

A Block-Centered Finite Difference Method for Slightly Compressible Darcy–Forchheimer Flow in Porous Media

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Abstract A block-centered finite difference method is introduced to solve an initial and boundary value problem for a nonlinear parabolic equation to model the slightly compressible flow in porous media, in which the velocity–pressure relation is described by Darcy–Forchheimer’s Law. The method can be thought as the lowest order Raviart–Thomas mixed element method with proper quadrature formulation. By using the method the velocity and pressure can be approximated simultaneously. We established the second-order error estimates for pressure and velocity in proper discrete norms on non-uniform rectangular grid. No time-step restriction is needed for the error estimates. The numerical experiments using the scheme show that the convergence rates of the method are in agreement with the theoretical analysis.

Keywords Block-centered finite difference · Darcy–Forchheimer flow · Compressible · Error estimate · Numerical experiment

Mathematics Subject Classification 65M06 · 65M12 · 65M15

1 Introduction

Darcy’s flow in porous media is of great interest in many fields such as oil recovery and groundwater pollution contamination. Darcy’s law describes the linear relationship between the velocity of creep flow and the gradient of pressure. The relationship is valid by experiment

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under the condition that the creeping velocity is low and the porosity and permeability is small enough [1]. A theoretical derivation of Darcy's law can be found in [2, 3].

In some cases, for example when the velocity is higher, a nonlinear relationship between the velocity and the pressure gradient is developed, suggested by Forchheimer [1], by adding a second order term to reach a modified equation. The Darcy–Forchheimer equation (or Forchheimer's law) is described as follows,

$$\mu K^{-1} \mathbf{u} + \beta \rho |\mathbf{u}| \mathbf{u} + \nabla(p - \rho gh) = 0. \quad (1)$$

A theoretical derivation of it can be found in [4].

Forchheimer's law mainly describes the inertial effects for high speed flow. The most important feature of Forchheimer's law is that it combines the monotonicity of the nonlinear term and the non-degenerate of the Darcy's part. There are several papers to analyze the analytical solution for the Forchheimer flow problem, see, for example, [5–7].

There are some papers consider the numerical methods for Forchheimer flow in porous media. Mixed element methods for generalized Forchheimer equation were first studied by Douglas et al. [8], then a mixed element method for general nonlinear elliptic problem was studied by Park [9]. A mixed element method with piecewise constant approximation for velocity and nonconforming piecewise linear approximation for pressure, called primal mixed element [10], was considered in [11, 12]. And a mixed element method with Raviart-Thomas element was considered in [13]. Mixed element methods for time-dependent compressible Forchheimer flow was considered by [14]. And a numerical well model with cell-centered finite difference and finite element method for non-Darcy flow was considered in [15]. In [14] only the semi-discrete mixed element method is considered. And since they used the inversive assumption, the lowest-order Raviart–Thomas mixed element is not included in their error estimates.

A series of work about generalized Forchheimer flow can be found in [16–18], in which both expanded mixed element form and the nonlinear degenerate parabolic form are used to discrete the model problem. Numerical analysis is based on the monotone properties possessed by Forchheimer operator, [19–21].

Block-centered finite difference methods can be thought as the lowest order Raviart-Thomas mixed element method with proper quadrature formulation and has been used widely in reservoir numerical simulation. By using it both the velocity and pressure can be approximated with second-order accuracy for linear elliptic problem with diagonal diffusion coefficient was considered, see [22]. Then in [23, 24] cell-centered finite differences for linear elliptic problem with tensor diffusion coefficients were considered. Another advantage of block-centered, or cell-centered, finite difference method is that it transfers the saddle point system of the mixed element method into symmetric positive definite system, which has been used in many papers, see, for example, [25, 26].

Recently we introduced and analyzed a blocked-centered finite difference method for the incompressible Forchheimer equation with constant coefficients [27] and variable coefficients [28]. We demonstrate that the proposed scheme is second-order accuracy both for velocity and pressure in some discrete norms. A two-grid finite difference method for the the problem is also considered [29].

In this paper we present a blocked-centered finite difference method for the slightly compressible Forchheimer flow problem in porous media. The problem is a kind of nonlinear parabolic problems where the diffusion coefficient depends on the pressure and the absolute value of the vector-valued velocity. It is an extension of our work in [27], where just the nonlinear elliptic Forchheimer problem with constant coefficients was considered. In the scheme, the pressure, the velocity in x -direction and the velocity in y -direction are defined

on staggered grids. One key problem to present the scheme is to give a proper approximation to the nonlinear diffusion coefficient, which depend on the pressure and the absolute function of the velocity. The approximation should have the second-order accuracy and reserve the monotonicity of the operator, see Lemma 6 below. We demonstrate that the proposed scheme has these two properties on non-uniform rectangular grid.

Usually for a priori error estimate of numerical solution of nonlinear time-dependent problem, one may employ the inductive assumption and inverse inequality to bound the numerical solution. This may result in a time-step restriction, see, for example, [30–34]. Li and Sun [35,36] developed a technique to remove the time-step restriction for a priori estimate. In this paper no time-step restriction is needed for the second-order error estimates. Here our technique is different from Li and Sun [35,36], we used the monotonicity of the nonlinear operator.

Some numerical examples are carried out using the presented scheme. The numerical results show that the convergence rates of our method are in agreement with the theoretical analysis.

The paper is organized as follows. In Sect. 2 we give the problem and some notations. In Sect. 3 we present the block-centered finite difference scheme. In Sect. 4 we give the corresponding numerical analysis. In Sect. 5 some numerical experiments are carried out.

Throughout the paper we use C , with or without subscript, to denote a positive constant, which can have different values in different appearances.

2 The Problem and Some Notations

In this section we present a slightly compressible flow model in porous media, in which the velocity–pressure relation is described by the Darcy–Forchheimer’s law.

Firstly, the equation for mass conservation is as follows

$$\frac{\partial(\phi\rho)}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = \rho q, \quad (2)$$

where ϕ is the porosity of the media, q is the source term and \mathbf{u} , ρ are the velocity and density of the fluid. This model can be simplified under the circumstance that the fluid is slightly compressible [1,37,38]. Set C_F be the coefficient of compressibility $C_F = \frac{1}{\rho} \frac{\partial\rho}{\partial p}$, then

$$\rho = \rho_0 \exp(C_F(p - p_0)), \quad (3)$$

and we have

$$\phi \frac{\partial\rho}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial\rho}{\partial p} \nabla p \cdot \mathbf{u} + \rho \nabla \cdot \mathbf{u} = \rho q. \quad (4)$$

The term $\frac{\partial\rho}{\partial p} \nabla p \cdot \mathbf{u}$ is effectively quadratic in the velocity, which in almost all of the domain can be neglected [38–40]. Thus we arrive the following equation,

$$\phi c_F \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = q. \quad (5)$$

Combining the mass conservation equation (5) with the velocity–pressure equation (1) we obtain the model describing the slightly compressible flow in porous media, in which the velocity–pressure relation obeys the Forchheimer’s law.

$$\begin{cases} \text{(a)} & \mu K^{-1} \mathbf{u} + \beta \rho(p) |\mathbf{u}| \mathbf{u} + \nabla p = \nabla(\rho(p) \mathbf{g}H), & \mathbf{z} \in \Omega \times J, \\ \text{(b)} & \phi C_F \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = f, & \mathbf{z} \in \Omega \times J, \\ \text{(c)} & \mathbf{u} \cdot \mathbf{n} = f_N, & \mathbf{z} \in \partial\Omega \times J. \\ \text{(d)} & p|_{t=0} = p_0, & \mathbf{z} \in \Omega. \end{cases} \tag{6}$$

Here p represents the pressure while \mathbf{u} the velocity of the fluid. Ω is a porous media domain and $J = (0, T)$ is the time interval. For simplicity we consider the problem in two dimensional space. \mathbf{n} represents the unit exterior normal vector to the boundary of Ω , $|\cdot|$ denotes the Euclidean norm, $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$. ρ, μ and β are scalar functions which represent the density of the fluid, its viscosity and the Forchheimer number, respectively. For compressible fluid, ρ depends on the pressure p , $\rho = \rho(p)$. ϕ represents the porosity. K is the permeability tensor function. For simplicity we suppose that $K = \bar{k}I$ where \bar{k} is a positive constant and I represents the unit matrix. $f \in L^2(\Omega)$, a scalar function, represents the source and sink of the systems. $\rho(p) \mathbf{g} \nabla H \in (L^2(\Omega))^d$, a vector function, is the gradient of the depth function $H \in H^1(\Omega)$. $f_N \in L^2(\partial\Omega)$, a scalar function, represents the Neumann boundary condition, or the flux through the boundary.

A more general compressible Forchheimer flow model was considered in [8] where the dependence of ρ on pressure p does not be described explicitly.

For simplicity of constructing the block-centered finite difference scheme we consider the problem in a two dimensional rectangular domain, $\Omega = (0, L_x) \times (0, L_y)$. We use the notation (x, y) to denote the coordinate of a point in the domain and denote the velocity by $u = (u^x, u^y)$. We suppose that the depth function H is a constant, then $\nabla H = 0$. Furthermore we suppose that the problem is with homogeneous boundary condition, $f_N = 0$.

With the above assumptions the problem (6) can be re-written as

$$\begin{cases} \text{(a)} & \left(\frac{\mu}{\bar{k}} + \beta \rho(p) |\mathbf{u}| \right) \mathbf{u} + \nabla p = 0, & \mathbf{z} \in \Omega \times J, \\ \text{(b)} & \phi C_F \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = f, & \mathbf{z} \in \Omega \times J, \\ \text{(c)} & \mathbf{u} \cdot \mathbf{n} = 0, & \mathbf{z} \in \partial\Omega \times J. \\ \text{(d)} & p|_{t=0} = p_0, & \mathbf{z} \in \Omega. \end{cases} \tag{7}$$

We will derive the block-centered finite difference method for the model problem (7).

