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Optimal Error Estimates of Semi-implicit Galerkin Method for Time-Dependent Nematic Liquid Crystal Flows

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Abstract This paper focuses on the optimal error estimates of a linearized semi-implicit scheme for the nematic liquid crystal flows, which is used to describe the time evolution of the materials under the influence of both the flow velocity and the microscopic orientation configurations of rod-like liquid crystal flows. Optimal error estimates of the scheme are proved without any restriction of time step by using an error splitting technique proposed by Li and Sun. Numerical results are provided to confirm the theoretical analysis and the stability of the semi-implicit scheme.

Keywords Nematic liquid crystal model · Linearized semi-implicit scheme · Finite element method · Optimal error estimates

Mathematics Subject Classification 65M12 · 65M60 · 35Q35

1 Introduction

In this paper, we consider the following hydrodynamics system modeling the flow of nematic liquid crystal material (see [28]):

$$\mathbf{u}_t - \mu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + \lambda \mathrm{div}\,(\nabla\mathbf{b}\odot\nabla\mathbf{b}) = \mathbf{f}, \tag{1.1}$$

$$\mathbf{b}_t - \gamma\Delta\mathbf{b} + (\mathbf{u}\cdot\nabla)\mathbf{b} - \gamma|\nabla\mathbf{b}|^2\mathbf{b} = 0, \tag{1.2}$$

$$\mathrm{div}\,\mathbf{u} = 0, \quad |\mathbf{b}| = 1. \tag{1.3}$$

at $x \in \Omega$ and $t \in [0, T]$ for some positive constant $T > 0$. Here $\Omega \subset \mathbb{R}^2$ is a bounded and convex domain with a smooth boundary $\partial\Omega$. The unknown $\mathbf{u}(x, t) : \Omega \times [0, T] \longrightarrow$

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$\mathbb{R}^2$ and $p(x,t) : \Omega \times [0,T] \longrightarrow \mathbb{R}$ represent the velocity and the pressure of the flows, respectively. The unknown $\mathbf{b}(x,t) : \Omega \times [0,T] \longrightarrow \mathbb{S}$, where $\mathbb{S}$ is the unit circle in $\mathbb{R}^2$, represents the macroscopic molecular orientation of the liquid crystal material. The vector $\mathbf{f} : \Omega \times [0,T] \longrightarrow \mathbb{R}^2$ represents a body force on the flow. The constants μ, λ and γ denote the viscosity, the competition between kinetic and potential energy, and the microscopic elastic relaxation time for the molecular orientation field, respectively. The term $\nabla\mathbf{b} \odot \nabla\mathbf{b}$ is a 2×2 matrix whose (i,j)-the entry is given by $(\nabla_i \mathbf{b}) \cdot (\nabla_j \mathbf{b})$ for $1 \le i, j \le 2$. It is noteworthy that if $\mathbf{b}$ is a constant map, the system (1.1)–(1.3) reduces to the incompressible Navier–Stokes equations [35]. If $\mathbf{u} = 0$, the system (1.1)–(1.3) reduces to the heat flow of harmonic maps [8]. In addition, the above system (1.1)–(1.3) should be completed by an appropriate initial and boundary condition. For the sake of simplicity, we consider the following initial and boundary conditions:

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{b}(x,0) = \mathbf{b}_0(x), \quad \text{in } \Omega, \tag{1.4}$$

$$\mathbf{u}(x,t) = 0, \quad \partial_{\mathbf{n}}\mathbf{b}(x,t) = 0, \quad \text{on } \partial\Omega \times [0,T], \tag{1.5}$$

where $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$. Here, we require that the initial vector functions $\mathbf{u}_0$ and $\mathbf{b}_0$ satisfy the compatibility condition div $\mathbf{u}_0 = 0$ and $|\mathbf{b}_0| = 1$.

The system (1.1)–(1.3) was firstly derived by Lin [28] as a simplified version of Ericksen–Leslie model for the hydrodynamics of nematic liquid crystal flows developed by Ericksen [10,11] and Leslie [22]. It is the macroscopic continuum description of the time evolution of the materials under the influence of both the flow velocity $\mathbf{u}$ and the microscopic orientation configurations $\mathbf{b}$ of rod-like liquid crystal flows. There have two major difficulties in studying the above system. One is the presence of div $(\nabla\mathbf{b} \odot \nabla\mathbf{b})$ such that the system (1.1)–(1.3) becomes a strongly nonlinear coupled system. The other comes from the nonlinear constraint $|\mathbf{b}| = 1$.

The mathematical analysis for (1.1)–(1.5) was initiated by Lin and Liu [29,30]. The nonlinear constraint $|\mathbf{b}| = 1$ was relaxed by introducing a Ginzburg–Landau penalty function $(1 - |\mathbf{b}|^2)\mathbf{b}/\varepsilon^2$ to replace $|\nabla\mathbf{b}|^2\mathbf{b}$ in (1.2), where $\varepsilon > 0$ is a small penalty parameter. For Ginzburg–Landau approximation problem, Lin and Liu [29] proved the local existence of the strong solution and the global existence of the weak solution. It also has been shown that the global strong solution exists in the case of the large viscosity μ. The partial regularity for the suitable weak solution was proved in [30]. Some regularity criterions for the global weak solution were studied for 3D bounded and smooth domain [15]. However, as pointed in [29, 30], since the estimates and arguments heavily depend on ε, it is still an open and challenging problem to study the limiting case as ε tends to zero. Recently, some new theoretical analysis for the original problem (1.1)–(1.5) have been studied. For example, Xu and Zhang [37] proved the global existence and regularity of weak solution for 2D Cauchy problem if $||\mathbf{u}_0||_{L^2}$ and $||\nabla\mathbf{b}_0||_{L^2}$ are small enough. For 3D model in a bounded and smooth domain, Li and Wang [27] established the existence and uniqueness of the local strong solution with large initial data and the global strong solution with small initial data. For the compressible nematic liquid crystal flows, Huang et al. [21] proved the local existence and uniqueness of strong solution provided that the initial data are sufficiently smooth and the pressure is a local Lipschitz continuous function with respect to the density function.

The numerical methods for Ginzburg–Landau approximation problem have been investigated in some previous works. For example, the first numerical method was studied by Liu and Walkington [31], where $\mathbb{Q}_3$ Hermite finite element was used for the approximation of the director. To avoid using Hermite finite element, a mixed finite element method was subsequently studied in [32]. In their works, the fully nonlinear implicit schemes were proposed.

Although these schemes are unconditional stable, however, one has to solve a nonlinear problem by Newton's iteration scheme at each time step. Becker et al. [6] studied a new mixed method by introducing $\mathbf{w} = -\Delta\mathbf{b} + \frac{|\mathbf{b}|^2 - 1}{\varepsilon^2}\mathbf{b}$. The fully discrete scheme proposed in [6] was nonlinear at the step of solving $\mathbf{w}$ numerically, and allowed them to establish a discrete energy law. With the help of this energy law, the authors showed the unconditional convergence of the numerical solution to the solution of Ginzburg–Landau approximation problem as h and τ tend to zero. But no error estimates were derived in [6]. Motivated by Becker–Feng–Prohl, Girault and González-Santacreu [14] introduced the auxiliary variable $\mathbf{w} = -\Delta\mathbf{b}$ to design a semiexplicit Euler scheme where the Ginzburg–Landau penalty function was explicitly discretizated. But the error estimate derived in [14] heavily depends on $O(e^{-1/\varepsilon^2})$. Other mixed FEM fully discrete schemes were developed by González-Santacreu [7,16] in views of a fully explicit time integration of the potential term and the projection time-stepping method for Navier–Stokes equations [9,35]. Based on a saddle-point strategy, Badia–González–Santacreu suggested a fully implicit scheme and a semi-implicit scheme for Ginzburg–Landau approximation problem in [3]. We observe that no error estimates are derived in [3,7,16]. The reader is referred to [2] for a survey of numerical methods on nematic liquid crystal flows and the Ginzburg–Landau approximation.

Instead of using the Ginzburg–Landau penalty function to relax the nonlinear constraint $|\mathbf{b}| = 1$, there have some existing works in studying the fully discrete schemes which directly approximate the original system (1.1)–(1.5). Becker et al. [6] investigated a fully discrete scheme which was conditionally stable under the time step restriction $\tau \leq O(h^3)$. The constraint $|\mathbf{b}| = 1$ are derived in the sense of $\mathbf{L}^2$-norm by the convergence of the numerical solution as h and τ tend to zero. The same approximation also has been used for Landau–Lifshitz equation in [12]. Inspired by the projection time-stepping method for Navier–Stokes equations [9,35] and Landau-Lifshitz equation [36], a time-stepping/projection scheme for the approximation of (1.1)–(1.5) is proposed by Prohl [33]. However, the error estimates derived in [33] are not optimal.

In this paper, we will propose a linearized semi-implicit finite element scheme for the approximation of the original system (1.1)–(1.5) and prove the optimal error estimates of this scheme. The derivations of the optimal error estimates are based upon the recent works by Li and Sun [23,24] (also see [20,25]), where the error estimates are split into the temporal error, the spatial error and the projection error by introducing a corresponding time-discrete parabolic system (or elliptic system). A key issue is that the regularities of the solutions to the discrete parabolic system need to be proved such that the uniform boundedness in different norms hold. With this boundedness, we can show that the spatial error analysis are bounded by Ch^σ for some $\sigma > 0$ and $C > 0$ independent of h and τ, from which the time step restriction can be removed. Meanwhile, optimal error estimates in the discrete $\mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$-norm and $\mathbf{L}^\infty(0, T; \mathbf{H}^1(\Omega))$-norm for $\mathbf{u}$ and in the discrete $\mathbf{L}^\infty(0, T; \mathbf{H}^1(\Omega))$-norm for $\mathbf{b}$ are established without any time step restriction.

The rest of the paper is organized as follows. In Sect. 2, we introduce some notations and recall some known results for the nematic liquid crystal model (1.1)–(1.5). The uncoupled and linearized semi-implicit Euler finite element scheme and the main results in this paper are presented in Sect. 3. Meanwhile, the discrete parabolic system corresponding to the original system is introduced. Moreover, from the regularity of $(\mathbf{u}, p)$, the lowest-order $P_1 - P_0$ stabilized finite elements are used to approximate the velocity and the pressure in (1.1). The temporal error and the spatial error are shown in Sects. 4 and 5, respectively. The regularities of the solution to the discrete parabolic system are established in Sect. 4. The numerical results are presented in Sect. 7 to confirm our theoretical analysis. The conclusions are summarized in final section.

2 Preliminaries

Standard Sobolev space notations are used in this paper [1]. We use the boldface Sobolev spaces $\mathbf{H}^m(\Omega)$, $\mathbf{W}^{m,p}(\Omega)$ and $\mathbf{L}^p(\Omega)$ to denote the vector Sobolev spaces $H^m(\Omega)^2$, $W^{m,p}(\Omega)^2$ and $L^p(\Omega)^2$ for $m \in \mathbb{N}$ and $1 \le p \le +\infty$, respectively. In particular, $(\cdot,\cdot)$ denotes the $\mathbf{L}^2(\Omega)$ inner product. The symbols $C, C_0, C_1, C_2, \ldots$ are used to denote a generic positive constant which may depends on $\mathbf{u}$, p, $\mathbf{b}$, $\mathbf{f}$ and μ, λ, γ and is independent of the mesh size h and the time step τ.

For the mathematical setting of the nematic liquid system model (1.1)–(1.5), we introduce the following spaces:

$$
\begin{aligned}
&\mathbf{H} = \{\mathbf{u} \in \mathbf{L}^2(\Omega),\ \mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},\ \mathbf{V} = \mathbf{H}_0^1(\Omega),\ \mathbf{X} = \mathbf{H}^1(\Omega),\\
&\mathbf{V}_0 = \{\mathbf{u} \in \mathbf{V},\ \mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega\},\ M = L_0^2(\Omega) = \{q \in L^2(\Omega),\ \int_\Omega q dx = 0\},\\
&\mathbf{H}(\mathrm{div}\,,\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega),\ \mathrm{div}\,\mathbf{u} \in L^2(\Omega)\}.
\end{aligned}
$$

It is well known that the norm $||\nabla \mathbf{v}||_{L^2}$ is equivalent to the standard $\mathbf{H}^1$ norm for any $\mathbf{v} \in \mathbf{V}$ due to Poincaré inequality. Define the following continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $\mathbf{V}\times\mathbf{V}$ and $\mathbf{V}\times M$, respectively, by

$$
\begin{aligned}
a(\mathbf{u},\mathbf{v}) &= \mu \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v} dx, \quad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{V},\\
d(\mathbf{v},q) &= \int_\Omega q \mathrm{div}\,\mathbf{v} dx, \quad \forall\, \mathbf{v} \in \mathbf{V}, q \in M,
\end{aligned}
$$

and a trilinear form on $\mathbf{X}\times\mathbf{X}\times\mathbf{X}$ by

$$
b(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} dx, \quad \forall\, \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{X}.
$$

Integrating by part, it is easy to check that

$$
b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \quad \forall\, \mathbf{u} \in \mathbf{V}_0, \mathbf{v} \in \mathbf{X}. \tag{2.1}
$$

Corresponding to (1.1), we recall Stokes operator A. Introduce the orthogonal projection operator $\mathbb{P}_\mathbf{H}$ from $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$ which satisfies (cf. [35])

$$
||\mathbb{P}_\mathbf{H}\mathbf{u}||_{H^1} \le C||\mathbf{u}||_{H^1}, \quad \forall\, \mathbf{u} \in \mathbf{H}^1(\Omega). \tag{2.2}
$$

Then Stokes operator A is defined by (cf. [35])

$$
A\mathbf{u} = -\mathbb{P}_\mathbf{H}\Delta\mathbf{u}, \quad \forall\, \mathbf{u} \in \mathbf{D}(A) = \mathbf{V}_0 \cap \mathbf{H}^2(\Omega). \tag{2.3}
$$

Now, we recall some known inequalities frequently used in this paper [1,35]:

$$||\mathbf{v}||_{L^r} \le C||\nabla\mathbf{v}||_{L^2}(2 \le r \le 6), \quad ||\mathbf{v}||_{L^4} \le C||\mathbf{v}||_{L^2}^{1/2}||\nabla\mathbf{v}||_{L^2}^{1/2}, \quad \forall\, \mathbf{v} \in \mathbf{V}, \tag{2.4}$$

$$||\mathbf{v}||_{H^2} \le C||A\mathbf{v}||_{L^2}, \quad ||\mathbf{v}||_{L^\infty} \le C||\mathbf{v}||_{L^2}^{1/2}||A\mathbf{v}||_{L^2}^{1/2}, \quad \forall\, \mathbf{v} \in \mathbf{D}(A), \tag{2.5}$$

$$||\mathbf{v}||_{L^r} \le C||\mathbf{v}||_{H^1}(2 \le r \le 6), \qquad ||\mathbf{v}||_{L^4} \le C||\mathbf{v}||_{L^2}^{1/2}||\mathbf{v}||_{H^1}^{1/2}, \quad \forall\, \mathbf{v} \in \mathbf{X}, \tag{2.6}$$

$$||\nabla^2\mathbf{v}||_{L^2} \le C||\Delta\mathbf{v}||_{L^2}, \quad \forall\, \mathbf{v} \in \mathbf{H}^2(\Omega) \text{ with } \partial_{\mathbf{n}}\mathbf{v}|_{\partial\Omega} = 0, \tag{2.7}$$

$$||\mathbf{v}||_{L^\infty} \le C||\mathbf{v}||_{L^2}^{1/2}(||\mathbf{v}||_{L^2}^2 + ||\Delta\mathbf{v}||_{L^2}^2)^{1/4}, \quad \forall\, \mathbf{v} \in \mathbf{H}^2(\Omega) \text{ with } \partial_{\mathbf{n}}\mathbf{v}|_{\partial\Omega} = 0, \tag{2.8}$$

$$||\nabla\mathbf{v}||_{L^3} \le C(||\nabla\mathbf{v}||_{L^2}^{2/3}||\Delta\mathbf{v}||_{L^2}^{1/3} + ||\nabla\mathbf{v}||_{L^2}), \quad \forall\, \mathbf{v} \in \mathbf{H}^2(\Omega). \tag{2.9}$$

Next, we give a regularity result for the solution to the problem (1.1)–(1.5) established in [21].

Theorem 2.1 *Let $\boldsymbol{u}_0 \in \boldsymbol{D}(A)$ and $\boldsymbol{b}_0 \in \boldsymbol{H}^3(\Omega)$ with $|\boldsymbol{b}_0| = 1$ in Ω. For given $\boldsymbol{f} \in \boldsymbol{L}^\infty(0, T; \boldsymbol{H}) \cap \boldsymbol{L}^2(0, T; \boldsymbol{L}^4(\Omega))$, then there exists some $T^\star < T$ such that the problem (1.1)–(1.5) admits a unique local strong solution $(\boldsymbol{u}, p, \boldsymbol{b})$ satisfying*

$$\boldsymbol{u} \in \boldsymbol{L}^2(0, T^\star; \boldsymbol{W}^{2,4}(\Omega)) \cap \boldsymbol{L}^\infty(0, T^\star; \boldsymbol{D}(A)), \tag{2.10}$$

$$\boldsymbol{u}_t \in \boldsymbol{L}^2(0, T^\star; \boldsymbol{V}) \cap \boldsymbol{L}^\infty(0, T^\star; \boldsymbol{H}), \tag{2.11}$$

$$\boldsymbol{b} \in \boldsymbol{L}^\infty(0, T^\star; \boldsymbol{H}^3(\Omega)) \cap \boldsymbol{L}^2(0, T^\star; \boldsymbol{H}^4(\Omega)), \tag{2.12}$$

$$\boldsymbol{b}_t \in \boldsymbol{L}^\infty(0, T^\star; \boldsymbol{H}^1(\Omega)) \cap \boldsymbol{L}^2(0, T^\star; \boldsymbol{H}^2(\Omega)), \tag{2.13}$$

$$p \in L^\infty(0, T^\star; H^1(\Omega) \cap M). \tag{2.14}$$

Remark 2.1 Although the authors investigated the compressible nematic liquid crystal model in [21], the regularity results derived in [21] also hold for the incompressible nematic liquid crystal model (1.1)–(1.3) with the initial and boundary conditions (1.4)–(1.5). The regularity (2.14) for the pressure is not derived in [21] because the pressure p depends on the density in the compressible nematic liquid crystal model. But it can be easily proved by using (2.10)–(2.12) and inf-sup condition.

Remark 2.2 We require that the initial value $\mathbf{u}_0$ and π is the solution to the following Stokes problem

$$\begin{cases} -\mu\Delta\mathbf{u}_0 + \nabla\pi = \mathbf{f}_0, & \text{in } \Omega, \\ \text{div } \mathbf{u}_0 = 0, & \text{in } \Omega, \\ \mathbf{u}_0 = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$

where $\mathbf{f}_0 = \mathbf{f}(x, 0) \in \mathbf{H}$. Then by the regularity result for Stokes problem [35], the solution $(\mathbf{u}_0, \pi)$ belongs to $\mathbf{D}(A) \times H^1(\Omega) \cap M$.

