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Unconditional Superconvergence Analysis of a Crank–Nicolson Galerkin FEM for Nonlinear Schrödinger Equation

Dongyang Shi[1] · **Junjun Wang**[1]

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Abstract A linearized Crank–Nicolson Galerkin finite element method with bilinear element for nonlinear Schrödinger equation is studied. By splitting the error into two parts which are called the temporal error and the spatial error, the unconditional superconvergence result is deduced. On one hand, the regularity for a time-discrete system is presented based on the proof of the temporal error. On the other hand, the classical Ritz projection is applied to get the spatial error with order $O(h^2)$ in L^2-norm, which plays an important role in getting rid of the restriction of τ. Then the superclose estimates of order $O(h^2+\tau^2)$ in H^1-norm is arrived at based on the relationship between the Ritz projection and the interpolated operator. At the same time, global superconvergence property is arrived at by the interpolated postprocessing technique. At last, three numerical examples are provided to confirm the theoretical analysis. Here, h is the subdivision parameter and τ is the time step.

Keywords Unconditional superconvergence results · NLSE · Linearized C–N Galerkin FEM · Temporal and spatial errors · Ritz projection and interpolated operators

Mathematics Subject Classification 65N15 · 65N30

1 Introduction

Consider the following nonlinear Schrödinger equation:

$$\begin{cases} iu_t + \Delta u + f(|u|^2)u = 0, & (X,t) \in \Omega \times (0,T], \\ u = 0, & (X,t) \in \partial\Omega \times (0,T], \\ u(X,0) = u_0(X), & X \in \Omega, \end{cases} \tag{1.1}$$

✉ Dongyang Shi
shi_dy@zzu.edu.cn

1 School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China

where $X = (x, y)$, $0 < T < \infty$, and $\Omega \subset \mathbb{R}^2$ is a rectangle with the boundary $\partial\Omega$. i is the imaginary unit, $u_0(X)$ is a known complex-valued function. Moreover, $f(s)$ is a real-valued nonlinear function which is twicely continuously differentiable with respective to s.

The NLSE plays an important role in describing physical phenomena, such as optical pulses, plasma physics and water waves and so on. Different numerical methods for the NLSE have been investigated extensively. For example, [1] discussed an iterative modification of the linearized scheme and proved second-order error estimates by use of Newtons method to linearize the equations at each time level. Continuous Galerkin methods were employed in [2] and optimal order error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$, and the corresponding superconvergence results at the temporal nodes t^n were obtained. [3] and [4] studied the normal Galerkin method and introduced the semi-discrete scheme and fully-discrete schemes for NLSE, respectively and both derived the superclose and superconvergence results in H^1-norm. A meshless local boundary integral equation method and two-grid mixed finite element method were proposed to solve the unsteady Schrödinger equation in [5] and [6], respectively. [7] and [8] researched the discontinuous Galerkin method and get optimal order error estimates. Finite difference method were also considered extensively in [9–12].

In fact, studying a nonlinear physical system often involves the boundedness of U_h^n in L^∞-norm or a stronger norm, where U_h^n is the numerical solution. The usual technique is employing the inverse inequality to deal with such issue, which will result in some time-step restrictions, such as $\tau = o(h^{\frac{1}{4}})$ and $\tau = O(h^2)/\tau = O(h)$ in [1] and [3], respectively. Moreover, such restrictions also arise in the studies on other nonlinear evolution equations, such as nonlinear hyperbolic equations [13,14], nonlinear parabolic equation [15–18], nonlinear Sobolev problems [19,20], Navier–Stokes equations [21,22], and so on. Therefore, how to get rid of such restriction becomes a hot topic and for this issue, a lot of efforts have been devoted. For instance, a corresponding time-discrete system was introduced in [23] to split the error into two parts, the temporal error and the spatial error, and the spatial error was reduced to the unconditional boundedness of numerical solution in L^∞-norm. Then the optimal L^2 error estimate without any time-step restrictions for the NLSE was obtained. Subsequently, this so-called splitting technique was also applied to other equations [24–30]. Especially, [31] used different technique from the above studies to get the unconditional superclose for Sobolev equation with conforming mixed FEM.

Different from [3] and [23], we discuss the unconditional superconvergence estimate for (1.1) with bilinear element[32]. A time-discrete system with solution U^n is developed to split the error $u^n - U_h^n$ into the temporal error $u^n - U^n$ and the spatial error $U^n - U_h^n$. On one hand, we obtain the temporal error $\|u^n - U^n\|_2 = O(\tau^2)$, which is one order higher than that of [23]. Then the boundedness of $\tilde{\partial}_{tt} U^n$, which plays an important role in the analysis of the spatial error, is arrived at. As it is shown in our paper, H^2 error estimate of the temporal error is important for getting rid of the restriction of τ. In the existing literature, there have also been other related works of H^2 error estimate for certain nonlinear PDEs, such as [32,33]. On the other hand, we introduce the classical Ritz prjection operator R_h to get the unconditional result of $\|R_h U^n - U_h^n\|_0$ with order $O(h^2)$, which implies the unconditional boundedness of $\|U_h^n\|_{0,\infty}$. Consequently, the superclose property of $\|R_h U^n - U_h^n\|_1$ with order $O(h^2 + \tau^2)$ is deduced on the basis of the above achievements. Furthermore, through the relationship between R_h and the corresponding interpolation operator I_h, we get $\|I_h u^n - U_h^n\|_1$ with order $O(h^2 + \tau^2)$ unconditionally. At the same time, we derive the global superconvergence by using the postprocessing operator in [31]. At last, some numerical results also show the validity of the theoretical analysis.

Throughout this paper, we denote the natural inner product in $L^2(\Omega)$ by $(\cdot, \cdot)$ and the norm by $\|\cdot\|_0$, and let $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. Further, we use the classical

Sobolev spaces $W^{m,p}(\Omega)$, $1 \leq p \leq \infty$, denoted by $W^{m,p}$, with norm $\|\cdot\|_{m,p}$. When $p = 2$, we simply write $\|\cdot\|_{m,p}$ as $\|\cdot\|_m$. Besides, we define the space $L^p(a, b; Y)$ with the norm $\|f\|_{L^p(a,b;Y)} = (\int_a^b \|f(\cdot, t)\|_Y^p dt)^{\frac{1}{p}}$, and if $p = \infty$, the integral is replaced by the essential supremum.

2 A Linearized Galerkin Approximation Scheme

Let Ω be a rectangle in (x, y) plane with edges parallel to the coordinate axes, Γ_h be a quasiuniform partition of Ω into rectangular π_h. Denote $h = \max\limits_{\pi_h \in \Gamma_h} \mathrm{diam}\pi_h$ the mesh size, V_h be the usual bilinear FE space, $V_{h0} = \{v_h \in V_h, v_h|_{\partial\Omega} = 0\}$. Let $R_h : H_0^1 \to V_{h0}$ be the associated Ritz projection operator on V_{h0} defined by

$$(\nabla(u - R_h u), \nabla v_h) = 0, \forall v_h \in V_{h0}. \tag{2.1}$$

It follows from [17] that

$$\|\nabla R_h u\|_0 \leq C\|\nabla u\|_0, \tag{2.2}$$

and

$$\|u - R_h u\|_0 \leq Ch^s\|u\|_s, s = 1, 2, \forall u \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2.3}$$

Moreover, for $u \in H^3(\Omega)$, we can found in [34] that

$$\|I_h u - R_h u\|_1 = O(h^2)\|u\|_3, \tag{2.4}$$

where I_h be the associated interpolated operator over V_{h0}.

Let $\{t_n : t_n = n\tau; 0 \leq n \leq N\}$ be a uniform partition of $[0, T]$ with the time step $\tau = T/N$, $t_{n-\frac{1}{2}} = \frac{1}{2}(t_n + t_{n-1})$ and $\sigma^n = \sigma(X, t_n)$. For a sequence of functions $\{\sigma^n\}_{n=0}^N$, we remark

$$\tilde{\partial}_t\sigma^n = \frac{\sigma^n - \sigma^{n-1}}{\tau}, \tilde{\partial}_{tt}\sigma^n = \frac{\tilde{\partial}_t\sigma^n - \tilde{\partial}_t\sigma^{n-1}}{\tau}, \tilde{\sigma}^n = \frac{\sigma^n + \sigma^{n-1}}{2}, \quad n = 1, 2, \ldots, N,$$
$$\hat{\sigma}^n = \frac{3}{2}\sigma^{n-1} - \frac{1}{2}\sigma^{n-2}, \quad n = 2, \ldots, N.$$

With these notations, we develop the linearized Galerkin FEM to problem (1.1): seek $U_h^n \in V_{h0}$, such that for $n \geq 2$,

$$i(\tilde{\partial}_t U_h^n, v_h) - (\nabla\tilde{U}_h^n, \nabla v_h) + (f(|\hat{U}_h^n|^2)\tilde{U}_h^n, v_h) = 0, \quad \forall v_h \in V_{h0}, \tag{2.5}$$

and we will analyze a predictor corrector method to determine U_h^1:

$$i\left(\frac{U_h^{1,0} - U_h^0}{\tau}, v_h\right) - \left(\frac{\nabla U_h^{1,0} + \nabla U_h^0}{2}, \nabla v_h\right) = \left(f(|U_h^0|^2)\frac{U_h^{1,0} + U_h^0}{2}, v_h\right), \tag{2.6}$$

followed by

$$i\left(\frac{U_h^1 - U_h^0}{\tau}, v_h\right) - \left(\frac{\nabla U_h^1 + \nabla U_h^0}{2}, \nabla v_h\right) = \left(f\left(\left|\frac{U_h^{1,0} + U_h^0}{2}\right|^2\right)\frac{U_h^1 + U_h^0}{2}, v_h\right), \tag{2.7}$$

where $U_h^0 = R_h u_0$. Obviously, only a linear system with certain constant coefficients need to be solved now.

3 Error Estimates for Time-Discrete System

In this section, we introduce the following time-discrete system:

$$\begin{cases} i\tilde{\partial}_t U^n + \Delta\tilde{U}^n + f(|\hat{U}^n|^2)\tilde{U}^n = 0, & (X,t)\in\Omega\times(0,T], n\geq 2 \\ U^n = 0, & (X,t)\in\ \partial\Omega\times(0,T], n\geq 1 \\ U(X,0) = u_0(X), & X\in\Omega, \end{cases} \tag{3.1}$$

When $n = 1$, we determine U^1 by

$$i\frac{U^{1,0}-U^0}{\tau} + \frac{\Delta U^{1,0}+\Delta U^0}{2} + f(|U^0|^2)\frac{U^{1,0}+U^0}{2} = 0 \tag{3.2}$$

and

$$i\frac{U^1-U^0}{\tau} + \frac{\Delta U^1+\Delta U^0}{2} + f\left(\left|\frac{U^{1,0}+U^0}{2}\right|^2\right)\frac{U^1+U^0}{2} = 0, \tag{3.3}$$

where $U^{1,0}|_{\partial\Omega} = 0$. The above system can be viewed as a system of linear elliptic equations, and the existence and uniqueness of solution can be proved immediately. In what follows, we will set $e^n = u^n - U^n (n = 0, 1, 2, \ldots N)$, analyze $\|u^n - U^n\|_i (i = 0, 1, 2)$ and give the regularity result of U^n.