Let $N > 0$ be a positive integer. Set

$$\Delta t = T/N; \quad t^n = n\Delta t \text{ for } n \leq T/N.$$

The domain $\Omega = (0, L_x) \times (0, L_y)$ is partitioned by $\delta_x \times \delta_y$, where

$$\begin{aligned} \delta_x : 0 &= x_{\frac{1}{2}} < x_{3/2} < \dots < x_{N_x - \frac{1}{2}} < x_{N_x + \frac{1}{2}} = L_x, \\ \delta_y : 0 &= y_{\frac{1}{2}} < y_{3/2} < \dots < y_{N_y - \frac{1}{2}} < y_{N_y + \frac{1}{2}} = L_y. \end{aligned}$$

For $i = 1, \dots, N_x$ and $j = 1, \dots, N_y$, define

$$\begin{aligned} x_i &= \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}, \\ h_i &= x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \quad h = \max_i h_i, \end{aligned}$$

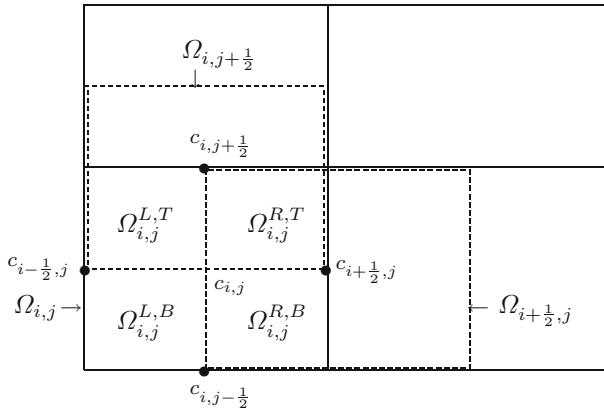


Fig. 1 A example of mesh partition

$$\begin{aligned}
 h_{i+\frac{1}{2}} &= \frac{h_{i+1} + h_i}{2} = x_{i+1} - x_i, \\
 y_j &= \frac{y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}}}{2}, \\
 k_j &= y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}, \quad k = \max_j k_j, \\
 k_{j+\frac{1}{2}} &= \frac{k_{j+1} + k_j}{2} = y_{j+1} - y_j, \\
 \Omega_{i,j} &= (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}), \\
 \Omega_{i+\frac{1}{2},j} &= (x_i, x_{i+1}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}), \\
 \Omega_{i,j+\frac{1}{2}} &= (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_j, y_{j+1}).
 \end{aligned}$$

We divide each $\Omega_{i,j}$ into 4 parts,

$$\begin{aligned}
 \Omega_{i,j}^{L,T} &= (x_{i-1/2}, x_i) \times (y_j, y_{j+1/2}), \quad \Omega_{i,j}^{R,T} = (x_i, x_{i+1/2}) \times (y_j, y_{j+1/2}), \\
 \Omega_{i,j}^{L,B} &= (x_{i-1/2}, x_i) \times (y_{j-1/2}, y_j) \quad \Omega_{i,j}^{R,B} = (x_i, x_{i+1/2}) \times (y_{j-1/2}, y_j).
 \end{aligned}$$

Here the superscript ‘L’, ‘R’, ‘T’ and ‘B’ means ‘Left’, ‘Right’, ‘Top’ and ‘Bottom’, respectively. It is clear that

$$\begin{aligned}
 \Omega_{i,j} &= \Omega_{i,j}^{L,T} \cup \Omega_{i,j}^{L,B} \cup \Omega_{i,j}^{R,T} \cup \Omega_{i,j}^{R,B}, \\
 \Omega_{i+1/2,j} &= \Omega_{i,j}^{R,T} \cup \Omega_{i,j}^{R,B} \cup \Omega_{i+1,j}^{L,T} \cup \Omega_{i+1,j}^{L,B}, \\
 \Omega_{i,j+1/2} &= \Omega_{i,j}^{L,T} \cup \Omega_{i,j}^{R,T} \cup \Omega_{i,j+1}^{L,B} \cup \Omega_{i,j+1}^{R,B}.
 \end{aligned}$$

The following Fig. 1 is a description of the dividing.

For a function $\theta(x, y, t)$, let $\theta_{l,m}^n$ denote $\theta(x_l, y_m, t^n)$ where l may take values $i, i + \frac{1}{2}$ for non-negative integers i , and m may take values $j, j + \frac{1}{2}$ for non-negative integers j . For discrete functions with values at proper discrete points, define

$$\begin{aligned}
 [d_t \theta]_{l,m}^n &= \frac{\theta_{l,m}^n - \theta_{l,m}^{n-1}}{\Delta t}, \\
 [d_x \theta]_{i+\frac{1}{2},j} &= \frac{\theta_{i+1,j} - \theta_{i,j}}{h_{i+\frac{1}{2}}}, \quad [d_y \theta]_{i,j+\frac{1}{2}} = \frac{\theta_{i,j+1} - \theta_{i,j}}{k_{j+\frac{1}{2}}}, \\
 [D_x \theta]_{i,j} &= \frac{\theta_{i+\frac{1}{2},j} - \theta_{i-\frac{1}{2},j}}{h_i}, \quad [D_y \theta]_{i,j} = \frac{\theta_{i,j+\frac{1}{2}} - \theta_{i,j-\frac{1}{2}}}{k_j},
 \end{aligned}$$

where for simplicity we omit the subscript n .

Also define the discrete inner products and norms and semi-norms as follows,

$$\begin{aligned}
 (\theta, \tau)_M &= (\theta, \tau)_{M_x, M_y} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} h_i k_j \theta_{i,j} \tau_{i,j}, \\
 (\theta, \tau)_x &= (\theta, \tau)_{T_x, M_y} = \sum_{i=2}^{N_x} \sum_{j=1}^{N_y} h_{i-\frac{1}{2}} k_j \theta_{i-\frac{1}{2},j} \tau_{i-\frac{1}{2},j}, \\
 (\theta, \tau)_y &= (\theta, \tau)_{M_x, T_y} = \sum_{i=1}^{N_x} \sum_{j=2}^{N_y} h_i k_{j-\frac{1}{2}} \theta_{i,j-\frac{1}{2}} \tau_{i,j-\frac{1}{2}}, \\
 \|\theta\|_M^2 &= (\theta, \theta)_{M_x, M_y}, \quad \|\theta\|_x^2 = (\theta, \theta)_x, \quad \|\theta\|_y^2 = (\theta, \theta)_y.
 \end{aligned}$$

3 A Block-Centered Finite Difference Method

In this section we present a block-centered finite difference method for the slightly compressible Forchheimer model.

For simplicity we use the following notation,

$$\alpha = \phi C_F, \quad a_1 = \frac{\mu}{\bar{k}}, \quad a_2 = a_2(p) = \beta \rho(p), \quad a(p, w) = a_1 + a_2(p)w. \tag{8}$$

Then the problem (7) can be written as

$$\left\{ \begin{array}{ll}
 \text{(i)} & (a_1 + a_2(p)|\mathbf{u}|)\mathbf{u} + \nabla p = 0, \quad \text{in } \Omega \times J, \\
 \text{(ii)} & \alpha \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = f, \quad (\mathbf{z}, t) \in \Omega \times J, \\
 \text{(iii)} & \mathbf{u} \cdot \mathbf{n} = 0, \quad (\mathbf{z}, t) \in \partial\Omega \times J. \\
 \text{(iv)} & p|_{t=0} = p_0, \quad \mathbf{z} \in \Omega.
 \end{array} \right. \tag{9}$$

Here $\partial\Omega$ is the boundary of Ω , and \mathbf{n} denotes the outward unit normal to $\partial\Omega$.

For slightly compressible flow in porous media μ, \bar{k}, ϕ and C_F are positive and bounded up and below, and β is non-negative. So a_1 and α are positive and bounded up and below, and a_2 is non-negative and bounded. For numerical analysis we make the following assumptions on the coefficients and analytical solution.

Assumption 1 a_1, a_2 and α are continuous functions, and there exist positive constants \underline{a} and \bar{a} such that,

$$0 < \underline{a} \leq a_1, \quad \alpha \leq \bar{a}, \quad a_2(p) \geq 0.$$

Assumption 2 The analytical solution (u, p) has the following regularity.

$$p \in L^\infty(0, T; W^{3,\infty}(\Omega)) \cap W^{1,\infty}(0, T; W^{2,\infty}(\Omega)) \cap W^{2,\infty}(0, T; L^\infty(\Omega));$$

$$u \in (W^{1,\infty}(0, T; W^{1,\infty}(\Omega)) \cap L^\infty(0, T; W^{3,\infty}(\Omega)))^2;$$

Assumption 3 $a_2(p) \in L^\infty(0, T; W^{2,\infty}(\Omega)) \cap W^{1,\infty}(0, T; W^{1,\infty}(\Omega))$.

The regularity analysis for the solution (u, p) of problem (7), or equivalently (9), can be found in articles and books such as [41–46]. For the regularity in Assumption 2 to hold some constraints on the initial value p_0 , the right-hand term f and the coefficients should be needed, see the references mentioned above.

For the definition of the scheme we define some interpolation operators. For a discrete function $\{q_{i,j}\}$ with value on nodal points $\{(x_i, y_j)\}$, define a piecewise-constant function $\Pi_h q_h$ on Ω such that,

$$\Pi_h q(x, y) = q_{i,j}, \quad (x, y) \in \Omega_{i,j}, \tag{10}$$

and a piecewise constant function $I_h q$ on $\Omega_{i,j}$ such that

$$I_h q = \begin{cases} \bar{I}_h q_{i+\frac{1}{4},j+\frac{1}{4}}, & (x, y) \in \Omega_{i,j}^{R,T}, \\ \bar{I}_h q_{i+\frac{1}{4},j-\frac{1}{4}}, & (x, y) \in \Omega_{i,j}^{R,B}, \\ \bar{I}_h q_{i-\frac{1}{4},j+\frac{1}{4}}, & (x, y) \in \Omega_{i,j}^{L,T}, \\ \bar{I}_h q_{i-\frac{1}{4},j-\frac{1}{4}}, & (x, y) \in \Omega_{i,j}^{L,B}. \end{cases} \tag{11}$$

Here the discrete interpolant function $\{\bar{I}_h q\}$ with values at $\{(x_i \pm \frac{h_i}{4}, y_j \pm \frac{k_j}{4})\}$ is as follows

$$\begin{cases} \bar{I}_h q_{i+\frac{1}{4},j\pm\frac{1}{4}} = \frac{1}{4h_{i+\frac{1}{2}}} (h_{i+1}q_{i,j} + h_i q_{i+1,j}) + \frac{1}{4k_{j\pm\frac{1}{2}}} (k_{j\pm 1}q_{i,j} + k_j q_{i,j\pm 1}), \\ \bar{I}_h q_{i-\frac{1}{4},j\pm\frac{1}{4}} = \frac{1}{4h_{i-\frac{1}{2}}} (h_{i-1}q_{i,j} + h_i q_{i-1,j}) + \frac{1}{4k_{j\pm\frac{1}{2}}} (k_{j\pm 1}q_{i,j} + k_j q_{i,j\pm 1}). \end{cases}$$