Suppose that $\mathbf{u}_0 \in \mathbf{D}(A)$ satisfies (2.15). Under the following non-local compatibility conditions:

$$\nabla p_0 = (\mu\Delta\mathbf{u}_0 + \mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - \lambda \text{div } (\nabla\mathbf{b}_0 \odot \nabla\mathbf{b}_0)), \quad \text{on } \partial\Omega,$$

$$\nabla(\gamma\Delta\mathbf{b}_0 + (\mathbf{u}_0 \cdot \nabla)\mathbf{b}_0 + \gamma|\nabla\mathbf{b}_0|^2\mathbf{b}_0) \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega,$$

where $p_0 \in H^1(\Omega) \cap M$ is the weak solution to

$$\begin{cases} \Delta p_0 = \text{div } (\mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - \lambda \text{div } (\nabla\mathbf{b}_0 \odot \nabla\mathbf{b}_0)), & \text{in } \Omega, \\ \partial_{\mathbf{n}} p_0 = (\mu\Delta\mathbf{u}_0 + \mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - \lambda \text{div } (\nabla\mathbf{b}_0 \odot \nabla\mathbf{b}_0)) \cdot \mathbf{n}, & \text{on } \partial\Omega, \end{cases}$$

the following regularities can be derived:

$$\mathbf{u}_t \in \mathbf{L}^\infty(0, T^\star; \mathbf{V}) \cap \mathbf{L}^2(0, T^\star; \mathbf{D}(A)), \quad \nabla p_t \in \mathbf{L}^2(0, T^\star; \mathbf{L}^2(\Omega)), \tag{2.16}$$

$$\mathbf{u}_{tt} \in \mathbf{L}^\infty(0, T^\star; \mathbf{V}') \cap \mathbf{L}^2(0, T^\star; \mathbf{H}), \quad \mathbf{b}_{tt} \in \mathbf{L}^2(0, T^\star; \mathbf{L}^2(\Omega)) \tag{2.17}$$

under $\mathbf{f}_t \in \mathbf{L}^\infty(0, T; \mathbf{V}') \cap \mathbf{L}^2(0, T; \mathbf{H})$, where $\mathbf{V}_0'$ is the dual space of $\mathbf{V}_0$. The proof of (2.16)–(2.17) is given in Appendix.

3 Main Results

In this paper, we assume that Ω is a bounded and convex domain with a smooth boundary $\partial\Omega$. Let T_h be a family of quasi-uniform triangular partition of Ω. The corresponding ordered triangles are denoted by $K_1, K_2, \ldots, K_M$. Let $h_i = \text{diam}(K_i), i = 1, \ldots, M$. Then we denote by $h = \max\{h_1, h_2, \ldots, h_M\}$ the mesh size. For a triangle K_j with two nodes on the boundary, we use $\widetilde{K}_j$ to denote the triangle with one curved edge with the same nodes as K_j. For interior element, we simply set $\widetilde{K}_j$ as K_j itself. Let $\Omega_h = \bigcup_1^M K_j$ and $x = G(\widetilde{x})$ be a map from Ω_h to Ω such that G and G^{-1} both are Lipschitz continuous, and G is the identity mapping for interior element K_j, and G maps K_j onto $\widetilde{K}_j$ smoothly for K_j at the boundary [13,38]. For a given partition of Ω, we define

$$\begin{aligned}
&\widehat{\mathbf{X}}_h = \{\mathbf{v}_h \in \mathcal{C}(\overline{\Omega}_h),\ \mathbf{v}_h \in \mathbf{P}_2(K),\ \forall\, K \in T_h\},\\
&\widehat{\mathbf{V}}_h = \{\mathbf{w}_h \in \mathcal{C}(\overline{\Omega}_h),\ \mathbf{w}_h \in \mathbf{P}_1(K),\ \forall\, K \in T_h \text{ and } \mathbf{w}_h = 0 \text{ on } \partial\Omega_h\},\\
&\widehat{M}_h = \{q_h \in L^2(\Omega_h),\ q_h \in P_0(K),\ \forall\, K \in T_h \text{ and } \int_{\Omega_h} q_h(\widetilde{x})|\det(J_G)|d\widetilde{x} = 0\},\\
&\widehat{\widetilde{M}}_h = \{\phi_h \in L^2(\Omega_h),\ \phi_h \in P_1(K),\ \forall\, K \in T_h \text{ and } \int_{\Omega_h} \phi_h(\widetilde{x})|\det(J_G)|d\widetilde{x} = 0\},
\end{aligned}$$

where J_G denotes the Jacobian of G, and $P_r(K)$ denotes the space of the polynomials on K of degree at most r for every $K \in T_h$ and a nonnegative integer r. For $x \in \Omega$, we define an operator $\mathcal{G}_X$ on $\widehat{\mathbf{X}}_h$ by $\mathcal{G}_X\mathbf{v}_h(x) = \mathbf{v}_h(G^{-1}(x))$, and an operator $\mathcal{G}_V$ on $\widehat{\mathbf{V}}_h$ by $\mathcal{G}_V\mathbf{w}_h(x) = \mathbf{w}_h(G^{-1}(x))$, and an operator $\mathcal{G}_M$ on $\widehat{M}_h$ by $\mathcal{G}_M q_h(x) = q_h(G^{-1}(x))$, and an operator $\mathcal{G}_{\widetilde{M}}$ on $\widehat{\widetilde{M}}_h$ by $\mathcal{G}_{\widetilde{M}}\phi_h(x) = \phi_h(G^{-1}(x))$. Then the finite element spaces are defined by

$$\begin{aligned}
\mathbf{X}_h &= \{\mathcal{G}_X\mathbf{v}_h : \mathbf{v}_h \in \widehat{\mathbf{X}}_h\}, \quad \mathbf{V}_h = \{\mathcal{G}_V\mathbf{w}_h : \mathbf{w}_h \in \widehat{\mathbf{V}}_h\},\\
M_h &= \{\mathcal{G}_M q_h : q_h \in \widehat{M}_h\}, \quad \widetilde{M}_h = \{\mathcal{G}_{\widetilde{M}}\phi_h : \phi_h \in \widehat{\widetilde{M}}_h\}.
\end{aligned}$$

It is clear that $\mathbf{X}_h$ is a finite element subspace of $\mathbf{X}$ and $\mathbf{V}_h$ is a finite element subspace of $\mathbf{V}$. Moreover, there holds $\int_\Omega \mathcal{G}_M q_h(x)dx = \int_\Omega \mathcal{G}_{\widetilde{M}}\phi_h(x)dx = 0$. Thus, M_h and $\widetilde{M}_h$ both are finite element spaces of M. For any $\mathbf{v} \in \mathbf{X}$, we define $\Pi_h^0\mathbf{v} = \mathcal{G}_X\widehat{\Pi}_h\mathcal{G}_X^{-1}\mathbf{v}$, where $\widehat{\Pi}_h : \mathcal{C}(\Omega_h) \to \widehat{X}_h$ is the Lagrange interpolation operator. Then for any $\mathbf{v} \in \mathbf{H}^3(\Omega)$, (cf. [13,26])

$$||\mathbf{v} - \Pi_h^0\mathbf{v}||_{L^2} + h||\mathbf{v} - \Pi_h^0\mathbf{v}||_{H^1} \le Ch^3||\mathbf{v}||_{H^3}. \tag{3.1}$$

Similarly, For any $\mathbf{w} \in \mathbf{V}$, we define $\mathbf{R}_h^0\mathbf{v} = \mathcal{G}_V\widehat{\mathbf{R}}_h\mathcal{G}_V^{-1}\mathbf{w}$, where $\widehat{\mathbf{R}}_h : \mathcal{C}(\Omega_h) \to \widehat{V}_h$ is the Lagrange interpolation operator. Then for any $\mathbf{w} \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$, (cf. [34])

$$||\mathbf{w} - \mathbf{R}_h^0\mathbf{w}||_{L^2} + h||\nabla(\mathbf{w} - \mathbf{R}_h^0\mathbf{w})||_{L^2} \le Ch^2||\mathbf{w}||_{H^2}. \tag{3.2}$$

Observe that $P_1 - P_0$ finite element space for velocity and pressure does not satisfy the so-called discrete inf-sup condition. Here, we use a stabilized technique proposed by Bochev et al. [4] for Stokes problem. Introduce the generalized bilinear forms defined by

$$\mathcal{B}(\mathbf{w}_h, r_h; \mathbf{v}_h, q_h) = a(\mathbf{w}_h, \mathbf{v}_h) - d(\mathbf{v}_h, r_h) + d(\mathbf{w}_h, q_h),$$
$$\mathcal{B}_h(\mathbf{w}_h, r_h; \mathbf{v}_h, q_h) = \mathcal{B}(\mathbf{w}_h, r_h; \mathbf{v}_h, q_h) + \alpha G(r_h, q_h),$$

for all $(\mathbf{w}_h, r_h), (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h$. Here $\alpha > 0$ is the stable parameter. The stable term $G(r_h, q_h)$ is defined by

$$G(r_h, q_h) = (r_h - \Pi_1 r_h, q_h - \Pi_1 q_h), \quad \forall\, r_h, q_h \in M_h,$$

where Π_1 is a continuous projection operator from M_h to $\widetilde{M}_h$.

Suppose that the solution $(\mathbf{u}, p, \mathbf{b})$ satisfies the regularities (2.10)–(2.14) and (2.16)–(2.17) in $[0, T]$ for some $T > 0$. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0, T]$ with time step $\tau = T/N$ and $t_n = n\tau$ for $0 \le n \le N$. Let

$$\mathbf{u}^n = \mathbf{u}(x, t_n), \quad p^n = p(x, t_n), \quad \mathbf{b}^n = \mathbf{b}(x, t_n), \quad \mathbf{f}^n = \mathbf{f}(x, t_n).$$

For any sequence $\{g^n\}_{n=0}^N$, denote $D_\tau g^{n+1} = \dfrac{g^{n+1} - g^n}{\tau}$ for $0 \le n \le N-1$.

Under the above notations, we propose a linearized semi-implicit Euler finite element scheme for the nematic liquid crystal model (1.1)–(1.5), which is to find $\mathbf{B}_h^{n+1} \in \mathbf{X}_h$ and $(\mathbf{U}_h^{n+1}, P_h^{n+1}) \in \mathbf{V}_h \times M_h$ for $n = 0, 1, \ldots, N-1$, such that

$$(D_\tau \mathbf{B}_h^{n+1}, \phi_h) + \gamma(\nabla \mathbf{B}_h^{n+1}, \nabla \phi_h) + b_h(\mathbf{U}_h^n, \mathbf{B}_h^{n+1}, \phi_h) = \gamma(|\nabla \mathbf{B}_h^n|^2 \mathbf{B}_h^n, \phi_h), \quad \forall\, \phi_h \in \mathbf{X}_h, \tag{3.3}$$

$$\begin{aligned}&(D_\tau \mathbf{U}_h^{n+1}, \mathbf{v}_h) + \mathcal{B}_h(\mathbf{U}_h^{n+1}, P_h^{n+1}; \mathbf{v}_h, q_h) + b_h(\mathbf{U}_h^n, \mathbf{U}_h^{n+1}, \mathbf{v}_h)\\ &\quad - \lambda(\nabla \mathbf{B}_h^{n+1} \odot \nabla \mathbf{B}_h^n, \nabla \mathbf{v}_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h), \quad \forall\, (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h,\end{aligned} \tag{3.4}$$

with $\mathbf{B}_h^0 = \Pi_h^0 \mathbf{b}_0 \in \mathbf{X}_h$, $\mathbf{U}_h^0 = \mathbf{R}_h^0 \mathbf{u}_0 \in \mathbf{V}_h$ and

$$b_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w}\, dx + \frac{1}{2}\int_\Omega (\text{div}\, \mathbf{u})\mathbf{v} \cdot \mathbf{w}\, dx. \tag{3.5}$$

The emphasis of this paper is to show optimal error estimates for the semi-implicit Euler scheme (3.3)–(3.4). The main result derived in this paper is presented in the following theorem.

Theorem 3.1 *Suppose* $\boldsymbol{u}_0 \in \boldsymbol{V}_0 \cap \boldsymbol{H}^2(\Omega)$, $\boldsymbol{b}_0 \in \boldsymbol{H}^3(\Omega)$ *with* $|\boldsymbol{b}_0| = 1$, $\boldsymbol{f} \in \boldsymbol{L}^\infty(0, T; \boldsymbol{H}) \cap \boldsymbol{L}^2(0, T; \boldsymbol{L}^4(\Omega))$ *and* $\boldsymbol{f}_t \in \boldsymbol{L}^\infty(0, T; \boldsymbol{V}_0') \cap \boldsymbol{L}^2(0, T; \boldsymbol{H})$. *Then the finite element semi-implicit discrete system* (3.3)–(3.4) *exists a unique solution* $(\boldsymbol{U}_h^n, P_h^n, \boldsymbol{B}_h^n) \in \boldsymbol{V}_h \times M_h \times \boldsymbol{X}_h$ *for* $n = 1, \ldots, N$. *Moreover, there have two constants* $h_0 > 0$ *and* $\tau_0 > 0$ *such that when* $h < h_0$ *and* $\tau < \tau_0$, *the following optimal error estimates hold:*

$$\max_{0 \le n \le N} \left(||\boldsymbol{b}^n - \boldsymbol{B}_h^n||_{H^1} + ||1 - |\boldsymbol{B}_h^n|^2||_{L^2} + ||\boldsymbol{u}^n - \boldsymbol{U}_h^n||_{L^2}\right) \le C_0(\tau + h^2), \tag{3.6}$$

$$\max_{0 \le n \le N} ||\nabla \boldsymbol{u}^n - \nabla \boldsymbol{U}_h^n||_{L^2} \le C_0(\tau + h). \tag{3.7}$$

To prove Theorem 3.1 by using the temporal-spatial error splitting method proposed by Li and Sun in [23,24], for $\mathbf{B}^0 = \mathbf{b}_0$ and $\mathbf{U}^0 = \mathbf{u}_0$ and $n = 0, 1, \ldots, N-1$, we define $\mathbf{B}^{n+1}$

and $(\mathbf{U}^{n+1}, P^{n+1})$ to be the solutions of the following discrete parabolic (or elliptic) system corresponding to the time-dependent system (1.1)–(1.5):

$$D_\tau \mathbf{B}^{n+1} - \gamma \Delta \mathbf{B}^{n+1} + (\mathbf{U}^n \cdot \nabla)\mathbf{B}^{n+1} = \gamma |\nabla \mathbf{B}^n|^2 \mathbf{B}^n \tag{3.8}$$

with homogeneous Neumann boundary condition $\partial_n \mathbf{B}^{n+1}| = 0$ on $\partial\Omega$, and

$$\begin{aligned} &D_\tau \mathbf{U}^{n+1} - \mu \Delta \mathbf{U}^{n+1} + (\mathbf{U}^n \cdot \nabla)\mathbf{U}^{n+1} + \nabla P^{n+1} \\ &\quad + \lambda \mathrm{div}\,(\nabla \mathbf{B}^{n+1} \odot \nabla \mathbf{B}^n) = \mathbf{f}^{n+1}, \qquad \mathrm{div}\, \mathbf{U}^{n+1} = 0, \end{aligned} \tag{3.9}$$

with homogeneous boundary condition $\mathbf{U}^{n+1} = 0$ on $\partial\Omega$.

Remark 3.1 For given $\mathbf{U}^n$ with div $\mathbf{U}^n = 0$ and $\mathbf{B}^n$, the existence and uniqueness of the weak solution $\mathbf{B}^{n+1}$ to the linear elliptic problem (3.8) with Neumann boundary condition follows from Lax–Milgram theorem by using $b(\mathbf{U}^n, \mathbf{B}^{n+1}, \mathbf{B}^{n+1}) = 0$. For given $\mathbf{U}^n$, $\mathbf{B}^n$ and $\mathbf{B}^{n+1}$, the existence and uniqueness of the weak solution $(\mathbf{U}^{n+1}, P^{n+1})$ to the linearized Navier–Stokes equations (3.9) with homogeneous boundary condition follows from the classical existence and uniqueness theorem for steady Navier–Stokes problem [35].

Let us denote

$$\mathbf{E}_\mathbf{u}^0 = \mathbf{R}_h^0 \mathbf{U}^0 - \mathbf{U}^0, \quad \mathbf{e}_{\mathbf{u}h}^0 = \mathbf{R}_h^0 \mathbf{U}^0 - \mathbf{U}_h^0, \quad \mathbf{E}_\mathbf{b}^0 = \Pi_h^0 \mathbf{B}^0 - \mathbf{B}^0, \quad \mathbf{e}_{\mathbf{b}h}^0 = \Pi_h^0 \mathbf{B}^0 - \mathbf{B}_h^0,$$

and for $1 \le n \le N$,

$$\begin{aligned} &\mathbf{e}_\mathbf{u}^n = \mathbf{u}^n - \mathbf{U}^n, \quad e_p^n = p^n - P^n, \quad \mathbf{e}_\mathbf{b}^n = \mathbf{b}^n - \mathbf{B}^n, \\ &\mathbf{e}_{\mathbf{u}h}^n = \mathbf{R}_h \mathbf{U}^n - \mathbf{U}_h^n, \quad e_{ph}^n = Q_h P^n - P_h^n, \quad \mathbf{e}_{\mathbf{b}h}^n = \Pi_h^n \mathbf{B}^n - \mathbf{B}_h^n, \\ &\mathbf{E}_\mathbf{u}^n = \mathbf{R}_h \mathbf{U}^n - \mathbf{U}^n, \quad E_p^n = Q_h P^n - P^n, \quad \mathbf{E}_\mathbf{b}^n = \Pi_h^n \mathbf{B}^n - \mathbf{B}^n, \end{aligned}$$

where $\mathbf{R}_h$, Q_h and Π_h^n are projection operators defined in Sect. 5. The proof of (3.6)–(3.7) is based upon the following error splitting in some norm $||\cdot||$:

$$\begin{aligned} ||\mathbf{u}^n - \mathbf{U}_h^n|| &\le ||\mathbf{e}_\mathbf{u}^n|| + ||\mathbf{E}_\mathbf{u}^n|| + ||\mathbf{e}_{\mathbf{u}h}^n||, \\ ||\mathbf{b}^n - \mathbf{B}_h^n|| &\le ||\mathbf{e}_\mathbf{b}^n|| + ||\mathbf{E}_\mathbf{b}^n|| + ||\mathbf{e}_{\mathbf{b}h}^n||, \\ ||p^n - P_h^n|| &\le ||e_p^n|| + ||E_p^n|| + ||e_{ph}^n||. \end{aligned}$$

Here $||\mathbf{e}_\mathbf{u}^n||$, $||\mathbf{e}_\mathbf{b}^n||$, $||e_p^n||$ are temporal errors, and $||\mathbf{e}_{\mathbf{u}h}^n||$, $||\mathbf{e}_{\mathbf{b}h}^n||$, $||e_{ph}^n||$ are spatial errors, and $||\mathbf{E}_\mathbf{u}^n||$, $||\mathbf{E}_\mathbf{b}^n||$, $||E_p^n||$ are projection errors.

Before the proof of Theorem 3.1, we recall the following inverse inequality which holds for $\mathbf{v}_h \in \mathbf{V}_h$ or $\mathbf{v}_h \in \mathbf{X}_h$ [5]:

$$||\mathbf{v}_h||_{W^{l,q_1}} \le Ch^{m-l+2\min\{\frac{1}{q_1}-\frac{1}{q_2},0\}} ||\mathbf{v}_h||_{W^{m,q_2}}, \quad \forall\, 1 \le q_1, q_2 \le \infty,\ 0 \le m \le l. \tag{3.10}$$

Finally, a discrete version of Gronwall's inequality established in [19] is frequently used in this paper.