Theorem 1 *Let u and $U^m (m = 0, 1, 2, \ldots N)$ be the solutions of* (1.1) *and* (3.1)–(3.3), *respectively, $u \in L^2(0,T;H^3(\Omega))$, $u_t \in L^\infty(0,T;H^2(\Omega))$, $u_{tt} \in L^\infty(0,T;H^2(\Omega))$, then for $m = 1,\ldots,N$, there exists τ_0 such that when $\tau \leq \tau_0$, we have*

$$\|\tilde{\partial}_t e^m\|_0 + \|e^m\|_2 \leq C_0\tau^2 \tag{3.4}$$

and

$$\left\|\frac{U^{1,0}-U^0}{\tau}\right\|_2 + \|\tilde{\partial}_{tt}U^m\|_2 \leq C_0. \tag{3.5}$$

Proof Setting $K_0 \triangleq 1 + \max_{1\leq m\leq N}(\|u^m\|_{0,\infty} + \|\tilde{\partial}_t u^m\|_{0,\infty})$. Then we begin to prove (3.4) and (3.5) by mathematical induction. When $m = 1$, we have the error equations by (1.1) and (3.2)–(3.3) as follows:

$$i\frac{e^{1,0}}{\tau} + \frac{\Delta e^{1,0}}{2} + f\left(\left|u^0\right|^2\right)\frac{e^{1,0}}{2} = S_1 + S_2 + S_3 \tag{3.6}$$

and

$$i\frac{e^1}{\tau} + \frac{\Delta e^1}{2} + P_1^1 = S_1 + S_2 + S_4, \tag{3.7}$$

where $S_1 = \frac{u^1-u^0}{\tau} - u_t^{\frac{1}{2}}$, $S_2 = \frac{\Delta u^1+\Delta u^0}{2} - \Delta u^{\frac{1}{2}}$, $S_3 = f(|u^0|^2)\frac{u^1+u^0}{2} - f(|u^{\frac{1}{2}}|^2)u^{\frac{1}{2}}$, $S_4 = f(|\frac{u^1+u^0}{2}|^2)\frac{u^1+u^0}{2} - f(|u^{\frac{1}{2}}|^2)u^{\frac{1}{2}}$ and $P_1^1 = f(|\frac{u^1+u^0}{2}|^2)\frac{u^1+u^0}{2} - f(|\frac{U^{1,0}+U^0}{2}|^2)\frac{U^1+U^0}{2}$. It is easy to see that $\|S_1\|_0 + \|S_2\|_0 + \|S_4\|_0 \leq C\tau^2$, $\|S_3\|_0 \leq C\tau$.

On one hand, multiplying (3.6) by $\frac{e^{1,0}}{\tau}$, integrating it over Ω and then we get

$$i\left\|\frac{e^{1,0}}{\tau}\right\|_0^2 - \frac{1}{2\tau}\|\nabla e^{1,0}\|_0^2 = -\left(f\left(\left|u^0\right|^2\right)\frac{e^{1,0}}{2}, \frac{e^{1,0}}{\tau}\right) + \left(S_1+S_2+S_3, \frac{e^{1,0}}{\tau}\right). \tag{3.8}$$

Taking the imaginary part of (3.8), it is easy to get

$$\left\| \frac{e^{1,0}}{\tau} \right\|_0^2 \le C\tau^2 + \frac{1}{2} \left\| \frac{e^{1,0}}{\tau} \right\|_0^2 + C \left\| e^{1,0} \right\|_0^2. \tag{3.9}$$

Then there exist τ_1, C_1, such that when $\tau \le \tau_1$, we have

$$\|e^{1,0}\|_0 \le C_1 \tau^2. \tag{3.10}$$

Again, multiplying (3.6) by $\frac{\Delta e^{1,0}}{\tau}$ and integrating it over Ω to yield

$$-i \left\| \frac{\nabla e^{1,0}}{\tau} \right\|_0^2 + \frac{1}{2\tau} \left\| \Delta e^{1,0} \right\|_0^2 = -\left(f\left(|u^0|^2\right) \frac{e^{1,0}}{2}, \frac{\Delta e^{1,0}}{\tau} \right) + \left(S_1 + S_2 + S_3, \frac{\Delta e^{1,0}}{\tau} \right). \tag{3.11}$$

Noting

$$\left| \left(f\left(|u^0|^2\right) \frac{e^{1,0}}{2}, \frac{\Delta e^{1,0}}{\tau} \right) \right| \le \frac{1}{8} \left\| \frac{\nabla e^{1,0}}{\tau} \right\|_0^2 + C \left\| \Delta e^{1,0} \right\|_0^2$$

and

$$\left| \left(S_1 + S_2 + S_3, \frac{\Delta e^{1,0}}{\tau} \right) \right| \le C\tau + \frac{1}{8\tau} \left\| \Delta e^{1,0} \right\|_0^2.$$

Then by taking the imaginary part and the real part of (3.11), and summing them together, we have

$$\left\| \frac{\nabla e^{1,0}}{\tau} \right\|_0^2 + \frac{1}{2\tau} \left\| \Delta e^{1,0} \right\|_0^2 \le C\tau + \frac{1}{4\tau} \left\| \Delta e^{1,0} \right\|_0^2 + \frac{1}{4} \left\| \frac{\nabla e^{1,0}}{\tau} \right\|_0^2 + C\|\Delta e^{1,0}\|_0^2. \tag{3.12}$$

Since $e^{1,0} \in H^2(\Omega) \cap H_0^1(\Omega)$, there exist τ_2, C_2, C_3, such that when $\tau \le \tau_2$, we have

$$\sqrt{\tau} \left\| \frac{e^{1,0}}{\tau} \right\|_1 + \left\| e^{1,0} \right\|_2 \le C_2 \tau, \tag{3.13}$$

which implies

$$\left\| \frac{U^{1,0} - U^0}{\tau} \right\|_2 \le C_3 \tag{3.14}$$

and

$$\|U^{1,0}\|_{0,\infty} \le \|e^{1,0}\|_{0,\infty} + \|u^1\|_{0,\infty} \le C\|e^{1,0}\|_2 + \|u^1\|_{0,\infty} \le CC_2\tau + \|u^1\|_{0,\infty} \le K_0, \tag{3.15}$$

where $\tau \le \tau_3 \le 1/CC_2$.

On the other hand, multiplying (3.7) by $\frac{e^1}{\tau}$, integrating it over Ω and then we get

$$i \left\| \frac{e^1}{\tau} \right\|_0^2 - \frac{1}{2\tau} \|\nabla e^1\|_0^2 = -\left(P_1^1, \frac{e^1}{\tau} \right) + \left(S_1 + S_2 + S_4, \frac{e^1}{\tau} \right). \tag{3.16}$$

By the help of (3.10) and (3.15), we get

$$\|P_1^1\|_0 = \left\| f\left(\left|\frac{U^{1,0}+U^0}{2}\right|^2\right)\frac{e^1}{2} + \frac{u^1+u^0}{2}\left(f\left(\left|\frac{u^1+u^0}{2}\right|^2\right) - f\left(\left|\frac{U^{1,0}+U^0}{2}\right|^2\right)\right)\right\|_0$$
$$\leq C\|e^1\|_0 + C\|e^{1,0}\|_0 \leq C\tau^2 + C\|e^1\|_0. \tag{3.17}$$

Taking the imaginary part of (3.16), it is obvious to see that there exist τ_4, C_4, such that $\tau \leq \tau_4$, it follows that

$$\left\|\frac{e^1}{\tau}\right\|_0 \leq C_4\tau^2. \tag{3.18}$$

Once more, multiplying (3.7) by $\frac{\Delta e^1}{\tau}$, integrating it over Ω and then we get

$$-i\left\|\frac{\nabla e^1}{\tau}\right\|_0^2 + \frac{1}{2\tau}\|\Delta e^1\|_0^2 = -\left(P_1^1, \frac{\Delta e^1}{\tau}\right) + \left(S_1+S_2+S_4, \frac{\Delta e^1}{\tau}\right). \tag{3.19}$$

Similarly to the estimates of $e^{1,0}$, we get

$$\left|\left(P_1^1, \frac{\Delta e^1}{\tau}\right)\right| = \left|\left(f\left(\left|\frac{U^{1,0}+U^0}{2}\right|^2\right)\frac{e^1}{2}, \frac{\Delta e^1}{\tau}\right) + \left(\frac{u^1+u^0}{2}\left(f\left(\left|\frac{u^1+u^0}{2}\right|^2\right)\right.\right.\right.$$
$$\left.\left.\left. - f\left(\left|\frac{U^{1,0}+U^0}{2}\right|^2\right)\right), \frac{\Delta e^1}{\tau}\right)\right|$$
$$\leq C\tau^3 + \frac{1}{8}\left\|\frac{\nabla e^1}{\tau}\right\|_0^2 + \frac{1}{16\tau}\|\Delta e^1\|_0^2 + C\|\Delta e^1\|_0^2$$

and

$$\left|\left(S_1+S_2+S_4, \frac{\Delta e^1}{\tau}\right)\right| \leq C\tau^3 + \frac{1}{16\tau}\|\Delta e^1\|_0^2,$$

which implies

$$\left\|\frac{\nabla e^1}{\tau}\right\|_0^2 + \frac{1}{2\tau}\|\Delta e^1\|_0^2 \leq C\tau^3 + \frac{1}{4}\left\|\frac{\nabla e^1}{\tau}\right\|_0^2 + \frac{1}{4\tau}\|\Delta e^1\|_0^2 + C\|\Delta e^1\|_0^2. \tag{3.20}$$

It is apparent to see that there exist τ_5, C_5, C_6, such that when $\tau \leq \tau_5$, we have

$$\|e^1\|_2 \leq C_5\tau^2, \tag{3.21}$$

which leads to

$$\|\tilde{\partial}_{tt}U^1\|_2 \leq C_6, \tag{3.22}$$

and

$$\|\tilde{\partial}_t U^1\|_{0,\infty} + \|U^1\|_{0,\infty} \leq \|\tilde{\partial}_t e^1\|_{0,\infty} + \|\tilde{\partial}_t u^1\|_{0,\infty} + \|e^1\|_{0,\infty} + \|u^1\|_{0,\infty}$$
$$\leq CC_5\tau + \|\tilde{\partial}_t u^1\|_{0,\infty} + \|u^1\|_{0,\infty} \leq K_0, \tag{3.23}$$

where $\tau \leq \tau_6 = 1/CC_5$.

By mathematical induction, we assume that (3.4) and (3.5) hold for $m \le n-1$. Then

$$\|\tilde{\partial}_t U^m\|_{0,\infty} + \|U^m\|_{0,\infty} \le \|\tilde{\partial}_t e^m\|_{0,\infty} + \|\tilde{\partial}_t u^m\|_{0,\infty} + \|e^m\|_{0,\infty} + \|u^m\|_{0,\infty}$$
$$\le CC_0\tau + \|\tilde{\partial}_t u^m\|_{0,\infty} + \|u^m\|_{0,\infty} \le K_0,$$

where $\tau \le \tau_7 = 1/CC_0$.