For a pair of discrete functions $\{V^x_{i+\frac{1}{2},j}\}$ and $\{V^y_{i,j+\frac{1}{2}}\}$ define the interpolant operator Π_2 as follows.

$$\Pi_2 V = (\Pi_x V^x, \Pi_y V^y) \tag{12}$$

where

$$\Pi_x V^x(x, y) = V^x_{i+\frac{1}{2},j}, \quad (x, y) \in \Omega_{i+\frac{1}{2},j}, \tag{13}$$

$$\Pi_y V^y(x, y) = V^y_{i,j+\frac{1}{2}}, \quad (x, y) \in \Omega_{i,j+\frac{1}{2}}. \tag{14}$$

Let $|(U, V)|$ be the norm function for a vector (U, V) . Direct calculation shows that

$$\begin{cases} |\Omega_{i,j}^{R,T}|^{-1} \int_{\Omega_{i,j}^{R,T}} |(\Pi_x V^x, \Pi_y V^y)| dx dy = |(V^x_{i+\frac{1}{2},j}, V^y_{i,j+\frac{1}{2}})|, \\ |\Omega_{i,j}^{R,B}|^{-1} \int_{\Omega_{i,j}^{R,B}} |(\Pi_x V^x, \Pi_y V^y)| dx dy = |(V^x_{i+\frac{1}{2},j}, V^y_{i,j-\frac{1}{2}})|, \\ |\Omega_{i+1,j}^{L,T}|^{-1} \int_{\Omega_{i+1,j}^{L,T}} |(\Pi_x V^x, \Pi_y V^y)| dx dy = |(V^x_{i+\frac{1}{2},j}, V^y_{i+1,j+\frac{1}{2}})|, \\ |\Omega_{i+1,j}^{L,B}|^{-1} \int_{\Omega_{i+1,j}^{L,B}} |(\Pi_x V^x, \Pi_y V^y)| dx dy = |(V^x_{i+\frac{1}{2},j}, V^y_{i+1,j-\frac{1}{2}})|. \end{cases} \tag{15}$$

For $w = p$, $I_h p$, define an interpolant $\bar{Q}(a_2(w), u)$ on $\Omega_{i,j}$ as follows.

$$\bar{Q}(a_2(w), u) = \begin{cases} a_2(w_{i+\frac{1}{4},j+\frac{1}{4}})|u_{i+\frac{1}{2},j}^x, u_{i,j+\frac{1}{2}}^y|, & (x, y) \in \Omega_{i,j}^{R,T}, \\ a_2(w_{i+\frac{1}{4},j-\frac{1}{4}})|u_{i+\frac{1}{2},j}^x, u_{i,j-\frac{1}{2}}^y|, & (x, y) \in \Omega_{i,j}^{R,B}, \\ a_2(w_{i-\frac{1}{4},j+\frac{1}{4}})|u_{i-\frac{1}{2},j}^x, u_{i,j+\frac{1}{2}}^y|, & (x, y) \in \Omega_{i,j}^{L,T}, \\ a_2(w_{i-\frac{1}{4},j-\frac{1}{4}})|u_{i-\frac{1}{2},j}^x, u_{i,j-\frac{1}{2}}^y|, & (x, y) \in \Omega_{i,j}^{L,B}. \end{cases} \tag{16}$$

Here it is remarkable that $\{I_h p_{i,j} \neq p_{i,j}\}$, then $\bar{Q}(a_2(p), u) \neq \bar{Q}(a_2(I_h p), u)$. Then define two square root averages as follows.

$$[Q(a_2(w), u)]_{i+\frac{1}{2},j} = \frac{1}{|\Omega_{i+\frac{1}{2},j}|} \int_{\Omega_{i+\frac{1}{2},j}} \bar{Q}(a_2(w), u) dx dy, \quad w = p, I_h p, \tag{17}$$

$$[Q(a_2(w), u)]_{i,j+\frac{1}{2}} = \frac{1}{|\Omega_{i,j+\frac{1}{2}}|} \int_{\Omega_{i,j+\frac{1}{2}}} \bar{Q}(a_2(w), u) dx dy, \quad w = p, I_h p. \tag{18}$$

Using the above notations the block-centered finite difference approximations $\{U_{i+\frac{1}{2},j}^x\}$, $\{U_{i,j+\frac{1}{2}}^y\}$ and $\{P_{i,j}\}$ to $\{u^x(x_{i+\frac{1}{2},j})\}$, $\{u^y(x_{i,j+\frac{1}{2}})\}$ and $\{p(x_{i,j})\}$, respectively, are chosen so that

$$\alpha [d_t P]_{i,j}^n + [D_x U^x]_{i,j}^n + [D_y U^y]_{i,j}^n = f_{i,j}^n, \tag{19}$$

$$\left(a_1 + [Q(a_2(I_h P), U)]_{i+\frac{1}{2},j}^n \right) U_{i+\frac{1}{2},j}^{x,n} = -[d_x P]_{i+\frac{1}{2},j}^n, \tag{20}$$

$$\left(a_1 + [Q(a_2(I_h P), U)]_{i,j+\frac{1}{2}}^n \right) U_{i,j+\frac{1}{2}}^{y,n} = -[d_y P]_{i,j+\frac{1}{2}}^n, \tag{21}$$

with boundary condition

$$U_{\frac{1}{2},j}^{x,n} = 0, \quad U_{N_x+\frac{1}{2},j}^{x,n} = 0, \quad j = 0, \dots, N_y, \tag{22}$$

$$U_{i,\frac{1}{2}}^{y,n} = 0, \quad U_{i,N_y+\frac{1}{2}}^{y,n} = 0, \quad i = 0, \dots, N_x. \tag{23}$$

Remark 3.1 One reason to define the interpolation operators I_h , Π_2 and Q is to ensure the monotonicity of the discrete nonlinear operator, see Lemma 6, which is necessary in convergence analysis. There are other possible definitions which can be used in real computation. Up to now we just proved the convergence with the presented interpolation.

Remark 3.2 The scheme is given suppose that the coefficients are continuous. It is clear that it can be used to solve the problem with piecewise continuous problem, provided that the coefficients are continuous in each cell.

The more complicated problem with tensor permeability are under consideration with the methods of [23,24].

Remark 3.3 Similar to [28] we know the approximate solution $\{P_{i,j}^n\}$, $\{U_{i+\frac{1}{2},j}^{x,n}\}$ and $\{U_{i,j+\frac{1}{2}}^{y,n}\}$ exist uniquely.

4 Error Estimates

In this section we verify that if the analytical solution u and p are sufficiently smooth, (U^x, U^y, P) is a second-order approximation to (u^x, u^y, p) .

For this purpose we present some lemmas. Set

$$\begin{aligned} \epsilon_{i+\frac{1}{2},j}^x(p) &= \frac{1}{2h_{i+\frac{1}{2}}} \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \left(\frac{h_{i+1}^2}{4} - (x - x_{i+1})^2 \right) \frac{\partial^3 p}{\partial x^3}(x, y_j, t) dx \\ &\quad - \frac{1}{2h_{i+\frac{1}{2}}} \int_{x_{i+\frac{1}{2}}}^{x_i} \left(\frac{h_i^2}{4} - (x - x_i)^2 \right) \frac{\partial^3 p}{\partial x^3}(x, y_j, t) dx. \end{aligned} \tag{24}$$

$$\begin{aligned} \epsilon_{i,j+\frac{1}{2}}^y(p) &= \frac{1}{2k_{j+\frac{1}{2}}} \int_{y_{j+\frac{1}{2}}}^{y_{j+1}} \left(\frac{k_{j+1}^2}{4} - (y - y_{j+1})^2 \right) \frac{\partial^3 p}{\partial x^3}(x_i, y, t) dx \\ &\quad - \frac{1}{2} \int_{y_{j+\frac{1}{2}}}^{y_j} \left(\frac{k_j^2}{4} - (y - y_j)^2 \right) \frac{\partial^3 p}{\partial x^3}(x_i, y, t) dx. \end{aligned} \tag{25}$$

The first lemma can be found in [47].

Lemma 1 *If $p \in W^{3,\infty}(\Omega)$, then there holds*

$$\begin{cases} \frac{\partial p_{i+\frac{1}{2},j}}{\partial x} = [d_x p]_{i+\frac{1}{2},j} - \frac{1}{8} \left[d_x \left(h^2 \frac{\partial^2 p}{\partial x^2} \right) \right]_{i+\frac{1}{2},j} + \epsilon_{i+\frac{1}{2},j}^x(p), \\ \frac{\partial p_{i,j+\frac{1}{2}}}{\partial y} = [d_y p]_{i,j+\frac{1}{2}} - \frac{1}{8} \left[d_y \left(k^2 \frac{\partial^2 p}{\partial y^2} \right) \right]_{i,j+\frac{1}{2}} + \epsilon_{i,j+\frac{1}{2}}^y(p), \end{cases} \tag{26}$$

with the approximation properties $\epsilon_{i+\frac{1}{2},j}^x(p) = O(h^2)$ and $\epsilon_{i,j+\frac{1}{2}}^y(p) = O(k^2)$.

Define

$$\delta_{i,j} = \left[\frac{h^2}{8} \frac{\partial^2 p}{\partial x^2} + \frac{k^2}{8} \frac{\partial^2 p}{\partial y^2} \right]_{i,j} = \frac{h_i^2}{8} \frac{\partial^2 p_{i,j}}{\partial x^2} + \frac{k_j^2}{8} \frac{\partial^2 p_{i,j}}{\partial y^2}, \tag{27}$$

and set

$$\tilde{\epsilon}_{i+\frac{1}{2},j}^x(p) = \epsilon_{i+\frac{1}{2},j}^x(p) + \left[d_x \left(\frac{k^2}{8} \frac{\partial^2 p}{\partial y^2} \right) \right]_{i+\frac{1}{2},j}. \tag{28}$$

$$\tilde{\epsilon}_{i,j+\frac{1}{2}}^y(p) = \epsilon_{i,j+\frac{1}{2}}^y(p) + \left[d_y \left(\frac{h^2}{8} \frac{\partial^2 p}{\partial x^2} \right) \right]_{i,j+\frac{1}{2}}, \tag{29}$$

Similar to Lemma 4.2 of [47] and Lemma 4.1 of [28] we have the following lemma.

