \Psi(0)\right)_{\mathcal{T}_h}\\ &\quad + \left(\mathbf{q}^0 - \mathbf{q}_h^0, \kappa^{-1}\Pi_p^{BDM}(\kappa\nabla\underline{\Psi}(0)) - \nabla I_h\underline{\Psi}(0)\right)_{\mathcal{T}_h}\\ &\quad + \int_0^T \left(\dot{\mathbf{e}}_h, \kappa^{-1}\Pi_p^{BDM}(\kappa\nabla\underline{\Psi}) - \nabla I_h\underline{\Psi}\right))_{\mathcal{T}_h}\\ &\quad + k\sum_{n=1}^{N-1}\left(\mathrm{D}_k^2(u^n - u_h^n), \rho I_h(\mathrm{I}_k\Psi)^n - P_{p-1}(\rho\Psi^n)\right)_{\mathcal{T}_h}\\ &\quad + \tfrac{k}{2}\left(\boldsymbol{\varepsilon}_{h,k}^{q,0}, \nabla I_h(\mathrm{I}_k\Psi)^0\right)_{\mathcal{T}_h} - k\sum_{n=1}^{N-1}\left(\mathrm{D}_k^2 u^n - \mathrm{A}_k\ddot{u}^n, I_h(\mathrm{I}_k\Psi)^n\right)_{\mathcal{T}_h}\\ &\quad - \tfrac{k^3}{12}\sum_{n=1}^{N-1}\left(\mathrm{D}_k^2\boldsymbol{\varepsilon}_{h,k}^{q,n}, \nabla I_h(\mathrm{I}_k\Psi)^n\right)_{\mathcal{T}_h},\end{aligned}$$

where I_h is any h-uniformly bounded interpolant from $L^2(\Omega)$ into $W_h \cap H_0^1(\Omega)$ and $\mathbf{e}_h$ is the linear interpolant of the values $\mathbf{e}_h^n := \mathbf{q}^n - \mathbf{q}_h^n$.

Proof By the definition of the solution Ψ of the dual problem (3.2), we can write

$$\begin{aligned}\left(P_{p-1}\varepsilon_{h,k}^{u,N}, \rho\theta\right)_{\mathcal{T}_h} &= \left(\rho P_{p-1}\varepsilon_{h,k}^{u}, \dot{\Psi}\right)_{\mathcal{T}_h}(T) = \left(\rho P_{p-1}\varepsilon_{h,k}^{u,0}, \dot{\Psi}(0)\right)_{\mathcal{T}_h}\\ &\quad + \int_0^T \frac{d}{dt}\left(\rho P_{p-1}\varepsilon_{h,k}^{u}, \dot{\Psi}\right)_{\mathcal{T}_h}\\ &= \left(\rho P_{p-1}\varepsilon_{h,k}^{u,0}, \dot{\Psi}(0)\right)_{\mathcal{T}_h} + T_1 + T_2,\end{aligned}$$

where $T_1 := \int_0^T (\rho P_{p-1}\dot{\varepsilon}_{h,k}^u, \dot{\Psi})_{\mathcal{T}_h}$ and $T_2 := \int_0^T (\rho P_{p-1}\varepsilon_{h,k}^u, \ddot{\Psi})_{\mathcal{T}_h}$.

Let us work on T_1. Note first that the definition of the HDG projection (2.3), and the error equations (5.1b) and (5.1d) imply that for $n \ge 1$,

$$\begin{aligned}\left(\rho\mathrm{D}_k^2\left(u^n - u_h^n\right), w\right)_{\mathcal{T}_h} &= \left(\rho\left(\mathrm{D}_k^2 u^n - \mathrm{A}_k\ddot{u}^n\right), w\right)_{\mathcal{T}_h}\\ &\quad + \left(\mathrm{A}_k\left(\mathbf{q}^n - \mathbf{q}_h^n\right), \nabla w\right)_{\mathcal{T}_h} \quad \forall w \in W_h \cap H_0^1(\Omega).\end{aligned}$$

Therefore, by Lemma A.1, using the fact that $(\Pi u^n - u^n, P_{p-1}v)_{\mathcal{T}_h} = 0$ for all v, and noting that $\Psi^N = \Psi(T) = 0$, we have

