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A Uniform Additive Schwarz Preconditioner for High-Order Discontinuous Galerkin Approximations of Elliptic Problems

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Abstract In this paper we design and analyze a uniform preconditioner for a class of high-order Discontinuous Galerkin schemes. The preconditioner is based on a space splitting involving the high-order conforming subspace and results from the interpretation of the problem as a nearly-singular problem. We show that the proposed preconditioner exhibits spectral bounds that are uniform with respect to the discretization parameters, i.e., the mesh size, the polynomial degree and the penalization coefficient. The theoretical estimates obtained are supported by numerical tests.

Keywords Discontinuous Galerkin method · High-order discretizations · Uniform preconditioning

1 Introduction

In the last years, the design of efficient solution techniques for the system of equations arising from Discontinuous Galerkin (DG) discretizations of elliptic partial differential equations

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has become an increasingly active field of research. On the one hand, DG methods are characterized by a great versatility in treating a variety of problems and handling, for instance, non-conforming grids and hp-adaptive strategies. On the other hand, the main drawback of DG methods is the larger number of degrees of freedom compared to (standard) conforming discretizations. In this respect, the case of high-order DG schemes is particularly representative, since the corresponding linear system of equations is very ill-conditioned: it can be proved that, for elliptic problems, the spectral condition number of the resulting stiffness matrix grows like $h^{-2}p^4$, h and p being the granularity of the underlying mesh and the polynomial approximation degree, respectively, cf. [6]. As a consequence, the design of effective tools for the solution of the linear system of equations arising from high-order DG discretizations becomes particularly challenging.

In the context of elliptic problems, Schwarz methods for low order DG schemes have been studied in [29], where overlapping and non-overlapping domain decomposition preconditioners are considered, and bounds of $O(H/\delta)$ and $O(H/h)$, respectively, are obtained for the condition number of the preconditioned operator. Here H, h and δ stand for the granularity of the coarse and fine grids and the size of the overlap, respectively. Further extensions including inexact local solvers, and the extension of two-level Schwarz methods to advection–diffusion and fourth-order problems can be found in [1–3,5,12,26,30,38]. In the field of Balancing Domain Decomposition (BDD) methods, a number of results exist in literature: exploiting a Neumann–Neumann type method, in [23,24] a conforming discretization is used on each subdomain combined with interior penalty method on non-conforming boundaries, thus obtaining a bound for the condition number of the resulting preconditioner of $O((1-\log(H/h))^2)$. In [21], using the unified framework of [11] a BDDC method is designed and analyzed for a wide range of DG methods. The *auxiliary space method* (ASM) (see e.g., [32,33,41,49]) is employed in the context of h-version DG methods to develop, for instance, the two-level preconditioners of [22] and the multilevel method of [15]. In both cases a stable splitting for the linear DG space is provided by a decomposition consisting of a conforming subspace and a correction, thus obtaining uniformly bounded preconditioners with respect to the mesh size.

All the previous results focus on low order (i.e., linear) DG methods. In the context of preconditioning high-order DG methods we mention [6,8], where a class of non-overlapping Schwarz preconditioners is introduced, and [4], where a quasi-optimal (with respect to h and p) preconditioner is designed in the framework of substructuring methods for hp-Nitsche-type discretizations. Multigrid methods for high-order DG discretizations have been analyzed in [9], cf. also [7] for the extension to genereal polygonal/polyhedral meshes. A study of a BDDC scheme in the case of hp-spectral DG methods is addressed in [19], where the DG framework is reduced to the conforming one via the ASM. The ASM framework is employed also in [14], where the high-order conforming space is employed as auxiliary subspace, and a uniform multilevel preconditioner is designed for hp-DG spectral element methods in the case of locally varying polynomial degree. To the best of our knowledge, this preconditioner is the only uniform preconditioner designed for high-order DG discretizations. We note that, in the framework of high-order methods, the decomposition involving a conforming subspace was already employed in the case of a-posteriori error analysis, see for example [17,35,51]. In this paper, we address the issue of preconditioning high-order DG methods by exploiting this kind of space splitting based on a high-order conforming space and a correction. However, in our case the space decomposition is suggested by the interpretation of the high-order DG scheme in terms of a nearly-singular problem, cf. [39]. Even though the space decomposition is similar to that of [14], the preconditioner and the analysis we present differs considerably since here we employ the abstract framework of subspace correction

methods provided by [50]. More precisely, we are able to show that a simple pointwise Jacobi method paired with an overlapping additive Schwarz method for the conforming subspace, gives uniform convergence with respect to all the discretization parameters, i.e., the mesh size, the polynomial order and the penalization coefficient appearing in the DG bilinear form.

The rest of the paper is organized as follows. In Sect. 2, we introduce the model problem and the corresponding discretization through a class of symmetric DG schemes. Section 3 is devoted to few auxiliary results regarding the Gauss–Legendre–Lobatto nodes, whose properties are fundamental to prove the stability of the space decomposition proposed in Sect. 4. The analysis of the preconditioner is presented in Sect. 5 and the theoretical results are supported by the numerical simulations of Sect. 6. We also test the performance of the proposed method in the case of more general diffusion equation, with isotropic discontinuous as well as anisotropic diffusion tensors.

2 Model Problem and High-Order DG Discretization

In this section we introduce the model problem and its discretization through several Discontinuous Galerkin schemes, see also [11].

Throughout the paper, we use the notation $x \lesssim y$ and $x \gtrsim y$ to denote the inequalities $x \leq Cy$ and $x \geq Cy$, respectively, C being a positive constant independent of the discretization parameters. Moreover, $x \approx y$ means that there exist constants $C_1, C_2 > 0$ such that $C_1 y \leq x \leq C_2 y$. When needed, the constants are written explicitly.

Given a convex polygonal/polyhedral domain $\Omega \in \mathbb{R}^d$, $d = 2, 3$, and $f \in L^2(\Omega)$, we consider the following weak formulation of the Poisson problem with homogeneous Dirichlet boundary conditions: find $u \in V := H_0^1(\Omega)$, such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in V. \tag{1}$$

Let $\mathcal{T}_h$ denote a shape-regular, conforming, locally quasi-uniform partition of Ω into shape-regular elements κ of diameter h_κ, and set $h := \max_{\kappa \in \mathcal{T}_h} h_\kappa$. We also assume that each element $\kappa \in \mathcal{T}_h$ results from the mapping, through an affine operator F_κ, of a reference element $\hat{\kappa}$, which is the open, unit d-hypercube in $\mathbb{R}^d$, $d = 2, 3$.

We denote by $\mathcal{F}_h^I$ and $\mathcal{F}_h^B$ the set of internal and boundary faces (for $d = 2$ "face" means "edge") of $\mathcal{T}_h$, respectively, and define $\mathcal{F}_h := \mathcal{F}_h^I \cup \mathcal{F}_h^B$. We associate to any $F \in \mathcal{F}_h$ a unit vector $\mathbf{n}_F$ orthogonal to the face itself and also denote by $\mathbf{n}_{F,\kappa}$ the outward normal vector to $F \subset \partial\kappa$ with respect to κ. We observe that for any $F \in \mathcal{F}_h^B$, $\mathbf{n}_{F,\kappa} = \mathbf{n}_F$, since F belongs to a unique element. For any $F \in \mathcal{F}_h^I$, we assume $\overline{F} = \partial\overline{\kappa^+} \cap \partial\overline{\kappa^-}$, where

$$\kappa^+ := \{\kappa \in \mathcal{T}_h : F \subset \partial\kappa,\ \mathbf{n}_F \cdot \mathbf{n}_{F,\kappa} > 0\},$$
$$\kappa^- := \{\kappa \in \mathcal{T}_h : F \subset \partial\kappa,\ \mathbf{n}_F \cdot \mathbf{n}_{F,\kappa} < 0\}.$$

For regular enough vector-valued and scalar functions $\boldsymbol{\tau}$ and v, we denote by $\boldsymbol{\tau}^\pm$ and $v^\pm$ the corresponding traces taken from the interior of $\kappa^\pm$, respectively, and define the *jumps* and *averages* across the face $F \in \mathcal{F}_h^I$ as follows

$$[\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}^+ \cdot \mathbf{n}_{F,\kappa^+} + \boldsymbol{\tau}^- \cdot \mathbf{n}_{F,\kappa^-}, \qquad \{\!\{\boldsymbol{\tau}\}\!\} := \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2},$$
$$[\![v]\!] := v^+ \mathbf{n}_{F,\kappa^+} + v^- \mathbf{n}_{F,\kappa^-}, \qquad \{\!\{v\}\!\} := \frac{v^+ + v^-}{2},$$

