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An Unconditionally Stable Quadratic Finite Volume Scheme over Triangular Meshes for Elliptic Equations

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Abstract In this note, we present and analyze a special quadratic finite volume scheme over triangular meshes for elliptic equations. The scheme is designed with the second degree Gauss points on the edges and the barycenters of the triangle elements. With a novel from-the-trial-to-test-space mapping, the inf–sup condition of the scheme is shown to hold independently of the minimal angle of the underlying mesh. As a direct consequence, the H^1 norm error of the finite volume solution is shown to converge with the optimal order.

Keywords Finite volume method · Inf–sup condition · Quadratic element

Mathematics Subject Classification Primary 65N30 · Secondary 45N08

1 Introduction

The finite volume method (FVM) enjoys a great popularity in scientific and engineering computations (see, e.g., [10,14,15,20–23]). However, its mathematical theory has not been developed satisfactorily (cf., [1,2,4,5,7,9,12,13,16,18,19,24]). In particular, the theory for high order FVMs is a challenging task.

This is one of our series work on the theory of high order FVMs. It is a common sense that the analysis of stability (or inf–sup condition in general) is challenging in the establishment of the FVM theory. The linear FV scheme over triangular meshes can be regarded as a small perturbation of its corresponding finite element (FE) scheme, its stability holds automatically

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when the mesh size is sufficient small, see e.g [3,11]. The stability analysis for high order finite volume schemes is much more complicated than that for linear schemes. Some earlier works adopt the so-called *elementwise-stiffness-matrix-analysis* by calculating eigenvalues of the stiffness matrix : the stability is established if all eigenvalues of the stiffness matrix are positive, see, e.g. [6,17,18,25]. This *elementwise-stiffness-matrix-analysis* technique is somehow artificial and often requires some additional conditions on the underlying meshes. For instances, to guarantee the inf–sup condition of their quadratic FV scheme over triangular mesh, it is required in [18] that the ratio between two adjacent edges of the triangle in the underlying mesh should be between $(\sqrt{\frac{2}{3}}, \sqrt{\frac{3}{2}})$. Although this restrictive condition on the mesh has been significantly relaxed in [6,17,25], a common sense is that the inf–sup condition depends heavily on the shape of the triangular mesh, namely the minimal angle of the triangles in the underlying mesh.

In a very recent work [26], it is shown that for a class of appropriately designed any order FV schemes over quadrilateral meshes, the inf–sup condition holds independently of the minimal angle of the mesh. A natural question is to ask whether we can construct some high order finite volume schemes over triangular meshes such that their corresponding inf–sup condition holds independently of the minimal angle of the mesh? In this work, we will give a confirmatory answer to this question.

Precisely, we will present and analyze a quadratic finite volume scheme of which the inf–sup condition holds over any shape regular triangular mesh in this paper. To explain our results, we begin with the construction of our quadratic FV schemes [8]. It is known that the construction of the FV schemes depends heavily on both the primal and dual mesh, and for a given primal triangular mesh, there are a variant of methods to construct the quadratic dual mesh, see e.g. [10,17,18], while a systematic approach was proposed in [25]. Although it is known that the different choice of dual meshes may influence the validity of the inf–sup condition of the corresponding FV scheme, the works in the literature focused on studying the influence of different primal meshes on the inf–sup condition, see e.g. [6,17,18,25]. In this paper, the dual mesh is constructed with the second-degree Gauss points on the edges and the barycenters of the triangular elements of the underlying mesh. With these delicate design of the dual mesh, we can define a special one-to-one mapping from the trial space to the test space(see Sect. 3.1). This mapping transfers the bilinear form defined on the trial-test spaces to a bilinear form on the trial space only, and thereby changes the the analysis framework from a Petrov–Galerkin method to a Galerkin finite element method. Since the coercivity of a Galerkin FEM is independent of the minimal angle of the mesh, the validity of inf–sup condition of our quadratic FVM is also independent of the minimal angle of the primal mesh. As byproducts, we obtain the existence, uniqueness and H^1 error estimate of the corresponding finite volume solution with a routine work.

The rest of the paper is organized as follows. In Sect. 2, we present a special quadratic finite volume over unstructured mesh. In Sect. 3, we analyze the stability and convergence properties of the proposed FV scheme. To this end, a novel from-trial-to-test mapping will be introduced. Some concluding remarks will be given in the final and fourth section.