$$\begin{aligned}T_1 &= -\left(P_{p-1}\delta_k\varepsilon_{h,k}^{u,0}, \rho\Psi^0\right)_{\mathcal{T}_h} - k\sum_{n=1}^{N-1}\left(\mathrm{D}_k^2(u^n - u_h^n), P_{p-1}\rho\Psi^n\right)_{\mathcal{T}_h}\\ &= -\left(P_{p-1}\delta_k\varepsilon_{h,k}^{u,0}, \rho\Psi^0\right)_{\mathcal{T}_h} + k\sum_{n=1}^{N-1}\left(\mathrm{D}_k^2\left(u^n - u_h^n\right), \rho I_h\mathrm{I}_k\Psi^n - P_{p-1}(\rho\Psi^n)\right)_{\mathcal{T}_h}\\ &\quad - k\sum_{n=1}^{N-1}\left(\rho\mathrm{D}_k^2\left(u^n - u_h^n\right), I_h\mathrm{I}_k\Psi^n\right)_{\mathcal{T}_h}\\ &= -(P_{p-1}\delta_k\varepsilon_{h,k}^{u,0}, \rho\Psi^0) + k\sum_{n=1}^{N-1}\left(\mathrm{D}_k^2\left(u^n - u_h^n\right), \rho I_h\mathrm{I}_k\Psi^n - P_{p-1}(\rho\Psi^n)\right)_{\mathcal{T}_h}\end{aligned}$$

$$
\begin{aligned}
&- k\sum_{n=1}^{N-1}\left(\mathrm{A}_k\left(\mathbf{q}^n-\mathbf{q}_h^n\right), \nabla I_h(\mathrm{I}_k\Psi)^n\right)_{\mathcal{T}_h} - k\sum_{n=1}^{N-1}\left(\mathrm{D}_k^2 u^n - \mathrm{A}_k\ddot{u}^n, I_h(\mathrm{I}_k\Psi)^n\right)_{\mathcal{T}_h}\\
&= -\left(P_{p-1}\delta_k\varepsilon_{h,k}^{u,0}, \rho\Psi^0\right)_{\mathcal{T}_h} + k\sum_{n=1}^{N-1}\left(\mathrm{D}_k^2\left(u^n-u_h^n\right), \rho I_h \mathrm{I}_k\Psi^n - P_{p-1}(\rho\Psi^n)\right)_{\mathcal{T}_h}\\
&- k\sum_{n=1}^{N-1}\left(\mathrm{D}_k^2 u^n - \mathrm{A}_k\ddot{u}^n, I_h(\mathrm{I}_k\Psi)^n\right)_{\mathcal{T}_h} - \tfrac{k^3}{12}\sum_{n=1}^{N-1}\left(\mathrm{D}_k^2\left(\mathbf{q}^n-\mathbf{q}_h^n\right), \nabla I_h(\mathrm{I}_k\Psi)^n\right)_{\mathcal{T}_h}\\
&- k\sum_{n=1}^{N-1}\left(\mathbf{q}^n-\mathbf{q}_h^n, \nabla I_h(\mathrm{I}_k\Psi)^n\right)_{\mathcal{T}_h},
\end{aligned}
$$

since $\mathrm{A}_k = \mathrm{I} + \frac{k^2}{12}\mathrm{D}_k^2$.

Now, let us work on T_2. First of all, using the error equations (5.1a) and (5.1d), it follows that

$$\left(\varepsilon_{h,k}^{u,n}, \nabla\cdot\Pi_p^{BDM}\mathbf{r}\right)_{\mathcal{T}_h} = \left(\kappa^{-1}\mathbf{e}_h^n, \Pi_p^{BDM}\mathbf{r}\right)_{\mathcal{T}_h} \tag{5.2}$$

for all $\mathbf{r}$. By the dual problem (3.2), the definition of $\underline{\Psi}$ (note that $\underline{\Psi}(T)=0$) and the well known commutativity property of the BDM projection Π_p^{BDM}, we get that

$$
\begin{aligned}
T_2 &= \int_0^T \left(\varepsilon_{h,k}^u, P_{p-1}(\nabla\cdot(\kappa\Psi))\right)_{\mathcal{T}_h}\\
&= (\varepsilon_{h,k}^{u,0}, P_{p-1}\nabla\cdot(\kappa\nabla\underline{\Psi}(0))_{\mathcal{T}_h} + \int_0^T \left(\dot{\varepsilon}_{h,k}^u, P_{p-1}\nabla\cdot(\kappa\nabla\underline{\Psi})\right)_{\mathcal{T}_h}\\
&= (\varepsilon_{h,k}^{u,0}, \nabla\cdot\Pi_p^{BDM}(\kappa\nabla\underline{\Psi}(0))_{\mathcal{T}_h} + \int_0^T \left(\dot{\varepsilon}_{h,k}^u, \nabla\cdot\Pi_p^{BDM}(\kappa\nabla\underline{\Psi})\right)_{\mathcal{T}_h}\\
&= (\mathbf{q}^0-\mathbf{q}_h^0, \kappa^{-1}\Pi_p^{BDM}(\kappa\nabla\underline{\Psi}(0))_{\mathcal{T}_h} + \int_0^T \left(\kappa^{-1}\dot{\mathbf{e}}_h, \nabla\cdot\Pi_p^{BDM}(\kappa\nabla\underline{\Psi})\right)_{\mathcal{T}_h},
\end{aligned}
$$

