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Hodge Decomposition Methods for a Quad-Curl Problem on Planar Domains

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Abstract We develop and analyze P_k Lagrange finite element methods for a quad-curl problem on planar domains that is based on the Hodge decomposition of divergence-free vector fields. Numerical results that illustrate the performance of the finite element methods are also presented.

Keywords Quad-curl problem · Hodge decomposition · Lagrange finite element

Mathematics Subject Classification 65N30 · 65N15 · 35Q60

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. The energy space for the quad-curl problem to be considered in this paper is

$$\mathbb{E} = \{\boldsymbol{v} \in [L_2(\Omega)]^2 : \text{curl}\, \boldsymbol{v} \in H_0^1(\Omega), \text{div}\, \boldsymbol{v} = 0 \text{ and } \boldsymbol{n} \times \boldsymbol{v} = 0 \text{ on } \partial\Omega\}, \tag{1.1}$$

with the norm $\|\cdot\|_{\mathbb{E}}$ given by

$$\|\boldsymbol{v}\|_{\mathbb{E}}^2 = \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + |\text{curl}\, \boldsymbol{v}|_{H^1(\Omega)}^2. \tag{1.2}$$

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Here and below we will follow standard notation for differential operators, function spaces and norms that can be found for example in [1,13,16,21,25].

We will consider the following problem: Find $\boldsymbol{u} \in \mathbb{E}$ such that

$$\big(\text{curl}\,(\text{curl}\,\boldsymbol{u}), \text{curl}\,(\text{curl}\,\boldsymbol{v})\big) + \beta(\text{curl}\,\boldsymbol{u}, \text{curl}\,\boldsymbol{v}) + \gamma(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \mathbb{E}, \quad (1.3)$$

where $(\cdot, \cdot)$ denotes the inner product for $L_2(\Omega)$ (or $[L_2(\Omega)]^2$), β and γ are nonnegative constants ($\gamma > 0$ if Ω is not simply connected), and $\boldsymbol{f} \in [L_2(\Omega)]^2$. Since the divergence-free condition is included in the definition of $\mathbb{E}$, the problem (1.3) provides an elliptic formulation for the quad-curl problem.

Remark 1.1 In two dimensions the curl of the vector field $\boldsymbol{v} = [v_1,\ v_2]^t$ is the scalar function $\text{curl}\,\boldsymbol{v} = (\partial v_2/\partial x_1) - (\partial v_1/\partial x_2)$, and the curl of a scalar function ϕ is the vector field $\text{curl}\,\phi = [\partial\phi/\partial x_2,\ -\partial\phi/\partial x_1]^t$. An alternative notation for $\text{curl}\,\phi$ is $\text{rot}\,\phi$.

The quad-curl problem is related to the Maxwell transmission eigenvalue problem (cf. [14, 26]) and mathematical models for magnetohydrodynamics with hyperresistivity (cf. [7,15]). Finite element methods for the quad-curl problem (based on a non-elliptic formulation) were recently developed in [23,29,31] using a nonconforming finite element method, a discontinuous Galerkin method and a mixed finite element method. In this paper we will use a Hodge decomposition approach to reduce (1.3) to second order elliptic boundary value problems that can be solved by simple H^1 conforming finite element methods.

We note that the Hodge decomposition approach to time harmonic Maxwell equations on planar domains was investigated in [9] for the perfectly conducting boundary condition and extended to general boundary conditions in [11] with applications to metamaterials. Adaptive and multigrid methods for these problems based on the Hodge decomposition approach were developed in [10,12,17]. Applications of the Hodge decomposition to other electromagnetic problems can also be found in [2,3,5].

The rest of the paper is organized as follows. We recall the Hodge decomposition for divergence-free vector fields in Sect. 2, where the well-posedness of (1.3) is also addressed. The reduction of (1.3) to second order elliptic boundary value problems is established in Sect. 3. Based on this reduction, we develop P_k finite element methods for (1.3) in Sect. 4, followed by a convergence analysis in Sect. 5. Numerical results are presented in Sect. 6, and we end with some concluding remarks in Sect. 7.

2 Hodge Decomposition for $\boldsymbol{H}(\textbf{div}^{\,0}; \boldsymbol{\Omega})$

The space $H(\text{div}^{\,0}; \Omega)$ of divergence-free vector fields is the orthogonal complement of $\text{grad}\, H_0^1(\Omega)$, i.e.,

$$H(\text{div}^{\,0}; \Omega) = \{\boldsymbol{v} \in [L_2(\Omega)]^2 : (\boldsymbol{v},\ \text{grad}\,\eta) = 0 \quad \forall\, \eta \in H_0^1(\Omega)\},$$

and $L_2^0(\Omega) = \{v \in L_2(\Omega) : (1, v) = 0\}$ is the zero-mean subspace of $L_2(\Omega)$. Given any $\boldsymbol{v} \in H(\text{div}^{\,0}; \Omega)$, we have a unique decomposition (cf. [9,21]):

$$\boldsymbol{v} = \text{curl}\,\psi + \sum_{j=1}^{m} d_j\, \text{grad}\,\varphi_j, \qquad (2.1)$$

where $\psi \in H^1(\Omega) \cap L_2^0(\Omega)$, the non-negative integer m is the Betti number for Ω ($m = 0$ if Ω is simply connected, cf. Fig. 1), d_j $(1 \le j \le m)$ are real numbers, and the harmonic functions $\varphi_1, \ldots, \varphi_m$ are defined as follows.

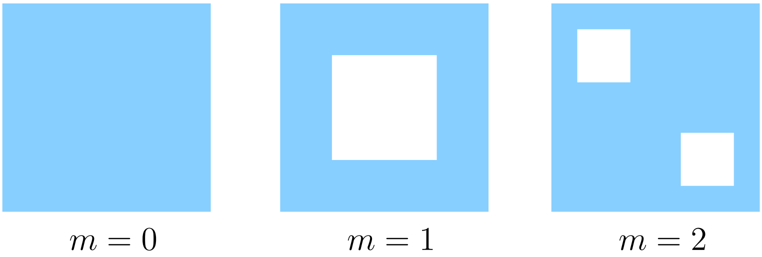

Fig. 1 Betti numbers

Let the outer boundary of Ω be denoted by Γ_0 and the m components of the inner boundary be denoted by $\Gamma_1, \ldots, \Gamma_m$. Then the harmonic functions φ_j are determined by

$$(\operatorname{grad} \varphi_j, \operatorname{grad} v) = 0 \quad \forall v \in H_0^1(\Omega), \tag{2.2a}$$

$$\varphi_j\big|_{\Gamma_0} = 0, \tag{2.2b}$$

$$\varphi_j\big|_{\Gamma_k} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \quad \text{for} \quad 1 \leq k \leq m. \tag{2.2c}$$

Remark 2.1 Note that $\operatorname{grad} \varphi_j$ belongs to $\mathbb{E}$ for $1 \leq j \leq m$ (cf. [9, Corollary 2.5]) and

$$(\operatorname{curl} \psi, \operatorname{grad} \varphi_j) = 0 \qquad \forall \psi \in H^1(\Omega) \tag{2.3}$$

(cf. [9, Lemma 2.4]).

2.1 Properties of the Space $\mathbb{E}$

Let $\boldsymbol{v} \in \mathbb{E}$ be represented by (2.1). We have

$$(\operatorname{curl} \psi, \operatorname{curl} \rho) = (\boldsymbol{v}, \operatorname{curl} \rho) \qquad \forall \rho \in H^1(\Omega)$$

by (2.3). Since $\mathbb{E}$ is a subspace of $H_0(\operatorname{curl}; \Omega)$, we also have (cf. [21, Theorems 2.2.11 and 2.2.12])

$$(\boldsymbol{v}, \operatorname{curl} \rho) = (\operatorname{curl} \boldsymbol{v}, \rho) \qquad \forall \rho \in H^1(\Omega).$$

It follows that the function $\psi \in H^1(\Omega) \cap L_2^0(\Omega)$ satisfies

$$(\operatorname{curl} \psi, \operatorname{curl} \rho) = (\operatorname{curl} \boldsymbol{v}, \rho) \qquad \forall \rho \in H^1(\Omega). \tag{2.4}$$

Remark 2.2 Since $\operatorname{curl} \phi = [\partial\phi/\partial x_2, \ -\partial\phi/\partial x_1]^t$, we have

$$(\operatorname{curl} \phi, \operatorname{curl} \rho) = (\operatorname{grad} \phi, \operatorname{grad} \rho) \qquad \forall \phi, \rho \in H^1(\Omega)$$

and $\|\operatorname{curl} \rho\|_{L_2(\Omega)} = |\rho|_{H^1(\Omega)}$ for all $\rho \in H^1(\Omega)$.

We can deduce properties of $\boldsymbol{v}$ from the following results (cf. [22, Sect. 5.1] and [18, Sect. 2.5]) for elliptic boundary value problems on polygonal domains, where ω is the largest interior angle at the corners of Ω.

