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Convergence Analysis of Two Numerical Schemes Applied to a Nonlinear Elliptic Problem

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Abstract For a given nonlinear problem discretized by standard finite elements, we propose two iterative schemes to solve the discrete problem. We prove the well-posedness of the corresponding problems and their convergence. Next, we construct error indicators and prove optimal a posteriori estimates where we treat separately the discretization and linearization errors. Some numerical experiments confirm the validity of the schemes and allow us to compare them.

Keywords Posteriori error estimation · Nonlinear problems · Iterative methods · Finite elements method

1 Introduction

Let Ω be an open polygon in $\mathbb{R}^2$, we consider the problem

$$-\Delta u + \lambda |u|^{2p} u = f \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

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where λ and p are two positive real numbers. The right-hand side f belongs to the dual space $H^{-1}(\Omega)$ of the Sobolev space $H_0^1(\Omega)$. The *a posteriori* error analysis of finite element approximations of the present model problem has been studied by Bernardi et al. [2]. In fact, let $V_h \subset H_0^1(\Omega)$ be the $\mathcal{P}_1$ finite element space associated with a regular family of triangulations of Ω, denoted by $\mathcal{T}_h$. Using $\mathcal{P}_1$ Lagrange finite elements, the discrete variational problem obtained by the Galerkin method amounts to (from now on, we denote by $(\cdot,\cdot)$ the scalar product of $L^2(\Omega)$).

Find $u_h \in V_h$ such that

$$\forall v_h \in V_h, \qquad \big(\nabla u_h, \nabla v_h\big) + \lambda\big(|u_h|^{2p} u_h, v_h\big) = \langle f, v_h\rangle. \tag{1.3}$$

In order to solve the discrete nonlinear problem (1.3), we introduced in [2] the following linear numerical scheme, called fixed-point algorithm:

Find $u_h^{i+1} \in V_h$ such that

$$\forall v_h \in V_h, \qquad \big(\nabla u_h^{i+1}, \nabla v_h\big) + \lambda\big(|u_h^i|^{2p} u_h^{i+1}, v_h\big) = \langle f, v_h\rangle. \tag{1.4}$$

This algorithm leads to a conditional convergence of the problem. In fact, the convergence of this numerical schemes depends on the parameters λ, p and f. Furthermore, the *a priori* estimate of the discrete variationel problem is presented in [2]. As well, the *a posteriori* analysis of the discretization is performed but requires that the discrete solution belongs to a neighborhood of the exact solution u.

As a new contribution to the previous work that we have carried out recently on the *a posteriori* analysis of the present nonlinear problem, see [2], we introduce in this paper two different convergent numerical schemes to solve this problem. In fact, the main idea is to introduce a parameter α which can be controlled in order to insure the convergence. Let u_h^0 be an initial guess, for $i \geq 0$ we introduce the following two algorithms:

First numerical scheme.
Find $u_h^{i+1} \in V_h$ such that

$$\forall v_h \in V_h, \qquad \alpha\big(u_h^{i+1} - u_h^i, v_h\big) + \big(\nabla u_h^{i+1}, \nabla v_h\big) + \lambda\big(|u_h^i|^{2p} u_h^{i+1}, v_h\big) = \langle f, v_h\rangle, \tag{1.5}$$

Second numerical scheme.
Find $u_h^{i+1} \in V_h$ such that

$$\forall v_h \in V_h, \qquad \alpha\big(\nabla u_h^{i+1} - \nabla u_h^i, \nabla v_h\big) + \big(\nabla u_h^{i+1}, \nabla v_h\big) + \lambda\big(|u_h^i|^{2p} u_h^{i+1}, v_h\big) = \langle f, v_h\rangle, \tag{1.6}$$

For a parameter α bigger than a specific constant that depends on λ, p and the data f, problem (1.5) and (1.6) always converge. Moreover, our objective is to derive an *a posteriori* error estimate distinguishing linearization and discretization errors.

In practice, the present problem (1.1)–(1.2) is solved using an iterative method involving a linearization process and approximated by the finite element method. Thus, two sources of error appear, namely linearization and discretization. The main result in [2] is a two-sided bound of the error distinguishing linearization and discretization errors in the context of an adaptive procedure. This type of analysis was introduced by Chaillou and Suri [3,4] for a general class of problems characterized by strongly monotone operators and developed by El Alaoui et al. [5] for a class of second-order monotone quasi-linear diffusion-type problems approximated by piecewise affine, continuous finite elements. We wish to extend these results to the problem that we consider and prove optimal estimates.

In the following, we summarize the differences between the scheme (1.4) (studied in [2]) and the schemes (1.5) and (1.6) studied in this work:

(1) (1.4) converges when the data $(\mathbf{f}, \lambda, p)$ verifies a condition called small condition (see [2], Theorem 4.1). But the two schemes presented in this paper [(1.5) and (1.6)] introduce a parameter α which can be calibrated to obtain the convergence for any data. Numerical simulations of comparison are listed in Sect. 5 and specially in Tables 1 and 2 to show the comparison between them.
(2) In [2] we show the *a posteriori* error corresponding to the fixed point scheme when the discrete iterative solution $\mathbf{u}_h^{i+1}$ is in a neighborhood of the exact solution $\mathbf{u}$. With the two schemes presented in this work, we derive *a posteriori* error estimates for any iterative solution u_h^{i+1} without the neighborhood constraints.

The paper is organized as follows:

- Section 2 describes the model problem.
- Section 3 is devoted to the study of the convergence of the schemes.
- Section 4 provides the *a posteriori* estimates for both problems.
- Section 5 is devoted to the numerical results.

2 Preliminaries

In this section, we describe the variational formulation associated with the nonlinear problem (1.1)–(1.2) and introduce and recall some corresponding properties which will be used later. We denote by $L^p(\Omega)$ the space of measurable functions summable with power p, and for all $v \in L^p(\Omega)$, the corresponding norm is defined by

$$\| v \|_{L^p(\Omega)} = \left(\int_\Omega |v(\mathbf{x})|^p d\mathbf{x} \right)^{1/p}.$$

In the case $p = 2$, we also denote this norm by $\|v\|_{0,\Omega} = \|v\|_{L^2(\Omega)}$. Throughout this paper, we constantly use the classical Sobolev space

$$H^1(\Omega) = \left\{ v \in L^2(\Omega);\ \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2} \in L^2(\Omega) \right\},$$

which is equipped respectively with the semi-norm and norm

$$|v|_{1,\Omega} = \left(\int_\Omega \left(\left| \frac{\partial v}{\partial x_1} \right|^2 + \left| \frac{\partial v}{\partial x_2} \right|^2 \right) d\mathbf{x} \right)^{1/2} \quad \text{and} \quad \| v \|_{1,\Omega} = \left(||v||_{0,\Omega}^2 + |v|_{1,\Omega}^2) \right)^{1/2}.$$

In particular, we consider the following space

$$H_0^1(\Omega) = \{ v \in H^1(\Omega),\ v_{|\partial\Omega} = 0 \},$$

and its dual space $H^{-1}(\Omega)$. We recall the Sobolev imbeddings (see Adams [1], Chapter 3).

Lemma 2.1 *For any bounded domain Ω in $\mathbb{R}^2$, for all j, $1 \le j < \infty$, there exists a positive constant S_j such that*

$$\forall v \in H_0^1(\Omega), \quad \| v \|_{L^j(\Omega)} \le S_j |v|_{1,\Omega}. \tag{2.1}$$

Remark 2.2 For domains Ω in $\mathbb{R}^3$, inequality (2.1) with standard definition of $H_0^1(\Omega)$ remains valid only for $j \leq 6$, whence the interest of working in dimension $d = 2$.
Setting $X = H_0^1(\Omega)$, the model problem (1.1)–(1.2) admits the equivalent variational formulation:
Find $u \in X$ such that

$$\forall v \in X, \qquad \int_\Omega \nabla u \nabla v d\mathbf{x} + \int_\Omega \lambda |u|^{2p} u v d\mathbf{x} = \langle f, v \rangle, \tag{2.2}$$

Theorem 2.3 [2] *Problem (2.2) admits a unique solution $u \in X$.*

We now introduce the following technical lemmas:

Lemma 2.4 *Let a, b and $p \geq 1$ be three real numbers. We have the following relation*

$$\big||a|^p - |b|^p\big| \leq p|a-b|\big(|a|^{p-1} + |b|^{p-1}\big).$$

Proof The result follows from applying the mean value theorem to $f(x) = x^p$ with $x \geq 0$. □

Remark 2.5 For a real positive $p < 1$ and for any real numbers a and b, the last lemma can be written as follow

$$\big||a|^p - |b|^p\big| \leq (p+1)|a-b|\big(|a|^{p-1} + |b|^{p-1}\big).$$

Lemma 2.6 *For all $x, y \in \mathbb{R}$ and $p \in \mathbb{R}^+$, we have*

$$(|x|^{2p}x - |y|^{2p}y)(x-y) \geq 0.$$

Remark 2.7 In the sequel, we denote by $C, C', \ldots$ generic constants that can vary from line to line but are always independent of all discretization parameters.