by (5.2). Therefore, applying integration by parts in the time variable,

$$
\begin{aligned}
T_2 &= (\mathbf{q}^0-\mathbf{q}_h^0, \kappa^{-1}\Pi_p^{BDM}(\kappa\nabla\underline{\Psi}(0))_{\mathcal{T}_h} + \int_0^T \left(\dot{\mathbf{e}}_h, \kappa^{-1}\Pi_p^{BDM}(\kappa\nabla\underline{\Psi}) - \nabla I_h\underline{\Psi}\right)_{\mathcal{T}_h}\\
&\quad + \int_0^T (\dot{\mathbf{e}}_h, \nabla I_h\underline{\Psi})_{\mathcal{T}_h}\\
&= \left(\mathbf{q}^0-\mathbf{q}_h^0, \kappa^{-1}\Pi_p^{BDM}(\kappa\nabla\underline{\Psi}(0)) - \nabla I_h\Psi(0)\right)_{\mathcal{T}_h}\\
&\quad + \int_0^T \left(\dot{\mathbf{e}}_h, \kappa^{-1}\Pi_k^{BDM}(\kappa\nabla\underline{\Psi}) - \nabla I_h\underline{\Psi}\right)_{\mathcal{T}_h} + \int_0^T (\mathbf{e}, \nabla I_h\Psi)_{\mathcal{T}_h}.
\end{aligned}
$$

Finally, using Lemma A.1 to rewrite the last term, we get that

$$
\begin{aligned}
T_2 &= \left(\mathbf{q}^0-\mathbf{q}_h^0, \kappa^{-1}\Pi_p^{BDM}(\kappa\nabla\underline{\Psi}) - \nabla I_h\underline{\Psi}(0)\right)_{\mathcal{T}_h} + \int_0^T (\dot{\mathbf{e}}_h, \kappa^{-1}\Pi_p^{BDM}(\kappa\nabla\underline{\Psi}) - \nabla I_h\underline{\Psi})_{\mathcal{T}_h}\\
&\quad + \tfrac{k}{2}\left(\boldsymbol{\varepsilon}_{h,k}^{q,0}, \nabla I_h(\mathrm{I}_k\Psi)^0\right)_{\mathcal{T}_h} + k\sum_{n=1}^{N-1}\left(\mathbf{q}^n-\mathbf{q}_h^n, \nabla I_h(\mathrm{I}_k\Psi)^n\right)_{\mathcal{T}_h}.
\end{aligned}
$$

This completes the proof. □

As a direct consequence of this result, we can obtain the last estimate of Theorem 2.3. Indeed, by the previous lemma, we have

$$
\begin{aligned}
\left|(P_{p-1}\varepsilon_{h,k}^{u,N}, \rho\theta)_{\mathcal{T}_h}\right| \leq & H_1\|P_{p-1}\varepsilon_{h,k}^{u,0}\|_\Omega + H_2\|P_{p-1}\delta_k\varepsilon_{h,k}^{u,1}\|_\Omega + H_3\|\mathbf{q}^0 - \mathbf{q}_h^0\|_\Omega \\
& + H_4 \max_{n=1,\ldots,N} \|\delta_k\left(\mathbf{q}^n - \mathbf{q}_h^n\right)\|_\Omega \\
& + H_5 \max_{n=1,\ldots,N-1} \|\mathrm{D}_k^2\left(u^n - u_h^n\right)\|_\Omega \\
& + \tfrac{k}{2} H_6\|\boldsymbol{\varepsilon}_{h,k}^{q,0}\|_\Omega + H_7 \max_{n=1,\ldots,N-1} \|\mathrm{D}_k^2 u^n - \mathrm{A}_k\ddot{u}^n\|_\Omega \\
& + \tfrac{k^2}{12} H_8 \max_{n=1,\ldots,N-1} \|\mathrm{D}_k^2\boldsymbol{\varepsilon}_{h,k}^{q,n}\|_\Omega,
\end{aligned}
$$

where H_i, $i = 1, \ldots, 5$, are defined exactly as in the semidiscrete case (see the paragraph after Lemma 3.5), and

$$
\begin{aligned}
H_6 = \|\nabla I_h(\mathrm{I}_k\Psi)^0\|_\Omega, \quad H_7 = T \max_{n=1,\ldots,N-1} \|I_h(\mathrm{I}_k\Psi)^n\|_\Omega, \quad H_8 \\
= T \max_{n=1,\ldots,N-1} \|\nabla I_h(\mathrm{I}_k\Psi)^n\|_\Omega.
\end{aligned}
$$

The result now follows by using the regularity estimates of Proposition 3.4. This completes the last estimates of Theorem 4.5.