Lemma 2.3 *Let* $\mu \in H_0^1(\Omega)$ *satisfy*

$$(\operatorname{grad}\mu,\ \operatorname{grad}\eta) + \beta(\mu,\eta) = (g,\eta) \qquad \forall\, \eta \in H_0^1(\Omega),$$

where $g \in H^1(\Omega)$. *Then we have* $\mu \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$ *for any* $\epsilon > 0$ *and*

$$\|\mu\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \le C_\epsilon \|g\|_{H^1(\Omega)}.$$

Lemma 2.4 *Let* $\lambda \in H^1(\Omega)$ *satisfy*

$$(\operatorname{grad}\lambda,\ \operatorname{grad}\psi) + (\lambda,1)(\psi,1) = (g,\psi) \qquad \forall\, \psi \in H^1(\Omega),$$

where $g \in H^1(\Omega)$. *Then we have* $\lambda \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$ *for any* $\epsilon > 0$ *and*

$$\|\lambda\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \le C_\epsilon \|g\|_{H^1(\Omega)}.$$

Since ψ belongs to $H^1(\Omega)\cap L_2^0(\Omega)$ and curl $\boldsymbol{v}$ belongs to $H_0^1(\Omega)$, we can apply Lemma 2.4 to (2.4) and conclude that ψ belongs to $H^{1+(\pi/\omega)-\epsilon}(\Omega)$ for any $\epsilon > 0$. Note that (2.2a), (2.2b), (2.2c) can be transformed to a problem of the form in Lemma 2.3 where $\beta = 0$ and $g \in C^\infty(\bar{\Omega})$. Hence we can apply Lemma 2.3 to conclude that φ_j belongs to $H^{1+(\pi/\omega)-\epsilon}(\Omega)$ for $1 \le j \le m$ and any $\epsilon > 0$. Then (2.1) implies that $\boldsymbol{v}$ belongs to $[H^{(\pi/\omega)-\epsilon}(\Omega)]^2$ and we have established the following result.

Theorem 2.5 *The space* $\mathbb{E}$ *is a subspace of* $[H^{(\pi/\omega)-\epsilon}(\Omega)]^2$ *for any* $\epsilon > 0$, *where* ω *is the largest angle at the corners of* Ω.

Remark 2.6 If Ω is a smooth domain, then we can apply the elliptic regularity theory for such domains [27] to conclude that ψ belongs to $H^3(\Omega)$ and φ_j belongs to $C^\infty(\bar{\Omega})$ for $1 \le j \le m$. It follows that $\mathbb{E}$ is a subspace of $[H^2(\Omega)]^2$.

2.2 Well-Posedness of (1.3)

Since $\mathbb{E}$ is compactly embedded in $[L_2(\Omega)]^2$ by Theorem 2.5 and the Rellich–Kondrachov Theorem [1], we can establish the well-posedness of (1.3) by the Fredholm theory [30]. It suffices to show that if $\boldsymbol{v} \in \mathbb{E}$ satisfies

$$\big(\operatorname{curl}(\operatorname{curl}\boldsymbol{v}), \operatorname{curl}(\operatorname{curl}\boldsymbol{w})\big) + \beta(\operatorname{curl}\boldsymbol{v}, \operatorname{curl}\boldsymbol{w}) + \gamma(\boldsymbol{v},\boldsymbol{w}) = 0 \qquad \forall\, \boldsymbol{w} \in \mathbb{E}, \tag{2.5}$$

then $\boldsymbol{v} = 0$.

This is obvious if $\gamma > 0$. In the case where $\gamma = 0$ and Ω is simply connected, we deduce from (2.5) that

$$\operatorname{curl}(\operatorname{curl}\boldsymbol{v}) = 0$$

and hence curl $\boldsymbol{v} = 0$ (because curl $\boldsymbol{v} \in H_0^1(\Omega)$). It then follows from (2.4) that $\psi = 0$ and hence $\boldsymbol{v} = 0$ by (2.1).

3 Reduction to Second Order Elliptic Boundary Value Problems

According to (2.1), we can write

$$\boldsymbol{u} = \operatorname{curl}\phi + \sum_{j=1}^{m} c_j \operatorname{grad}\varphi_j, \tag{3.1}$$

where $\phi \in H^1(\Omega) \cap L_2^0(\Omega)$ and $c_1, \ldots, c_m$ are real numbers. The idea of the Hodge decomposition approach is to find ϕ and $c_1, \ldots, c_m$, and then recover $\boldsymbol{u}$ by (3.1).

Below we will find problems that determine the function ϕ and the coefficients $c_1, \ldots, c_m$ in the decomposition (3.1).

3.1 A Problem for ϕ

It follows from (2.4) and (3.1) that

$$(\operatorname{curl} \phi, \operatorname{curl} \psi) = (\xi, \psi) \qquad \forall\, \psi \in H^1(\Omega), \tag{3.2}$$

where $\xi = \operatorname{curl} \boldsymbol{u} \in H_0^1(\Omega)$. Note that $\boldsymbol{n} \times \boldsymbol{u} = 0$ on $\partial\Omega$ implies $(1, \xi) = 0$ and hence the singular Neumann boundary value problem (3.2) has a unique solution $\phi \in H^1(\Omega) \cap L_2^0(\Omega)$. An equivalent formulation that avoids the zero-mean constraint is to find $\phi \in H^1(\Omega)$ such that

$$(\operatorname{curl} \phi, \operatorname{curl} \psi) + (\phi, 1)(\psi, 1) = (\xi, \psi) \qquad \forall\, \psi \in H^1(\Omega). \tag{3.3}$$

It only remains to find a problem that determines ξ.

3.2 A Problem for ξ

We begin with a lemma.

Lemma 3.1 *The function* $\xi = \operatorname{curl} \boldsymbol{u} \in H_0^1(\Omega) \cap L_2^0(\Omega)$ *satisfies*

$$\big(\operatorname{curl} \xi, \operatorname{curl} (\operatorname{curl} \boldsymbol{\zeta})\big) + \beta(\xi, \operatorname{curl} \boldsymbol{\zeta}) + \gamma(\boldsymbol{u}, \boldsymbol{\zeta}) = (Q\boldsymbol{f}, \boldsymbol{\zeta}) \qquad \forall\, \boldsymbol{\zeta} \in [C_c^\infty(\Omega)]^2, \tag{3.4}$$

where Q is the orthogonal projection from $[L_2(\Omega)]^2$ *onto* $H(\operatorname{div}^0; \Omega)$.

Proof Since $\boldsymbol{\zeta} - Q\boldsymbol{\zeta}$ belongs to $\operatorname{grad}(H_0^1(\Omega))$, we have $\operatorname{curl} Q\boldsymbol{\zeta} = \operatorname{curl} \boldsymbol{\zeta} \in H_0^1(\Omega)$, $\boldsymbol{n} \times Q\boldsymbol{\zeta} = \boldsymbol{n} \times \boldsymbol{\zeta} = 0$ on $\partial\Omega$ (cf. [9, Corollary 2.5]) and hence $Q\boldsymbol{\zeta} \in \mathbb{E}$. Therefore (3.4) follows from (1.3):

$$\begin{aligned}
&\big(\operatorname{curl} \xi, \operatorname{curl} (\operatorname{curl} \boldsymbol{\zeta})\big) + \beta(\xi, \operatorname{curl} \boldsymbol{\zeta}) + \gamma(\boldsymbol{u}, \boldsymbol{\zeta}) \\
&\quad = \big(\operatorname{curl} (\operatorname{curl} \boldsymbol{u}), \operatorname{curl} (\operatorname{curl} \boldsymbol{\zeta})\big) + \beta(\operatorname{curl} \boldsymbol{u}, \operatorname{curl} \boldsymbol{\zeta}) + \gamma(\boldsymbol{u}, \boldsymbol{\zeta}) \\
&\quad = \big(\operatorname{curl} (\operatorname{curl} \boldsymbol{u}), \operatorname{curl} (\operatorname{curl} Q\boldsymbol{\zeta})\big) + \beta(\operatorname{curl} \boldsymbol{u}, \operatorname{curl} Q\boldsymbol{\zeta}) + \gamma(\boldsymbol{u}, Q\boldsymbol{\zeta}) \\
&\quad = (Q\boldsymbol{f}, \boldsymbol{\zeta})
\end{aligned}$$

□

It follows from (3.4) that

$$\operatorname{curl} (-\Delta\xi) = -\beta \operatorname{curl} \xi - \gamma \boldsymbol{u} + Q\boldsymbol{f} \tag{3.5}$$

in the sense of distributions.

We will exploit (3.5) through the following lemma due to Nečas [8,20,28].