Lemma 3.1 *Let a_k, b_k, c_k and γ_k, for integers $k \ge 0$, be the nonnegative numbers such that*

$$a_n + \tau \sum_{k=0}^{n} b_k \le \tau \sum_{k=0}^{n} \gamma_k a_k + \tau \sum_{k=0}^{n} c_k + B \qquad \textit{for } n \ge 0. \tag{3.11}$$

Suppose that $\tau\gamma_k < 1$, for all k, and set $\sigma_k = (1 - \tau\gamma_k)^{-1}$. Then

$$a_n + \tau \sum_{k=0}^{n} b_k \le \exp\left(\tau \sum_{k=0}^{n} \gamma_k \sigma_k\right)\left(\tau \sum_{k=0}^{n} c_k + B\right) \qquad \textit{for } n \ge 0. \tag{3.12}$$

Remark 3.2 If the first sum on the right in (3.11) extends only up to $n-1$, then the estimate (3.12) holds for all $\tau > 0$ with $\sigma_k = 1$.

4 Temporal Error Analysis

In this section, we begin to estimate temporal errors $\mathbf{e}_\mathbf{u}^n$, $\mathbf{e}_\mathbf{b}^n$ and $\mathbf{e}_p^n$ for $1 \le n \le N$. Meanwhile, some regularities of solutions $\mathbf{B}^n$ and $(\mathbf{U}^n, P^n)$ to the discrete parabolic system (3.8)–(3.9) are derived. For $0 \le n \le N-1$, we take $t = (n+1)\tau$ at (1.1)–(1.5) to yield

$$\begin{aligned} &D_\tau \mathbf{u}^{n+1} - \mu\Delta\mathbf{u}^{n+1} + (\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} \\ &\quad + \lambda \mathrm{div}\,(\nabla\mathbf{b}^{n+1} \odot \nabla\mathbf{b}^{n+1}) = \mathbf{f}^{n+1} - \mathbf{R}_\mathbf{u}^{n+1}, \qquad (4.1) \\ &D_\tau \mathbf{b}^{n+1} - \gamma\Delta\mathbf{b}^{n+1} + (\mathbf{u}^{n+1}\cdot\nabla)\mathbf{b}^{n+1} = \gamma|\nabla\mathbf{b}^{n+1}|^2\mathbf{b}^{n+1} - \mathbf{R}_\mathbf{b}^{n+1}, \qquad (4.2) \\ &\mathrm{div}\,\mathbf{u}^{n+1} = 0, \quad |\mathbf{b}^{n+1}| = 1, \qquad (4.3) \end{aligned}$$

with boundary conditions $\mathbf{u}^{n+1} = 0$ and $\partial_\mathbf{n}\mathbf{b}^{n+1} = 0$ on $\partial\Omega$, where

$$\mathbf{R}_\mathbf{u}^{n+1} = \frac{1}{\tau}\int_{t_n}^{t_{n+1}} (s-t_n)\partial_{tt}\mathbf{u}(s)ds, \quad \mathbf{R}_\mathbf{b}^{n+1} = \frac{1}{\tau}\int_{t_n}^{t_{n+1}} (s-t_n)\partial_{tt}\mathbf{b}(s)ds.$$

It follows from Hölder's inequality and (2.17) that

$$\tau\sum_{n=0}^{N-1} ||\mathbf{R}_\mathbf{u}^{n+1}||_{L^2}^2 + \tau\sum_{n=0}^{N-1} ||\mathbf{R}_\mathbf{b}^{n+1}||_{L^2}^2 \le C\tau^2. \qquad (4.4)$$

Subtracting (3.8) from (4.2), and (3.9) from (4.1), (4.3) leads to

$$\begin{aligned} D_\tau \mathbf{e}_\mathbf{b}^{n+1} - \gamma\Delta\mathbf{e}_\mathbf{b}^{n+1} &= \gamma(|\nabla\mathbf{b}^{n+1}|^2\mathbf{b}^{n+1} - |\nabla\mathbf{B}^n|^2\mathbf{B}^n) - \mathbf{R}_\mathbf{b}^{n+1} \\ &\quad -((\mathbf{u}^{n+1}\cdot\nabla)\mathbf{b}^{n+1} - (\mathbf{U}^n\cdot\nabla)\mathbf{B}^{n+1}), \qquad (4.5) \end{aligned}$$

and

$$\begin{aligned} &D_\tau \mathbf{e}_\mathbf{u}^{n+1} - \mu\Delta\mathbf{e}_\mathbf{u}^{n+1} + \nabla e_p^{n+1} = -((\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^{n+1} - (\mathbf{U}^n\cdot\nabla)\mathbf{U}^{n+1}) \\ &\quad -\lambda(\mathrm{div}\,(\nabla\mathbf{b}^{n+1}\odot\nabla\mathbf{b}^{n+1}) - \mathrm{div}\,(\nabla\mathbf{B}^{n+1}\odot\nabla\mathbf{B}^n)) - \mathbf{R}_\mathbf{u}^{n+1}, \quad \mathrm{div}\,\mathbf{e}_\mathbf{u}^{n+1} = 0 \qquad (4.6) \end{aligned}$$

with boundary conditions $\mathbf{e}_\mathbf{u}^{n+1} = 0$ and $\partial_\mathbf{n}\mathbf{e}_\mathbf{b}^{n+1} = 0$ on $\partial\Omega$.

First, we prove the following temporal errors.

Lemma 4.1 *Suppose that the solution* $(\boldsymbol{u}, p, \boldsymbol{b})$ *to* (1.1)–(1.5) *satisfies the regularities* (2.10)–(2.14) *and* (2.16)–(2.17) *in* $[0, T]$. *For* $0 \le n \le N-1$, *there exists some* $\tau_1 > 0$ *such that when* $\tau < \tau_1$, *there hold*

$$\max_{0\le m\le n+1}\left(||\boldsymbol{e}_\boldsymbol{b}^m||_{H^1}^2 + ||\nabla\boldsymbol{e}_\boldsymbol{u}^m||_{L^2}^2 + \tau\sum_{k=0}^m ||\boldsymbol{e}_\boldsymbol{b}^k||_{H^2}^2 + \tau\sum_{k=0}^m ||A\boldsymbol{e}_\boldsymbol{u}^k||_{L^2}^2\right) \le \frac{C_0^2}{16}\tau^2, \qquad (4.7)$$

$$\max_{1\le m\le n+1}\left(||D_\tau\boldsymbol{e}_\boldsymbol{b}^m||_{H^1}^2 + ||\boldsymbol{e}_\boldsymbol{b}^m||_{H^2}^2 + \tau\sum_{k=1}^m ||D_\tau\boldsymbol{e}_\boldsymbol{b}^k||_{H^2}^2\right) \le C_1 \qquad (4.8)$$

$$\max_{1\le m\le n+1}\left(||\nabla D_\tau\boldsymbol{e}_\boldsymbol{u}^m||_{L^2}^2 + \tau\sum_{k=1}^m ||AD_\tau\boldsymbol{e}_\boldsymbol{u}^k||_{L^2}^2\right) \le C_1. \qquad (4.9)$$

Proof Due to $\mathbf{u}^0 = \mathbf{U}^0 = \mathbf{u}_0$ and $\mathbf{b}^0 = \mathbf{B}^0 = \mathbf{b}_0$, the inequalities (4.7)–(4.9) obviously hold for $m = 0$. Suppose that (4.7)–(4.9) hold for $m \le n$, then we need to show that these inequalities also hold for $m \le n+1$. Multiplying (4.5) by $\mathbf{e}_\mathbf{b}^{n+1}$ and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}D_\tau||\mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2 + \frac{1}{2\tau}||\mathbf{e}_\mathbf{b}^{n+1} - \mathbf{e}_\mathbf{b}^{n}||_{L^2}^2 + \gamma||\nabla\mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2 \\
&\quad \le \gamma|(|\nabla\mathbf{b}^{n+1}|^2\mathbf{b}^{n+1} - |\nabla\mathbf{B}^n|^2\mathbf{B}^n, \mathbf{e}_\mathbf{b}^{n+1})| + |(\mathbf{R}_\mathbf{b}^{n+1}, \mathbf{e}_\mathbf{b}^{n+1})| \\
&\qquad + |b(\mathbf{u}^{n+1}, \mathbf{b}^{n+1}, \mathbf{e}_\mathbf{b}^{n+1}) - b(\mathbf{U}^n, \mathbf{B}^{n+1}, \mathbf{e}_\mathbf{b}^{n+1})| = I_1 + I_2 + I_3.
\end{aligned} \tag{4.10}$$

Rewrite $|\nabla\mathbf{b}^{n+1}|^2\mathbf{b}^{n+1} - |\nabla\mathbf{B}^n|^2\mathbf{B}^n$ as

$$\begin{aligned}
&|\nabla\mathbf{b}^{n+1}|^2\mathbf{b}^{n+1} - |\nabla\mathbf{B}^n|^2\mathbf{B}^n \\
&\quad = (\nabla\mathbf{b}^{n+1} - \nabla\mathbf{b}^n)\cdot(\nabla\mathbf{b}^{n+1} + \nabla\mathbf{b}^n)\mathbf{b}^{n+1} + |\nabla\mathbf{b}^n|^2(\mathbf{b}^{n+1} - \mathbf{b}^n) \\
&\qquad + |\nabla\mathbf{b}^n|^2\mathbf{e}_\mathbf{b}^n + 2(\nabla\mathbf{e}_b^n\cdot\nabla\mathbf{b}^n)\mathbf{b}^n - 2(\nabla\mathbf{e}_b^n\cdot\nabla\mathbf{b}^n)\mathbf{e}_\mathbf{b}^n + |\nabla\mathbf{e}_\mathbf{b}^n|^2\mathbf{e}_\mathbf{b}^n - |\nabla\mathbf{e}_\mathbf{b}^n|^2\mathbf{b}^n.
\end{aligned}$$

By using (2.12), (2.13), (2.6), (4.7) for $m \le n$, Hölder's inequality and Young's inequality, I_1 is bounded by

$$\begin{aligned}
I_1 &\le \gamma(||\nabla\mathbf{b}^{n+1} + \nabla\mathbf{b}^n||_{L^\infty} + ||\nabla\mathbf{b}^n||_{L^\infty}^2)||\mathbf{b}^{n+1} - \mathbf{b}^n||_{H^1}||\mathbf{e}_\mathbf{b}^{n+1}||_{L^2} \\
&\quad + \gamma(||\nabla\mathbf{b}^n||_{L^\infty}^2||\mathbf{e}_\mathbf{b}^n||_{L^2} + ||\nabla\mathbf{b}^n||_{L^\infty}||\mathbf{b}^n||_{L^\infty}||\nabla\mathbf{e}_\mathbf{b}^n||_{L^2})||\mathbf{e}_\mathbf{b}^{n+1}||_{L^2} \\
&\quad + \gamma||\nabla\mathbf{b}^n||_{L^\infty}||\nabla\mathbf{e}_\mathbf{b}^n||_{L^2}||\mathbf{e}_\mathbf{b}^n||_{H^2}||\mathbf{e}_\mathbf{b}^{n+1}||_{L^2} \\
&\quad + \gamma(||\mathbf{e}_\mathbf{b}^n||_{H^2} + ||\mathbf{b}^n||_{L^\infty})||\nabla\mathbf{e}_\mathbf{b}^n||_{L^4}^2||\mathbf{e}_\mathbf{b}^{n+1}||_{L^2} \\
&\le C_2(||\mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2 + ||\mathbf{e}_\mathbf{b}^n||_{L^2}^2 + \tau^2) + \frac{\gamma}{4}||\mathbf{e}_\mathbf{b}^n||_{H^1}^2 + \frac{C_2C_0^2}{8}\tau^2||\mathbf{e}_\mathbf{b}^n||_{H^2}^2,
\end{aligned}$$

where $C_2 > 0$ is independent of C_0. Moreover, we use

$$||\mathbf{b}^{n+1} - \mathbf{b}^n||_{H^1}^2 = ||\int_{t_n}^{t_{n+1}}\mathbf{b}_t(t)dt||_{H^1}^2 \le \tau^2||\mathbf{b}_t||_{L^\infty(0,T;H^1(\Omega))}^2. \tag{4.11}$$

From Hölder's inequality and Young's inequality again, we estimate I_2 as

$$I_2 \le C_2(||\mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2 + ||\mathbf{R}_\mathbf{b}^{n+1}||_{L^2}^2).$$

An alternative to I_3 is

$$I_3 = |b(\mathbf{u}^{n+1} - \mathbf{u}^n, \mathbf{b}^{n+1}, \mathbf{e}_\mathbf{b}^{n+1}) + b(\mathbf{u}^n - \mathbf{U}^n, \mathbf{b}^{n+1}, \mathbf{e}_\mathbf{b}^{n+1})|.$$

Then, from (2.11) and (2.12), we get

$$\begin{aligned}
I_3 &\le (||\mathbf{u}^{n+1} - \mathbf{u}^n||_{L^2} + ||\mathbf{e}_\mathbf{u}^n||_{L^2})||\nabla\mathbf{b}^{n+1}||_{L^\infty}||\mathbf{e}_\mathbf{b}^{n+1}||_{L^2} \\
&\le C_2(||\mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2 + ||\mathbf{e}_\mathbf{u}^n||_{L^2}^2 + \tau^2),
\end{aligned}$$

where we use

$$||\mathbf{u}^{n+1} - \mathbf{u}^n||_{L^2}^2 = ||\int_{t_n}^{t_{n+1}}\mathbf{u}_t(t)dt||_{L^2}^2 \le \tau^2||\mathbf{u}_t||_{L^\infty(0,T;H)}^2. \tag{4.12}$$

Then combining these estimates for I_1, I_2 and I_3 into (4.10), we obtain

$$
\begin{aligned}
&D_\tau ||\mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2 + \frac{1}{\tau}||\mathbf{e}_\mathbf{b}^{n+1} - \mathbf{e}_\mathbf{b}^n||_{L^2}^2 + \gamma||\nabla \mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2 \\
&\quad \le C_2(||\mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2 + ||\mathbf{e}_\mathbf{b}^n||_{L^2}^2 + ||\mathbf{e}_\mathbf{u}^n||_{L^2}^2 + ||\mathbf{R}_\mathbf{b}^{n+1}||_{L^2}^2 + \tau^2) + \frac{\gamma}{2}||\mathbf{e}_\mathbf{b}^n||_{H^1}^2 \\
&\qquad + \frac{C_2C_0^2}{4}\tau^2||\mathbf{e}_\mathbf{b}^n||_{H^2}^2.
\end{aligned} \tag{4.13}
$$

Testing (4.6) by $\mathbf{e}_\mathbf{u}^{n+1}$ yields

$$
\begin{aligned}
&\frac{1}{2}D_\tau ||\mathbf{e}_\mathbf{u}^{n+1}||_{L^2}^2 + \frac{1}{2\tau}||\mathbf{e}_\mathbf{u}^{n+1} - \mathbf{e}_\mathbf{u}^n||_{L^2}^2 + \mu||\nabla \mathbf{e}_\mathbf{u}^{n+1}||_{L^2}^2 \\
&\quad \le |b(\mathbf{U}^n, \mathbf{U}^{n+1}, \mathbf{e}_\mathbf{u}^{n+1}) - b(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{e}_\mathbf{u}^{n+1})| + |(\mathbf{R}_\mathbf{u}^{n+1}, \mathbf{e}_\mathbf{u}^{n+1})| \\
&\qquad + |\lambda(\nabla \mathbf{b}^{n+1} \odot \nabla \mathbf{b}^{n+1} - \nabla \mathbf{B}^{n+1} \odot \nabla \mathbf{B}^n, \nabla \mathbf{e}_\mathbf{u}^{n+1})| = I_4 + I_5 + I_6.
\end{aligned} \tag{4.14}
$$

From Hölder's inequality and Young's inequality, I_4 satisfies

$$
\begin{aligned}
I_4 &= |b(\mathbf{e}_\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{e}_\mathbf{u}^{n+1}) - b(\mathbf{e}_\mathbf{u}^n, \mathbf{u}^{n+1}, \mathbf{e}_\mathbf{u}^{n+1})| \\
&\le (||\mathbf{e}_\mathbf{u}^n||_{L^2}||\mathbf{u}^{n+1}||_{W^{2,4}} + ||\mathbf{e}_\mathbf{u}^{n+1}||_{L^2}||\nabla \mathbf{u}^{n+1}||_{L^2})||\nabla \mathbf{e}_\mathbf{u}^{n+1}||_{L^2} \\
&\le \frac{\mu}{4}||\nabla \mathbf{e}_\mathbf{u}^{n+1}||_{L^2}^2 + C_2||\mathbf{e}_\mathbf{u}^{n+1}||_{L^2}^2 + C_2||\mathbf{e}_\mathbf{u}^n||_{L^2}^2||\mathbf{u}^{n+1}||_{W^{2,4}}^2.
\end{aligned}
$$

Similarly, it is easy to show that

$$
I_5 \le C_2(||\mathbf{e}_\mathbf{u}^{n+1}||_{L^2}^2 + ||\mathbf{R}_\mathbf{u}^{n+1}||_{L^2}^2).
$$

We rewrite I_6 as

$$
\begin{aligned}
I_6 = &|\lambda(\nabla \mathbf{e}_\mathbf{b}^{n+1} \odot \nabla \mathbf{b}^{n+1}, \nabla \mathbf{e}_\mathbf{u}^{n+1}) + \lambda(\nabla \mathbf{e}_\mathbf{b}^{n+1} \odot \nabla(\mathbf{b}^{n+1} - \mathbf{b}^n), \nabla \mathbf{e}_\mathbf{u}^{n+1}) \\
&- \lambda(\nabla \mathbf{b}^{n+1} \odot \nabla(\mathbf{b}^{n+1} - \mathbf{b}^n), \nabla \mathbf{e}_\mathbf{u}^{n+1}) + \lambda(\nabla \mathbf{e}_\mathbf{b}^{n+1} \odot \nabla \mathbf{e}_\mathbf{b}^n, \nabla \mathbf{e}_\mathbf{u}^{n+1}) \\
&- \lambda(\nabla \mathbf{b}^{n+1} \odot \nabla \mathbf{e}_\mathbf{b}^n, \nabla \mathbf{e}_\mathbf{u}^{n+1})|
\end{aligned}
$$