3 Finite Element Discretization and Convergence

In this section, we begin to collect some useful notation concerning the discrete setting and the *a priori* estimate. Then, we show the convergence of the schemes (1.5) and (1.6).
Let $(\mathcal{T}_h)_h$ be a regular family of triangulations of Ω, in the sense that, for each h:

- The union of all elements of $\mathcal{T}_h$ is equal to $\overline{\Omega}$.
- The intersection of two different elements of $\mathcal{T}_h$, if not empty, is a vertex or a whole edge of both triangles.
- The ratio of the diameter h_K of any element K of $\mathcal{T}_h$ to the diameter of its inscribed circle δ_K is smaller than a constant independent of h.
  As usual, h stands for the maximum of the diameters h_K of the element $K \in \mathcal{T}_h$. Let $V_h \subset H_0^1(\Omega)$ be the Lagrange $\mathcal{P}_\ell$ finite element space associated with $\mathcal{T}_h$, more precisely

$$V_h = \Big\{ v_h \in H_0^1(\Omega);\ \forall K \in \mathcal{T}_h,\ v_{h_{|K}} \in \mathcal{P}_\ell(K) \Big\},$$

  where $\mathcal{P}_\ell(K)$ stands for the space of restrictions to K of polynomial functions of degree $\leq \ell$ on $\mathbb{R}^2$.

Remark 3.1 (Inverse inequality) There exists a constant $S_I > 0$ such that for all $v_h \in V_h$ and $K \in \mathcal{T}_h$, we have

$$|v_h|_{1,K} \leq S_I h_K^{-1} ||v_h||_{0,K}. \tag{3.1}$$

Theorem 3.2 [2] *Let u be the solution of (2.2). Then, Problem (1.3) has a unique solution u_h. Moreover, if $u \in H^2(\Omega)$, the following estimate holds*

$$\| u_h - u \|_{1,\Omega} \leq Ch \| u \|_{2,\Omega} .$$

In the following, we investigate the convergence of the schemes (1.5) and (1.6).

Theorem 3.3 *Problem (1.5) admits a unique solution. Furthermore, if the initial value u_h^0 satisfies the condition*

$$||u_h^0||_{0,\Omega} \leq S_2 ||f||_{-1,\Omega}, \tag{3.2}$$

then the solution of the problem (1.5) satisfies the estimates

$$||u_h^{i+1}||_{0,\Omega} \leq S_2 ||f||_{-1,\Omega} \quad \textit{and} \quad |u_h^{i+1}|_{1,\Omega} \leq \sqrt{1+\alpha S_2^2} ||f||_{-1,\Omega}. \tag{3.3}$$

Proof It is readily checked that problem (1.5) has a unique solution as a consequence of the coercivity of the bilinear form.
We consider the Eq. (1.5) with $v_h = u_h^{i+1}$ and we obtain:

$$\frac{\alpha}{2}||u_h^{i+1}||_{0,\Omega}^2 - \frac{\alpha}{2}||u_h^i||_{0,\Omega}^2 + \frac{\alpha}{2}||u_h^{i+1} - u_h^i||_{0,\Omega}^2 + |u_h^{i+1}|_{1,\Omega}^2$$
$$+\lambda(|u_h^i|^{2p}\, u_h^{i+1}, u_h^{i+1}) = (f, u_h^{i+1}).$$

By using the inequality

$$(f, u_h^{i+1}) \leq \frac{1}{2}||f||_{-1,\Omega}^2 + \frac{1}{2}|u_h^{i+1}|_{1,\Omega}^2,$$

we deduce the relation

$$\frac{\alpha}{2}||u_h^{i+1}||_{0,\Omega}^2 - \frac{\alpha}{2}||u_h^i||_{0,\Omega}^2 + \frac{\alpha}{2}||u_h^{i+1} - u_h^i||_{0,\Omega}^2$$
$$+\frac{1}{2}|u_h^{i+1}|_{1,\Omega}^2 + \lambda(|u_h^i|^{2p}\, u_h^{i+1}, u_h^{i+1}) \leq \frac{1}{2}||f||_{-1,\Omega}^2. \tag{3.4}$$

We now prove the first estimate in (3.3) by induction on i. Starting with the relation (3.2), we suppose that we have

$$||u_h^i||_{0,\Omega} \leq S_2 ||f||_{-1,\Omega}.$$

We are in one of the following two situations:

- We have $||u_h^{i+1}||_{0,\Omega} \leq ||u_h^i||_{0,\Omega}$. We obviously deduce the bound

  $$||u_h^{i+1}||_{0,\Omega} \leq S_2 ||f||_{-1,\Omega}$$

  from the induction hypothesis.
- We have $||u_h^{i+1}||_{0,\Omega} \geq ||u_h^i||_{0,\Omega}$. The Eq. (3.4) gives

  $$|u_h^{i+1}|_{1,\Omega}^2 \leq ||f||_{-1,\Omega}^2$$

  and we deduce the inequality

  $$||u_h^{i+1}||_{0,\Omega}^2 \leq S_2^2 |u_h^{i+1}|_{1,\Omega}^2$$
  $$\leq S_2^2 ||f||_{-1,\Omega}^2.$$

This gives the first part of (3.3). We now check the second part. We have from (3.4)

$$|u_h^{i+1}|_{1,\Omega}^2 \le ||f||_{-1,\Omega}^2 + \alpha ||u_h^i||_{0,\Omega}^2 \le (1+\alpha S_2^2)||f||_{-1,\Omega}^2,$$

whence the desired result.

Theorem 3.4 *Problem (1.6) admits a unique solution. Furthermore, if the initial value u_h^0 verifies the condition*

$$|u_h^0|_{1,\Omega} \le ||f||_{-1,\Omega}, \tag{3.5}$$

then the solution of Problem (1.6) satisfies the estimate

$$|u_h^{i+1}|_{1,\Omega} \le ||f||_{-1,\Omega}. \tag{3.6}$$

Proof We follow the same proof as for Theorem 3.3. It is readily checked that problem (1.6) has a unique solution as a consequence of the coercivity of the bilinear form.
We consider the Eq. (1.6) with $v_h = u_h^{i+1}$ and we obtain:

$$\frac{\alpha}{2}|u_h^{i+1}|_{1,\Omega}^2 - \frac{\alpha}{2}|u_h^i|_{1,\Omega}^2 + \frac{\alpha}{2}|u_h^{i+1} - u_h^i|_{1,\Omega}^2 + |u_h^{i+1}|_{1,\Omega}^2 + \lambda(|u_h^i|^{2p}\, u_h^{i+1}, u_h^{i+1}) = (f, u_h^{i+1}).$$

We deduce the relation

$$\begin{aligned} &\frac{\alpha}{2}|u_h^{i+1}|_{1,\Omega}^2 - \frac{\alpha}{2}|u_h^i|_{1,\Omega}^2 + \frac{\alpha}{2}|u_h^{i+1} - u_h^i|_{1,\Omega}^2 \\ &+\frac{1}{2}|u_h^{i+1}|_{1,\Omega}^2 + \lambda(|u_h^i|^{2p}\, u_h^{i+1}, u_h^{i+1}) \le \frac{1}{2}||f||_{-1,\Omega}^2. \end{aligned} \tag{3.7}$$

We prove the relation (3.6) recursively. Starting with (3.5), we suppose that we have

$$|u_h^i|_{1,\Omega} \le ||f||_{-1,\Omega}.$$

We are in one of the following two situations:

- We have $|u_h^{i+1}|_{1,\Omega} \le |u_h^i|_{1,\Omega}$. We deduce the bound

$$|u_h^{i+1}|_{1,\Omega} \le ||f||_{-1,\Omega}.$$

- We have $|u_h^{i+1}|_{1,\Omega} \ge |u_h^i|_{1,\Omega}$. It follows from (3.7) that

$$|u_h^{i+1}|_{1,\Omega}^2 \le ||f||_{-1,\Omega}^2.$$

We conclude the proof of the theorem.

Unfortunately the proof of the next result is much more technical.

Theorem 3.5 *Assume that there exists $\beta > 0$ such that, for every element $K \in \mathcal{T}_h$, we have*

$$h_K \ge \beta h,$$

(which means that the family of triangulations is uniformly regular). Under the assumptions of Theorem 3.3 and for

$$\alpha > C^2 p^2 \lambda^2 h^{-4p} \tag{3.8}$$

where

$$C = 4 S_4 S_8 S_{8(2p-1)}^{2p-1} \frac{S_I^{2p}}{\beta^{2p}} S_2^{2p} ||f||_{-1,\Omega}^{2p},$$

the sequence of solutions (u_h^i) of Problem (1.5) converges in $H_0^1(\Omega)$ to the solution u_h of Problem (1.3).

