

Interior Penalty Discontinuous Galerkin Methods for Second Order Linear Non-divergence Form Elliptic PDEs

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Abstract This paper develops interior penalty discontinuous Galerkin (IP-DG) methods to approximate $W^{2,p}$ strong solutions of second order linear elliptic partial differential equations (PDEs) in non-divergence form with continuous coefficients. The proposed IP-DG methods are closely related to the IP-DG methods for advection-diffusion equations, and they are easy to implement on existing standard IP-DG software platforms. It is proved that the proposed IP-DG methods have unique solutions and converge with optimal rate to the $W^{2,p}$ strong solution in a discrete $W^{2,p}$ -norm. The crux of the analysis is to establish a DG discrete counterpart of the Calderon–Zygmund estimate and to adapt a freezing coefficient technique used for the PDE analysis at the discrete level. To obtain such a crucial estimate, we need to establish broken $W^{1,p}$ -norm error estimates for IP-DG approximations of constant coefficient elliptic PDEs, which is also of independent interest. Numerical experiments are provided to gauge the performance of the proposed IP-DG methods and to validate the theoretical convergence results.

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1 Introduction

In this paper we develop interior penalty discontinuous Galerkin methods for approximating the $W^{2,p}$ strong solution to the following second order linear elliptic PDE in non-divergence form:

$$\mathcal{L}u(x) := -A(x) : D^2u(x) = f(x) \quad x \in \Omega, \quad (1.1a)$$

$$u(x) = 0 \quad x \in \partial\Omega, \quad (1.1b)$$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded domain with boundary $\partial\Omega$, $f \in L^p(\Omega)$ with $1 < p < \infty$, and $A \in [C^0(\overline{\Omega})]^{n \times n}$ is positive definite in $\overline{\Omega}$.

Non-divergence form elliptic PDEs arrive naturally from many applications such as stochastic optimal control and game theory [12]; they are also encountered in the linearization of fully nonlinear PDEs such as Monge–Ampère-type equations [4]. If A is differentiable, then it is easy to check that equation (1.1a) can be rewritten as a diffusion-convection equation with A as the diffusion coefficient and $\nabla \cdot A$ as the convection coefficient. However, if $A \in [C^0(\overline{\Omega})]^{n \times n}$, then this formulation is not possible since $\nabla \cdot A$ does not exist as a function, but rather only as a measure. We recall that in the literature there are three well-established PDE theories for non-divergence form elliptic PDEs depending on the smoothness of A and f (as well as the smoothness of the boundary $\partial\Omega$ which we assume sufficiently smooth here to ease the presentation). The first theory is the classical solution (or Schauder’s) theory [13, Chapter 6] which seeks solutions in the Hölder space $C^{2,\alpha}(\overline{\Omega})$ for $0 < \alpha < 1$ when $A \in [C^\alpha(\overline{\Omega})]^{n \times n}$ and $f \in C^\alpha(\overline{\Omega})$. The second one is the $W^{2,p}$ (strong) solution theory [13, Chapter 9] which seeks solutions in the Sobolev space $W^{2,p}(\Omega)$ for $1 < p < \infty$ that satisfy the PDE almost everywhere in Ω under the assumptions that $A \in [C^0(\overline{\Omega})]^{n \times n}$ and $f \in L^p(\Omega)$. If the coefficient matrix satisfies the Cordes condition and if the domain is convex, then the notion of strong solutions may be extended to the case $A \in [L^\infty(\Omega)]^{n \times n}$ [16,26]. The third one is the viscosity (weak) solution theory [7] which seeks solutions in $C^0(\Omega)$ (or even in $B(\Omega)$, the space of bounded functions in Ω) that satisfy the PDE in the viscosity sense under the assumptions that $A \in [L^\infty(\Omega)]^{n \times n}$ and $f \in L^\infty(\Omega)$. We note that all the three solution concepts and PDE theories are non-variational; this is a main difference between divergence form PDEs [13, Chapter 8] and non-divergence form PDEs. We also note that in the case of the viscosity solution theory, the uniqueness and regularity of solutions had been the main focus of the study for second order non-divergence form PDEs (see [7,17,23,26] and the references therein).

In contrast to the advances of the PDE analysis, almost no progress on numerical methods and numerical analysis was achieved until very recently for second order elliptic PDEs in non-divergence form with non-differentiable coefficient matrix A (cf. [8,10,15,18,20,26,28]). The main difficulty is caused by the non-divergence structure of the PDEs which prevents any straightforward application of Galerkin-type numerical methodologies such as finite element methods, discontinuous Galerkin methods and spectral methods. Moreover, the non-variational nature of the strong and viscosity solution concepts make convergence analysis and/or error estimates of any convergent numerical methods very delicate and difficult. Nonstandard numerical techniques are often required to do the job (cf. [10,18,20,26,28]).

The primary goal of this paper is to develop convergent interior penalty discontinuous Galerkin (IP-DG) methods for approximating the $W^{2,p}$ strong solution of problem (1.1) under the assumption that $A \in [C^0(\overline{\Omega})]^{n \times n}$ and its solution satisfies the Calderon–Zygmund estimate [13, Chapter 9]. The reason for developing IP-DG methods is twofold. First, we

intend to take advantage of some of the features of DG methods such as simplicity and ease of computation, and flexibility of mesh and ease for adaptivity, to design better numerical methods for problem (1.1a)–(1.1b). Second, we intend to develop new numerical analysis techniques and machineries for non-divergence form PDEs. It is expected that the DG convergence analysis is more involved because of the difficulty caused by the non-conformity of the DG finite element space and by their complicate mesh-dependent bilinear forms. The crux of the analysis of this paper is to establish a discrete Calderon-Zygmund estimate for the proposed IP-DG methods and to adapt a freezing coefficient technique used in the PDE analysis at the discrete level. Moreover, in order to prove the desired discrete Calderon-Zygmund estimate, we need to establish the $W_h^{1,p}$ stability and error estimates for the IP-DG approximations of constant coefficient elliptic PDEs. Such estimates seem to be new and have independent interest.

The remainder of this paper is organized as follows. In Sect. 2 we provide the notation and a collection of preliminary estimates, in particular, some properties of DG functions are either cited or proved. Section 3 analyzes the IP-DG methods for constant coefficient elliptic PDEs. The $W_h^{1,p}$ stability and error estimates are derived, which in turn lead to global $W_h^{2,p}$ stability estimates. The latter estimate can be regarded as a discrete Calderon-Zygmund estimate for the proposed IP-DG methods. To the best of our knowledge, such an estimate is new and of independent interest. Sect. 4 is devoted to the formulation, stability and convergence analysis and error estimate for the proposed IP-DG methods. Here, using the continuity of the coefficient matrix A , the freezing coefficient technique is adapted to establish local stability (or a left-side inf-sup condition) for the IP-DG discrete operators, which together with a covering argument leads to a global Gårding-type inequality for the formal adjoint operators of the IP-DG operators. Next, using a duality argument we obtain a global left-side inf-sup condition for the IP-DG discrete operators. Finally, by employing another duality argument, we derive the desired discrete Calderon-Zygmund estimate for the DG discrete operators. Once this stability estimate is shown, well-posedness and convergence of the IP-DG methods follow easily. In Section 5 we present a number of numerical experiments to verify our theoretical results and to gauge the performance of the proposed IP-DG methods, even for the case $A \in [L^\infty(\bar{\Omega})]^{n \times n}$ which is not covered by our convergence theory.

This paper is a condensed version of [11] where one can find the detailed proofs of those lemmas which are omitted here due to page limitation.

2 Preliminary Results

2.1 Notation

Let Ω be an open and bounded domain in \mathbb{R}^n . For a subdomain D of Ω with boundary ∂D , let $L^p(D)$ and $W^{s,p}(D)$ for $s \geq 0$ and $1 \leq p \leq \infty$ denote the standard Lebesgue and Sobolev spaces respectively, and $W_0^{1,p}(D)$ be the closure of $C_c^\infty(D)$ in $W^{1,p}(D)$. Let $(\cdot, \cdot)_D$ be the L^2 inner product on D and $(\cdot, \cdot) := (\cdot, \cdot)_\Omega$. To improve the readability of the paper, we adopt the convention that $a \lesssim b$ stands for $a \leq Cb$ for some $C > 0$ which does not depend on any discretization parameters.

Let \mathcal{T}_h be a shape-regular and conforming triangulation of Ω . Let \mathcal{E}_h^I and \mathcal{E}_h^B denote respectively the sets of all interior and boundary edges/faces of \mathcal{T}_h , and set $\mathcal{E}_h := \mathcal{E}_h^I \cup \mathcal{E}_h^B$. We introduce the broken Sobolev spaces

$$W^{s,p}(\mathcal{T}_h) := \prod_{T \in \mathcal{T}_h} W^{s,p}(T), \quad L^p(\mathcal{T}_h) := W^{0,p}(\mathcal{T}_h),$$

$$W_h^{s,p}(D) := W^{s,p}(\mathcal{T}_h)|_D, \quad L_h^p(D) := L^p(\mathcal{T}_h)|_D.$$

For any interior edge/face $e = \partial T^+ \cap \partial T^- \in \mathcal{E}_h^I$ we define the jump and average of a scalar or vector valued function v as $[v]|_e := v^+ - v^-$, $\{v\}|_e := \frac{1}{2}(v^+ + v^-)$, where $v^\pm = v|_{T^\pm}$. On a boundary edge/face $e \in \mathcal{E}_h^B$ with $e = \partial T^+ \cap \partial \Omega$, we set $[v]|_e = \{v\}|_e = v^+$.

For any $e \in \mathcal{E}_h^I$ we use ν_e to denote the unit outward normal vector pointing in the direction of the element with the smaller global index. For $e \in \mathcal{E}_h^B$ we set ν_e to be the outward normal to $\partial \Omega$ restricted to e . The standard DG finite element space is defined as

$$V_h := \left\{ v_h \in W^{2,p}(\mathcal{T}_h); v_h|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h \right\},$$

where $\mathbb{P}_k(T)$ denotes the set of all polynomials of degree less than or equal to k on T . We also introduce for any $D \subset \Omega$

$$V_h(D) := \left\{ v \in V_h; v|_{\Omega \setminus \bar{D}} \equiv 0 \right\}.$$

Note that $V_h(D)$ is nontrivial provided that there exists an inscribed ball B with radius $r \geq h$ such that $B \subset D$. We also adopt the convention $V_h(\Omega) = V_h$.

For each $e \in \mathcal{E}_h$, let $\gamma_e > 0$ be constant on e . We define the following mesh-dependent norms on $W_h^{2,p}(D)$ and $W_h^{1,p}(D)$:

$$\|v\|_{W_h^{2,p}(D)} := \|D_h^2 v\|_{L^p(D)} + \left(\sum_{e \in \mathcal{E}_h^I} h_e^{1-p} \|[\nabla v]\|_{L^p(e \cap \bar{D})}^p \right)^{\frac{1}{p}} \tag{2.1}$$

$$+ \left(\sum_{e \in \mathcal{E}_h} \gamma_e^p h_e^{1-2p} \|[v]\|_{L^p(e \cap \bar{D})}^p \right)^{\frac{1}{p}},$$

$$\|v\|_{W_h^{1,p}(D)} := \|\nabla_h v\|_{L^p(D)} + \left(\sum_{e \in \mathcal{E}_h} \gamma_e^p h_e^{1-p} \|[v]\|_{L^p(e \cap \bar{D})}^p \right)^{\frac{1}{p}} \tag{2.2}$$

$$+ \left(\sum_{e \in \mathcal{E}_h} h_e \|\{\nabla v\}\|_{L^p(e \cap \bar{D})}^p \right)^{\frac{1}{p}},$$

where $\nabla_h v$ and $D_h^2 v$ denote the piecewise gradient and Hessian of v .

In addition, we define the discrete $W_h^{-2,p}$ -norm and $W_h^{-1,p}$ -norm as follows:

$$\|q\|_{W_h^{-2,p}(D)} := \sup_{0 \neq v_h \in V_h} \frac{(q, v_h)_D}{\|v_h\|_{W_h^{2,p'}(D)}}, \tag{2.3}$$

$$\|q\|_{W_h^{-1,p}(D)} := \sup_{0 \neq v \in W_h^{1,p'}(D)} \frac{(q, v)_D}{\|v\|_{W_h^{1,p'}(D)}}, \tag{2.4}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Finally, for any domain $D \subseteq \Omega$ and any $w \in L_h^p(D)$, we introduce the following mesh-dependent semi-norm

$$\|w\|_{L_h^p(D)} := \sup_{0 \neq v_h \in V_h(D)} \frac{(w, v_h)_D}{\|v_h\|_{L^{p'}(D)}}. \tag{2.5}$$

It can be proved that (cf. [10])

$$\|w_h\|_{L^p(\Omega)} \lesssim \|w_h\|_{L_h^p(\Omega)} \quad \forall w_h \in V_h. \tag{2.6}$$

2.2 Properties of the DG Space V_h

In this subsection we collect some technical lemmas that cover the basic properties of functions in the DG space V_h . These facts will be used many times in the later sections. We first state the standard trace inequalities for broken Sobolev functions, a proof of this lemma can be found in [3].