Lemma 2 *If $p \in W^{3,\infty}(\Omega)$, then there holds*

$$\begin{cases} [(a_1 + a_2(p)|u|)u^x]_{i+\frac{1}{2},j} = -[d_x(p - \delta)]_{i+\frac{1}{2},j} - \tilde{\epsilon}_{i+\frac{1}{2},j}^x(p) \\ [(a_1 + a_2(p)|u|)u^y]_{i,j+\frac{1}{2}} = -[d_y(p - \delta)]_{i,j+\frac{1}{2}} - \tilde{\epsilon}_{i,j+\frac{1}{2}}^y(p), \end{cases} \tag{30}$$

with the following approximate properties

$$\tilde{\epsilon}_{i+\frac{1}{2},j}^x(p) = O(h^2 + k^2), \quad \tilde{\epsilon}_{i,j+\frac{1}{2}}^y(p) = O(h^2 + k^2). \tag{31}$$

Define

$$\begin{aligned} \tilde{\delta}_{i,j} &= \frac{a_2(p_{i,j})}{4} \frac{u_{i,j}^x u_{i,j}^y}{|u_{i,j}|} \frac{\partial u_{i,j}^y}{\partial x} h_i^2 + \frac{a_2(p_{i,j})}{4} \frac{u_{i,j}^x u_{i,j}^y}{|u_{i,j}|} \frac{\partial u_{i,j}^x}{\partial y} k_j^2 \\ &\quad + \frac{1}{8} \frac{\partial a_2(p)_{i,j}}{\partial x} |u_{i,j}| u_{i,j}^x h_i^2 + \frac{1}{8} \frac{\partial a_2(p)_{i,j}}{\partial y} |u_{i,j}| u_{i,j}^y k_j^2. \end{aligned} \tag{32}$$

$$\eta_{i,j} = \delta_{i,j} + \tilde{\delta}_{i,j}. \tag{33}$$

Similar to Lemma 4.2 of [27] we can prove the following lemma.

Lemma 3 *If $p \in L^\infty(0, T; W^{3,\infty}(\Omega))$, $u \in (L^\infty(0, T; L^\infty(\Omega)))^2$ and $a_2(p) \in L^\infty(0, T; W^{2,\infty}(\Omega))$, then we have that*

$$(a_1 + [Q(a_2(p), u)]_{i+\frac{1}{2},j}) u_{i+\frac{1}{2},j}^x = -[d_x(p - \eta)]_{i+\frac{1}{2},j} - \tilde{\epsilon}_{i+\frac{1}{2},j}^x(p), \tag{34}$$

$$(a_1 + [Q(a_2(p), u)]_{i,j+\frac{1}{2}}) u_{i,j+\frac{1}{2}}^y = -[d_y(p - \eta)]_{i,j+\frac{1}{2}} - \tilde{\epsilon}_{i,j+\frac{1}{2}}^y(p), \tag{35}$$

with the following approximate properties

$$\tilde{\epsilon}_{i+\frac{1}{2},j}^x(p) = O(h^2 + k^2), \quad \tilde{\epsilon}_{i,j+\frac{1}{2}}^y(p) = O(h^2 + k^2). \tag{36}$$

Lemma 4 *Under the condition of Lemma 3, we have that*

$$(a_1 + [Q(a_2(I_h p), u)]_{i+\frac{1}{2},j}) u_{i+\frac{1}{2},j}^x = -[d_x(p - \eta)]_{i+\frac{1}{2},j} - \epsilon_{i+\frac{1}{2},j}^x(p), \tag{37}$$

$$(a_1 + [Q(a_2(I_h p), u)]_{i,j+\frac{1}{2}}) u_{i,j+\frac{1}{2}}^y = -[d_y(p - \eta)]_{i,j+\frac{1}{2}} - \epsilon_{i,j+\frac{1}{2}}^y(p), \tag{38}$$

with the following approximate properties

$$\epsilon_{i+\frac{1}{2},j}^x(p) = O(h^2 + k^2), \quad \epsilon_{i,j+\frac{1}{2}}^y(p) = O(h^2 + k^2). \tag{39}$$

Proof Direct calculation shows that

$$\begin{aligned} &\left(a_1 + [Q(a_2(I_h p), u)]_{i+\frac{1}{2},j} \right) u_{i+\frac{1}{2},j}^x \\ &= \left(a_1 + [Q(a_2(p), u)]_{i+\frac{1}{2},j} \right) u_{i+\frac{1}{2},j}^x \\ &\quad + \left([Q(a_2(I_h p), u)]_{i+\frac{1}{2},j} - [Q(a_2(p), u)]_{i+\frac{1}{2},j} \right) u_{i+\frac{1}{2},j}^x \\ &= -[d_x(p - \eta)]_{i+\frac{1}{2},j} - \tilde{\epsilon}_{i+\frac{1}{2},j}^x(p) \\ &\quad + \left([Q(a_2(I_h p), u)]_{i+\frac{1}{2},j} - [Q(a_2(p), u)]_{i+\frac{1}{2},j} \right) u_{i+\frac{1}{2},j}^x \\ &\equiv -[d_x(p - \eta)]_{i+\frac{1}{2},j} - \epsilon_{i+\frac{1}{2},j}^x(p). \end{aligned} \tag{40}$$

Here the last equivalence is the definition of $\epsilon_{i+\frac{1}{2},j}^x(p)$. By the definition of $I_h p$ it is clear that

$$I_h p_{i\pm\frac{1}{4},j\pm\frac{1}{4}} - p_{i\pm\frac{1}{4},j\pm\frac{1}{4}} = O(h^2 + k^2),$$

Noticing the symmetric structures in the definition (17) we have that

$$[Q(a_2(I_h p), u)]_{i+\frac{1}{2},j} - [Q(a_2(p), u)]_{i+\frac{1}{2},j} = O(h^2 + k^2).$$

Therefore

$$\begin{aligned} \varepsilon_{i+\frac{1}{2},j}^x(p) &= \tilde{\varepsilon}_{i+\frac{1}{2},j}^x(p) - ([Q(a_2(I_h p), u)] - [Q(a_2(p), u)])_{i+\frac{1}{2},j} u_{i+\frac{1}{2},j}^x \\ &= O(h^2 + k^2). \end{aligned} \tag{41}$$

Similarly we can prove

$$\begin{aligned} \varepsilon_{i,j+\frac{1}{2}}^y(p) &= \tilde{\varepsilon}_{i,j+\frac{1}{2}}^y(p) - ([Q(a_2(I_h p), u)] - [Q(a_2(p), u)])_{i,j+\frac{1}{2}} u_{i,j+\frac{1}{2}}^y \\ &= O(h^2 + k^2), \end{aligned} \tag{42}$$

and

$$\left(a_1 + [Q(a_2(I_h p), u)]_{i,j+\frac{1}{2}}\right) u_{i,j+\frac{1}{2}}^y = -[d_y(p - \eta)]_{i,j+\frac{1}{2}} - \varepsilon_{i,j+\frac{1}{2}}^y(p),$$

which complete the proof. □

The following two lemmas can be found in [5, 6], see Lemma 2.3 of [6] and Proposition III.6 of [5], or in [27].

Lemma 5 *Let $\mathbf{z}, \mathbf{h} \in \mathbb{R}^d$. The vector-valued function $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is defined as $\mathbf{f}(\mathbf{z}) = |\mathbf{z}|\mathbf{z}$. Then there exists a positive constant C_0 such that*

$$C_0(|\mathbf{z}| + |\mathbf{z} + \mathbf{h}|)|\mathbf{h}|^2 \leq (\mathbf{f}(\mathbf{z} + \mathbf{h}) - \mathbf{f}(\mathbf{z})) \cdot \mathbf{h}. \tag{43}$$

Lemma 6 *For any vector-valued functions $\mathbf{V} = (V^x, V^y)$ and $\mathbf{W} = (W^x, W^y)$ we have that $(|\mathbf{V}|\mathbf{V} - |\mathbf{W}|\mathbf{W}, \mathbf{V} - \mathbf{W}) \geq 0$, and further*

$$(a(|\mathbf{V}|\mathbf{V} - a(|\mathbf{W}|\mathbf{W}, \mathbf{V} - \mathbf{W})) \geq a_1 \|\mathbf{V} - \mathbf{W}\|^2. \tag{44}$$

The result of the following lemma is obvious, see [27].

Lemma 7 *Let $\{V_{i+\frac{1}{2},j}^x\}, \{V_{i,j+\frac{1}{2}}^y\}, \{W_{i+\frac{1}{2},j}^x\}, \{W_{i,j+\frac{1}{2}}^y\}$ and $\{q_{i,j}^x\}, \{q_{i,j}^y\}$ be discrete functions with $W_{\frac{1}{2},j}^x = W_{N_x+\frac{1}{2},j}^x = W_{i,\frac{1}{2}}^y = W_{i,N_y+\frac{1}{2}}^y = 0$. Then there hold,*

$$(-d_x q^x, W^x)_x = (q^x, D_x W^x)_M, \quad (-d_y q^y, W^y)_y = (q^y, D_y W^y)_M. \tag{45}$$

Now we consider the error estimate.

Theorem 1 *Suppose the coefficients μ, α, \bar{k} are continuous functions and are bounded up and below. Suppose also that Assumptions 1–3 are hold. For the solution of the block-centered finite difference scheme when the discretization parameters $\Delta t, h$ and k are sufficiently small there exists a positive constant C independent of $\Delta t, h$ and k such that for $m \leq \frac{T}{\Delta t}$,*

$$\begin{aligned} \|(P - p)^m\|_M &\leq C(\Delta t + h^2 + k^2) \\ \sum_{n=1}^m \Delta t \left(\|(U - u)^{x,n}\|_x^2 + \|(U - u)^{y,n}\|_y^2 \right)^{\frac{1}{2}} &\leq C(\Delta t + h^2 + k^2). \end{aligned}$$

Proof From Lemma 4, Eqs. (20), (21) and (9) we have that

$$\begin{aligned} & a_1[U^x - u^x]_{i+\frac{1}{2},j}^n + [Q(a_2(I_h P^n), U^n)]_{i+\frac{1}{2},j} U_{i+\frac{1}{2},j}^{x,n} \\ & \quad - [Q(a_2(I_h P^n), u^n)]_{i+\frac{1}{2},j} u_{i+\frac{1}{2},j}^{x,n} \\ & = - [d_x(P - p + \eta)]_{i+\frac{1}{2},j}^n + \varepsilon_{i+\frac{1}{2},j}^{x,n}(p). \end{aligned} \tag{46}$$

$$\begin{aligned} & a_1[U^y - u^y]_{i,j+\frac{1}{2}}^n + [Q(a_2(I_h P^n), U^n)]_{i,j+\frac{1}{2}} U_{i,j+\frac{1}{2}}^{y,n} \\ & \quad - Q(a_2(I_h P^n), u^n)]_{i,j+\frac{1}{2}} u_{i,j+\frac{1}{2}}^{y,n} \\ & = - [d_y(P - p + \eta)]_{i,j+\frac{1}{2}}^n + \varepsilon_{i,j+\frac{1}{2}}^{y,n}(p). \end{aligned} \tag{47}$$