Lemma 3.2 *If τ, $\partial\tau/\partial x_1$ and $\partial\tau/\partial x_2$ belong to $H^{-1}(\Omega)$, then τ belongs to $L_2(\Omega)$.*

Since $-\Delta\xi$ belongs to $H^{-1}(\Omega) = [H_0^1(\Omega)]'$ and the right-hand side of (3.5) belongs to $[L_2(\Omega)]^2$, we can apply Lemma 3.2 to conclude that $-\Delta\xi$ belongs to $L_2(\Omega)$. Then (3.5) implies

$$-\Delta\xi \in H^1(\Omega). \tag{3.6}$$

Let $\rho \in H^1(\Omega) \cap L_2^0(\Omega)$ be defined by the following consistent singular Neumann boundary value problem

$$(\text{curl}\,\rho, \text{curl}\,\psi) = -\gamma(\boldsymbol{u}, \text{curl}\,\psi) + (Q\boldsymbol{f}, \text{curl}\,\psi) \\ = -\gamma(\xi, \psi) + (\boldsymbol{f}, \text{curl}\,\psi) \qquad \forall \psi \in H^1(\Omega). \tag{3.7}$$

Since the relations (3.5)–(3.7) imply $-\Delta\xi + \beta\xi \in H^1(\Omega)$ and

$$(\text{curl}\,(-\Delta\xi + \beta\xi), \text{curl}\,\psi) = (\text{curl}\,\rho, \text{curl}\,\psi) \qquad \forall \psi \in H^1(\Omega),$$

we have

$$-\Delta\xi + \beta\xi = \rho + c$$

for some constant c, and hence

$$(\text{curl}\,\xi, \text{curl}\,\eta) + \beta(\xi, \eta) = (\rho, \eta) \qquad \forall \eta \in H_0^1(\Omega) \cap L_2^0(\Omega). \tag{3.8}$$

Below we will show that the function $\xi \in H^1(\Omega) \cap L_2^0(\Omega)$ is determined by (3.7) and (3.8).

3.2.1 The Case Where $\gamma = 0$

When γ is 0 (and Ω is simply connected), the two equations (3.7) and (3.8) are decoupled. We can first solve (3.7) for ρ and then solve (3.8) for ξ.

In this case (3.7) becomes a consistent singular Neumann boundary value problem: Find $\rho \in H^1(\Omega) \cap L_2^0(\Omega)$ such that

$$(\text{curl}\,\rho, \text{curl}\,\psi) = (\boldsymbol{f}, \text{curl}\,\psi) \qquad \forall \psi \in H^1(\Omega). \tag{3.9}$$

An equivalent formulation without the zero-mean constraint is to find $\rho \in H^1(\Omega)$ such that

$$(\text{curl}\,\rho, \text{curl}\,\psi) + (\rho, 1)(\psi, 1) = (\boldsymbol{f}, \text{curl}\,\psi) \qquad \forall \psi \in H^1(\Omega). \tag{3.10}$$

Once we have found $\rho \in H^1(\Omega) \cap L_2^0(\Omega)$, $\xi \in H_0^1(\Omega) \cap L_2^0(\Omega)$ is determined by the well-posed (nonstandard) elliptic boundary value problem (3.8). We can also determine ξ through standard boundary value problems that do not involve the zero-mean constraint.

Lemma 3.3 *The solution ξ of* (3.8) *is given by*

$$\xi = \xi_0 - \frac{(1, \xi_0)}{(1, \xi_1)}\xi_1, \tag{3.11}$$

where $\xi_0, \xi_1 \in H_0^1(\Omega)$ satisfy

$$(\text{curl}\,\xi_0, \text{curl}\,\eta) + \beta(\xi_0, \eta) = (\rho, \eta) \qquad \forall \eta \in H_0^1(\Omega), \tag{3.12}$$

$$(\text{curl}\,\xi_1, \text{curl}\,\eta) + \beta(\xi_1, \eta) = (1, \eta) \qquad \forall \eta \in H_0^1(\Omega). \tag{3.13}$$

Proof First we note that (3.12) and (3.13) are standard elliptic boundary value problems and that (3.13) implies $(1, \xi_1) > 0$.

By construction, the function ξ belongs to $H_0^1(\Omega) \cap L_2^0(\Omega)$ and

$$(\text{curl}\,\xi, \text{curl}\,\eta) + \beta(\xi, \eta) = (\rho, \eta) - \frac{(1, \xi_0)}{(1, \xi_1)}(1, \eta) \qquad \forall \eta \in H_0^1(\Omega),$$

which implies (3.8). □

3.2.2 The Case Where $\gamma > 0$

When γ is positive, the problems (3.7) and (3.8) are coupled and we can reformulate them as the following problem:

Find $(\zeta, \xi) \in [H^1(\Omega) \cap L_2^0(\Omega)] \times [H_0^1(\Omega) \cap L_2^0(\Omega)]$ such that

$$(\operatorname{curl} \zeta, \operatorname{curl} \psi) + \gamma^{\frac{1}{2}}(\psi, \xi) = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \operatorname{curl} \psi) \quad \forall \psi \in H^1(\Omega) \cap L_2^0(\Omega), \tag{3.14a}$$

$$-\gamma^{\frac{1}{2}}(\zeta, \eta) + (\operatorname{curl} \xi, \operatorname{curl} \eta) + \beta(\xi, \eta) = 0 \quad \forall \eta \in H_0^1(\Omega) \cap L_2^0(\Omega), \tag{3.14b}$$

where $\zeta = \gamma^{-\frac{1}{2}}\rho$.

We can also write (3.14a), (3.14b) concisely as

$$\mathcal{A}\big((\zeta, \xi), (\psi, \eta)\big) = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \operatorname{curl} \psi) \qquad \forall \psi \in H^1(\Omega) \cap L_2^0(\Omega), \eta \in H_0^1(\Omega) \cap L_2^0(\Omega), \tag{3.15}$$

where the bilinear form $\mathcal{A}(\cdot, \cdot)$ on $H^1(\Omega) \times H_0^1(\Omega)$ is defined by

$$\mathcal{A}\big((\zeta, \xi), (\psi, \eta)\big) = (\operatorname{curl} \zeta, \operatorname{curl} \psi) + \gamma^{\frac{1}{2}}(\psi, \xi) - \gamma^{\frac{1}{2}}(\zeta, \eta) + (\operatorname{curl} \xi, \operatorname{curl} \eta) + \beta(\xi, \eta). \tag{3.16}$$

The bilinear form $\mathcal{A}(\cdot, \cdot)$ is clearly bounded on $H^1(\Omega) \times H_0^1(\Omega)$, and it follows from the identity

$$\mathcal{A}\big((\psi, \eta), (\psi, \eta)\big) = (\operatorname{curl} \psi, \operatorname{curl} \psi) + (\operatorname{curl} \eta, \operatorname{curl} \eta) + \beta(\eta, \eta) \tag{3.17}$$

and standard Poincaré-Friedrichs inequalities [27] that $\mathcal{A}(\cdot, \cdot)$ is coercive on $[H^1(\Omega) \cap L_2^0(\Omega)] \times H_0^1(\Omega)$. Therefore the problem (3.14a), (3.14b) is well-posed by the Lax–Milgram theorem [24].

We can also determine (ζ, ξ) through problems that do not involve the zero-mean constraint.

Lemma 3.4 *The solution (ζ, ξ) of (3.14a), (3.14b) is given by*

$$(\zeta, \xi) = (\zeta_0, \xi_0) - \frac{(1, \xi_0)}{(1, \xi_1)}(\zeta_1, \xi_1), \tag{3.18}$$

where $(\zeta_0, \xi_0), (\zeta_1, \xi_1) \in H^1(\Omega) \times H_0^1(\Omega)$ are defined by

$$\mathcal{A}\big((\zeta_0, \xi_0), (\psi, \eta)\big) + (\zeta_0, 1)(\psi, 1) = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \operatorname{curl} \psi) \qquad \forall (\psi, \eta) \in H^1(\Omega) \times H_0^1(\Omega), \tag{3.19}$$

$$\mathcal{A}\big((\zeta_1, \xi_1), (\psi, \eta)\big) + (\zeta_1, 1)(\psi, 1) = (1, \eta) \qquad \forall (\psi, \eta) \in H^1(\Omega) \times H_0^1(\Omega). \tag{3.20}$$

Proof First we note that (3.17) implies

$$\begin{aligned} &\mathcal{A}\big(\psi, \eta), (\psi, \eta)\big) + (\psi, 1)(\psi, 1) \\ &\quad = (\operatorname{curl} \psi, \operatorname{curl} \psi) + (\psi, 1)(\psi, 1) + (\operatorname{curl} \eta, \operatorname{curl} \eta) + \beta(\eta, \eta) \end{aligned} \tag{3.21}$$

and hence, by standard Poincaré-Friedrichs inequalities, the problems (3.19) and (3.20) are well-posed. Moreover (3.20) and (3.21) imply that $(1, \xi_1) > 0$.