In the rest of this paper, "$A \lesssim B$" means that A can be bounded by B multiplied by a constant which is independent of the parameters which A and B may depend on. "$A \sim B$" means "$A \lesssim B$" and "$B \lesssim A$".

2 A Special Quadratic Finite Volume Scheme

We consider the following second-order elliptic boundary value problem

$$-\bigtriangledown \cdot (\alpha \bigtriangledown u) = f \quad \text{in} \quad \Omega, \tag{2.1}$$

$$u = 0 \quad \text{on} \quad \Gamma, \tag{2.2}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal bounded domain, $\Gamma = \partial\Omega$ and α is a bounded and piecewise continuous function that is bounded below: There exists a constant $\alpha_0 > 0$ such that $\alpha(x) \geq \alpha_0$ for almost all $x \in \Omega$.

Let $\mathcal{T}_h$ be a conforming and shape-regular *triangulation* of Ω. With respect to $\mathcal{T}_h$, we define a finite element space

$$\mathcal{U}_h^k = \left\{ v \in C(\overline{\Omega}) : v|_\tau \in \mathbf{P}_k, \ \text{ for all } \tau \in \mathcal{T}_h, \ \ v|_{\partial\Omega} = 0 \right\} \tag{2.3}$$

where $\mathbf{P}_k$ is the set of all polynomials of degree equal or less than k. Apparently, $\mathcal{U}_h^k \subset H_0^1(\Omega)$. For simplicity, we denote $\mathcal{U}_h = \mathcal{U}_h^2$.

We next explain the construction of *control volumes* associated with the primal mesh $\mathcal{T}_h$. Let $\mathcal{N}_h$ and $\mathcal{M}_h$ be the set of interior vertices and the set of mid-points of the internal edges respectively. Generally speaking, a *control volume* is a polygon K_{p_0} surrounding a vertex $p_0 \in \mathcal{N}_h$ or a polygon K_m surrounding $m \in \mathcal{M}_h$. Let us now give some details on their construction. For each triangle $\tau = \Delta p_1 p_2 p_3 \in \mathcal{T}_h$, we denote by O its barycenter and by m_1, m_2, m_3 the midpoints of the edges $\overline{p_2 p_3}, \overline{p_3 p_1}, \overline{p_1 p_2}$, respectively. Let g_{ij}, g_{ji} be the two second-degree Gauss points on the edge $\overline{p_i p_j}$, $i, j = 1, 2, 3$ such that

$$\frac{|p_i g_{ij}|}{|p_i p_j|} = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right), \quad \frac{|p_i g_{ji}|}{|p_i p_j|} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right). \tag{2.4}$$

Note that by this definition, g_{ij} is closer to p_i than to p_j. We denote by q_1, q_2, q_3 the mid-points of the segments $\overline{g_{12} g_{13}}, \overline{g_{21} g_{23}}, \overline{g_{31} g_{32}}$, respectively. By connecting with O to q_i, $i = 1, 2, 3$, the element τ is split into six portions, see Fig. 1. With this construction, it is easy to verify that

$$|V_{p_i} \cap \tau| = \left(\frac{1}{3} - \frac{1}{2\sqrt{3}}\right)|\tau|, \quad |V_{m_i} \cap \tau| = \frac{1}{2\sqrt{3}}|\tau|. \tag{2.5}$$

Based on this splitting, we are now ready to construct our control volumes.

First, the control volume V_{p_0} associated with a vertex $p_0 \in \mathcal{N}_h$ is a polygon surrounding p_0 by successively connecting the Gauss points $g_{01}, g_{02}, \ldots, g_{0m}, g_{01}$, where $p_i (i = 1, 2, \ldots, m)$ are the adjacent vertices of p_0, see Fig. 2 for an example of V_{p_0} where $m = 6$. Secondly the control volume V_m associated with a midpoint $m \in \mathcal{M}_h$ is a polygon surrounding m which can be described as below. Suppose m is the midpoint of the common side of two adjacent triangular elements $\tau_1 = \Delta p_1 p_2 p_3$ and $\tau_2 = \Delta p_1 p_2 p_4$. We denote by O_1 and O_2 the barycenter of τ_1 and τ_2 respectively. Let $q_{11}, q_{12}, q_{21}, q_{22}$ be the midpoints of $\overline{g_{12} g_{13}}, \overline{g_{21} g_{23}}, \overline{g_{12} g_{14}}$ and $\overline{g_{21} g_{24}}$, respectively. The volume V_m is a polygon surrounding m obtained by connecting successively the points $q_{11}, g_{12}, q_{21}, O_2, q_{22}, g_{21}, q_{12}, O_1, q_{11}$ with straight segments, see Fig. 3.