Define

$$\begin{cases} e_{i,j}^{p,n} = (P - p)_{i,j}^n, \\ e_{i+\frac{1}{2},j}^{x,n} = (U^x - u^x)_{i+\frac{1}{2},j}^n, & e_{i,j+\frac{1}{2}}^{y,n} = (U^y - u^y)_{i,j+\frac{1}{2}}^n. \end{cases} \tag{48}$$

Then

$$\begin{aligned} & a_1 e_{i+\frac{1}{2},j}^{x,n} + [Q(a_2(I_h P), U)]_{i+\frac{1}{2},j}^n U_{i+\frac{1}{2},j}^{x,n} - [Q(a_2(I_h p), u)]_{i+\frac{1}{2},j}^n u_{i+\frac{1}{2},j}^{x,n} \\ & = - [d_x(e^p + \eta)]_{i+\frac{1}{2},j}^n + \varepsilon_{i+\frac{1}{2},j}^{x,n}(p). \end{aligned} \tag{49}$$

$$\begin{aligned} & a_1 e_{i,j+\frac{1}{2}}^{y,n} + [Q(a_2(I_h P), U)]_{i,j+\frac{1}{2}}^n U_{i,j+\frac{1}{2}}^{y,n} - [Q(a_2(I_h p), u)]_{i,j+\frac{1}{2}}^n u_{i,j+\frac{1}{2}}^{y,n} \\ & = - [d_y(e^p + \eta)]_{i,j+\frac{1}{2}}^n + \varepsilon_{i,j+\frac{1}{2}}^{y,n}(p). \end{aligned} \tag{50}$$

From Eq. (9) we have that

$$\alpha d_t p_{i,j}^n + [D_x u^x]_{i,j}^n + [D_y u^y]_{i,j}^n = f_{i,j}^n + \epsilon_{i,j}^{1,n}, \tag{51}$$

where

$$\begin{aligned} \epsilon_{i,j}^{1,n} &= \alpha d_t p_{i,j}^n - \alpha \frac{\partial p_{i,j}^n}{\partial t} + [D_x u^x]_{i,j}^n - \frac{\partial u_{i,j}^{x,n}}{\partial x} + [D_y u^y]_{i,j}^n - \frac{\partial u_{i,j}^{y,n}}{\partial y} \\ &= O(\Delta t + h^2 + k^2). \end{aligned} \tag{52}$$

Here we have used the fact that x_i is the midpoint of $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and y_j is the midpoint of $(y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$.

From (51) and (19) we have that

$$\alpha d_t (P - p)_{i,j}^n + [D_x (U^x - u^x)]_{i,j}^n + [D_y (U^y - u^y)]_{i,j}^n = -\epsilon_{i,j}^{1,n}. \tag{53}$$

Denote by

$$\begin{aligned} \epsilon_{i,j}^{2,n} &= d_t \eta_{i,j}^n \\ &= \frac{h_i^2}{8} d_t \left(\frac{\partial^2 p}{\partial x^2} \right)_{i,j}^n + \frac{k_j^2}{8} d_t \left(\frac{\partial^2 p}{\partial y^2} \right)_{i,j}^n \\ & \quad + \frac{h_i^2}{4} d_t \left(a_2(p) \frac{u^x u^y}{|u|} \frac{\partial u^y}{\partial x} \right)_{i,j}^n + \frac{k_j^2}{4} \left(a_2(p) \frac{u^x u^y}{|u|} \frac{\partial u^x}{\partial y} \right)_{i,j}^n \\ & \quad + \frac{h_i^2}{8} d_t \left(\frac{\partial a_2(p)}{\partial x} |u| u^x \right)_{i,j}^n + \frac{k_j^2}{8} d_t \left(\frac{\partial a_2(p)}{\partial y} |u| u^y \right)_{i,j}^n. \end{aligned} \tag{54}$$

When p and u are sufficiently smooth, it is clear that

$$\epsilon_{i,j}^{2,n} = O(h^2 + k^2). \tag{55}$$

From (53) we have that

$$d_t(e^p + \eta)_{i,j}^n + [D_x e^x]_{i,j}^n + [D_y e^y]_{i,j}^n = -\epsilon_{i,j}^{1,n} + \epsilon_{i,j}^{2,n}. \tag{56}$$

Multiplying (56) by $(e^p + \eta)_{i,j}^n h_i k_j$ and summing for $i, j, 1 \leq i \leq N_x, 1 \leq j \leq N_y$, we have that

$$\begin{aligned} &(d_t(e^p + \eta)^n, (e^p + \eta)^n)_M + (D_x e^{x,n}, (e^p + \eta)^n)_M \\ &+ (D_y e^{y,n}, (e^p + \eta)^n)_M = (-\epsilon^{1,n} + \epsilon^{2,n}, (e^p + \eta)^n)_M. \end{aligned} \tag{57}$$

By Lemma 7 we have that

$$\begin{aligned} &(d_t(e^p + \eta)^n, (e^p + \eta)^n)_M - (e^{x,n}, d_x(e^p + \eta)^n)_x - (e^{y,n}, d_y(e^p + \eta)^n)_y \\ &= (-\epsilon^{1,n} + \epsilon^{2,n}, (e^p + \eta)^n)_M. \end{aligned} \tag{58}$$

Using (49) and (50) we have that

$$\begin{aligned} &\frac{1}{2} d_t \| (e^p + \eta)^n \|_M^2 + \frac{\Delta t}{2} \| d_t (e^p + \eta)^n \|_M^2 + a_1 \left(\| e^{x,n} \|_x^2 + \| e^{y,n} \|_y^2 \right) \\ &+ ([Q(a_2(I_h P^n), U^n)] U^{x,n} - Q(a_2(I_h P^n), u^n)] u^{x,n}, e^{x,n})_x \\ &+ ([Q(a_2(I_h P^n), U^n)] U^{y,n} - Q(a_2(I_h P^n), u^n)] u^{y,n}, e^{y,n})_y \\ &= (-\epsilon^{1,n} + \epsilon^{2,n}, (e^p + \eta)^n)_M + (e^{x,n}, \mathcal{E}^{x,n}(p))_x + (e^{y,n}, \mathcal{E}^{y,n}(p))_y. \end{aligned} \tag{59}$$

$$\begin{aligned} &\frac{1}{2} d_t \| (e^p + \eta)^n \|_M^2 + \frac{\Delta t}{2} \| d_t (e^p + \eta)^n \|_M^2 + a_1 \left(\| e^{x,n} \|_x^2 + \| e^{y,n} \|_y^2 \right) \\ &+ ([Q(a_2(I_h P^n), U^n)] U^{x,n} - [Q(a_2(I_h P^n), u^n)] u^{x,n}, e^{x,n})_x \\ &+ ([Q(a_2(I_h P^n), U^n)] U^{y,n} - [Q(a_2(I_h P^n), u^n)] u^{y,n}, e^{y,n})_y \\ &= (-\epsilon^{1,n} + \epsilon^{2,n}, (e^p + \eta)^n)_M + (e^{x,n}, \mathcal{E}^{x,n}(p))_x + (e^{y,n}, \mathcal{E}^{y,n}(p))_y \\ &- ([Q(a_2(I_h P^n), U^n)] - Q(a_2(I_h P^n), u^n)] u^{x,n}, e^{x,n})_x \\ &- ([Q(a_2(I_h P^n), U^n)] - Q(a_2(I_h P^n), u^n)] u^{y,n}, e^{y,n})_y. \end{aligned} \tag{60}$$

The third combining with the fourth terms on the left hand sides can be estimated as follows.

$$\begin{aligned} &([Q(a_2(I_h P^n), U^n)] U^{x,n} - [Q(a_2(I_h P^n), u^n)] u^{x,n}, e^{x,n})_x \\ &+ ([Q(a_2(I_h P^n), U^n)] U^{y,n} - [Q(a_2(I_h P^n), u^n)] u^{y,n}, e^{y,n})_y \\ &\equiv \sum_{i,j} (I_{i,j}^1 + I_{i,j}^2 + I_{i,j}^3 + I_{i,j}^4), \end{aligned} \tag{61}$$

where

$$I_{i,j}^1 = \int_{\Omega_{i,j}^{R,T}} \left\{ \bar{Q}(a_2(I_h P^n), U^n) \left(U_{i+\frac{1}{2},j}^{x,n} e_{i+\frac{1}{2},j}^{x,n} + U_{i,j+\frac{1}{2}}^{y,n} e_{i,j+\frac{1}{2}}^{y,n} \right) - \bar{Q}(a_2(I_h P^n), u^n) \left(u_{i+\frac{1}{2},j}^{x,n} e_{i+\frac{1}{2},j}^{x,n} + u_{i,j+\frac{1}{2}}^{y,n} e_{i,j+\frac{1}{2}}^{y,n} \right) \right\} dx dy, \tag{62}$$

$$I_{i,j}^2 = \int_{\Omega_{i,j}^{L,B}} \left\{ \bar{Q}(a_2(I_h P^n), U^n) \left(U_{i+\frac{1}{2},j}^{x,n} e_{i+\frac{1}{2},j}^{x,n} + U_{i,j-\frac{1}{2}}^{y,n} e_{i,j-\frac{1}{2}}^{y,n} \right) - \bar{Q}(a_2(I_h P^n), u^n) \left(u_{i+\frac{1}{2},j}^{x,n} e_{i+\frac{1}{2},j}^{x,n} + u_{i,j-\frac{1}{2}}^{y,n} e_{i,j-\frac{1}{2}}^{y,n} \right) \right\} dx dy, \tag{63}$$

$$I_{i,j}^3 = \int_{\Omega_{i+1,j}^{L,T}} \left\{ \bar{Q}(a_2(I_h P^n), U^n) \left(U_{i+\frac{1}{2},j}^{x,n} e_{i+\frac{1}{2},j}^{x,n} + U_{i+1,j+\frac{1}{2}}^{y,n} e_{i+1,j+\frac{1}{2}}^{y,n} \right) - \bar{Q}(a_2(I_h P^n), u^n) \left(u_{i+\frac{1}{2},j}^{x,n} e_{i+\frac{1}{2},j}^{x,n} + u_{i+1,j+\frac{1}{2}}^{y,n} e_{i+1,j+\frac{1}{2}}^{y,n} \right) \right\} dx dy, \tag{64}$$

$$I_{i,j}^4 = \int_{\Omega_{i+1,j}^{L,B}} \left\{ \bar{Q}(a_2(I_h P^n), U^n) \left(U_{i+\frac{1}{2},j}^{x,n} e_{i+\frac{1}{2},j}^{x,n} + U_{i+1,j-\frac{1}{2}}^{y,n} e_{i+1,j-\frac{1}{2}}^{y,n} \right) - \bar{Q}(a_2(I_h P^n), u^n) \left(u_{i+\frac{1}{2},j}^{x,n} e_{i+\frac{1}{2},j}^{x,n} + u_{i+1,j-\frac{1}{2}}^{y,n} e_{i+1,j-\frac{1}{2}}^{y,n} \right) \right\} dx dy. \tag{65}$$