For $F \in \mathcal{F}_h^B$, the previous definitions reduce to $[\![v]\!] := v\mathbf{n}_F$ and $\{\{\boldsymbol{\tau}\}\} := \boldsymbol{\tau}$.
We now associate to the partition $\mathcal{T}_h$, the high-order Discontinuous Galerkin finite element space V_{hp} defined as

$$V_{hp} := \{v \in L^2(\Omega) : v \circ \mathsf{F}_\kappa \in \mathbb{Q}^p(\hat{\kappa}) \quad \forall \kappa \in \mathcal{T}_h\},$$

with $\mathbb{Q}^p$ denoting the space of all tensor-product polynomials on $\hat{\kappa}$ of degree $p > 1$ in each coordinate direction. We define the lifting operators $\mathcal{R}(\boldsymbol{\tau}) := \sum_{F\in\mathcal{F}_h} r_F(\boldsymbol{\tau})$ and $\mathcal{L}(v) := \sum_{F\in\mathcal{F}_h^I} l_F(v)$, where

$$r_F : [L^2(F)]^d \to [V_{hp}]^d, \quad \int_\Omega r_F(\boldsymbol{\tau}) \cdot \boldsymbol{\eta}\, dx := -\int_F \boldsymbol{\tau} \cdot \{\{\boldsymbol{\eta}\}\}\, ds \qquad \forall F \in \mathcal{F}_h.$$

$$l_F : L^2(F) \to [V_{hp}]^d, \quad \int_\Omega l_F(v) \cdot \boldsymbol{\eta}\, dx := -\int_F v[\![\boldsymbol{\eta}]\!]\, ds \qquad \forall F \in \mathcal{F}_h^I,$$

for any $\boldsymbol{\eta} \in [V_{hp}]^d$.
We then introduce the DG finite element formulation: find $u \in V_{hp}$ such that

$$\mathcal{A}(u, v) = \int_\Omega f v\, dx \quad \forall v \in V_{hp}, \tag{2}$$

with $\mathcal{A}(\cdot,\cdot) : V_{hp} \times V_{hp} \to \mathbb{R}$ defined as

$$\begin{aligned}\mathcal{A}(u, v) := &\sum_{\kappa\in\mathcal{T}_h} \int_\kappa \nabla u \cdot \nabla v\, dx + \sum_{\kappa\in\mathcal{T}_h} \int_\kappa \nabla u \cdot (\mathcal{R}([\![v]\!]) + \mathcal{L}(\boldsymbol{\beta} \cdot [\![v]\!]))\, dx \\ &+ \sum_{\kappa\in\mathcal{T}_h} \int_\kappa (\mathcal{R}([\![u]\!]) + \mathcal{L}(\boldsymbol{\beta} \cdot [\![u]\!])) \cdot \nabla v\, dx + \sum_{F\in\mathcal{F}_h} \int_F \sigma [\![u]\!] \cdot [\![v]\!] ds \\ &+ \theta \int_\Omega (\mathcal{R}([\![u]\!]) + \mathcal{L}(\boldsymbol{\beta} \cdot [\![u]\!])) \cdot (\mathcal{R}([\![v]\!]) + \mathcal{L}(\boldsymbol{\beta} \cdot [\![v]\!]))dx,\end{aligned} \tag{3}$$

where $\theta = 0$ for the SIPG method of [10] and $\theta = 1$ for the LDG method of [20]. With regard to the vector function $\boldsymbol{\beta}$, we have $\boldsymbol{\beta} = \mathbf{0}$ for the SIPG method, while $\boldsymbol{\beta} \in \mathbb{R}^d$ is a uniformly bounded (and possibly null) vector for the LDG method. The penalization function $\sigma \in L^\infty(\mathcal{F}_h)$ is defined as

$$\sigma|_F := \alpha \frac{p^2}{\min(h_{\kappa^+}, h_{\kappa^-})}, \quad F \in \mathcal{F}_h^I, \quad \sigma|_F := \alpha \frac{p^2}{h_\kappa} \quad F \in \mathcal{F}_h^B,$$

being $\alpha \geq 1$ and $h_{\kappa^\pm}$ the diameters of the neighboring elements $\kappa^\pm \in \mathcal{T}_h$ sharing the face $F \in \mathcal{F}_h^I$.
We endow the DG space V_{hp} with the following norm

$$\|v\|_{DG}^2 := \sum_{\kappa\in\mathcal{T}_h} \|\nabla v\|_{L^2(\kappa)}^2 + \sum_{F\in\mathcal{F}_h} \|\sigma^{1/2}[\![v]\!]\|_{L^2(F)}^2,$$

and state the following result, cf. [6,36,44,45].

Lemma 1 *The following results hold*

$$\begin{aligned}&\mathcal{A}(u, v) \lesssim \|u\|_{DG}\|v\|_{DG} && \forall u, v \in V_{hp}, \\ &\mathcal{A}(u, u) \gtrsim \|u\|_{DG}^2 && \forall u \in V_{hp}.\end{aligned} \tag{4}$$

For the SIPG formulation coercivity holds provided the penalization coefficient α is chosen large enough.

From Lemma 1 and using the Poincaré inequality for piecewise H^1 functions of [13], the following spectral bounds hold, cf. [6].

Lemma 2 *For any $u \in V_{hp}$ it holds that*

$$\sum_{\kappa\in\mathcal{T}_h} \|u\|^2_{L^2(\kappa)} \lesssim \mathcal{A}(u,u) \lesssim \sum_{\kappa\in\mathcal{T}_h} \alpha \frac{p^4}{h_\kappa^2} \|u\|^2_{L^2(\kappa)}. \tag{5}$$

3 Gauss–Legendre–Lobatto Nodes and Quadrature Rule

In this section we provide some details regarding the choice of the basis functions spanning the space V_{hp} and the corresponding degrees of freedom. On the reference d-hypercube $[-1,1]^d$, we choose the basis obtained by the tensor product of the one-dimensional Lagrange polynomials on the reference interval $[-1,1]$, based on Gauss–Legendre–Lobatto (GLL) nodes. We denote by $\mathcal{N}_{\mathrm{I}}(\hat\kappa)$ ($\mathcal{N}_{\mathrm{B}}(\hat\kappa)$) the set of interior (boundary) nodes of $\hat\kappa$, and define $\mathcal{N}(\hat\kappa) := \mathcal{N}_{\mathrm{I}}(\hat\kappa) \cup \mathcal{N}_{\mathrm{B}}(\hat\kappa)$. The analogous sets in the physical frame are denoted by $\mathcal{N}_{\mathrm{I}}(\kappa)$, $\mathcal{N}_{\mathrm{B}}(\kappa)$ and $\mathcal{N}(\kappa)$, where any $\xi_p \in \mathcal{N}(\kappa)$ is obtained by applying the linear mapping $\mathsf{F}_\kappa : \hat\kappa \to \kappa$ to the corresponding $\hat\xi_p \in \mathcal{N}(\hat\kappa)$. The choice of GLL points as degrees of freedom allow us to exploit the properties of the associated quadrature rule. We recall that, given $(p+1)^d$ GLL quadrature nodes $\{\hat\xi_p\}$ and weights $\{\hat{\mathsf{w}}_{\xi_p}\}$, we have

$$\sum_{\hat\xi_p\in\mathcal{N}(\hat\kappa)} v(\hat\xi_p)\hat{\mathsf{w}}_{\xi_p} = \int_{\hat\kappa} v\,dx \qquad \forall v \in \mathbb{Q}^{2p-1}(\hat\kappa),$$

which implies that

$$\sum_{\hat\xi_p\in\mathcal{N}(\hat\kappa)} v(\hat\xi_p)^2\hat{\mathsf{w}}_{\xi_p} \neq \int_{\hat\kappa} v^2\,dx \qquad \forall v \in \mathbb{Q}^{p}(\hat\kappa).$$

However, by defining, for $v \in \mathbb{Q}^p(\hat\kappa)$, the following norm

$$\|v\|^2_{0,p,\hat\kappa} := \sum_{\xi_p\in\mathcal{N}(\hat\kappa)} v(\hat\xi_p)^2\hat{\mathsf{w}}_{\xi_p},$$

it can be proved that

$$\|v\|^2_{0,p,\hat\kappa} \approx \|v\|^2_{L^2(\hat\kappa)}, \tag{6}$$

cf. [18, Section 5.3]. The same result holds for the physical frame κ, i.e., $\|v\|^2_{0,p,\kappa} \approx \|v\|^2_{L^2(\kappa)}$. Considering the Lagrange basis $\{\phi_{\xi_p}\}$, $\xi_p \in \bigcup_{\kappa\in\mathcal{T}_h}\mathcal{N}(\kappa)$, we can write any $v \in V_{hp}$ as

$$v = \sum_{\kappa\in\mathcal{T}_h}\sum_{\xi_p\in\mathcal{N}(\kappa)} v(\xi_p)\phi_{\xi_p} = \sum_{\kappa\in\mathcal{T}_h}\sum_{\xi_p\in\mathcal{N}(\kappa)} v^{\xi_p}, \tag{7}$$

where we note that $v^{\xi_p} = v(\xi_p)\phi_{\xi_p}$.

Lemma 3 *For any $v \in V_{hp}$, given the decomposition (7), the following equivalence holds*

$$\|v\|^2_{L^2(\Omega)} \approx \sum_{\kappa\in\mathcal{T}_h}\sum_{\xi_p\in\mathcal{N}(\kappa)} \|v^{\xi_p}\|^2_{L^2(\kappa)}.$$

Proof The proof can be restricted to the case of a single element $\kappa \in \mathcal{T}_h$. We write $v \in V_{hp}$ as in (7), and observe that

$$\|v^{\xi_p}\|^2_{0,p,\kappa} = \sum_{\xi'_p \in \mathcal{N}(\kappa)} v^{\xi_p}(\xi'_p)^2 \mathbf{w}_{\xi_p} = v^{\xi_p}(\xi_p)^2 \mathbf{w}_{\xi_p},$$

hence, by (6),

$$\begin{aligned}\|v\|^2_{L^2(\kappa)} &\approx \sum_{\xi_p \in \mathcal{N}(\kappa)} v(\xi_p)^2 \mathbf{w}_{\xi_p} = \sum_{\xi_p \in \mathcal{N}(\kappa)} v^{\xi_p}(\xi_p)^2 \mathbf{w}_{\xi_p} \\ &= \sum_{\xi_p \in \mathcal{N}(\kappa)} \|v^{\xi_p}\|^2_{0,p,\kappa} \approx \sum_{\xi_p \in \mathcal{N}(\kappa)} \|v^{\xi_p}\|^2_{L^2(\kappa)},\end{aligned}$$

and the thesis follows summing over all $\kappa \in \mathcal{T}_h$. □

4 Space Decomposition for High-Order DG Methods

The design of our preconditioner is based on a two-stage space decomposition: we first split the high-order DG space as $V_{hp} = V^B_{hp} + V^C_{hp}$, with V^B_{hp} denoting a proper subspace of V_{hp}, to be defined later, and V^C_{hp} denoting the high-order conforming subspace. As a second step, both spaces are further decomposed to build two corresponding additive Schwarz methods in each of the subspaces. The final preconditioner on V_{hp} is then obtained by combining the two subspace preconditioners. The first space splitting is suggested by the interpretation of the high-order DG formulation (2) as a nearly-singular problem. To present the motivation behind this choice, we briefly introduce the theoretical framework of [39] regarding space decomposition methods for this class of equations. Given a finite dimensional Hilbert space V, we consider the following problem: find $U \in V$ such that

$$AU = (A_0 + \epsilon A_1)U = F, \tag{8}$$

where F is a given vector and where A_0 is symmetric and positive semi-definite and A_1 is symmetric and positive definite. As a consequence, if $\epsilon = 0$, the problem is singular, but here we are interested in the case $\epsilon > 0$ (with ϵ small, $\epsilon \ll 1$), i.e., (8) is nearly-singular. In general, the conditioning of problem (8) degenerates for decreasing ϵ, and this affects the performance of standard preconditioned iterative methods, unless proper initial guess are chosen. In the framework of space decomposition methods, in order to obtain a ϵ-uniform preconditioner, a key assumption on the space splitting $V_{hp} = \sum_{i=1}^N V_i$ is needed.