Proof We take the difference between the Eqs. (1.5) and (1.3) with $v_h = u_h^{i+1} - u_h$ and we obtain the equation

$$\frac{\alpha}{2}||u_h^{i+1} - u_h||_{0,\Omega}^2 - \frac{\alpha}{2}||u_h^i - u_h||_{0,\Omega}^2 + \frac{\alpha}{2}||u_h^{i+1} - u_h^i||_{0,\Omega}^2 + |u_h^{i+1} - u_h|_{1,\Omega}^2 \\ +\lambda(|u_h^i|^{2p}u_h^{i+1} - |u_h|^{2p}u_h, u_h^{i+1} - u_h) = 0.$$

The last term in the previous equation, denoted by T, can be decomposed as

$$T = \lambda((|u_h^i|^{2p} - |u_h^{i+1}|^{2p})u_h^{i+1}, u_h^{i+1} - u_h) + \lambda(|u_h^{i+1}|^{2p}u_h^{i+1} - |u_h|^{2p}u_h, u_h^{i+1} - u_h).$$

We denote by T_1 and T_2, respectively, the first and the second terms in the right-hand side of the last equation. Using Lemma 2.6, we have $T_2 \geq 0$. Then we derive by using Lemma 2.4 (with p replaced by $2p$)

$$\begin{aligned}
&\frac{\alpha}{2}||u_h^{i+1} - u_h||_{0,\Omega}^2 - \frac{\alpha}{2}||u_h^i - u_h||_{0,\Omega}^2 + \frac{\alpha}{2}||u_h^{i+1} - u_h^i||_{0,\Omega}^2 + |u_h^{i+1} - u_h|_{1,\Omega}^2 + T_2 = -T_1 \\
&\leq 2p\lambda \int_\Omega (|u_h^{i+1}|^{2p-1} + |u_h^i|^{2p-1})|u_h^{i+1} - u_h^i|\,|u_h^{i+1}|\,|u_h^{i+1} - u_h| d\mathbf{x} \\
&\leq 2p\lambda||u_h^{i+1} - u_h^i||_{0,\Omega}\,|||u_h^{i+1}|^{2p-1} + |u_h^i|^{2p-1}||_{L^8(\Omega)}\,||u_h^{i+1}||_{L^8(\Omega)}\,||u_h^{i+1} - u_h||_{L^4(\Omega)} \\
&\leq 2p\lambda S_4 S_8 S_{8(2p-1)}^{2p-1}(|u_h^{i+1}|_{1,\Omega}^{2p-1} + |u_h^i|_{1,\Omega}^{2p-1})|u_h^{i+1}|_{1,\Omega}||u_h^{i+1} - u_h^i||_{0,\Omega}|u_h^{i+1} - u_h|_{1,\Omega} \\
&\leq 2p\lambda S_4 S_8 S_{8(2p-1)}^{2p-1}\frac{S_I^{2p}}{\beta^{2p}}h^{-2p}||u_h^{i+1}||_{0,\Omega}^{2p-1} + ||u_h^i||_{0,\Omega}^{2p-1}||u_h^{i+1}||_{0,\Omega}||u_h^{i+1} - u_h^i||_{0,\Omega}|u_h^{i+1} \\
&\quad -u_h|_{1,\Omega} \\
&\leq 4p\lambda S_4 S_8 S_{8(2p-1)}^{2p-1}\frac{S_I^{2p}}{\beta^{2p}}S_2^{2p}h^{-2p}||f||_{-1,\Omega}^{2p}||u_h^{i+1} - u_h^i||_{0,\Omega}|u_h^{i+1} - u_h|_{1,\Omega}
\end{aligned}$$

We denote by $C = 4S_4S_8S_{8(2p-1)}^{2p-1}\frac{S_I^{2p}}{\beta^{2p}}S_2^{2p}||f||_{-1,\Omega}^{2p}$ and we use the Cauchy's inequality $ab \leq \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$ (with $\varepsilon = \frac{1}{Cp\lambda h^{-2p}}$) to obtain the following bound

$$\frac{\alpha}{2}||u_h^{i+1} - u_h||_{0,\Omega}^2 - \frac{\alpha}{2}||u_h^i - u_h||_{0,\Omega}^2 + \frac{\alpha}{2}||u_h^{i+1} \\ -u_h^i||_{0,\Omega}^2 + \frac{1}{2}|u_h^{i+1} - u_h|_{1,\Omega}^2 + T_2 \leq \frac{C^2p^2\lambda^2}{2}h^{-4p}||u_h^{i+1} - u_h^i||_{0,\Omega}^2.$$

We choice $\alpha > C^2p^2\lambda^2h^{-4p}$, denote by $C_1 = \frac{\alpha - C^2p^2\lambda^2h^{-4p}}{2}$ and obtain

$$\frac{\alpha}{2}||u_h^{i+1} - u_h||_{0,\Omega}^2 - \frac{\alpha}{2}||u_h^i - u_h||_{0,\Omega}^2 + C_1||u_h^{i+1} - u_h^i||_{0,\Omega}^2 + \frac{1}{2}|u_h^{i+1} - u_h|_{1,\Omega}^2 + T_2 \leq 0. \tag{3.9}$$

We deduce that, for all $i \geq 1$, we have (if $||u_h^i - u_h||_{0,\Omega} \neq 0$)

$$||u_h^{i+1} - u_h||_{0,\Omega} < ||u_h^i - u_h||_{0,\Omega},$$

and we deduce the convergence of the sequence $(u_h^{i+1} - u_h)$ in $L^2(\Omega)$ and the the convergence of the sequence u_h^i in $L^2(\Omega)$. By taking the limit of (3.9) we get

$$\lim_{i\to+\infty}\left(\frac{1}{2}|u_h^{i+1} - u_h|_{1,\Omega}^2 + T_2\right) \leq 0.$$

As $T_2 \geq 0$, we deduce that $|u_h^{i+1} - u_h|_{1,\Omega}$ converges to 0 and u_h^{i+1} converges to u_h in $H_0^1(\Omega)$.

Theorem 3.6 *Under the assumptions of Theorem 3.4 and for*

$$\alpha > \left(4S_2S_4S_8S_{8(2p-1)}^{2p-1}||f||_{-1,\Omega}^{2p}\right)^2 p^2\lambda^2, \tag{3.10}$$

the sequence of solutions (u_h^i) *of Problem (1.6) converges in* $H_0^1(\Omega)$ *to the solution* u_h *of Problem (1.3).*

Proof We take the difference between the Eqs. (1.6) and (1.3) with $v_h = u_h^{i+1} - u_h$ and we obtain the equation

$$\frac{\alpha}{2}|u_h^{i+1} - u_h|_{1,\Omega}^2 - \frac{\alpha}{2}|u_h^i - u_h|_{1,\Omega}^2 + \frac{\alpha}{2}|u_h^{i+1} - u_h^i|_{1,\Omega}^2 + |u_h^{i+1} - u_h|_{1,\Omega}^2$$
$$+\lambda(|u_h^i|^{2p}u_h^{i+1} - |u_h|^{2p}u_h, u_h^{i+1} - u_h) = 0.$$

The last term in the previous equation, denoted by T, can be decomposed as

$$T = \lambda((|u_h^i|^{2p} - |u_h^{i+1}|^{2p})u_h^{i+1}, u_h^{i+1} - u_h) + \lambda(|u_h^{i+1}|^{2p}u_h^{i+1} - |u_h|^{2p}u_h, u_h^{i+1} - u_h).$$

We denote by T_1 and T_2 respectively the first and the second terms in the right-hand side of the last equation. Using Lemma 2.6, we have $T_2 \geq 0$. Then we have by using Lemma 2.4

$$\frac{\alpha}{2}|u_h^{i+1} - u_h|_{1,\Omega}^2 - \frac{\alpha}{2}|u_h^i - u_h|_{1,\Omega}^2 + \frac{\alpha}{2}|u_h^{i+1} - u_h^i|_{1,\Omega}^2 + |u_h^{i+1} - u_h|_{1,\Omega}^2 + T_2 = -T_1$$
$$\leq 2p\lambda\big((|u_h^{i+1}|^{2p-1} + |u_h^i|^{2p-1}), |u_h^{i+1} - u_h^i|\,|u_h^{i+1}|\,|u_h^{i+1} - u_h|\big)$$
$$\leq 2p\lambda||u_h^{i+1} - u_h^i||_{0,\Omega}\,||(|u_h^{i+1}|^{2p-1} + |u_h^i|^{2p-1})||_{L^8(\Omega)}\,||u_h^{i+1}||_{L^8(\Omega)}\,||u_h^{i+1} - u_h||_{L^4(\Omega)}$$
$$\leq 4p\lambda S_2S_4S_8S_{8(2p-1)}^{2p-1}||f||_{-1,\Omega}^{2p}|u_h^{i+1} - u_h^i|_{1,\Omega}|u_h^{i+1} - u_h|_{1,\Omega}.$$