Then I_6 is bounded by

$$
\begin{aligned}
I_6 \le\ &\lambda||\nabla \mathbf{e}_\mathbf{b}^{n+1}||_{L^2}||\nabla \mathbf{b}^{n+1}||_{L^\infty}||\nabla \mathbf{e}_\mathbf{u}^{n+1}||_{L^2} + \lambda||\nabla \mathbf{e}_\mathbf{b}^{n+1}||_{L^4}||\nabla(\mathbf{b}^{n+1} - \mathbf{b}^n)||_{L^4}||\nabla \mathbf{e}_\mathbf{u}^{n+1}||_{L^2} \\
&+ \lambda||\nabla \mathbf{b}^{n+1}||_{L^\infty}||\nabla(\mathbf{b}^{n+1} - \mathbf{b}^n)||_{L^2}||\nabla \mathbf{e}_\mathbf{u}^{n+1}||_{L^2} + \lambda||\nabla \mathbf{e}_\mathbf{b}^{n+1}||_{L^4}||\nabla \mathbf{e}_\mathbf{b}^n||_{L^4}||\nabla \mathbf{e}_\mathbf{u}^{n+1}||_{L^2} \\
&+ \lambda||\nabla \mathbf{b}^{n+1}||_{L^\infty}||\nabla \mathbf{e}_\mathbf{b}^n||_{L^2}||\nabla \mathbf{e}_\mathbf{u}^{n+1}||_{L^2} \\
\le\ &\frac{\mu}{4}||\nabla \mathbf{e}_\mathbf{u}^{n+1}||_{L^2}^2 + C_2(||\nabla \mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2 + ||\nabla \mathbf{e}_\mathbf{b}^n||_{L^2}^2 + ||\nabla(\mathbf{b}^{n+1} - \mathbf{b}^n)||_{L^2}^2) \\
&+ C_2||\nabla(\mathbf{b}^{n+1} - \mathbf{b}^n)||_{L^2}^2||\mathbf{e}_\mathbf{b}^{n+1}||_{H^2}^2 + C_2||\mathbf{b}^{n+1} - \mathbf{b}^n||_{H^2}^2||\nabla \mathbf{b}^{n+1}||_{L^2}^2 \\
&+ C_2||\nabla \mathbf{e}_\mathbf{b}^n||_{L^2}||\mathbf{e}_\mathbf{b}^n||_{H^2}||\mathbf{e}_\mathbf{b}^{n+1}||_{H^2}^2 \\
\le\ &\frac{\mu}{4}||\nabla \mathbf{e}_\mathbf{u}^{n+1}||_{L^2}^2 + C_2(||\nabla \mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2 + ||\nabla \mathbf{e}_\mathbf{b}^n||_{L^2}^2 + \tau^2) \\
&+ C_2\tau^2||\mathbf{e}_\mathbf{b}^{n+1}||_{H^2}^2 + C_2\tau||\nabla \mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2\int_{t_n}^{t_{n+1}}||\mathbf{b}_t(t)||_{H^2}^2 dt + \frac{C_0C_1C_2}{4}\tau||\mathbf{e}_\mathbf{b}^{n+1}||_{H^2}^2,
\end{aligned}
$$

where we use (4.7)–(4.8) for $m \le n$ and

$$
||\mathbf{b}^{n+1} - \mathbf{b}^n||_{H^2}^2 = ||\int_{t_n}^{t_{n+1}} \mathbf{b}_t(t)dt||_{H^2}^2 \le \tau\int_{t_n}^{t_{n+1}}||\mathbf{b}_t(t)||_{H^2}^2 dt. \tag{4.15}
$$

Combining these estimates into (4.14), we obtain

$$
\begin{aligned}
&D_\tau||\mathbf{e}_\mathbf{u}^{n+1}||_{L^2}^2+\frac{1}{\tau}||\mathbf{e}_\mathbf{u}^{n+1}-\mathbf{e}_\mathbf{u}^{n}||_{L^2}^2+\mu||\nabla\mathbf{e}_\mathbf{u}^{n+1}||_{L^2}^2\\
&\quad\le C_2||\mathbf{e}_\mathbf{u}^{n+1}||_{L^2}^2+C_2(||\mathbf{e}_\mathbf{u}^{n}||_{L^2}^2||\mathbf{u}^{n+1}||_{W^{2,4}}^2+||\mathbf{R}_\mathbf{u}^{n+1}||_{L^2}^2+||\nabla\mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2+||\nabla\mathbf{e}_\mathbf{b}^{n}||_{L^2}^2+\tau^2)\\
&\qquad+C_2\tau^2||\mathbf{e}_\mathbf{b}^{n+1}||_{H^2}^2+C_2\tau||\nabla\mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2\int_{t_n}^{t_{n+1}}||\mathbf{b}_t(t)||_{H^2}^2dt+\frac{C_0C_1C_2}{2}\tau||\mathbf{e}_\mathbf{b}^{n+1}||_{H^2}^2.
\end{aligned}
\tag{4.16}
$$

Testing (4.5) by $-\Delta\mathbf{e}_\mathbf{b}^{n+1}$ and using a similar argument for (4.13), we can get

$$
\begin{aligned}
&D_\tau||\nabla\mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2+\gamma||\Delta\mathbf{e}_\mathbf{b}^{n+1}||_{L^2}^2\\
&\quad\le C_2(||\mathbf{e}_\mathbf{b}^{n}||_{L^2}^2+||\mathbf{e}_\mathbf{u}^{n}||_{L^2}^2+||\mathbf{R}_\mathbf{b}^{n+1}||_{L^2}+\tau^2)+\frac{\gamma}{2}||\mathbf{e}_\mathbf{b}^{n}||_{H^1}^2\\
&\qquad+\frac{C_2C_0^2}{4}\tau^2||\mathbf{e}_\mathbf{b}^{n}||_{H^2}^2.
\end{aligned}
\tag{4.17}
$$

Taking sufficiently small τ to satisfy

$$\tau\max\{4C_2||\mathbf{b}_t||_{L^2(0,T;H^2)},2C_0C_1C_2,C_2C_0^2\tau\}<\gamma,$$

and summing up the inequalities (4.13), (4.16), (4.17) and using discrete Gronwall's inequality (see Lemma 5.1 in [19]), there exists some $C_3>0$ and $\tau_{11}>0$ such that when $\tau<\tau_{11}$, there holds

$$||\mathbf{e}_\mathbf{b}^{n+1}||_{H^1}^2+||\mathbf{e}_\mathbf{u}^{n+1}||_{L^2}^2+\tau\sum_{k=0}^{n+1}||\mathbf{e}_\mathbf{b}^{k}||_{H^2}^2+\tau\sum_{k=0}^{n+1}||\nabla\mathbf{e}_\mathbf{u}^{k}||_{L^2}^2\le\exp(2TC_3)\tau^2. \tag{4.18}$$

Then temporal errors of the director in (4.7) hold if we choose $C_0>4\exp(TC_3)$.

For $1\le k\le N$, by the definition $\mathbf{e}_\mathbf{b}^k$, we have

$$||D_\tau\mathbf{B}^k||_{H^i}\le||D_\tau\mathbf{e}_\mathbf{b}^k||_{H^i}+||D_\tau\mathbf{b}^k||_{H^i},\quad i=1,2.$$

Following (2.13) and (4.18), we get

$$\max_{1\le n\le N}\left(||D_\tau\mathbf{B}^n||_{H^1}+||\mathbf{B}^n||_{H^2}+||\nabla\mathbf{U}^n||_{L^2}\right)\le C.$$

For $0\le n\le N-1$, rewrite (3.8) as

$$-\gamma\Delta\mathbf{B}^{n+1}=\mathbf{F}^n,$$

where $\mathbf{F}^n=\gamma|\nabla\mathbf{B}^n|^2\mathbf{B}^n-(\mathbf{U}^n\cdot\nabla)\mathbf{B}^{n+1}-D_\tau\mathbf{B}^{n+1}$. Then we have

$$
\begin{aligned}
||\mathbf{F}^n||_{L^4}&\le\gamma|||\nabla\mathbf{B}^n|^2\mathbf{B}^n||_{L^4}+||(\mathbf{U}^n\cdot\nabla)\mathbf{B}^{n+1}||_{L^4}+||D_\tau\mathbf{B}^{n+1}||_{L^4}\\
&\le C||\nabla\mathbf{B}^n||_{L^6}||\nabla\mathbf{B}^n||_{L^{12}}||\mathbf{B}^n||_{L^\infty}+C||\mathbf{U}^n||_{L^6}||\nabla\mathbf{B}^{n+1}||_{L^{12}}\\
&\quad+C||D_\tau\mathbf{B}^{n+1}||_{H^1}\le C,
\end{aligned}
$$

where we use Sobolev imbedding $\mathbf{H}^2(\Omega)\subset\mathbf{W}^{1,p}(\Omega)$ for any $1\le p<+\infty$. Thus, by classical regularity theory of elliptic problem, we have $||\mathbf{B}^{n+1}||_{W^{2,4}}\le C$, which implies $||\mathbf{B}^{n+1}||_{W^{1,\infty}}\le C$ if we use Sobolev imbedding $\mathbf{W}^{2,4}(\Omega)\subset\mathbf{W}^{1,\infty}(\Omega)$. In this case, $\nabla\mathbf{F}^n$ in $\mathbf{L}^2$-norm can be bounded by

$$
\begin{aligned}
||\nabla \mathbf{F}^n||_{L^2} &\le \gamma |||\nabla(\nabla \mathbf{B}^n|^2 \mathbf{B}^n)||_{L^2} + ||\nabla((\mathbf{U}^n \cdot \nabla)\mathbf{B}^{n+1})||_{L^2} + ||\nabla D_\tau \mathbf{B}^{n+1}||_{L^2} \\
&\le C||\mathbf{B}^n||^3_{W^{1,6}} + C||\mathbf{B}^n||_{H^2}||\nabla \mathbf{B}^n||_{L^\infty}||\mathbf{B}^n||_{L^\infty} + C||\nabla \mathbf{U}^n||_{L^2}||\nabla \mathbf{B}^{n+1}||_{L^\infty} \\
&\quad + C||\mathbf{U}^n||_{L^4}||\mathbf{B}^{n+1}||_{W^{2,4}} + C||D_\tau \mathbf{B}^{n+1}||_{H^1} \le C.
\end{aligned}
$$

Thus $\mathbf{F}^n \in \mathbf{H}^1(\Omega)$. From classical regularity theory of elliptic problem, again, we obtain

$$
\max_{1\le n\le N} ||\mathbf{B}^n||_{H^3} \le C. \tag{4.19}
$$

The above estimate with (2.12) implies that

$$
\max_{1\le n\le N} ||\mathbf{e}^n_{\mathbf{b}}||_{H^3} \le C. \tag{4.20}
$$

Testing (4.6) by $A\mathbf{e}^{n+1}_{\mathbf{u}}$, we get

$$
\begin{aligned}
&\frac{1}{2} D_\tau ||\nabla \mathbf{e}^{n+1}_{\mathbf{u}}||^2_{L^2} + \frac{1}{2\tau}||\nabla \mathbf{e}^{n+1}_{\mathbf{u}} - \nabla \mathbf{e}^n_{\mathbf{u}}||^2_{L^2} + \mu ||A\mathbf{e}^{n+1}_{\mathbf{u}}||^2_{L^2} \\
&\quad \le |b(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, A\mathbf{e}^{n+1}_{\mathbf{u}}) - b(\mathbf{U}^n, \mathbf{U}^{n+1}, A\mathbf{e}^{n+1}_{\mathbf{u}})| + |(\mathbf{R}^{n+1}_{\mathbf{u}}, A\mathbf{e}^{n+1}_{\mathbf{u}})| \\
&\qquad + |\lambda(\mathrm{div}\,(\nabla \mathbf{b}^{n+1} \odot \nabla \mathbf{b}^{n+1}) - \mathrm{div}\,(\nabla \mathbf{B}^{n+1} \odot \nabla \mathbf{B}^n), A\mathbf{e}^{n+1}_{\mathbf{u}})| \\
&\quad = I_7 + I_8 + I_9.
\end{aligned} \tag{4.21}
$$

An alternative to I_7 is

$$
\begin{aligned}
I_7 = |&b(\mathbf{u}^{n+1}, \mathbf{e}^{n+1}_{\mathbf{u}}, A\mathbf{e}^{n+1}_{\mathbf{u}}) - b(\mathbf{u}^{n+1} - \mathbf{u}^n, \mathbf{e}^{n+1}_{\mathbf{u}}, A\mathbf{e}^{n+1}_{\mathbf{u}}) + b(\mathbf{u}^{n+1} - \mathbf{u}^n, \mathbf{u}^{n+1}, A\mathbf{e}^{n+1}_{\mathbf{u}}) \\
&- b(\mathbf{e}^n_{\mathbf{u}}, \mathbf{e}^{n+1}_{\mathbf{u}}, A\mathbf{e}^{n+1}_{\mathbf{u}}) + b(\mathbf{e}^n_{\mathbf{u}}, \mathbf{u}^{n+1}, A\mathbf{e}^{n+1}_{\mathbf{u}})|,
\end{aligned}
$$

which can be bounded by

$$
\begin{aligned}
I_7 &\le (||\mathbf{u}^{n+1}||_{L^\infty} + ||\mathbf{u}^n||_{L^\infty})||\nabla \mathbf{e}^{n+1}_{\mathbf{u}}||_{L^2}||A\mathbf{e}^{n+1}_{\mathbf{u}}||_{L^2} \\
&\quad + ||\nabla \mathbf{u}^{n+1}||_{L^\infty}||\mathbf{u}^{n+1} - \mathbf{u}^n||_{L^2}||A\mathbf{e}^{n+1}_{\mathbf{u}}||_{L^2} \\
&\quad + ||\mathbf{e}^n_{\mathbf{u}}||_{L^4}||\nabla \mathbf{e}^{n+1}_{\mathbf{u}}||_{L^4}||A\mathbf{e}^{n+1}_{\mathbf{u}}||_{L^2} + ||\nabla \mathbf{u}^{n+1}||_{L^\infty}||\mathbf{e}^n_{\mathbf{u}}||_{L^2}||A\mathbf{e}^{n+1}_{\mathbf{u}}||_{L^2} \\
&\le \frac{\mu}{4}||A\mathbf{e}^{n+1}_{\mathbf{u}}||^2_{L^2} + C_4(||\nabla \mathbf{e}^{n+1}_{\mathbf{u}}||^2_{L^2} + ||\mathbf{u}^{n+1} - \mathbf{u}^n||^2_{L^2} + ||\mathbf{e}^n_{\mathbf{u}}||^2_{L^2}) \\
&\quad + C_4(||\mathbf{e}^n_{\mathbf{u}}||^2_{L^2}||\nabla \mathbf{e}^{n+1}_{\mathbf{u}}||^2_{L^2} + ||\nabla \mathbf{e}^n_{\mathbf{u}}||^2_{L^2}||\mathbf{e}^{n+1}_{\mathbf{u}}||^2_{L^2}).
\end{aligned}
$$

From Young's inequality, we bound I_8 as

$$
I_8 \le \frac{\mu}{4}||A\mathbf{e}^{n+1}_{\mathbf{u}}||^2_{L^2} + C_4||\mathbf{R}^{n+1}_{\mathbf{u}}||^2_{L^2}.
$$

An alternative to I_9 is

$$
\begin{aligned}
I_9 &= \lambda(\mathrm{div}\,(\nabla \mathbf{e}^{n+1}_{\mathbf{b}} \odot \nabla \mathbf{b}^{n+1}), A\mathbf{e}^{n+1}_{\mathbf{u}}) + \lambda(\mathrm{div}\,(\nabla \mathbf{e}^{n+1}_{\mathbf{b}} \odot \nabla(\mathbf{b}^{n+1} - \mathbf{b}^n)), A\mathbf{e}^{n+1}_{\mathbf{u}}) \\
&\quad - \lambda(\mathrm{div}\,(\nabla \mathbf{b}^{n+1} \odot \nabla(\mathbf{b}^{n+1} - \mathbf{b}^n)), A\mathbf{e}^{n+1}_{\mathbf{u}}) + \lambda(\mathrm{div}\,(\nabla \mathbf{e}^{n+1}_{\mathbf{b}} \odot \nabla \mathbf{e}^n_{\mathbf{b}}), A\mathbf{e}^{n+1}_{\mathbf{u}}) \\
&\quad - \lambda(\mathrm{div}\,(\nabla \mathbf{b}^{n+1} \odot \nabla \mathbf{e}^n_{\mathbf{b}}), A\mathbf{e}^{n+1}_{\mathbf{u}})| \\
&= I_9^1 + I_9^2 + I_9^3 + I_9^4 + I_9^5.
\end{aligned}
$$

By Hölder's inequality and (4.20), all terms in the right-hand side of the above inequality can be estimated, respectively, by

$$
\begin{aligned}
I_9^1 &\le C_4(||\mathbf{e_b}^{n+1}||_{H^2}||\mathbf{b}^{n+1}||_{H^3} + ||\nabla \mathbf{e_b}^{n+1}||_{L^2}||\mathbf{b}^{n+1}||_{H^4})||A\mathbf{e_u}^{n+1}||_{L^2},\\
I_9^2 &\le C_4(||\mathbf{e_b}^{n+1}||_{H^2}||\mathbf{b}^{n+1} - \mathbf{b}^n||_{H^3} + ||\nabla \mathbf{e_b}^{n+1}||_{L^2}||\mathbf{b}^{n+1} - \mathbf{b}^n||_{H^4})||A\mathbf{e_u}^{n+1}||_{L^2},\\
I_9^3 &\le C_4(||\mathbf{b}^{n+1}||_{H^4}||\nabla \mathbf{b}^{n+1} - \nabla \mathbf{b}^n||_{L^2} + ||\mathbf{b}^{n+1}||_{H^3}||\mathbf{b}^{n+1} - \mathbf{b}^n||_{H^2})||A\mathbf{e_u}^{n+1}||_{L^2},\\
I_9^4 &\le C_4(||\mathbf{e_b}^{n+1}||_{H^2}||\mathbf{e_b}^{n}||_{H^3} + ||\mathbf{e_b}^{n}||_{H^2}||\mathbf{e_b}^{n+1}||_{H^3})||A\mathbf{e_u}^{n+1}||_{L^2},\\
I_9^5 &\le C_4(||\mathbf{e_b}^{n}||_{H^2}||\mathbf{b}^{n+1}||_{H^3} + ||\nabla \mathbf{e_b}^{n}||_{L^2}||\mathbf{b}^{n+1}||_{H^4})||A\mathbf{e_u}^{n+1}||_{L^2}.
\end{aligned}
$$

Combining these estimates into (4.21), and using Young's inequality and the discrete Gronwall's inequality, we conclude that there exists some $C_5 > 0$ and $\tau_{12} > 0$ such that when $\tau < \tau_{12}$, there holds

$$
||\nabla \mathbf{e_u}^{n+1}||_{L^2}^2 + \tau \sum_{k=0}^{n+1} ||A\mathbf{e_u}^{k}||_{L^2}^2 \le \exp(2TC_5)\tau^2. \tag{4.22}
$$

Thus, (4.7) holds if we choose $C_0 > 4\max\{\exp(TC_3), \exp(TC_5)\}$. As a direct result of (4.7), we have

$$
\max_{0\le m\le n+1} ||D_\tau \mathbf{e_b}^{m}||_{L^2} \le C. \tag{4.23}
$$

From the arguments in estimating I_1 to I_3, we have

$$
||\gamma(|\nabla \mathbf{b}^{n+1}|^2\mathbf{b}^{n+1} - |\nabla \mathbf{B}^n|^2\mathbf{B}^n) - \mathbf{R_b}^{n+1} - ((\mathbf{u}^{n+1}\cdot\nabla)\mathbf{b}^{n+1} - (\mathbf{U}^n\cdot\nabla)\mathbf{B}^{n+1})||_{L^2} \le C.
$$

In terms of (4.23) and $\mathbf{H}^2$ regularity for linear elliptic problem, it can be shown $||\mathbf{e_b}^{n+1}||_{H^2} \le C_1$. Thus, we complete the proof of

$$
\max_{0\le m\le n+1} ||\mathbf{e_b}^{m}||_{H^2} \le C_1, \quad \max_{0\le m\le n+1} ||\mathbf{B}^m||_{H^2} = \max_{0\le m\le n+1} ||\mathbf{b}^m - \mathbf{e_b}^{m}||_{H^2} \le C. \tag{4.24}
$$

Other estimates in (4.8) and (4.9) are from (4.7) by a simple calculation. □

For $0 \le n \le N-1$, in order to estimate $||e_p^{n+1}||_{L^2}$ by the inf-sup condition, we need to estimate $||D_\tau \mathbf{e_u}^{n+1}||_{L^2}$. Testing (4.6) by $\tau D_\tau \mathbf{e_u}^{n+1}$ and using a similar proof for (4.21), we can easily obtain

$$
\tau \sum_{n=0}^{N-1} ||D_\tau \mathbf{e_u}^{n+1}||_{L^2}^2 \le C\tau^2. \tag{4.25}
$$

As a direct consequence of (4.7) and inf-sup condition, we immediately obtain

$$
\tau \sum_{n=0}^{N-1} ||e_p^{n+1}||_{L^2}^2 \le C\tau^2. \tag{4.26}
$$

Other regularities of $\mathbf{B}^n$ and $(\mathbf{U}^n, P^n)$ are derived in next lemma.