Assumption 1 ([39]) The decomposition $V_{hp} = \sum_{i=1}^N V_i$ satisfies

$$\ker(A_0) = \sum_{i=1}^N (V_i \cap \ker(A_0)),$$

where $\ker(A_0)$ is the kernel of A_0.

We now turn to our DG framework, and show that a high-order DG formulation can be indeed read as a nearly-singular problem with a suitable choice of ϵ. For the sake of simplicity, and without any loss of generality, we retrieve Eq. (8) working directly on a bilinear form that is spectrally equivalent to $\mathcal{A}(\cdot,\cdot)$. To this aim, let the bilinear forms $\mathcal{A}_\nabla(\cdot,\cdot)$, $\mathcal{A}_J(\cdot,\cdot)$ and $\widetilde{\mathcal{A}}(\cdot,\cdot)$ be defined as

$$\mathcal{A}_\nabla(u,v) := \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \nabla u\cdot\nabla v\,dx,$$

$$\mathcal{A}_J(u,v) := \sum_{F\in\mathcal{F}_h}\int_F [\![u]\!]\cdot[\![v]\!]\,ds,$$

$$\widetilde{\mathcal{A}}(u,v) := \mathcal{A}_\nabla(u,v) + \alpha\frac{p^2}{h}\mathcal{A}_J(u,v),$$

and let A_∇, A_J, and $\widetilde{A}$ be their corresponding operators. Clearly, A_∇ and A_J are both symmetric and positive semi-definite, and $\widetilde{A}$ is symmetric and positive definite. Moreover, thanks to Lemma 1, and the local quasi-uniformity of the partition, the following spectral equivalence result holds

$$\mathcal{A}(u,u) \approx \|u\|^2_{DG} \approx \widetilde{\mathcal{A}}(u,u).$$

From the spectral equivalence shown above, it is immediate to conclude that any optimal preconditioner for $\widetilde{A}$ is automatically an optimal preconditioner for A. Therefore, in the following we focus our attention on the following problem

$$\widetilde{A}\widetilde{u} = \left(A_\nabla + \frac{1}{\epsilon}A_J\right)\widetilde{u} = f, \tag{9}$$

with $\epsilon := h/(\alpha p^2) < 1$. After some simple calculations, we can write (9) as

$$[\epsilon(A_\nabla + A_J) + (1-\epsilon)A_J]\,\widetilde{u} = \epsilon f,$$

which corresponds to (8) with $A_1 = A_\nabla + A_J$ and $A_0 = (1-\epsilon)A_J$. In order to obtain a suitable space splitting satisfying Assumption 1, we observe that, according to the definition above, the kernel of A_0 is given by the space of continuous polynomial functions of degree p vanishing on the boundary $\partial\Omega$. We then derive the first space decomposition

$$V_{hp} = V^B_{hp} + V^C_{hp}, \tag{10}$$

with

$$V^B_{hp} := \left\{ v\in V_{hp} : v(\xi_p) = 0 \quad \forall \xi_p \in \bigcup_{\kappa\in\mathcal{T}_h}\mathcal{N}_1(\kappa)\right\},$$

$$V^C_{hp} := \{v\in C^0(\overline{\Omega}) : v\circ \mathsf{F}_\kappa \in \mathbb{Q}^p(\hat{\kappa}) \quad \forall\kappa\in\mathcal{T}_h,\ v|_{\partial\Omega} = 0\} \subseteq H^1_0(\Omega),$$

i.e., V^B_{hp} consists of the functions in V_{hp} that are null in any degree of freedom in the interior of any $\kappa\in\mathcal{T}_h$. Moreover, we observe that $V^B_{hp}\subset V_{hp}$, and $V^B_{hp}\cap V^C_{hp}\subset V^C_{hp}$, hence Assumption 1 is satisfied by decomposition (10), which is the basis to develop the analysis of our preconditioner for problem (2).

4.1 Technical Results

In this subsection we present several results, which are fundamental for the forthcoming analysis. We introduce a suitable interpolation operator $\mathsf{Q}_\mathsf{h} : V_{hp}\to V^C_{hp}$, consisting of the Oswald operator, cf. [16,17,28,34,37]. For any $v\in V_{hp}$, we can define on each $\kappa\in\mathcal{T}_h$ the action of the operator Q_h, by prescribing the value of $\mathsf{Q}_\mathsf{h}v$ in any $\xi_p\in\mathcal{N}(\kappa)$:

$$\mathsf{Q_h} v(\xi_p) := \begin{cases} 0 & \text{if } \xi_p \in \partial\Omega, \\ \dfrac{1}{\text{card}(\mathcal{T}_{\xi_p})} \displaystyle\sum_{\kappa \in \mathcal{T}_{\xi_p}} v|_\kappa(\xi_p) & \text{otherwise,} \end{cases} \tag{11}$$

with $\mathcal{T}_{\xi_p} := \{\kappa' \in \mathcal{T}_h : \xi_p \in \kappa'\}$. Note that from the above definition it follows that $v - \mathsf{Q_h} v \in V_{hp}^B$, for any $v \in V_{hp}$. In addition to the space of polynomials $\mathbb{Q}^p(\kappa)$, we define $\mathbb{Q}_0^p(\kappa)$ as

$$\mathbb{Q}_0^p(\kappa) := \{v \in \mathbb{Q}^p(\kappa) : v(\xi_p) = 0 \quad \forall \xi_p \in \mathcal{N}_1(\kappa)\},$$

and state the following trace and inverse trace inequalities.

Lemma 4 ([17, Lemma 3.1]) *The following trace and inverse trace inequalities hold*

$$\begin{aligned} \|v\|_{L^2(\partial\kappa)}^2 &\lesssim \frac{p^2}{h_\kappa} \|v\|_{L^2(\kappa)}^2 \ \forall v \in \mathbb{Q}^p(\kappa), \\ \|v\|_{L^2(\kappa)}^2 &\lesssim \frac{h_\kappa}{p^2} \|v\|_{L^2(\partial\kappa)}^2 \ \forall v \in \mathbb{Q}_0^p(\kappa). \end{aligned} \tag{12}$$

The next result is a keypoint for the forthcoming analysis, and can be found in [34] and [17, Lemma 3.2].

Lemma 5 ([34] and [17, Lemma 3.2]) *For any $v \in V_{hp}$, the following estimate holds*

$$\|v - \mathsf{Q_h} v\|_{L^2(\kappa)}^2 \lesssim \frac{h_\kappa}{p^2} \sum_{F \in \mathcal{F}_h(\kappa)} \|[\![v]\!]\|_{L^2(F)}^2,$$

with $\mathcal{F}_h(\kappa) := \{F \in \mathcal{F}_h : F \cap \kappa \neq \emptyset\}$.

Thanks to Lemma 5 we can prove the following theorem.

Theorem 1 *For any $v \in V_{hp}$, it holds that*

$$\mathcal{A}(v - \mathsf{Q_h} v, v - \mathsf{Q_h} v) + \mathcal{A}(\mathsf{Q_h} v, \mathsf{Q_h} v) \lesssim \mathcal{A}(v, v), \tag{13}$$

where $\mathsf{Q_h} v \in V_{hp}^C$ is defined as in (11). Then the space decomposition defined in (10) is stable.

Proof We observe that, from (5), the quasi-uniformity of the mesh and Lemma 5, we obtain

$$\begin{aligned} \mathcal{A}(v - \mathsf{Q_h} v, v - \mathsf{Q_h} v) &\lesssim \sum_{\kappa \in \mathcal{T}_h} \alpha \frac{p^4}{h_\kappa^2} \|v - \mathsf{Q_h} v\|_{L^2(\kappa)}^2 \lesssim \alpha \sum_{\kappa \in \mathcal{T}_h} \frac{p^4}{h_\kappa^2} \frac{h_\kappa}{p^2} \sum_{F \in \mathcal{F}_h(\kappa)} \|[\![v]\!]\|_{L^2(F)}^2 \\ &\lesssim \sum_{F \in \mathcal{F}_h} \|\sigma^{1/2} [\![v]\!]\|_{L^2(F)}^2 \lesssim \mathcal{A}(v, v). \end{aligned}$$

The upper bound (13) follows from the triangle inequality and the above estimate

$$\mathcal{A}(\mathsf{Q_h} v, \mathsf{Q_h} v) \leq \mathcal{A}(v - \mathsf{Q_h} v, v - \mathsf{Q_h} v) + \mathcal{A}(v, v) \lesssim \mathcal{A}(v, v).$$

For any $v \in V_{hp}$, we recall that $v - \mathsf{Q_h} v \in V_{hp}^B$, which implies

$$\inf_{\substack{v^B \in V_{hp}^B, v^C \in V_{hp}^C \\ v^B + v^C = v}} \mathcal{A}(v^B, v^B) + \mathcal{A}(v^C, v^C) \leq \mathcal{A}(v - \mathsf{Q_h} v, v - \mathsf{Q_h} v) + \mathcal{A}(\mathsf{Q_h} v, \mathsf{Q_h} v) \lesssim \mathcal{A}(v, v).$$

□

5 Construction and Analysis of the Preconditioner

In this section we introduce our preconditioner and analyze the condition number of the preconditioned system. Employing the nomencalture of [48], the preconditioner is a parallel subspace correction method (also known as additive Schwarz preconditioner, see. e.g., [27,40,47]). Our construction uses a decomposition in two subspaces, cf. (10), and inexact subspace solvers. Each of the subspace solvers is a parallel subspace correction method itself.