We define the dual partition $\mathcal{T}_h' = \{V_p, V_m : p \in \mathcal{N}_h, m \in \mathcal{M}_h\}$. The test function space $\mathcal{V}_T'$ contains all the piecewise constant functions with respect to $\mathcal{T}'$:

$$\mathcal{V}_h' = \text{span}\left\{\psi_p, \psi_m : p \in \mathcal{N}_h, m \in \mathcal{M}_h\right\}, \tag{2.6}$$

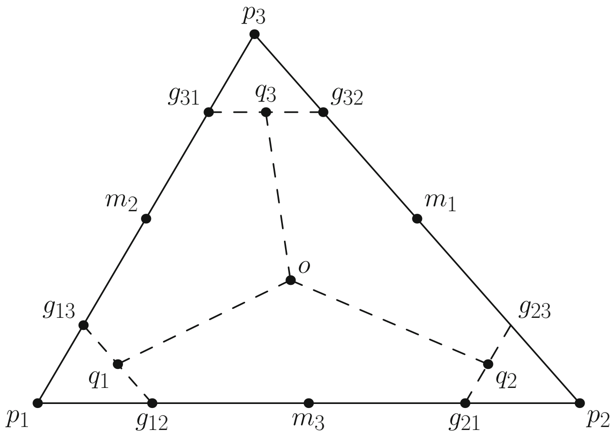
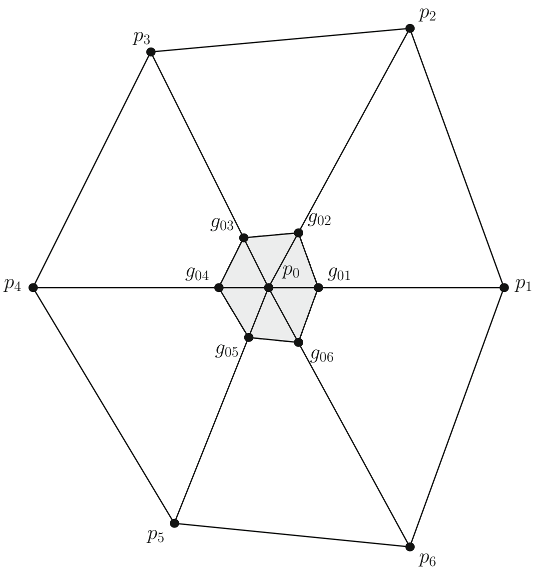


Fig. 1 Partition of a primal triangle

Fig. 2 Control volume V_{p_0}



where ψ_P and ψ_M are characteristic functions of V_p and V_m, respectively.

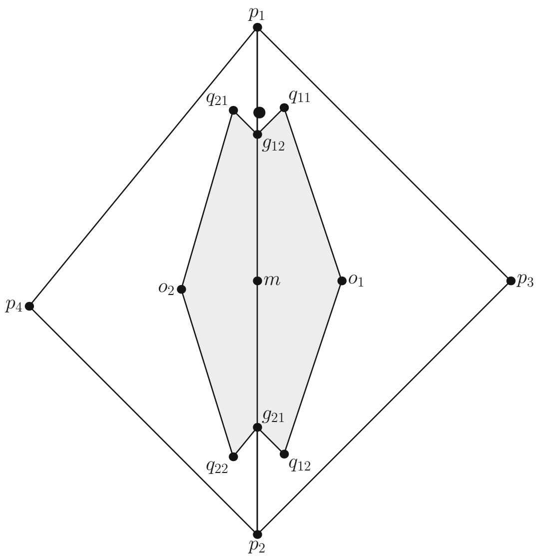
The quadratic finite volume solution of (2.1)–(2.2) is a function $u_h \in \mathcal{U}_h$ satisfying the following local conservation laws:

$$-\int_{\partial V_p} \alpha \frac{\partial u_h}{\partial \mathbf{n}} = \int_{V_p} f, \quad \forall\, p \in \mathcal{N}_h; \qquad -\int_{\partial V_m} \alpha \frac{\partial u_h}{\partial \mathbf{n}} = \int_{V_m} f, \quad \forall m \in \mathcal{M}_h, \tag{2.7}$$

where $\mathbf{n}$ is the outward unit normal to ∂D_P or ∂D_M. If we define the bilinear form for all $u \in H_0^1(\Omega)$, $v \in \mathcal{V}_h'$ as

$$a_h(u, v) = -\sum_{p\in\mathcal{N}_h} \int_{\partial V_p} \alpha \frac{\partial u}{\partial \mathbf{n}}\, v(p) ds - \sum_{m\in\mathcal{M}_h} \int_{\partial V_m} \alpha \frac{\partial u}{\partial \mathbf{n}}\, v(m) ds, \tag{2.8}$$