Lemma 4.2 *Under the assumptions in Lemma 4.1, we have*

$$\max_{1\le m\le N}\left(\tau\sum_{k=1}^{m}||D_\tau \boldsymbol{B}^k||_{H^2}^2\right)\le C, \tag{4.27}$$

$$\max_{1\le m\le N}\left(||\nabla D_\tau \boldsymbol{U}^m||_{L^2}+\tau\sum_{k=1}^{m}||AD_\tau \boldsymbol{U}^k||_{L^2}^2\right)\le C, \tag{4.28}$$

$$\max_{0\le m\le N}\left(||A\boldsymbol{U}^m||_{L^2}+||P^m||_{H^1}+\tau\sum_{k=0}^{m}||A\boldsymbol{U}^k||_{L^4}^2\right)\le C. \tag{4.29}$$

Proof Following (2.13) and (4.8), we get

$$\max_{1\le m\le N}\left(\tau\sum_{k=0}^{m}||D_\tau \mathbf{B}^k||_{H^2}^2\right)\le C.$$

This completes the proof of (4.27). The proof of (4.28) can be easily completed by using (2.16) and (4.9). From (4.19), we have

$$\begin{aligned}||\text{div}\,(\nabla\mathbf{B}^{n+1}\odot\nabla\mathbf{B}^n)||_{L^2}&\le C||\nabla^2\mathbf{B}^{n+1}||_{L^2}||\nabla\mathbf{B}^n||_{L^\infty}+C||\nabla^2\mathbf{B}^n||_{L^2}||\nabla\mathbf{B}^{n+1}||_{L^\infty}\\&\le C||\mathbf{B}^{n+1}||_{H^3}||\mathbf{B}^n||_{H^3}\le C,\end{aligned}$$

and

$$\begin{aligned}||\text{div}\,(\nabla\mathbf{B}^{n+1}\odot\nabla\mathbf{B}^n)||_{L^4}&\le C||\text{div}\,(\nabla\mathbf{B}^{n+1}\odot\nabla\mathbf{B}^n)||_{L^2}\\&\quad+C||\text{div}\,(\nabla\mathbf{B}^{n+1}\odot\nabla\mathbf{B}^n)||_{L^2}^{1/2}||\nabla(\text{div}\,(\nabla\mathbf{B}^{n+1}\odot\nabla\mathbf{B}^n))||_{L^2}^{1/2}\\&\le C+||\nabla(\text{div}\,(\nabla\mathbf{B}^{n+1}\odot\nabla\mathbf{B}^n))||_{L^2}^{1/2}\le C\\&\quad+||\mathbf{B}^{n+1}||_{H^3}||\mathbf{B}^{n+1}||_{H^3}\le C,\end{aligned}$$

By using $\mathbf{f}\in\mathbf{L}^\infty(0,T;\mathbf{H})\cap\mathbf{L}^2(0,T;\mathbf{L}^4(\Omega))$, we derive

$$\max_{1\le m\le N}\left(||A\mathbf{U}^m||_{L^2}+\tau\sum_{k=0}^{m}||A\mathbf{U}^k||_{L^4}^2\right)\le C$$

from the regularity result for steady Navier–Stokes equations. For $0\le n\le N-1$, the estimate for P^{n+1} is derived from (3.9) by

$$\begin{aligned}||\nabla P^{n+1}||_{L^2}&\le C||A\mathbf{U}^{n+1}||_{L^2}+C||D_\tau\mathbf{U}^{n+1}||_{L^2}+C||A\mathbf{U}^n||_{L^2}||\nabla\mathbf{U}^{n+1}||_{L^2}\\&\quad+C||\text{div}\,(\nabla\mathbf{B}^{n+1}\odot\nabla\mathbf{B}^n)||_{L^2}+C||\mathbf{f}^{n+1}||_{L^2}\le C.\end{aligned}$$

Observing $||p||_{L^2}\le C||\nabla p||_{L^2}$ for any $p\in M$, we complete the proof of this lemma. □

5 Spatial Error Analysis

In this section, for $1\le n\le N$, we begin to estimate spatial errors $\mathbf{e}_{\mathbf{u}h}^n$, $\mathbf{e}_{\mathbf{b}h}^n$ and $\mathbf{e}_{ph}^n$ under the regularities of $\mathbf{U}^n$, $\mathbf{B}^n$ and P^n derived in Lemma 4.2. In order to derive the spatial error estimates, we need to introduce the following projection $(\mathbf{R}_h, Q_h):\mathbf{V}\times M\longrightarrow\mathbf{V}_h\times M_h$ defined by

$$\mathcal{B}_h(\mathbf{R}_h\mathbf{w},Q_hr;\mathbf{w}_h,r_h)=\mathcal{B}(\mathbf{w},r;\mathbf{w}_h,r_h) \tag{5.1}$$

for each $(\mathbf{w}, r) \in \mathbf{V} \times M$ and all $(\mathbf{w}_h, r_h) \in \mathbf{V}_h \times M_h$. According to Theorem 4.1 in [4], it is easy to check that $(\mathbf{R}_h\mathbf{w}, Q_h r)$ is well defined. By the definition of $\mathcal{B}_h$ and $\mathcal{B}$, there holds

$$\mathcal{B}_h(\mathbf{R}_h\mathbf{w}, Q_h r; \mathbf{w}_h, r_h) = \mathcal{B}_h(\mathbf{w}, r; \mathbf{w}_h, r_h) - \alpha G(r, r_h). \tag{5.2}$$

For $1 \le n \le N$, by a classical argument as in [17] the following approximation properties hold:

$$||\mathbf{U}^n - \mathbf{R}_h\mathbf{U}^n||_{L^2} + h||\nabla\mathbf{U}^n - \nabla\mathbf{R}_h\mathbf{U}^n||_{L^2} \le Ch^2||A\mathbf{U}^n||_{L^2}, \tag{5.3}$$

$$||D_\tau\mathbf{U}^n - \mathbf{R}_h D_\tau\mathbf{U}^n||_{L^2} \le Ch^2||AD_\tau\mathbf{U}^n||_{L^2}, \tag{5.4}$$

$$||D_\tau\mathbf{U}^n - \mathbf{R}_h D_\tau\mathbf{U}^n||_{L^2} \le Ch||D_\tau\mathbf{U}^n||_{H^1}, \tag{5.5}$$

$$||P^n - Q_h P^n||_{L^2} \le Ch||P^n||_{H^1}. \tag{5.6}$$

For given $1 \le n \le N$ and $\mathbf{U}^{n-1} \in \mathbf{V}$, define $\Pi_h^n\mathbf{B}^n \in \mathbf{X}_h$ by

$$\gamma(\nabla(\Pi_h^n\mathbf{B}^n - \mathbf{B}^n), \nabla\mathbf{w}_h) + \gamma(\Pi_h^n\mathbf{B}^n - \mathbf{B}^n, \mathbf{w}_h) + b_h(\mathbf{U}^{n-1}, \Pi_h^n\mathbf{B}^n - \mathbf{B}^n, \mathbf{w}_h) = 0,$$
$$\forall\, \mathbf{w}_h \in \mathbf{X}_h.$$

From the classical finite element theory for elliptic problem [5], we have

$$||\mathbf{B}^n - \Pi_h^n\mathbf{B}^n||_{L^2} + h||\mathbf{B}^n - \Pi_h^n\mathbf{B}^n||_{H^1} \le Ch^3||\mathbf{B}^n||_{H^3}, \tag{5.7}$$

$$||D_\tau\mathbf{B}^n - \Pi_h^n D_\tau\mathbf{B}^n||_{L^2} \le Ch^2||\Delta\mathbf{D}_\tau\mathbf{B}^n||_{L^2}, \tag{5.8}$$

$$\text{for } m = 0, 1,\ 2 < p \le +\infty,\ ||\Pi_h^n\mathbf{B}^n||_{W^{m,p}} \le C||\mathbf{B}^n||_{W^{m,p}}, \tag{5.9}$$

$$\text{for } 2 \le p \le 6,\ ||\mathbf{B}^n - \Pi_h^n\mathbf{B}^n||_{W^{1,p}} \le Ch||\mathbf{B}^n||_{W^{2,p}}. \tag{5.10}$$

Multiplying (3.8) by $\phi_h \in \mathbf{X}_h$ and (3.9) by $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h$, and subtracting the resulting equations from (3.3) and (3.4), respectively, we get

$$\begin{aligned}(D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1}, \phi_h) + \gamma(\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}, \nabla\phi_h) &= (D_\tau\mathbf{E}_{\mathbf{b}}^{n+1}, \phi_h) - (\nabla\mathbf{E}_{\mathbf{b}}^{n+1}, \nabla\phi_h)\\ &\quad + b_h(\mathbf{U}_h^n, \mathbf{B}_h^{n+1}, \phi_h) - b_h(\mathbf{U}^n, \Pi_h^{n+1}\mathbf{B}^{n+1}, \phi_h)\\ &\quad + \gamma(|\nabla\mathbf{B}^n|^2\mathbf{B}^n - |\nabla\mathbf{B}_h^n|^2\mathbf{B}_h^n, \phi_h),\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}&(D_\tau\mathbf{e}_{\mathbf{u}h}^{n+1}, \mathbf{v}_h) + \mathcal{B}_h(\mathbf{e}_{\mathbf{u}h}^{n+1}, e_{ph}^{n+1}; \mathbf{v}_h, q_h)\\ &= (D_\tau\mathbf{E}_{\mathbf{u}}^{n+1}, \mathbf{v}_h) + b_h(\mathbf{U}_h^n, \mathbf{U}_h^{n+1}, \mathbf{v}_h) - b_h(\mathbf{U}^n, \mathbf{U}^{n+1}, \mathbf{v}_h)\\ &\quad + \lambda(\nabla\mathbf{B}^{n+1} \odot \nabla\mathbf{B}^n - \nabla\mathbf{B}_h^{n+1} \odot \nabla\mathbf{B}_h^n, \nabla\mathbf{v}_h),\end{aligned} \tag{5.12}$$

where we use the definitions of Π_h^n and $(\mathbf{R}_h, Q_h)$.

The main results in this section are summarized in the following lemma.

Lemma 5.1 *Under the assumptions in Theorem 3.1 and Lemma 4.1, for $0 \le n \le N-1$, there exist some $h_2 > 0$ and $\tau_2 > 0$ such that when $h < h_2$ and $\tau < \tau_2$, there hold*

$$\max_{0\le m\le n+1}\left(||\boldsymbol{e}_{\boldsymbol{b}h}^m||_{L^2}^2 + \tau\sum_{k=0}^m||\nabla\boldsymbol{e}_{\boldsymbol{b}h}^k||_{L^2}^2\right) \le \frac{C_0^2}{16}h^4, \tag{5.13}$$

$$\max_{0\le m\le n+1}\left(||\boldsymbol{e}_{\boldsymbol{u}h}^m||_{L^2}^2 + \tau\sum_{k=0}^m||\nabla\boldsymbol{e}_{\boldsymbol{u}h}^k||_{L^2}^2\right) \le \frac{C_0^2}{16}h^4, \tag{5.14}$$

$$\max_{0\le m\le n+1}||\nabla\boldsymbol{e}_{\boldsymbol{b}h}^m||_{L^2} \le \frac{C_0}{4}h^2. \tag{5.15}$$

Proof Since $\mathbf{e}_{\mathbf{u}h}^0 = \mathbf{R}_h^0\mathbf{U}^0 - \mathbf{U}_h^0 = 0$ and $\mathbf{e}_{\mathbf{b}h}^0 = \Pi_h^0\mathbf{B}^0 - \mathbf{B}_h^0 = 0$, the inequalities (5.13) and (5.15) obviously hold for $m = 0$. Now, we suppose that (5.13) and (5.15) hold for $m \le n$. Then we need to show that these inequalities also hold for $m \le n+1$. By inverse inequality (3.10), we have

$$\max_{0\le m\le n} ||\mathbf{e}_{\mathbf{b}h}^m||_{W^{1,\infty}} \le \frac{C_0C_6}{4}h. \tag{5.16}$$

Taking $\phi_h = \mathbf{e}_{\mathbf{b}h}^{n+1}$ in (5.11) leads to

$$\begin{aligned}
&\frac{1}{2}D_\tau||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + \frac{1}{2\tau}||\mathbf{e}_{\mathbf{b}h}^{n+1} - \mathbf{e}_{\mathbf{b}h}^n||_{L^2}^2 + \gamma||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2\\
&\quad\le |(D_\tau\mathbf{E}_{\mathbf{b}}^{n+1}, \mathbf{e}_{\mathbf{b}h}^{n+1})| + \gamma|(\mathbf{E}_{\mathbf{b}}^{n+1}, \mathbf{e}_{\mathbf{b}h}^{n+1})|\\
&\quad\quad + |b_h(\mathbf{U}_h^n, \mathbf{B}_h^{n+1}, \mathbf{e}_{\mathbf{b}h}^{n+1}) - b_h(\mathbf{U}^n, \Pi_h^{n+1}\mathbf{B}^{n+1}, \mathbf{e}_{\mathbf{b}h}^{n+1})|\\
&\quad\quad + \gamma|(|\nabla\mathbf{B}^n|^2\mathbf{B}^n - |\nabla\mathbf{B}_h^n|^2\mathbf{B}_h^n, \mathbf{e}_{\mathbf{b}h}^{n+1})| = J_1 + J_2 + J_3 + J_4.
\end{aligned}\tag{5.17}$$

By using (5.7), (5.8) and (4.19), it is easy to bound J_1 and J_2, respectively, as

$$J_1 \le ||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + C_7||D_\tau\mathbf{E}_{\mathbf{b}}^{n+1}||_{L^2}^2 \le ||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + C_7h^4||D_\tau\mathbf{B}^{n+1}||_{H^2}^2,$$

and

$$J_2 \le ||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + C_7||\mathbf{E}_{\mathbf{b}}^{n+1}||_{L^2}^2 \le ||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + C_7h^4.$$

From (3.5), (5.3), (5.9) and (5.14) for $m \le n$, J_3 satisfies

$$\begin{aligned}
J_3 &= |b_h(\mathbf{E}_{\mathbf{u}}^n - \mathbf{e}_{\mathbf{u}h}^n, \Pi_h^{n+1}\mathbf{B}^{n+1}, \mathbf{e}_{\mathbf{b}h}^{n+1})|\\
&\le (||\mathbf{E}_{\mathbf{u}}^n||_{L^2} + ||\mathbf{e}_{\mathbf{u}h}^n||_{L^2})||\mathbf{B}^{n+1}||_{W^{1,3}}||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}\\
&\le \frac{\gamma}{4}||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + C_7(h^4 + ||\mathbf{e}_{\mathbf{u}h}^n||_{L^2}^2).
\end{aligned}$$

We rewrite J_4 according to the following equation:

$$\begin{aligned}
&|\nabla\mathbf{B}^n|^2\mathbf{B}^n - |\nabla\mathbf{B}_h^n|^2\mathbf{B}_h^n\\
&\quad= |\nabla\mathbf{B}^n|^2(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n) - 2\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)\cdot\nabla\mathbf{B}^n(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n) + 2\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)\cdot\nabla\mathbf{B}^n\mathbf{B}^n\\
&\quad\quad + |\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)|^2(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n) - |\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)|^2\mathbf{B}^n = J_4^1 + \cdots + J_4^5.
\end{aligned}$$

Then from the following estimates:

$$\begin{aligned}
\gamma(J_4^1, \mathbf{e}_{\mathbf{b}h}^{n+1}) &\le \gamma||\nabla\mathbf{B}^n||_{L^\infty}^2||\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n||_{L^2}||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2} \le C_7(h^2 + ||\mathbf{e}_{\mathbf{b}h}^n||_{L^2})||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}\\
\gamma(J_4^2, \mathbf{e}_{\mathbf{b}h}^{n+1}) &\le \gamma||\nabla\mathbf{B}^n||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^2}||\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n||_{L^\infty}||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}\\
&\le C_7C_6C_0h(h^2 + ||\nabla\mathbf{e}_{\mathbf{b}h}^n||_{L^2})||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}\\
\gamma(J_4^3, \mathbf{e}_{\mathbf{b}h}^{n+1}) &\le \gamma||\nabla\mathbf{B}^n||_{L^\infty}||\mathbf{B}^n||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^2}||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}\\
&\le C_7(h^2 + ||\nabla\mathbf{e}_{\mathbf{b}h}^n||_{L^2})||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}\\
\gamma(J_4^4, \mathbf{e}_{\mathbf{b}h}^{n+1}) &\le \gamma||\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^2}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^3}||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^6}\\
&\le C_7C_6C_0h(||\nabla\mathbf{e}_{\mathbf{b}h}^n||_{L^2} + h^2)(h^{-1/3}||\nabla\mathbf{e}_{\mathbf{b}h}^n||_{L^2} + h)||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}\\
&\le C_7C_6^2C_0^2h^{5/3}(h^2 + ||\nabla\mathbf{e}_{\mathbf{b}h}^n||_{L^2})||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}
\end{aligned}$$

$$\gamma(J_4^5, \mathbf{e}_{\mathbf{b}h}^{n+1}) \le \gamma||\mathbf{B}^n||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^2}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^3}||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^6}$$
$$\le C_7(||\nabla\mathbf{e}_{\mathbf{b}h}^n||_{L^2} + h^2)(h^{-1/3}||\nabla\mathbf{e}_{\mathbf{b}h}^n||_{L^2} + h)||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}$$
$$\le C_7C_6C_0h^{2/3}(||\nabla\mathbf{e}_{\mathbf{b}h}^n||_{L^2} + h^2)||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2},$$

J_4 can be bounded by

$$J_4 \le \frac{\gamma}{4}||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + C_8||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + (\gamma C_9^2C_0^2h^{4/3} + C_9^4C_0^4h^{10/3})(h^4 + ||\nabla\mathbf{e}_{\mathbf{b}h}^n||_{L^2}^2).$$

For sufficiently small h such that $(\gamma C_9^2C_0^2h^{4/3} + C_9^4C_0^4h^{10/3}) < \gamma/4$. Then combining these estimates into (5.17) leads to

$$D_\tau||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + \frac{1}{\tau}||\mathbf{e}_{\mathbf{b}h}^{n+1} - \mathbf{e}_{\mathbf{b}h}^n||_{L^2}^2 + \gamma||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2$$
$$\le C_{10}(h^4 + h^4||D_\tau\mathbf{B}^{n+1}||_{H^2}^2 + ||\mathbf{e}_{\mathbf{u}h}^n||_{L^2}^2) + C_{11}||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + \frac{\gamma}{2}||\nabla\mathbf{e}_{\mathbf{b}h}^n||_{L^2}^2. \quad (5.18)$$