Lemma 2.1 *For any $T \in \mathcal{T}_h$ and $1 < p < \infty$, there holds*

$$\|v\|_{L^p(\partial T)}^p \lesssim \left(h_T^{p-1} \|\nabla v\|_{L^p(T)}^p + h_T^{-1} \|v\|_{L^p(T)}^p \right) \quad \forall v \in W^{1,p}(T). \tag{2.7}$$

Therefore by scaling we have

$$\sum_{e \in \mathcal{E}_h^I} h_e \|v\|_{L^p(e \cap \bar{D})}^p \lesssim \begin{cases} \|v\|_{L^p(D)}^p & \forall v \in V_h(D), \\ \|v\|_{L^p(D)}^p + h^p \|\nabla v\|_{L^p(D)}^p & \forall v \in W^{2,p}(\mathcal{T}_h); v|_{\Omega \setminus D} = 0. \end{cases} \tag{2.8}$$

Next, we state two inverse inequalities between the mesh-dependent norms defined above. We refer to [11] for their proofs.

Lemma 2.2 *For any $v_h \in V_h$, $D \subseteq \Omega$, there hold for $1 < p < \infty$*

$$\|v_h\|_{W_h^{2,p}(D)} \lesssim h^{-1} \|v_h\|_{W_h^{1,p}(D_h)}, \tag{2.9}$$

$$\|v_h\|_{L^p(D)} \lesssim h^{-1} \|v_h\|_{W_h^{-1,p}(D)}, \tag{2.10}$$

where $D_h = \{x \in \Omega; \text{dist}(x, D) \leq h\}$.

The following lemma shows that the broken Sobolev norms are controlled by their corresponding Sobolev norms.

Lemma 2.3 *For any $1 < p < \infty$ there holds the following inequality:*

$$\|\varphi\|_{W_h^{2,p}(\Omega)} \leq \|\varphi\|_{W^{2,p}(\Omega)} \quad \forall \varphi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Proof Since the inequality holds for all $\varphi \in C^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ and it can be extended to all $\varphi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ by a density argument. □

The next lemma establishes a Poincaré–Friedrichs’ inequality for DG functions. Again, we omit the proof to save space and refer the reader to [11] for its proof.

Lemma 2.4 *Let $D \subset \Omega$ such that $V_h(D) \neq \{0\}$ and $\text{diam}(D) \geq h$. Then for any $v_h \in V_h(D)$ there hold the following inequalities:*

$$\|v_h\|_{L^p(D)} \lesssim \text{diam}(D) \|v_h\|_{W_h^{1,p}(D)}, \tag{2.11}$$

$$\|v_h\|_{W_h^{1,p}(D)} \lesssim \text{diam}(D) \|v_h\|_{W_h^{2,p}(D)}. \tag{2.12}$$

The last lemma of this section establishes some local super approximation estimates for the DG nodal interpolation in various discrete norms. The proof of the lemma is standard (cf. [19]); it can be given in [11].

Lemma 2.5 *Let $I_h : C^0(\mathcal{T}_h) := \Pi_{T \in \mathcal{T}_h} C^0(\bar{T}) \rightarrow V_h$ denote the nodal interpolation operator, and $\eta \in C^\infty(\Omega)$ with $|\eta|_{W^{j,\infty}(\Omega)} \lesssim d^{-j}$ for $0 \leq j \leq k$. Then for any $v_h \in V_h$ and $D \subseteq \Omega$ we have*

$$\|\eta v_h - I_h(\eta v_h)\|_{L^p(D)} \lesssim \frac{h}{d} \|v_h\|_{L^p(D_h)}, \tag{2.13}$$

$$h \|\nabla_h(\eta v_h - I_h(\eta v_h))\|_{L^p(D)} \lesssim \frac{h}{d} \|v_h\|_{L^p(D_h)}, \tag{2.14}$$

$$h^2 \|D_h^2(\eta v_h - I_h(\eta v_h))\|_{L^p(D)} \lesssim \frac{h}{d} \|v_h\|_{L^p(D_h)}, \tag{2.15}$$

$$\|\eta v_h - I_h(\eta v_h)\|_{W_h^{2,p}(D)} \lesssim \frac{1}{d^2} (\|v_h\|_{L^p(D_h)} + \|\nabla_h v_h\|_{L^p(D_h)}), \tag{2.16}$$

where D_h is the same as in Lemma 2.2. Moreover, there holds

$$\|\eta v_h - I_h(\eta v_h)\|_{W_h^{2,p}(D)} \lesssim \frac{h}{d^3} \|v_h\|_{W_h^{2,p}(D_h)} \tag{2.17}$$

if the polynomial degree k is greater than or equal to two.

3 DG Discrete $W^{1,p}$ and Calderon–Zygmund Estimates for PDEs with Constant Coefficients

In this section we consider the constant coefficient case, that is, $A(x) \equiv A_0 \in \mathbb{R}^{n \times n}$ on Ω . We define three interior-penalty discontinuous Galerkin discretizations $\mathcal{L}_{0,h}^\varepsilon$ to the PDE operator \mathcal{L} and extend their domains to the broken Sobolev space $W^{2,p}(\mathcal{T}_h)$. Our goal in this subsection is to prove global stability estimates for $\mathcal{L}_{0,h}^\varepsilon$ which will be crucially used in the next section. The final global stability estimate given in Theorem 3.6 can be regarded as a DG discrete Calderon–Zygmund estimate for $\mathcal{L}_{0,h}^\varepsilon$.

Let A_0 be a constant, positive-definite matrix in $\mathbb{R}^{n \times n}$ and define

$$\mathcal{L}_0 w := -A_0 : D^2 w = -\nabla \cdot (A_0 \nabla w). \tag{3.1}$$

From this we gather the standard PDE weak form:

$$a_0(w, v) := \int_\Omega A_0 \nabla w \cdot \nabla v \, dx \quad \forall w, v \in H_0^1(\Omega). \tag{3.2}$$

The Lax-Milgram theorem [3] yields the existence and boundedness of $\mathcal{L}_0^{-1} : H^{-1}(\Omega) \rightarrow H_0^1(\Omega)$. Moreover if $\partial\Omega \in C^{1,1}$ we have from Calderon-Zygmund theory [13] that $\mathcal{L}_0^{-1} : L^p(\Omega) \rightarrow W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ exists and

$$\|\mathcal{L}_0^{-1} \varphi\|_{W^{2,p}(\Omega)} \lesssim \|\varphi\|_{L^p(\Omega)} \quad \forall \varphi \in L^p(\Omega),$$

and therefore

$$\|w\|_{W^{2,p}(\Omega)} \lesssim \|\mathcal{L}_0 w\|_{L^p(\Omega)} \quad \forall w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Define $\mathcal{L}_{0,h}^\varepsilon : V_h \rightarrow V_h$ by

$$(\mathcal{L}_{0,h}^\varepsilon w_h, v_h) := a_{0,h}^\varepsilon(w_h, v_h) \quad \forall v_h, w_h \in V_h, \tag{3.3}$$

where the IP-DG bilinear form is defined by

$$\begin{aligned}
 a_{0,h}(w_h, v_h) := & \int_{\Omega} A_0 \nabla_h w_h \cdot \nabla_h v_h \, dx - \sum_{e \in \mathcal{E}_h} \int_e \{A_0 \nabla w_h \cdot \nu_e\} [v_h] \, dS \\
 & - \varepsilon \sum_{e \in \mathcal{E}_h} \int_e \{A_0 \nabla v_h \cdot \nu_e\} [w_h] \, dS + \sum_{e \in \mathcal{E}_h} \int_e \frac{\gamma_e}{h_e} [w_h] [v_h] \, dS, \tag{3.4}
 \end{aligned}$$

and $\gamma_e > 0$ is a penalization parameter. The parameter choices $\varepsilon \in \{1, 0, -1\}$ give respectively the SIP-DG, IIP-DG, and NIP-DG formulations. For the sake of clarity and readability we shall assume for the rest of the paper that ε may be either 1, 0, or -1 unless otherwise stated.

We recall the following well-known DG integration by parts formula:

$$\int_{\Omega} \tau \cdot \nabla_h v \, dx = - \int_{\Omega} (\nabla_h \cdot \tau) v \, dx + \sum_{e \in \mathcal{E}_h^i} \int_e [\tau \cdot \nu_e] \{v\} \, dS + \sum_{e \in \mathcal{E}_h} \int_e \{\tau \cdot \nu_e\} [v] \, dS, \tag{3.5}$$

which holds for any piecewise scalar-valued function v and vector-valued function τ . Applying (3.5) to the first term on the right-hand side of (3.4) yields

$$\begin{aligned}
 a_{0,h}^{\varepsilon}(w_h, v_h) = & - \int_{\Omega} (A_0 : D_h^2 w_h) v_h \, dx + \sum_{e \in \mathcal{E}_h^i} \int_e [A_0 \nabla w_h \cdot \nu_e] \{v_h\} \, dS \\
 & - \varepsilon \sum_{e \in \mathcal{E}_h} \int_e \{A_0 \nabla v_h \cdot \nu_e\} [w_h] \, dS + \sum_{e \in \mathcal{E}_h} \int_e \frac{\gamma_e}{h_e} [w_h] [v_h] \, dS \tag{3.6}
 \end{aligned}$$

for any $w_h, v_h \in V_h$. By Hölder’s inequality, it is easy to check that the above new form of $a_{0,h}^{\varepsilon}(\cdot, \cdot)$ is also well-defined on $W^{2,p}(\mathcal{T}_h) \times W^{2,p'}(\mathcal{T}_h)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. As a result, this new form enables us to extend the domain of $a_{0,h}^{\varepsilon}(\cdot, \cdot)$ to $W^{2,p}(\mathcal{T}_h) \times W^{2,p'}(\mathcal{T}_h)$ and $\mathcal{L}_{0,h}^{\varepsilon} : W^{2,p}(\mathcal{T}_h) \rightarrow (W^{2,p}(\mathcal{T}_h))^*$.

3.1 DG Discrete $W^{1,p}$ Error Estimates

From the standard IP-DG theory [22], there exists $\gamma^* = \gamma^*(\|A_0\|_{L^\infty(\Omega)}, \mathcal{T}_h) > 0$ depending only on the shape regularity of the mesh and on $\|A_0\|_{L^\infty(\Omega)}$ such that $\mathcal{L}_{0,h}^{\varepsilon}$ is invertible on V_h provided $\gamma_e \geq \gamma^*$; in the non-symmetric case $\varepsilon = -1$, γ_* can be any positive number. Moreover, if $w \in W^{2,2}(\mathcal{T}_h) \cap H_0^1(\Omega)$ and $w_h \in V_h$ satisfy

$$a_{0,h}^{\varepsilon}(w - w_h, v_h) = 0 \quad \forall v_h \in V_h, \tag{3.7}$$

then the quasi-optimal error estimate

$$\|w - w_h\|_{W_h^{1,2}(\Omega)} \lesssim \inf_{v_h \in V_h} \|w - v_h\|_{W_h^{1,2}(\Omega)} \tag{3.8}$$

is satisfied. The goal of this subsection is to generalize this result to general exponent $p \in (1, \infty)$ for the SIP-DG method. In particular, we have

Theorem 3.1 *Suppose $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ ($1 < p < \infty$) and $w_h \in V_h$ satisfy (3.7) with $\varepsilon = 1$. Then there holds*

$$\|w - w_h\|_{W_h^{1,p}(\Omega)} \lesssim h |\log h|^t \|w\|_{W^{2,p}(\Omega)}, \tag{3.9}$$

where $t = (p + 1)/p$ if $k = 1$ and $t = 0$ if $k \geq 2$.

To prove Theorem 3.1 we introduce some notation given in [5] (also see [25]). For given $z \in \overline{\Omega}$, we define the weight function σ_z as

$$\sigma_z(x) = \frac{h}{|x - z| + h}. \tag{3.10}$$

For $p \in [1, \infty)$, and $s \in \mathbb{R}$, we define the following weighted norms

$$\begin{aligned} \|v\|_{L^p(D),z,s} &= \left(\int_D |\sigma_z^s(x)v(x)|^p \, dx \right)^{1/p}, \\ \|v\|_{W^{1,p}(D),z,s} &= \|v\|_{L^p(D),z,s} + \|\nabla_h v\|_{L^p(D),z,s}, \\ \|v\|_{W_h^{1,p}(D),z,s} &= \|v\|_{W^{1,p}(D),z,s} + \left(\sum_{e \in \mathcal{E}_h} h_e^{1-p} \|\sigma_z^s[v]\|_{L^p(e \cap \overline{D})}^p \right)^{1/p} \\ &\quad + \left(\sum_{e \in \mathcal{E}_h} h_e \|\sigma_z^s\{\nabla_h v\}\|_{L^p(e \cap \overline{D})}^p \right)^{1/p}. \end{aligned} \tag{3.11}$$

The weighted norms in the case $p = \infty$ are defined analogously.