5.1 Canonical Representation of a Parallel Subspace Correction Method

The main ingredients needed for the analysis of the parallel subspace correction (PSC) preconditioners are suitable space splittings and the corresponding subspace solvers (see [27,31,40,46–48,50]). In our analysis we use the notation and the general setting from [50]. We have the following abstract result.

Lemma 6 ([50, Lemma 2.4]) *Let V be a Hilbert space which is decomposed as $V = \sum_{i=1}^{N} V_i$, $V_i \subset V$, $i = 1, \ldots, N$, and $T_i : V \to V_i$, $i = 1, \ldots, N$ be operators whose restrictions on V_i are symmetric and positive definite. For $T := \sum_{i=1}^{N} T_i$ the following identity holds*

$$\mathcal{A}(T^{-1}v, v) = \inf_{\substack{v_i \in V_i \\ \sum v_i = v}} \sum_{i=1}^{N} \mathcal{A}(T_i^{-1} v_i, v_i). \tag{14}$$

According to the above lemma, to show a bound on the condition number of the preconditioned system we need to show that there exist positive constants c and C such that

$$c\mathcal{A}(v, v) \leq \mathcal{A}(T^{-1}v, v) \leq C\mathcal{A}(v, v).$$

Remark 1 In many cases we have $T_i = P_i$, $i = 1, \ldots, N$, where $P_i : V \to V_i$ are the elliptic projections defined as follows: for $v \in V$, its projection $P_i v$ is the unique element of V_i satisfying $\mathcal{A}(P_i v, v_i) := \mathcal{A}(v, v_i)$, for all $v_i \in V_i$. Note that by definition, P_i is the identity on V_i, namely, $P_i v_i = v_i = P_i^{-1} v_i$, for all $v_i \in V_i$. Hence, for $T = \sum_{i=1}^{N} P_i$, the relation (14) gives

$$\mathcal{A}(T^{-1}v, v) = \inf_{\substack{v_i \in V_i \\ \sum v_i = v}} \sum_{i=1}^{N} \mathcal{A}(v_i, v_i). \tag{15}$$

5.2 Space Splitting and Subspace Solvers

To fix the notation, let us point out that in what follows we use T (with subscript when necessary) to denote (sub)space solvers and preconditioners. Accordingly, P with subscript or superscript denotes elliptic projection on the corresponding subspace.

We now define the space splitting and the corresponding subspace solvers. We recall the space decomposition from Sect. 4, $V_{hp} = V_{hp}^B + V_{hp}^C$, where V_{hp}^B are all functions in V_{hp} for which the degrees of freedom in the interior of any $\kappa \in \mathcal{T}_h$ vanish, and V_{hp}^C is the space of high-order continuous polynomials vanishing on $\partial\Omega$. Note that $V_{hp}^B \cap V_{hp}^C \neq \{0\}$, and that V_{hp}^B contains non-smooth and oscillatory functions, while V_{hp}^C contains the smooth part of the space V_{hp}. Next, on each of these subspaces we define approximate solvers $T_B : V_{hp} \to V_{hp}^B$ and $T_C : V_{hp} \to V_{hp}^C$.

First, we decompose V_{hp}^B as follows

$$V_{hp}^B = \sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}_{\mathrm{B}}(\kappa)} V^{\xi_p}, \tag{16}$$

where

$$V^{\xi_p} := \left\{ v \in V_{hp}^B : v(\xi_p{}') = 0 \text{ for any } \xi_p{}' \in \left(\bigcup_{\kappa \in \mathcal{T}_h} \mathcal{N}_{\mathrm{B}}(\kappa) \right) \setminus \{\xi_p\} \right\}.$$

The approximate solver on V_B then is a simple Jacobi method, defined as

$$T_B : V_{hp} \to V_{hp}^B, \quad T_B := \left[\sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}_{\mathrm{B}}(\kappa)} P^{\xi_p} \right] P_B.$$

where P_B and P^{ξ_p} are the elliptic projections on V_{hp}^B and V^{ξ_p}, respectively. Note that T_B is defined on all of V_{hp} and is also an isomorphism when restricted to V_{hp}^B, because the elliptic projection P_B and P^{ξ_p} are the identity on V_{hp}^B and V^{ξ_p}, respectively. In addition, the splitting is a direct sum, and, hence, any $v \in V_{hp}^B$ is uniquely represented as $v = \sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}_{\mathrm{B}}(\kappa)} v^{\xi_p}$, $v^{\xi_p} \in V^{\xi_p}$. Then, taking $P_i = P^{\xi_p} P_B : V_{hp} \to V^{\xi_p}$, from (15), we have

$$\mathcal{A}(T_B^{-1} v^B, v^B) = \sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}_{\mathrm{B}}(\kappa)} \mathcal{A}(v^{\xi_p}, v^{\xi_p}), \quad \forall \ v^B \in V_{hp}^B. \tag{17}$$

Next, we introduce the preconditioner T_C on V_{hp}^C. This is the two-level overlapping additive Schwarz method introduced in [42,43] for high-order *conforming* discretizations. If we denote by N_V the number of interior vertices of $\mathcal{T}_h$, then this preconditioner corresponds to the following decomposition of V_{hp}^C:

$$V_{hp}^C = \sum_{i=0}^{N_V} V_i^C. \tag{18}$$

Here V_0^C is the (coarse) space of continuous piecewise linear functions on $\mathcal{T}_h$, and for $i = 1, \ldots, N_V$, $V_i^C := V_{hp}^C \cap H_0^1(\Omega_i)$, where Ω_i is the union of the elements sharing the i-th vertex (see Fig. 1 for a two-dimensional example). We recall that, in the case of Neumann and mixed boundary conditions, in order to obtain a uniform preconditioner, the decomposition (18) should be enriched with the subdomains associated to those vertices not lying on a Dirichlet boundary, see [42,43] for details.

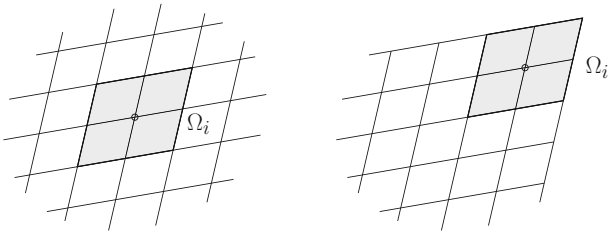

Fig. 1 Examples of subdomains in a two-dimensional setting

Then, for any $V_i^C, i = 0, \ldots, N_V$, we denote by $P_i^C : V_{hp}^C \to V_i^C$ the elliptic projections on V_i^C and define the two-level overlapping additive Schwarz operator as

$$T_C : V_{hp} \to V_{hp}^C, \quad T_C := \left[P_0^C + \sum_{i=1}^{N_V} P_i^C \right] P_C = (P_0 + P_V) P_C, \tag{19}$$

where P_C is the elliptic projection on V_{hp}^C. As in the case of T_B, we have that the restriction of T_C on V_{hp}^C is an isomorphism. In addition, from (15) with $P_i = P_i^C P_C : V_{hp} \to V_i^C$, we have

$$\mathcal{A}(T_C^{-1} v, v) = \inf_{\substack{v_i \in V_i^C \\ \sum v_i = v}} \sum_{i=0}^{N_V} \mathcal{A}(v_i, v_i). \tag{20}$$

Observe that, especially for three dimensional problems, the dimension of the coarse space can be quite large, for example when the underlying mesh is very fine. In these cases, the solution of the corresponding coarse subproblem through a direct method can be computationally unfeasible. However, since the coarse level is the piece-wise linear conforming subspace, the associated linear system of equations can be efficiently solved by further preconditioning it with one of the efficient techniques already available for this class of problems, for example (Algebraic) Multigrid methods.

5.3 Definition of the Global Preconditioner

Finally, we define the global preconditioner on V_{hp} by setting

$$T_{DG} : V_{hp} \to V_{hp}, \quad T_{DG} := T_B + T_C, \tag{21}$$

We remark that from Lemma 6, with $N = 2$, $T_1 = T_B$, $V_1 = V_{hp}^B$, $T_2 = T_C$, $V_2 = V_{hp}^C$, we have

$$\mathcal{A}(T_{DG}^{-1} v, v) = \inf_{\substack{v^B \in V_{hp}^B, v^C \in V_{hp}^C \\ v^B + v^C = v}} \left[\mathcal{A}(T_B^{-1} v^B, v^B) + \mathcal{A}(T_C^{-1} v^C, v^C) \right]. \tag{22}$$

5.4 Condition Number Estimates: Subspace Solvers

We now show the estimates on the conditioning of the subspace solvers needed to bound the condition number of T_{DG}. The first result that we prove is on the conditioning of T_B.