Taking $\mathbf{v}_h = \mathbf{e}_{\mathbf{u}h}^{n+1}$ and $q_h = e_{ph}^{n+1}$ in (5.12) leads to

$$\frac{1}{2}D_\tau||\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}^2 + \frac{1}{2\tau}||\mathbf{e}_{\mathbf{u}h}^{n+1} - \mathbf{e}_{\mathbf{u}h}^n||_{L^2}^2 + \mu||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}^2 + \alpha||(I - \Pi_1)e_{ph}^{n+1}||_{L^2}^2$$
$$\le |(D_\tau\mathbf{E}_{\mathbf{u}}^{n+1}, \mathbf{e}_{\mathbf{u}h}^{n+1})| + |b_h(\mathbf{U}_h^n, \mathbf{U}_h^{n+1}, \mathbf{e}_{\mathbf{u}h}^{n+1}) - b_h(\mathbf{U}^n, \mathbf{U}^{n+1}, \mathbf{e}_{\mathbf{u}h}^{n+1})|$$
$$+ \lambda|(\nabla\mathbf{B}^{n+1} \odot \nabla\mathbf{B}^n - \nabla\mathbf{B}_h^{n+1} \odot \nabla\mathbf{B}_h^n, \nabla\mathbf{e}_{\mathbf{u}h}^{n+1})|. = J_5 + J_6 + J_7. \quad (5.19)$$

From (5.4) and (4.28), J_5 is bounded by

$$J_5 \le ||\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}^2 + C_{12}||D_\tau\mathbf{E}_{\mathbf{u}}^{n+1}||_{L^2}^2 \le ||\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}^2 + C_{12}h^4||AD_\tau\mathbf{U}^{n+1}||_{L^2}^2.$$

An alternative to J_6 is

$$J_6 = |b_h(\mathbf{E}_{\mathbf{u}}^n - \mathbf{e}_{\mathbf{u}h}^n, \mathbf{E}_{\mathbf{u}}^{n+1}, \mathbf{e}_{\mathbf{u}h}^{n+1}) - b_h(\mathbf{U}^n, \mathbf{e}_{\mathbf{u}h}^{n+1}, \mathbf{E}_{\mathbf{u}}^{n+1})$$
$$+ b_h(\mathbf{E}_{\mathbf{u}}^n - \mathbf{e}_{\mathbf{u}h}^n, \mathbf{U}^{n+1}, \mathbf{e}_{\mathbf{u}h}^{n+1})|.$$

Then we have

$$J_6 \le ||\mathbf{E}_{\mathbf{u}}^n - \mathbf{e}_{\mathbf{u}h}^n||_{L^2}||\nabla\mathbf{E}_{\mathbf{u}}^{n+1}||_{L^3}||\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^6} + ||A\mathbf{U}^n||_{L^2}||\mathbf{E}_{\mathbf{u}}^{n+1}||_{L^2}||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}$$
$$+ ||\mathbf{E}_{\mathbf{u}}^n - \mathbf{e}_{\mathbf{u}h}^n||_{L^2}||\nabla\mathbf{U}^{n+1}||_{L^3}||\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^6} \le \frac{\mu}{4}||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}^2 + C_{12}(h^4 + ||\mathbf{e}_{\mathbf{u}h}^n||_{L^2}^2).$$

Observing the following identity

$$\nabla\mathbf{B}^{n+1} \odot \nabla\mathbf{B}^n - \nabla\mathbf{B}_h^{n+1} \odot \nabla\mathbf{B}_h^n$$
$$= \nabla(\mathbf{e}_{\mathbf{b}h}^{n+1} - \mathbf{E}_{\mathbf{b}}^{n+1}) \odot \nabla\mathbf{B}^n - \nabla\mathbf{e}_{\mathbf{b}h}^{n+1} \odot \nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)$$
$$+ \nabla\mathbf{E}_{\mathbf{b}}^{n+1} \odot \nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n) + \nabla\mathbf{B}^{n+1} \odot \nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)$$
$$= J_7^1 + J_7^2 + J_7^3 + J_7^4,$$

and using the following estimates:

$$\lambda(J_7^1, \nabla\mathbf{e}_{\mathbf{u}h}^{n+1}) \le \lambda||\nabla\mathbf{B}^n||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^{n+1} - \mathbf{E}_{\mathbf{b}}^{n+1})||_{L^2}||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}$$
$$\le C_{12}(h^2 + ||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2})||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}$$
$$\lambda(J_7^2, \nabla\mathbf{e}_{\mathbf{u}h}^{n+1}) \le \lambda||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^2}||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}$$
$$\le C_{12}h^{-2}||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^2}||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}$$

$$
\begin{aligned}
&\le \frac{C_0C_6C_{12}}{4}||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}\\
\lambda(J_7^3, \nabla\mathbf{e}_{\mathbf{u}h}^{n+1}) &\le \lambda(||\nabla\mathbf{E}_{\mathbf{b}}^{n+1}||_{L^4}||\nabla\mathbf{E}_{\mathbf{b}}^{n}||_{L^4} + ||\nabla\mathbf{E}_{\mathbf{b}}^{n+1}||_{L^2}||\nabla\mathbf{e}_{\mathbf{b}h}^{n}||_{L^\infty})||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}\\
&\le C_{12}(||\nabla\mathbf{E}_{\mathbf{b}}^{n+1}||_{L^4}||\nabla\mathbf{E}_{\mathbf{b}}^{n}||_{L^4} + C_6^2h^{-2}||\nabla\mathbf{E}_{\mathbf{b}}^{n+1}||_{L^2}||\mathbf{e}_{\mathbf{b}h}^{n}||_{L^2})||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}\\
&\le C_{12}C_6^2(h^2 + ||\mathbf{e}_{\mathbf{b}h}^{n}||_{L^2})||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}\\
\lambda(J_7^4, \nabla\mathbf{e}_{\mathbf{u}h}^{n+1}) &\le \lambda||\nabla\mathbf{B}^{n+1}||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^{n} - \mathbf{E}_{\mathbf{b}}^{n})||_{L^2}||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}\\
&\le C_{12}(h^2 + ||\nabla\mathbf{e}_{\mathbf{b}h}^{n}||_{L^2})||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2},
\end{aligned}
$$

we estimate J_7 as

$$
\begin{aligned}
J_7 \le{}& \frac{\mu}{4}||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}^2 + C_{12}(h^4 + ||\nabla\mathbf{e}_{\mathbf{b}h}^{n}||_{L^2}^2 + ||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + ||\mathbf{e}_{\mathbf{b}h}^{n}||_{L^2})\\
&+ C_{13}C_0^2C_6^2||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2.
\end{aligned}
$$

Combining these estimates into (5.19) leads to

$$
\begin{aligned}
&D_\tau||\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}^2 + \frac{1}{\tau}||\mathbf{e}_{\mathbf{u}h}^{n+1} - \mathbf{e}_{\mathbf{u}h}^{n}||_{L^2}^2 + \mu||\nabla\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}^2\\
&\le C_{12}(||\mathbf{e}_{\mathbf{u}h}^{n+1}||_{L^2}^2 + h^4 + h^4||AD_\tau\mathbf{U}^{n+1}||_{L^2}^2 + ||\mathbf{e}_{\mathbf{u}h}^{n}||_{L^2}^2)\\
&\quad + C_{12}(||\nabla\mathbf{e}_{\mathbf{b}h}^{n}||_{L^2}^2 + ||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + ||\mathbf{e}_{\mathbf{b}h}^{n}||_{L^2}) + C_{13}C_0^2C_6^2||\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2. \qquad (5.20)
\end{aligned}
$$

To prove (5.15), we set $\phi_h = D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1}$ in (5.11) to get

$$
\begin{aligned}
&\frac{1}{2}||D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + \frac{1}{2}D_\tau||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + \frac{1}{2\tau}||\nabla\mathbf{e}_{\mathbf{b}h}^{n+1} - \nabla\mathbf{e}_{\mathbf{b}h}^{n}||_{L^2}^2\\
&\quad = (D_\tau\mathbf{E}_{\mathbf{b}}^{n+1}, D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1}) - \gamma(\mathbf{E}_{\mathbf{b}}^{n+1}, D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1})\\
&\qquad + b_h(\mathbf{U}_h^n, \mathbf{B}_h^{n+1}, D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1}) - b_h(\mathbf{U}^n, \mathbf{B}^{n+1}, D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1})\\
&\qquad + \gamma(|\nabla\mathbf{B}^n|^2\mathbf{B}^n - |\nabla\mathbf{B}_h^n|^2\mathbf{B}_h^n, D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1})\\
&\quad = J_8 + J_9 + J_{10} + J_{11}. \qquad (5.21)
\end{aligned}
$$

By the similar arguments for J_1 and J_2, we have

$$
J_8 \le \frac{1}{8}||D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + C_{14}h^4||D_\tau\mathbf{B}^{n+1}||_{H^2}^2,
$$

and

$$
J_9 \le \frac{1}{8}||D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + C_{14}h^4.
$$

An alternative to J_{10} is

$$
\begin{aligned}
J_{10} ={}& b_h(\mathbf{E}_{\mathbf{u}}^n - \mathbf{e}_{\mathbf{u}h}^n, \mathbf{e}_{\mathbf{b}h}^n, D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1}) - b_h(\mathbf{U}^n, \mathbf{e}_{\mathbf{b}h}^n, D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1})\\
&+ b_h(\mathbf{E}_{\mathbf{u}}^n - \mathbf{e}_{\mathbf{u}h}^n, \Pi_h^{n+1}\mathbf{B}^{n+1}, D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1}) = J_{10}^1 + J_{10}^2 + J_{10}^3,
\end{aligned}
$$

which are estimated, respectively, by

$$
\begin{aligned}
J_{10}^1 &\le ||\mathbf{E}_{\mathbf{u}}^n - \mathbf{e}_{\mathbf{u}h}^n||_{L^2}||\nabla\mathbf{e}_{\mathbf{b}h}^n||_{L^\infty}||D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}\\
&\le C_{14}h^{-2}||\mathbf{E}_{\mathbf{u}}^n - \mathbf{e}_{\mathbf{u}h}^n||_{L^2}||\mathbf{e}_{\mathbf{b}h}^n||_{L^2}||D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}\\
&\le \frac{C_0C_6C_{14}}{4}(h^2 + ||\mathbf{e}_{\mathbf{u}h}^n||_{L^2})||D_\tau\mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2},
\end{aligned}
$$

$$J_{10}^2 \le ||\mathbf{U}^n||_{L^\infty}||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^2}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2} \le C_{14}||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^2}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2},$$

$$J_{10}^3 \le ||\mathbf{E}_{\mathbf{u}}^n - \mathbf{e}_{\mathbf{u}h}^n||_{L^2}||\nabla \Pi_h^{n+1}\mathbf{B}^{n+1}||_{L^\infty}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}$$
$$\le C_{14}||\mathbf{E}_{\mathbf{u}}^n - \mathbf{e}_{\mathbf{u}h}^n||_{L^2}||\mathbf{B}^{n+1}||_{H^3}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2} \le C_{14}(h^2 + ||\mathbf{e}_{\mathbf{u}h}^n||_{L^2})||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}.$$

Then J_{10} is bounded by

$$J_{10} \le \frac{1}{8}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + C_{14}\left(h^4 + ||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^2}^2 + \frac{C_0^2C_6^2}{16}||\mathbf{e}_{\mathbf{u}h}^n||_{L^2}^2\right).$$

To estimate J_{11}, we rewrite it as

$$\begin{aligned}
&|\nabla \mathbf{B}^n|^2\mathbf{B}^n - |\nabla \mathbf{B}_h^n|^2\mathbf{B}_h^n \\
&= |\nabla \mathbf{B}^n|^2(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n) - \nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)\cdot\nabla \mathbf{B}^n\mathbf{e}_{\mathbf{b}h}^n + \nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)\cdot\nabla \mathbf{B}^n\Pi_h^n\mathbf{B}^n \\
&\quad + \nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)\cdot\nabla \mathbf{e}_{\mathbf{b}h}^n\mathbf{e}_{\mathbf{b}h}^n - \nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)\cdot\nabla \mathbf{e}_{\mathbf{b}h}^n\Pi_h^n\mathbf{B}^n \\
&\quad - \nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)\cdot\nabla \Pi_h^n\mathbf{B}^n\mathbf{e}_{\mathbf{b}h}^n + \nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)\cdot\nabla \Pi_h^n\mathbf{B}^n\Pi_h^n\mathbf{B}^n \\
&= J_{11}^1 + \cdots + J_{11}^7.
\end{aligned}$$

Then all terms in J_{11} are bounded by

$$\begin{aligned}
\gamma(J_{11}^1, D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}) &\le \gamma||\nabla \mathbf{B}^n||_{L^\infty}^2||\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n||_{L^2}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2} \le C_{14}h^2||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}, \\
\gamma(J_{11}^2, D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}) &\le \gamma||\nabla \mathbf{B}^n||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^2}||\mathbf{e}_{\mathbf{b}h}^n||_{L^\infty}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2} \\
&\le \frac{C_0C_6C_{14}}{4}h(h^2 + ||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^2})||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}, \\
\gamma(J_{11}^3, D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}) &\le \gamma||\nabla \mathbf{B}^n||_{L^\infty}||\Pi_h^n\mathbf{B}^n||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^2}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2} \\
&\le C_{14}(h^2 + ||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^2})||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}, \\
\gamma(J_{11}^4, D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}) &\le \gamma||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^\infty}||\mathbf{e}_{\mathbf{b}h}^n||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^2}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2} \\
&\le \frac{C_0^2C_6^2C_{14}}{16}h^2(h^2 + ||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^2})||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}, \\
\gamma(J_{11}^5, D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}) &\le \gamma||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^\infty}||\Pi_h^n\mathbf{B}^n||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^2}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2} \\
&\le \frac{C_0C_6C_{14}}{4}h(h^2 + ||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^2})||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}, \\
\gamma(J_{11}^6, D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}) &\le \gamma||\mathbf{e}_{\mathbf{b}h}^n||_{L^\infty}||\nabla \Pi_h^n\mathbf{B}^n||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^2}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2} \\
&\le \frac{C_0C_6C_{14}}{4}h(h^2 + ||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^2})||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}, \\
\gamma(J_{11}^7, D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}) &\le \gamma||\Pi_h^n\mathbf{B}^n||_{L^\infty}||\nabla \Pi_h^n\mathbf{B}^n||_{L^\infty}||\nabla(\mathbf{e}_{\mathbf{b}h}^n - \mathbf{E}_{\mathbf{b}}^n)||_{L^2}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2} \\
&\le C_{14}(h^2 + ||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^2})||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}.
\end{aligned}$$

For sufficiently small h such that $\frac{C_0^2C_6^2}{16}h^2 < 1$, then J_{11} satisfies

$$J_{11} \le \frac{1}{8}||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + C_{14}(h^4 + ||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^2}^2).$$

Combining these estimates into (5.21) leads to

$$\begin{aligned}
&||D_\tau \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + D_\tau||\nabla \mathbf{e}_{\mathbf{b}h}^{n+1}||_{L^2}^2 + \tau||\nabla \mathbf{e}_{\mathbf{b}h}^{n+1} - \nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^2}^2 \\
&\quad \le C_{14}(h^4 + h^4||D_\tau \mathbf{B}^{n+1}||_{H^2}^2 + ||\nabla \mathbf{e}_{\mathbf{b}h}^n||_{L^2}^2 + C_{15}C_0^2C_6^2||\mathbf{e}_{\mathbf{u}h}^n||_{L^2}^2). \qquad (5.22)
\end{aligned}$$

Summing up three inequalities (5.18), (5.20), (5.22), for sufficiently small τ such that

$$\tau \max\{C_{11}, C_{13}C_0^2C_6^2, C_{15}C_0^2C_6^2\} < 1,$$

then from discrete Gronwall's inequality we derive

$$||\mathbf{e}_{\mathbf{b}h}^m||_{L^2}^2 + ||\mathbf{e}_{\mathbf{u}h}^m||_{L^2}^2 + ||\nabla\mathbf{e}_{\mathbf{b}h}^m||_{L^2}^2 + \tau\sum_{k=0}^{m}(||\nabla\mathbf{e}_{\mathbf{b}h}^k||_{L^2}^2 + ||\nabla\mathbf{e}_{\mathbf{u}h}^k||_{L^2}^2) \le C_{16}\exp(2TC_{17})h^4,$$

which proves Lemma 5.1 if we take $C_0^2 > 16C_{16}\exp(TC_{17})$. □

6 Proof of Theorem 3.1

First, we show the existence and uniqueness of the solution to the finite element discrete system (3.3)–(3.4). For $0 \le n \le N-1$, an alternative to (3.3) is

$$\begin{aligned}(\mathbf{B}_h^{n+1}, \phi_h) + \gamma\tau(\nabla\mathbf{B}_h^{n+1}, \nabla\phi_h) + \tau b_h(\mathbf{U}_h^n, \mathbf{B}_h^{n+1}, \phi_h)\\ = (\mathbf{B}_h^n, \phi_h) + \gamma\tau(|\nabla\mathbf{B}_h^n|^2\mathbf{B}_h^n, \phi_h), \quad \forall\, \phi_h \in \mathbf{X}_h.\end{aligned}$$

Taking $\phi_h = \mathbf{B}_h^{n+1}$ in the above equation leads to

$$\begin{aligned}(\mathbf{B}_h^{n+1}, \mathbf{B}_h^{n+1}) + \gamma\tau(\nabla\mathbf{B}_h^{n+1}, \nabla\mathbf{B}_h^{n+1}) + \tau b_h(\mathbf{U}_h^n, \mathbf{B}_h^{n+1}, \mathbf{B}_h^{n+1})\\ = ||\mathbf{B}_h^{n+1}||_{L^2}^2 + \gamma\tau||\nabla\mathbf{B}_h^{n+1}||_{L^2}^2 \ge \min\{1, \gamma\tau\}||\mathbf{B}_h^{n+1}||_{H^1}^2.\end{aligned}$$

Then, the existence and uniqueness of $\mathbf{B}_h^{n+1}$ for $0 \le n \le N-1$ follows from Lax–Milgram theorem. The existence and uniqueness of the solution to (3.4) can be shown by using Lax–Milgram theorem for $\mathbf{U}_h^{n+1}$ and Theorem 4.1 in [4] for P_h^{n+1}. Error estimates $||\mathbf{b}^n - \mathbf{B}_h^n||_{H^1}$ and $||\mathbf{u}^n - \mathbf{U}_h^n||_{L^2}$ follow from the error splitting and (4.7), (5.3), (5.7), (5.14) and (5.15). By inverse inequality (3.10), (4.7) and (5.3), we derive

$$||\nabla\mathbf{u}^n - \nabla\mathbf{U}_h^n||_{L^2} \le ||\nabla\mathbf{e}_{\mathbf{u}}^n||_{L^2} + ||\nabla\mathbf{E}_{\mathbf{u}}^n||_{L^2} + Ch^{-1}||\mathbf{e}_{\mathbf{u}h}^n||_{L^2} \le C_0(\tau + h)$$

for sufficiently large C_0. On the other hand, by using

$$||\mathbf{B}_h^n||_{L^\infty} \le ||\Pi_h^n\mathbf{B}^n||_{L^\infty} + ||\mathbf{e}_{\mathbf{b}h}^n||_{L^\infty} \le C||\mathbf{B}^n||_{L^\infty} + Ch \le C,$$

we have

$$||1 - |\mathbf{B}_h^n|^2||_{L^2} = |||\mathbf{b}^n|^2 - |\mathbf{B}_h^n|^2||_{L^2} = ||\mathbf{b}^n - \mathbf{B}_h^n||_{L^2}||\mathbf{b}^n + \mathbf{B}_h^n||_{L^\infty} \le C_0(\tau + h^2)$$

for sufficiently large C_0. We completes the proof of Theorem 3.1 if we take $\tau_0 = \min\{\tau_1, \tau_2\}$ and $h_0 = h_2$.