Now we prove (3.4) and (3.5) also hold for $m = n$. To estimate e^n, we subtract (3.1) from (1.1) to obtain

$$i\tilde{\partial}_t e^n + \Delta\tilde{e}^n + P_1^n = R_1^n + R_2^n + R_3^n, \tag{3.24}$$

where $R_1^n = i(\tilde{\partial}_t u^n - u_t^{n-\frac{1}{2}})$, $R_2^n = \Delta\tilde{u}^n - \Delta u^{n-\frac{1}{2}}$, $R_3^n = f(|\hat{u}^n|^2)\tilde{u}^n - f(|u^{n-\frac{1}{2}}|^2)u^{n-\frac{1}{2}}$ and $P_1^n = f(|\hat{u}^n|^2)\tilde{u}^n - f(|\hat{U}^n|^2)\tilde{U}^n$. By Taylor's expansion, we have

$$\|R_1^n\|_0 + \|R_2^n\|_0 + \|R_3^n\|_0 = O(\tau^2). \tag{3.25}$$

We multiply (3.24) by $\tilde{\partial}_t \Delta e^n$ and integrate it over Ω to get

$$-i\|\tilde{\partial}_t \nabla e^n\|_0^2 + (\Delta\tilde{e}^n, \tilde{\partial}_t\Delta e^n) = -(P_1^n, \tilde{\partial}_t\Delta e^n) + (R_1^n + R_2^n + R_3^n, \tilde{\partial}_t\Delta e^n). \tag{3.26}$$

Taking the real part, the left hand can be rewritten as

$$Re(\Delta\tilde{e}^n, \tilde{\partial}_t\Delta e^n) = \frac{1}{2\tau}(\|\Delta e^n\|_0^2 - \|\Delta e^{n-1}\|_0^2). \tag{3.27}$$

As to the right hand of (3.26), we need to transfer τ from one part of the inner product to the other, for there is no term concerning with $\tilde{\partial}_t\Delta e^n$ on the left hand. Define $\hat{u}^1 = \tilde{u}^1$, $\hat{U}^1 = \tilde{U}^1$ and $\hat{e}^1 = \tilde{e}^1$, rewrite $(P_1^n, \tilde{\partial}_t\Delta e^n)$ by

$$(P_1^n, \tilde{\partial}_t\Delta e^n) = -(\tilde{\partial}_t P_1^n, \Delta e^{n-1}) + \tilde{\partial}_t(P_1^n, \Delta e^n). \tag{3.28}$$

Indeed, by the assumption of the mathematical induction, we have

$$\begin{aligned}
\|\tilde{\partial}_t P_1^n\|_0 &= \left\| \frac{(f(|\hat{u}^n|^2)\tilde{u}^n - f(|\hat{U}^n|^2)\tilde{U}^n) - (f(|\hat{u}^{n-1}|^2)\tilde{u}^{n-1} - f(|\hat{U}^{n-1}|^2)\tilde{U}^{n-1})}{\tau} \right\|_0 \\
&= \Bigg\| f(|\hat{U}^{n-1}|^2)\tilde{\partial}_t\tilde{e}^n + \tilde{\partial}_t\tilde{u}^n(f(|\hat{u}^{n-1}|^2) - f(|\hat{U}^{n-1}|^2)) \\
&\quad + \frac{(f(|\hat{U}^n|^2) - f(|\hat{U}^{n-1}|^2))\tilde{e}^n}{\tau} \\
&\quad + \frac{\tilde{u}^n((f(|\hat{u}^n|^2) - f(|\hat{u}^{n-1}|^2)) - (f(|\hat{U}^n|^2) - f(|\hat{U}^{n-1}|^2)))}{\tau} \Bigg\|_0 \\
&\le C\|\tilde{\partial}_t\tilde{e}^n\|_0 + C\|\hat{e}^{n-1}\|_0 + C\|\tilde{e}^n\|_0 \\
&\quad + C\left\| \frac{(f(|\hat{u}^n|^2) - f(|\hat{u}^{n-1}|^2)) - (f(|\hat{U}^n|^2) - f(|\hat{U}^{n-1}|^2))}{\tau} \right\|_0 .
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{(f(|\hat{u}^n|^2) - f(|\hat{u}^{n-1}|^2)) - (f(|\hat{U}^n|^2) - f(|\hat{U}^{n-1}|^2))}{\tau} \\
&= \frac{(f'(|\hat{u}^{n-1}|^2)(|\hat{u}^n|^2 - |\hat{u}^{n-1}|^2) + \frac{1}{2}f''(\mu_1^n)(|\hat{u}^n|^2 - |\hat{u}^{n-1}|^2)^2)}{\tau}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(f'(|\hat{U}^{n-1}|^2)(|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2) + \frac{1}{2}f''(\mu_2^n)(|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2)^2)}{\tau} \\
&= \frac{f'(|\hat{U}^{n-1}|^2)((|\hat{u}^n|^2 - |\hat{u}^{n-1}|^2) - (|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2))}{\tau} \\
&+ \frac{(|\hat{u}^n|^2 - |\hat{u}^{n-1}|^2)(f'(|\hat{u}^{n-1}|^2) - f'(|\hat{U}^{n-1}|^2))}{\tau} \\
&+ \frac{1}{2}\frac{f''(\mu_2^n)((|\hat{u}^n|^2 - |\hat{u}^{n-1}|^2)^2 - (|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2)^2)}{\tau} \\
&+ \frac{1}{2}\frac{(|\hat{u}^n|^2 - |\hat{u}^{n-1}|^2)^2(f''(\mu_1^n) - f''(\mu_2^n))}{\tau},
\end{aligned}
$$

where

$$
\mu_1^n = |\hat{u}^{n-1}|^2 + \lambda_1^n(|\hat{u}^n|^2 - |\hat{u}^{n-1}|^2), \mu_2^n = |\hat{U}^{n-1}|^2 + \lambda_2^n(|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2).
$$

We find that

$$
\begin{aligned}
&\left\| \frac{(|\hat{u}^n|^2 - |\hat{u}^{n-1}|^2) - (|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2)}{\tau} \right\|_0 \\
&= \left\| \frac{(\hat{u}^n\bar{\hat{u}}^n - \hat{u}^{n-1}\bar{\hat{u}}^{n-1}) - (\hat{U}^n\bar{\hat{U}}^n - \hat{U}^{n-1}\bar{\hat{U}}^{n-1})}{\tau} \right\|_0 \\
&= \|(\hat{u}^{n-1}\tilde{\partial}_t\bar{\hat{u}}^n + \bar{\hat{u}}^n\tilde{\partial}_t\hat{u}^n) - (\hat{U}^{n-1}\tilde{\partial}_t\bar{\hat{U}}^n + \bar{\hat{U}}^n\tilde{\partial}_t\hat{U}^n)\|_0 \\
&= \|\hat{U}^{n-1}\tilde{\partial}_t\bar{\hat{e}}^n + \tilde{\partial}_t\bar{\hat{u}}^n\hat{e}^{n-1} + \bar{\hat{U}}^n\tilde{\partial}_t\hat{e}^n + \tilde{\partial}_t\hat{u}^n\bar{\hat{e}}^n\|_0 \\
&\le C\|\tilde{\partial}_t\hat{e}^n\|_0 + C\|\hat{e}^{n-1}\|_0 + C\|\hat{e}^n\|_0,
\end{aligned} \tag{3.29}
$$

which implies

$$
\begin{aligned}
&\left\| \frac{(|\hat{u}^n|^2 - |\hat{u}^{n-1}|^2)^2 - (|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2)^2}{\tau} \right\|_0 \\
&= \left\| \frac{((|\hat{u}^n|^2 - |\hat{u}^{n-1}|^2) - (|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2))((|\hat{u}^n|^2 - |\hat{u}^{n-1}|^2) + (|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2))}{\tau} \right\|_0 \\
&\le C\|\tilde{\partial}_t\hat{e}^n\|_0 + C\|\hat{e}^{n-1}\|_0 + C\|\hat{e}^n\|_0.
\end{aligned} \tag{3.30}
$$

Moreover

$$
\left\| \frac{\mu_1^n - \mu_2^n}{\tau} \right\|_{0,\infty} = \left\| \frac{|\hat{u}^{n-1}|^2 - |\hat{U}^{n-1}|^2 + \lambda_1^n(|\hat{u}^n|^2 - |\hat{u}^{n-1}|^2) - \lambda_2^n(|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2)}{\tau} \right\|_{0,\infty} \le C. \tag{3.31}
$$

Allocating (3.29)–(3.31), we have

$$
\begin{aligned}
&\left\| \frac{(f(|\hat{u}^{n-1}|^2) - f(|\hat{u}^{n-2}|^2)) - (f(|\hat{U}^{n-1}|^2) - f(|\hat{U}^{n-2}|^2))}{\tau} \right\|_0 \\
&\le C\tau^2 + C\|\tilde{\partial}_t\hat{e}^n\|_0 + C\|\hat{e}^{n-1}\|_0 + C\|\hat{e}^n\|_0,
\end{aligned}
$$

which leads to

$$
\|\tilde{\partial}_t P_1^n\|_0 \le C\tau^2 + C\|\tilde{\partial}_t\hat{e}^n\|_0 + C\|\hat{e}^{n-1}\|_0 + C\|\hat{e}^n\|_0 + C\|\tilde{\partial}_t\tilde{e}^n\|_0 + C\|\tilde{e}^n\|_0. \tag{3.32}
$$

Thus

$$(P_1^n, \tilde{\partial}_t \Delta e^n) \le C\tau^2 + C\|\tilde{\partial}_t \hat{e}^n\|_0 + C\|\hat{e}^{n-1}\|_0 + C\|\hat{e}^n\|_0 + C\|\tilde{\partial}_t \tilde{e}^n\|_0 + C\|\tilde{e}^n\|_0 + C\|\Delta e^{n-1}\|_0^2 + \tilde{\partial}_t (P_1^n, \Delta e^n). \tag{3.33}$$

Similar to (3.28), we rewrite $(R_1^n + R_2^n + R_3^n, \tilde{\partial}_t \Delta e^n)$ as

$$(R_1^n + R_2^n + R_3^n, \tilde{\partial}_t \Delta e^n) = -(\tilde{\partial}_t R_1^n + \tilde{\partial}_t R_2^n + \tilde{\partial}_t R_3^n, \Delta e^{n-1}) + \tilde{\partial}_t (R_1^n + R_2^n + R_3^n, \Delta e^n). \tag{3.34}$$

It is not difficult to check that

$$\|\tilde{\partial}_t R_1^n\|_0 = \left\| \frac{(\tilde{\partial}_t u^n - u_t^n) - (\tilde{\partial}_t u^{n-1} - u_t^{n-1})}{\tau} \right\|_0 \le C\tau^2, \tag{3.35}$$

and

$$\|\tilde{\partial}_t R_2^n\|_0 = \left\| \frac{(\Delta \tilde{u}^n - \Delta u^{n-\frac{1}{2}}) - (\Delta \tilde{u}^{n-1} - \Delta u^{n-1-\frac{1}{2}})}{\tau} \right\|_0 \le C\tau^2. \tag{3.36}$$

Note that

$$\begin{aligned}
&\left\| \tilde{\partial}_t R_3^n \|_0 = \| \frac{(f(|\hat{u}^n|^2)\tilde{u}^n - f(|u^{n-\frac{1}{2}}|^2)u^{n-\frac{1}{2}}) - (f(|\hat{u}^{n-1}|^2)\tilde{u}^{n-1} - f(|u^{n-1-\frac{1}{2}}|^2)u^{n-1-\frac{1}{2}})}{\tau} \right\|_0 \\
&= \left\| \frac{f(|u^{n-1-\frac{1}{2}}|^2)((\tilde{u}^n - \tilde{u}^{n-1}) - (u^{n-\frac{1}{2}} - u^{n-1-\frac{1}{2}}))}{\tau} + \frac{(\tilde{u}^n - \tilde{u}^{n-1})(f(|\hat{u}^{n-1}|^2) - f(|u^{n-1-\frac{1}{2}}|^2))}{\tau} \right. \\
&\quad + \frac{u^{n-\frac{1}{2}}((f(|\hat{u}^n|^2) - f(|\hat{u}^{n-1}|^2)) - (f(|u^{n-\frac{1}{2}}|^2) - f(|u^{n-1-\frac{1}{2}}|^2)))}{\tau} \\
&\quad \left. + \frac{(f(|\hat{u}^n|^2) - f(|\hat{u}^{n-1}|^2))(\tilde{u}^n - u^{n-\frac{1}{2}})}{\tau} \right\|_0 \\
&\le C\tau^2.
\end{aligned} \tag{3.37}$$

Therefore,

$$(R_1^n + R_2^n + R_3^n, \tilde{\partial}_t \Delta e^n) \le C\tau^4 + C\|\Delta e^{n-1}\|_0^2 + \tilde{\partial}_t (R_1^n + R_2^n + R_3^n, \Delta e^n). \tag{3.38}$$

Allocating all the estimates above to get

$$\begin{aligned}
\frac{1}{2\tau}(\|\Delta e^n\|_0^2 - \|\Delta e^{n-1}\|_0^2) \le{}& C\tau^4 + C\|\tilde{\partial}_t e^n\|_0^2 + C\|\tilde{\partial}_t e^{n-1}\|_0^2 + C\|\tilde{\partial}_t \hat{e}^n\|_0^2 \\
&+ C\|\Delta \hat{e}^n\|_0^2 + C\|\Delta \hat{e}^{n-1}\|_0^2 + C\|\Delta e^n\|_0^2 + C\|\Delta e^{n-1}\|_0^2 \\
&+ \tilde{\partial}_t (P_1^n, \Delta e^n) + \tilde{\partial}_t (R_1^n + R_2^n + R_3^n, \Delta e^n).
\end{aligned} \tag{3.39}$$