Lemma 7 *Let T_B denote the Jacobi preconditioner defined in (5.2). Then there exist two positive constants C_1^J and C_2^J, independent of the granularity of the mesh h, the polynomial approximation degree p and the penalization coefficient α, such that*

$$\mathcal{A}(T_B^{-1} v^B, v^B) \geq \mathsf{C}_1^J \mathcal{A}(v^B, v^B) \quad \forall v^B \in V_{hp}^B \tag{23}$$

$$\mathcal{A}(T_B^{-1}(v - \mathsf{Q}_\mathsf{h} v), v - \mathsf{Q}_\mathsf{h} v) \leq \mathsf{C}_2^J \mathcal{A}(v - \mathsf{Q}_\mathsf{h} v, v - \mathsf{Q}_\mathsf{h} v) \quad \forall v \in V_{hp}, \tag{24}$$

with $\mathsf{Q}_\mathsf{h} v$ defined in (11).

Proof We refer to the space decomposition (16) and write

$$v^B = \sum_{\kappa \in \mathcal{T}_h} \sum_{\xi_p \in \mathcal{N}_\mathrm{B}(\kappa)} v^{\xi_p}.$$

For the lower bound (23), we employ the eigenvalue estimate (5) and Lemma 3, thus obtaining

$$\mathcal{A}(v^B, v^B) \lesssim \sum_{\kappa\in\mathcal{T}_h} \alpha \frac{p^4}{h_\kappa^2} \|v^B\|^2_{L^2(\kappa)} \lesssim \sum_{\kappa\in\mathcal{T}_h} \alpha \frac{p^4}{h_\kappa^2} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \|v^{\xi_p}\|^2_{L^2(\kappa)}.$$

We now observe that for any $\xi_p \in \mathcal{N}_{\mathrm{B}}(\kappa)$, $v^{\xi_p} \in \mathbb{Q}_0^p(\kappa)$, and we can thus apply the inverse trace inequality (12) to obtain

$$\mathcal{A}(v^B, v^B) \lesssim \sum_{\kappa\in\mathcal{T}_h} \alpha \frac{p^4}{h_\kappa^2} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \|v^{\xi_p}\|^2_{L^2(\kappa)} \lesssim \sum_{\kappa\in\mathcal{T}_h} \alpha \frac{p^2}{h_\kappa} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \|v^{\xi_p}\|^2_{L^2(\partial\kappa)}.$$

Noting that $\|v^{\xi_p}\|^2_{L^2(\partial\kappa)} = \|[\![v^{\xi_p}]\!]\|^2_{L^2(\partial\kappa)}$, it follows that

$$\begin{aligned}\mathcal{A}(v^B, v^B) &\lesssim \sum_{\kappa\in\mathcal{T}_h} \alpha \frac{p^2}{h_\kappa} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \|v^{\xi_p}\|^2_{L^2(\partial\kappa)} \lesssim \sum_{\kappa\in\mathcal{T}_h} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \|\sigma^{1/2}[\![v^{\xi_p}]\!]\|^2_{L^2(\partial\kappa)} \\ &\lesssim \sum_{\kappa\in\mathcal{T}_h} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \|v^{\xi_p}\|^2_{DG},\end{aligned}$$

and the thesis follows from the coercivity bound (4) and (17). With regard to the upper bound (24), for the sake of simplicity we denote $w = (I - \mathsf{Q}_\mathsf{h})v$, and observe that $w = (I - \mathsf{Q}_\mathsf{h})w$. Since $w \in V^B_{hp}$, we write

$$w = \sum_{\kappa\in\mathcal{T}_h} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} w^{\xi_p},$$

and, from (17),

$$\mathcal{A}(T_B^{-1}w, w) = \sum_{\kappa\in\mathcal{T}_h} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \mathcal{A}(w^{\xi_p}, w^{\xi_p}).$$

Applying again the estimate (5) and Lemma 3, we obtain

$$\begin{aligned}\sum_{\kappa\in\mathcal{T}_h} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \mathcal{A}(w^{\xi_p}, w^{\xi_p}) &\lesssim \sum_{\kappa\in\mathcal{T}_h} \sum_{\xi_p\in\mathcal{N}_{\mathrm{B}}(\kappa)} \alpha \frac{p^4}{h_\kappa^2} \|w^{\xi_p}\|^2_{L^2(\kappa)} \lesssim \sum_{\kappa\in\mathcal{T}_h} \alpha \frac{p^4}{h_\kappa^2} \|w\|^2_{L^2(\kappa)} \\ &\lesssim \sum_{\kappa\in\mathcal{T}_h} \alpha \frac{p^4}{h_\kappa^2} \|(I - \mathsf{Q}_\mathsf{h})w\|^2_{L^2(\kappa)} \lesssim \mathcal{A}(w, w),\end{aligned}$$

where the last steps follows from Lemma 5 and the quasi-uniformity of the mesh. □

For the analysis of the additive preconditioner T_C given in (19), we need several preliminary results (see [42,43] for additional details). First of all, given the decomposition

$$v = v_0 + \sum_{i=1}^{N_V} v_i \qquad \forall v \in V^C_{hp},\ v_0 \in V^C_0,\ v_i \in V^C_i, \tag{25}$$

we define the coarse function v_0 as the L^2-projection on the space V^C_0, i.e., $v_0 := \mathcal{I}_0 v$ with $\mathcal{I}_0 v$ satisfying

$$\|v - {}_0v\|^2_{L^2(\Omega)} \lesssim h^2 |v|^2_{H^1(\Omega)}, \tag{26}$$

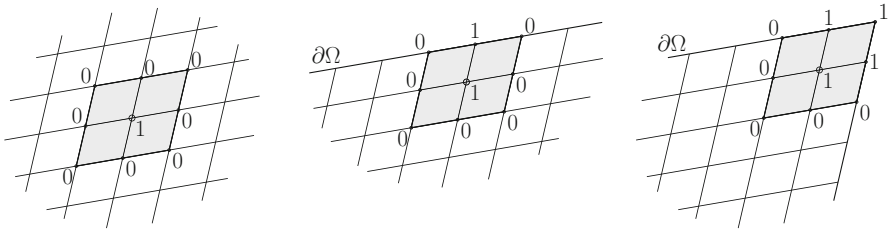

Fig. 2 Values of the partition of unity θ_i for $d = 2$

$$|\mathcal{I}_0 v|^2_{H^1(\Omega)} \lesssim |v|^2_{H^1(\Omega)}, \tag{27}$$

for any $v \in H^1_0(\Omega)$. For any $i = 1, \ldots, N_V$, the functions v_i appearing in (25) are defined as

$$v_i := I_p(\theta_i(v - v_0)),$$

where θ_i is a proper partition of unity and I_p is an interpolation operator, described in the following. For any Ω_i, $i = 1, \ldots, N_V$, the partition of unity θ_i is such that $\theta_i \in V^C_{h1}$ and it can be defined by prescribing its values at the vertices $\{\mathsf{v}\}$ belonging to $\overline{\Omega}_i$, and imposing it to be zero on $\Omega \setminus \overline{\Omega}_i$, see Fig. 2 for $d = 2$. More precisely,

$$\theta_i(\mathsf{v}) = \begin{cases} 1 & \text{if } \mathsf{v} \text{ is the internal vertex or } \mathcal{F}_\mathsf{v} \subset \mathcal{F}^B_h, \\ 0 & \text{otherwise,} \end{cases}$$

with $\mathcal{F}_\mathsf{v} := \{F \in \mathcal{F}_h, F \subseteq \partial\Omega_i : \mathsf{v} \in F\}$.

It follows that:

$$\text{supp}(\theta_i) = \Omega_i, \qquad 0 \le \theta_i \le 1, \qquad \sum_{i=1}^{N_V} \theta_i = 1, \qquad |\nabla \theta_i| \lesssim \frac{1}{h}. \tag{28}$$

As interpolation operator I_p, we make use of the operator defined in [42,43]: setting $z := v - v_0$, we define

$$I_p(\theta_i z)(\xi_p) = (\theta_i z)(\xi_p) \quad \forall \xi_p \in \mathcal{N}(\kappa), \forall \kappa \in \Omega_i. \tag{29}$$

Notice that, despite defined locally, $I_p(\theta_i z)$ belongs to V^C_i since the interelement continuity is guaranteed by the fact that the $(p+1)^{d-1}$ GLL points on a face uniquely determine a tensor product polynomial of degree p defined on that face. The following result holds.

Lemma 8 ([42, Lemma 3.1, Lemma 3.3]) *The interpolation operator $I_p : \mathbb{Q}^{p+1}(\hat{\kappa}) \to \mathbb{Q}^p(\hat{\kappa})$, defined in (29), is bounded uniformly in the H^1 seminorm, i.e.,*

$$|I_p(u)|_{H^1(\hat{\kappa})} \lesssim |u|_{H^1(\hat{\kappa})} \qquad \forall u \in \mathbb{Q}^{p+1}(\hat{\kappa}). \tag{30}$$

Once the partition of unity and the interpolation operator are defined, we are able to complete the analysis of T_C. In analogy to Lemma 7, which is based on (17), we now use (20) and the above auxiliary results to show the following lemma.

Lemma 9 *Let T_C denote the two-level overlapping additive Schwarz preconditioner defined in (19). Then there exist two positive constants C^C_1 and C^C_2, independent of the discretization*

parameters, i.e., the granularity of the mesh h and the polynomial approximation degree p, such that

$$\mathcal{A}(T_C^{-1}v, v) \geq \mathsf{C}_1^C \mathcal{A}(v, v) \tag{31}$$

$$\mathcal{A}(T_C^{-1}v, v) \leq \mathsf{C}_2^C \mathcal{A}(v, v), \tag{32}$$

for any $v \in V_{hp}^C$.