The derivation of $W^{1,p}$ error estimates of DG approximations is based on the work [5], where localized pointwise estimates of DG approximations are obtained. There it was shown that if $w \in W^{2,\infty}(\Omega)$ and $w_h \in V_h$ satisfy (3.7) with $\varepsilon = 1$, then

$$|\nabla(w - w_h)(z)| \lesssim \inf_{v_h \in V_h} \|w - v_h\|_{W_h^{1,\infty}(\Omega),z,s} \quad 0 \leq s < k \tag{3.12}$$

for all $z \in \overline{\Omega}$. Similar to pointwise estimates of finite element approximations (e.g., [3,25]), the ingredients to prove (3.12) include duality arguments and DG approximation estimates of regularized Green functions in a weighted (discrete) $W^{1,1}$ -norm. These results are rather technical and involve dyadic decompositions of Ω , local DG error estimates, and Green function estimates.

Here, we follow a similar argument to derive $W^{1,p}$ estimates; the main difference being that we derive DG approximation estimates of regularized Green functions in a weighted (discrete) $W^{1,p'}$ -norm with $1/p + 1/p' = 1$ (cf. Lemma 3.4). Using these estimates and applying similar arguments in [5,25] yield the estimate

$$|\nabla(w - w_h)(z)|^p \lesssim h^{-n} \inf_{v_h \in V_h} \|w - v_h\|_{W_h^{1,p}(\Omega),z,s}^p$$

for certain values of s . Integrating this expression with respect to z and applying Fubini’s theorem (cf. Lemma 3.2) then yields L^p estimates of the piecewise gradient error.

Unfortunately, the strategy just described does not immediately give us estimates for the terms $h_e^{1-p} \|[w - w_h]\|_{L^p(e)}^p$ appearing in the $W_h^{1,p}$ -norm. To bypass this difficulty, we first use the trace inequality

$$\sum_{e \in \mathcal{E}_h} h_e^{1-p} \|[w - w_h]\|_{L^p(e)}^p \lesssim \|\nabla_h(w - w_h)\|_{L^p(\Omega)}^p + h^{-p} \|w - w_h\|_{L^p(\Omega)}^p,$$

and then derive estimates for $h^{-p} \|w - w_h\|_{L^p(\Omega)}^p$. We note that the standard duality argument to derive L^p estimates yields

$$\|w - w_h\|_{L^p(\Omega)} \lesssim h \|w - w_h\|_{W_h^{1,p}(\Omega)},$$

which is of little benefit. Rather, our strategy is to modify the arguments given in [5, Theorem 5.1] and estimate $|(w - w_h)(z)|$ in terms of $\inf_{v_h \in V_h} \|w - v_h\|_{W_h^{1,p}(\Omega),z,s}$ (cf. Lemma 3.3),

integrate the estimate with respect to z , and then apply Fubini’s theorem. We note that it is due to this term that the $|\log h|^t$ factor appears in Theorem 3.1.

To summarize, the derivation of $W^{1,p}$ error estimates of DG approximations consists of three main ingredients. The first result (Lemma 3.2) essentially follows from Fubini’s Theorem. The second result (Lemma 3.3) derives pointwise estimates of the error and is an extension of [5, Theorem 5.1] where the case $p = \infty$ is given. The third result (Lemma 3.4) gives DG error estimates of regularized Green functions and is a generalization of [5, Lemma 5.4] where the case $p' = 1$ is shown. The proofs of these results can be found in [11].

Lemma 3.2 *Let $p \in [2, \infty)$ and $v \in L^p(\Omega)$. Let $z \in \Omega$ and $T_z \in \mathcal{T}_h$ such that $z \in T_z$. Then there holds*

$$\int_{\Omega} \int_{T_z} |v(x)|^p \, dx \, dz \lesssim h^n \|v\|_{L^p(\Omega)}^p. \tag{3.13}$$

Moreover for any $s > n/p$ and $w \in W^{2,p}(\mathcal{T}_h)$, there holds

$$\int_{\Omega} \|v\|_{W_h^{1,p}(\Omega),z,s}^p \, dz \lesssim \frac{h^n}{ps - n} (\|\nabla_h v\|_{L^p(\Omega)}^p + h^{-p} \|v\|_{L^p(\Omega)}^p + h^p \|D_h^2 v\|_{L^p(\Omega)}^p). \tag{3.14}$$

If $s = n/p$, then we have

$$\int_{\Omega} \|v\|_{W_h^{1,p}(\Omega),z,n/p}^p \, dz \lesssim |\log h| h^n (\|\nabla_h v\|_{L^p(\Omega)}^p + h^{-p} \|v\|_{L^p(\Omega)}^p + h^p \|D_h^2 v\|_{L^p(\Omega)}^p). \tag{3.15}$$

Lemma 3.3 *Let $w \in W^{2,p}(\mathcal{T}_h)$ ($2 \leq p \leq \infty$) and $w_h \in V_h$ satisfy (3.7) with $\varepsilon = 1$. Then for any $0 \leq s \leq k - 1 + n/p$ and $z \in \bar{\Omega}$,*

$$|(w - w_h)(z)| \lesssim h^{1-n/p} |\log h|^{\bar{s}(p)} \inf_{v_h \in V_h} \|w - v_h\|_{W_h^{1,p}(\Omega),z,s},$$

where $\bar{s}(p) = 1$ if $k = s + 1 - n/p$ and $\bar{s}(p) = 0$ for $k > s + 1 - n/p$.

Lemma 3.4 *Let z and T_z be as in Lemma 3.2. For arbitrary $\varphi \in C_0^\infty(T_z)$, with $\|\varphi\|_{W^{1,2}(T_z)} = 1$, we extend φ to Ω by zero, and let \hat{g}_z be the solution to*

$$\mathcal{L}_0^* \hat{g}_z = h^{-n/2-1} \partial\varphi/\partial x_i \text{ in } \Omega, \quad \hat{g}_z = 0 \text{ on } \partial\Omega.$$

Let $\hat{g}_{z,h} \in V_h$ satisfy the discrete adjoint problem

$$a_{0,h}(v_h, \hat{g}_{z,h}) = h^{-n/2-1} \int_{\Omega} (\partial\varphi/\partial x_i) v_h \, dx \quad \forall v_h \in V_h,$$

where we have dropped the superscript of the bilinear form for notational simplicity. Let $p \in [2, \infty]$, $p' \in [1, 2]$ such that $1/p + 1/p' = 1$. Then for any $0 \leq s \leq k + n/p$ there holds

$$\|\hat{g}_z - \hat{g}_{z,h}\|_{W_h^{1,p'}(\Omega),z,-s} \lesssim |\log h|^{\bar{s}(p)} h^{-n/p},$$

where $\bar{s}(p) = 1$ if $s = k + n/p$ and $\bar{s}(p) = 0$ otherwise.

Lemma 3.5 *There holds, for $p' \in [2, \infty)$,*

$$\|v_h\|_{W_h^{1,p'}(\Omega)} \lesssim |\log h|^{t'} \sup_{0 \neq z_h \in V_h} \frac{a_{0,h}(v_h, z_h)}{\|z_h\|_{W_h^{1,p}(\Omega)}} \quad \forall v_h \in V_h,$$

where $p \in (1, 2]$ satisfies $1/p + 1/p' = 1$ and $t' = (p' + 1)/p'$ if $k = 1$ and $t' = 0$ if $k \geq 2$.

3.1.1 Proof of Theorem 3.1 for $p \geq 2$

We now prove Theorem 3.1 in the case $p \in [2, \infty)$. To this end, let $z \in \Omega$ and $T_z \in \mathcal{T}_h$ such that $z \in T_z$. Using an inverse estimate, (2.10), and the triangle inequality we obtain

$$\begin{aligned} |\partial w_h(z)/\partial x_i| &\lesssim h^{-n/2} \|\partial w_h/\partial x_i\|_{L^2(T_z)} \\ &\lesssim h^{-n/2-1} \|\partial w_h/\partial x_i\|_{W^{-1,2}(T_z)} \\ &\lesssim h^{-n/2-1} \left(\|\partial(w - w_h)/\partial x_i\|_{W^{-1,2}(T_z)} + \|\partial w/\partial x_i\|_{W^{-1,2}(T_z)} \right). \end{aligned} \tag{3.17}$$

Note that, by the Poincaré-Friedrichs and Hölder inequalities,

$$\begin{aligned} \|\partial w/\partial x_i\|_{W^{-1,2}(T_z)} &= \sup_{\substack{\varphi \in C_0^\infty(T_z) \\ \|\varphi\|_{W^{1,2}(T_z)}=1}} (\partial w/\partial x_i, \varphi)_{T_z} \\ &\lesssim \sup_{\substack{\varphi \in C_0^\infty(T_z) \\ \|\varphi\|_{W^{1,2}(T_z)}=1}} |T_z|^{\frac{p-2}{2p}} \|\partial w/\partial x_i\|_{L^p(T_z)} \|\varphi\|_{L^2(T_z)} \\ &\lesssim |T_z|^{\frac{p-2}{2p}} \text{diam}(T_z) \|\partial w/\partial x_i\|_{L^p(T_z)} \lesssim h^{1+n/2-n/p} \|\partial w/\partial x_i\|_{L^p(T_z)}. \end{aligned}$$

Inserting this estimate into (3.17) yields

$$|\partial w_h(z)/\partial x_i| \lesssim h^{-n/p} \|\partial w/\partial x_i\|_{L^p(T_z)} + h^{-n/2-1} \|\partial(w - w_h)/\partial x_i\|_{W^{-1,2}(T_z)}.$$

Replacing w by $w - v_h$ and w_h by $w_h - v_h$ for some $v_h \in V_h$ in the argument above, we conclude

$$\begin{aligned} |\partial(w_h - v_h)(z)/\partial x_i| &\lesssim h^{-n/p} \|\partial(w - v_h)/\partial x_i\|_{L^p(T_z)} \\ &\quad + h^{-n/2-1} \|\partial(w - w_h)/\partial x_i\|_{W^{-1,2}(T_z)}. \end{aligned} \tag{3.18}$$

Let φ, \hat{g}_z and $\hat{g}_{z,h}$ be as in Lemma 3.4. Setting $\hat{e}_z = \hat{g}_z - \hat{g}_{z,h}$, we have for arbitrary $v_h \in V_h$

$$\begin{aligned} h^{-n/2-1} \int_{T_z} (w - w_h) \partial \varphi / \partial x_i \, dx &= a_{0,h}(w - v_h, \hat{e}_z) \\ &\lesssim \|w - v_h\|_{W_h^{1,p}(\Omega),z,s} \|\hat{e}_z\|_{W_h^{1,p'}(\Omega),z,-s} \lesssim \|w - v_h\|_{W_h^{1,p}(\Omega),z,s} |\log h|^{\bar{s}(p)} h^{-n/p}, \end{aligned}$$

where $\bar{s}(p)$ is defined in Lemma 3.4.

Applying this last estimate into (3.18) yields

$$\begin{aligned} |\nabla(w_h - v_h)(z)| &\lesssim h^{-n/p} \|\nabla(w - v_h)\|_{L^p(T_z)} \\ &\quad + h^{-n/p} |\log h|^{\bar{s}(p)} \|w - v_h\|_{W_h^{1,p}(\Omega),z,s}. \end{aligned} \tag{3.19}$$

Raising (3.19) by the power p and integrating over Ω with respect to z , we conclude

$$\begin{aligned} \|\nabla_h(w_h - v_h)\|_{L^p(\Omega)} &\lesssim \left(h^{-n} \int_{\Omega} \|\nabla_h(w - v_h)\|_{L^p(T_z)}^p \, dz \right)^{1/p} \\ &\quad + \left(h^{-n} |\log h|^{\bar{s}(p)p} \int_{\Omega} \|w - v_h\|_{W_h^{1,p}(\Omega),z,s}^p \, dz \right)^{1/p}. \end{aligned}$$

Next, we choose s such that $n/p < s < k + n/p$. Then $\bar{s}(p) = 0$, and by (3.13)–(3.14)

$$\|\nabla_h(w_h - v_h)\|_{L^p(\Omega)}^p \lesssim \|\nabla_h(w - v_h)\|_{L^p(\Omega)}^p + h^{-p} \|w - v_h\|_{L^p(\Omega)}^p + h^p \|D_h^2(w - v_h)\|_{L^p(\Omega)}^p,$$

and thus by the triangle inequality and taking $v_h = I_h w$, the nodal interpolant of w ,

$$\|\nabla_h(w - w_h)\|_{L^p(\Omega)} \lesssim h \|w\|_{W^{2,p}(\Omega)}. \tag{3.20}$$

Next we bound the jumps $\| [w - w_h] \|_{L^p(e)}$. First, by the trace inequalities stated in Lemma 2.1 we have

$$\sum_{e \in \mathcal{E}_h} h_e^{1-p} \| [w - w_h] \|_{L^p(e)}^p \lesssim C \left(\|\nabla_h(w - w_h)\|_{L^p(\Omega)}^p + h^{-p} \|w - w_h\|_{L^p(\Omega)}^p \right). \tag{3.21}$$