Step 4: Error estimates for the starting functions

Lemma 5.5 *For $p \geq 1$, we have*

$$
\|\varepsilon_{h,k}^{u,0}\|_\Omega \leq Ch(h^{p+1} + k^4), \qquad \|\delta_k\varepsilon_{h,k}^{u,1}\|_\Omega \leq Ck^5
$$

and

$$
\mathbb{E}_{h,k}^{1/2} \leq C(h^{p+1} + k^4)^2, \qquad \mathbb{F}_{h,k}^1 \leq C(h^{p+1} + k^4)^2.
$$

Proof The standard results on HDG methods for steady-state diffusion problems in [7] yield

$$
\begin{aligned}
&\|\boldsymbol{\Pi}\mathbf{q}_A - \mathbf{q}_{h,A}\|_{\kappa^{-1}}^2 + 2|(\Pi u_A - u_{h,A}) - (Pu_A - \widehat{u}_{h,A})|_\tau^2 \leq \|\mathbf{q}_A - \boldsymbol{\Pi}\mathbf{q}_A\|_{\kappa^{-1}}^2, \\
&\|\Pi u_A - u_{h,A}\|_\Omega \leq Ch^{\min\{p,1\}}\|\mathbf{q}_A - \boldsymbol{\Pi}\mathbf{q}_A\|_\Omega.
\end{aligned}
$$

By (4.9) and (4.10), we have $(\rho\, u_{D^2,h}, w)_{\mathcal{T}_h} = (\mathrm{A}_k f^1 + \nabla\cdot(\kappa \mathrm{A}_k \nabla \mathrm{T} u^1), w)_{\mathcal{T}_h}$, which implies that

$$
\begin{aligned}
\left(\rho\,(u_{D^2,h} - \Pi\mathrm{D}_k^2 u^1), w\right)_{\mathcal{T}_h} &= \left(\mathrm{A}_k f^1 + \nabla\cdot(\kappa\mathrm{A}_k\nabla u^1) - \rho\,\Pi\mathrm{D}_k^2 u^1\right. \\
&\quad \left. + \nabla\cdot(\kappa\mathrm{A}_k\nabla\mathrm{T}u^1) - \nabla\cdot(\kappa\mathrm{A}_k\nabla u^1), w\right)_{\mathcal{T}_h} \\
&= \left(\rho\,\left(\mathrm{A}_k\ddot{u}^1 - \mathrm{D}_k^2 u^1\right) + \rho\,(\mathrm{D}_k^2 u^1 - \Pi\mathrm{D}_k^2 u^1)\right. \\
&\quad \left. + \nabla\cdot(\kappa\mathrm{A}_k\nabla\mathrm{T}u^1) - \nabla\cdot(\kappa\mathrm{A}_k\nabla u^1), w\right)_{\mathcal{T}_h},
\end{aligned}
$$

and $\|u_{D^2,h} - \Pi\mathrm{D}_k^2 u^1\|_\rho \leq C(h^{p+1} + k^4)$. Comparing system (4.8) with (4.9) and (4.10), it is not difficult to see $(u_{D^2,h}, \mathbf{q}_{h,A}, u_{h,A}, \widehat{u}_{h,A}) = (\mathrm{D}_k^2 u_h^1, \mathrm{A}_k\mathbf{q}_h, \mathrm{A}_k u_h, \mathrm{A}_k\widehat{u}_h)$, and recalling the definition of $(\mathbf{q}_A, \mathbf{u}_A)$, we have $\|\mathrm{A}_k\mathbf{q}^1 - \mathbf{q}_A\|_\Omega \leq Ck^5$ and $\|\mathrm{A}_k u^1 - u_A\|_\Omega \leq Ck^5$. We then obtain

$$
\|\mathrm{A}_k\varepsilon_{h,k}^{u,1}\|_\Omega \leq Ch^{\min\{p,1\}}(h^{p+1} + k^5), \qquad \|\mathrm{D}_k^2\varepsilon_{h,k}^{u,1}\|_\rho \leq C(h^{p+1} + k^4).
$$

Table 1 Experimental CFL condition for the Stormer-Numerov HDG schemes with stabilization parameter $\tau = 1$

	$p = 1$	$p = 2$	$p = 3$
$d = 1$	0.5	0.3	0.15
$d = 2$	0.35	0.2	0.1

By the definition of u_h^0, we then have $\|\delta_k \varepsilon_{h,k}^{u,1}\|_\Omega = \|(\Pi u^1 - \Pi \mathrm{T} u^1)/k\|_\Omega \leq Ck^4$, and then