Replacing n by i in (3.39), then summing it from 2 to n, it follows that

$$\begin{aligned}
\|\Delta e^n\|_0^2 \le{}& \|\Delta e^1\|_0^2 + C\tau^4 + C\tau \sum_{i=1}^{n} (\|\tilde{\partial}_t e^i\|_0^2 + \|\Delta e^i\|_0^2) + C\|\tilde{\partial}_t \hat{e}^2\|_0^2 + C\|\Delta \hat{e}^1\|_0^2 \\
&+ (P_1^n, \Delta e^n) - (P_1^1, \Delta e^1) + (R_1^n + R_2^n + R_3^n, \Delta e^n) - (R_1^1 + R_2^1 + R_3^1, \Delta e^1).
\end{aligned} \tag{3.40}$$

Since

$$\|P_1^n\|_0^2 = \|f(|\hat{U}^{n-1}|^2)\tilde{e}^n + \tilde{u}^n(f(|\hat{u}^{n-1}|^2) - f(|\hat{U}^{n-1}|^2))\|_0^2$$
$$\le C\|\tilde{e}^n\|_0^2 + C\|\hat{e}^{n-1}\|_0^2 = C\tau^2 \left\| \sum_{i=1}^{n} \bar{\partial}_t e^i \right\|_0^2 \le C\tau \sum_{i=1}^{n} \|\bar{\partial}_t e^i\|_0^2, \tag{3.41}$$

together with (3.18) and (3.21), we have

$$\|\Delta e^n\|_0^2 \le C\tau^4 + C\tau \sum_{i=1}^{n} (\|\bar{\partial}_t e^i\|_0^2 + \|\Delta e^i\|_0^2). \tag{3.42}$$

In order to estimate $\|\bar{\partial}_t e^n\|_0$, we take difference between two time levels n and $n-1$ of (3.24), and multiply it by $\frac{1}{\tau}$ on both sides, then there holds

$$i\tilde{\partial}_{tt}e^n + \tilde{\partial}_t \Delta\tilde{e}^n + \tilde{\partial}_t P_1^n = \tilde{\partial}_t R_1^n + \tilde{\partial}_t R_2^n + \tilde{\partial}_t R_3^n. \tag{3.43}$$

On the other hand, multiplying (3.43) by $\tilde{\partial}_t\tilde{e}^n$, integrating it over Ω and then it follows that

$$i(\tilde{\partial}_{tt}e^n, \tilde{\partial}_t\tilde{e}^n) - \|\tilde{\partial}_t\nabla\tilde{e}^n\|_0^2 + (\tilde{\partial}_t P_1^n, \tilde{\partial}_t\tilde{e}^n) = (\tilde{\partial}_t R_1^n, \tilde{\partial}_t\tilde{e}^n) + (\tilde{\partial}_t R_2^n, \tilde{\partial}_t\tilde{e}^n) + (\tilde{\partial}_t R_3^n, \tilde{\partial}_t\tilde{e}^n). \tag{3.44}$$

Then taking the impartial part of (3.44) and using (3.32), (3.35)–(3.37), it follows that

$$\frac{1}{2\tau}(\|\tilde{\partial}_t e^n\|_0^2 - \|\tilde{\partial}_t e^{n-1}\|_0^2) = -Im(\tilde{\partial}_t P_1^n, \tilde{\partial}_t\tilde{e}^n) + Im(\tilde{\partial}_t R_1^n, \tilde{\partial}_t\tilde{e}^n)$$
$$+ Im(\tilde{\partial}_t R_2^n, \tilde{\partial}_t\tilde{e}^n) + Im(\tilde{\partial}_t R_3^n, \tilde{\partial}_t\tilde{e}^n)$$
$$\le C\tau^4 + C\|\tilde{\partial}_t e^n\|_0^2 + C\|\tilde{\partial}_t e^{n-1}\|_0^2 + C\|\Delta e^n\|_0^2$$
$$+ C\|\Delta e^{n-1}\|_0^2 + C\|\Delta e^{n-2}\|_0^2. \tag{3.45}$$

Replacing n by i in (3.45), then summing it from 2 to n, with the result of e^1, we get

$$\|\tilde{\partial}_t e^n\|_0^2 \le C\tau^4 + C\tau \sum_{i=1}^{n} (\|\tilde{\partial}_t e^i\|_0^2 + \|\Delta e^i\|_0^2). \tag{3.46}$$

Combining (3.42) and (3.46), we have

$$\|\tilde{\partial}_t e^n\|_0^2 + \|\Delta e^n\|_0^2 \le C\tau^4 + C\tau \sum_{i=1}^{n} (\|\tilde{\partial}_t e^i\|_0^2 + \|\Delta e^i\|_0^2). \tag{3.47}$$

Applying the Gronwall's inequality to (3.47), there exist τ_8, C_7, C_8, such that when $\tau \le \tau_8$, there holds

$$\|\tilde{\partial}_t e^n\|_0 + \|e^n\|_2 \le C_7\tau^2, \tag{3.48}$$

which implies

$$\|\tilde{\partial}_{tt} U^n\|_2 \le C_8, \tag{3.49}$$

and

$$\|U^n\|_{0,\infty} + \|\tilde{\partial}_t U^n\|_{0,\infty} \le \|e^n\|_{0,\infty} + \|\tilde{\partial}_t e^n\|_{0,\infty} + C\|u^n\|_{0,\infty} + \|\tilde{\partial}_t u^n\|_{0,\infty}$$
$$\le CC_7\tau + C\|u^n\|_{0,\infty} + \|\tilde{\partial}_t u^n\|_{0,\infty} \le K_0, \tag{3.50}$$

where $\tau \leq \tau_9 \leq 1/CC_7$. It can be seen that C_7 and C_8 have nothing to do with C_0. Then (3.2) and (3.3) hold for $m = n$ if we take $C_0 \geq \sum_{i=1}^{8} C_i$ and $\tau_0 \leq \min_{1\leq i\leq 9} \tau_i$. □

Remark 1 It can be seen that the result of $\|e^n\|_2 = O(\tau^2)$ is one order higher that in [23], which leads to $\|\tilde{\partial}_{tt} U^m\|_2 \leq C_0$. This will play an important role in the foregoing superconvergence analysis.

Remark 2 If we use a fully explicit method for the nonlinear term in (3.1)–(3.3), the unconditional convergence analysis is still valid by an H^2 error estimate for the time semi-discrete. The idea is very similar and the process of proof is much easier.

4 Superconvergence Results for the Fully Discrete System

In this section, we will establish an estimate for $\|R_h U^n - U_h^n\|_0 = O(h^2)$, which results in the unconditional boundedness of $\|U_h^n\|_{0,\infty}$. Then $\|\nabla(R_h U^n - U_h^n)\|_0$ with order $O(h^2+\tau^2)$ is deduced which will result in the superclose results $\|\nabla(I_h u^n - U_h^n)\|_0$ with order $O(h^2 + \tau^2)$ unconditionally on the basis of the relationship between I_h and R_h. At last, the global superconvergence is deduced through the interpolated postprocessing technique. A pervading strategy throughout the error analysis in the rest of this paper is splitting the error to a sum of two terms:

$$U^i - U_h^i = U^i - R_h U^i + R_h U^i - U_h^i \triangleq r^i + \theta^i, i = 0, 1, 2, \ldots, N. \tag{4.1}$$

Theorem 2 *Let u and U_h^m be the solutions of* (1.1) *and* (2.5)–(2.7) *respectively, for $m = 1, 2, \ldots, N$, under the conditions in Theorem 1, we have*

$$\|\nabla(I_h u^m - U_h^m)\|_0 = O(h^2 + \tau^2). \tag{4.2}$$

Proof Since $\|R_h U^{1,0}\|_0 + \|R_h U^{1,0}\|_{0,\infty} + \|R_h U^m\|_{0,\infty} \leq C\|U^0\|_2 + C\|U^{1,0}\|_2 + C\|U^m\|_2 \leq C$, let $K_0^{'} \triangleq 1 + \|R_h U^{1,0}\|_{0,\infty} + \max_{0\leq i\leq N} \|R_h U^i\|_{0,\infty}$. First of all, we obtain the result that there exist $\tau_0^{'}$ and $h_0^{'}$, when $\tau \leq \tau_0^{'}$ and $h \leq h_0^{'}$, it follows

$$\|\theta^m\|_0 \leq C_0^{'} h^2, \tag{4.3}$$

which bounds $\|U_h^m\|_{0,\infty}$ unconditionally. For $m = 1$, we have $\|U_h^0\|_{0,\infty} = \|R_h U^0\|_{0,\infty} \leq K_0^{'}$. Using (2.6) and (3.2), the error equation is deduced by

$$\begin{aligned} i\left(\frac{\theta^{1,0}}{\tau}, v_h\right) - \left(\frac{\nabla\theta^{1,0}}{2}, \nabla v_h\right) =& -i\left(\frac{r^{1,0} - r^0}{\tau}, v_h\right) + \left(\frac{\nabla r^{1,0} + \nabla r^0}{2}, \nabla v_h\right) \\ &+ \left(f(|U^0|^2)\frac{U^{1,0} + U^0}{2} - f(|U_h^0|^2)\frac{U_h^{1,0} + U_h^0}{2}, v_h\right). \end{aligned} \tag{4.4}$$

Substituting $v_h = \frac{\theta^{1,0}}{\tau}$ in (4.4), we get

$$i\left\|\frac{\theta^{1,0}}{\tau}\right\|_0^2 - \frac{1}{2\tau}\|\nabla\theta^{1,0}\|_0^2 = -i\left(\frac{r^{1,0}-r^0}{\tau}, \frac{\theta^{1,0}}{\tau}\right) + \left(\frac{\nabla r^{1,0}+\nabla r^0}{2}, \frac{\nabla\theta^{1,0}}{\tau}\right) + \left(f(|U^0|^2)\frac{U^{1,0}+U^0}{2} - f(|U_h^0|^2)\frac{U_h^{1,0}+U_h^0}{2}, \frac{\theta^{1,0}}{\tau}\right). \tag{4.5}$$

By (2.1) and (2.3), it follows that

$$\left|\left(\frac{r^{1,0}-r^0}{\tau}, \frac{\theta^{1,0}}{\tau}\right)\right| \le Ch^2\left\|\frac{U^{1,0}-U^0}{\tau}\right\|_2\left\|\frac{\theta^{1,0}}{\tau}\right\|_0 \le Ch^4 + \frac{1}{8}\left\|\frac{\theta^{1,0}}{\tau}\right\|_0^2$$

$$\left(\frac{\nabla r^{1,0}+\nabla r^0}{2}, \frac{\nabla\theta^{1,0}}{\tau}\right) = 0$$

$$\begin{aligned}&\left|\left(f(|U^0|^2)\frac{U^{1,0}+U^0}{2} - f(|U_h^0|^2)\frac{U_h^{1,0}+U_h^0}{2}, \frac{\theta^{1,0}}{\tau}\right)\right|\\ &\quad= \left|\left(f(|U_h^0|^2)\left(\frac{\theta^{1,0}}{2} + \frac{r^{1,0}+r^0}{2}\right), \frac{\theta^{1,0}}{\tau}\right) + \left(\frac{U^{1,0}+U^0}{2}(f(|U^0|^2) - f(|U_h^0|^2)), \frac{\theta^{1,0}}{\tau}\right)\right|\\ &\quad\le Ch^4 + C\|\theta^{1,0}\|_0^2 + \frac{1}{8}\left\|\frac{\theta^{1,0}}{\tau}\right\|_0^2.\end{aligned}$$

Taking the imaginary part and the real part, respectively, summing them together, then we get

$$\left\|\frac{\theta^{1,0}}{\tau}\right\|_0^2 + \frac{1}{2\tau}\|\nabla\theta^{1,0}\|_0^2 \le Ch^4 + C\|\theta^{1,0}\|_0^2 + \frac{1}{2}\left\|\frac{\theta^{1,0}}{\tau}\right\|_0^2. \tag{4.6}$$