Proof We first prove the lower bound (31), and given the decomposition (25), we can write

$$\mathcal{A}(v, v) = \sum_{i,j=0}^{N_V} \mathcal{A}(v_i, v_j) \lesssim \mathcal{A}(v_0, v_0) + \sum_{i,j=1}^{N_V} \mathcal{A}(v_i, v_j).$$

We now note that $\mathcal{A}(v_i, v_j) \neq 0$ only if $i = j$ and $\Omega_i \cap \Omega_j \neq \emptyset$, and since each Ω_i is overlapped by a limited number of neighboring subdomains, we conclude that

$$\mathcal{A}(v, v) \lesssim \mathcal{A}(v_0, v_0) + \sum_{i,j=1}^{N_V} \mathcal{A}(v_i, v_j) \lesssim \mathcal{A}(v_0, v_0) + \sum_{i=1}^{N_V} \mathcal{A}(v_i, v_i).$$

Inequality (31) follows from the bound above and (20), denoting with C_1^C the hidden constant. Note that, from (20), the upper bound (32) is proved provided the following inequality holds

$$\sum_{i=0}^{N_V} \mathcal{A}(v_i, v_i) \leq \mathsf{C}_2^C \mathcal{A}(v, v) \qquad \forall v \in V_{hp}^C. \tag{33}$$

We recall that $v_0 = \mathcal{I}_0 v$, and from (27) it follows that

$$\mathcal{A}(v_0, v_0) = \mathcal{A}({}_0v, {}_0v) \lesssim \mathcal{A}(v, v). \tag{34}$$

For $i = 1, \ldots, N_V$, we have $v_i = I_p(\theta_i z)$, with $z = v - v_0$, and by (30), we obtain

$$|v_i|^2_{H^1(\kappa')} \lesssim |\theta_i z|^2_{H^1(\kappa')} \lesssim \sum_{j=1}^{d} \left\| \frac{\partial \theta_i}{\partial x_j} z + \theta_i \frac{\partial z}{\partial x_j} \right\|^2_{L^2(\kappa')},$$

for any $\kappa' \in \Omega_i$. By (28) it holds that

$$|\nabla \theta_i| \lesssim \frac{1}{h}, \qquad \|\theta_i\|_{L^\infty} \leq 1,$$

hence,

$$|v_i|^2_{H^1(\kappa')} \lesssim \frac{1}{h^2}\|z\|^2_{L^2(\kappa')} + \sum_{j=1}^{d} \|\frac{\partial z}{\partial x_j}\|^2_{L^2(\kappa')} \lesssim \frac{1}{h^2}\|v - v_0\|^2_{L^2(\kappa')} + |v - v_0|^2_{H^1(\kappa')}.$$

On any element κ', a limited number of components v_i are different from zero (at most four for $d = 2$, and eight for $d = 3$), which implies that we can sum over all the components v_i, $i = 1, \ldots, N_V$, and then over all the elements, thus obtaining

$$\sum_{i=1}^{N_V} |v_i|^2_{H^1(\Omega)} \lesssim \frac{1}{h^2}\|v - v_0\|^2_{L^2(\Omega)} + |v - v_0|^2_{H^1(\Omega)} \lesssim |v|^2_{H^1(\Omega)},$$

where the last step follows from (26) and (27). The addition of the above result and (34), gives (33), denoting with C_2^C the resulting hidden constant. □

5.5 Condition Number Estimates: Global Preconditioner

We are now ready to prove the main result of the paper regarding the condition number of the preconditioned problem.

Theorem 2 *Let T_{DG} be defined as in (21). Then, for any $v \in V_{hp}$, it holds that*

$$\mathcal{A}(v, v) \lesssim \mathcal{A}(T_{DG}^{-1} v, v) \lesssim \mathcal{A}(v, v), \tag{35}$$

where the hidden constants are independent of the discretization parameters, i.e., the mesh size h, the polynomial approximation degree p, and the penalization coefficient α.

Before proving Theorem 2, we note the following: As the operator T_{DG} is symmetric and positive definite, from Theorem 2 it immediately follows that the spectral condition number of the preconditioned operator T_{DG}

$$\kappa(T_{DG}) := \frac{\lambda_{\max}(T_{DG})}{\lambda_{\min}(T_{DG})} \tag{36}$$

is uniformly bounded, i.e. $\kappa(T_{DG}) \lesssim 1$, where the hidden constant is independent of the the mesh size h, the polynomial approximation degree p, and the penalization coefficient α. In (36) $\lambda_{\max}(T_{DG})$ and $\lambda_{\min}(T_{DG})$ denotes the extremal eigenvalues of T_{DG} defined as

$$\lambda_{\max}(T_{DG}) := \sup_{0 \neq v \in V_{hp}} \frac{\mathcal{A}(T_{DG} v, v)}{\mathcal{A}(v, v)},$$
$$\lambda_{\min}(T_{DG}) := \inf_{0 \neq v \in V_{hp}} \frac{\mathcal{A}(T_{DG} v, v)}{\mathcal{A}(v, v)}.$$

As a consequence, the conjugate gradient algorithm can be employed to solve the preconditioned linear system of equations and the number of iterations needed to reduce the norm of the (relative) residual below a given threshold is expected to be uniformly constant independently of the size of the underlying matrix, see, e.g., [46] for more details.

Proof (Proof of Theorem 2). To prove the upper bound, we first consider the identity (22). Recalling that, by definition (11), $v - \mathsf{Q}_\mathsf{h} v \in V_{hp}^B$, for any $v \in V_{hp}$, we obtain

$$\begin{aligned}\mathcal{A}(T_{DG}^{-1} v, v) &= \inf_{\substack{v^B \in V_{hp}^B, v^C \in V_{hp}^C \\ v^B + v^C = v}} \left[\mathcal{A}(T_B^{-1} v^B, v^B) + \mathcal{A}(T_C^{-1} v^C, v^C) \right] \\ &\leq \mathcal{A}(T_B^{-1}(v - \mathsf{Q}_\mathsf{h} v), v - \mathsf{Q}_\mathsf{h} v) + \mathcal{A}(T_C^{-1} \mathsf{Q}_\mathsf{h} v, \mathsf{Q}_\mathsf{h} v).\end{aligned}$$

From the bounds (24) and (32) for $\mathsf{Q}_\mathsf{h} v$, it follows that

$$\begin{aligned}\mathcal{A}(T_{DG}^{-1} v, v) &\leq \mathcal{A}(T_B^{-1}(v - \mathsf{Q}_\mathsf{h} v), v - \mathsf{Q}_\mathsf{h} v) + \mathcal{A}(T_C^{-1} \mathsf{Q}_\mathsf{h} v, \mathsf{Q}_\mathsf{h} v) \\ &\lesssim \mathcal{A}(v - \mathsf{Q}_\mathsf{h} v, v - \mathsf{Q}_\mathsf{h} v) + \mathcal{A}(\mathsf{Q}_\mathsf{h} v, \mathsf{Q}_\mathsf{h} v) \lesssim \mathcal{A}(v, v),\end{aligned}$$

where the last step follows from (13). The lower bound follows from (22), the bounds (23) and (31), and a triangle inequality

$$\begin{aligned}\mathcal{A}(T_{DG}^{-1}v, v) &= \inf_{\substack{v^B \in V_{hp}^B, v^C \in V_{hp}^C \\ v^B + v^C = v}} \left[\mathcal{A}(T_B^{-1}v^B, v^B) + \mathcal{A}(T_C^{-1}v^C, v^C)\right] \\ &\gtrsim \inf_{\substack{v^B \in V_{hp}^B, v^C \in V_{hp}^C \\ v^B + v^C = v}} \left[\mathcal{A}(v^B, v^B) + \mathcal{A}(v^C, v^C)\right] \gtrsim \mathcal{A}(v, v).\end{aligned}$$

□

6 Numerical Experiments

In this section we present some numerical tests to verify the theoretical estimates provided in Lemma 7, Lemma 9 and Theorem 2, as well as to test the performance of our preconditioner in the case of a variable diffusion tensor. To this aim we present several tests with anisotropies and/or heterogeneities in the diffusion coefficients following the lines of those proposed in [25]. As these tests show, the preconditioner is robust for problems with tensor coefficients, even when strong anisotropies are presented, as long as the variation of these coefficients throughout the domain is bounded. As expected, a deterioration in the condition number of the preconditioned system is observed when the coefficients have large jumps. This behavior is due to the use of a standard conforming coarse space. More complex constructions of coarse spaces depending on the coefficient variation are possible, but they are beyond the scope of our considerations here.

Before presenting the numerical results assessing the practical performance of our preconditioner, we briefly provide some implementation details. The preconditioner is composed by the sum of two operators: *i)* a (block) Jacobi preconditioner acting on the degrees of freedom (dofs) of the DG space with the exclusion of the dofs associated to the interior of the elements (i.e., the bubble modes) and *ii)* a classical two-level overlapping Schwarz preconditioner acting on the space of high-order continuous polynomials vanishing on $\partial\Omega$. Implementing the action of the first operator is straightforward as it requires only a multiplication by a (block) diagonal matrix associated to the vertex/edge/face dofs. Regarding the action of the second operator, associated to the smooth part of the space V_{hp}, it is easily seen that the computationally demanding part is the construction of the projector from the discontinuous space V_{hp} to the conforming space. For example, in two-dimensions, it consists in constructing a matrix that averages the values at the vertex and edge dofs. Clearly, the projection matrix has a highly sparse structure. Once the conforming stiffness matrix is available, it is enough to apply the classical overlapping Schwarz preconditioner of [42,43], with coarse space from piecewise continuous linear polynomials, for which efficient implementations are available.

Finally, let us briefly comment on the actual cost of the proposed preconditioner relative to other strategies. Since the proposed approach is based on an overlapping partition of the computational domain, each application of the preconditioner is more expensive than the Schwarz preconditioners proposed in [1,2,6,8], that feature a high-level of scalability and load balancing because they are built using a non-overlapping approach. On the other hand, our preconditioner is uniform with respect to the polynomial approximation degree p whereas the preconditioner of [6,8] exhibit spectral bounds that depend on p.