$$
\begin{aligned}
\|\varepsilon_{h,k}^{u,1}\|_\Omega &= \|\mathrm{A}_k \varepsilon_{h,k}^{u,1} - \tfrac{k^2}{12}\delta_k \varepsilon_{h,k}^{u,1}\|_\Omega && \leq Ch^{\min\{p,1\}}(h^{p+1} + k^5)\\
&\leq Ch(h^{p+1} + k^5),\\
\|\varepsilon_{h,k}^{u,0}\|_\Omega &= \|\varepsilon_{h,k}^{u,1} - k\delta_k \varepsilon_{h,k}^{u,1}\|_\Omega && \leq Ch^{\min\{p,1\}}(h^{p+1} + k^5)\\
&\leq Ch(h^{p+1} + k^5),\\
\|\delta_k \varepsilon_{h,k}^{u,2}\|_\Omega &= \|\delta_k \varepsilon_{h,k}^{u,1} + k\mathrm{D}_k^2 \varepsilon_{h,k}^{u,1}\| && \leq Ch^{\min\{p,1\}}(h^{p+1} + k^5)\\
&\leq Ch(h^{p+1} + k^5),
\end{aligned}
$$

for $p \geq 1$. Now we use Proposition 4.2, noticing that $(\boldsymbol{\varepsilon}_{h,k}^{q,1}, \widehat{\varepsilon}_{h,k}^{u,n}) = \boldsymbol{\mathcal{S}}(\mathbf{a}_h^1, \varepsilon_{h,k}^{u,1}, 0)$ for $n \geq 0$, to get that

$$
\|\boldsymbol{\varepsilon}_{h,k}^{q,1}\|_{\kappa^{-1}}^2 + |\varepsilon_{h,k}^{u,1} - \widehat{\varepsilon}_{h,k}^{u,1}|_\tau^2 \leq 4\,C_h^2\,\|\varepsilon_{h,k}^{u,1}\|_\rho^2 + 2\,\|\mathbf{a}_h^1\|_{\kappa^{-1}}^2 \leq C(h^{p+1} + k^5)^2.
$$

Similarly, we have

$$
\begin{aligned}
\|\boldsymbol{\varepsilon}_{h,k}^{q,0}\|_{\kappa^{-1}}^2 + |\varepsilon_{h,k}^{u,0} - \widehat{\varepsilon}_{h,k}^{u,0}|_\tau^2 &\leq 4\,C_h^2\,\|\varepsilon_{h,k}^{u,0}\|_\rho^2 + 2\,\|\mathbf{a}_h^0\|_{\kappa^{-1}}^2 \leq C(h^{p+1} + k^5)^2,\\
\|\delta_k \boldsymbol{\varepsilon}_{h,k}^{q,1}\|_{\kappa^{-1}}^2 + |\delta_k \varepsilon_{h,k}^{u,1} - \delta_k \widehat{\varepsilon}_{h,k}^{u,1}|_\tau^2 &\leq 4\,C_h^2\,\|\delta_k \varepsilon_{h,k}^{u,1}\|_\rho^2 + 2\,\|\delta_k \mathbf{a}_h^1\|_{\kappa^{-1}}^2 \leq C(h^{p+1} + k^3)^2,\\
\|\delta_k \boldsymbol{\varepsilon}_{h,k}^{q,2}\|_{\kappa^{-1}}^2 + |\delta_k \varepsilon_{h,k}^{u,2} - \delta_k \widehat{\varepsilon}_{h,k}^{u,2}|_\tau^2 &\leq 4\,C_h^2\,\|\delta_k \varepsilon_{h,k}^{u,2}\|_\rho^2 + 2\,\|\delta_k \mathbf{a}_h^2\|_{\kappa^{-1}}^2 \leq C(h^{p+1} + k^5)^2.
\end{aligned}
$$

We obtain the estimates by combining all the results above. This complete the proof. □

Step 5: *Conclusion* Applying the estimates obtained in the previous steps, and using the approximation properties of the auxiliary HDG projection Theorem 2.2, we obtain the second and third estimate of Theorem 4.6. To obtain the first estimate we use the fact that

$$
\|u^n - u_h^n\|_\rho \leq \sum_{m=1}^{n} k\|\delta_k u^m - \delta_k u_h^m\|_\rho + \|u^0 - u_h^0\|_\rho.
$$

The error estimate of $u - u_h^*$ can be proven in essentially the same way as in [9]. This concludes the proof of Theorem 4.6.

6 Numerical Examples

In this section, we present two numerical examples illustrating the convergence and conservative properties of our scheme. We numerically found CFL conditions for the schemes in $d = 1$ and $d = 2$ dimension. These are presented in Table 1.