Thus there exist $\tau_1^{'}, C_1^{'}$, such that when $\tau \le \tau_1^{'}$, we have

$$\frac{1}{\tau}\|\theta^{1,0}\|_0 + \|\nabla\theta^{1,0}\|_0 \le C_1^{'}h^2, \tag{4.7}$$

which implies

$$\|U_h^{1,0}\|_{0,\infty} \le Ch^{-1}\|\theta^{1,0}\|_0 + \|R_hU^{1,0}\|_{0,\infty} \le CC_1^{'}h + \|R_hU^{1,0}\|_{0,\infty} \le K_0^{'}, \tag{4.8}$$

where $h \le h_1^{'} \le 1/CC_1^{'}$. Making use of (2.7) and (3.3) to deduce the error equation and setting $v_h = \frac{\theta^1}{\tau}$, then we have

$$i\left\|\frac{\theta^1}{\tau}\right\|_0^2 - \frac{1}{2\tau}\|\nabla\theta^1\|_0^2 = -i\left(\frac{r^1-r^0}{\tau}, \frac{\theta^1}{\tau}\right) + \left(\frac{\nabla r^1+\nabla r^0}{2}, \frac{\nabla\theta^1}{\tau}\right) + \left(f\left(\left|\frac{U^{1,0}+U^0}{2}\right|^2\right)\frac{U^1+U^0}{2} - f\left(\left|\frac{U_h^{1,0}+U_h^0}{2}\right|^2\right)\frac{U_h^1+U_h^0}{2}, \frac{\theta^1}{\tau}\right). \tag{4.9}$$

Similar to the proof of $\theta^{1,0}$, we get

$$\left|\left(\frac{r^1-r^0}{\tau},\frac{\theta^1}{\tau}\right)\right| \le Ch^2\left\|\frac{U^1-U^0}{\tau}\right\|_2\left\|\frac{\theta^1}{\tau}\right\|_0 \le Ch^4+\frac{1}{8}\left\|\frac{\theta^1}{\tau}\right\|_0^2,$$

$$\left(\frac{\nabla r^1+\nabla r^0}{2},\frac{\nabla\theta^1}{\tau}\right)=0,$$

$$\begin{aligned}&\left|\left(f\left(\left|\frac{U^{1,0}+U^0}{2}\right|^2\right)\frac{U^1+U^0}{2}-f\left(\left|\frac{U_h^{1,0}+U_h^0}{2}\right|^2\right)\frac{U_h^1+U_h^0}{2},\frac{\theta^1}{\tau}\right)\right|\\
&=\left|\left(f\left(\left|\frac{U_h^{1,0}+U_h^0}{2}\right|^2\right)\left(\frac{\theta^1}{2}+\frac{r^1+r^0}{2}\right),\frac{\theta^1}{\tau}\right)\right.\\
&\quad\left.+\frac{U^1+U^0}{2}\left(f\left(\left|\frac{U^{1,0}+U^0}{2}\right|^2\right)-f\left(\left|\frac{U_h^{1,0}+U_h^0}{2}\right|^2\right),\frac{\theta^1}{\tau}\right)\right|\\
&\le Ch^4+C\|\theta^1\|_0^2+C\|\theta^{1,0}\|_0^2+\frac{1}{8}\left\|\frac{\theta^1}{\tau}\right\|_0^2 \le Ch^4+C\|\theta^1\|_0^2+\frac{1}{8}\left\|\frac{\theta^1}{\tau}\right\|_0^2,\end{aligned}$$

where the last step is deduced by the help of (4.7). Also, taking the imaginary part and the real part, respectively, summing them together, then we get

$$\left\|\frac{\theta^1}{\tau}\right\|_0^2+\frac{1}{2\tau}\|\nabla\theta^1\|_0^2 \le Ch^4+C\|\theta^1\|_0^2+\frac{1}{2}\left\|\frac{\theta^1}{\tau}\right\|_0^2. \tag{4.10}$$

Thus there exist $\tau_2^{'}, C_2^{'}$, such that when $\tau \le \tau_2^{'}$, we have

$$\left\|\frac{\theta^1}{\tau}\right\|_0+\|\nabla\theta^1\|_0 \le C_2^{'}h^2, \tag{4.11}$$

which implies

$$\|U_h^1\|_{0,\infty} \le Ch^{-1}\|\theta^1\|_0+\|R_hU^1\|_{0,\infty} \le CC_2^{'}h+\|R_hU^1\|_{0,\infty} \le K_0^{'}, \tag{4.12}$$

where $h \le h_2^{'} \le 1/CC_2^{'}$. By mathematical induction, we assume that (4.3) holds for $m \le n-1$, then we have

$$\|U_h^m\|_{0,\infty} \le Ch^{-1}\|\theta^m\|_0+\|R_hU^m\|_{0,\infty} \le CC_0^{'}h+\|R_hU^m\|_{0,\infty} \le K_0^{'}, \tag{4.13}$$

where $h \le h_3^{'} \le 1/CC_0^{'}$.

Then when $m=n$, setting $P_2^n=f(|\hat{U}^n|^2)\tilde{U}^n-f(|\hat{U}_h^n|^2)\tilde{U}_h^n$, we get the error equation from (2.5) and (3.1) as follows:

$$i(\bar{\partial}_t\theta^n,v_h)-(\nabla\tilde{\theta}^n,\nabla v_h)=-i(\bar{\partial}_t r^n,v_h)+(\nabla\tilde{r}^n,\nabla v_h)-(P_2^n,v_h). \tag{4.14}$$

Choosing $v_h=\tilde{\theta}^n$ in (4.14), the impartial part results in

$$\frac{1}{2\tau}(\|\theta^n\|_0^2-\|\theta^{n-1}\|_0^2)=-Re(\bar{\partial}_t r^n,\tilde{\theta}^n)+Im(\nabla r^n,\nabla\tilde{\theta}^n)-Im(P_2^n,\tilde{\theta}^n). \tag{4.15}$$

We bound P_2^n as

$$\|P_2^n\|_0 = \|f(|\hat{U}_h^n|^2)(\tilde{\theta}^n + \tilde{r}^n) + \tilde{U}^n f^{'}(\mu_6^n)(|\hat{U}^n|^2 - |\hat{U}_h^n|^2)\|_0$$
$$\leq C\|\tilde{\theta}^n\|_0 + C\|\tilde{r}^n\|_0 + C\|\hat{\theta}^n\|_0 + C\|\hat{r}^n\|_0 \leq C\|\tilde{\theta}^n\|_0 + C\|\hat{\theta}^n\|_0 + Ch^2,$$

where $\mu_6^n = |\hat{U}^{n-1}|^2 + \lambda_6^n(|\hat{U}_h^{n-1}|^2 - |\hat{U}^{n-1}|^2)$.

Thus,

$$\frac{1}{2\tau}(\|\theta^n\|_0^2 - \|\theta^{n-1}\|_0^2) \leq C\|\theta^n\|_0^2 + C\|\theta^{n-1}\|_0^2 + C\|\theta^{n-2}\|_0^2 + Ch^4. \tag{4.16}$$

Summing (4.16) up gives

$$\|\theta^n\|_0^2 \leq \|\theta^1\|_0^2 + C\tau\sum_{i=1}^{n}\|\theta^i\|_0^2 + Ch^4. \tag{4.17}$$

Applying the Gronwall's inequality to (4.17), together with (4.11), there exist $\tau_3^{'}, C_3^{'}$, when $\tau \leq \tau_3^{'}$, we have

$$\|\theta^n\|_0 \leq C_3^{'}h^2, \tag{4.18}$$

which leads to

$$\|U_h^n\|_{0,\infty} \leq Ch^{-1}\|\theta^n\|_0 + \|R_hU^n\|_{0,\infty} \leq CC_3^{'}h + \|R_hU^n\|_{0,\infty} \leq K_0^{'}, \tag{4.19}$$

where $h \leq h_4^{'} \leq 1/CC_3^{'}$. Clearly, $C_3^{'}$ has nothing to do with $C_0^{'}$. Thus (4.3) holds for $m = n$, if we take $C_0^{'} \geq \sum_{i=1}^{3} C_i^{'}$, $\tau_0^{'} \leq \min_{1\leq\tau\leq 3}\tau_i^{'}$ and $h_0^{'} \leq \min_{1\leq\tau\leq 4}h_i^{'}$.

Secondly, we will give the result

$$\|\nabla\theta^m\|_0 \leq C(h^2 + \tau^2) \tag{4.20}$$

unconditionally. Because of (4.11), it is apparent to see that (4.20) holds for $m = 1$. When $m = n$, $(n \geq 2)$, choosing $v_h = \tilde{\partial}_t\theta^n$ in (4.14) and taking the real part result in

$$\frac{1}{2\tau}(\|\nabla\theta^n\|_0^2 - \|\nabla\theta^{n-1}\|_0^2) = Im(\tilde{\partial}_t r^n, \tilde{\partial}_t\theta^n) + Re(\nabla r^n, \nabla\tilde{\partial}_t\theta^n) - Re(P_2^n, \tilde{\partial}_t\theta^n). \tag{4.21}$$

Then (4.21) leads to

$$\frac{1}{2\tau}(\|\nabla\theta^n\|_0^2 - \|\nabla\theta^{n-1}\|_0^2) \leq Ch^4 + C\|\tilde{\partial}_t\theta^n\|_0^2. \tag{4.22}$$

Summing (4.22) from 2 to n, we obtain

$$\|\nabla\theta^n\|_0^2 \leq Ch^4 + C\tau\sum_{i=1}^{n}\|\tilde{\partial}_t\theta^i\|_0^2. \tag{4.23}$$

Obviously, to obtain the estimate of $\|\nabla\theta^n\|_0$, we need the boundedness of $\|\tilde{\partial}_t\theta^i\|_0$. Taking difference between two time levels n and $n-1$ of (4.14), with $\hat{U}_h^1 = \tilde{U}_h^1, \hat{r}^1 = \tilde{r}^1, \hat{\theta}^1 = \tilde{\theta}^1$ we have

$$i(\tilde{\partial}_{tt}\theta^n, v_h) - (\nabla\tilde{\partial}_t\tilde{\theta}^n, \nabla v_h) = -i(\tilde{\partial}_{tt}r^n, v_h) + (\nabla\tilde{\partial}_t\tilde{r}^n, \nabla v_h) - (\tilde{\partial}_t P_2^n, v_h). \tag{4.24}$$

Setting $v_h = \tilde{\partial}_t \tilde{\theta}^n$ in (4.24), the imaginary part gives

$$\frac{1}{2\tau}(\|\tilde{\partial}_t \theta^n\|_0^2 - \|\tilde{\partial}_t \theta^{n-1}\|_0^2) = -Re(\tilde{\partial}_{tt} r^n, \tilde{\partial}_t \tilde{\theta}^n) + Im(\nabla \tilde{\partial}_t \tilde{r}^n, \nabla \tilde{\partial}_t \tilde{\theta}^n) - Im(\tilde{\partial}_t P_2^n, \tilde{\partial}_t \tilde{\theta}^n). \tag{4.25}$$

Similar to the estimate of θ^n, it is not difficult to check that

$$|(\tilde{\partial}_{tt} r^n, \tilde{\partial}_t \tilde{\theta}^n)| \leq Ch^2 \|\tilde{\partial}_{tt} U^n\|_2 \|\tilde{\partial}_t \tilde{\theta}^n\|_0 \leq Ch^4 + C\|\tilde{\partial}_t \tilde{\theta}^n\|_0^2, \tag{4.26}$$

$$(\nabla \tilde{\partial}_t \tilde{r}^n, \nabla \tilde{\partial}_t \tilde{\theta}^n) = 0. \tag{4.27}$$