By construction, we have $(1, \xi) = 0$ and

$$\mathcal{A}\big((\zeta, \xi), (\psi, \eta)\big) + (\zeta, 1)(\psi, 1) = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \operatorname{curl}\psi) - \frac{(1, \xi_0)}{(1, \xi_1)}(1, \eta) \tag{3.22}$$

for all $(\psi, \eta) \in H^1(\Omega) \times H_0^1(\Omega)$, which implies (3.15). We can also recover the zero-mean condition $(1, \zeta) = 0$ by taking $(\psi, \eta) = (1, 0)$ in (3.22). □

3.3 A Problem for $c_1, \ldots, c_m$

If Ω is not simply connected, then γ is positive. According to Remark 2.1, we can take $\boldsymbol{v} = \operatorname{grad}\varphi_i$ in (1.3) to obtain

$$\gamma(\boldsymbol{u}, \operatorname{grad}\varphi_i) = (\boldsymbol{f}, \operatorname{grad}\varphi_i) \quad \text{for} \quad 1 \le i \le m.$$

It then follows from (2.3) and (3.1) that the coefficients $c_1, \ldots, c_m$ in the Hodge decomposition (3.1) are determined by the $m \times m$ system

$$\sum_{j=1}^{m}(\operatorname{grad}\varphi_i, \operatorname{grad}\varphi_j)c_j = \frac{1}{\gamma}(\boldsymbol{f}, \operatorname{grad}\varphi_i) \quad \text{for} \quad 1 \le i \le m. \tag{3.23}$$

Note that (3.23) is symmetric positive definite because of (2.2b).

3.4 Regularity of $\boldsymbol{u}$

First we observe that ρ belongs to $H^1(\Omega)$ by construction. Then (3.8) and Lemma 2.3 imply that $\xi \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$ for any $\epsilon > 0$, and (3.2) and Lemma 2.4 imply that ϕ belongs to $H^{1+(\pi/\omega)-\epsilon}(\Omega)$ for any $\epsilon > 0$. Moreover the harmonic functions φ_j $(1 \le j \le m)$ also belong to $H^{1+(\pi/\omega)-\epsilon}(\Omega)$ by Lemma 2.3. It follows that $\boldsymbol{u}$ belongs to $[H^{(\pi/\omega)-\epsilon}(\Omega)]^2$ for any $\epsilon > 0$. Thus the regularity of $\boldsymbol{u}$ is better than H^1 for a convex polygon and worse than H^1 for a nonconvex polygon.

Remark 3.5 Note that the regularity of $\xi = \operatorname{curl}\boldsymbol{u}$ is better than $H^1(\Omega)$. But the regularity of $\boldsymbol{u}$ is the same as the one in Theorem 2.5 for $\mathbb{E}$. This is due to the presence of singularities at the corners of Ω that prevents full elliptic regularity for $\boldsymbol{u}$. In contrast, for a smooth Ω, $\rho \in H^1(\Omega)$ implies $\xi \in H^3(\Omega)$ by (3.8), which in turn implies $\phi \in H^5(\Omega)$ by (3.2). Since the harmonic functions φ_j for $1 \le j \le m$ belong to $C^\infty(\bar{\Omega})$, $\boldsymbol{u}$ belongs to $[H^4(\Omega)]^2$ by (3.1), which is two orders higher than the regularity of the vector fields in $\mathbb{E}$ (cf. Remark 2.6).

4 P_k Finite Element Methods

The reduction in Sect. 3 leads to the following numerical procedure for (1.3).

(1) Find an approximation $\tilde{\xi} \in H_0^1(\Omega) \cap L_2^0(\Omega)$ for ξ numerically. In the case where $\gamma = 0$ (and Ω is simply connected), one can first solve (3.9) numerically to find an approximation $\tilde{\rho} \in H_0^1(\Omega) \cap L_2^0(\Omega)$ for ρ, and then solve (3.8) (with ρ replaced by $\tilde{\rho}$) numerically to find an approximation $\tilde{\xi}$ for ξ (cf. Sect. 3.2.1). In the case where $\gamma > 0$, we can obtain $\tilde{\xi}$ by solving the coupled problem (3.14a), (3.14b) numerically (cf. Sect. 3.2.2).
(2) Solve (3.2) (with ξ replaced by $\tilde{\xi}$) numerically to find an approximation $\tilde{\phi} \in H^1(\Omega) \cap L_2^0(\Omega)$ for ϕ.

(3) In the case where Ω is not simply connected (and $\gamma > 0$), solve numerically the boundary value problems in (2.2a), (2.2b), (2.2c) to find approximations $\tilde{\varphi}_j \in H^1(\Omega)$ for φ_j ($1 \leq j \leq m$) and then solve (3.23) (with φ_j replaced by $\tilde{\varphi}_j$) numerically to find approximations $\tilde{c}_1, \ldots, \tilde{c}_m$ for $c_1, \ldots, c_m$. Note that the computation of $\tilde{\varphi}_j$ ($1 \leq j \leq m$) only involves Ω and hence can be carried out in advance.

(4) The approximation $\tilde{\boldsymbol{u}}$ of $\boldsymbol{u}$ is given by

$$\tilde{\boldsymbol{u}} = \operatorname{curl} \tilde{\phi} + \sum_{j=1}^{m} \tilde{c}_j \operatorname{grad} \tilde{\varphi}_j.$$

Therefore any numerical method that works for second order elliptic boundary value problems can also be applied to (1.3). Here we will consider P_k Lagrange finite element methods.

Let $\mathcal{T}_h$ be a quasi-uniform simplicial triangulation of Ω. We denote by $V_h (\subset H^1(\Omega))$ the P_k ($k \geq 1$) Lagrange finite element space [13,16] associated with $\mathcal{T}_h$ and by $\mathring{V}_h (\subset H_0^1(\Omega))$ the subspace of V_h whose members vanish on $\partial\Omega$.

4.1 The P_k Finite Element Method for the Approximation of ξ

We consider two separate cases depending on whether γ is 0 or positive.

4.1.1 The Case Where $\gamma = 0$

Following the discussion in Sect. 3.2.1, we can first compute ρ and then ξ.

The P_k finite element method for (3.10) is to find $\rho_h \in V_h$ such that

$$(\operatorname{curl} \rho_h, \operatorname{curl} \psi) + (\rho, 1)(\psi, 1) = (\boldsymbol{f}, \operatorname{curl} \psi) \qquad \forall\, \psi \in V_h. \tag{4.1}$$

The P_k finite element method for (3.8) [cf. (3.11)–(3.13)] is to find

$$\xi_h = \xi_{0,h} - \frac{(1, \xi_{0,h})}{(1, \xi_{1,h})} \xi_{1,h}, \tag{4.2}$$

where $\xi_{0,h}, \xi_{1,h} \in \mathring{V}_h$ satisfy

$$(\operatorname{curl} \xi_{0,h}, \operatorname{curl} \eta) + \beta(\xi_{0,h}, \eta) = (\rho_h, \eta) \qquad \forall\, \eta \in \mathring{V}_h, \tag{4.3}$$

$$(\operatorname{curl} \xi_{1,h}, \operatorname{curl} \eta) + \beta(\xi_{1,h}, \eta) = (1, \eta) \qquad \forall\, \eta \in \mathring{V}_h. \tag{4.4}$$

4.1.2 The Case Where $\gamma > 0$

The P_k finite element method for (3.14a), (3.14b) [cf. (3.18)–(3.20)] is to find

$$(\zeta_h, \xi_h) = (\zeta_{0,h}, \xi_{0,h}) - \frac{(1, \xi_{0,h})}{(1, \xi_{1,h})} (\zeta_{1,h}, \xi_{1,h}), \tag{4.5}$$

where $(\zeta_{0,h}, \xi_{0,h}), (\zeta_{1,h}, \xi_{1,h}) \in V_h \times \mathring{V}_h$ satisfy

$$\mathcal{A}\big((\zeta_{0,h}, \xi_{0,h}), (\psi, \eta)\big) + (\zeta_{0,h}, 1)(\psi, 1) = \gamma^{-\frac{1}{2}} (\boldsymbol{f}, \operatorname{curl} \psi) \qquad \forall\, (\psi, \eta) \in V_h \times \mathring{V}_h, \tag{4.6}$$

$$\mathcal{A}\big((\zeta_{1,h}, \xi_{1,h}), (\psi, \eta)\big) + (\zeta_{1,h}, 1)(\psi, 1) = (1, \eta) \qquad \forall\, (\psi, \eta) \in V_h \times \mathring{V}_h. \tag{4.7}$$

Remark 4.1 The function ξ_h is an approximation of $\operatorname{curl} \boldsymbol{u}$.