From the definition of the interpolant and Lemma 6 we have that

$$I_{i,j}^1 = a_2 \left(I_h P^n_{i+\frac{1}{2},j+\frac{1}{4}} \right) \int_{\Omega_{i,j}^{R,T}} \left\{ \left| \left(U_{i+\frac{1}{2},j}^{x,n}, U_{i,j+\frac{1}{2}}^{y,n} \right) \right| \left(U_{i+\frac{1}{2},j}^{x,n} e_{i+\frac{1}{2},j}^{x,n} + U_{i,j+\frac{1}{2}}^{y,n} e_{i,j+\frac{1}{2}}^{y,n} \right) - \left| \left(u_{i+\frac{1}{2},j}^{x,n}, u_{i,j+\frac{1}{2}}^{y,n} \right) \right| \left(u_{i+\frac{1}{2},j}^{x,n} e_{i+\frac{1}{2},j}^{x,n} + u_{i,j+\frac{1}{2}}^{y,n} e_{i,j+\frac{1}{2}}^{y,n} \right) \right\} dx dy \geq 0. \tag{66}$$

Similarly we deal with other terms $I_{i,j}^2$ to $I_{i,j}^4$.

Then from (60) and (61) we have that

$$\begin{aligned} & \frac{1}{2} d_t \|(e^p + \eta)^n\|_M^2 + \frac{\Delta t}{2} \|d_t(e^p + \eta)^n\|_M^2 + a_1 \left(\|e^{x,n}\|_x^2 + \|e^{y,n}\|_y^2 \right) \\ & \leq (-\epsilon^{1,n} + \epsilon^{2,n}, (e^p + \eta)^n)_M + (e^{x,n}, \varepsilon^{x,n}(p))_x + (e^{y,n}, \varepsilon^{y,n}(p))_y \\ & \quad - ([Q(a_2(I_h P^n), u^n) - Q(a_2(I_h P^n), u^n)]u^{x,n}, e^{x,n})_x \\ & \quad - ([Q(a_2(I_h P^n), u^n) - Q(a_2(I_h P^n), u^n)]u^{y,n}, e^{y,n})_y \\ & \equiv \sum_{l=1}^5 I_l. \end{aligned} \tag{67}$$

By Schwarz’s inequality it is clear that

$$I_1 + I_2 + I_3 \leq \|(e^p + \eta)^n\|_M^2 + \frac{a_1}{4} (\|e^{x,n}\|_x^2 + \|e^{y,n}\|_y^2) + C_1 (\|\epsilon^{1,n} + \epsilon^{2,n}\|_M^2 + \|\varepsilon^{x,n}(p)\|_x^2 + \|\varepsilon^{y,n}(p)\|_y^2). \tag{68}$$

Because we use $\{P_{i,j}^n\}$ and $\{p_{i,j}^n\}$ in the definitions of $\{I_h P_{i\pm\frac{1}{4},j\pm\frac{1}{4}}^n\}$ and $\{I_h p_{i\pm\frac{1}{4},j\pm\frac{1}{4}}^n\}$, from some simple calculations we can get

$$\begin{aligned} I_4 &\leq C \|P^n - p^n\|_M \|e^{x,n}\|_x \\ &\leq C (\|(e^P + \eta)^n\|_M + \|\eta^n\|_M) \|e^{x,n}\|_x \\ &\leq \frac{a_1}{4} \|e^{x,n}\|_x^2 + C_2 (\|(e^P + \eta)^n\|_M^2 + \|\eta^n\|_M^2). \end{aligned} \tag{69}$$

Similarly

$$I_5 \leq \frac{a_1}{4} \|e^{y,n}\|_y^2 + C_3 (\|(e^P + \eta)^n\|_M^2 + \|\eta^n\|_M^2). \tag{70}$$

Combining (67) with (68), (69) and (70) we have that

$$\begin{aligned} &\frac{1}{2} \Delta t \|(e^P + \eta)^n\|_M^2 + \frac{\Delta t}{2} \|d_t(e^P + \eta)^n\|_M^2 + \frac{a_1}{2} (\|e^{x,n}\|_x^2 + \|e^{y,n}\|_y^2) \\ &\leq C_4 \|(e^P + \eta)^n\|_M^2 + C_4 (\|\epsilon^{1,n}\|_M^2 + \|\epsilon^{2,n}\|_M^2 + \|\eta^n\|_M^2 + \|\epsilon^{x,n}\|_x^2 + \|\epsilon^{y,n}\|_y^2). \end{aligned} \tag{71}$$

Summing (71) for n from 1 to m , $m \leq \frac{T}{\Delta t}$, and using the estimates of $\epsilon^{1,n}$, $\epsilon^{2,n}$, $\epsilon^{x,n}(p)$, $\epsilon^{y,n}(p)$ and η^n we have that

$$\begin{aligned} &\|(e^P + \eta)^m\|_M^2 + \sum_{n=1}^m \Delta t^2 \|(d_t(e^P + \eta))^n\|_M^2 + a_1 (\|e^{x,n}\|_x^2 + \|e^{y,n}\|_y^2) \\ &\leq 2C_4 \sum_{n=1}^m \Delta t \|(e^P + \eta)^n\|_M^2 + \|(e^P + \eta)^0\|_M^2 \\ &\quad + 2C_4 \sum_{n=1}^m \Delta t (\|\epsilon^{1,n}\|_M^2 + \|\epsilon^{2,n}\|_M^2 + \|\eta^n\|_M^2 + \|\epsilon^{x,n}(p)\|_x^2 + \|\epsilon^{y,n}(p)\|_y^2) \\ &\leq 2C_4 \sum_{n=1}^m \Delta t \|(e^P + \eta)^n\|_M^2 + C(\Delta t + h^2 + k^2). \end{aligned} \tag{72}$$

By Gronwall’s inequality when Δt is sufficiently small we have that

$$\begin{aligned} &\|(P - p + \eta)^m\|_M^2 + \sum_{n=1}^m \Delta t^2 \|d_t(P - p + \eta)^n\|_M^2 \\ &\quad + a_1 \sum_{n=1}^m \Delta t (\|(U - u)^{x,n}\|_x^2 + \|(U - u)^{y,n}\|_y^2) \\ &\leq C(\Delta t^2 + h^4 + k^4). \end{aligned} \tag{73}$$

Combining (73) with the estimate for η results in

$$\begin{aligned} &\|(P - p)^m\|_M^2 + \sum_{n=1}^m \Delta t (\|(U - u)^{x,n}\|_x^2 + \|(U - u)^{y,n}\|_y^2) \\ &\leq C(\Delta t^2 + h^4 + k^4), \end{aligned}$$

which completes the proof of the first estimate. It is well-known that

$$\begin{aligned}
 & \sum_{n=1}^m \Delta t (\| (U - u)^{x,n} \|_x^2 + \| (U - u)^{y,n} \|_y^2)^{\frac{1}{2}} \\
 & \leq \left(\sum_{n=1}^m \Delta t \right)^{\frac{1}{2}} \left(\sum_{n=1}^m \Delta t (\| (U - u)^{x,n} \|_x^2 + \| (U - u)^{y,n} \|_y^2) \right)^{\frac{1}{2}} \\
 & \leq C(\Delta t + h^2 + k^2),
 \end{aligned} \tag{74}$$

which completes the proof of the second one. □

Remark 4.1 In this paper we just considered the homogeneous Neumann boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$. For non-homogeneous boundary condition the present interpolant operators does not have second-order accuracy near the boundary. We will improve the result next.

5 Numerical Experiment

In this section we carry out some numerical experiments using the block-centered finite difference scheme in two dimensional region. For simplicity, the region are selected as unit square, i.e. $\Omega = [0, 1] \times [0, 1]$. The time interval is $(0, T] = (0, 1]$. The permeability, the viscosity, density and Forchheimer number β are all constants. For simplicity, take $\mu = 2, \bar{k} = 4, \rho = 1, \beta = 5$. We use an iterative procedure to solve the nonlinear system obtained from the finite difference discretization.

We test Examples 1 and 2 to verify the convergence rates of the presented scheme. The initial partition is 10×10 grid. And then the grid is refined 4 times. For each refining we take $\frac{\Delta t}{h^2}$ to be a constant. A grid with degree of freedom is plotted in Fig. 2.