We denote by $C = 4S_2S_4S_8S_{8(2p-1)}^{2p-1}||f||_{-1,\Omega}^{2p}$ and we use the Cauchy's inequality $ab \leq \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$ (with $\varepsilon = \frac{1}{Cp\lambda}$) to obtain the following bound

$$\frac{\alpha}{2}|u_h^{i+1} - u_h|_{1,\Omega}^2 - \frac{\alpha}{2}|u_h^i - u_h|_{1,\Omega}^2 + \frac{\alpha}{2}|u_h^{i+1} - u_h^i|_{1,\Omega}^2$$
$$+\frac{1}{2}|u_h^{i+1} - u_h|_{1,\Omega}^2 + T_2 \leq \frac{C^2p^2\lambda^2}{2}|u_h^{i+1} - u_h^i|_{1,\Omega}^2.$$

We choose $\alpha > C^2p^2\lambda^2$, denote by $C_1 = \frac{\alpha - C^2p^2\lambda^2}{2}$ and obtain

$$\frac{\alpha}{2}|u_h^{i+1} - u_h|_{1,\Omega}^2 - \frac{\alpha}{2}|u_h^i - u_h|_{1,\Omega}^2$$
$$+C_1|u_h^{i+1} - u_h^i|_{1,\Omega}^2 + \frac{1}{2}|u_h^{i+1} - u_h|_{1,\Omega}^2 + T_2 \leq 0. \tag{3.11}$$

We derive that, for all $i \geq 1$, we have

$$|u_h^{i+1} - u_h|_{1,\Omega} < |u_h^i - u_h|_{1,\Omega},$$

we obtain the convergence of the sequence $(u_h^{i+1} - u_h)$ in $H^1(\Omega)$ and then the convergence of u_h^i in $H^1(\Omega)$, by taking the limit of (3.11) we get

$$\lim_{i\to+\infty}\left(\frac{1}{2}|u_h^{i+1} - u_h|_{1,\Omega}^2 + T_2\right) \leq 0.$$

As $T_2 \geq 0$, we deduce that $|u_h^{i+1} - u_h|_{1,\Omega}$ converges to 0 and u_h^{i+1} converges to u_h in $H_0^1(\Omega)$. □

Remark 3.7 The conditions (3.2) and (3.5) suppose that the initial values of the algorithms are small related to the data f. We can always take $u_h^0 = 0$.

Remark 3.8 The previous two theorems bring to light a first difference between the two schemes (1.5) and (1.6): in opposite to (1.5), the convergence of (1.6) is proved when α is larger than a constant independent of h (and does not require the uniform regularity of the family of triangulations).

4 A Posteriori Error Analysis

We start this section by introducing some additional notation which is needed for constructing and analyzing the error indicators in the sequel.
For any triangle $K \in \mathcal{T}_h$ we denote by $\mathcal{E}(K)$ and $\mathcal{N}(K)$ the set of its edges and vertices, respectively, and we set

$$\mathcal{E}_h = \bigcup_{K\in\mathcal{T}_h} \mathcal{E}(K) \quad \text{and} \quad \mathcal{N}_h = \bigcup_{K\in\mathcal{T}_h} \mathcal{N}(K).$$

With any edge $e \in \mathcal{E}_h$ we associate a unit vector n such that n is orthogonal to e. We split $\mathcal{E}_h$ and $\mathcal{N}_h$ in the form

$$\mathcal{E}_h = \mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,\partial\Omega} \quad \text{and} \quad \mathcal{N}_h = \mathcal{N}_{h,\Omega} \cup \mathcal{N}_{h,\partial\Omega}$$

where $\mathcal{E}_{h,\partial\Omega}$ is the set of edges in $\mathcal{E}_h$ that lie on $\partial\Omega$ and $\mathcal{E}_{h,\Omega} = \mathcal{E}_h \backslash \mathcal{E}_{h,\partial\Omega}$. The same goes for $\mathcal{N}_{h,\partial\Omega}$.
Furthermore, for $K \in \mathcal{T}_h$ and $e \in \mathcal{E}_h$, let h_K and h_e be their diameter and length, respectively. An important tool in the construction of an upper bound for the total error is Clément's interpolation operator $\mathcal{R}_h$ with values in V_h. The operator $\mathcal{R}_h$ satisfies, for all $v \in H_0^1(\Omega)$, the following local approximation properties (see Verfürth [7], Chapter 1):

$$\| v - R_h v \|_{L^2(K)} \le C h_K |v|_{1,\Delta_K},$$
$$\| v - R_h v \|_{L^2(e)} \le C h_e^{1/2} |v|_{1,\Delta_e},$$

where Δ_K and Δ_e are the following sets:

$$\Delta_K = \bigcup \left\{ K' \in \mathcal{T}_h; \quad K' \cap K \neq \emptyset \right\} \quad \text{and} \quad \Delta_e = \bigcup \left\{ K' \in \mathcal{T}_h; \quad K' \cap e \neq \emptyset \right\}.$$

We now recall the following properties (see Verfürth [7], Chapter 1):

Proposition 4.1 *Let r be a positive integer. For all $v \in \mathcal{P}_r(K)$, the following properties hold*

$$C \| v \|_{L^2(K)} \le \| v\psi_K^{1/2} \|_{L^2(K)} \le \| v \|_{L^2(K)} \tag{4.1}$$
$$|v|_{1,K} \le C h_K^{-1} \| v \|_{L^2(K)} . \tag{4.2}$$

where ψ_K is the triangle-bubble function (equal to the product of the barycentric coordinates associated with the vertices of K).

We also introduce a lifting operator: For each $K \in \mathcal{T}_h$ and any edge e of K, $L_{e,K}$ maps polynomials of fixed degree on e vanishing on ∂e into polynomials on K vanishing on $\partial K \backslash e$ and is constructed by affine transformation from a fixed lifting operator on the reference triangle. For a positive integer r, we denote by $\mathcal{P}_r(e)$ the space of restrictions to e of polynomial functions of degree $\le r$ on $\mathbb{R}^2$.

Proposition 4.2 *Let r be a positive integer. For all $v \in \mathcal{P}_r(e)$, we have the following property*

$$C \parallel v \parallel_{L^2(e)} \leq \parallel v\psi_e^{1/2} \parallel_{L^2(e)} \leq \parallel v \parallel_{L^2(e)}, \tag{4.3}$$

where ψ_e is the bubble function on the edge e, and for all $v \in \mathcal{P}_r(e)$ vanishing on ∂e, we have

$$||L_{e,\kappa}v||_{L^2(\kappa)} + h_e|L_{e,\kappa}v|_{1,\kappa} \;\leq\; Ch_e^{1/2} \parallel v \parallel_{L^2(e)}, \tag{4.4}$$

where κ is a triangle of edge e.

Finally, we denote by $[v_h]$ the jump of v_h across the common edge e of two adjacent elements $K, K' \in \mathcal{T}_h$. We have now provided all prerequisites to establish an upper bound and lower bound for the total error. Let u_h^{i+1} and u be the solution of the iterative problem (1.5) or (1.6) and the continuous problem, respectively. They satisfy the identity

$$\int_\Omega \nabla(u_h^{i+1} - u)\nabla v d\mathbf{x} = \int_\Omega \nabla u_h^{i+1}\nabla v d\mathbf{x} + \lambda \int_\Omega |u|^{2p} u v d\mathbf{x} - \int_\Omega f v d\mathbf{x}. \tag{4.5}$$

We now start the *a posteriori* analysis of our algorithms.