Fig. 3 Control volume V_m



The quadratic finite volume solution satisfies the following Petrov–Galerkin scheme

$$a_h(u_h, v) = (f, v), \quad \forall v \in \mathcal{V}_h'. \tag{2.9}$$

3 Analysis

Following a technique in [25], the theoretic results in this section will be obtained using the following strategy: We first analyze our problem in a simple case that the coefficient α is piecewise constant with respect to $\mathcal{T}_h$ and then extend our analysis to a general case that α is piecewise in $W^{1,\infty}$.

3.1 A Novel from-Trial-to-Test-Space Mapping

The finite volume method is a Petrov–Galerkin method due to the fact that its test space is different from its test space. Since the analysis for a Petrov–Galerkin method is often more complicated than that for a Galerkin method, we transform the FVM to a Galerkin method by establishing a mapping from the trial space to test space. The choice of this from-trial-to-test-space mapping has a significant influence on the analysis of the FVM. Unlike the classic one proposed in [18], here we propose a novel mapping which can be roughly described as below.

Let Π map a $v \in \mathcal{U}_h$ to $v^* = \Pi v \in \mathcal{V}_h'$ such that for each vertex $p \in \mathcal{N}_h$,

$$v^*(p) = v(p), \tag{3.1}$$

and for each midpoint $m \in \mathcal{M}_h$,

$$v^*(m) = \frac{v(p_1) + v(p_2)}{2}\left(1 - \frac{2}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}}v(m), \tag{3.2}$$

where the edge $\overline{p_1 p_2} \in \mathcal{E}_h$ has m as its midpoint. Note that the classic from-trial-to-test mapping requires [18]

$$v^*(m) = v(m)$$

instead of (3.2) for the midpoint value. We denote by $\mathbf{t}$ the unit tangent direction pointing from p_1 to p_2, by g_{12} the second degree Gauss point on $\overline{p_1 p_2}$ which is closer to p_1 than to p_2, and by $w_{12} = \frac{1}{2}|p_1 p_2|$ the weight of Gauss quadrature. Then, a direct calculation yields that

$$v^*(m) = v^*(p_1) + \omega_{12} \frac{\partial v}{\partial \mathbf{t}}(g_{12}).$$

In other words, the definition of Π here is consistent with that defined in [26].

With this definition, it is easy to verify that the mapping Π has the following properties.

Theorem 3.1 *Let $\mathcal{E}_h$ be the set of all internal edges of $\mathcal{T}_h$. Then for all $E \in \mathcal{E}_h$,*

$$\int_E (v - v^*) = 0, \quad \forall v \in \mathcal{U}_h^2, \tag{3.3}$$

$$\int_E v(w - w^*) = 0, \quad \forall v \in \mathcal{U}_h^1, w \in \mathcal{U}_h^2. \tag{3.4}$$

And for all $\tau \in \mathcal{T}_h$,

$$\int_\tau (w - w^*) = 0, \quad \forall w \in \mathcal{U}_h^2. \tag{3.5}$$

Proof We first prove (3.4) and (3.3). For all $E \in \mathcal{E}_h$, we denote by m_E the midpoint and by p_1, p_2 the two vertices of E. Let λ_1, λ_2 be the two barycenter coordinates corresponding to p_1, p_2, respectively.