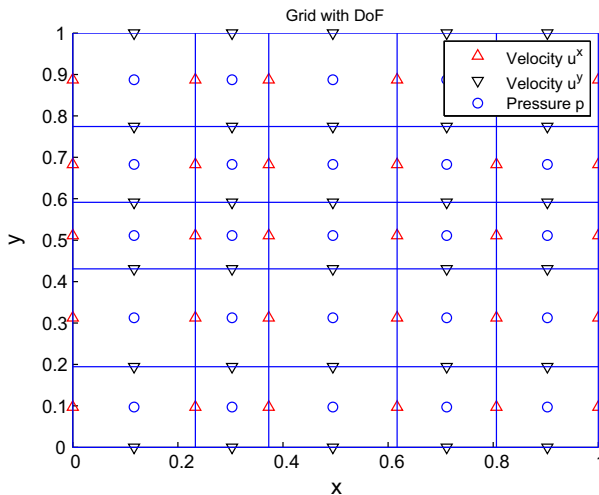


Fig. 2 Grid with degree of freedom of first level

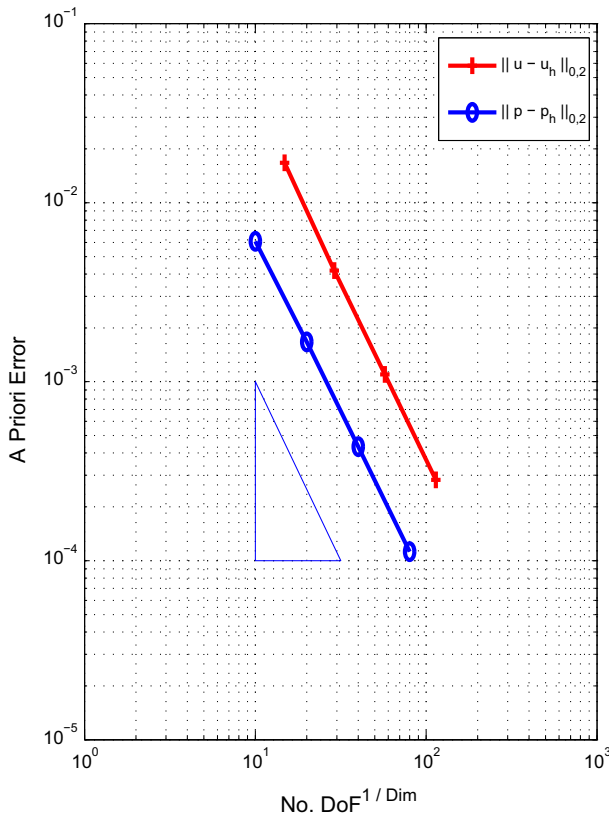


Fig. 3 Convergence rates of Example 1 (The tangent of the triangle is 2)

The numerical results are listed in Figs. 3 and 4 and Tables 1 and 2. In the tables $\frac{h_{max}}{h_{min}}$ and $\frac{k_{max}}{k_{min}}$ are listed to show the non-uniformity of the grid, where h_{max} (h_{min}) is the maximum (minimum) meshsize in x -direction and k_{max} (k_{min}) is the maximum (minimum) meshsize in y -direction. The discrete l^2 norms of the errors are defined as follows.

$$E_{u,l^2} = \max_{m \leq T/\Delta t} \left(\|(U - u)^{x,m}\|_x^2 + \|(U - u)^{y,m}\|_y^2 \right)^{\frac{1}{2}},$$

$$E_{p,l^2} = \max_{m \leq T/\Delta t} \|(P - p)^m\|_M.$$

Example 1 An example with homogeneous Neumann boundary condition is as below. The flux on the boundary condition, $\mathbf{u} \cdot \mathbf{n}$, is computed according to the analytic solution given as below.

$$\begin{cases} p(x, y, t) = (x - x^2)(y - y^2), \\ \mathbf{u}(x, y, t) = (\sin \pi t \sin \pi x \cos \pi y, \sin \pi t \cos \pi x \sin \pi y)^T. \end{cases}$$

The numerical results are listed in Fig. 3 and Table 1.

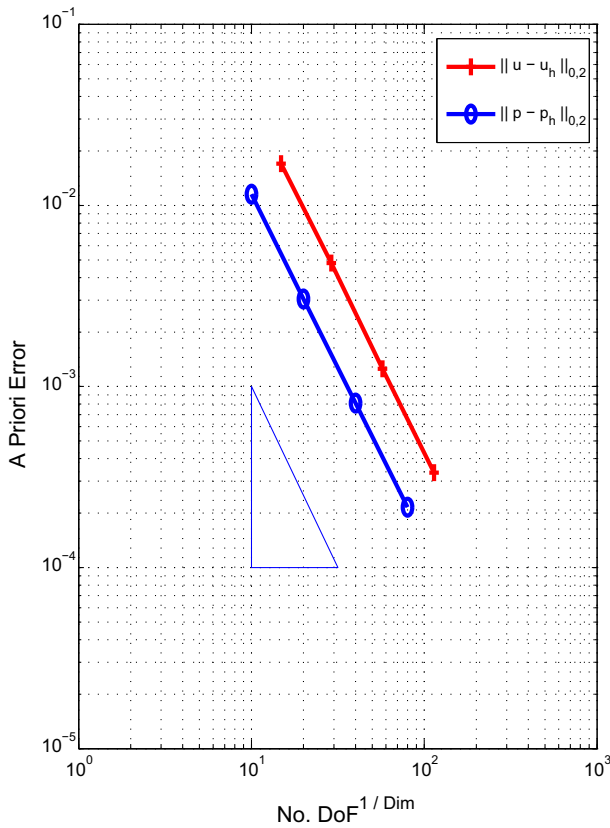


Fig. 4 Convergence rates of Example 2 (The tangent of the triangle is 2.)

Table 1 A priori error and convergence rates for Example 1 at $t = 1$

Partition	E_{u,l^2}	Rate	E_{p,l^2}	Rate	$\frac{h_{max}}{h_{min}}$	$\frac{k_{max}}{k_{min}}$
10×10	1.6715E-2	–	6.0956E-3	–	4.8019	3.6283
20×20	4.2766E-3	-1.9387	1.6701E-3	-1.8678	5.4701	4.9100
40×40	1.0998E-3	-1.9251	4.3426E-4	-1.9433	6.2039	5.2358
80×80	2.8301E-4	-1.9584	1.1222E-4	-1.9522	7.2772	6.7445

Table 2 A priori error and convergence rates for Example 2 at $t = 1$

Partition	E_{u,l^2}	Rate	E_{p,l^2}	Rate	$\frac{h_{max}}{h_{min}}$	$\frac{k_{max}}{k_{min}}$
10×10	1.7040E-2	–	1.1541E-2	–	1.5662	2.1596
20×20	4.8135E-3	-1.8238	3.0429E-3	-1.9233	1.7328	2.3355
40×40	1.2523E-3	-1.9425	8.0992E-4	-1.9096	2.4673	4.8795
80×80	3.3451E-4	-1.9045	2.1579E-4	-1.9082	3.0891	5.2573

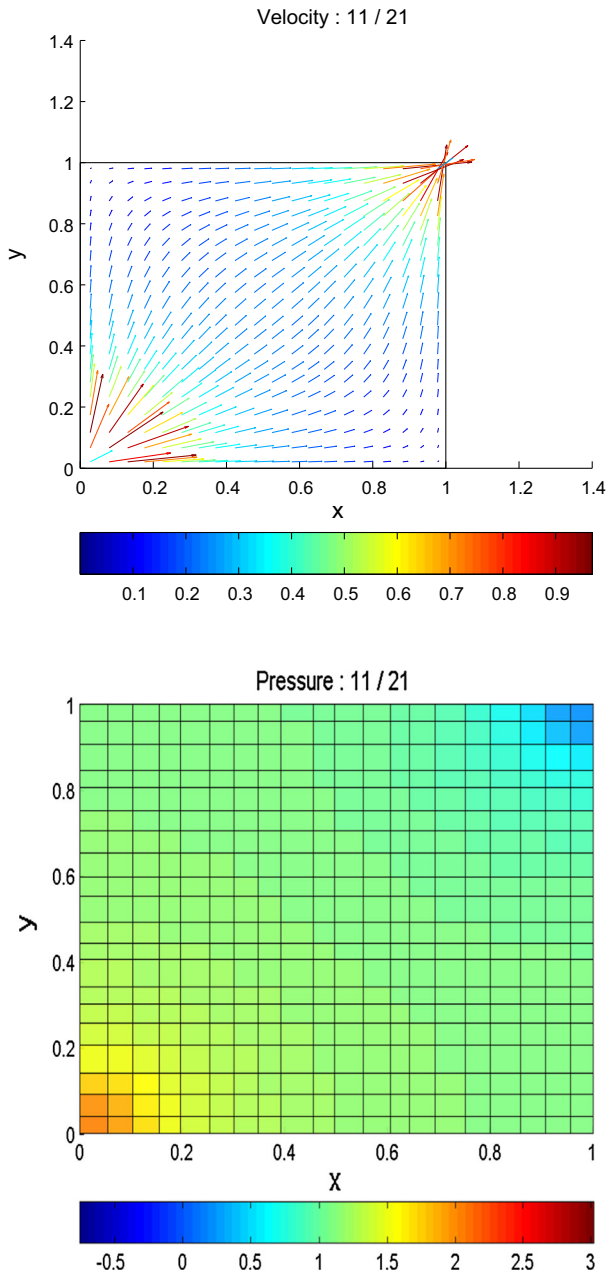


Fig. 5 Velocity quiver and pressure distribution for Example 3, $t = 0.5$

Example 2 An example with homogeneous Neumann boundary condition is as below. The flux on the boundary condition, $\mathbf{u} \cdot \mathbf{n}$, is computed according to the analytic solution given as below.

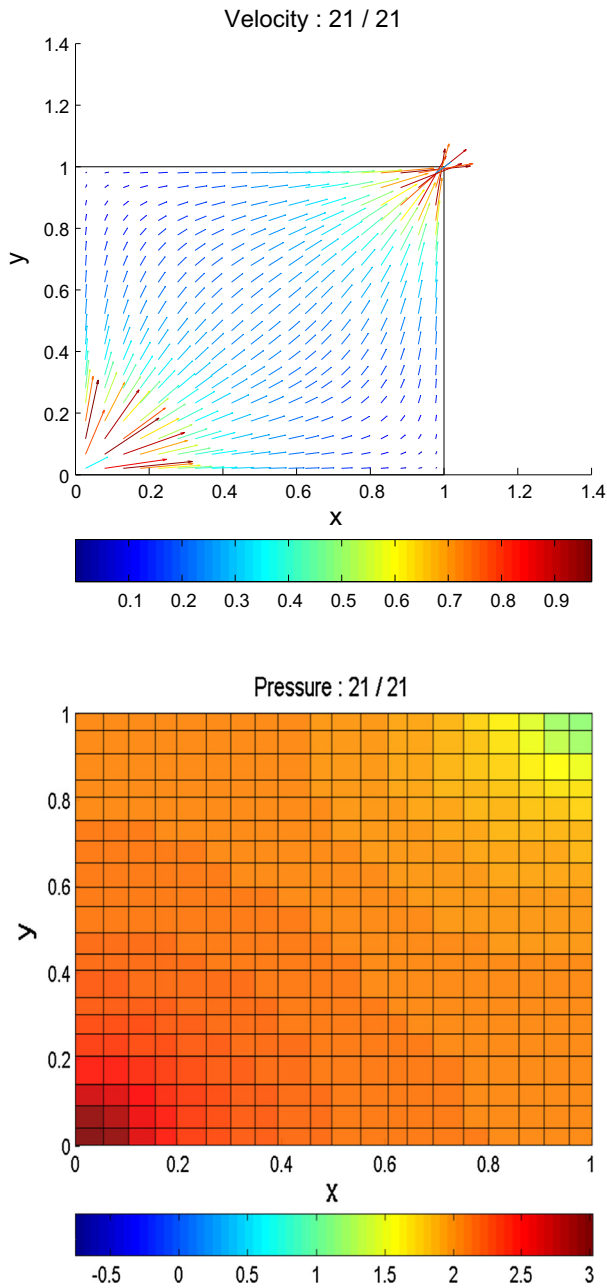


Fig. 6 Velocity quiver and pressure distribution for Example 3, $t = 1.0$

$$\begin{cases} p(x, y, t) = \arctan(x + y - t - 1), \\ \mathbf{u}(x, y, t) = (e^{-t}y \sin \pi x, e^{-t}x \sin \pi y)^T. \end{cases}$$

The numerical results are listed in Fig. 4 and Table 2.