4.1 Algorithm (1.5)

In order to prove an upper bound of the error, we introduce an approximation f_h of the data f which is constant on each element K of $\mathcal{T}_h$. We first write the residual equation

$$\begin{aligned}
&\int_\Omega \nabla u \nabla v d\mathbf{x} + \lambda \int_\Omega |u|^{2p} u v d\mathbf{x} - \int_\Omega \nabla u_h^{i+1} \nabla v d\mathbf{x} - \lambda \int_\Omega |u_h^i|^{2p} u_h^{i+1} v d\mathbf{x} \\
&\quad = \int_\Omega (f - f_h)(v - v_h) d\mathbf{x} + \sum_{K\in\mathcal{T}_h} \Big\{ \int_K (f_h + \Delta u_h^{i+1} - \lambda |u_h^i|^{2p} u_h^{i+1} \\
&\qquad - \alpha(u_h^{i+1} - u_h^i))(v - v_h) d\mathbf{x} \\
&\qquad - \frac{1}{2} \sum_{e\in\mathcal{E}(K)} \int_e [\frac{\partial u_h^{i+1}}{\partial n}](v - v_h) d\tau \Big\} + \alpha \sum_{K\in\mathcal{T}_h} \int_K (u_h^{i+1} - u_h^i) v d\mathbf{x}.
\end{aligned} \tag{4.6}$$

By adding and subtracting $\lambda \int_\Omega |u_h^{i+1}|^{2p} u_h^{i+1} v d\mathbf{x}$, we obtain

$$\begin{aligned}
&\int_\Omega \nabla u \nabla v d\mathbf{x} + \lambda \int_\Omega |u|^{2p} u v d\mathbf{x} - \int_\Omega \nabla u_h^{i+1} \nabla v d\mathbf{x} - \lambda \int_\Omega |u_h^{i+1}|^{2p} u_h^{i+1} v d\mathbf{x} \\
&\quad = \int_\Omega (f - f_h)(v - v_h) d\mathbf{x} + \sum_{K\in\mathcal{T}_h} \Big\{ \int_K (f_h + \Delta u_h^{i+1} \\
&\qquad - \lambda |u_h^i|^{2p} u_h^{i+1} - \alpha(u_h^{i+1} - u_h^i))(v - v_h) d\mathbf{x} \\
&\qquad - \frac{1}{2} \sum_{e\in\mathcal{E}(K)} \int_e [\frac{\partial u_h^{i+1}}{\partial n}](v - v_h) d\tau \Big\} \\
&\qquad + \lambda \int_\Omega \big(|u_h^i|^{2p} - |u_h^{i+1}|^{2p}\big) u_h^{i+1} v d\mathbf{x} + \alpha \sum_{K\in\mathcal{T}_h} \int_K (u_h^{i+1} - u_h^i) v d\mathbf{x}.
\end{aligned} \tag{4.7}$$

We now define the local linearization indicator $\eta_{K,i}^{(L)}$ and the local discretization indicator $\eta_{K,i}^{(D)}$ at each iteration i by:

$$\begin{aligned}\eta_{K,i}^{(L)} &= ||u_h^{i+1} - u_h^i||_{1,K},\\ \big(\eta_{K,i}^{(D)}\big)^2 &= h_K^2 \parallel f_h + \Delta u_h^{i+1} - \lambda |u_h^i|^{2p} u_h^{i+1} - \alpha (u_h^{i+1} - u_h^i) \parallel^2_{L^2(K)}\\ &\quad + \sum_{e\in\mathcal{E}(K)} h_e \parallel [\frac{\partial u_h^{i+1}}{\partial n}] \parallel^2_{L^2(e)}.\end{aligned}$$

We are in a position to state the first result of this section:

Theorem 4.3 Upper bound. *Let u_h^{i+1} and u be the solution of the iterative problem (1.5) and the exact problem* (2.2) *respectively. We have the following a posteriori error estimate*

$$\begin{aligned}&|u_h^{i+1} - u|_{1,\Omega}\\ &\le C\left(\left(\sum_{K\in\mathcal{T}_h}\left(\left(\eta_{K,i}^{(D)}\right)^2 + h_K^2 \parallel f - f_h \parallel^2_{L^2(K)}\right)\right)^{1/2} + \left(\sum_{K\in\mathcal{T}_h}\left(\eta_{K,i}^{(L)}\right)^2\right)^{1/2}\right).\end{aligned}$$

Proof We consider Eq. (4.7) with $v = u - u_h^{i+1}$ and we obtain

$$\begin{aligned}&\int_\Omega \nabla(u - u_h^{i+1})^2 d\mathbf{x} + \lambda \int_\Omega (|u|^{2p}u - |u_h^{i+1}|^{2p}u_h^{i+1})(u - u_h^{i+1})d\mathbf{x}\\ &= \sum_{K\in\mathcal{T}_h}\int_K (f - f_h)(v - v_h)d\mathbf{x} + \sum_{K\in\mathcal{T}_h}\Bigg\{\int_K (f_h + \Delta u_h^{i+1} - \lambda|u_h^i|^{2p}u_h^{i+1}\\ &\quad - \alpha(u_h^{i+1} - u_h^i))(v - v_h)d\mathbf{x}\\ &\quad - \frac{1}{2}\sum_{e\in\mathcal{E}(K)}\int_e [\frac{\partial u_h^{i+1}}{\partial n}](v - v_h)d\tau\Bigg\} + \lambda\int_\Omega \big(|u_h^i|^{2p}\\ &\quad - |u_h^{i+1}|^{2p}\big)u_h^{i+1} v d\mathbf{x} + \alpha\sum_{K\in\mathcal{T}_h}\int_K (u_h^{i+1} - u_h^i) v d\mathbf{x}. \qquad (4.8)\end{aligned}$$

Then we have by using Lemmas 2.4 and 2.6

$$\begin{aligned}|u - u_h^{i+1}|^2_{1,\Omega} &\le \sum_{K\in\mathcal{T}_h} ||f - f_h||_{L^2(K)}||v - v_h||_{L^2(K)}\\ &+ \sum_{K\in\mathcal{T}_h}(||f_h + \Delta u_h^{i+1} - \lambda|u_h^i|^{2p}u_h^{i+1} - \alpha(u_h^{i+1} - u_h^i)||_{L^2(K)}||v - v_h||_{L^2(K)}\\ &+ \frac{1}{2}\sum_{e\in\mathcal{E}(K)} ||[\frac{\partial u_h^{i+1}}{\partial n}]||_{L^2(e)}||v - v_h||_{L^2(e)}\\ &+ \lambda\int_\Omega 2p|u_h^i - u_h^{i+1}|\big(|u_h^i|^{2p-1} + |u_h^{i+1}|^{2p-1}\big)|u_h^{i+1}||v|d\mathbf{x}\\ &+ \alpha\sum_{K\in\mathcal{T}_h} ||u_h^{i+1} - u_h^i||_{L^2(K)}||v||_{L^2(K)}\end{aligned}$$

We choose $v_h = R_h v$, the image of v by the Clément operator and we obtain

$$
\begin{aligned}
|u - u_h^{i+1}|_{1,\Omega}^2 \le C \sum_{K\in\mathcal{T}_h} h_K ||f - f_h||_{L^2(K)} |v|_{1,\Delta_K} \\
+ \sum_{K\in\mathcal{T}_h} (C h_K ||f_h + \Delta u_h^{i+1} - \lambda |u_h^i|^{2p} u_h^{i+1} - \alpha(u_h^{i+1} - u_h^i)||_{L^2(K)} |v|_{1,\Delta_K} \\
+ \frac{C}{2} \sum_{e\in\mathcal{E}(K)} h_e^{\frac{1}{2}} ||[\frac{\partial u_h^{i+1}}{\partial n}]||_{L^2(e)} |v|_{1,\Delta_e}) \\
+ \lambda \int_\Omega 2p |u_h^i - u_h^{i+1}| \big(|u_h^i|^{2p-1} + |u_h^{i+1}|^{2p-1}\big) |u_h^{i+1}| |v| d\mathbf{x} \\
+ \alpha \sum_{K\in\mathcal{T}_h} ||u_h^{i+1} - u_h^i||_{L^2(K)} ||v||_{L^2(K)}
\end{aligned}
$$

We begin by bounding the second term of the right-hand side of the last inequality and we obtain by using Theorem 3.3

$$
\begin{aligned}
&\lambda \int_\Omega 2p |u_h^i - u_h^{i+1}| \big(|u_h^i|^{2p-1} + |u_h^{i+1}|^{2p-1}\big) |u_h^{i+1}| |v| d\mathbf{x} \\
&\le 2\lambda p || \, |u_h^i|^{2p-1} + |u_h^{i+1}|^{2p-1} ||_{L^8(\Omega)} \big||u_h^i - u_h^{i+1}||_{L^8(\Omega)} ||u_h^{i+1}||_{L^4(\Omega)} ||v||_{L^2(\Omega)} \\
&\le 2\lambda p S_2 S_4 S_8 \big(|| \, |u_h^i|^{2p-1} ||_{L^8(\Omega)} + || \, |u_h^{i+1}|^{2p-1} ||_{L^8(\Omega)} \big) |u_h^i - u_h^{i+1}|_{1,\Omega} |u_h^{i+1}|_{1,\Omega} |v|_{1,\Omega} \\
&\le 2\lambda p S_2 S_4 S_8 S_{8(2p-1)}^{2p-1} (|u_h^i|_{1,\Omega}^{2p-1} + |u_h^{i+1}|_{1,\Omega}^{2p-1}) \big|u_h^i - u_h^{i+1}|_{1,\Omega} |u_h^{i+1}|_{1,\Omega} |v|_{1,\Omega} \\
&\le 4\lambda p (1 + \alpha S_2^2)^p S_2 S_4 S_8 S_{8(2p-1)}^{2p-1} ||f||_{-1,\Omega}^{2p} \big|u_h^i - u_h^{i+1}|_{1,\Omega} |v|_{1,\Omega}.
\end{aligned}
$$