We next show (3.3). On the edge E, each $v \in \mathcal{U}_h^2$ can be represented as

$$v = v(p_1)\lambda_1 + v(p_2)\lambda_2 + \left(v(m_E) - \frac{v(p_1) + v(p_2)}{2} \right) 4\lambda_1\lambda_2.$$

Therefore,

$$\int_E v = \frac{v(p_1) + v(p_2)}{6}|E| + \frac{2}{3}v(m_E)|E|.$$

On the other hand, the facts that

$$v^* = v(p_1) \text{ on } \overline{p_1 g_{12}},$$
$$v^* = v(p_2) \text{ on } \overline{p_2 g_{21}},$$

and

$$v^* = v^*(m_E) = \frac{v(p_1) + v(p_2)}{2}\left(1 - \frac{2}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}}v(m_E) \text{ on } \overline{g_{12} g_{21}}$$

yield that

$$\begin{aligned}
\int_E v^* &= \frac{v(p_1) + v(p_2)}{2}\left(1 - \frac{1}{\sqrt{3}}\right)|E| \\
&\quad + \left(\frac{v(p_1) + v(p_2)}{2}\left(1 - \frac{2}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}}v(m_E) \right) \cdot \frac{1}{\sqrt{3}}|E| \\
&= \frac{v(p_1) + v(p_2)}{6}|E| + \frac{2}{3}v(m_E)|E|.
\end{aligned}$$

Namely, (3.3) is verified. We next show (3.4). The fact that for all $i, j \in \{1, 2\}, i \neq j$,

$$\lambda_i^*(p_i) = 1,$$
$$\lambda_i^*(p_j) = 0,$$
$$\lambda_i^*(m_E) = \frac{1}{2}$$

yields that for $i, j \in \{1, 2\}$,

$$\int_E \lambda_i \lambda_j = \int_E \lambda_i \lambda_j^* = \begin{cases} \frac{1}{3}|E|, \; i = j, \\ \frac{1}{6}|E|, \; i \neq j. \end{cases}$$

Thus the equality (3.4) is valid for all $v, w \in \mathcal{U}_h^1$. To prove (3.4) for all $v \in \mathcal{U}_h^1$, $w \in \mathcal{U}_h^2$, it is sufficient to verify (3.4) for $v = \lambda_1$ and $w = \lambda_1\lambda_2$. Note that for this special w, we have

$$w^* = 0 \text{ on } \overline{p_1 g_{12}} \cup \overline{p_2 g_{21}},$$

and

$$w^* = \frac{1}{2\sqrt{3}} \text{ on } \overline{g_{12} g_{21}}.$$

Then,

$$\int_E \lambda_1 (\lambda_1 \lambda_2)^* = \frac{1}{2\sqrt{3}} \int_{\overline{g_{12} g_{21}}} \lambda_1 = \frac{1}{12} |p_1 p_2|.$$

Since we also have

$$\int_E \lambda_1^2 \lambda_2 = \frac{1}{12} |p_1 p_2|,$$

(3.4) is valid for $v = \lambda_1$ and $w = \lambda_1\lambda_2$. Consequently, (3.4) is valid for all $v \in \mathcal{U}_h^1$ and $w \in \mathcal{U}_h^2$.

Next we prove (3.5). Let $\tau = \Delta p_1 p_2 p_3$ be an arbitrary element in $\mathcal{T}$. As in Fig. 1, we use m_i to denote the midpoint of the edge opposing the vertex p_i. For all $i = 1, 2, 3$, we have

$$\int_\tau \lambda_i = \frac{1}{3}|\tau| = \int_\tau \lambda_i^*.$$

Then (3.5) is valid for $w \in \mathcal{U}_h^1$. Since $\mathcal{U}_h^2 = \text{Span}\{\lambda_1, \lambda_2, \lambda_3, 4\lambda_1\lambda_2, 4\lambda_2\lambda_3, 4\lambda_3\lambda_1\}$ in τ, to prove the validity of (3.5) for all $w \in \mathcal{U}_h^2$, we only need to verify (3.5) for the special function $w = 4\lambda_1\lambda_2$. It is easy to check that

$$\int_\tau w = 4 \int_\tau \lambda_1 \lambda_2 = \frac{1}{3}|\tau|.$$

By the definition of the mapping Π, we have that

$$w^*(m_3) = \frac{2}{\sqrt{3}},$$

and

$$w^*(p_i) = w^*(m_j) = 0, \quad \forall i = 1, 2, 3; \; j = 1, 2.$$

Therefore, by (2.5)

$$\int_\tau w^* = \int_{\tau\cap V_{m_3}} w^*$$
$$= \frac{2}{\sqrt{3}}|\tau \cap V_{m_3}|$$
$$= \frac{2}{\sqrt{3}} \cdot \frac{1}{2\sqrt{3}}|\tau| = \frac{1}{3}|\tau|.$$

In other words, (3.5) holds for $w = 4\lambda_1\lambda_2$. By the same arguments, (3.5) holds also for $w = 4\lambda_2\lambda_3$ or $w = 4\lambda_2\lambda_3$. Consequently, (3.5) is valid for all $w \in \mathcal{U}_h^2$.