4.2 The P_k Finite Element Method for the Approximation of ϕ

The finite element method for (3.3) is to find $\phi_h \in V_h \cap L_2^0(\Omega)$ such that

$$(\operatorname{curl}\phi, \operatorname{curl}\psi) + (\phi, 1)(\psi, 1) = (\xi_h, \psi) \qquad \forall\, \psi \in V_h. \tag{4.8}$$

4.3 The Approximation of $\boldsymbol{u}$

We take

$$\boldsymbol{u}_h = \operatorname{curl}\phi_h + \sum_{j=1}^{m} c_{j,h}\varphi_{j,h} \tag{4.9}$$

to be the approximation of $\boldsymbol{u}$, where $c_{1,h}, \ldots, c_{m,h}$ are determined by

$$\sum_{j=1}^{m}(\operatorname{grad}\varphi_{i,h},\ \operatorname{grad}\varphi_{j,h})c_{j,h} = \gamma^{-1}(\boldsymbol{f},\ \operatorname{grad}\varphi_{i,h}) \quad \text{for} \quad 1 \le i \le m, \tag{4.10}$$

and the discrete harmonic functions $\varphi_{1,h}, \ldots, \varphi_{m,h}$ are determined by (cf. (2.2a), (2.2b), (2.2c))

$$(\operatorname{grad}\varphi_{j,h},\ \operatorname{grad} v) = 0 \qquad \forall\, v \in \mathring{V}_h, \tag{4.11a}$$

$$\varphi_{j,h}\big|_{\Gamma_0} = 0, \tag{4.11b}$$

$$\varphi_{j,h}\big|_{\Gamma_k} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \qquad \text{for} \quad 1 \le k \le m. \tag{4.11c}$$

For a simply connected Ω, the approximation for $\boldsymbol{u}$ is simplified to $\boldsymbol{u}_h = \operatorname{curl}\phi_h$.

5 Convergence Analysis

Since the error analysis for the discrete harmonic functions $\varphi_{1,h}, \ldots, \varphi_{m,h}$ and the coefficients $c_{1,h}, \ldots, c_{m,h}$ has already been carried out in [9], we only need to focus on the error analysis for ξ_h and ϕ_h.

We will use the following standard polynomial approximation result [13,16,19].

Lemma 5.1 *Given any $\delta > 0$, there exists a positive constant C independent of h such that*

$$\inf_{\psi \in V_h} \|\lambda - \psi\|_{H^1(\Omega)} \le Ch^{\min(\delta,k)}\|\lambda\|_{H^{1+\delta}(\Omega)} \qquad \forall\, \lambda \in H^{1+\delta}(\Omega),$$

$$\inf_{\eta \in \mathring{V}_h} \|\mu - \psi\|_{H^1(\Omega)} \le Ch^{\min(\delta,k)}\|\mu\|_{H^{1+\delta}(\Omega)} \qquad \forall\, \mu \in H^{1+\delta}(\Omega) \cap H_0^1(\Omega).$$

From here on we will use C (with or without subscript) to denote a generic positive constant that is independent of h. The error estimates below will depend on ω, the largest interior angle at the corners of Ω.

5.1 Error Analysis for ξ_h

Our goal is to establish the following result.

Lemma 5.2 *For any* $\epsilon > 0$*, there exists a positive constant* C_ϵ *independent of* h *such that*

$$|\xi - \xi_h|_{H^1(\Omega)} \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)}. \tag{5.1}$$

We will consider the two cases $\gamma = 0$ and $\gamma > 0$ separately.

5.1.1 The Case Where $\gamma = 0$

We first estimate the error for ρ_h in the norm of $[H^1(\Omega)]'$ by a duality argument.

Lemma 5.3 *For any* $\epsilon > 0$*, there exists a positive constant* C_ϵ *independent of* h *such that*

$$|(\rho - \rho_h, \chi)| \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)} \|\chi\|_{H^1(\Omega)} \qquad \forall\, \chi \in H^1(\Omega). \tag{5.2}$$

Proof In view of (3.10), (4.1) and the fact that $\rho, \rho_h \in L_2^0(\Omega)$, we have

$$\|\mathrm{curl}\, \rho\|_{L_2(\Omega)} \le \|\boldsymbol{f}\|_{L_2(\Omega)}, \quad \|\mathrm{curl}\, \rho_h\|_{L_2(\Omega)} \le \|\boldsymbol{f}\|_{L_2(\Omega)}, \tag{5.3}$$

and a Galerkin orthogonality relation

$$\big(\mathrm{curl}\, (\rho - \rho_h), \mathrm{curl}\, \psi\big) = 0 \qquad \forall\, \psi \in V_h. \tag{5.4}$$

Let $\chi \in H^1(\Omega)$ be arbitrary and $\lambda \in H^1(\Omega)$ be defined by

$$(\mathrm{curl}\, \psi, \mathrm{curl}\, \lambda) + (\psi, 1)(\lambda, 1) = (\psi, \chi) \qquad \forall\, \psi \in H^1(\Omega). \tag{5.5}$$

Then we have

$$(\rho - \rho_h, \chi) = \big(\mathrm{curl}\, (\rho - \rho_h), \mathrm{curl}\, \lambda\big) = \big(\mathrm{curl}\, (\rho - \rho_h), \mathrm{curl}\, (\lambda - \psi)\big) \qquad \forall\, \psi \in V_h$$

by (5.4), (5.5) and the fact that $\rho, \rho_h \in L_2^0(\Omega)$, which implies

$$|(\rho - \rho_h, \chi)| \le \|\mathrm{curl}\, (\rho - \rho_h)\|_{L_2(\Omega)} \inf_{\psi \in V_h} |\lambda - \psi|_{H^1(\Omega)}. \tag{5.6}$$

According to Lemma 2.4 and (5.5), we have $\lambda \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$ for any $\epsilon > 0$ and also $\|\lambda\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \le C_\epsilon \|\chi\|_{H^1(\Omega)}$. Lemma 5.1 then implies

$$\inf_{\psi \in V_h} |\lambda - \psi|_{H^1(\Omega)} \le C h^{\min((\pi/\omega)-\epsilon,k)} \|\lambda\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)} \|\chi\|_{H^1(\Omega)}. \tag{5.7}$$

The estimate (5.2) follows from (5.3), (5.6) and (5.7). □

Next we estimate $|\xi - \xi_h|_{H^1(\Omega)}$.

Let $\tilde{\xi}_{0,h} \in \mathring{V}_h$ be defined by

$$(\mathrm{curl}\, \tilde{\xi}_{0,h}, \mathrm{curl}\, \eta) + \beta(\tilde{\xi}_{0,h}, \eta) = (\rho, \eta) \qquad \forall\, \eta \in \mathring{V}_h. \tag{5.8}$$

On one hand we have

$$\big(\mathrm{curl}\, (\tilde{\xi}_{0,h} - \xi_{0,h}), \mathrm{curl}\, \eta\big) + \beta\big(\tilde{\xi}_{0,h} - \xi_{0,h}, \eta\big) = (\rho - \rho_h, \eta) \qquad \forall\, \eta \in \mathring{V}_h$$

by comparing (4.3) and (5.8). It follows that

$$|\tilde{\xi}_{0,h} - \xi_{0,h}|^2_{H^1(\Omega)} \le (\rho - \rho_h, \tilde{\xi}_{0,h} - \xi_{0,h}),$$

which together with (5.2) and a standard Poncaré-Friedrichs inequality implies

$$|\tilde{\xi}_{0,h} - \xi_{0,h}|_{H^1(\Omega)} \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)}. \tag{5.9}$$

On the other hand $\tilde{\xi}_{0,h}$ is the Galerkin finite element approximation of ξ_0 (cf. (3.12) and (5.8)). Therefore we have

$$|\xi_0 - \tilde{\xi}_{0,h}|_{H^1(\Omega)} \le C \inf_{\eta \in \mathring{V}_h} |\xi_0 - \eta|_{H^1(\Omega)} \tag{5.10}$$

by Céa's lemma [13,16].

According to Lemma 2.3 and (3.12), we have $\xi_0 \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$ and $\|\xi_0\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \le C_\epsilon \|\rho\|_{H^1(\Omega)}$. It then follows from Lemma 5.1 that

$$\inf_{\eta \in \mathring{V}_h} |\xi_0 - \eta|_{H^1(\Omega)} \le C h^{\min((\pi/\omega)-\epsilon,k)} \|\xi_0\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)}. \tag{5.11}$$

Putting (5.9)–(5.11) together we obtain

$$|\xi_0 - \xi_{0,h}|_{H^1(\Omega)} \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)}. \tag{5.12}$$

Similarly, since $\xi_{1,h}$ is the Galerkin finite element approximation of ξ_1 (cf. (3.13) and (4.4)), we have

$$|\xi_1 - \xi_{1,h}|_{H^1(\Omega)} \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)}. \tag{5.13}$$

The estimate (5.1) follows from (3.11), (4.2), (5.12) and (5.13).

5.1.2 The Case Where $\gamma > 0$

The error analysis follows the ideas in Sect. 5.1.1 within the setting of the coupled problem (3.14a), (3.14b).