7 Numerical Results

In this section, we present the numerical results by using the linearized semi-implicit scheme (3.5)–(3.6) to verify the optimal error estimates derived in Theorem 3.1. All programs are implemented by the free finite element software FreeFem++ [18]. We consider the nematic liquid crystal model in the unit circle $\Omega = \{(x, y) : x^2 + y^2 < 1\}$. The initial data are taken as

$$\mathbf{u}_0 = \mathbf{0},\ \mathbf{f} = \mathbf{0}, \quad \mathbf{b}_0 = (\sin(a), \cos(a)), \quad a = \pi(x^2 + y^2)^2.$$

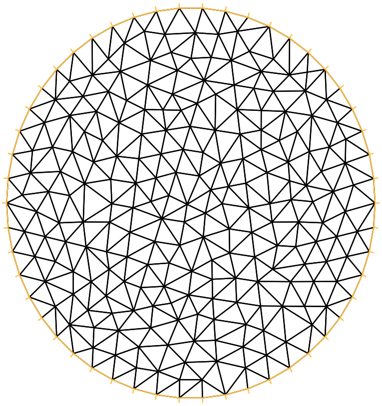

Fig. 1 The FEM meshes of the unit circle with $M = 50$

Table 1 Numerical errors and convergence rates at $T = 1$ for $\tau = 9/M^2$

$\tau = 9/M^2$	$\|\|\mathbf{b}(\cdot, 1) - \mathbf{B}_h^J\|\|_{H^1}$	Rate	$\|\|1 - \|\mathbf{B}_h^J\|^2\|\|_{L^2}$	Rate
$M = 50$	3.34614E−01		5.78705E−01	
$M = 100$	7.96421E−02	2.0709	1.36172E−01	2.0874
$M = 150$	3.20721E−02	2.2432	5.62595E−02	2.1796

Table 2 Numerical errors and convergence rates at $T = 1$ for $\tau = 1/M^2$

$\tau = 1/M^2$	$\|\|\mathbf{b}(\cdot, 1) - \mathbf{B}_h^J\|\|_{H^1}$	Rate	$\|\|1 - \|\mathbf{B}_h^J\|^2\|\|_{L^2}$	Rate
$M = 50$	2.65892E−02		5.15313E−02	
$M = 100$	6.53797E−03	2.0239	1.34739E−02	1.9353
$M = 150$	2.51260E−03	2.3585	6.07014E−03	1.9666

Table 3 Numerical errors and convergence rates for different τ

τ	$\|\|\mathbf{b}(\cdot, 0.5) - \mathbf{B}_h^J\|\|_{H^1}$	$\|\|\mathbf{u}(\cdot, 0.5) - \mathbf{U}_h^J\|\|_{L^2}$	$\|\|1 - \|\mathbf{B}_h^J\|^2\|\|_{L^2}$
1.00×10^{-3}	8.97006E−02	2.08193E−01	1.38704E−01
5.00×10^{-4}	4.30171E−02	9.69920E−02	6.54445E−02
2.50×10^{-4}	2.08904E−02	4.64736E−02	3.18394E−02
1.25×10^{-4}	1.01993E−02	2.25480E−02	1.58644E−02
6.25×10^{-5}	4.97587E−03	1.09681E−02	8.12155E−03
Rate	1.0430	1.0616	1.0235

Parameters are set as $\alpha = \lambda = \gamma = 1$ and $\mu = 1$. We take a uniform triangular partition with M nodes on $\partial\Omega$. Then a class of uniform meshes of the unit circle is made by a mesh generator in FreeFem++; see Fig. 1 for illustration.

Since no exact solution exists, to verify the optimal convergence rates, the reference solution is taken as the numerical solution corresponding to $M = 300$. The time step τ is required to satisfy $\tau = O(1/M^2)$. Therefore, from the error estimates derived in Theorem 3.1, we

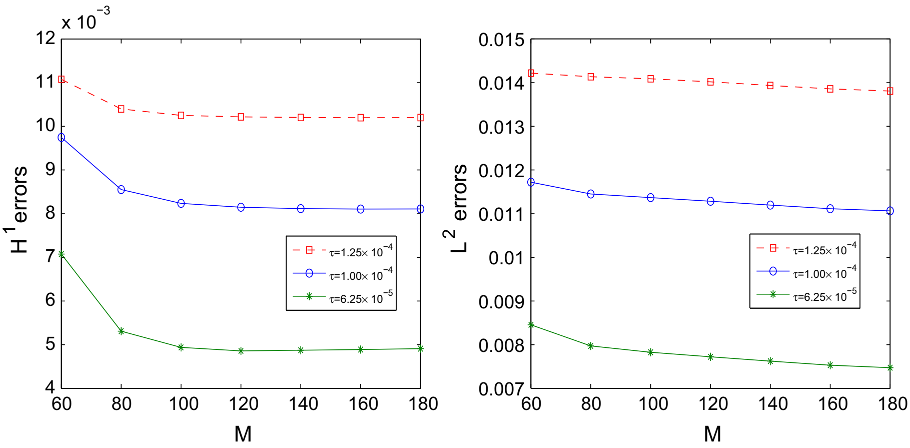

Fig. 2 H^1 and L^2 errors of the director

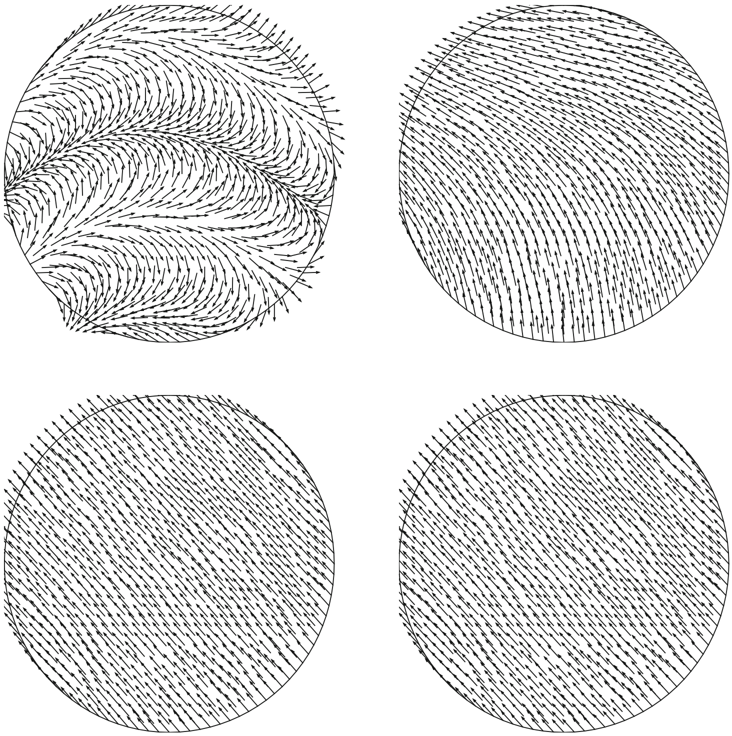

Fig. 3 Evolution of director fields: $T = 0$ (*top left*), $T = 0.6$ (*top right*), $T = 1.5$ (*bottom left*), $T = 2.74375$ (*bottom right*)

have the second-order convergence rate for the errors $||\mathbf{b}^n - \mathbf{B}_h^n||_{H^1}$ and $||1 - |\mathbf{B}_h^n|^2||_{L^2}$. To verify the optimal convergence rates, we use several mesh pairs $M = 50$, 100 and 150 with different time step $\tau = 9/M^2$ and $1/M^2$. The numerical results are displayed in Tables 1 and 2, from which we can see that the numerical convergence rates for orientation of molec-

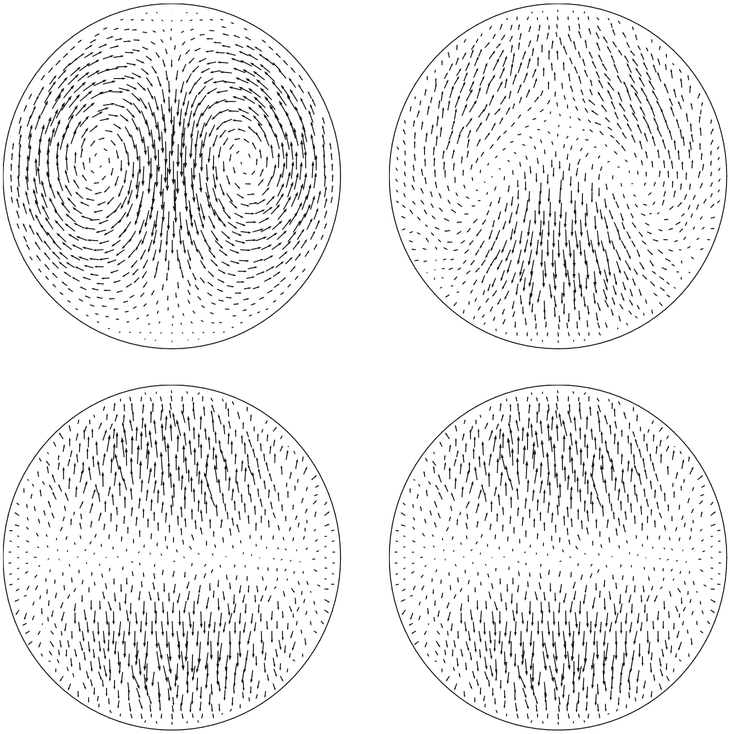

Fig. 4 Evolution of velocity fields: $T = 0.1$ (*top left*), $T = 0.6$ (*top right*), $T = 1.5$ (*bottom left*), $T = 2.74375$ (*bottom right*)

ular coincide with the ones predicted by theoretical analysis in Theorem 3.2, although the numerical errors with $\tau = 9/M^2$ seem not very accurate.

To confirm the temporal error is of the first-order convergence rate, we take the reference solution corresponding to $\tau = 3.125 \times 10^{-6}$ and $M = 120$. For different time step $\tau_{i+1} = 0.5\tau_i$ for $i = 1, 2, 4$ with $\tau_1 = 10^{-3}$, the numerical errors of velocity, director and pressure are displayed in Table 3. It can be observed that the semi-implicit scheme gives the convergence rates of the order $O(\tau)$ on the temporal errors which coincide with the ones predicted in Lemma 4.1.

To confirm the stability of the semi-implicit scheme without any restriction of the time step τ, the reference solution is taken as the numerical solution corresponding to $\tau = 3.125 \times 10^{-6}$ and $M = 200$. We solve the semi-implicit scheme (3.3) and (3.4) with three different time step $\tau = 1.25 \times 10^{-4}, 1.00 \times 10^{-4}$ and 6.25×10^{-5} on gradually refined meshes with $M = 20i, i = 2, \ldots, 9$. The H^1 errors $||\mathbf{b}(\cdot, 0.5) - \mathbf{B}_h^J||_{H^1}$ and L^2 errors $||1 - |\mathbf{B}_h^J|^2||_{L^2}$ are displayed in Fig. 2, from which we can see that for a fixed τ, when refining the mesh gradually, the H^1 errors converge to a small constant and the proposed semi-implicit scheme (3.3) and (3.4) is stable and convergent without any restriction of the time step.

Fix $M = 100$ and $\tau = 1/M^2$. The evolutions of the director fields and the velocity fields at different times are displayed in Figs. 3 and 4, where $T = 2.74375$ is the stoping time of iteration. The stopping criterion used is to require $||\mathbf{U}_h^n - \mathbf{U}_h^{n-1}||_{L^2} + ||\mathbf{B}_h^n - \mathbf{B}_h^{n-1}||_{L^2}$ to be less than 10^{-6}.

8 Conclusion

In this paper, we show optimal error estimates for an linearized semi-implicit Euler finite element scheme for the approximation of the nematic liquid crystals flows. To the best of our knowledge, no optimal error estimates have been given in previous works. More important, the semi-implicit scheme proposed in this paper is linear. Optimal error estimates are proved without any restriction of time step τ by using the error splitting technique. The numerical results show the efficiency of the scheme and confirm our theoretical analysis. In addition, the technique presented in this paper can be applied to analyze linearized semi-implicit Galerkin FEM for 3D nematic liquid crystal flows.

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9 Appendix

To get the regularities (2.16)–(2.17) of $\mathbf{u}_{tt}$, p_t and $\mathbf{b}_{tt}$, where $(\mathbf{u}, p, \mathbf{b})$ is the solution to the problem (1.1)–(1.5) and satisfies (2.10)–(2.14), we need to show that $||\nabla \mathbf{u}_t(x,t)||_{L^2}$ and $||\mathbf{b}(x,t)||_{H^1}$ remain bounded as $t \longrightarrow 0$. In this case, a non-local compatibility condition is needed and can be derived as follows. First, we begin to show that the problem

$$\begin{cases} \Delta p_0 = \mathrm{div}\,(\mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - \lambda \mathrm{div}\,(\nabla \mathbf{b}_0 \odot \nabla \mathbf{b}_0)), & \text{in } \Omega, \\ \partial_{\mathbf{n}} p_0 = (\mu \Delta \mathbf{u}_0 + \mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - \lambda \mathrm{div}\,(\nabla \mathbf{b}_0 \odot \nabla \mathbf{b}_0)) \cdot \mathbf{n}, & \text{on } \partial\Omega, \end{cases} \tag{9.1}$$

exists a unique $p_0 \in H^1(\Omega) \cap M$. In fact, from $\mathbf{f}_0 \in \mathbf{H}$, $\mathbf{u}_0 \in \mathbf{D}(A)$ and $\mathbf{b}_0 \in \mathbf{H}^3(\Omega)$, it can be shown that $\mathrm{div}\,(\mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - \lambda \mathrm{div}\,(\nabla \mathbf{b}_0 \odot \nabla \mathbf{b}_0)) \in L^2(\Omega)$ and $\mu \Delta \mathbf{u}_0 + \mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - \lambda \mathrm{div}\,(\nabla \mathbf{b}_0 \odot \nabla \mathbf{b}_0) \in \mathbf{H}(\mathrm{div}\,, \Omega)$. Thus, one has $(\mu \Delta \mathbf{u}_0 + \mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - \lambda \mathrm{div}\,(\nabla \mathbf{b}_0 \odot \nabla \mathbf{b}_0)) \cdot \mathbf{n}|_{\partial\Omega} \in H^{-1/2}(\Omega)$. Finally, we note that the following compatibility condition is satisfied:

$$\int_\Omega \mathrm{div}\,(\mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - \lambda \mathrm{div}\,(\nabla \mathbf{b}_0 \odot \nabla \mathbf{b}_0))dx$$
$$= \int_{\partial\Omega} (\mu \Delta \mathbf{u}_0 + \mathbf{f}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - \lambda \mathrm{div}\,(\nabla \mathbf{b}_0 \odot \nabla \mathbf{b}_0)) \cdot \mathbf{n} ds$$

due to the fact $\mathrm{div}\,(\Delta \mathbf{u}_0) = 0$. From these observations, it follows that the problem (9.1) exists a unique weak solution $p_0 \in H^1(\Omega) \cap M$. Moreover, the solution p_0 is the limit of the pressure $p(x,t)$ in $H^1(\Omega) \cap M$ as $t \longrightarrow 0$. To make our point precise, we give the following lemma:

Lemma 9.1 *Let the initial values* $\boldsymbol{u}_0 \in \boldsymbol{D}(A)$ *and* $\boldsymbol{b}_0 \in \boldsymbol{H}^3(\Omega)$ *with* $|\boldsymbol{b}_0| = 1$ *in* Ω. *Suppose that the solution* $(\boldsymbol{u}, p, \boldsymbol{b})$ *to the problem* (1.1)–(1.5) *satisfies* $||A\boldsymbol{u}(x,t) - A\boldsymbol{u}_0(x)||_{L^2} \longrightarrow 0$ *and* $||\boldsymbol{b}(x,t) - \boldsymbol{b}_0(x)||_{H^3} \longrightarrow 0$ *as* $t \longrightarrow 0$. *Then the pressure* $p(x,t)$ *tends to the solution* p_0 *to the problem* (9.1) *in the sense that*

$$||\nabla p(x,t) - \nabla p_0(x)||_{L^2} \longrightarrow 0 \quad as\ t \longrightarrow 0.$$

Proof For $t > 0$, it follows from (1.1) and div $\mathbf{u} = 0$ that the pressure $p \in H^1(\Omega) \cap M$ is the weak solution to the problem

$$\begin{cases} \Delta p = \text{div}\,(\mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \lambda\text{div}\,(\nabla\mathbf{b}\odot\nabla\mathbf{b})), & \text{in } \Omega, \\ \partial_{\mathbf{n}} p = (\mu\Delta\mathbf{u} + \mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \lambda\text{div}\,(\nabla\mathbf{b}\odot\nabla\mathbf{b}))\cdot\mathbf{n}, & \text{on } \partial\Omega. \end{cases} \tag{9.2}$$

From $||A\mathbf{u}(x,t) - A\mathbf{u}_0(x)||_{L^2} \longrightarrow 0$ and $||\mathbf{b}(x,t) - \mathbf{b}_0(x)||_{H^3} \longrightarrow 0$ as $t \longrightarrow 0$, we can see that

$$\begin{aligned} &\text{div}\,(\mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \lambda\text{div}\,(\nabla\mathbf{b}\odot\nabla\mathbf{b})) \\ &\quad\longrightarrow \text{div}\,(\mathbf{f}_0 - (\mathbf{u}_0\cdot\nabla)\mathbf{u}_0 - \lambda\text{div}\,(\nabla\mathbf{b}_0\odot\nabla\mathbf{b}_0)) \end{aligned}$$

in $L^2(\Omega)$, and

$$\begin{aligned} &(\mu\Delta\mathbf{u} + \mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \lambda\text{div}\,(\nabla\mathbf{b}\odot\nabla\mathbf{b}))\cdot\mathbf{n} \\ &\quad\longrightarrow (\mu\Delta\mathbf{u}_0 + \mathbf{f}_0 - (\mathbf{u}_0\cdot\nabla)\mathbf{u}_0 - \lambda\text{div}\,(\nabla\mathbf{b}_0\odot\nabla\mathbf{b}_0))\cdot\mathbf{n} \end{aligned}$$

in $H^{-1/2}(\partial\Omega)$ as $t \longrightarrow 0$. These facts imply the desired result. □

Let $p_0 \in H^1(\Omega) \cap M$ be defined by the problem (9.1). Then the non-local compatibility conditions are concluded in the following lemma.