By Lemma 3.3 we have for any $z \in \bar{\Omega}$ and $v_h \in V_h$,

$$|(w - w_h)(z)|^p \lesssim h^{p-n} |\log h|^{p\bar{s}(p)} \|w - v_h\|_{W_h^{1,p}(\Omega),z,s}^p,$$

where $\bar{s}(p) = 1$ if $k = s + 1 - n/p$ and $\bar{s}(p) = 0$ for $k > s + 1 - n/p$. Integrating this expression with respect to z yields

$$\|w - w_h\|_{L^p(\Omega)}^p \lesssim h^{p-n} |\log h|^{p\bar{s}(p)} \int_{\Omega} \|w - v_h\|_{W_h^{1,p}(\Omega),z,s}^p \, dz. \tag{3.22}$$

If $k = 1$, then we set $s = n/p$, so that $\bar{s}(p) = 1$, and by (3.15) with $v_h = I_h w$,

$$\begin{aligned} \|w - w_h\|_{L^p(\Omega)}^p &\lesssim h^{p-n} |\log h|^p \int_{\Omega} \|w - v_h\|_{W_h^{1,p}(\Omega),z,n/p}^p \, dz \\ &\lesssim h^{2p} |\log h|^{p+1} \|w\|_{W^{2,p}(\Omega)}^p. \end{aligned} \tag{3.23}$$

On the other hand, if $k \geq 2$, then we choose s such that $n/p < s < k - 1 + n/p$. Then $\bar{s}(p) = 0$, and by (3.22) and (3.14),

$$\|w - w_h\|_{L^p(\Omega)}^p \lesssim h^{2p} \|w\|_{W^{2,p}(\Omega)}^p. \tag{3.24}$$

Combining (3.21) with (3.20), (3.23) and (3.24) then yields,

$$\sum_{e \in \mathcal{E}_h} h_e^{1-p} \| [w - w_h] \|_{L^p(e)}^p \lesssim |\log h|^{p+1} h^p \|w\|_{W^{2,p}(\Omega)}^p. \tag{3.25}$$

Finally combining (3.20), (3.25) and applying standard scaling arguments yields (3.9). This completes the proof of Theorem 3.1 in the case $p \geq 2$. □

3.1.2 Proof of Theorem 3.1 for $1 < p < 2$

Next we prove Theorem 3.1 for $1 < p < 2$. To this end, for $w_h \in V_h$ and $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying (3.7), let $v_h \in V_h$ be the unique solution to

$$a_{0,h}(v_h, z_h) = \int_{\Omega} |\nabla_h w_h|^{p-2} \nabla_h w_h \cdot \nabla_h z_h \, dx + \sum_{e \in \mathcal{E}_h} h_e^{1-p} \int_e |[w_h]|^{p-2} [w_h] [z_h] \, dS$$

for all $z_h \in V_h$. Setting $z_h = w_h$ and using a scaling argument yields

$$\|w_h\|_{W_h^{1,p}(\Omega)}^p \lesssim a_{0,h}(v_h, w_h).$$

Moreover, Lemma 3.5 and Hölder’s inequality gets

$$\|v_h\|_{W_h^{1,p'}(\Omega)} \lesssim |\log h|^{t'} \|w_h\|_{W_h^{1,p}(\Omega)}^{p-1}.$$

Consequently,

$$\|w_h\|_{W_h^{1,p}(\Omega)} = \frac{\|w_h\|_{W_h^{1,p}(\Omega)}^p}{\|w_h\|_{W_h^{1,p}(\Omega)}^{p-1}} \lesssim |\log h|^{t'} \frac{a_{0,h}(v_h, w_h)}{\|v_h\|_{W_h^{1,p'}(\Omega)}} \lesssim |\log h|^{t'} \|w\|_{W_h^{1,p}(\Omega)}.$$

Standard arguments then show that this estimate implies

$$\|w - w_h\|_{W_h^{1,p}(\Omega)} \lesssim |\log h|^{t'} h \|w\|_{W^{2,p}(\Omega)} \quad 1 < p < 2. \tag{3.26}$$

This completes the proof of Theorem 3.1 upon noting that $t' = (p' + 1)/p' = (2p - 1)/p \leq (p + 1)/p = t$ for $p \in (1, 2]$. □

3.2 DG Discrete Calderon–Zygmund Estimates for PDEs with Constant Coefficients

The goal of this subsection is to establish a stability result for the operator $\mathcal{L}_{0,h}^\varepsilon$ in the $W_h^{2,p}$ -norm, which is a discrete counterpart of (3.3). Such an estimate can be regarded as a DG discrete Calderon-Zygmund estimate for $\mathcal{L}_{0,h}^\varepsilon$.

Theorem 3.6 (i) For $\varepsilon = 1$ and $1 < p < \infty$ we have

$$\|w_h\|_{W_h^{2,p}(\Omega)} \lesssim |\log h|^t \|\mathcal{L}_{0,h}^\varepsilon w_h\|_{L^p(\Omega)} \quad \forall w_h \in V_h, \tag{3.27}$$

where $t = (p + 1)/p$ if $k = 1$ and $t = 0$ if $k \geq 2$.

(ii) (3.27) also holds with $t = 0$ for $\varepsilon \in \{1, 0, -1\}$ and $p = 2$.

Proof (i) We observe that (3.27) is equivalent to showing

$$\|(\mathcal{L}_{0,h}^\varepsilon)^{-1} \varphi_h\|_{W_h^{2,p}(\Omega)} \lesssim |\log h|^t \|\varphi_h\|_{L^p(\Omega)} \quad \forall \varphi_h \in V_h. \tag{3.28}$$

For any $\varphi_h \in V_h$, let $w := \mathcal{L}_0^{-1} \varphi_h \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $w_h := (\mathcal{L}_{0,h}^\varepsilon)^{-1} \varphi_h \in V_h$. Since $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ we have

$$a_{0,h}^\varepsilon(w, v_h) = (\varphi_h, v_h) = a_{0,h}^\varepsilon(w_h, v_h) \quad \forall v_h \in V_h.$$

Thus w_h is the IP-DG approximate solution to w . Applying Theorem 3.1 and the elliptic regularity estimate, we obtain

$$\|w - w_h\|_{W_h^{1,p}(\Omega)} \lesssim |\log h|^t h \|w\|_{W^{2,p}(\Omega)} \lesssim |\log h|^t h \|\varphi_h\|_{L^p(\Omega)}. \tag{3.29}$$

Moreover, by Lemma 2.3 and the Calderon–Zygmund estimate for \mathcal{L}_0 we have

$$\|w\|_{W_h^{2,p}(\Omega)} \leq \|w\|_{W^{2,p}(\Omega)} \lesssim \|\varphi_h\|_{L^p(\Omega)}. \tag{3.30}$$

Denote by $I_h : C^0(\Omega) \rightarrow V_h$ the nodal interpolation operator onto V_h . By finite element interpolation theory [6] we have

$$h^{-1} \|w - I_h w\|_{W_h^{1,p}(\Omega)} + \|w - I_h w\|_{W_h^{2,p}(\Omega)} \lesssim \|w\|_{W^{2,p}(\Omega)}. \tag{3.31}$$

Therefore by the triangle inequality, an inverse estimate, Lemma 2.3, (3.29), and (3.30), we obtain

$$\begin{aligned} \|w_h\|_{W_h^{2,p}(\Omega)} &\leq \|w - I_h w\|_{W_h^{2,p}(\Omega)} + \|I_h w - w_h\|_{W_h^{2,p}(\Omega)} + \|w\|_{W_h^{2,p}(\Omega)} \\ &\lesssim h^{-1} \|I_h w - w_h\|_{W_h^{1,p}(\Omega)} + \|\varphi_h\|_{L^p(\Omega)} \\ &\leq h^{-1} (\|w - w_h\|_{W_h^{1,p}(\Omega)} + \|w - I_h w\|_{W_h^{1,p}(\Omega)}) + \|\varphi_h\|_{L^p(\Omega)} \\ &\lesssim |\log h|^t \|\varphi_h\|_{L^p(\Omega)} = |\log h|^t \|\mathcal{L}_{0,h} w_h\|_{L^p(\Omega)}. \end{aligned}$$

(ii) The proof of this part is exactly same as that of Part (i), the only difference is that now (3.8), instead of (3.9), should be called in the proof. □

4 IP-DG Methods and Their Convergence Analysis

As mentioned in Sect. 1, our primary goal in this paper to develop convergent IP-DG methods for approximating the $W^{2,p}$ strong solution to the boundary value problem (1.1). We assume that Eq. (1.1a) is uniformly elliptic, precisely, we assume $A \in [C^0(\overline{\Omega})]^{n \times n}$ is positive definite, that is, there exist constants $\Lambda > \lambda > 0$ such that

$$\lambda |\xi|^2 \leq A(x) \leq \Lambda |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } x \in \overline{\Omega}. \tag{4.1}$$

We also assume that the solution u satisfies the following Calderon–Zygmund estimate:

$$\|u\|_{W^{2,p}(\Omega)} \lesssim \|f\|_{L^p(\Omega)}. \tag{4.2}$$

It is well-known [13, Chapter 9] that the above estimate holds for any $f \in L^p(\Omega)$ with $1 < p < \infty$ if $\partial\Omega \in C^{1,1}$. Moreover, when $n, p = 2$, (4.2) also holds if Ω is a convex domain [2, 21].

4.1 Formulation of IP-DG Methods

We follow the same recipe as in the constant coefficient case to build our IP-DG methods. To this end, we momentarily assume $A \in [C^1(\Omega)]^{n \times n}$, so that we can rewrite the PDE (1.1a) in divergence form as follows:

$$-\nabla \cdot (A \nabla u) + (\nabla \cdot A) \cdot \nabla u = f, \tag{4.3}$$

where $\nabla \cdot A$ is defined row-wise. We then define the following (standard) IP-DG methods for problem (4.3) by seeking $u_h \in V_h$ such that

$$\begin{aligned} &\int_{\Omega} (A \nabla_h u_h) \cdot \nabla_h v_h \, dx + \int_{\Omega} ((\nabla \cdot A) \cdot \nabla_h u_h) v_h \, dx \\ &- \sum_{e \in \mathcal{E}_h} \int_e \{A \nabla u_h \cdot \nu_e\} [v_h] \, dS - \varepsilon \sum_{e \in \mathcal{E}_h} \int_e \{A \nabla v_h \cdot \nu_e\} [u_h] \, dS \\ &+ \sum_{e \in \mathcal{E}_h} \int_e \frac{\gamma_e}{h_e} [u_h] [v_h] \, dS = \int_{\Omega} f v_h \, dx, \end{aligned} \tag{4.4}$$

where $\gamma_e \geq \gamma_*(\|A\|_{L^\infty(\Omega)}, \mathcal{T}_h) > 0$. We emphasize that γ^* is independent of the derivatives of A .

Now come back to our case in hand with $A \in [C^0(\Omega)]^{n \times n}$. Clearly, the term $\nabla \cdot A$ does not exist as a function (it is in fact a Radon measure), so the above formulation is not defined for the case we are considering. To overcome this difficulty, our idea is to apply the DG integration by parts formula (3.5) to the first term on the left-hand side of (4.4), yielding

$$\begin{aligned}
 a_h(w_h, v_h) := & - \int_{\Omega} (A : D_h^2 w_h) v_h \, dx + \sum_{e \in \mathcal{E}_h^I} \int_e [A \nabla w_h \cdot \nu_e] \{v_h\} \, dS \\
 & - \varepsilon \sum_{e \in \mathcal{E}_h} \int_e \{A \nabla v_h \cdot \nu_e\} [w_h] \, dS + \sum_{e \in \mathcal{E}_h} \int_e \frac{\gamma_e}{h_e} [w_h] [v_h] \, dS. \tag{4.5}
 \end{aligned}$$

No derivative of A appears in the above new form of $a_h^\varepsilon(\cdot, \cdot)$; thus, it is well-defined on $V_h \times V_h$. This leads to the following definition.

Definition 4.1 Our IP-DG methods are defined by seeking $u_h \in V_h$ such that

$$a_h^\varepsilon(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h, \quad \varepsilon \in \{1, 0, -1\}. \tag{4.6}$$

When $\varepsilon = 1$ we will refer to the method as ‘‘symmetrically induced’’ even though the bilinear form is not symmetric. Likewise, $\varepsilon = 0$ and $\varepsilon = -1$ yields an ‘‘incompletely induced’’ and ‘‘non-symmetrically induced’’ method, respectively.

4.2 Stability Analysis

As in Section 3 we can define the IP-DG approximation $\mathcal{L}_h^\varepsilon$ of \mathcal{L} on V_h using the bilinear form $a_h^\varepsilon(\cdot, \cdot)$; precisely, we define $\mathcal{L}_h^\varepsilon : V_h \rightarrow V_h$ by

$$(\mathcal{L}_h^\varepsilon w_h, v_h) := a_h^\varepsilon(w_h, v_h) \quad \forall w_h, v_h \in V_h. \tag{4.7}$$

Since we can extend the domain of $a_h^\varepsilon(\cdot, \cdot)$ to $W^{2,p}(\mathcal{T}_h) \times W^{2,p'}(\mathcal{T}_h)$, then the domain and co-domain of \mathcal{L}_h can be extended to the broken Sobolev spaces $W^{2,p}(\mathcal{T}_h)$ and $(W^{2,p'}(\mathcal{T}_h))^*$ respectively.