First we use a duality argument to estimate the error for $\zeta_{0,h}$ in the norm of $[H^1(\Omega)]'$. In view of (3.19), (3.21) and (4.6), we have

$$\|\zeta_0\|_{H^1(\Omega)} + |\xi_0|_{H^1(\Omega)} \le C\|\boldsymbol{f}\|_{L_2(\Omega)}, \quad \|\zeta_{0,h}\|_{H^1(\Omega)} + |\xi_{0,h}|_{H^1(\Omega)} \le C\|\boldsymbol{f}\|_{L_2(\Omega)}, \tag{5.14}$$

and the Galerkin orthogonality relation

$$\mathcal{A}\big((\zeta_0 - \zeta_{0,h}, \xi_0 - \xi_{0,h}), (\psi, \eta)\big) + (\zeta_0 - \zeta_{0,h}, 1)(\psi, 1) = 0 \qquad \forall\, (\psi, \eta) \in V_h \times \mathring{V}_h. \tag{5.15}$$

Let $\chi \in H^1(\Omega)$ be arbitrary and $(\lambda, \mu) \in H^1(\Omega) \times H_0^1(\Omega)$ be defined by

$$\mathcal{A}\big((\psi, \eta), (\lambda, \mu)\big) + (\psi, 1)(\lambda, 1) = (\psi, \chi) \qquad \forall\, (\psi, \eta) \in H^1(\Omega) \times H_0^1(\Omega). \tag{5.16}$$

Then the function $\zeta_0 - \zeta_{0,h} \in H^1(\Omega)$ satisfies, by (5.15) and (5.16),

$$\begin{aligned}(\zeta_0 - \zeta_{0,h}, \chi) &= \mathcal{A}\big((\zeta_0 - \zeta_{0,h}, \xi_0 - \xi_{0,h}), (\lambda, \mu)\big) + (\zeta - \zeta_{0,h}, 1)(\lambda, 1)\\ &= \mathcal{A}\big((\zeta_0 - \zeta_{0,h}, \xi_0 - \xi_{0,h}), (\lambda - \psi, \mu - \eta)\big) + (\zeta - \zeta_{0,h}, 1)(\lambda - \psi, 1)\end{aligned}$$

for all $(\psi, \eta) \in V_h \times \mathring{V}_h$, and hence

$$\begin{aligned}|(\zeta_0 - \zeta_{0,h}, \chi)| \le\; &C\big(\|\zeta_0 - \zeta_{0,h}\|_{H^1(\Omega)} + |\xi_0 - \xi_{0,h}|_{H^1(\Omega)}\big)\\ &\times \inf_{(\psi,\eta) \in V_h \times \mathring{V}_h} \big(\|\lambda - \psi\|_{H^1(\Omega)} + |\mu - \eta|_{H^1(\Omega)}\big).\end{aligned} \tag{5.17}$$

Observe that the well-posedness of (5.16) implies

$$\|\lambda\|_{H^1(\Omega)} + \|\mu\|_{H^1(\Omega)} \le C\|\chi\|_{L_2(\Omega)}. \tag{5.18}$$

It then follows from Lemmas 2.3, 2.4 and the relations (cf. (3.16) and (5.16))

$$(\mathrm{curl}\,\psi, \mathrm{curl}\,\lambda) + (\psi, 1)(\lambda, 1) = \gamma^{\frac{1}{2}}(\psi, \mu) + (\psi, \chi) \qquad \forall\,\psi \in H^1(\Omega),$$
$$(\mathrm{curl}\,\eta, \mathrm{curl}\,\mu) + \beta(\eta, \mu) = -\gamma^{\frac{1}{2}}(\lambda, \eta) \qquad \forall\,\eta \in H_0^1(\Omega),$$

that $(\lambda, \mu) \in H^{1+(\pi/\omega)-\epsilon}(\Omega) \times H^{1+(\pi/\omega)-\epsilon}(\Omega)$ and, because of (5.18),

$$\|\lambda\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} + \|\mu\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \le C_\epsilon\|\chi\|_{H^1(\Omega)}.$$

Hence Lemma 5.1 implies

$$\inf_{(\psi,\eta)\in V_h\times \mathring{V}_h} \big(\|\lambda-\psi\|_{H^1(\Omega)} + |\mu-\eta|_{H^1(\Omega)}\big) \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)}\|\chi\|_{H^1(\Omega)}. \tag{5.19}$$

Putting (5.14), (5.17) and (5.19) together, we see that

$$|(\zeta_0 - \zeta_{0,h}, \chi)| \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)}\|\chi\|_{H^1(\Omega)} \qquad \forall\,\chi \in H^1(\Omega). \tag{5.20}$$

Next we compare the equation

$$(\mathrm{curl}\,\xi_0, \mathrm{curl}\,\eta) + \beta(\xi_0, \eta) = \gamma^{\frac{1}{2}}(\zeta, \eta) \qquad \forall\,\eta \in H_0^1(\Omega),$$

that is a part of (3.19) with the equation

$$(\mathrm{curl}\,\xi_{0,h}, \mathrm{curl}\,\eta) + \beta(\xi_{0,h}, \eta) = \gamma^{\frac{1}{2}}(\zeta_{0,h}, \eta) \qquad \forall\,\eta \in \mathring{V}_h$$

that is a part of (4.6). Using (5.20) and the arguments in the derivation of(5.12), we find

$$|\xi_h - \xi_{0,h}|_{H^1(\Omega)} \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)}. \tag{5.21}$$

Similarly, we have

$$|\xi_1 - \xi_{1,h}|_{H^1(\Omega)} \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)} \tag{5.22}$$

by comparing (3.20) and (4.7).

The estimate (5.1) follows from (3.18), (4.5), (5.21) and (5.22).

5.2 Error Analysis for ϕ_h

The error analysis for ϕ_h is similar to the error analysis for ξ_h in Sect. 5.1.1.

Lemma 5.4 *For any $\epsilon > 0$, there exists a positive constant C_ϵ independent of h such that*

$$|\phi - \phi_h|_{H^1(\Omega)} \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)}. \tag{5.23}$$

Proof It follows from Lemma 2.4 and (3.3) that $\phi \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$ for any $\epsilon > 0$ and $\|\phi\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \le C_\epsilon\|\xi\|_{H^1(\Omega)}$.

Let the function $\tilde{\phi}_h \in V_h$ be defined by

$$(\mathrm{curl}\,\tilde{\phi}_h, \mathrm{curl}\,\psi) + (\tilde{\phi}_h, 1)(\psi, 1) = (\xi, \psi) \qquad \forall\,\psi \in V_h. \tag{5.24}$$

On one hand we have, by comparing (3.3) and (5.24),

$$|\tilde{\phi}_h - \phi_h|_{H^1(\Omega)} \le C\|\xi - \xi_h\|_{L_2(\Omega)}. \tag{5.25}$$

On the other hand $\tilde{\phi}_h \in V_h$ is the Galerkin P_k finite element approximation of ϕ, and the orthogonality relation

$$\big(\text{curl}\,(\phi - \tilde{\phi}_h), \text{curl}\,\psi\big) = 0 \qquad \forall\,\psi \in V_h$$

together with Lemma 5.1 implies that

$$\begin{aligned}|\phi - \tilde{\phi}_h|_{H^1(\Omega)} &\le \inf_{\psi \in V_h} |\phi - \psi|_{H^1(\Omega)} \\ &\le Ch^{\min((\pi/\omega)-\epsilon,k)} \|\phi\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)} \|\xi\|_{H^1(\Omega)}. \quad (5.26)\end{aligned}$$

The estimate (5.23) follows from (5.25), (5.26) and the estimate in Lemma 5.2 for $\xi - \xi_h \in H_0^1(\Omega)$. □

5.3 Error Analysis for $\varphi_{h,j}$ and $c_{h,j}$

The following result can be found in [9, Lemmas 4.6 and 4.7].

Lemma 5.5 *In the case where Ω is not simply connected, we have*

$$|\varphi_j - \varphi_{j,h}|_{H^1(\Omega)} + |c_j - c_{j,h}| \le Ch^{\pi/\omega} \qquad \textit{for } 1 \le j \le m.$$

5.4 Convergence Results

In view of Lemmas 5.2, 5.4, 5.5, (3.1) and (4.9), we immediately have the following result.

Theorem 5.6 *The approximations ξ_h and $\boldsymbol{u}_h$ obtained by the P_k finite element method satisfy*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L_2(\Omega)} + |\text{curl}\,\boldsymbol{u} - \xi_h|_{H^1(\Omega)} \le C_\epsilon h^{\min((\pi/\omega)-\epsilon,k)}, \quad (5.27)$$

for any $\epsilon > 0$, where ω is the largest angle at the corners of Ω.

Remark 5.7 In the case where (π/ω) is not an integer, a more detailed analysis that takes into account the nature of the singularities at the corners of Ω (cf. [6]) shows that the ϵ in (5.27) can be removed.

6 Numerical Results

In this section we report the results of numerical experiments for three different domains: the unit square, a nonconvex but simply connected domain and a domain whose Betti number is 1. We use quasi-uniform meshes in all the experiments.

Experiment 6.1 In the first experiment the domain Ω is the unit square $(0, 1) \times (0, 1)$. We take $\beta = \gamma = 0$ and the exact solution to be $\boldsymbol{u} = \text{curl}\,\phi$ where

$$\phi(x) = \sin^3(\pi x_1)\sin^3(\pi x_2).$$

We solve (1.3) by the P_1 and P_2 finite element methods. The results are presented in Tables 1 and 2. They agree with Theorem 5.6 with $\omega = \pi/2$.