Table 1 Left and middle: numerical evaluation of the constants C_1^J and C_2^J of Lemma 7 as a function of p for the SIPG and LDG methods; right: numerical evaluation of the constants C_1^C and C_2^C of Lemma 9 as a function of p

	SIPG ($\alpha = 10, \boldsymbol{\beta} = \mathbf{0}$)		LDG ($\alpha = 10, \boldsymbol{\beta} = \mathbf{1}$)			
	C_1^J	C_2^J	C_1^J	C_2^J	C_1^C	C_2^C
$p = 2$	0.4036	3.0084	0.3844	3.6393	0.2500	1.1606
$p = 3$	0.4343	2.9133	0.4129	3.3232	0.2500	1.0742
$p = 4$	0.4502	2.8304	0.4298	3.1487	0.2500	1.0934
$p = 5$	0.4605	2.7633	0.4410	3.0321	0.2500	1.0820
$p = 6$	0.4674	2.7088	0.4489	2.9467	0.2500	1.0854

Table 2 Condition number of the unpreconditioned ($\mathsf{K}(A)$) and preconditioned ($\mathsf{K}(T_{DG})$) linear systems of equations and corresponding CG (N_{iter}^{CG}) and PCG (N_{iter}^{PCG}) iteration counts as a function of p for the SIPG and LDG methods

	SIPG ($\alpha = 10, \boldsymbol{\beta} = \mathbf{0}$)				LDG ($\alpha = 10, \boldsymbol{\beta} = \mathbf{1}$)			
	$\mathsf{K}(A)$	N_{iter}^{CG}	$\mathsf{K}(T_{DG})$	N_{iter}^{PCG}	$\mathsf{K}(A)$	N_{iter}^{CG}	$\mathsf{K}(T_{DG})$	N_{iter}^{PCG}
$p = 2$	5.26×10^3	284	14.26	27	8.88×10^3	392	35.02	36
$p = 3$	1.52×10^4	450	14.22	25	2.29×10^4	556	38.29	31
$p = 4$	3.38×10^4	684	14.72	26	4.89×10^4	851	37.74	33
$p = 5$	6.27×10^4	919	15.35	24	8.83×10^4	1137	38.37	30
$p = 6$	1.05×10^5	1200	15.98	25	1.45×10^5	1482	42.65	32

6.1 Example 1

We consider problem (2) in the two dimensional case with $\Omega = (-1, 1)^2$ and SIPG and LDG discretizations. For the first experiment, we set $h = 0.0625$, the penalization parameter $\alpha = 10$ and $\boldsymbol{\beta} = \mathbf{1}$ for the LDG method. In Table 1, we show the numerical evaluation of the constants C_1^J and C_2^J of Lemma 7 and C_1^C and C_2^C of Lemma 9, as a function of the polynomial order employed in the discretization: the constants are independent of p, as expected from theory. With regard to the constants C_1^C and C_2^C, we observe that the values are the same for both the SIPG and LDG methods, since the preconditioner on the conforming subspace reduces to the same operator regardless of the DG scheme employed.

Table 2 shows a comparison of the spectral condition number of the original system ($\mathsf{K}(A)$) and of the preconditioned one ($\mathsf{K}(T_{DG})$). While the former grows as p^4, cf. [6], the latter is constant with p, as stated in (35). The theoretical results are further confirmed by the number of iterations N_{iter}^{PCG} and N_{iter}^{CG} of the Preconditioned Conjugate Gradient (PCG) and the Conjugate Gradient (CG), respectively, needed to reduce the initial relative residual of a factor of 10^{-8}.

In the second numerical experiment, we consider the same test case presented above, but we now fix the polynomial approximation degree $p = 6$ and decrease the mesh-size h. The computed results obtained with the SIPG ($\boldsymbol{\beta} = \mathbf{0}, \alpha = 10$) and LDG ($\boldsymbol{\beta} = \mathbf{1}, \alpha = 10$) methods are shown in Table 3. For the sake of comparison, Table 3 also shows the computed spectral condition numbers of the unpreconditioned system as well as the number of CG iterations

Table 3 Condition number of the unpreconditioned ($\mathsf{K}(A)$) and preconditioned ($\mathsf{K}(T_{DG})$) linear systems of equations and corresponding CG (N_{iter}^{CG}) and PCG (N_{iter}^{PCG}) iteration counts as a function of the mesh-size h for the SIPG and LDG methods for $p = 6$

	SIPG ($\alpha = 10$, $\boldsymbol{\beta} = \mathbf{0}$)				LDG ($\alpha = 10$, $\boldsymbol{\beta} = \mathbf{1}$)			
	$\mathsf{K}(A)$	N_{iter}^{CG}	$\mathsf{K}(T_{DG})$	N_{iter}^{PCG}	$\mathsf{K}(A)$	N_{iter}^{CG}	$\mathsf{K}(T_{DG})$	N_{iter}^{PCG}
$h = 1/4$	6.62×10^3	275	14.25	25	9.26×10^3	347	30.70	31
$h = 1/8$	2.64×10^4	593	15.60	25	3.64×10^4	739	35.37	32
$h = 1/16$	1.05×10^5	1199	15.98	25	1.45×10^5	1482	36.67	32

Table 4 Condition number of the unpreconditioned ($\mathsf{K}(A)$) and preconditioned ($\mathsf{K}(T_{DG})$) linear systems of equations and corresponding CG (N_{iter}^{CG}) and PCG (N_{iter}^{PCG}) iteration counts as a function of α for the SIPG and LDG methods

	SIPG ($p = 2$, $\boldsymbol{\beta} = \mathbf{0}$)				LDG ($p = 2$, $\boldsymbol{\beta} = \mathbf{1}$)			
	$\mathsf{K}(A)$	N_{iter}^{CG}	$\mathsf{K}(T_{DG})$	N_{iter}^{PCG}	$\mathsf{K}(A)$	N_{iter}^{CG}	$\mathsf{K}(T_{DG})$	N_{iter}^{PCG}
$\alpha = 2$	1.04×10^3	137	12.66	28	4.55×10^3	297	62.54	47
$\alpha = 5$	2.62×10^3	205	13.02	28	6.17×10^3	338	41.94	39
$\alpha = 10$	5.26×10^3	284	14.26	27	8.88×10^3	392	35.02	36
$\alpha = 10^2$	5.41×10^4	690	15.73	28	5.78×10^4	717	29.32	31
$\alpha = 10^3$	5.44×10^5	1116	15.90	28	5.47×10^5	1142	28.92	30
$\alpha = 10^4$	5.44×10^6	1509	15.91	28	5.44×10^6	1518	28.89	30

needed to reduced the relative residual below the given tolerance 10^{-8}. As predicted from our theoretical results, the condition number of the preconditioned system is insensitive to the the mesh-size, whereas $\mathsf{K}(A)$ grows quadratically as h tends to zero.

The last numerical experiment of this section aims at verifying the uniformity of the proposed preconditioner with respect to the penalization coefficient α. In this case, we consider the same test case presented above, but we now fix the polynomial approximation degree $p = 2$ and increase α. The numerical data obtained are reported in Table 4: as done before, we compare the spectral condition numbers of the unpreconditioned and preconditioned systems and the iteration counts of the CG and PCG methods. As predicted from theory, while $\mathsf{K}(A)$ grows like α, the values of $\mathsf{K}(T_{DG})$ are constant.

6.2 Example 2

We now address a more general model problem by introducing a diffusion tensor ρ in (1). We then rewrite the weak formulation (1) as follows: find $u \in V$, such that

$$\int_\Omega \rho \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in V,$$

where, for simplicity we assume that

$$\rho \equiv \rho(x) = \begin{bmatrix} \rho_{11}(x) & 0 \\ 0 & \rho_{22}(x) \end{bmatrix},$$

with $\rho_{11}(x)$ and $\rho_{22}(x)$ (possibly discontinuous) positive constants a.e. in Ω. We also assume that there exists an initial partition $\mathcal{T}_{h_0}$ such that the (possible) discontinuities of $\rho_{11}(x)$ and $\rho_{22}(x)$ are aligned with $\mathcal{T}_{h_0}$. We then modify the bilinear form (3) as follows, restricting, for simplicity, ourselves to the SIPG bilinear form, i.e., $\theta = 0$ and $\boldsymbol{\beta} = \mathbf{0}$,

$$\mathcal{A}_\rho(u, v) := \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \rho\nabla u\cdot\nabla v\,dx + \sum_{\kappa\in\mathcal{T}_h}\int_\kappa (\nabla u\cdot\mathcal{R}(\{\{\rho\}\}_{\mathrm{A}}[\![v]\!]) + \mathcal{R}(\{\{\rho\}\}_{\mathrm{A}}[\![u]\!])\cdot\nabla v)\,dx$$
$$+ \sum_{F\in\mathcal{F}_h}\int_F \sigma\{\{\rho\}\}_{\mathrm{A}}[\![u]\!]\cdot[\![v]\!]ds$$

where, on each face $F \in \mathcal{F}_h$, $\{\{\rho\}\}_{\mathrm{A}}$ is defined as

$$\{\{\rho\}\}_{\mathrm{A}} := \begin{cases} 2\dfrac{(\mathbf{n}^T_{F,\kappa^+}\rho^+\mathbf{n}_{F,\kappa^+})(\mathbf{n}^T_{F,\kappa^-}\rho^-\mathbf{n}_{F,\kappa^-})}{\mathbf{n}^T_{F,\kappa^+}\rho^+\mathbf{n}_{F,\kappa^+} + \mathbf{n}^T_{F,\kappa^-}\rho^-\mathbf{n}_{F,\kappa^-}} & \text{if } F\in\mathcal{F}_h^I, \\ \mathbf{n}_F^T\rho\mathbf{n}_F & \text{if } F\in\mathcal{F}_h^B, \end{cases}$$

see, e.g., [25]. We then consider the following DG formulation:

$$\mathcal{A}_\rho(u, v) = \int_\Omega f v\,dx \quad \forall v\in V_{hp}. \tag{37}$$

The numerical results reported in this section have been obtained choosing piecewise quadratic elements, i.e., $p = 2$ and the penalization parameter $\alpha = 10$.