Based on the achievements above, it follows that

$$\begin{aligned}
\|\tilde{\partial}_t P_2^n\|_0 &= \left\| \frac{(f(|\hat{U}^n|^2)\tilde{U}^n - f(|\hat{U}^{n-1}|^2)\tilde{U}^{n-1}) - (f(|\hat{U}_h^n|^2)\tilde{U}_h^n) - f(|\hat{U}_h^{n-1}|^2)\tilde{U}_h^{n-1})}{\tau} \right\|_0 \\
&= \Big\| f(|\hat{U}^{n-1}|^2)\tilde{\partial}_t \tilde{U}^n - f(|\hat{U}_h^{n-1}|^2)\tilde{\partial}_t \tilde{U}_h^n \\
&\quad + \frac{\tilde{U}^n(f(|\hat{U}^n|^2) - f(|\hat{U}^{n-1}|^2)) - \tilde{U}_h^n(f(|\hat{U}_h^n|^2) - f(|\hat{U}_h^{n-1}|^2))}{\tau} \Big\|_0 \\
&\leq \|f(|\hat{U}_h^{n-1}|^2)(\tilde{\partial}_t \tilde{\theta}^n + \tilde{\partial}_t \tilde{r}^n) + \tilde{\partial}_t \tilde{U}^n (f(|\hat{U}^{n-1}|^2) - f(|\hat{U}_h^{n-1}|^2))\|_0 \\
&\quad + C \left\| \frac{(f(|\hat{U}^n|^2) - f(|\hat{U}^{n-1}|^2)) - (f(|\hat{U}_h^n|^2) - f(|\hat{U}_h^{n-1}|^2))}{\tau} \right\|_0 \\
&\quad + C \left\| \frac{(f(|\hat{U}^n|^2) - f(|\hat{U}^{n-1}|^2))(\tilde{U}^n - \tilde{U}_h^n)}{\tau} \right\|_0 .
\end{aligned}$$

Note that

$$\begin{aligned}
&\left\| \frac{(f(|\hat{U}^n|^2) - f(|\hat{U}^{n-1}|^2)) - (f(|\hat{U}_h^n|^2) - f(|\hat{U}_h^{n-1}|^2))}{\tau} \right\|_0 \\
&= \Big\| f'(|\hat{U}_h^{n-1}|^2) \frac{(|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2) - (|\hat{U}_h^n|^2 - |\hat{U}_h^{n-1}|^2)}{\tau} \\
&\quad + \frac{(|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2)}{\tau}(f'(|\hat{U}^{n-1}|^2) - f'(|\hat{U}_h^{n-1}|^2)) \\
&\quad + \frac{1}{2} f''(\mu_9^n) \frac{(|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2)^2 - (|\hat{U}_h^n|^2 - |\hat{U}_h^{n-1}|^2)^2)}{\tau} \\
&\quad + \frac{(|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2)^2}{2\tau}(f''(\mu_8^n) - f''(\mu_9^n)) \Big\|_0 ,
\end{aligned}$$

where

$$\mu_8^n = |\hat{U}^{n-1}|^2 + \lambda_8^n(|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2), \mu_9^n = |\hat{U}_h^{n-1}|^2 + \lambda_9^n(|\hat{U}_h^n|^2 - |\hat{U}_h^{n-1}|^2).$$

In fact,

$$
\begin{aligned}
&\left\| \frac{(|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2) - (|\hat{U}_h^n|^2 - |\hat{U}_h^{n-1}|^2)}{\tau} \right\|_0 \\
&= \left\| \frac{(\hat{U}^n \bar{\hat{U}}^n - \hat{U}^{n-1}\bar{\hat{U}}^{n-1}) - (\hat{U}_h^n \bar{\hat{U}}_h^n - \hat{U}_h^{n-1}\bar{\hat{U}}_h^{n-1})}{\tau} \right\|_0 \\
&= \|\hat{U}^{n-1}\tilde{\partial}_t \bar{\hat{U}}^n - \hat{U}_h^{n-1}\tilde{\partial}_t \bar{\hat{U}}_h^n + \bar{\hat{U}}^n \tilde{\partial}_t \hat{U}^n - \bar{\hat{U}}_h^n \tilde{\partial}_t \hat{U}_h^n\|_0 \\
&= \|\hat{U}_h^{n-1}(\tilde{\partial}_t \bar{\hat{U}}^n - \tilde{\partial}_t \bar{\hat{U}}_h^n) + \tilde{\partial}_t \bar{\hat{U}}^n(\hat{U}^{n-1} - \hat{U}_h^{n-1}) + \bar{\hat{U}}_h^n(\tilde{\partial}_t \hat{U}^n - \tilde{\partial}_t \hat{U}_h^n) + \tilde{\partial}_t \hat{U}^n(\bar{\hat{U}}^n - \bar{\hat{U}}_h^n)\|_0 \\
&\le C\|\tilde{\partial}_t \hat{\theta}^n\|_0^2 + \|\tilde{\partial}_t \hat{r}^n\|_0^2 + C\|\hat{\theta}^n\|_0^2 + \|\hat{r}^n\|_0^2 + C\|\hat{\theta}^{n-1}\|_0^2 + \|\hat{r}^{n-1}\|_0^2 \\
&\le Ch^4 + C\|\tilde{\partial}_t \hat{\theta}^n\|_0^2 + C\|\hat{\theta}^n\|_0^2 + C\|\hat{\theta}^{n-1}\|_0^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \frac{(|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2)^2 - (|\hat{U}_h^n|^2 - |\hat{U}_h^{n-1}|^2)^2)}{\tau} \right\|_0 \\
&= \left\| (|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2) + (|\hat{U}_h^n|^2 - |\hat{U}_h^{n-1}|^2)) \frac{(|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2) - (|\hat{U}_h^n|^2 - |\hat{U}_h^{n-1}|^2))}{\tau} \right\|_0 \\
&\le Ch^4 + C\|\tilde{\partial}_t \hat{\theta}^n\|_0^2 + C\|\hat{\theta}^n\|_0^2 + C\|\hat{\theta}^{n-1}\|_0^2.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\|f''(\mu_8^n) - f''(\mu_9^n)\|_0 &= \||\hat{U}^{n-1}|^2 - |\hat{U}_h^{n-1}|^2 + \lambda_9^n((|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2) \\
&\quad - (|\hat{U}_h^n|^2 - |\hat{U}_h^{n-1}|^2)) + (|\hat{U}^n|^2 - |\hat{U}^{n-1}|^2)(\lambda_8^n - \lambda_9^n)\|_0 \\
&\le C\|\hat{\theta}^n\|_0 + C\|\hat{\theta}^{n-1}\|_0 + Ch^2 + C\tau.
\end{aligned}
$$

Therefore

$$
|(\tilde{\partial}_t P_2^n, \tilde{\partial}_t \theta^n)| \le Ch^4 + C\tau^4 + \|\tilde{\partial}_t \tilde{\theta}^n\|_0 + C\|\tilde{\partial}_t \hat{\theta}^n\|_0^2 + C\|\hat{\theta}^n\|_0^2 + C\|\hat{\theta}^{n-1}\|_0^2. \tag{4.28}
$$

Recalling (4.26)–(4.28), it follows that

$$
\frac{1}{2\tau}(\|\tilde{\partial}_t \theta^n\|_0^2 - \|\tilde{\partial}_t \theta^{n-1}\|_0^2) \le Ch^4 + C\tau^4 + \|\tilde{\partial}_t \tilde{\theta}^n\|_0 + C\|\tilde{\partial}_t \hat{\theta}^n\|_0^2 + C\|\hat{\theta}^n\|_0^2 + C\|\hat{\theta}^{n-1}\|_0^2. \tag{4.29}
$$

Summing (4.29), together with (4.23), it gives that

$$
\|\tilde{\partial}_t \theta^n\|_0^2 \le Ch^4 + C\tau^4 + C\tau \sum_{i=1}^{n} \|\tilde{\partial}_t \theta^i\|_0^2. \tag{4.30}
$$

Applying the Gronwall's inequality to (4.30), we have

$$
\|\tilde{\partial}_t \theta^n\|_0^2 \le Ch^4 + C\tau^4, \tag{4.31}
$$

which results in

$$
\|\nabla \theta^n\|_0 \le Ch^2 + C\tau^2. \tag{4.32}
$$

At last, by the help of (2.2)–(2.4), it reduces to

$$\begin{aligned}\|\nabla(I_h u^n - U_h^n)\|_0 &\le C\|\nabla(I_h u^n - R_h u^n)\|_0 + C\|\nabla(R_h u^n - R_h U^n)\|_0 \\ &\quad + C\|\nabla(R_h U^n - U_h^n)\|_0 \\ &\le Ch^2\|u^n\|_3 + C\|e^n\|_2 + C\|\nabla\theta^n\|_0 \\ &\le Ch^2 + C\tau^2.\end{aligned}$$

□

Remark 3 It is worthy to note that Theorem 2 can not be obtained by I_h alone. In order to keep the order of $\|I_h U^n - U_h^n\|_0$ and $\|\nabla(I_h U^n - U_h^n)\|_0$, we should employ $(\nabla(u^n - I_h u^n), \nabla v_h) = O(h^2)\|u^n\|_3\|v_h\|_1$ or $(\nabla(u^n - I_h u^n), \nabla v_h) = O(h^2)\|u^n\|_4\|v_h\|_0$ as that in [3]. Thus the regularity of U^n and u^n should be much stricter. However, we can only bound $\|U^n\|_2$ under the assumption that Ω is a rectangle. On the other hand, we take different approach to bound $\|\tilde{\partial}_t P_2^n\|_0$, comparing with that in [3], then we avoid the appearance of the boundedness about $\|\tilde{\partial}_t U_h^n\|_{0,\infty}$. Thus we get our final result unconditionally, which improves the conclusion of [3].

Based on Theorem 2 and interpolated postprocessing operator I_{2h}^2 constructed in [31], we can deduce the following global superconvergence easily.

Theorem 3 *Let u and U_h^m be the solutions of* (1.1) *and* (2.2)–(2.4) *respectively, for $m = 1, \ldots, N$, under the conditions of Theorem* 1*, we have*

$$\|u^m - I_{2h}^2 U_h^m\|_1 = O(h^2 + \tau^2). \tag{4.33}$$

5 Numerical Results

In this section, we present three numerical examples to confirm our theoretical analysis.

Example 1 Considering the cubic Schrödinger equation [23] with $\Omega = [0, 1] \times [0, 1]$, we set $f(s) = s$, $u = 5e^{it}(1+2t^2)(1-x)(1-y)\sin(x)\sin(y)$ and $g(X, t)$ is chosen corresponding to the exact solution. A uniform rectangular partition with $M + 1$ nodes in each direction is used in our computation.