The goal of this subsection is to establish a DG discrete Calderon–Zygmund estimate similar to (3.27) for the operator $\mathcal{L}_h^\varepsilon$. To this end, our main idea is to mimic, at the discrete level, the ‘‘freezing the coefficients’’ technique and the covering argument found in Schauder theory and $W^{2,p}$ strong solution theory [13, Chapters 6 and 9]. Since A is continuous, we show that in a small ball $B_\delta \subset \Omega$ A behaves as if it were constant. This allows us to conclude that $\mathcal{L}_h^\varepsilon$ is locally very close to $\mathcal{L}_{0,h}^\varepsilon$ in the ball $B_\delta(x_0)$ for any $x_0 \in \Omega$. By applying the above mentioned ‘‘freezing the coefficients’’ technique and covering argument to the formal adjoint of $\mathcal{L}_h^\varepsilon$, we are able to prove a global left-side inf-sup condition for $\mathcal{L}_h^\varepsilon$. Then by employing a duality argument, we derive the desired discrete Calderon–Zygmund estimate for $\mathcal{L}_h^\varepsilon$.

We now proceed to establish a few auxiliary lemmas which will be needed to show the desired estimate.

Lemma 4.1 For all $\delta > 0$, there exists $R_\delta > 0$ and $h_\delta > 0$ such that for all $x_0 \in \Omega$ and $A_0 \equiv A(x_0)$

$$\|(\mathcal{L}_h^\varepsilon - \mathcal{L}_{0,h}^\varepsilon)w\|_{L_h^p(B_{R_\delta}(x_0))} \lesssim \delta \|w\|_{W_h^{2,p}(B_{R_\delta}(x_0))} \quad \forall w \in W^{2,p}(\mathcal{T}_h), \quad \forall h \leq h_\delta. \tag{4.8}$$

Here, $B_{R_\delta}(x_0) := \{x \in \Omega : |x - x_0| < R_\delta\}$ denotes the ball with center x_0 and radius R_δ .

Proof Since A is continuous on $\overline{\Omega}$, then it is uniformly continuous. Therefore, for every $\delta > 0$ there exists $R_\delta > 0$ such that if $x, y \in \Omega$ satisfies $|x - y| < R_\delta$, we have $|A(x) - A(y)| < \delta$. Consequently, for any $x_0 \in \Omega$

$$\|A - A_0\|_{L^\infty(B_{R_\delta})} \leq \delta, \tag{4.9}$$

where we have used the shorthand notation $B_{R_\delta} := B_{R_\delta}(x_0)$.

Set $h_\delta = \min\{h_0, \frac{R_\delta}{4}\}$ and let $0 < h < h_\delta$, $w \in W^{2,p}(\mathcal{T}_h)$, and $v_h \in V_h(B_{R_\delta})$. Since $(\mathcal{L}_{0,h}^\varepsilon - \mathcal{L}_h^\varepsilon)w \in W^{2,p}(\mathcal{T}_h)$, it follows from (3.6) and (4.5) that for every $v_h \in V_h(B_{R_\delta})$ we have

$$\begin{aligned} & ((\mathcal{L}_{0,h}^\varepsilon - \mathcal{L}_h^\varepsilon)w, v_h) = - \int_{\Omega \cap B_{R_\delta}} ((A_0 - A) : D_h^2 w) v_h \, dx \\ & + \sum_{e \in \mathcal{E}_h^I} \int_{e \cap \overline{B_{R_\delta}}} [(A_0 - A) \nabla w \cdot \nu_e] \{v_h\} \, dS - \varepsilon \sum_{e \in \mathcal{E}_h} \int_{e \cap \overline{B_{R_\delta}}} \{(A - A_0) \nabla v_h \cdot \nu_e\} [w_h] \, dS \\ & \leq \|A - A_0\|_{L^\infty(B_{R_\delta})} \left(\|D_h^2 w\|_{L^p(\Omega \cap B_{R_\delta})} \|v_h\|_{L^{p'}(\Omega \cap B_{R_\delta})} \right. \\ & + \left(\sum_{e \in \mathcal{E}_h} h_e^{1-2p} \|[w]\|_{L^p(e \cap \overline{B_{R_\delta}})}^p \right)^{\frac{1}{p}} \left(\sum_{e \in \mathcal{E}_h} h_e h_e^{p'} \|\{\nabla v_h \cdot \nu_e\}\|_{L^{p'}(e \cap \overline{B_{R_\delta}})}^{p'} \right)^{\frac{1}{p'}} \\ & + \left. \left(\sum_{e \in \mathcal{E}_h^I} h_e^{1-p} \|\nabla w\|_{L^p(e \cap \overline{B_{R_\delta}})}^p \right)^{\frac{1}{p}} \left(\sum_{e \in \mathcal{E}_h^I} h_e \|\{v_h\}\|_{L^{p'}(e \cap \overline{B_{R_\delta}})}^{p'} \right)^{\frac{1}{p'}} \right) \\ & \lesssim \|A - A_0\|_{L^\infty(B_{R_\delta})} \|w\|_{W_h^{2,p}(B_{R_\delta})} \left(\|v_h\|_{L^{p'}(B_{R_\delta})} + h \|\nabla_h v_h\|_{L^{p'}(B_{R_\delta})} \right) \\ & \lesssim \delta \|w\|_{W_h^{2,p}(B_{R_\delta})} \|v_h\|_{L^{p'}(B_{R_\delta})} = \delta \|w\|_{W_h^{2,p}(B_{R_\delta})} \|v_h\|_{L^{p'}(B_{R_\delta})}. \end{aligned}$$

Dividing both sides by $\|v_h\|_{L^{p'}(B_{R_\delta})}$ yields the estimate. The proof is complete. □

The next lemma shows that $\mathcal{L}_h^\varepsilon$ is locally a bounded operator on $W^{2,p}(\mathcal{T}_h)$. We omit its proof to save space and refer the reader to [11] for a detailed proof.

Lemma 4.2 *For any $x_0 \in \Omega$ and $R \geq h$, there holds*

$$\|\mathcal{L}_h^\varepsilon w\|_{L_h^p(B_R(x_0))} \lesssim \|w\|_{W_h^{2,p}(B_R(x_0))} \quad \forall w \in W^{2,p}(\mathcal{T}_h). \tag{4.10}$$

Our last lemma establishes a left-side inf-sup condition for $\mathcal{L}_h^\varepsilon$. This estimate relies on the formal adjoint operator $\mathcal{L}_h^* := (\mathcal{L}_h^\varepsilon)^*$ and some techniques from [24].

Lemma 4.3 *There exists an $h_0 > 0$ such that for all $h \leq h_0$ and $k \geq 2$ we have*

$$\|v_h\|_{L^{p'}(\Omega)} \lesssim \sup_{0 \neq w_h \in V_h} \frac{(\mathcal{L}_h^\varepsilon w_h, v_h)}{\|w_h\|_{W_h^{2,p}(\Omega)}} \quad \forall v_h \in V_h, \tag{4.11}$$

where $1 < p < \infty$ if $\varepsilon = 1$ and $p = 2$ if $\varepsilon \in \{0, -1\}$.

Proof Note that (4.11) is equivalent to

$$\|v_h\|_{L^{p'}(\Omega)} \lesssim \sup_{0 \neq w_h \in V_h} \frac{(\mathcal{L}_h^\varepsilon w_h, v_h)}{\|w_h\|_{W_h^{2,p}(\Omega)}} = \sup_{0 \neq w_h \in V_h} \frac{(\mathcal{L}_h^* v_h, w_h)}{\|w_h\|_{W_h^{2,p}(\Omega)}} = \|\mathcal{L}_h^* v_h\|_{W_h^{-2,p'}(\Omega)} \tag{4.12}$$

for all $v_h \in V_h$. We divide the remaining proof into three steps.

Step 1: Local estimates Let $x_0 \in \Omega$, $A_0 \equiv A(x_0)$, $\delta_0, h_{\delta_0}, R_{\delta_0}, R_1 := (1/3)R_{\delta_0}$, and $B_1 := B_{R_1}(x_0)$ be as in Lemma 4.1 with $\delta_0 > 0$ to be determined, and set $h \leq h_{\delta_0}$.

By the elliptic regularity of \mathcal{L} , for any $v_h \in V_h(B_1)$, there exists $\varphi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\mathcal{L}\varphi = v_h|v_h|^{p-2}$ in Ω and satisfies the estimate

$$\|\varphi\|_{W^{2,p}(\Omega)} \lesssim \|v_h\|_{L^{p'}(\Omega)}^{p'-1} = \|v_h\|_{L^{p'}(B_1)}^{p'-1}. \tag{4.13}$$

Since $\mathcal{L}_h^\varepsilon$ is consistent with \mathcal{L} for any $\varphi_h \in V_h$ we have

$$\begin{aligned} \|v_h\|_{L^{p'}(B_1)} &= \|v_h\|_{L^{p'}(\Omega)} = (\mathcal{L}\varphi, v_h) = (\mathcal{L}_h^\varepsilon\varphi, v_h) \\ &= (\mathcal{L}_h^*v_h, \varphi_h) + (\mathcal{L}_{0,h}^\varepsilon(\varphi - \varphi_h), v_h) + ((\mathcal{L}_h^\varepsilon - \mathcal{L}_{0,h}^\varepsilon)(\varphi - \varphi_h), v_h). \end{aligned} \tag{4.14}$$

From the existence-uniqueness of the IP-DG scheme (3.4), there exists $\varphi_h \in V_h$ such that

$$(\mathcal{L}_{0,h}^\varepsilon(\varphi - \varphi_h), w_h) = 0 \quad \forall w_h \in V_h.$$

Combining Galerkin orthogonality, Theorem 3.6, and (4.13) gives us the solution estimate

$$\|\varphi_h\|_{W_h^{2,p}(\Omega)} \lesssim \|\mathcal{L}_{0,h}^\varepsilon\varphi_h\|_{L_h^p(\Omega)} = \|\mathcal{L}_{0,h}^\varepsilon\varphi\|_{L_h^p(\Omega)} \lesssim \|\varphi\|_{W^{2,p}(\Omega)} \lesssim \|v_h\|_{L^{p'}(B_1)}^{p'-1}. \tag{4.15}$$

Using Lemma 4.1 and (4.13)–(4.15) we have

$$\begin{aligned} \|v_h\|_{L^{p'}(B_1)}^{p'} &= (\mathcal{L}_h^*v_h, \varphi_h) + ((\mathcal{L}_h^\varepsilon - \mathcal{L}_{0,h}^\varepsilon)(\varphi - \varphi_h), v_h) \\ &\lesssim \|\mathcal{L}_h^*v_h\|_{W_h^{-2,p'}(\Omega)} \|v_h\|_{L^{p'}(B_1)}^{p'-1} + \delta_0 \|v_h\|_{L^{p'}(B_1)}. \end{aligned}$$

Taking δ_0 sufficiently small to move the right hand term to the left side and dividing by $\|v_h\|_{L^{p'}(B_1)}^{p'-1}$ gives us the local estimate

$$\|v_h\|_{L^{p'}(B_1)} \lesssim \|\mathcal{L}_h^*v_h\|_{W_h^{-2,p}(B_1)} \quad \forall v_h \in V_h(B_1). \tag{4.16}$$

Step 2: A Gårding type inequality by a covering argument Given R_1 from Step 1, let $R_2 = 2R_1$ and $R_3 = 3R_1$. Let $\eta \in C^3(\Omega)$ be a cutoff function satisfying

$$0 \leq \eta \leq 1, \quad \eta|_{B_1} = 1, \quad \eta|_{\Omega \setminus B_2} = 0, \quad |\eta|_{W^{m,\infty}(\Omega)} = O(R_1^{-m}). \tag{4.17}$$

For any $v_h \in V_h$, we have by (4.16),

$$\begin{aligned} \|v_h\|_{L^{p'}(B_1)} &= \|\eta v_h\|_{L^{p'}(B_1)} \leq \|\eta v_h - I_h(\eta v_h)\|_{L^{p'}(B_1)} + \|I_h(\eta v_h)\|_{L^{p'}(B_1)} \\ &\lesssim \|\eta v_h - I_h(\eta v_h)\|_{L^{p'}(B_1)} + \|\mathcal{L}_h^*(I_h(\eta v_h) - \eta v_h)\|_{W_h^{-2,p'}(B_1)} + \|\mathcal{L}_h^*(\eta v_h)\|_{W_h^{-2,p'}(B_1)}. \end{aligned} \tag{4.18}$$