Let $S = 4\lambda p (1 + \alpha S_2^2)^p S_2 S_4 S_8 S_{8(2p-1)}^{2p-1} ||f||_{-1,\Omega}^{2p}$, then we have

$$
\begin{aligned}
|u - u_h^{i+1}|_{1,\Omega}^2 \le C \sum_{K\in\mathcal{T}_h} h_K ||f - f_h||_{L^2(K)} |v|_{1,\Delta_K} \\
+ \sum_{K\in\mathcal{T}_h} (C h_K ||f_h + \Delta u_h^{i+1} - \lambda |u_h^i|^{2p} u_h^{i+1} - \alpha(u_h^{i+1} - u_h^i)||_{L^2(K)} |v|_{1,\Delta_K} \\
+ \frac{C}{2} \sum_{e\in\mathcal{E}_{(K)}} h_e^{\frac{1}{2}} ||[\frac{\partial u_h^{i+1}}{\partial n}]||_{L^2(e)} |v|_{1,\Delta_e}) \\
+ S |u_h^i - u_h^{i+1}|_{1,\Omega} |v|_{1,\Omega} + \alpha \sum_{K\in\mathcal{T}_h} ||u_h^{i+1} - u_h^i||_{L^2(K)} ||v||_{L^2(K)}.
\end{aligned}
$$

By using the formula $ab \le \frac{1}{2\varepsilon} a^2 + \frac{\varepsilon}{2} b^2$, we obtain

$$
\begin{aligned}
|u - u_h^{i+1}|_{1,\Omega}^2 \le \frac{C_1 \varepsilon_1}{2} \sum_{K\in\mathcal{T}_h} h_K^2 ||f - f_h||_{L^2(K)}^2 + \frac{C_1}{2\varepsilon_1} \sum_{K\in\mathcal{T}_h} |v|_{1,K}^2 \\
+ \frac{C_2 \varepsilon_2}{2} \sum_{K\in\mathcal{T}_h} h_K^2 ||f_h + \Delta u_h^{i+1} - \lambda |u_h^i|^{2p} u_h^{i+1} - \alpha(u_h^{i+1} - u_h^i)||_{L^2(K)}^2 \\
+ \frac{C_2}{2\varepsilon_2} \sum_{K\in\mathcal{T}_h} |v|_{1,K}^2 + \frac{C_3 \varepsilon_3}{4} \sum_{K\in\mathcal{T}_h} \sum_{E\in\mathcal{E}_{(K)}} h_E ||[\frac{\partial u_h^{i+1}}{\partial n}]||_{L^2(E)}^2 + \frac{C_3}{4\varepsilon_3} \sum_{K\in\mathcal{T}_h} |v|_{1,K}^2
\end{aligned}
$$

$$+\frac{S\varepsilon_4}{2}\sum_{K\in\mathcal{T}_h}||u_h^i-u_h^{i+1}||_{1,K}^2+\frac{S}{2\varepsilon_4}\sum_{K\in\mathcal{T}_h}|v|_{1,K}^2$$
$$+\frac{\alpha\varepsilon_5}{2}\sum_{K\in\mathcal{T}_h}||u_h^{i+1}-u_h^i||_{1,K}^2+\frac{\alpha}{2\varepsilon_5}||v||_{L^2(\Omega)}^2$$

We choose $\varepsilon_1=8C_1$, $\varepsilon_2=8C_2$, $\varepsilon_3=4C_3$, $\varepsilon_4=8S$ et $\varepsilon_5=8\alpha S_2^2$ to obtain

$$\begin{aligned}|u-u_h^{i+1}|_{1,\Omega}^2\le C\Big(&\sum_{K\in\mathcal{T}_h}h_K^2||f-f_h||_{L^2(K)}^2+\sum_{K\in\mathcal{T}_h}\sum_{e\in\mathcal{E}(K)}h_e||[\frac{\partial u_h^{i+1}}{\partial n}]||_{L^2(e)}^2\\&+\sum_{K\in\mathcal{T}_h}h_K^2||f_h+\Delta u_h^{i+1}-\lambda|u_h^i|^{2p}u_h^{i+1}-\alpha(u_h^{i+1}-u_h^i)||_{L^2(K)}^2\\&+\sum_{K\in\mathcal{T}_h}||u_h^i-u_h^{i+1}||_{1,K}^2+\sum_{K\in\mathcal{T}_h}||u_h^{i+1}-u_h^i||_{1,K}^2\Big)+\frac{5}{16}|v|_{1,\Omega}^2,\end{aligned}$$

and then

$$|u_h^{i+1}-u|_{1,\Omega}\le C((\sum_{K\in\mathcal{T}_h}((\eta_{K,i}^{(D)})^2+h_K^2||f-f_h||_{L^2(K)}^2))^{\frac{1}{2}}+(\sum_{K\in\mathcal{T}_h}(\eta_{K,i}^{(L)})^2)^{\frac{1}{2}}).$$

We conclude the proof of the theorem.
We address now the efficiency of the previous indicators. □

Theorem 4.4 Lower bound. *For each $K\in\mathcal{T}_h$, there holds*

$$\eta_{K,i}^{(L)}\le\|u_h^i-u\|_{1,K}+\|u_h^{i+1}-u\|_{1,K},$$
$$\eta_{K,i}^{(D)}\le C\sum_{\kappa\subset\omega_K}\left(||u-u_h^{i+1}||_{1,\kappa}+\eta_{\kappa,i}^{(L)}+h_\kappa\|f-f_h\|_{L^2(\kappa)}\right),$$

where ω_K is the union of the triangles sharing at least one edge with K.

Proof The estimation of the linearization indicator follows easily from the triangle inequality by introducing u in $\eta_{K,i}^{(L)}$. We now estimate the discretization indicator $\eta_{K,i}^{(D)}$. We proceed in two steps:
(i) We start by adding and subtracting $\lambda\int_\Omega|u_h^{i+1}|^{2p}u_h^{i+1}vd\mathbf{x}$ in (4.6). Taking $v_h=0$, we derive

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\int_K(f_h+\Delta u_h^{i+1}-\lambda|u_h^i|^{2p}u_h^{i+1}-\alpha(u_h^{i+1}-u_h^i))vd\mathbf{x}\\&=\int_\Omega\nabla(u-u_h^{i+1})\nabla vd\mathbf{x}-\sum_{K\in\mathcal{T}_h}\int_K(f-f_h)vd\mathbf{x}\\&+\frac{1}{2}\sum_{K\in\mathcal{T}_h}\sum_{e\in\mathcal{E}(K)}\int_e[\frac{\partial u_h^{i+1}}{\partial n}]vd\tau+\lambda\int_\Omega(|u|^{2p}u-|u_h^{i+1}|^{2p}u_h^{i+1})vd\mathbf{x}\\&+\lambda\int_\Omega u_h^{i+1}(|u_h^{i+1}|^{2p}-|u_h^i|^{2p})vd\mathbf{x}-\alpha\sum_{K\in\mathcal{T}_h}\int_K(u_h^{i+1}-u_h^i)vd\mathbf{x}.\end{aligned}\qquad(4.9)$$

We choose $v = v_K$ such that

$$v_K = \begin{cases} (f_h + \Delta u_h^{i+1} - \lambda |u_h^i|^{2p} u_h^{i+1} - \alpha(u_h^{i+1} - u_h^i))\psi_K & \text{in } K \\ 0 & \text{in } \Omega\backslash K \end{cases}$$

where ψ_K is the triangle-bubble function.
Using Cauchy–Schwarz inequality, (2.1), (4.1) and (4.2) we obtain

$$\begin{aligned} &\| f_h + \Delta u_h^{i+1} - \lambda |u_h^i|^{2p} u_h^{i+1} - \alpha(u_h^{i+1} - u_h^i) \|_{L^2(K)}^2 \\ &\le (1 + \lambda C \| f \|_{-1,\Omega}^{2p}) \| u - u_h^{i+1} \|_{1,K} |v_K|_{1,K} + \| f - f_h \|_{L^2(K)} \| v_K \|_{L^2(K)} \\ &\quad + \lambda C \| u_h^i - u_h^{i+1} \|_{1,K} |v_K|_{1,K} + \alpha ||u_h^{i+1} - u_h^i||_{L^2(K)} ||v_K||_{L^2(K)}. \end{aligned}$$