Experiment 6.2 The domain Ω for the second experiment is also the unit square $(0, 1) \times (0, 1)$. We take $\beta = \gamma = 0$ and

$$\boldsymbol{f} = \begin{bmatrix} (x_1^2 + 1)\sin x_1 + x_1 x_2^3 + 2 \\ (x_2^2 + 1)\cos x_1 + x_1^3 x_2^2 - 1 \end{bmatrix}.$$

Table 1 Results for the P_1 finite element method for Experiment 6.1

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}$	Order	$\|\text{curl}\,\boldsymbol{u}-\xi_h\|_{H^1(\Omega)}$	Order
1/10	3.58137×10^{-1}	–	3.48606×10^{1}	–
1/20	1.68292×10^{-1}	1.064	1.74349×10^{1}	1.001
1/40	8.25728×10^{-2}	1.019	8.72558×10^{0}	0.999
1/80	4.10892×10^{-2}	1.004	4.36509×10^{0}	0.999
1/160	2.05210×10^{-2}	1.001	2.18300×10^{0}	1.000

Table 2 Results for the P_2 finite element method for Experiment 6.1

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L_2(\Omega)}$	Order	$\|\text{curl}\,\boldsymbol{u}-\xi_h\|_{H^1(\Omega)}$	Order
1/10	3.09934×10^{-2}	–	3.74032×10^{0}	–
1/20	7.85594×10^{-3}	1.980	9.52019×10^{-1}	1.974
1/40	1.97684×10^{-3}	1.991	2.39679×10^{-1}	1.990
1/80	4.95696×10^{-4}	1.996	6.00980×10^{-2}	1.996
1/160	1.24101×10^{-4}	1.998	1.50448×10^{-2}	1.998

Table 3 Results for the P_1 finite element method for Experiment 6.2

h	$\frac{\|\boldsymbol{u}_{h,i}-\boldsymbol{u}_{h,i+1}\|_{L_2(\Omega)}}{\|\boldsymbol{u}_{h,i+1}\|_{L_2(\Omega)}}$	Order	$\frac{\|\xi_{i,h}-\xi_{h,i+1}\|_{H^1(\Omega)}}{\|\xi_{h,i+1}\|_{H^1(\Omega)}}$	Order
1/20	6.51821×10^{-2}	–	1.83399×10^{-1}	–
1/40	3.06374×10^{-2}	1.064	9.16403×10^{-2}	1.001
1/80	1.50366×10^{-2}	1.019	4.62392×10^{-2}	0.987
1/160	7.48116×10^{-3}	1.005	2.30384×10^{-2}	1.005

Table 4 Results for the P_2 finite element method for Experiment 6.2

h	$\frac{\|\boldsymbol{u}_{h,i}-\boldsymbol{u}_{h,i+1}\|_{L_2(\Omega)}}{\|\boldsymbol{u}_{h,i+1}\|_{L_2(\Omega)}}$	Order	$\frac{\|\xi_{i,h}-\xi_{h,i+1}\|_{H^1(\Omega)}}{\|\xi_{h,i+1}\|_{H^1(\Omega)}}$	Order
1/20	5.73968×10^{-3}	–	1.68212×10^{-2}	–
1/40	1.47911×10^{-3}	1.956	4.72439×10^{-3}	1.832
1/80	3.73962×10^{-4}	1.983	1.29630×10^{-3}	1.866
1/160	9.39210×10^{-5}	1.993	3.50361×10^{-4}	1.887

We solve (1.3) by the P_1 and P_2 finite element methods and report the results in Tables 3 and 4. Since the exact solution is not known, the relative errors are estimated by comparing the numerical solutions on consecutive refinement levels. The results agree with Theorem 5.6 with $\omega = \pi/2$.

Experiment 6.3 In the third experiment we solve (1.3) on the nonconvex domain (cf. Fig. 2) whose vertices are $(0, 0)$, $(.5, 0)$, $(.5, .7)$, $(1, .7)$, $(1, 1)$, $(0, 1)$, $(1, .75)$, $(.25, .75)$, $(.25, .625)$ and $(0, .625)$.

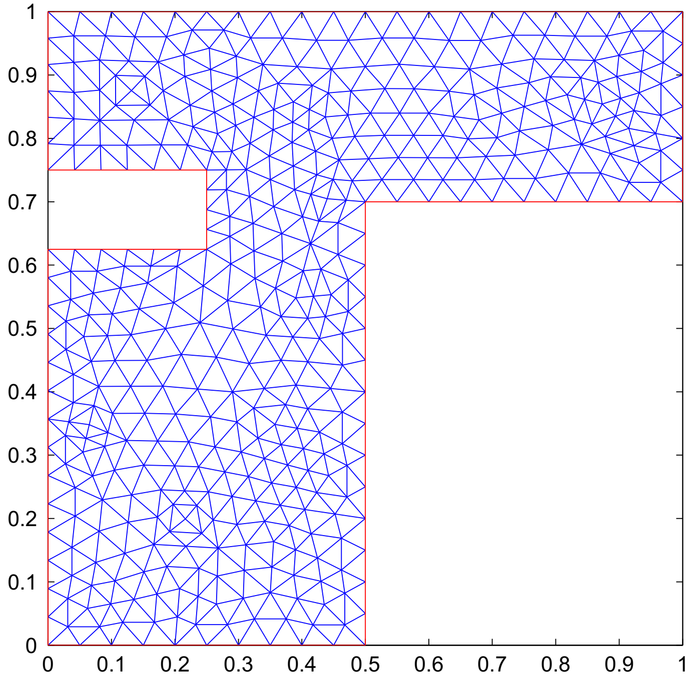

Fig. 2 Domain for Experiment 6.3

Table 5 Results for the P_1 finite element method for Experiment 6.3

h	$\frac{\|\boldsymbol{u}_{h,i}-\boldsymbol{u}_{h,i+1}\|_{L_2(\Omega)}}{\|\boldsymbol{u}_{h,i+1}\|_{L_2(\Omega)}}$	Order	$\frac{\|\xi_{i,h}-\xi_{h,i+1}\|_{H^1(\Omega)}}{\|\xi_{h,i+1}\|_{H^1(\Omega)}}$	Order
1/20	2.05797×10^{-1}	–	2.67889×10^{-1}	–
1/40	1.31128×10^{-1}	0.650	1.42644×10^{-1}	0.909
1/80	8.28659×10^{-2}	0.662	7.38178×10^{-2}	0.950
1/160	5.21382×10^{-2}	0.668	3.80373×10^{-2}	0.957
1/320	3.27776×10^{-2}	0.670	1.97576×10^{-2}	0.945

We take $\beta = \gamma = 0$ and use a piecewise constant vector field $\boldsymbol{f}$ defined by

$$\boldsymbol{f}(x) = \begin{cases} \left[\frac{1}{4}\ ,\ \frac{5}{4}\right]^{\mathrm{t}} & |x| < 2^{-1/2} \\ \left[\frac{1}{2}\ ,\ \frac{3}{2}\right]^{\mathrm{t}} & 2^{-1/2} \le |x| < 1 \\ [\,1\ ,\ 2\,]^{\mathrm{t}} & |x| \ge 1 \end{cases}.$$

The estimated relative errors for the P_1 and P_2 finite element methods are displayed in Tables 5 and 6.

For this problem the order of convergence predicted by Theorem 5.6 is $2/3$ (since $\omega = 3\pi/2$). This is observed in Table 6 for the P_2 finite element method, and also in Table 5 for the P_1 finite element method with respect to the convergence of $\boldsymbol{u}_h$. On the other hand the convergence observed in Table 5 for ξ_h is pre-asymptotic.

Table 6 Results for the P_2 finite element method for Experiment 6.3

h	$\frac{\\|\boldsymbol{u}_{h,i}-\boldsymbol{u}_{h,i+1}\\|_{L_2(\Omega)}}{\\|\boldsymbol{u}_{h,i+1}\\|_{L_2(\Omega)}}$	Order	$\frac{\|\xi_{i,h}-\xi_{h,i+1}\|_{H^1(\Omega)}}{\|\xi_{h,i+1}\|_{H^1(\Omega)}}$	Order
1/20	5.95103×10^{-2}	–	5.65277×10^{-2}	–
1/40	3.72674×10^{-2}	0.675	2.09547×10^{-2}	1.432
1/80	2.34229×10^{-2}	0.670	1.00626×10^{-2}	1.058
1/160	1.47409×10^{-2}	0.668	5.86889×10^{-3}	0.778
1/320	9.28116×10^{-3}	0.667	3.64280×10^{-3}	0.688

Table 7 Results for the P_1 finite element method for Experiment 6.4

h	$\frac{\\|\boldsymbol{u}_{h,i}-\boldsymbol{u}_{h,i+1}\\|_{L_2(\Omega)}}{\\|\boldsymbol{u}_{h,i+1}\\|_{L_2(\Omega)}}$	Order	$c_{i,h}$	$\frac{\|c_{h,i}-c_{h,i+1}\|}{\|c_{h,i+1}\|}$	Order	$\frac{\|\xi_{i,h}-\xi_{h,i+1}\|_{H^1(\Omega)}}{\|\xi_{h,i+1}\|_{H^1(\Omega)}}$	Order
1/20	$1.21342 \times 10^{-}1$	–	−0.15098	7.29352×10^{-3}	–	2.81481×10^{-1}	–
1/40	7.68130×10^{-2}	0.660	−0.15143	3.02568×10^{-3}	1.269	1.52447×10^{-1}	0.885
1/80	4.85896×10^{-2}	0.661	−0.15162	1.23262×10^{-3}	1.296	8.43960×10^{-2}	0.853
1/160	3.06974×10^{-2}	0.663	−0.15170	4.96890×10^{-4}	1.311	4.65719×10^{-2}	0.858

Experiment 6.4 The domain for the fourth experiment is

$$\Omega = (0, 1) \times (0, 1) \setminus [1/4, 3/4] \times [1/4, 3/4]$$

whose Betti number is 1. We take $\beta = \gamma = 1$ and use the same $\boldsymbol{f}$ in Experiment 6.2.