We now bound the second term on the right hand side of (4.18). By the definition of $\|\cdot\|_{W_h^{-2,p}}$, Lemma 4.2 and (2.6), for any $w_h \in V_h$ we have

$$\begin{aligned} \|\mathcal{L}_h^*(I_h(\eta v_h) - \eta v_h)\|_{W_h^{-2,p'}(B_1)} &= \sup_{0 \neq w_h \in V_h} \frac{(\mathcal{L}_h^*(I_h(\eta v_h) - \eta v_h), w_h)}{\|w_h\|_{W_h^{2,p}(B_1)}} \\ &\leq \sup_{w_h \in V_h} \frac{(\mathcal{L}_h^\varepsilon w_h, I_h(\eta v_h) - \eta v_h)}{\|w_h\|_{W_h^{2,p}(B_1)}} \lesssim \sup_{w_h \in V_h} \frac{\|\mathcal{L}_h^\varepsilon w_h\|_{L_h^p(B_1)} \|I_h(\eta v_h) - \eta v_h\|_{L^{p'}(B_1)}}{\|w_h\|_{W_h^{2,p}(B_1)}} \\ &\lesssim \sup_{w_h \in V_h} \frac{\|w_h\|_{W_h^{2,p}(B_1)} \|I_h(\eta v_h) - \eta v_h\|_{L^{p'}(B_1)}}{\|w_h\|_{W_h^{2,p}(B_1)}} = \|I_h(\eta v_h) - \eta v_h\|_{L^{p'}(B_1)}. \end{aligned}$$

Thus (4.18) becomes

$$\|v_h\|_{L^{p'}(B_1)} \lesssim \|\eta v_h - I_h(\eta v_h)\|_{L^{p'}(B_1)} + \|\mathcal{L}_h^*(\eta v_h)\|_{W_h^{-2,p'}(B_1)}. \tag{4.19}$$

Using Lemmas 2.2, 2.5, and 4.2 with (4.19) yields

$$\begin{aligned} \|v_h\|_{L^{p'}(B_1)} &\lesssim \frac{h}{R_1} \|v_h\|_{L^{p'}(B_3)} + \|\mathcal{L}_h^*(\eta v_h)\|_{W_h^{-2,p'}(B_3)} \\ &\lesssim \frac{1}{R_1} \|v_h\|_{W^{-1,p'}(B_3)} + \|\mathcal{L}_h^*(\eta v_h)\|_{W_h^{-2,p'}(B_3)}. \end{aligned} \tag{4.20}$$

We now want to remove the cutoff function η from the adjoint operator appearing in the right-hand side of (4.20). For $w_h \in V_h(B_3)$, we break up $\mathcal{L}_h^*(\eta v_h)$ as follows:

$$\begin{aligned} (\mathcal{L}_h^*(\eta v_h), w_h) &= (\mathcal{L}_h^\varepsilon w_h, \eta v_h) = (\mathcal{L}_h^\varepsilon w_h \eta, v_h) + \left[(\mathcal{L}_h^\varepsilon w_h, \eta v_h) - (\mathcal{L}_h^\varepsilon w_h \eta, v_h) \right] \\ &= (\mathcal{L}_h^\varepsilon (I_h(w_h \eta)), v_h) + (\mathcal{L}_h^\varepsilon (w_h \eta - I_h(w_h \eta)), v_h) \\ &\quad + \left[(\mathcal{L}_h^\varepsilon w_h, \eta v_h) - (\mathcal{L}_h^\varepsilon w_h \eta, v_h) \right] =: I_1 + I_2 + I_3 \end{aligned} \tag{4.21}$$

We then seek to bound each I in order. To bound I_1 , we will use the definition of $\|\cdot\|_{W_h^{-2,p}}$, the stability of I_h , and Lemma 2.4 to obtain

$$\begin{aligned} I_1 &= (\mathcal{L}_h^* v_h, I_h(w_h \eta)) \lesssim \|\mathcal{L}_h^* v_h\|_{W_h^{-2,p'}(B_3)} \|I_h(\eta w_h)\|_{W_h^{2,p}(B_3)} \\ &\lesssim \|\mathcal{L}_h^* v_h\|_{W_h^{-2,p'}(B_3)} \|\eta w_h\|_{W_h^{2,p}(B_3)} \lesssim \frac{1}{R_1^2} \|\mathcal{L}_h^* v_h\|_{W_h^{-2,p'}(B_3)} \|w_h\|_{W_h^{2,p}(B_3)}. \end{aligned} \tag{4.22}$$

For I_2 we use Lemmas 2.2, 2.5, 4.2 to get

$$\begin{aligned} I_2 &= (\mathcal{L}_h^\varepsilon (w_h \eta - I_h(w_h \eta)), v_h) \lesssim \|w_h \eta - I_h(w_h \eta)\|_{W_h^{2,p}(B_3)} \|v_h\|_{L^{p'}(B_3)} \\ &\lesssim \frac{h}{R_1^3} \|w_h\|_{W_h^{2,p}(B_3)} \|v_h\|_{L^{p'}(B_3)} \lesssim \frac{1}{R_1^3} \|w_h\|_{W_h^{2,p}(B_3)} \|v_h\|_{W^{-1,p'}(B_3)}. \end{aligned} \tag{4.23}$$

To bound I_3 we introduce the operator $\mathcal{L}_{0,h}^\varepsilon$. For $e \in \mathcal{E}_h$ let $e_3 := e \cap B_3$, and define $\tilde{A} := A - A_0$. We then write

$$\begin{aligned} I_3 &= (\mathcal{L}_h^\varepsilon w_h, \eta v_h) - (\mathcal{L}_h^\varepsilon w_h \eta, v_h) \\ &= (\mathcal{L}_{0,h}^\varepsilon w_h, \eta v_h) - (\mathcal{L}_{0,h}^\varepsilon w_h \eta, v_h) \\ &\quad + \left[(\mathcal{L}_h^\varepsilon w_h, \eta v_h) - (\mathcal{L}_h^\varepsilon w_h \eta, v_h) - (\mathcal{L}_{0,h}^\varepsilon w_h, \eta v_h) + (\mathcal{L}_{0,h}^\varepsilon w_h \eta, v_h) \right] \\ &= - \int_{B_3} \left(w_h A_0 : D^2 \eta + (A_0 + A_0^T) \nabla \eta \cdot \nabla_h w_h \right) v_h \, dx \\ &\quad - \varepsilon \sum_{e \in \mathcal{E}_h} \int_{e_3} (A_0 \nabla \eta \cdot \nu_e) \{v_h\} [w_h] \, dS \\ &\quad - \int_{B_3} \left(w_h (\tilde{A} : D^2 \eta + (\tilde{A} + \tilde{A}^T) \nabla \eta \cdot \nabla_h w_h) \right) v_h \, dx \\ &\quad - \varepsilon \sum_{e \in \mathcal{E}_h} \int_{e_3} (\tilde{A} \nabla \eta \cdot \nu_e) \{v_h\} [w_h] \, dS =: K_1 + K_2 + K_3 + K_4. \end{aligned} \tag{4.24}$$

We now must bound each K_i . To bound K_1 we use the definition of $\|\cdot\|_{W_h^{-1,p}(B_3)}$ and Lemma 2.2 to get

$$\begin{aligned} K_1 &\lesssim \left(\|w_h A_0 : D^2 \eta\|_{W_h^{1,p}(B_3)} + \|(A_0 + A_0^T) \nabla \eta \cdot \nabla_h w_h\|_{W_h^{1,p}(B_3)} \right) \|v_h\|_{W_h^{-1,p'}(B_3)} \\ &\lesssim \frac{1}{R_1^3} \|w_h\|_{W_h^{2,p}(B_3)} \|v_h\|_{W_h^{-1,p'}(B_3)}. \end{aligned} \tag{4.25}$$

The bound of K_2 uses Lemmas 2.1, 2.4 to obtain

$$\begin{aligned} K_2 &\lesssim \frac{1}{R_1} \left(\sum_{e \in \mathcal{E}_h} h_e^{1-2p} \|[w_h]\|_{L^p(e_3)}^p \right)^{\frac{1}{p}} \left(\sum_{e \in \mathcal{E}_h} h_e h_e^{p'} \|\{v_h\}\|_{L^{p'}(e_3)}^{p'} \right)^{\frac{1}{p'}} \\ &\lesssim \frac{1}{R_1} \|w_h\|_{W_h^{2,p}(B_3)} \|v_h\|_{W_h^{-1,p'}(B_3)}. \end{aligned} \tag{4.26}$$

We use similar techniques as (4.25), (4.26) and the fact that $\|\tilde{A}\|_{L^\infty(B_3)} \leq \delta_0$ to get

$$\begin{aligned} K_3 &\lesssim \left(\|w_h \tilde{A} : D^2 \eta\|_{L^p(B_3)} + \|(\tilde{A} + \tilde{A}^T) \nabla \eta \cdot \nabla_h w_h\|_{L^p(B_3)} \right) \|v_h\|_{L^{p'}(B_3)} \\ &\lesssim \delta_0 \left(\frac{1}{R_1^2} \|w_h\|_{L^p(B_3)} + \frac{1}{R_1} \|w_h\|_{W_h^{1,p}(B_3)} \right) \|v_h\|_{L^{p'}(B_3)} \\ &\lesssim \delta_0 \|w_h\|_{W_h^{2,p}(B_3)} \|v_h\|_{L^{p'}(B_3)}, \end{aligned} \tag{4.27}$$

where we have used Lemma 2.4 to derive the last inequality. Likewise, we find

$$\begin{aligned} K_4 &\lesssim \frac{1}{R_1} \left(\sum_{e \in \mathcal{E}_h} h_e^{1-2p} \|[w_h]\|_{L^p(e_3)}^p \right)^{\frac{1}{p}} \left(\sum_{e \in \mathcal{E}_h} h_e h_e^{p'} \|\{v_h\}\|_{L^{p'}(e_3)}^{p'} \right)^{\frac{1}{p'}} \\ &\lesssim \delta_0 \|w_h\|_{W_h^{2,p}(B_3)} \|v_h\|_{L^{p'}(B_3)}, \end{aligned} \tag{4.28}$$

where we have used the inequality $h \leq R_1$. Combining (4.24)–(4.28) we get

$$I_3 \lesssim \frac{1}{R_1^3} \|w_h\|_{W_h^{2,p}(B_3)} \|v_h\|_{W_h^{-1,p'}(B_3)} + \delta_0 \|w_h\|_{W_h^{2,p}(B_3)} \|v_h\|_{L^{p'}(B_3)}, \tag{4.29}$$

and bringing together (4.21)–(4.23), and (4.29) gives us

$$(\mathcal{L}_h^*(\eta v_h), w_h) \lesssim \frac{1}{R_1^3} \left(\|\mathcal{L}_h^* v_h\|_{W_h^{-2,p'}(B_3)} + \|v_h\|_{W_h^{-1,p'}(B_3)} \right) \|w_h\|_{W_h^{2,p}(B_3)} \tag{4.30}$$

$$+ \delta_0 \|w_h\|_{W_h^{2,p}(B_3)} \|v_h\|_{L^{p'}(B_3)}. \tag{4.31}$$

By the definition of $\|\cdot\|_{W_h^{-2,p'}(B_3)}$ and (4.30) we get

$$\|\mathcal{L}_h^*(\eta w_h)\|_{W_h^{-2,p'}(B_3)} \lesssim \frac{1}{R_1^3} \left(\|\mathcal{L}_h^* v_h\|_{W_h^{-2,p'}(B_3)} + \|v_h\|_{W_h^{-1,p'}(B_3)} \right) + \delta_0 \|v_h\|_{L^{p'}(B_3)}. \tag{4.32}$$

Using (4.20) and (4.32) gives us

$$\|v_h\|_{L^{p'}(B_1)} \lesssim \frac{1}{R_1^3} \left(\|\mathcal{L}_h^* v_h\|_{W_h^{-2,p'}(B_3)} + \|v_h\|_{W_h^{-1,p'}(B_3)} \right) + \delta_0 \|v_h\|_{L^{p'}(B_3)}.$$

Since $\bar{\Omega}$ is compact, employing a covering argument (cf. [10, 13]) then yields

$$\|v_h\|_{L^{p'}(\Omega)} \lesssim \|\mathcal{L}_h^* v_h\|_{W_h^{-2,p'}(\Omega)} + \|v_h\|_{W_h^{-1,p'}(\Omega)} + \delta_0 \|v_h\|_{L^{p'}(\Omega)}.$$

Because δ_0 is small, we can absorb the last term on the right-hand side to the left-hand side to arrive at the global estimate

$$\|v_h\|_{L^{p'}(\Omega)} \lesssim \|\mathcal{L}_h^* v_h\|_{W_h^{-2,p'}(\Omega)} + \|v_h\|_{W_h^{-1,p'}(\Omega)}, \tag{4.33}$$

which is a Gårding-type inequality.

Step 3: Duality argument on the adjoint operator To control the last term in (4.33) we now use a duality argument for \mathcal{L}_h^* . This argument uses the regularity estimate of the original problem \mathcal{L} .