Table 2 History of convergence of the numerical approximations of the wave equation with exact solution $u(t,x,y) = (1/(\sqrt{2}\pi))\sin(\pi x)\sin(\pi y)\cos(\sqrt{2}\pi t)$ by the scheme Stormer-Numerov HDG scheme

p	l	u_h		q_h		u_h^*	
		Error	e.o.c.	Error	e.o.c.	Error	e.o.c.
1	1	7.1e−2	–	1.8e−1	–	3.2e−2	–
	2	2.1e−2	1.78	3.7e−2	2.29	8.2e−3	1.96
	3	3.8e−3	2.47	6.4e−3	2.52	8.0e−4	3.36
	4	7.8e−4	2.27	1.4e−3	2.15	9.1e−5	3.14
	5	1.9e−4	2.07	3.6e−4	2.01	1.1e−5	3.11
2	1	1.7e−2	–	3.7e−2	–	3.5e−3	–
	2	1.8e−3	3.24	3.2e−3	3.51	1.2e−4	4.84
	3	1.5e−4	3.62	3.2e−4	3.35	5.1e−6	4.57
	4	2.0e−5	2.87	4.1e−5	2.96	2.9e−7	4.12
	5	2.4e−6	3.08	5.0e−6	3.02	1.8e−8	4.03
3	1	2.7e−3	–	4.8e−3	–	2.4e−4	–
	2	1.3e−4	4.40	2.4e−4	4.31	5.5e−6	5.43
	3	6.6e−6	4.29	1.4e−5	4.13	1.6e−7	5.05
	4	4.0e−7	4.04	8.7e−7	4.00	5.1e−9	5.00
	5	2.5e−8	4.01	5.4e−8	4.00	1.6e−10	5.00

Computations were performed up to a final time $T_f = 1.0$, $\tau = 1$, time steps $k = .35h$, $k = 0.2h$ and $k = 0.1h$, for $p = 1$, $p = 2$ and $p = 3$, respectively, and mesh parameters $h = 2^{-l}$, for $l = 1, 2, 3, 4, 5$

6.1 Convergence and Superconvergence Test

We consider the following exact solution of the two dimensional acoustic wave equation

$$u(t,x,y) = \frac{1}{\sqrt{2}\pi}\sin(\pi x)\sin(\pi y)\cos(\sqrt{2}\pi t), \quad x, y \in (0,1)^2,\ t \in \big(0, T_f\big],$$

with parameters $\rho = 1$ and $\kappa = 1$ and Dirichlet boundary conditions. We report in Table 2 the L^2-errors and estimated orders of convergence (e.o.c.) of the approximations by the schemes Stormer-Numerov HDG(p), with polynomial degree $p = 1, 2, 3$. We observe optimal convergence of order $p+1$ for the errors of the approximations u_h and q_h, and a superconvergent order of $p+2$ for the post-processed approximation u_h^*. Note that for the case $p = 3$ we observe a superconvergent order of $p + 2 = 5$, instead of the fourth-order of the time-stepping scheme. We argue that this is due to the small time step $k = 0.1h$ and short final time $T_f = 1.0$. We provide another example where the post-processed approximation converges with an order 4, the same order of the time marching scheme, for polynomials of degree $p = 3$. See Table 3. We also report for this example the history of convergence when the time step k is of order $h^{5/4}$. As we anticipated before, we observe the superconvergence for this time step.

Table 3 History of convergence of the numerical approximations of the wave equation with exact solution $u(t, x) = \sin(2\pi(x - t))$ by the scheme Stormer-Numerov HDG scheme

p	k	l	u_h		q_h		u_h^*	
			Error	e.o.c.	Error	e.o.c.	Error	e.o.c.
3	$.15h$	1	4.6e−2	–	3.1e−1	–	4.4e−2	–
		2	1.1e−3	5.44	1.2e−2	4.73	3.8e−4	6.85
		3	6.6e−5	4.00	6.8e−4	4.08	1.3e−5	4.91
		4	4.1e−6	4.00	4.3e−5	4.00	6.0e−7	4.40
		5	2.6e−7	4.00	2.7e−6	3.99	3.5e−8	4.13
3	$.15h^{5/4}$	1	4.5e−2	–	2.9e−1	–	4.3e−02	–
		2	1.1e−3	5.41	1.1e−2	4.73	3.2e−04	7.07
		3	6.6e−5	4.00	6.8e−4	4.02	8.9e−06	5.17
		4	4.1e−6	4.00	4.3e−5	3.99	2.7e−07	5.02
		5	2.6e−7	4.00	2.7e−6	3.99	8.6e−09	4.99

Computations were performed up to a final time $T_f = 5.0$, time steps $k = 0.15h$ and $k = .15h^{5/4}$, for $p = 3$, and parameters $h = 2^{-l}$, for $l = 1, 2, 3, 4, 5$, and $\tau = 1$

6.2 Conservation Properties Test

We consider the following travelling wave solution of the one dimensional acoustic wave equation with periodic boundary conditions

$$u(t, x) = \sin(12\pi(x - t)), \quad x \in (0, 1),\ t \in \left(0, T_f\right],$$

with parameters $\rho = 1$ and $\kappa = 1$. We compare the approximate solution of the fully discrete scheme HDG-Stormer-Numerov scheme presented in this paper with an alternative scheme consisting in the second-order semi-discrete HDG formulation in space and a diagonally implicit Runge–Kutta–Nyström (DIRKN) method in time. This method is implemented in a similar fashion to the DIRK-HDG methods in [22]. We utilize a fourth-order DIRKN method matching the order of the Stormer-Numerov scheme. The coefficients of the DIRKN scheme are detailed in [24] (see Eq. (4.8)). We remark that we also obtain optimal convergence results