Therefore, we derive the following estimate of the first term of the local discretization estimator $\eta_{K,i}^{(D)}$

$$\begin{aligned} &h_K \| f_h + \Delta u_h^{i+1} - \lambda |u_h^i|^{2p} u_h^{i+1} - \alpha(u_h^{i+1} - u_h^i) \|_{L^2(K)} \\ &\le C(\| u - u_h^{i+1} \|_{1,K} + h_K \| f - f_h \|_{L^2(K)}) + C' \eta_{K,i}^{(L)}. \end{aligned} \tag{4.10}$$

(ii) Now we estimate the second term of $\eta_{K,i}^{(D)}$. Similarly, using (4.9) we infer

$$\begin{aligned} &\frac{1}{2} \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}(K)} \int_e [\frac{\partial u_h^{i+1}}{\partial n}] v \, d\tau = \int_\Omega \nabla(u_h^{i+1} - u) \nabla v d\mathbf{x} \\ &+ \sum_{K \in \mathcal{T}_h} \int_K (f_h + \Delta u_h^{i+1} - \lambda |u_h^i|^{2p} u_h^{i+1} - \alpha(u_h^{i+1} - u_h^i)) v d\mathbf{x} + \int_\Omega (f - f_h) v d\mathbf{x} \\ &- \lambda \int_\Omega (|u_h^{i+1}|^{2p} u_h^{i+1} - |u_h^i|^{2p} u_h^{i+1}) v d\mathbf{x} - \lambda \int_\Omega (|u|^{2p} u - |u_h^{i+1}|^{2p} u_h^{i+1}) v d\mathbf{x} \\ &+ \alpha \sum_{K \in \mathcal{T}_h} \int_K (u_h^{i+1} - u_h^i) v d\mathbf{x}. \end{aligned} \tag{4.11}$$

We choose $v = v_e$ such that

$$v_e = \begin{cases} L_{e,\kappa} \left(\left[\frac{\partial u_h^{i+1}}{\partial n} \right] \psi_e \right) & \kappa \in \{K, K'\} \\ 0 & \text{in } \Omega\backslash(K \cup K') \end{cases}$$

where ψ_e is the edge-bubble function, K' denotes the other element of $\mathcal{T}_h$ that share e with K (the operator $L_{e,K}$ was introduced above Proposition 4.2).
Using Cauchy–Schwarz inequality, (2.1), (4.3) and (4.4) we derive

$$\begin{aligned} h_e^{1/2} \| [\frac{\partial u_h^{i+1}}{\partial n}] \|_{L^2(e)}^2 &\le (1 + \lambda C \| f \|_{-1,\Omega}^{2p}) \| u - u_h^{i+1} \|_{1,K \cup K'} \| v_e \|_{L^2(e)} \\ &\quad + h_e \| f - f_h \|_{L^2(K \cup K')} \| v_e \|_{L^2(e)} + h_e \| f_h + \Delta u_h^{i+1} \\ &\quad - \lambda |u_h^i|^{2p} u_h^{i+1} - \alpha(u_h^{i+1} - u_h^i) \|_{L^2(K \cup K')} \| v_e \|_{L^2(e)} \\ &\quad + C'(eta_{K,1}^{(L)} + \eta_{K',i}^{(L)}) ||v_e||_{L^2(e)}. \end{aligned} \tag{4.12}$$

Collecting the two bounds above leads to the following estimation

$$\eta_{K,i}^{(D)} \leq C \sum_{\kappa\subset\omega_K} \left(||u - u_h^{i+1}||_{1,\kappa} + \eta_{\kappa,i}^{(L)} + h_\kappa \parallel f - f_h \parallel_{L^2(\kappa)} \right)$$

These estimates of the local linearization and discretization indicators are fully optimal. □

4.2 Algorithm (1.6)

The same calculation is followed as before but in (4.6) and (4.7) we have $\alpha \sum_{K\in\mathcal{T}_h} \int_K \nabla(u_h^{i+1} - u_h^i)\nabla v$ instead of $\alpha \sum_{K\in\mathcal{T}_h} \int_K (u_h^{i+1} - u_h^i)v$. We are led to define the modified discretization error indicator $\bar{\eta}_{K,i}^{(D)}$ by

$$\begin{aligned}(\bar{\eta}_{K,i}^{(D)})^2 &= h_K^2 || f_h + \Delta u_h^{i+1} - \lambda |u_h^i|^{2p} u_h^{i+1} + \alpha\Delta(u_h^{i+1} - u_h^i)||_{L^2(K)}^2 \\ &\quad + \sum_{e\in\mathcal{E}(K)} h_e ||[\frac{\partial u_h^{i+1}}{\partial n} - \alpha \frac{\partial(u_h^{i+1} - u_h^i)}{\partial n}]||_{L^2(e)}^2.\end{aligned}$$

The rest of the calculation is similar. We skip the proofs since they are exactly the same as for Theorems 4.3 and 4.4.

Theorem 4.5 Upper bound. *Let u_h^{i+1} and u be the solution of the iterative problem (1.6) and the exact problem* (2.2) *respectively. We have the following a posteriori error estimate*

$$\begin{aligned}&|u_h^{i+1} - u|_{1,\Omega} \\ &\leq C\left(\left(\sum_{K\in\mathcal{T}_h}\left(\left(\bar{\eta}_{K,i}^{(D)}\right)^2 + h_K^2 \parallel f - f_h \parallel_{L^2(K)}^2\right)\right)^{1/2} + \left(\sum_{K\in\mathcal{T}_h}\left(\eta_{K,i}^{(L)}\right)^2\right)^{1/2}\right).\end{aligned}$$

Theorem 4.6 Lower bound. *For each $K \in \mathcal{T}_h$, there holds*

$$\begin{aligned}\eta_{K,i}^{(L)} &\leq \parallel u_h^i - u \parallel_{1,K} + \parallel u_h^{i+1} - u \parallel_{1,K}, \\ \bar{\eta}_{K,i}^{(D)} &\leq C \sum_{\kappa\subset\omega_K} \left(||u - u_h^{i+1}||_{1,\kappa} + \eta_{\kappa,i}^{(L)} + h_\kappa \parallel f - f_h \parallel_{L^2(\kappa)} \right),\end{aligned}$$

where ω_K is the union of the triangles sharing at least one edge with K.

5 Numerical Results

In this section, we present numerical experiments for our nonlinear problem. These simulations have been performed using the code FreeFem++ due to Hecht and Pironneau [6]. For all the numerical investigations and for simplicity, we use the finite element of degree $\ell = 1$.

5.1 A Priori Estimation

We consider the domain $\Omega =]-1, 1[^2$, each edge is divided into N equal segments so that Ω is divided into N^2 equal squares and finally into $2N^2$ equal triangles . We consider the exact solution $u = e^{-5(x^2+y^2)}$ where $f = -\Delta u + \lambda|u|^{2p}u$.

Table 1 Convergence of algorithms (1.5) and (1.6) with respect of α

α	0.01	0.5	0.76	0.77	1	10
Algo (1.5)	Div	Div	Div	Div	Div	Div
Algo (1.6)	Div	Div	Div	0.0581902	0.0581908	0.0581905
α	20	21.81	21.82	22	50	100
Algo (1.5)	Div	Div	0.0581906	0.0581904	0.0581897	0.0581886
Algo (1.6)	0.0581906	0.0581907	0.0581907	0.0581907	0.0581919	0.0581973

Table 2 Comparison of the convergence of algorithms (1.5), (1.6) (for $\alpha = 22$) with (1.4)

	$\lambda = 1$ $p = 1$	$\lambda = 2$ $p = 10$	$\lambda = 5$ $p = 10$	$\lambda = 5$ $p = 50$	$\lambda = 10$ $p = 10$
Algo (1.4)	0.0581392	0.0580725	Div	Div	Div
Algo (1.5)	0.0580467	0.058072	0.0581392	0.0581284	0.0582106
Algo (1.6)	0.0580458	0.0580717	0.0581399	0.0581292	0.0582111
	$\lambda = 10$ $p = 50$	$\lambda = 10$ $p = 100$	$\lambda = 50$ $p = 50$	$\lambda = 100$ $p = 50$	$\lambda = 100$ $p = 100$
Algo (1.4)	Div	Div	Div	Div	Div
Algo (1.5)	0.0581904	0.0581759	Div	Div	Div
Algo (1.6)	0.0581907	0.0581764	0.0582894	0.0583124	0.0582956

For the convergence, we use the classical stopping criterion $err_L \leq 10^{-5}$, where err_L is defined by

$$err_L = \frac{|u_h^{i+1} - u_h^i|_{1,\Omega}}{|u_h^{i+1}|_{1,\Omega}}.$$

We consider $\lambda = 10$, $p = 50$ and $N = 50$. Table 1 shows the error

$$Err = \frac{|u_h^i - u|_{1,\Omega}}{|u|_{1,\Omega}},$$

which describes the convergence of the algorithms (1.5) and (1.6) with respect of α. We remark that the algorithm (1.5) converges for $\alpha \geq 21.82$ and the algorithm (1.6) converges for $\alpha \geq 0.77$.