Since the domain is not simply connected, the solution $\boldsymbol{u}$ of (1.3) is given by

$$\boldsymbol{u} = \operatorname{curl} \phi + c \operatorname{grad} \varphi, \tag{6.1}$$

where φ is the harmonic function that vanishes on the outer boundary of Ω and equals 1 on the inner boundary of Ω. The approximation for $\boldsymbol{u}$ is given by

$$\boldsymbol{u}_h = \operatorname{curl} \phi_h + c_h \operatorname{grad} \varphi_h, \tag{6.2}$$

where φ_h is the discrete analog of φ.

We solve (1.3) by the P_1 finite element method and report the results in Table 7. The order of convergence for $\boldsymbol{u}_h$ is observed to be $2/3$, which agrees with Theorem 5.6 with $\omega = 3\pi/2$. The convergence of ξ_h is pre-asymptotic.

Note that the order of convergence for c_h is better than $2/3$, which is due to the fact that $\boldsymbol{f}$ is smooth (cf. [9, Remark 4.8 and Table 4.3]).

7 Concluding Remarks

We have designed and analyzed P_k Lagrange finite element methods for a quad-curl problem that are based on the Hodge decomposition approach. For simplicity we only considered quasi-uniform meshes and the performance of the methods suffer from the existence of reentrant corners. But optimal convergence rates can be recovered if we would use properly graded meshes (cf. [17] for the case of the Maxwell equations).

Our convergence analysis does not require H^2 (or higher) regularity for the exact solution that is assumed in [23,29,31]. The computational cost of our approach also compares favorably to those for the methods in [23,29,31].

Below are some related topics that can also benefit from the Hodge decomposition approach.

- As in the case of the Maxwell equations, the Hodge decomposition approach lends itself naturally to the development of fast solvers for the quad-curl problem.
- The Hodge decomposition approach can also be applied to the following eigenvalue problems on two dimensional domains. The first eigenvalue problem is to find $(\boldsymbol{u}, \lambda) \in \mathbb{E} \times \mathbb{R}$ such that

$$\big(\text{curl}\,(\text{curl}\,\boldsymbol{u}), \text{curl}\,(\text{curl}\,\boldsymbol{v})\big) = \lambda(\boldsymbol{u}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \mathbb{E} \quad \text{and} \quad \boldsymbol{u} \neq \mathbf{0}.$$

  The second eigenvalue problem is to find $(\boldsymbol{u}, \lambda) \in \mathbb{E} \times \mathbb{R}$ such that

$$\big(\text{curl}\,(\text{curl}\,\boldsymbol{u}), \text{curl}\,(\text{curl}\,\boldsymbol{v})\big) = \lambda(\text{curl}\,\boldsymbol{u}, \text{curl}\,\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \mathbb{E} \quad \text{and} \quad \boldsymbol{u} \neq \mathbf{0}.$$

  For both problems the Hodge decomposition approach reduces an elliptic eigenvalue problem for vector fields to an elliptic eigenvalue problem for scalar functions that can be solved by standard H^1 conforming finite elements.
- For three dimensional domains, the Hodge decomposition approach reduces the quad-curl problem to problems that can be solved numerically by standard $H(\text{curl})$, $H(\text{div})$ and H^1 conforming finite elements. Moreover these problems have already been analyzed in [4].

These topics are being investigated in our ongoing projects.

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References

1. Adams, R.A., Fournier, J.J.F.: Sobolev Spaces (Second Edition). Academic Press, Amsterdam (2003)
2. Alonso, A., Fernandes, P., Valli, A.: Weak and strong formulations for the time-harmonic eddy-current problem in general multi-connected domains. Eur. J. Appl. Math. **14**, 387–406 (2003)
3. Alonso-Rodríguez, A., Valli, A., Vázquez-Hernández, R.: A formulation of the eddy current problem in the presence of electric ports. Numer. Math. **113**, 643–672 (2009)
4. Amrouche, C., Bernardi, C., Dauge, M., Girault, V.: Vector potentials in three-dimensional non-smooth domains. Math. Methods Appl. Sci. **21**, 823–864 (1998)
5. Assous, F., Michaeli, M.: Hodge decomposition to solve singular static Maxwell's equations in a non-convex polygon. Appl. Numer. Math. **60**, 432–441 (2010)
6. Babuška, I., Suri, M.: The h-p version of the finite element method with quasiuniform meshes. M2AN Math. Model. Numer. Anal. **21**, 199–238 (1987)
7. Biskamp, D.: Magnetic Reconnection in Plasmas. Cambridge University Press, Cambridge (2000)
8. Bramble, J.H.: A proof of the inf-sup condition for the Stokes equations on Lipschitz domains. Math. Models Methods Appl. Sci. **13**, 361–371 (2003)
9. Brenner, S.C., Cui, J., Nan, Z., Sung, L.-Y.: Hodge decomposition for divergence-free vector fields and two-dimensional Maxwell's equations. Math. Comput. **81**, 643–659 (2012)
10. Brenner, S.C., Gedicke, J., Sung, L.-Y.: An adaptive P_1 finite element method for two-dimensional Maxwell's equations. J. Sci. Comput. **55**, 738–754 (2013)
11. Brenner, S.C., Gedicke, J., Sung, L.-Y.: Hodge decomposition for two-dimensional time harmonic Maxwell's equations: impedance boundary condition. Math. Methods Appl. Sci. **40**, 370–390 (2017). doi:10.1002/mma.3398

12. Brenner, S.C., Gedicke, J., Sung, L.-Y.: An adaptive P_1 finite element method for two-dimensional transverse magnetic time harmonic Maxwell's equations with general material properties and general boundary conditions. J. Sci. Comput. **68**, 848–863 (2016)
13. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods (Third Edition). Springer, New York (2008)
14. Cakoni, F., Colton, D., Monk, P., Sun, J.: The inverse electromagnetic scattering problem for anisotropic media. Inverse Probl. **26**, 074004 (2010)
15. Chacón, L., Simakov, A.N., Zocco, A.: Steady-state properties of driven magnetic reconnection in 2D electron magnetohydrodynamics. Phys. Rev. Lett. **99**, 235001 (2007)
16. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
17. Cui, J.: Multigrid methods for two-dimensional Maxwell's equations on graded meshes. J. Comput. Appl. Math. **255**, 231–247 (2014)
18. Dauge, M.: Elliptic Boundary Value Problems on Corner Domains. Lecture Notes in Mathematics, vol. 1341. Springer, Berlin (1988)
19. Dupont, T., Scott, R.: Polynomial approximation of functions in Sobolev spaces. Math. Comput. **34**, 441–463 (1980)
20. Duvaut, G., Lions, J.L.: Inequalities in Mechanics and Physics. Springer, Berlin (1976)
21. Girault, V., Raviart, P.-A.: Finite Element Methods for Navier–Stokes Equations. Theory and Algorithms. Springer, Berlin (1986)
22. Grisvard, P.: Elliptic Problems in Non Smooth Domains. Pitman, Boston (1985)
23. Hong, Q., Hu, J., Shu, S., Xu, J.: A discontinuous Galerkin method for the fourth-order curl problem. J. Comput. Math. **30**, 565–578 (2012)
24. Lax, P.D.: Functional Analysis. Wiley-Interscience, New York (2002)
25. Monk, P.: Finite Element Methods for Maxwell's Equations. Oxford University Press, New York (2003)
26. Monk, P., Sun, J.: Finite element methods for Maxwell's transmission eigenvalues. SIAM J. Sci. Comput. **34**, B247–B264 (2012)
27. Nečas, J.: Direct methods in the theory of elliptic equations. Springer, Heidelberg (2012)
28. Nečas, J.: Equations aux Dérivées Partielles. Presse de l'Université Montréal, Montreal (1965)
29. Sun, J.: A mixed FEM for the quad-curl eigenvalue problem. Numer. Math. **132**, 185–200 (2016)
30. Yosida, K.: *Functional Analysis* Classics in Mathematics. Springer, Berlin (1995). Reprint of the sixth (1980) edition
31. Zheng, B., Hu, Q., Xu, J.: A nonconforming finite element method for fourth order curl equations in $\mathbb{R}^3$. Math. Comput. **80**, 1871–1886 (2011)