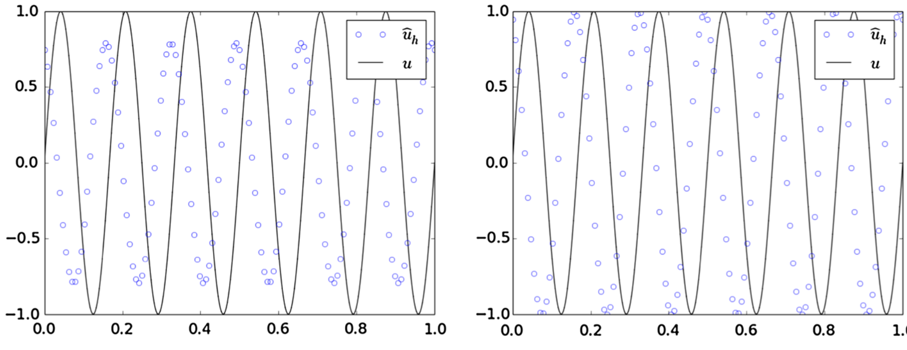

Fig. 1 *Left* exact solution (*black line*) and approximate solution $\widehat{u}_h$ (*blue circle*) by the DIRKN-HDG scheme. *Right* exact solution (*black line*) and approximate solution $\widehat{u}_h$ (*blue circle*) by the Stormer-Numerov HDG scheme. We computed with $p = 1$, $h = 2^{-7}$, $k = .5h$, $\tau = 20$, and up to $T_f = 200$ (Color figure online)

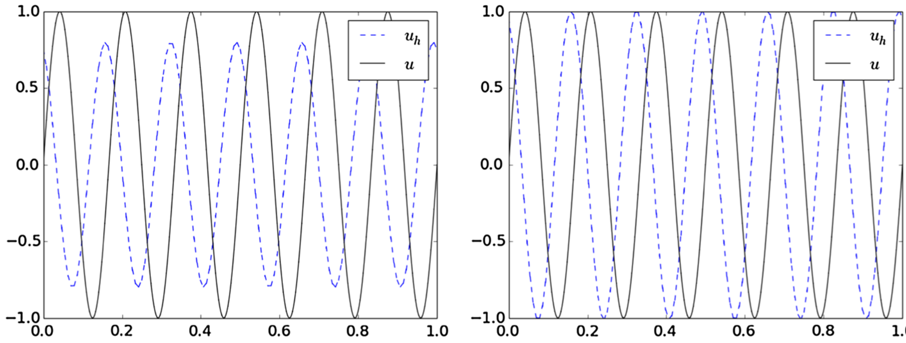

Fig. 2 *Left* exact solution (*black line*) and approximate solution u_h (*blue dashed line*) by the DIRKN-HDG scheme. *Right* exact solution (*black line*) and approximate solution u_h (*blue dashed line*) by the Stormer-Numerov HDG scheme. We computed with $p = 1$, $h = 2^{-7}$, $k = .5h$, $\tau = 20$, and up to $T_f = 200$ (Color figure online)

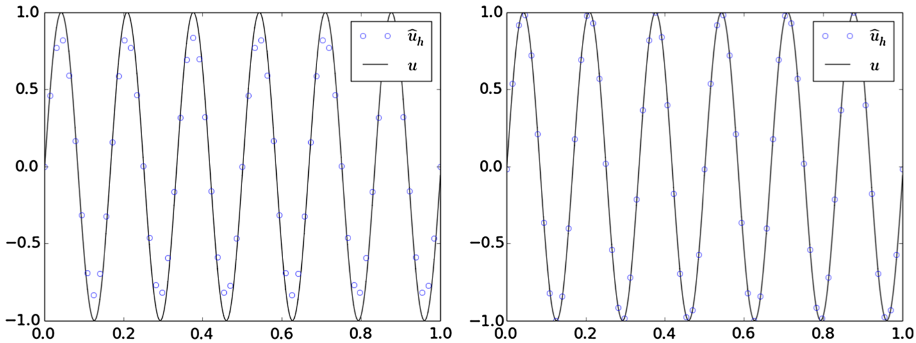

Fig. 3 *Left* exact solution (*black line*) and approximate solution $\widehat{u}_h$ (*blue circle*) by the DIRKN-HDG scheme. *Right* exact solution (*black line*) and approximate solution $\widehat{u}_h$ (*blue circle*) by the Stormer-Numerov HDG scheme. We computed with $p = 2$, $h = 2^{-6}$, $k = .3h$, $\tau = 20$, and up to $T_f = 200$ (Color figure online)