Define the set $X = \{g \in W_h^{1,p}(\Omega); \|g\|_{W_h^{1,p}(\Omega)} = 1\}$. By the discrete Poincaré inequality, with constant $C = C(p, \Omega)$, we have for all $g \in X$

$$\|g\|_{L^p(\Omega)} \leq C \|g\|_{W_h^{1,p}(\Omega)} < \infty,$$

since X is bounded in $W_h^{1,p}(\Omega)$. Thus, X is precompact in $L^p(\Omega)$ by Sobolev embedding. Next we define the set $W = \{\varphi := \mathcal{L}^{-1}g; g \in X\}$. Note that $\mathcal{L}^{-1} : L^p(\Omega) \rightarrow W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \subset W^{2,p}(\mathcal{T}_h)$ is well defined by well-posedness of the PDE. Also since \mathcal{L}^{-1} is linear and satisfies the estimate

$$\|\mathcal{L}^{-1}g\|_{W_h^{2,p}(\Omega)} = \|\varphi\|_{W_h^{2,p}(\Omega)} \leq \|\varphi\|_{W^{2,p}(\Omega)} \lesssim \|g\|_{L^p(\Omega)},$$

it is bounded in $W^{2,p}(\mathcal{T}_h)$. Thus W is precompact in $W^{2,p}(\mathcal{T}_h)$. From [24, Lemma 5], for every $\tau > 0$ there exists $h_* > 0$ that only depends on τ and \bar{W} such that for each $\varphi \in W$ and $0 < h \leq h_*$ there is a $\varphi_h \in V_h$ such that if $k \geq 2$ we have

$$\|\varphi - \varphi_h\|_{W_h^{2,p}(\Omega)} \leq \tau. \tag{4.34}$$

Note by the reverse triangle inequality and (4.34) we have

$$\|\varphi_h\|_{W_h^{2,p}(\Omega)} \leq \|\varphi\|_{W_h^{2,p}(\Omega)} \lesssim \|g\|_{L^p(\Omega)} \leq C$$

and hence $\{\varphi_h \in V_h; |\varphi_h - \varphi| \leq \tau\}$ is uniformly bounded in φ and h . Let $g \in X$ and choose $\varphi_g = \mathcal{L}^{-1}g \in W$ which tells us that $\mathcal{L}\varphi_g = g$. Let $v_h \in V_h$ and $\varphi_h \in V_h$. By Lemma 4.2 and the definition of $\|\cdot\|_{W_h^{-2,p'}(\Omega)}$ we have

$$\begin{aligned} \int_{\Omega} v_h g \, dx &= (\mathcal{L}_h^e \varphi_g, v_h) = (\mathcal{L}_h^e \varphi_h, v_h) + (\mathcal{L}_h^e (\varphi_g - \varphi_h), v_h) \\ &\lesssim \|\mathcal{L}_h^* v_h\|_{W_h^{-2,p'}(\Omega)} \|\varphi_h\|_{W_h^{2,p}(\Omega)} + \|\varphi_g - \varphi_h\|_{W_h^{2,p}(\Omega)} \|v_h\|_{L^{p'}(\Omega)}. \end{aligned}$$

Selecting φ_h to satisfy (4.34) and taking the supremum on g gives us

$$\|v_h\|_{W^{-1,p}(\Omega)} \lesssim \|\mathcal{L}_h^* v_h\|_{W_h^{-2,p'}(\Omega)} \|\varphi_h\|_{W_h^{2,p}(\Omega)} + \tau \|v_h\|_{L^{p'}(\Omega)}. \tag{4.35}$$

Combining (4.33) and (4.35) yields

$$\|v_h\|_{L^{p'}(\Omega)} \lesssim \|\mathcal{L}_h^* v_h\|_{W_h^{-2,p'}(\Omega)} + \tau \|v_h\|_{L^{p'}(\Omega)}. \tag{4.36}$$

By choosing τ sufficiently small to kick back the right-most term we have (4.12). This completes the proof upon taking $h_0 = \min\{h_{\delta_0}, h_*\}$. □

We are now ready to prove the global stability of the operator $\mathcal{L}_h^\varepsilon$.

Theorem 4.4 *Suppose that $h \leq h_0$ and $k \geq 2$. Then there holds the following stability estimate:*

$$\|w_h\|_{W_h^{2,p}(\Omega)} \lesssim \|\mathcal{L}_h^\varepsilon w_h\|_{L_h^p(\Omega)} \quad \forall w_h \in V_h, \tag{4.37}$$

where $1 < p < \infty$ if $\varepsilon = 1$, and $p = 2$ if $\varepsilon \in \{0, -1\}$.

Proof For $w_h \in V_h$, consider the auxiliary problem of finding $q_h \in V_h$ such that

$$\begin{aligned} (v_h, \mathcal{L}_h^* q_h) &= (\mathcal{L}_h^\varepsilon v_h, q_h) = \int_\Omega |D_h^2 w_h|^{p-2} D_h^2 w_h : D^2 v_h \, dx \\ &+ \sum_{e \in \mathcal{E}_h^I} h_e^{1-p} \int_e |[\nabla w_h]|^{p-2} [\nabla w_h] \cdot [\nabla v_h] \, dS \\ &+ \sum_{e \in \mathcal{E}_h} h_e^{1-2p} \int_e [|w_h]|^{p-2} [w_h][v_h] \, dS \quad \forall v_h \in V_h. \end{aligned} \tag{4.38}$$

Since V_h is finite dimensional and the operator is linear, the existence is equivalent to the uniqueness. To show the uniqueness, let $q_h^{(1)}$ and $q_h^{(2)}$ both solve (4.38). Then by Lemma 4.3 we get

$$\|q_h^{(1)} - q_h^{(2)}\|_{L^{p'}(\Omega)} \lesssim \sup_{0 \neq v_h \in V_h} \frac{(\mathcal{L}_h^\varepsilon v_h, q_h^{(1)} - q_h^{(2)})}{\|v_h\|_{W_h^{2,p}(\Omega)}} = 0.$$

Hence, (4.38) has a unique solution $q_h \in V_h$. Also by Lemma 4.3 and Hölder’s inequality,

$$\|q_h\|_{L^{p'}(\Omega)} \lesssim \sup_{0 \neq v_h \in V_h} \frac{(\mathcal{L}_h^\varepsilon v_h, q_h)}{\|v_h\|_{W_h^{2,p}(\Omega)}} \lesssim \|w_h\|_{W_h^{2,p}(\Omega)}^{p-1}.$$

Consequently, we find

$$\begin{aligned} \|w_h\|_{W_h^{2,p}(\Omega)}^p &\lesssim (w_h, \mathcal{L}_h^* q_h) = (\mathcal{L}_h w_h, q_h) \lesssim \|\mathcal{L}_h w_h\|_{L_h^p(\Omega)} \|q_h\|_{L^{p'}(\Omega)} \\ &\lesssim \|\mathcal{L}_h w_h\|_{L_h^p(\Omega)} \|w_h\|_{W_h^{2,p}(\Omega)}^{p-1}. \end{aligned}$$

Dividing by $\|w_h\|_{W_h^{2,p}(\Omega)}^{p-1}$ now yields the desired result. □

4.3 Well-Posedness and Error Estimates

The goals of this subsection are to establish the well-posedness for the IP-DG scheme (4.6) and to derive the optimal order error estimates in $W_h^{2,p}$ -norm for the IP-DG solutions.

Theorem 4.5 *Under the assumptions of Lemma 4.3, the IP-DG scheme (4.6) has a unique solution $u_h \in V_h$ such that*

$$\|u_h\|_{W_h^{2,p}(\Omega)} \lesssim \|f\|_{L^p(\Omega)}. \tag{4.39}$$

Proof Since (4.6) is equivalent to a linear system, hence it suffices to prove the uniqueness. To show the uniqueness, we first prove (4.39).

Let $u_h \in V_h$ be a solution of (4.6), then from (4.37) and the definition of $\|\cdot\|_{L^p_h(\Omega)}$ we have

$$\|u_h\|_{W^{2,p}_h(\Omega)} \lesssim \|\mathcal{L}_h^\varepsilon u_h\|_{L^p_h(\Omega)} = \sup_{v_h \in V_h} \frac{(\mathcal{L}_h^\varepsilon u_h, v_h)}{\|v_h\|_{L^{p'}(\Omega)}} = \sup_{v_h \in V_h} \frac{(f, v_h)}{\|v_h\|_{L^{p'}(\Omega)}} \leq \|f\|_{L^p(\Omega)}.$$

Hence, (4.39) holds.

Suppose that $u_h^1, u_h^2 \in V_h$ solve (4.6). Let $\tilde{u}_h = u_h^1 - u_h^2$. Then by (4.39) we have

$$\|\tilde{u}_h\|_{W^{2,p}_h(\Omega)} \leq \|0\|_{L^p(\Omega)} = 0.$$

Since $\tilde{u}_h \in V_h$ with $\|\tilde{u}_h\|_{W^{2,p}_h(\Omega)} = 0$ we conclude that $\tilde{u}_h \in C^1(\Omega)$, $\tilde{u}_h|_{\partial\Omega} = 0$, and $D_h^2 \tilde{u}_h = 0$ in Ω . The only way this can happen is if $\tilde{u}_h = 0$. Thus, the IP-DG solution must be unique. The proof is complete. \square

Next we show a Céa-type lemma for the IP-DG scheme, which immediately deduces the optimal order error estimates in the $W^{2,p}_h$ -norm.

Theorem 4.6 *Suppose that $h \leq h_0$ and $k \geq 2$. Let $u \in W^{2,p} \cap W_0^{1,p}(\Omega)$ be the solution of problem (1.1) and $u_h \in V_h$ solve (4.6). Then*

$$\|u - u_h\|_{W^{2,p}_h(\Omega)} \lesssim \inf_{w_h \in V_h} \|u - w_h\|_{W^{2,p}_h(\Omega)}. \tag{4.40}$$

Moreover, if $u \in W^{s,p}(\Omega)$ for some $s \geq 2$, we have

$$\|u - u_h\|_{W^{2,p}_h(\Omega)} \lesssim h^{r-2} \|u\|_{W^{r,p}(\Omega)}, \quad r = \min\{s, k + 1\}. \tag{4.41}$$

Proof By the consistency of $\mathcal{L}_h^\varepsilon$ we have the following Galerkin orthogonality:

$$(\mathcal{L}_h^\varepsilon(u - u_h), v_h) = 0 \quad \forall v_h \in V_h. \tag{4.42}$$

Let $w_h \in V_h$, Theorem 4.4, Lemma 4.2, (4.42), and the definition of $\|\cdot\|_{L^p_h(\Omega)}$ yield

$$\begin{aligned} \|u_h - w_h\|_{W^{2,p}_h(\Omega)} &\lesssim \|\mathcal{L}_h^\varepsilon(u_h - w_h)\|_{L^p_h(\Omega)} = \sup_{0 \neq v_h \in V_h} \frac{(\mathcal{L}_h^\varepsilon(u_h - w_h), v_h)}{\|v_h\|_{L^{p'}(\Omega)}} \\ &= \sup_{0 \neq v_h \in V_h} \frac{(\mathcal{L}_h^\varepsilon(u - w_h), v_h)}{\|v_h\|_{L^{p'}(\Omega)}} = \|\mathcal{L}_h^\varepsilon(u - w_h)\|_{L^p_h(\Omega)} \lesssim \|u - w_h\|_{W^{2,p}_h(\Omega)}. \end{aligned} \tag{4.43}$$

Thus by (4.43) and the triangle inequality we get

$$\|u - u_h\|_{W^{2,p}_h(\Omega)} \leq \|u - w_h\|_{W^{2,p}_h(\Omega)} + \|u_h - w_h\|_{W^{2,p}_h(\Omega)} \lesssim \|u - w_h\|_{W^{2,p}_h(\Omega)}. \tag{4.44}$$

Taking the infimum on both sides over all $w_h \in V_h$ yields (4.40). Finally, (4.41) follows from taking $w_h = I_h u$ and using the finite element interpolation theory [3]. The proof is complete. \square

5 Numerical Experiments

In this section we present a number of 2-D numerical tests to verify our error estimate and to gauge the performance of our IP-DG methods. In particular, we shall compare our IP-DG methods to the related conforming finite element counterpart developed in [10].

Moreover, we shall also perform numerical tests which are not covered by our convergence theory, this includes the cases when the coefficient matrix is either discontinuous or degenerate.

5.1 Hölder Continuous Coefficient

For this test we take A as the following Hölder continuous matrix-valued function:

$$A(x) = \begin{bmatrix} |x|^{1/2} + 1 & -|x|^{1/2} \\ -|x|^{1/2} & 5|x|^{1/2} + 1 \end{bmatrix}, \quad x \in \mathbb{R}^2.$$

Let $\Omega = (-1/2, 1/2)^2$ and choose f such that the exact solution is given by

$$u(x_1, x_2) = \sin(2\pi x_1) \sin(2\pi x_2) \exp(x_1 \cos(x_2)),$$

which has zero trace on the boundary.