3.2 FV and FE Bilinear Forms

In this subsection, we study the relationship between the FV bilinear form $a_h(\cdot, \cdot)$ and the classic Galerkin bilinear form (finite element method bilinear form) defined for all $v, w \in H_0^1(\Omega)$ by

$$a(v, w) = \int_\Omega \alpha \nabla v \cdot \nabla w. \tag{3.6}$$

Theorem 3.2 *Let the coefficient α be piecewise constant with respect to $\mathcal{T}_h$. Then*

$$a_h(v, w^*) = a(v, w), \quad \forall v, w \in \mathcal{U}_h. \tag{3.7}$$

Proof We will prove the theorem by comparing the element-wise bilinear forms defined for all $\tau \in \mathcal{T}$ by

$$a^\tau(v, w) = \int_\tau \alpha \nabla v \cdot \nabla w \tag{3.8}$$

and

$$a_h^\tau(v, w) = -\sum_{p\in\mathcal{N}_h} \int_{\partial V_p\cap\tau} \alpha \frac{\partial v}{\partial \mathbf{n}}\, w(p)ds - \sum_{m\in\mathcal{M}_h} \int_{\partial V_m\cap\tau} \alpha \frac{\partial v}{\partial \mathbf{n}}\, w(m)ds, \tag{3.9}$$

respectively.

Since α is piecewise constant with respect to $\mathcal{T}_h$, it is easy to use the Green's formulae to rewrite these two elementwise forms as

$$a^\tau(v, w) = -\alpha \int_\tau (\Delta v)w + \alpha \int_{\partial\tau} \frac{\partial v}{\partial \mathbf{n}} w,$$
$$a_h^\tau(v, w^*) = -\alpha \int_\tau (\Delta v)w^* + \alpha \int_{\partial\tau} \frac{\partial v}{\partial \mathbf{n}} w^*.$$

Note that for all $v \in \mathcal{U}_h = \mathcal{U}_h^2$, Δv is a constant in τ and $\frac{\partial v}{\partial \mathbf{n}}$ is linear along each edge of τ. Then by (3.4) and (3.5), we have

$$\int_\tau (\Delta v)w^* = \int_\tau (\Delta v)w \quad \forall v, w \in \mathcal{U}_h$$

and

$$\int_{\partial\tau} \frac{\partial v}{\partial n} w^* = \int_{\partial\tau} \frac{\partial v}{\partial n} w, \quad \forall v, w \in \mathcal{U}_h$$

respectively. Namely, we obtain the identity

$$a_h^\tau(v, w^*) = a^\tau(v, w), \quad \forall v, w \in \mathcal{U}_h$$

for all $\tau \in \mathcal{T}_h$. Thus, (3.7) is valid. □

The theorem implies that if α is piecewise constant, the stiffness matrix of the quadratic FVM can be obtained by multiplying the corresponding quadratic FEM stiffness matrix with a given matrix. In fact, it is known that on each triangle $\tau = \Delta p_1 p_2 p_3$, the nodal basis are given by $\phi_{p_i} = 2\lambda_i^2 - \lambda_i$ and $\phi_{m_i} = 4\lambda_{i+1}\lambda_{i+2}, i = 1, 2, 3$ with $\lambda_{i+3} = \lambda_i, i = 1, 2, 3$. Then the stiffness matrix of the quadratic finite element method reads as