Lemma 9.2 *Under the assumptions of Lemma 9.1, if* $||\nabla \boldsymbol{u}_t(x,t)||_{L^2}$ *and* $||\boldsymbol{b}(x,t)||_{H^1}$ *remain bounded as* $t \longrightarrow 0$*, then there must hold*

$$\nabla p_0 = (\mu\Delta\boldsymbol{u}_0 + \boldsymbol{f}_0 - (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0 - \lambda div\,(\nabla\boldsymbol{b}_0\odot\nabla\boldsymbol{b}_0)), \quad on\ \partial\Omega, \tag{9.3}$$

$$\nabla(\gamma\Delta\boldsymbol{b}_0 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{b}_0 + \gamma|\nabla\boldsymbol{b}_0|^2\boldsymbol{b}_0)\cdot\boldsymbol{n} = 0, \quad on\ \partial\Omega. \tag{9.4}$$

Proof As $t \longrightarrow 0$, it follows from $||A\mathbf{u}(x,t) - A\mathbf{u}_0(x)||_{L^2} \longrightarrow 0$ and $||\mathbf{b}(x,t) - \mathbf{b}_0(x)||_{H^3} \longrightarrow 0$ that

$$\mathbf{u}_t(x,t) \longrightarrow \mu\Delta\mathbf{u}_0 - \nabla p_0 + \mathbf{f}_0 - (\mathbf{u}_0\cdot\nabla)\mathbf{u}_0 - \lambda\text{div}\,(\nabla\mathbf{b}_0\odot\nabla\mathbf{b}_0) \quad \text{in } \mathbf{L}^2(\Omega), \tag{9.5}$$

$$\mathbf{b}_t(x,t) \longrightarrow \gamma\Delta\mathbf{b}_0 + (\mathbf{u}_0\cdot\nabla)\mathbf{b}_0 + \gamma|\nabla\mathbf{b}_0|^2\mathbf{b}_0 \quad \text{in } \mathbf{H}^1(\Omega). \tag{9.6}$$

If $||\nabla\mathbf{u}_t(x,t)||_{L^2}$ and $||\mathbf{b}(x,t)||_{H^1}$ remain bounded as $t \longrightarrow 0$, then the convergences (9.5) and (9.6) hold weakly in $\mathbf{H}^1(\Omega)$ and $\mathbf{H}^2(\Omega)$, respectively. Thus,

$$\begin{aligned} &\mathbf{u}_t(x,t)|_{\partial\Omega} \longrightarrow (\mu\Delta\mathbf{u}_0 - \nabla p_0 + \mathbf{f}_0 - (\mathbf{u}_0\cdot\nabla)\mathbf{u}_0 - \lambda\text{div}\,(\nabla\mathbf{b}_0\odot\nabla\mathbf{b}_0))|_{\partial\Omega}, \\ &\nabla\mathbf{b}_t\cdot\mathbf{n}|_{\partial\Omega} \longrightarrow \nabla(\gamma\Delta\mathbf{b}_0 + (\mathbf{u}_0\cdot\nabla)\mathbf{b}_0 + \gamma|\nabla\mathbf{b}_0|^2\mathbf{b}_0)\cdot\mathbf{n}|_{\partial\Omega}, \end{aligned}$$

hold weakly in $\mathbf{H}^{1/2}(\partial\Omega)$ as $t \longrightarrow 0$. The facts $\mathbf{u}_t|_{\partial\Omega} = 0$ and $\nabla\mathbf{b}_t\cdot\mathbf{n}|_{\partial\Omega} = 0$ for any $t > 0$ imply the desired non-local compatibility conditions (9.3) and (9.4). □

Under these non-local compatibility conditions, we can estimate $\mathbf{u}_t(0)$ and $\mathbf{b}_t(0)$ in $\mathbf{H}^1(\Omega)$ as follows.

Lemma 9.3 *Let* $\boldsymbol{f}_0 \in \boldsymbol{H}$ *and* $(\boldsymbol{u}_0, \pi) \in \boldsymbol{D}(A) \times H^1(\Omega) \cap M$ *be determined by the Stokes problem (2.15). Under the assumptions of Lemma 9.1, there holds that* $\boldsymbol{u}_t(0)$ *and* $\boldsymbol{b}_t(0)$ *belong to* $\boldsymbol{H}^1(\Omega)$.

Proof Taking $t = 0$ at (1.1) deduces to

$$\mathbf{u}_t(0) - \mu\Delta\mathbf{u}_0 + (\mathbf{u}_0\cdot\nabla)\mathbf{u}_0 + \nabla p_0 + \lambda\text{div}\,(\nabla\mathbf{b}_0\odot\nabla\mathbf{b}_0) = \mathbf{f}_0. \tag{9.7}$$

Since $(\mathbf{u}_0, \pi) \in \mathbf{D}(A) \times H^1(\Omega) \cap M$ is determined by the Stokes problem (2.15), then (9.7) can be rewritten as

$$\mathbf{u}_t(0) + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \nabla(p_0 - \pi) + \lambda \text{div}\,(\nabla \mathbf{b}_0 \odot \nabla \mathbf{b}_0) = 0.$$

By using the following formula:

$$\text{div}\,(\nabla \mathbf{b} \odot \nabla \mathbf{b}) = \Delta \mathbf{b} \cdot \nabla \mathbf{b} + \frac{1}{2}\nabla(|\nabla \mathbf{b}|^2),$$

and applying $\mathbb{P}_{\mathbf{H}}$ to the resulting equation, we obtain

$$\mathbf{u}_t(0) = -\mathbb{P}_{\mathbf{H}}((\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \lambda \Delta \mathbf{b}_0 \cdot \nabla \mathbf{b}_0).$$

It follows from (2.2) that

$$\begin{aligned} ||\mathbf{u}_t(0)||_{H^1} &= ||\mathbb{P}_{\mathbf{H}}((\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \lambda \Delta \mathbf{b}_0 \cdot \nabla \mathbf{b}_0)||_{H^1} \\ &\le C||(\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \lambda \Delta \mathbf{b}_0 \cdot \nabla \mathbf{b}_0||_{H^1}. \end{aligned}$$

Note $\mathbf{u}_0 \in \mathbf{D}(A)$ and $\mathbf{b}_0 \in \mathbf{H}^3(\Omega)$. Then we have

$$\begin{aligned} &||(\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \lambda \Delta \mathbf{b}_0 \cdot \nabla \mathbf{b}_0||_{L^2} \\ &\quad \le C||\mathbf{u}_0||_{L^\infty}||\nabla \mathbf{u}_0||_{L^2} + C||\nabla \mathbf{b}_0||_{L^\infty}||\Delta \mathbf{b}_0||_{L^2} \\ &\quad \le C||A\mathbf{u}_0||_{L^2}||\nabla \mathbf{u}_0||_{L^2} + C||\mathbf{b}_0||_{H^3}||\Delta \mathbf{b}_0||_{L^2} \le C, \end{aligned}$$

and

$$\begin{aligned} &||\nabla((\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \lambda \Delta \mathbf{b}_0 \cdot \nabla \mathbf{b}_0)||_{L^2} \\ &\quad \le ||\nabla \mathbf{u}_0||_{L^4}^2 + ||\mathbf{u}_0||_{L^\infty}||\nabla^2 \mathbf{u}_0||_{L^2} + ||\nabla \mathbf{b}_0||_{L^\infty}||\nabla \Delta \mathbf{b}_0||_{L^2} + ||\nabla^2 \mathbf{b}_0||_{L^2}||\Delta \mathbf{b}_0||_{L^2} \\ &\quad \le C(||A\mathbf{u}_0||_{L^2}^2 + ||\mathbf{b}_0||_{H^3}^2) \le C. \end{aligned}$$

The above two estimates imply $\mathbf{u}_t(0) \in \mathbf{H}^1(\Omega)$. Taking $t = 0$ in (1.2) leads to

$$\mathbf{b}_t(0) = \gamma \Delta \mathbf{b}_0 - (\mathbf{u}_0 \cdot \nabla)\mathbf{b}_0 + \gamma |\nabla \mathbf{b}_0|^2 \mathbf{b}_0.$$

By using a similar method, we can prove

$$\begin{aligned} ||\mathbf{b}_t(0)||_{L^2} &\le \gamma ||\Delta \mathbf{b}_0||_{L^2} + ||(\mathbf{u}_0 \cdot \nabla)\mathbf{b}_0||_{L^2} + \gamma |||\nabla \mathbf{b}_0|^2 \mathbf{b}_0||_{L^2} \\ &\le \gamma ||\Delta \mathbf{b}_0||_{L^2} + ||\mathbf{u}_0||_{L^\infty}||\nabla \mathbf{b}_0||_{L^2} + \gamma ||\nabla \mathbf{b}_0||_{L^4}^2 \le C, \end{aligned}$$

and

$$\begin{aligned} ||\nabla \mathbf{b}_t(0)||_{L^2} &\le \gamma ||\nabla \Delta \mathbf{b}_0||_{L^2} + ||\nabla \mathbf{u}_0||_{L^4}||\nabla \mathbf{b}_0||_{L^4} + ||\mathbf{u}_0||_{L^\infty}||\nabla^2 \mathbf{b}_0||_{L^2} \\ &\quad + \gamma ||\nabla \mathbf{b}_0||_{L^\infty}||\nabla^2 \mathbf{b}_0||_{L^2}||\mathbf{b}_0||_{L^\infty} + \gamma ||\nabla \mathbf{b}_0||_{L^6}^3 \le C, \end{aligned}$$

which imply $\mathbf{b}_t(0) \in \mathbf{H}^1(\Omega)$. □

Based on the results derived in Lemma 9.3, we can show some regularities of $\mathbf{u}_{tt}$, p_t and $\mathbf{b}_{tt}$ in next theorem.

Theorem 9.1 *Under the assumptions of Theorem 2.1 and Lemma 9.3, suppose $\boldsymbol{f}_t \in \boldsymbol{L}^\infty(0, T; \boldsymbol{V}_0') \cap \boldsymbol{L}^2(0, T; \boldsymbol{H})$, then we have*

$$\begin{aligned} \boldsymbol{u}_t &\in \boldsymbol{L}^\infty(0, T^\star; \boldsymbol{V}) \cap \boldsymbol{L}^2(0, T^\star; \boldsymbol{D}(A)),\ \nabla p_t \in \boldsymbol{L}^2(0, T^\star; \boldsymbol{L}^2(\Omega)), \\ \boldsymbol{u}_{tt} &\in \boldsymbol{L}^\infty(0, T^\star; \boldsymbol{V}_0') \cap \boldsymbol{L}^2(0, T^\star; \boldsymbol{H}),\ \boldsymbol{b}_{tt} \in \boldsymbol{L}^2(0, T^\star; \boldsymbol{L}^2(\Omega)), \end{aligned}$$

where $T^\star$ is defined in Theorem 2.1.

Proof Differentiating (1.1) with respect to t, we get

$$\mathbf{u}_{tt} - \mu\Delta\mathbf{u}_t + (\mathbf{u}_t \cdot \nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}_t + \nabla p_t + 2\lambda \text{div}\,(\nabla\mathbf{b}_t \odot \nabla\mathbf{b}) = \mathbf{f}_t. \tag{9.8}$$

Multiplying (9.8) by $\mathbf{u}_{tt}$ and integrating over Ω, we have

$$\begin{aligned}
&||\mathbf{u}_{tt}||_{L^2}^2 + \frac{\mu}{2}\frac{d}{dt}||\nabla\mathbf{u}_t||_{L^2}^2 \\
&\quad= (\mathbf{f}_t, \mathbf{u}_{tt}) - b(\mathbf{u}_t, \mathbf{u}, \mathbf{u}_{tt}) - b(\mathbf{u}, \mathbf{u}_t, \mathbf{u}_{tt}) - 2\lambda(\text{div}\,(\nabla\mathbf{b}_t \odot \nabla\mathbf{b}), \mathbf{u}_{tt}) \\
&\quad\le (||\mathbf{f}_t||_{L^2} + ||\nabla\mathbf{u}||_{L^\infty}||\mathbf{u}_t||_{L^2} + ||\mathbf{u}||_{L^\infty}||\nabla\mathbf{u}_t||_{L^2} \\
&\qquad + 2\lambda||\nabla\mathbf{b}||_{L^\infty}||\mathbf{b}_t||_{H^2} + 2\lambda||\nabla^2\mathbf{b}||_{L^\infty}||\nabla\mathbf{b}_t||_{L^2})||\mathbf{u}_{tt}||_{L^2} \\
&\quad\le \frac{1}{2}||\mathbf{u}_{tt}||_{L^2}^2 + C(||\mathbf{f}_t||_{L^2}^2 + ||\mathbf{u}||_{W^{2,4}}^2||\mathbf{u}_t||_{L^2}^2 + ||A\mathbf{u}||_{L^2}^2||\nabla\mathbf{u}_t||_{L^2}^2 \\
&\qquad + ||\mathbf{b}||_{H^3}^2||\mathbf{b}_t||_{H^2}^2 + ||\mathbf{b}||_{H^4}^2||\mathbf{b}_t||_{H^1}^2),
\end{aligned}$$

where we use div $\mathbf{u}_{tt} = 0$. It follows from (2.10)–(2.13) that $\mathbf{u}_{tt} \in \mathbf{L}^2(0, T^\star; \mathbf{H})$ and $\mathbf{u}_t \in \mathbf{L}^\infty(0, T^\star; \mathbf{V})$ if we integrate the above inequality with respect to t from 0 to $t \le T^\star$ and note that $\mathbf{u}_t(0) \in \mathbf{H}^1(\Omega)$. By using a similar method, multiplying (9.8) by $A\mathbf{u}_t$ and integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla\mathbf{u}_t||_{L^2}^2 + \mu||A\mathbf{u}_t||_{L^2}^2 \\
&\quad= (\mathbf{f}_t, A\mathbf{u}_t) - b(\mathbf{u}_t, \mathbf{u}, A\mathbf{u}_t) - b(\mathbf{u}, \mathbf{u}_t, A\mathbf{u}_t) - 2\lambda(\text{div}\,(\nabla\mathbf{b}_t \odot \nabla\mathbf{b}), A\mathbf{u}_t) \\
&\quad\le \frac{\mu}{2}||A\mathbf{u}_t||_{L^2}^2 + \frac{C}{\mu}(||\mathbf{f}_t||_{L^2}^2 + ||\mathbf{u}||_{W^{2,4}}^2||\mathbf{u}_t||_{L^2}^2 + ||A\mathbf{u}||_{L^2}^2||\nabla\mathbf{u}_t||_{L^2}^2 \\
&\qquad + ||\mathbf{b}||_{H^3}^2||\mathbf{b}_t||_{H^2}^2 + ||\mathbf{b}||_{H^4}^2||\mathbf{b}_t||_{H^1}^2),
\end{aligned}$$

which implies $\mathbf{u}_t \in L^2(0, T^\star; \mathbf{D}(A))$. Multiplying (9.8) by $\mathbf{v} \in \mathbf{V}_0$ leads to

$$(\mathbf{u}_{tt}, \mathbf{v}) + \mu(\nabla\mathbf{u}_t, \nabla\mathbf{v}) + b(\mathbf{u}_t, \mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}_t, \mathbf{v}) - 2\lambda(\nabla\mathbf{b}_t \odot \nabla\mathbf{b}, \nabla\mathbf{v}) = (\mathbf{f}_t, \mathbf{v}).$$

Then it is easy to show

$$||\mathbf{u}_{tt}||_{V_0'} \le C(||\nabla\mathbf{u}_t||_{L^2} + ||\nabla\mathbf{u}||_{L^2}||\nabla\mathbf{u}_t||_{L^2} + ||\nabla\mathbf{b}||_{L^\infty}||\nabla\mathbf{b}_t||_{L^2} + ||\mathbf{f}_t||_{V_0'}),$$

which implies $\mathbf{u}_{tt} \in \mathbf{L}^\infty(0, T^\star; \mathbf{V}_0')$. To estimate ∇p_t, we use (9.8) to deduce that

$$\begin{aligned}
||\nabla p_t||_{L^2} &\le ||\mathbf{u}_{tt}||_{L^2} + \mu||\Delta\mathbf{u}_t||_{L^2} + ||(\mathbf{u}_t\cdot\nabla)\mathbf{u}||_{L^2} + ||(\mathbf{u}\cdot\nabla)\mathbf{u}_t||_{L^2} \\
&\quad + 2\lambda||\text{div}\,(\nabla\mathbf{b}_t \odot \nabla\mathbf{b})||_{L^2} + ||\mathbf{f}_t||_{L^2} \\
&\le C(||\mathbf{u}_{tt}||_{L^2} + ||A\mathbf{u}_t||_{L^2} + |\mathbf{f}_t||_{L^2} + ||\nabla\mathbf{u}||_{L^\infty}||\mathbf{u}_t||_{L^2} + ||\mathbf{u}||_{L^\infty}||\nabla\mathbf{u}_t||_{L^2} \\
&\quad + ||\nabla\mathbf{b}||_{L^\infty}||\mathbf{b}_t||_{H^2} + ||\nabla^2\mathbf{b}||_{L^\infty}||\nabla\mathbf{b}_t||_{L^2}).
\end{aligned}$$

From $\mathbf{u}_{tt} \in \mathbf{L}^2(0, T^\star; \mathbf{H})$ and $\mathbf{u}_t \in L^2(0, T^\star; \mathbf{D}(A))$ shown in the previous paragraph, it is easily seen that $\nabla p_t \in \mathbf{L}^2(0, T^\star; \mathbf{L}^2(\Omega))$ after integrating the above inequality from 0 to $t \le T^\star$ and using (2.10)–(2.13).

Differentiating (1.2) with respect to t yields

$$\mathbf{b}_{tt} - \gamma\Delta\mathbf{b}_t + (\mathbf{u}_t\cdot\nabla)\mathbf{b} + (\mathbf{u}\cdot\nabla)\mathbf{b}_t = \gamma|\nabla\mathbf{b}|^2\mathbf{b}_t + 2\gamma(\nabla\mathbf{b}\cdot\nabla\mathbf{b}_t)\mathbf{b}.$$

It follows from (2.4)–(2.9) that

$$\begin{aligned}||\mathbf{b}_{tt}||_{L^2} &\leq \gamma||\Delta\mathbf{b}_t||_{L^2} + ||(\mathbf{u}_t\cdot\nabla)\mathbf{b}||_{L^2} + ||(\mathbf{u}\cdot\nabla)\mathbf{b}_t||_{L^2}\\ &\quad + \gamma|||\nabla\mathbf{b}|^2\mathbf{b}_t||_{L^2} + 2\gamma||(\nabla\mathbf{b}\cdot\nabla\mathbf{b}_t)\mathbf{b}||_{L^2}\\ &\leq C(||\mathbf{b}_t||_{H^2} + ||\mathbf{u}_t||_{L^2}||\nabla\mathbf{b}||_{L^\infty} + ||\mathbf{u}||_{L^\infty}||\nabla\mathbf{b}_t||_{L^2}\\ &\quad + ||\nabla\mathbf{b}||^2_{L^\infty}||\mathbf{b}_t||_{L^2} + ||\nabla\mathbf{b}||_{L^\infty}||\nabla\mathbf{b}_t||_{L^2})\\ &\leq C(||\mathbf{b}_t||_{H^2} + ||\mathbf{u}_t||_{L^2}||\mathbf{b}||_{H^3} + ||A\mathbf{u}||_{L^2}||\nabla\mathbf{b}_t||_{L^2}\\ &\quad + ||\mathbf{b}||^2_{H^3}||\mathbf{b}_t||_{L^2} + ||\mathbf{b}||_{H^3}||\nabla\mathbf{b}_t||_{L^2}),\end{aligned}$$

which completes the proof of $\mathbf{b}_{tt} \in \mathbf{L}^2(0, T^\star; \mathbf{L}^2(\Omega))$ by using (2.10)–(2.13). □

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