We solve the system by the linearized Galerkin method with bilinear element. To confirm our error estimates in H^1-norm, we choose $\tau = h$ and the numerical results with respect to time $t = 0.25, 0.5, 0.75, 1.0$ are listed in the following Tables 1, 2, 3 and 4 respectively. We can see clearly from them that when $h \to 0$, $\|u^n - U_h^n\|_1$ is convergent at an optimal rate

Table 1 Numerical results at $t = 0.25$ with $\tau = h$

$M \times M$	$\|u^n - U_h^n\|_1$	Order	$\|U_h^n - I_h u^n\|_1$	Order	$\|u^n - I_{2h}^2 U_h^n\|_1$	Order
10×10	7.4992×10^{-2}	–	8.1712×10^{-3}	–	7.4341×10^{-2}	–
20×20	3.7372×10^{-2}	1.0048	1.8003×10^{-3}	2.1823	1.8550×10^{-2}	2.0027
40×40	1.8670×10^{-2}	1.0013	4.4767×10^{-4}	2.0077	4.6333×10^{-3}	2.0013
80×80	9.3329×10^{-3}	1.0003	1.0724×10^{-4}	2.0616	1.1571×10^{-3}	2.0015

Table 2 Numerical results at $t = 0.5$ with $\tau = h$

$M \times M$	$\|u^n - U_h^n\|_1$	Order	$\|U_h^n - I_h u^n\|_1$	Order	$\|u^n - I_{2h}^2 U_h^n\|_1$	Order
10×10	9.9799×10^{-2}	–	1.3817×10^{-2}	–	9.9293×10^{-2}	–
20×20	4.9808×10^{-2}	1.0026	3.3910×10^{-3}	2.0267	2.4764×10^{-2}	2.0035
40×40	2.4891×10^{-2}	1.0008	8.2439×10^{-4}	2.0403	6.1742×10^{-3}	2.0039
80×80	1.2444×10^{-2}	1.0002	2.0720×10^{-4}	1.9923	1.5417×10^{-3}	2.0017

Table 3 Numerical results at $t = 0.75$ with $\tau = h$

$M \times M$	$\|u^n - U_h^n\|_1$	Order	$\|U_h^n - I_h u^n\|_1$	Order	$\|u^n - I_{2h}^2 U_h^n\|_1$	Order
10×10	1.4121×10^{-1}	–	1.4179×10^{-2}	–	1.4021×10^{-1}	–
20×20	7.0536×10^{-2}	1.0014	4.2319×10^{-3}	1.7444	3.5022×10^{-2}	2.0013
40×40	3.5259×10^{-2}	1.0004	1.0843×10^{-3}	1.9646	8.7352×10^{-3}	2.0033
80×80	1.7628×10^{-2}	1.0001	2.7280×10^{-4}	1.9908	2.1809×10^{-3}	2.0019

Table 4 Numerical results at $t = 1.0$ with $\tau = h$

$M \times M$	$\|u^n - U_h^n\|_1$	Order	$\|U_h^n - I_h u^n\|_1$	Order	$\|u^n - I_{2h}^2 U_h^n\|_1$	Order
10×10	1.9940×10^{-1}	–	1.2256×10^{-2}	–	1.9713×10^{-1}	–
20×20	9.9574×10^{-2}	1.0018	3.9617×10^{-3}	1.6293	4.9356×10^{-2}	1.9978
40×40	4.9775×10^{-2}	1.0003	1.0630×10^{-3}	1.8980	1.2318×10^{-2}	2.0025
80×80	2.4886×10^{-2}	1.0000	2.7281×10^{-4}	1.9621	3.0760×10^{-3}	2.0016

Table 5 Convergence results of $\|u^n - U_h^n\|_1$ with $h = \frac{1}{160}$ and $\tau = kh$

t	$k = 1$	$k = 5$	$k = 10$	$k = 20$	$k = 40$
0.25	9.33290×10^{-3}	9.33431×10^{-3}	9.38386×10^{-3}	1.13252×10^{-2}	3.42397×10^{-2}
0.50	1.24436×10^{-2}	1.24498×10^{-2}	1.26254×10^{-2}	1.51977×10^{-2}	1.64216×10^{-2}
0.75	1.76280×10^{-2}	1.76395×10^{-2}	1.78389×10^{-2}	1.82913×10^{-2}	4.99408×10^{-2}
1.00	2.48864×10^{-2}	2.48983×10^{-2}	2.49922×10^{-2}	2.80435×10^{-2}	4.20744×10^{-2}

$O(h)$, and $\|U_h^n - I_h u^n\|_1$, $\|u^n - I_{2h}^2 U_h^n\|_1$ are superconvergent at $O(h^2)$, which coincide with our theoretical analysis. To show the unconditional stability, we choose $h = 1/128$ and the large time steps $\tau = h, 4h, 8h, 16h$, respectively. We present the numerical results in Table 5, which suggest that the scheme is stable for large time steps.We also describe the error reduction results at $t = 0.25, 0.5, 0.75, 1.0$ in Figs. 1, 2, 3 and 4 respectively, where $E_h^1 = \|u^n - U_h^n\|_1$, $E_h^2 = \|U_h^n - I_h u^n\|_1$, $E_h^3 = \|u^n - I_{2h}^2 U_h^n\|_1$.

Example 2 We consider the Schrödinger equation with $\Omega = [0, 1] \times [0, 1]$, $f(s) = -s^2 + s$ and $u = e^{(i+1)t} x^3 y^3 (1 - x)(1 - y)$. $g(X, t)$ is chosen corresponding to the exact solution. A uniform rectangular partition with $M + 1$ nodes in each direction is used in our computation.

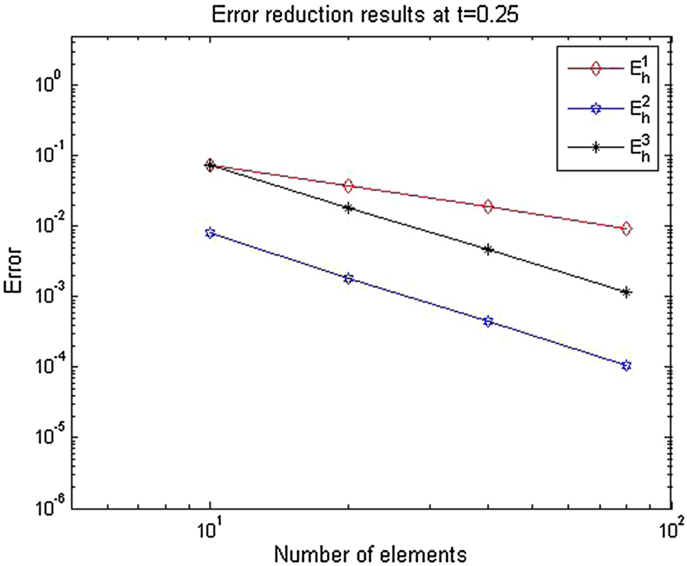

Fig. 1 Error reduction results at $t = 0.25$

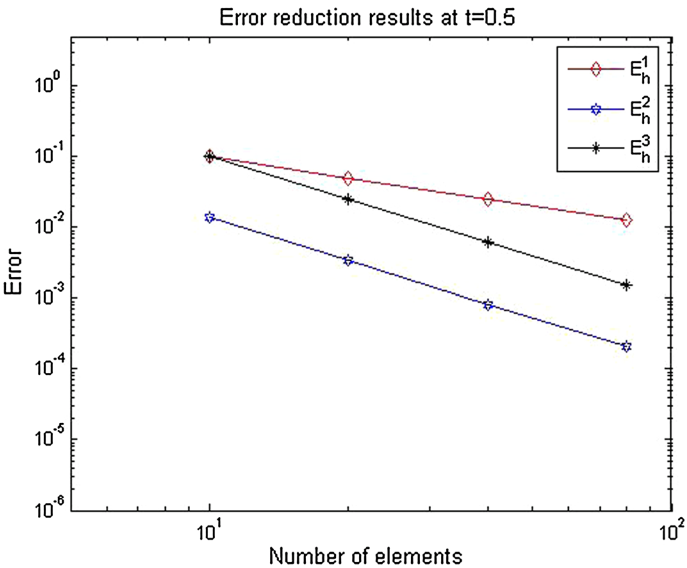

Fig. 2 Error reduction results at $t = 0.5$

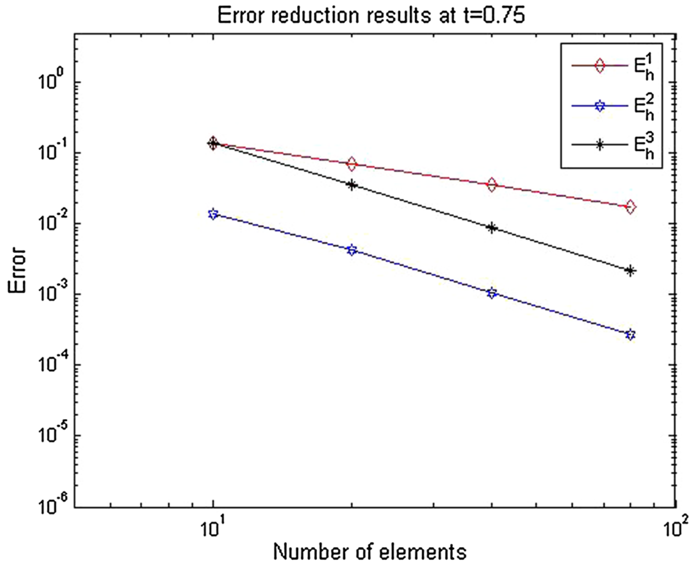

Fig. 3 Error reduction results at $t = 0.75$

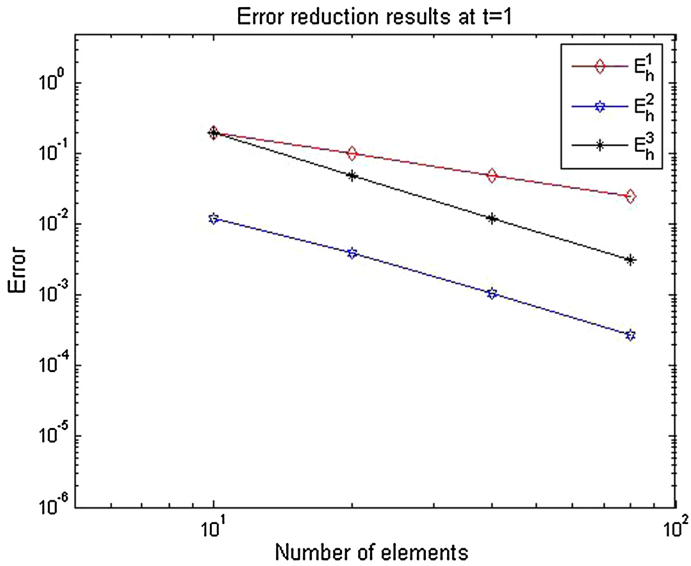

Fig. 4 Error reduction results at $t = 1.0$

Similar to Example 1, we can see from Tables 6, 7, 8, 9 and 10 and Figs. 5, 6, 7 and 8 that all these results are in good agreement with our theoretical analysis.

Table 6 Numerical results at $t = 0.25$ with $\tau = h$

$M \times M$	$\|u^n - U_h^n\|_1$	Order	$\|U_h^n - I_h u^n\|_1$	Order	$\|u^n - I_{2h}^2 U_h^n\|_1$	Order
10×10	7.2234×10^{-3}	–	8.7373×10^{-4}	–	1.0448×10^{-2}	–
20×20	3.6163×10^{-3}	0.9982	2.6497×10^{-4}	1.7214	2.9768×10^{-3}	1.8114
40×40	1.8085×10^{-3}	0.9997	4.8180×10^{-5}	2.4593	7.7611×10^{-4}	1.9394
80×80	9.0432×10^{-4}	0.9999	9.8289×10^{-6}	2.2933	1.9729×10^{-4}	1.9760

Table 7 Numerical results at $t = 0.5$ with $\tau = h$

$M \times M$	$\|u^n - U_h^n\|_1$	Order	$\|U_h^n - I_h u^n\|_1$	Order	$\|u^n - I_{2h}^2 U_h^n\|_1$	Order
10×10	9.2642×10^{-3}	–	1.1238×10^{-3}	–	1.3491×10^{-2}	–
20×20	4.6418×10^{-3}	0.9970	2.8240×10^{-4}	1.9925	3.8251×10^{-3}	1.8184
40×40	2.3220×10^{-3}	0.9993	5.9749×10^{-5}	2.2408	9.9645×10^{-4}	1.9406
80×80	1.1611×10^{-3}	0.9998	1.5509×10^{-5}	1.9458	2.5310×10^{-4}	1.9771

Table 8 Numerical results at $t = 0.75$ with $\tau = h$

$M \times M$	$\|u^n - U_h^n\|_1$	Order	$\|U_h^n - I_h u^n\|_1$	Order	$\|u^n - I_{2h}^2 U_h^n\|_1$	Order
10×10	1.1885×10^{-2}	–	1.5383×10^{-3}	–	1.7244×10^{-2}	–
20×20	5.9590×10^{-3}	0.9960	3.9513×10^{-4}	1.9610	4.9064×10^{-3}	1.8133
40×40	2.9814×10^{-3}	0.9991	6.8124×10^{-5}	2.5361	1.2800×10^{-3}	1.9385
80×80	1.4909×10^{-3}	0.9998	2.2895×10^{-5}	1.5731	3.2462×10^{-4}	1.9793