Example 3 An example with homogeneous Neumann boundary condition, point-source and point-sink flux is simulated. The right hand side equals zero except at the injection and production wells, where the injection flow rate, q_i , and production flow rate, q_p , at wells are,

$$q_i(x, y, t) = \delta(0, 0), \quad q_p(x, y) = -\delta(1, 1).$$

The numerical results are listed in Figs. 5 and 6.

From Figs. 3 and 4 and Tables 1 and 2, we can see that the block-centered finite difference approximations for pressure and velocity have the second order accuracy in discrete L^2 -norms. These results are in consistent with the error estimates in Theorem 1. Figures 5 and 6 show that the pressure approximation and velocity approximation are reasonable for the point-source and point-sink problem.

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References

1. Aziz, K., Settari, A.: Petroleum Reservoir Simulation. Applied Science Publishers LTD, London (1979)
2. Neuman, S.P.: Theoretical derivation of Darcy's law. *Acta Mech.* **25**(3), 153–170 (1977)
3. Whitaker, S.: Flow in porous media I: a theoretical derivation of Darcy's law. *Transp. Porous Media* **1**(1), 3–25 (1986)
4. Ruth, D., Ma, H.: On the derivation of the Forchheimer equation by means of the averaging theorem. *Transp. Porous Media* **7**(3), 255–264 (1992)
5. Aulisa, E., Bloshanskaya, L., Hoang, L., Lbragimov, A.: Analysis of generalized forchheimer flows of compressible fluids in porous media. *J. Math. Phys.* **50**(10), 103102 (2009)
6. Aulisa, E., Lbragimov, A., Volko, P.P., Walton, J.R.: Mathematical framework of the well productivity index for fast Forchheimer (non-Darcy) flow in porous media. *Math. Model. Methods Appl. Sci.* **19**(9), 1241–1275 (2009)
7. Fabrie, P.: Regularity of the solution of Darcy–Forchheimer's equation. *Nonlinear Anal. Theory Methods Appl.* **13**(9), 1025–1049 (1989)
8. Douglas, J.J., Paes-Leme, P.J., Giorgi, T.: Generalized Forchheimer flow in porous media. In: *Boundary Value Problems for Partial Differential Equations and Applications*. RMA Research Notes in Research Notes in Applied Mathematics, 29, Masson, Paris, pp. 99–111 (1993)
9. Park, E.J.: Mixed finite element method for nonlinear second order elliptic problems. *SIAM J. Numer. Anal.* **32**, 865–885 (1995)
10. Roberts, J.E., Thomas, J.M.: Mixed and Hybrid Methods. In: *Finite Element Methods (Part 1)*, Handbook of Numerical Analysis, vol. 2, pp. 523–639, Elsevier Science Publishers B.V., North-Holland, Amsterdam (1991)
11. Girault, V., Wheeler, M.F.: Numerical discretization of a Darcy–Forchheimer model. *Numer. Math.* **110**(2), 161–198 (2008)
12. Lopez, H., Molina, B., Jose, J.S.: Comparison between different numerical discretizations for a Darcy–Forchheimer model. *Electron. Trans. Numer. Anal.* **34**, 187–203 (2009)
13. Pan, H., Rui, H.: mixed element method for two-dimensional Darcy–Forchheimer model. *J. Sci. Comput.* **52**, 563–587 (2012)
14. Park, E.J.: Mixed finite element method for generalized Forchheimer flow in porous media. *Numer. Methods Part. Differ. Equ.* **21**, 213–228 (2005)
15. Ewing, R.E., Lazarov, R.D., Lyons, S.L., Papavassiliou, D.V., Pasciak, J., Qin, G.: Numerical well model for non-Darcy flow through isotropic porous media. *Comput. Geosci.* **3**(3–4), 184–204 (1999)
16. Ibragimov, A., Kieu, T.: An expanded mixed finite element method for generalized Forchheimer flows in porous media. *Comput. Math. Appl.* **72**, 1467–1483 (2016)

17. Kieu, T.: Numerical analysis for generalized Forchheimer flows of slightly compressible fluids in porous media. (2015). [arXiv:1508.00294](https://arxiv.org/abs/1508.00294)
18. Kieu, T.: Analysis of expanded mixed finite element methods for the generalized Forchheimer equations. *Numer. Methods Part. Differ. Equ.* **32**, 60–85 (2016)
19. Celik, E., Hoang, L., Kieu, T.: Doubly nonlinear parabolic equations for a general class of Forchheimer gas flows in porous media. (2016). [arXiv:1601.00703](https://arxiv.org/abs/1601.00703)
20. Hoang, L., Kieu, T.: Global estimates for generalized Forchheimer flows of slightly compressible fluids. (2015). [arXiv:1502.04732](https://arxiv.org/abs/1502.04732)
21. Hoang, L., Kieu, T.: Interior estimates for generalized Forchheimer flows of slightly compressible fluids. (2015). [arXiv:1404.6517](https://arxiv.org/abs/1404.6517)
22. Weiser, A., Wheeler, M.F.: On convergence of block-centered finite difference for elliptic problems. *SIAM J. Numer. Anal.* **25**, 351–375 (1988)
23. Arbogast, T., Dawson, C.N., Keenan, P.T., Wheeler, M.F., Yotov, I.: Enhanced cell-centered finite differences for elliptic equations on general geometry. *SIAM J. Sci. Comput.* **19**, 404–425 (1998)
24. Arbogast, T., Wheeler, M.F., Yotov, I.: Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences. *SIAM J. Numer. Anal.* **34**, 828–852 (1997)
25. Wheeler, M.F., Xue, G., Yotov, I.: A multipoint flux mixed finite element method on distorted quadrilaterals and hexahedra. *Numer. Math.* **121**, 165–204 (2012)
26. Wheeler, M.F., Yotov, I.: A multipoint flux mixed finite element method. *SIAM J. Numer. Anal.* **44**, 2082–20106 (2006)
27. Rui, H., Pan, H.: A block-centered finite difference method for the Darcy–Forchheimer model. *SIAM J. Numer. Anal.* **50**, 2612–2651 (2012)
28. Rui, H., Zhao, D., Pan, H.: Block-centered finite difference methods for Darcy–Forchheimer model with variable Forchheimer number. *Numer. Methods Part. Differ. Equ.* **31**, 1603–1622 (2015)
29. Rui, H., Liu, W.: A two-grid block-centered finite difference methods for Darcy–Forchheimer flow in porous media. *SIAM J. Numer. Anal.* **53**, 1941–1962 (2015)
30. Achdou, Y., Guermond, J.L.: Convergence analysis of a finite element projection/Lagrange–Galerkin methods for the incompressible Navier–Stokes equations. *SIAM J. Numer. Anal.* **37**, 799–826 (2000)
31. He, Y.: The Euler implicit/explicit scheme for the 2D time-dependent Navier–Stokes equations with smooth and non-smooth initial data. *Math. Comput.* **77**, 2097–2124 (2008)
32. Elliot, C.C., Larsson, S.: A finite element model for the time-dependent Joule heating problem. *Math. Comput.* **64**, 1433–1453 (1995)
33. Sun, W., Sun, Z.: Finite difference methods for a nonlinear and strongly coupled heat and moisture transportsystem in textile materials. *Numer. Math.* **120**, 153–187 (2012)
34. Wu, H., Ma, H., Li, H.: Optimal order error estimates for the Chebyshev–Legendre spectral method for solving the generalized Burgers equation. *SIAM J. Numer. Anal.* **41**, 659–672 (2003)
35. Li, B., Sun, W.: Error analysis of linearized semi-implicit Galerkin finite element methods for nonlinear Parabolic equations. *Inter. J. Numer. Anal. Model.* **10**, 622–633 (2013)
36. Li, B., Sun, W.: Unconditionally convergence and optimal error analysis for a Galerkin-mixed FEM for incompressible miscible flow in porous media. *SIAM Numer. Anal.* **51**, 1959–1977 (2013)
37. Bear, J.: *Dynamics of Fluids in Porous Media*. Elsevier, New York (1972)
38. Douglas Jr., J., Roberts, J.E.: Numerical methods for a model for compressible miscible displacement in porous media. *Math. Comput.* **41**(164), 441–459 (1983)
39. Aulisa, E., Bloshanskaya, L., Hoang, L., Ibragimov, A.: Analysis of generalized forchheimer flows of compressible fluids in porous media. Technical report. Institute for Mathematics and its Applications (2009)
40. Hoang, L., Ibragimov, A.: Structural stability of generalized Forchheimer equations for comopressible fluids in porous media. Technical report. Institute for Mathematics and its Applications (2010)
41. Dibenedetto, E.: *Degenerate Parabolic Equations*. Springer, Berlin (1993)
42. Dibenedetto, E., Gianazza, U., Vespri, V.: Forward, backward and elliptic harnack inequalities for non-negative solutions to certain singular parabolic partial differential equations. *Annali Della Scuola Normale Superiore Di Pisa Classe Di Scienze* **9**(2), 385–422 (2010)
43. Hoang, L., Ibragimov, A., Kieu, T., Sobol, Z.: Stability of solutions to generalized Forchheimer equations of any degree. *J. Math. Sci.* **210**(4), 1–69 (2015)
44. Ladyzhenskaya, O.A., Solonnikov, V.A., Uraltseva, N.N.: *Linear and quasilinear equations of parabolic type*. Translated from the Russian. *Translations of Mathematical Monographs*, vol. 23. American Mathematical Society (1968)
45. Luan, H., Ibragimov, A.: Qualitative study of generalized forchheimer flows with the flux boundary condition. *Adv. Differ. Equ.* **17**(17), 511–556 (2012)

46. Showalter, R.E.: *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. American Mathematical Society, Providence (2013)
47. Rui, H., Pan, H.: Block-centered finite difference methods for parabolic equation with time-dependent coefficient. *Jpn. J. Indus. Appl. Math.* **30**, 681–699 (2013)