Figure 1 shows the errors in the $L^2(\Omega)$, $W_h^{1,2}(\Omega)$, and $W_h^{2,2}(\Omega)$ norms of both the symmetrically and incompletely induced methods. The convergence rates observed for the symmetrically induced method are

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega)} &= \mathcal{O}(h^{k+1}) \quad \text{for all } k, \\ \|\nabla_h(u - u_h)\|_{L^2(\Omega)} &= \mathcal{O}(h^k) \quad \text{for all } k, \\ \|D_h^2(u - u_h)\|_{L^2(\Omega)} &= \mathcal{O}(h^{k-1}) \quad \text{for } k = 2, 3. \end{aligned}$$

As expected, these convergence rates are optimal. However, for the incompletely induced method we find that the rate of convergence in the L^2 -norm is sub-optimal for even degree polynomials and optimal with all other norms and degrees. This is expected since the incomplete scheme is sub-optimal even for smooth A [22].

5.2 Uniformly Continuous Coefficients

In this test we take $\Omega = (0, 1/2)^2$ and let

$$A(x) = \begin{bmatrix} -\frac{5}{\log(|x|)} + 15 & 1 \\ 1 & -\frac{1}{\log(|x|)} + 3 \end{bmatrix}.$$

f is chosen such that $u(x) = |x|^{7/4}$ is the exact solution. From [10] we see that the expected convergence rates are

$$\begin{aligned} \|\nabla_h(u - u_h)\|_{L^2(\Omega)} &= \mathcal{O}(h^{\min\{k, 7/4-\delta\}}) \quad \text{for all } k, \\ \|D_h^2(u - u_h)\|_{L^2(\Omega)} &= \mathcal{O}(h^{\min\{k, 7/4-\delta\}-1}) \quad \text{for } k = 2, 3 \end{aligned}$$

for any $\delta > 0$.

Figure 2 gives the computed results for both the symmetrically and incompletely induced schemes which match exactly the expected rates of convergence.

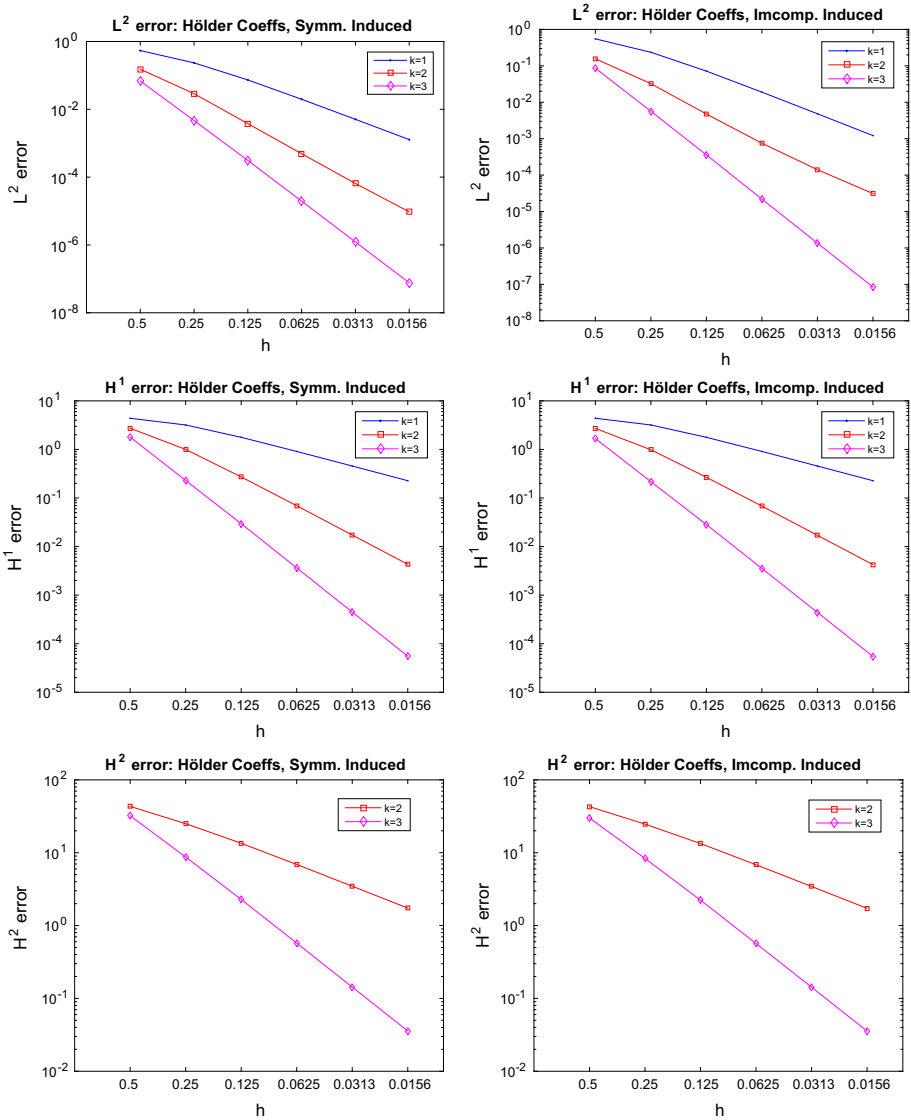


Fig. 1 The L^2 (top), piecewise H^1 (middle), and piecewise H^2 (bottom) errors for both the symmetrically (left) and incompletely (right) induced schemes with polynomial degree $k = 1, 2, 3$. $\gamma_e \equiv 100$ is used as the penalty parameter

5.3 Degenerate Coefficients

In this test we take $\Omega = (0, 1)^2$ and the matrix

$$A(x) = \frac{16}{9} \begin{bmatrix} x_1^{2/3} & -x_1^{1/3} x_2^{1/3} \\ -x_1^{1/3} x_2^{1/3} & x_2^{2/3} \end{bmatrix}.$$

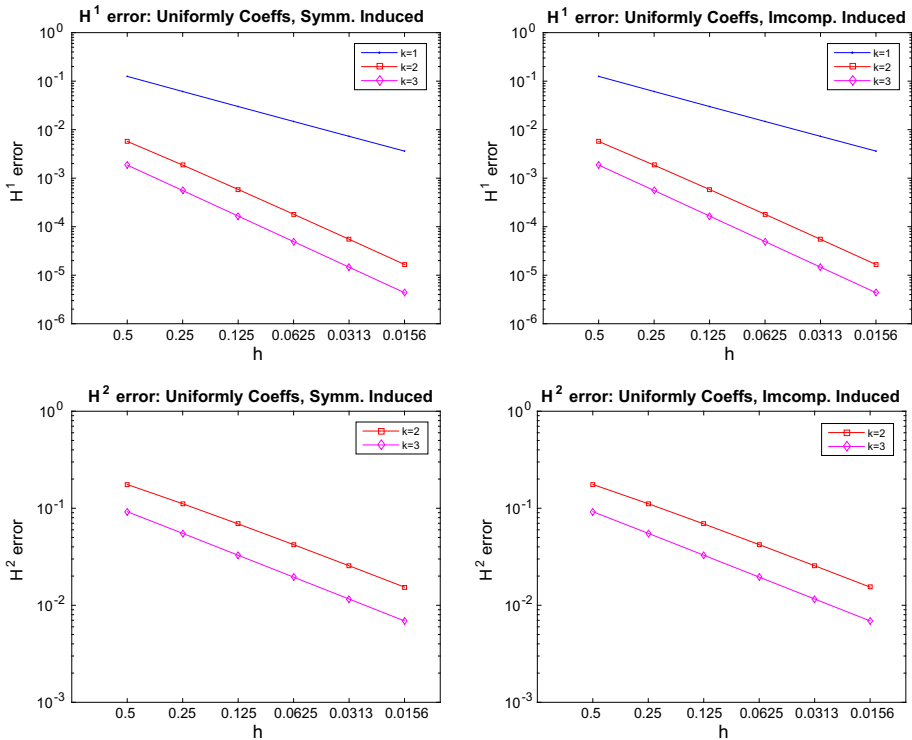


Fig. 2 The piecewise H^1 (top) and piecewise H^2 (bottom) errors for both the symmetrically (left) and incompletely (right) induced schemes with polynomial degree $k = 1, 2, 3$. $\gamma_e \equiv 1000$ is used as the penalty parameter

$f = 0$ and the exact solution $u(x) = x_1^{4/3} - x_2^{4/3}$. For an explanation for this example we refer to [10]. Note that $\det(A) = 0$ for every $x \in \Omega$ so this PDE is degenerate everywhere and is outside the case considered in this paper. We also observe that $u \in W^{m,p}(\Omega)$ provided $(4 - 3m)p > -1$.

Figure 3 shows the L^2 and piecewise H^1 errors for both the symmetrically and incompletely induced methods. The numerical results suggest the following rates of convergence:

$$\|u - u_h\|_{L^2(\Omega)} = \mathcal{O}(h^{4/3}), \quad \|\nabla_h(u - u_h)\|_{L^2(\Omega)} = \mathcal{O}(h^{5/6})$$

for $k = 1, 2, 3$. These rates are consistent with the results of the related conforming finite element method given in [10].

5.4 L^∞ Cordès Coefficients

Our next test is taken from [26,28] where a different DG method and a weak Galerkin method were used to solve this problem. Let $\Omega = [-1, 1]^2$ and

$$A(x) = \frac{16}{9} \begin{bmatrix} 2 & x_1x_2/|x_1x_2| \\ x_1x_2/|x_1x_2| & 2 \end{bmatrix}.$$

f is chosen so that the exact solution is $u(x) = x_1x_2(1 - e^{-|x_1|})(1 - e^{-|x_2|})$. Notice that the matrix A is discontinuous across the x_1 -axis and x_2 -axis, and it satisfies the Cordès

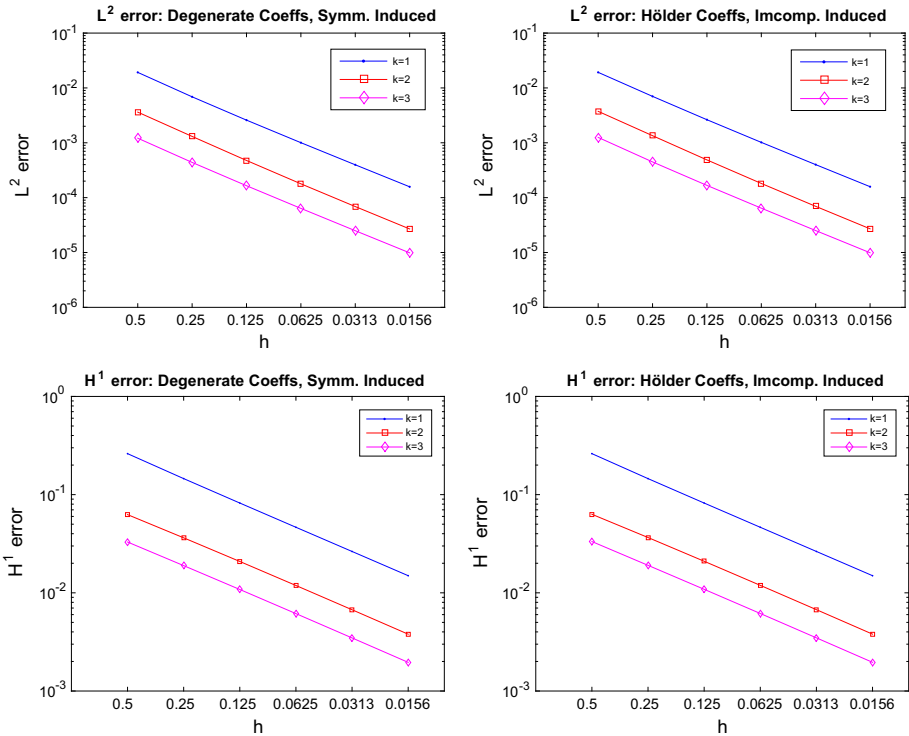


Fig. 3 The L^2 (top) and piecewise H^1 (bottom) errors for both the symmetrically (left) and incompletely (right) induced schemes with polynomial degree $k = 1, 2, 3$. $\gamma_e \equiv 100$ is used as the penalty parameter

Table 1 The L^2 errors and rates for the symmetrically induced method

h	$k = 1$		$k = 2$		$k = 3$	
	$\ u - u_h\ _{L^2(\Omega)}$	Rate	$\ u - u_h\ _{L^2(\Omega)}$	Rate	$\ u - u_h\ _{L^2(\Omega)}$	Rate
1	1.3e-1	–	7.7e-2	–	2.6e-2	–
1/2	8.9e-2	0.58	1.8e-2	2.09	1.5e-3	4.12
1/4	4.6e-2	0.95	2.9e-3	2.62	7.6e-4	4.27
1/8	1.9e-2	1.22	4.8e-3	2.62	4.2e-6	4.19
1/16	7.6e-3	1.35	8.0e-5	2.57	3.3e-7	3.65
1/32	2.9e-3	1.41	1.4e-5	2.54	3.2e-8	3.36

The rates for the incompletely induced method are similar. $\gamma_e \equiv 10000$ is used as the penalty parameter

condition. While our convergence theory does not apply to this example, we still compute the numerical solution on a uniform triangulation that has edges on all discontinuities of A . Due to its inconsistent behavior we list the L^2 error and convergence rates in Table 1. The following H^1 semi-norm rates are observed:

$$\|\nabla_h(u - u_h)\|_{L^2(\Omega)} = \mathcal{O}(h^k) \quad \text{for } k = 1, 2, 3.$$

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