Table 9 Numerical results at $t = 1.0$ with $\tau = h$

$M \times M$	$\|u^n - U_h^n\|_1$	Order	$\|U_h^n - I_h u^n\|_1$	Order	$\|u^n - I_{2h}^2 U_h^n\|_1$	Order
10×10	1.5255×10^{-2}	–	1.5360×10^{-3}	–	2.2180×10^{-2}	–
20×20	7.6504×10^{-3}	0.9957	3.7801×10^{-4}	2.0227	6.3027×10^{-3}	1.8152
40×40	3.8281×10^{-3}	0.9989	1.0225×10^{-4}	1.8863	1.6411×10^{-3}	1.9414
80×80	1.9144×10^{-3}	0.9997	2.9618×10^{-5}	1.7876	4.1648×10^{-4}	1.9783

Table 10 Convergence results of $\|u^n - U_h^n\|_1$ with $h = \frac{1}{160}$ and $\tau = kh$

t	$k = 1$	$k = 5$	$k = 10$	$k = 20$	$k = 40$
0.25	9.04315×10^{-4}	9.04401×10^{-4}	9.05601×10^{-4}	9.14301×10^{-4}	1.29022×10^{-3}
0.50	1.16114×10^{-3}	1.16126×10^{-3}	1.16265×10^{-3}	1.17703×10^{-3}	1.21668×10^{-3}
0.75	1.49092×10^{-3}	1.49098×10^{-3}	1.49184×10^{-3}	1.51410×10^{-3}	1.91354×10^{-3}
1.00	1.91436×10^{-3}	1.91443×10^{-3}	1.91539×10^{-3}	1.94401×10^{-3}	2.11468×10^{-3}

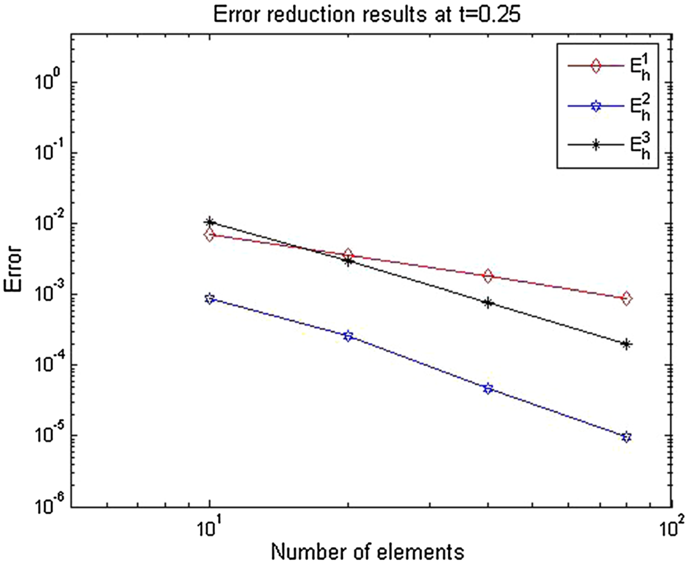

Fig. 5 Error reduction results at $t = 0.25$

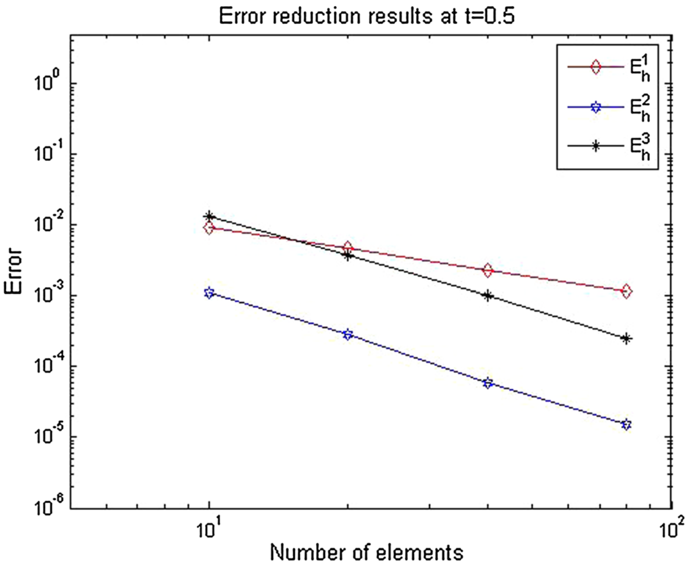

Fig. 6 Error reduction results at $t = 0.5$

Example 3 We consider the example [35] describing the dynamics of Bose-Einstein Condensate at extremely low temperature, reads

$$\begin{cases} i\frac{\partial u(x,y,t)}{\partial t} = -\frac{1}{2}\triangle u + V(x,y)u + |u|^2 u, (x,y) \in [0,2\pi] \times [0,2\pi], \\ u_0(x,y) = \sin x \sin y, \end{cases}$$

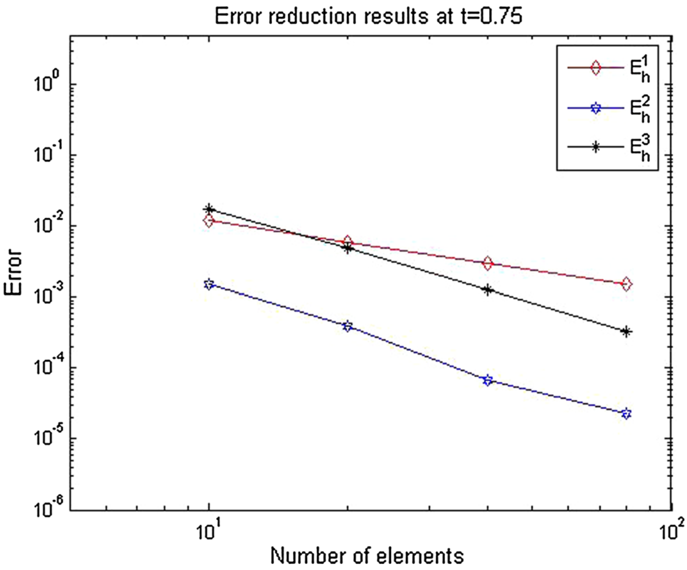

Fig. 7 Error reduction results at $t = 0.75$

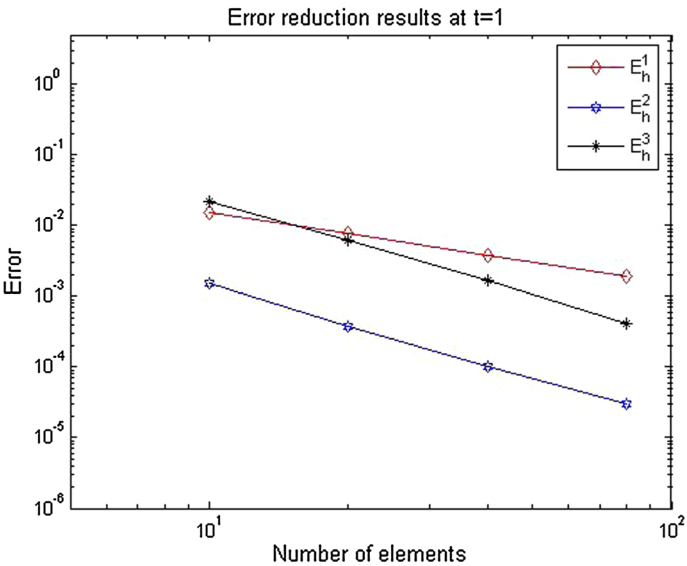

Fig. 8 Error reduction results at $t = 1.0$

where $V(x, y) = 1 - \sin^2 x \sin^2 y$ and $\Omega = [0, 1] \times [0, 1]$. The exact solution for the problem is $u = e^{-2ti} \sin x \sin y$. A uniform rectangular partition with $M + 1$ nodes in each direction is used in our computation.

Tables 11, 12 and 13 and Figs. 9 and 10 confirm our theoretical analysis.

Table 11 Numerical results at $t = 0.5$ with $\tau = h$

$M \times M$	$\|u^n - U_h^n\|_1$	Order	$\|U_h^n - I_h u^n\|_1$	Order	$\|u^n - I_{2h}^2 U_h^n\|_1$	Order
10×10	0.8429	–	0.0851	–	1.4949	–
20×20	0.4078	1.0473	0.0220	1.9537	0.3891	1.9419
40×40	0.2021	1.0131	0.0055	1.9886	0.0982	1.9859
80×80	0.1008	1.0033	0.0014	1.9972	0.0246	1.9965

Table 12 Numerical results at $t = 1.0$ with $\tau = h$

$M \times M$	$\|u^n - U_h^n\|_1$	Order	$\|U_h^n - I_h u^n\|_1$	Order	$\|u^n - I_{2h}^2 U_h^n\|_1$	Order
10×10	0.8562	–	0.1702	–	1.5027	–
20×20	0.4096	1.0636	0.0439	1.9537	0.3910	1.9424
40×40	0.2023	1.0177	0.0111	1.9886	0.0987	1.9860
80×80	0.1008	1.0046	0.0028	1.9972	0.0247	1.9965

Table 13 Convergence results of $\|u^n - U_h^n\|_1$ with $h = \frac{\pi}{80}$ and $\tau = kh$

t	$k = \frac{1}{\pi}$	$k = \frac{5}{\pi}$	$k = \frac{10}{\pi}$	$k = \frac{20}{\pi}$
0.25	0.1008017	0.1008401	0.1016795	0.1143344
0.50	0.1008063	0.1009596	0.1042730	0.1476723
0.75	0.1008140	0.1011586	0.1084578	0.1906936
1.00	0.1008248	0.1014366	0.1140587	0.2382053

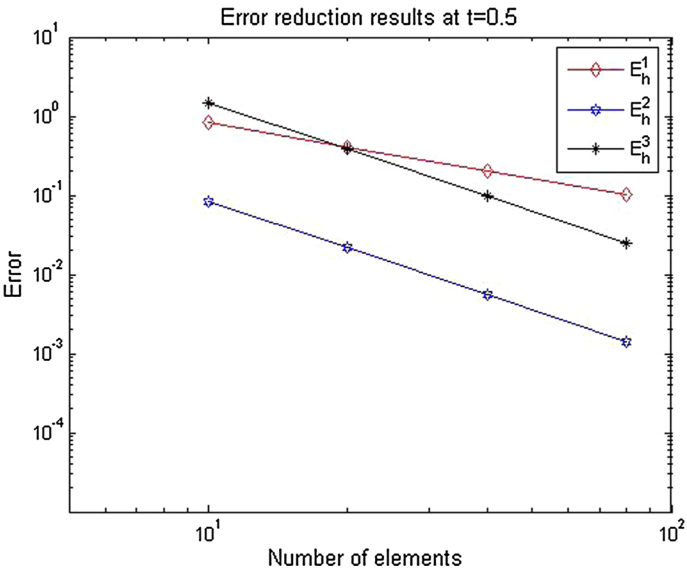

Fig. 9 Error reduction results at $t = 0.5$

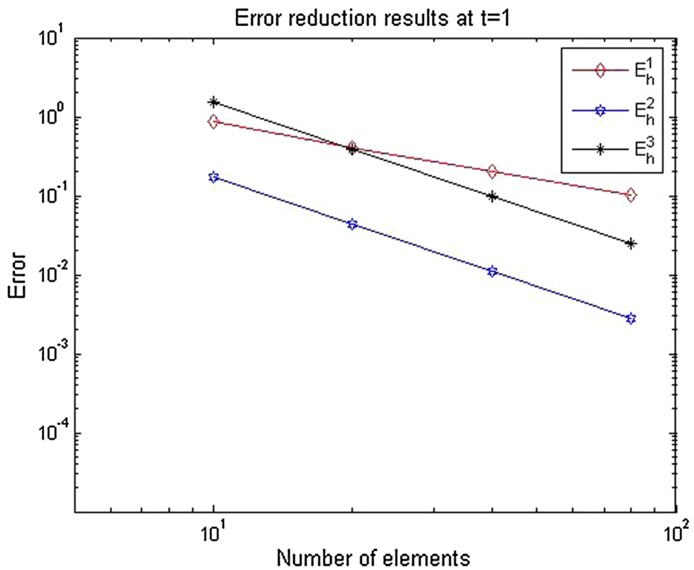

Fig. 10 Error reduction results at $t = 1.0$

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