In order to compare our algorithms (1.5) and (1.6) with (1.4), Table 2 shows the convergence for $N = 50$ and a fixed $\alpha = 22$ in our algorithms. In fact, for big values of λ and p, the algorithm (1.4) diverges. We mention that for λ and p where (1.5) and (1.6) diverge, we must take a bigger values of α to obtain the convergence. Figure 1 shows in logarithmic scale the error Err with respect to h (algorithm 1.5 in the left and algorithm 1.6 in the right). The slope of the error corresponding to (1.5) and (1.6) are respectively 0.92 and 0.96, which validates Theorem 3.2.

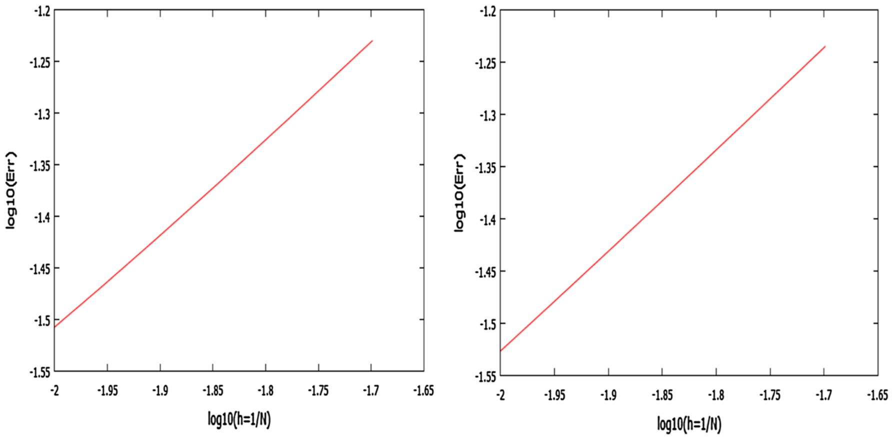

Fig. 1 A priori error with respect of $h = 1/N$: *left* (algo 1.5) and *right* (algo 1.6)

5.2 A Posteriori Analysis

In this section, we test our *a posteriori* error estimates on our model problem. We consider the same domain Ω with the theoretical solution now given by $u = \mathrm{e}^{-100(x^2+y^2)}$. and we choose $\lambda = 10$ and $p = 50$.
In [2] and for the adaptive strategy, we define the global indicators (introduced in [5]):

$$\eta_i^{(D)} = \left(\sum_{K \in \mathcal{T}_h} \left(\eta_{K,i}^{(D)} \right)^2 \right)^{1/2} \quad \text{and} \quad \eta_i^{(L)} = \left(\sum_{K \in \mathcal{T}_h} \left(\eta_{K,i}^{(L)} \right)^2 \right)^{1/2},$$

and we introduce two kinds of stopping criteria:

$$\eta_i^{(L)} \leq 10^{-5} \quad \text{Classical stopping criterion}, \tag{5.1}$$

and

$$\eta_i^{(L)} \leq \gamma \eta_i^{(D)} \quad \text{New stopping criterion}, \tag{5.2}$$

where γ is a parameter which balances the discretization and linearization errors. We studied in [2] the comparison between these two types of stopping criterion and we showed the efficiency of the new one which is considered in this paper with $\gamma = 0.001$.

For our numerical investigations, we follow the algorithm described in [2]. The evolution of the meshes with the new stopping criterion looks like the figures 3 and 4 in [2]. We note that for $\lambda = 10$ and $p = 50$, the algorithm (1.4) diverges.

Figure 2 gives a comparison in logarithmic scale of the error between the uniform and adaptive methods using the algorithms (1.5) and (1.6) with respect of the number of vertices. We can easily see that the algorithms (1.5) and (1.6) give comparable results but the adaptive method is more powerful than the uniform one.

Figure 3 shows the dependencies of the algorithms (1.5) and (1.6) with respect of γ in logarithmic scale. We remark that for the algorithm (1.5), the curves are similar for $\gamma \geq 0.1$ and we have approximately the same precision but it is much sensible with the variation of γ for the algorithm (1.6).

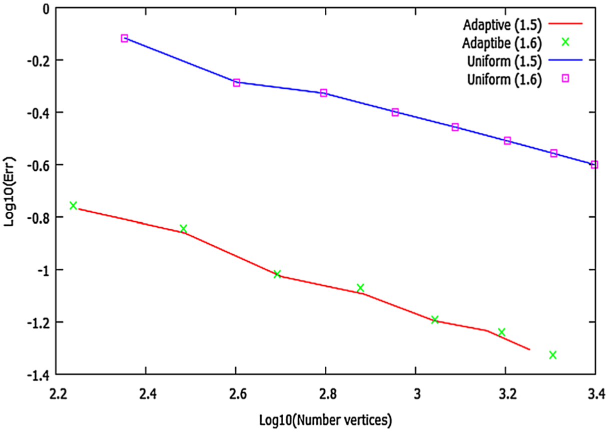

Fig. 2 Error versus number of vertices in logarithmic scale for adaptive and uniform methods with algorithms (1.5) and (1.6)

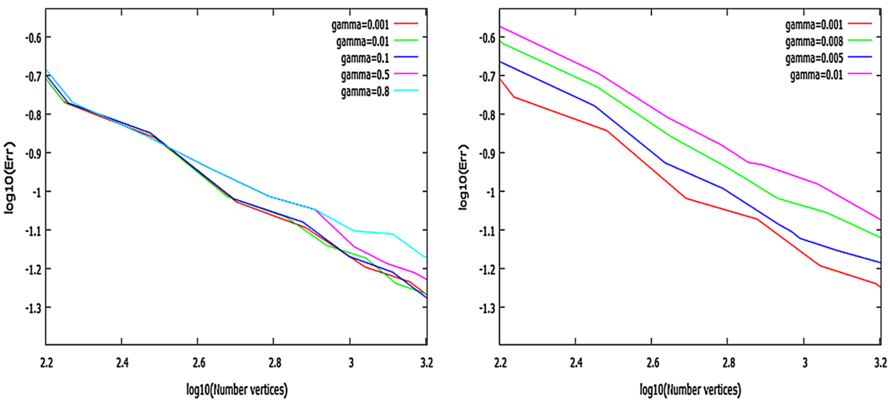

Fig. 3 Error versus number of vertices in logarithmic scale for adaptive method with respect of γ: *left* (algo 1.5) and *right* (algo 1.6)

Table 3 Comparison of the precision and the CPU time between the algorithm (1.5) and (1.6) with respect of α

α	22	30	40	50
Algo (1.5)	Time = 5.186 s	Time = 4.952 s	Time = 5.505 s	Time = 5.51 s
	Error = 0.0487858	Error = 0.0484306	Error = 0.048256	Error = 0.0478781
Algo (1.6)	Time = 56.625 s	Time = 59.499 s	Time = 66.231 s	Time = 61.268 s
	Error = 0.0475732	Error = 0.0486143	Error = 0.0469877	Error = 0.0494427

Table 3 shows comparisons, for approximatively the same precision, of the CPU time between the algorithm (1.5) and (1.6) with respect of α. We remark that algorithm (1.5) is faster than (1.6).

Table 4 Repartition of errors and indicators during the refinement levels (Itn): Left [algorithm (1.5)] and right [algorithm (1.6)]

Itn	Err	err_I	$C = \frac{err_I}{Err}$	Itn	Err	err_I	$C = \frac{err_I}{Err}$
1	0.426417	1.31249	3.07796	1	0.443195	1.29673	2.92587
2	0.169927	0.524469	3.08644	2	0.175596	0.536429	3.05491
3	0.138091	0.372655	2.69862	3	0.143715	0.38263	2.66242
4	0.093948	0.278664	2.96615	4	0.0959932	0.290036	3.02142
5	0.0806374	0.222064	2.75386	5	0.0848493	0.229053	2.69953
6	0.063787	0.186245	2.91979	6	0.0643159	0.191061	2.97066
7	0.0583991	0.160167	2.74262	7	0.0577646	0.157273	2.72265
8	0.049479	0.14235	2.87698	8	0.0474228	0.137726	2.90422

In order to have an idea of the constant on the upper bound in Theorem 4.3, Table 4 shows the repartition of the error Err and the sum of the indicators

$$err_I = \frac{((\eta_i^{(D)})^2 + (\eta_i^{(L)})^2)^{1/2}}{|u|_{1,\Omega}} \simeq \frac{\eta_i^{(D)}}{|u|_{1,\Omega}},$$

during the refinement level and after the convergence on each one. Even if the errors regularly decrease (for instance from 1 to 0.14 for err_I) with respect to the number of adaptive refinement levels which is consistent with adapted mesh method, the constant remains stable and can be approximated by 2.85.

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