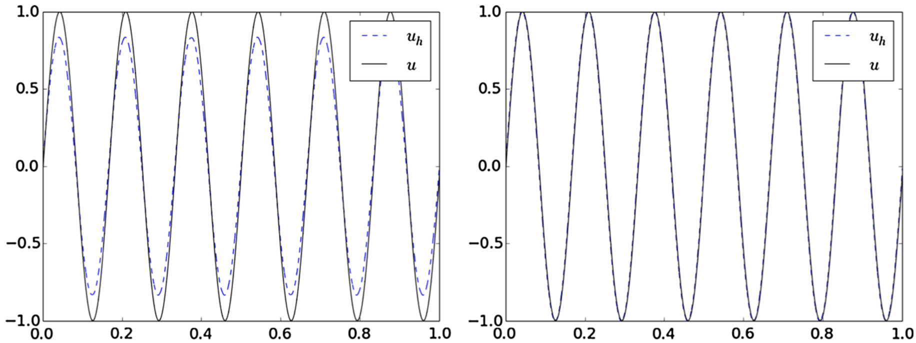

Fig. 4 *Left* exact solution (*black line*) and approximate solution u_h (*blue dashed line*) by the DIRKN-HDG scheme. *Right* exact solution (*black line*) and approximate solution u_h (*blue dashed line*) by the Stormer-Numerov HDG scheme. We computed with $p = 2$, $h = 2^{-6}$, $k = .3h$, $\tau = 20$, and up to $T_f = 200$ (Color figure online)

for the DIRKN-HDG scheme. We compare the approximate solutions by the two schemes for a long term computation $T_f = 200$. We observe that, thanks to the conservative properties of the Stormer-Numerov scheme, its approximate solution almost does not dissipate. On the other hand, dissipation is observed for the approximate solution by the DIRKN-HDG scheme. Computation are performed for polynomial degree $p = 1, 2$. Plots of the approximate solutions are provided in Figs. 1 and 2 for $p = 1$ and Figs. 3 and 4 for $p = 2$. Finally, we observe in the case $p = 1$ a phase-lag behaviour in both approximate solutions.

7 Concluding Remarks

We can obtain the very same results if we use the SDG method [5], or any of the HDG or mixed methods (for diffusion problems) obtained by the theory of M-decompositions recently introduced in [6].

A An Identity Used in the Duality Argument

Lemma A.1 *Suppose η is the continuous piecewise-linear Lagrange interpolation of the values $\{\eta^n\}_{n=0}^N$ and $\mu : [0, T] \to \mathbb{R}$ is a function. Then we have*

$$\int_0^T \eta(t)\mu(t)dt = \tfrac{k}{2}\eta(0)\mathrm{I}_k\mu(0) + k\sum_{n=1}^{N-1}\eta(t_n)\mathrm{I}_k\mu(t_n) + \tfrac{k}{2}\eta(0)\mathrm{I}_k\mu(0),$$

$$\int_0^T \dot\eta(t)\dot\mu(t)dt = \delta_k\eta^N\mu(T) - k\sum_{n=1}^{N-1}\mathrm{D}_k^2\eta^n\mu(t_n) - \delta_k\eta^1\mu(0).$$

There result also holds for functions taking values in an inner product space, when the pointwise product is substituted by the inner product.

Proof We have

$$\begin{aligned}\int_0^T \eta(t)\mu(t)dt &= \sum_{n=1}^N\left(\eta^n\int_{t_{n-1}}^{t_n}\tfrac{1}{k}(t-t_{n-1})\mu(t)dt + \eta^{n-1}\int_{t_{n-1}}^{t_n}\tfrac{1}{k}(t_n-t)\mu(t)dt\right)\\ &= \eta^N\int_{T-k}^T\tfrac{1}{k}(t-(T-k))\mu(t)dt + \sum_{n=1}^{N-1}\eta^n\int_{t_{n-1}}^{t_n}\tfrac{1}{k}(t-t_{n-1})\mu(t)dt\\ &\quad+\sum_{n=2}^N\eta^{n-1}\int_{t_{n-1}}^{t_n}\tfrac{1}{k}(t_n-t)\mu(t)dt + \eta^0\int_0^k\tfrac{1}{k}(k-t)\mu(t)dt.\end{aligned}$$

After using the definition of I_k, we get the first identity.

To obtain the second identity, note that we have

$$\int_0^T \dot\eta(t)\dot\mu(t)dt = \sum_{n=1}^N\int_{t_{n-1}}^{t_n}\dot\eta(t)\dot\mu(t)dt = \sum_{n=1}^N\delta_k\eta^n(\mu(t_n)-\mu(t_{n-1})),$$

and the result follows after simple rearrangements. □

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