For the first test we choose an elementwise discontinuous isotropic diffusion coefficient, i.e.

$$\rho(x) = \begin{bmatrix} \rho_{11}(x) & 0 \\ 0 & \rho_{11}(x) \end{bmatrix},$$

where $\rho_{11}(x)$ follows an elementwise red–black partitioning of the computational domain, which means $\rho_{11} = 1$ in red regions and $\rho_{11} = 10^\gamma$, $\gamma = 0, 1, \ldots, 7$, in the black ones. In Fig. 3, we numerically evaluate the constant appearing in the inequality

$$\mathcal{A}_\rho(\mathsf{Q}_h v, \mathsf{Q}_h v) \le \mathsf{C}_{\mathsf{Q}_h}\mathcal{A}_\rho(v, v), \tag{38}$$

which is the analogous of (13), and observe that the presence of a discontinuous diffusion coefficient introduces a linear dependence of $\mathsf{C}_{\mathsf{Q}_h}$ on the ratio between the maximum and minimum value of ρ. In Table 5 we report the estimated values of the constants C_1^J and C_2^J appearing in Lemma 7, and C_1^C and C_2^C appearing in Lemma 9, respectively, as a function of ρ_{11}. Here, we note that the two preconditioners, built on the corresponding subspaces, are robust with respect to the contrast in the diffusion tensor. From the results obtained by these numerical tests we can conclude that the condition number of the preconditioned operator T_{DG} depends linearly on the jumps of the diffusion coefficient, as further confirmed by the numerical tests reported in Table 6 where the spectral condition number of T_{DG} as well as the PCG iteration counts as a function of ρ_{11} are shown. For the sake of comparison, the same quantities are also shown for the unpreconditioned system, showing that, as expected also the unpreconditioned operator has a spectral condition number that depends linearly on the contrast of the diffusion coefficient.

Next, we investigate the behavior of the proposed technique whenever the diffusion tensor features a strong anisotropy. For this set of experiments we set

$$\rho(x) = \begin{bmatrix} 1 & 0 \\ 0 & \rho_{22} \end{bmatrix},$$

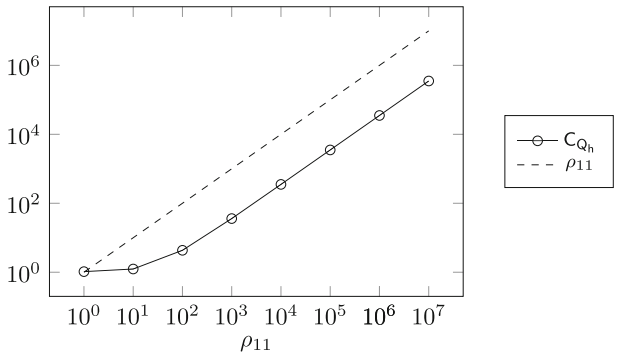

Fig. 3 Numerical evaluation of the constant $\mathsf{C}_{\mathsf{Q}_h}$ in (38) as a function of ρ_{11}. SIPG method (37) ($p = 2$, $\alpha = 10$)

Table 5 Left: numerical evaluation of the constants C_1^J and C_2^J of Lemma 7 as a function of ρ_{11} for the SIPG method (37) ($p = 2, \alpha = 10$); right: numerical evaluation of the constants C_1^C and C_2^C of Lemma 9 as a function of ρ_{11}

	C_1^J	C_2^J	C_1^C	C_2^C
$\rho_{11} = 1$	0.4036	3.0084	0.2500	1.1606
$\rho_{11} = 10$	0.4277	2.2562	0.2500	1.6160
$\rho_{11} = 10^2$	0.5497	1.9874	0.2500	2.0718
$\rho_{11} = 10^3$	0.6732	2.0199	0.2500	2.1414
$\rho_{11} = 10^4$	0.6989	2.0366	0.2500	2.1487
$\rho_{11} = 10^5$	0.7024	2.0390	0.2500	2.1494
$\rho_{11} = 10^6$	0.7028	2.0393	0.2500	2.1495
$\rho_{11} = 10^7$	0.7028	2.0393	0.2500	2.1495

Table 6 Condition number of the unpreconditioned ($\mathsf{K}(A)$) and preconditioned ($\mathsf{K}(T_{DG})$) linear systems of equations and corresponding CG (N_{iter}^{CG}) and PCG (N_{iter}^{PCG}) iteration counts as a function of ρ_{11} for the SIPG method (37) ($p = 2, \alpha = 10$)

	$\mathsf{K}(A)$	N_{iter}^{CG}	$\mathsf{K}(T_{DG})$	N_{iter}^{PCG}
$\rho_{11} = 1$	5.26×10^3	260	1.43×10^1	25
$\rho_{11} = 10$	7.02×10^3	293	1.26×10^2	32
$\rho_{11} = 10^2$	2.51×10^4	401	7.94×10^2	52
$\rho_{11} = 10^3$	1.96×10^5	703	7.04×10^3	84
$\rho_{11} = 10^4$	1.90×10^6	908	6.96×10^4	109
$\rho_{11} = 10^5$	1.89×10^7	1064	6.95×10^5	135
$\rho_{11} = 10^6$	1.89×10^8	1197	6.95×10^6	160
$\rho_{11} = 10^7$	1.89×10^9	1400	6.95×10^7	185

where ρ_{22} is a positive constant over the whole computational domain. We have repeated the same set of experiments as before, again with $p = 2$. In Fig. 4 we report the estimated constant $\mathsf{C}_{\mathsf{Q}_h}$ in (38) as a function of $\rho_{22} = 10^\gamma$, $\gamma = 0, 1, \ldots, 7$. We observe that $\mathsf{C}_{\mathsf{Q}_h}$ seems to be independent of the anisotropy of $\rho(x)$.

The computed constants C_1^J and C_2^J of Lemma 7 and C_1^C and C_2^C of Lemma 9 as a function of ρ_{22} are reported in Table 7. We note that C_2^J and C_2^C seem to be asymptotically independent of ρ_{22}.

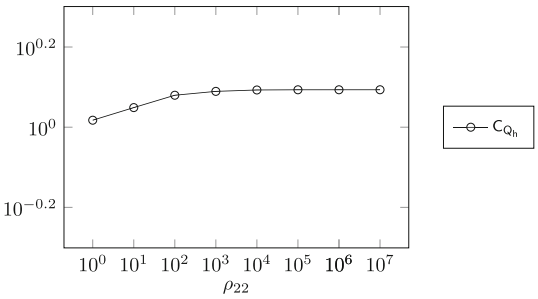

Fig. 4 Numerical evaluation of the constant $\mathsf{C}_{\mathsf{Q}_h}$ in (38) as a function of ρ_{22}. SIPG method (37) ($p = 2, \alpha = 10$)

Table 7 Left: numerical evaluation of the constants C_1^J and C_2^J of Lemma 7 as a function of ρ_{22} for the SIPG method in (37) ($p = 2$); right: numerical evaluation of the constants C_1^C and C_2^C of Lemma 9 as a function of ρ_{22}

	C_1^J	C_2^J	C_1^C	C_2^C
$\rho_{22} = 10^0$	0.4036	3.01×10^0	0.2500	1.1606
$\rho_{22} = 10^1$	0.4036	8.24×10^0	0.2294	2.2934
$\rho_{22} = 10^2$	0.4035	3.70×10^1	0.2103	10.8578
$\rho_{22} = 10^3$	0.4035	2.77×10^2	0.2072	33.6278
$\rho_{22} = 10^4$	0.4035	1.32×10^3	0.2069	51.2373
$\rho_{22} = 10^5$	0.4035	2.47×10^3	0.2069	57.6207
$\rho_{22} = 10^6$	0.4035	2.86×10^3	0.2069	58.3918
$\rho_{22} = 10^7$	0.4035	2.91×10^3	0.2069	58.4702

Table 8 Condition number of the unpreconditioned ($\mathsf{K}(A)$) and preconditioned ($\mathsf{K}(T_{DG})$) linear systems of equations and corresponding CG (N_{iter}^{CG}) and PCG (N_{iter}^{PCG}) iteration counts as a function of ρ_{22} for the SIPG method

	$\mathsf{K}(A)$	N_{iter}^{CG}	$\mathsf{K}(T_{DG})$	N_{iter}^{PCG}
$\rho_{22} = 10^0$	5.26×10^3	284	1.43×10^1	27
$\rho_{22} = 10^1$	9.28×10^3	460	4.24×10^1	43
$\rho_{22} = 10^2$	1.09×10^4	588	2.11×10^2	87
$\rho_{22} = 10^3$	1.98×10^4	654	1.68×10^3	164
$\rho_{22} = 10^4$	3.66×10^4	860	8.69×10^3	332
$\rho_{22} = 10^5$	4.11×10^4	684	1.72×10^4	418
$\rho_{22} = 10^6$	4.17×10^4	497	2.02×10^4	489
$\rho_{22} = 10^7$	4.18×10^4	387	2.06×10^4	490

The numerical data shown in Fig. 4 and Table 7 again suggest the expected behavior of the final preconditioner. Indeed the condition number $\mathsf{K}(T_{DG})$ seems to be asymptotically constant as ρ_{22} grows, as confirmed by the results reported in Table 8. Notice however that also the unpreconditioned DG discretization seems to be itself asymptotically robust with respect to the anisotropy.

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