$$A_\tau^e = \begin{pmatrix} a^\tau(\phi_{p_1}, \phi_{p_1}) & a^\tau(\phi_{p_1}, \phi_{p_2}) & a^\tau(\phi_{p_1}, \phi_{p_3}) & a^\tau(\phi_{p_1}, \phi_{m_1}) & a^\tau(\phi_{p_1}, \phi_{m_2}) & a^\tau(\phi_{p_1}, \phi_{m_3}) \\ a^\tau(\phi_{p_2}, \phi_{p_1}) & a^\tau(\phi_{p_2}, \phi_{p_2}) & a^\tau(\phi_{p_2}, \phi_{p_3}) & a^\tau(\phi_{p_2}, \phi_{m_1}) & a^\tau(\phi_{p_2}, \phi_{m_2}) & a^\tau(\phi_{p_2}, \phi_{m_3}) \\ a^\tau(\phi_{p_3}, \phi_{p_1}) & a^\tau(\phi_{p_3}, \phi_{p_2}) & a^\tau(\phi_{p_3}, \phi_{p_3}) & a^\tau(\phi_{p_3}, \phi_{m_1}) & a^\tau(\phi_{p_3}, \phi_{m_2}) & a^\tau(\phi_{p_3}, \phi_{m_3}) \\ a^\tau(\phi_{m_1}, \phi_{p_1}) & a^\tau(\phi_{m_1}, \phi_{p_2}) & a^\tau(\phi_{m_1}, \phi_{p_3}) & a^\tau(\phi_{m_1}, \phi_{m_1}) & a^\tau(\phi_{m_1}, \phi_{m_2}) & a^\tau(\phi_{m_1}, \phi_{m_3}) \\ a^\tau(\phi_{m_2}, \phi_{p_1}) & a^\tau(\phi_{m_2}, \phi_{p_2}) & a^\tau(\phi_{m_2}, \phi_{p_3}) & a^\tau(\phi_{m_2}, \phi_{m_1}) & a^\tau(\phi_{m_2}, \phi_{m_2}) & a^\tau(\phi_{m_2}, \phi_{m_3}) \\ a^\tau(\phi_{m_3}, \phi_{p_1}) & a^\tau(\phi_{m_3}, \phi_{p_2}) & a^\tau(\phi_{m_3}, \phi_{p_3}) & a^\tau(\phi_{m_3}, \phi_{m_1}) & a^\tau(\phi_{m_3}, \phi_{m_2}) & a^\tau(\phi_{m_3}, \phi_{m_3}) \end{pmatrix}$$

For all subset $A \subset \Omega$, let ψ_A be the characteristic function defined by $\psi_A(x) = 1$, if $x \in A$ and $\psi_A(x) = 0$ if $x \notin A$. Then the stiffness matrix of the quadratic finite volume reads as

$$A_\tau^v = \begin{pmatrix} a_h^\tau(\phi_{p_1}, \psi_{p_1}) & a_h^\tau(\phi_{p_1}, \psi_{p_2}) & a_h^\tau(\phi_{p_1}, \psi_{p_3}) & a_h^\tau(\phi_{p_1}, \psi_{m_1}) & a_h^\tau(\phi_{p_1}, \psi_{m_2}) & a_h^\tau(\phi_{p_1}, \psi_{m_3}) \\ a_h^\tau(\phi_{p_2}, \psi_{p_1}) & a_h^\tau(\phi_{p_2}, \psi_{p_2}) & a_h^\tau(\phi_{p_2}, \psi_{p_3}) & a_h^\tau(\phi_{p_2}, \psi_{m_1}) & a_h^\tau(\phi_{p_2}, \psi_{m_2}) & a_h^\tau(\phi_{p_2}, \psi_{m_3}) \\ a_h^\tau(\phi_{p_3}, \psi_{p_1}) & a_h^\tau(\phi_{p_3}, \psi_{p_2}) & a_h^\tau(\phi_{p_3}, \psi_{p_3}) & a_h^\tau(\phi_{p_3}, \psi_{m_1}) & a_h^\tau(\phi_{p_3}, \psi_{m_2}) & a_h^\tau(\phi_{p_3}, \psi_{m_3}) \\ a_h^\tau(\phi_{m_1}, \psi_{p_1}) & a_h^\tau(\phi_{m_1}, \psi_{p_2}) & a_h^\tau(\phi_{m_1}, \psi_{p_3}) & a_h^\tau(\phi_{m_1}, \psi_{m_1}) & a_h^\tau(\phi_{m_1}, \psi_{m_2}) & a_h^\tau(\phi_{m_1}, \psi_{m_3}) \\ a_h^\tau(\phi_{m_2}, \psi_{p_1}) & a_h^\tau(\phi_{m_2}, \psi_{p_2}) & a_h^\tau(\phi_{m_2}, \psi_{p_3}) & a_h^\tau(\phi_{m_2}, \psi_{m_1}) & a_h^\tau(\phi_{m_2}, \psi_{m_2}) & a_h^\tau(\phi_{m_2}, \psi_{m_3}) \\ a_h^\tau(\phi_{m_3}, \psi_{p_1}) & a_h^\tau(\phi_{m_3}, \psi_{p_2}) & a_h^\tau(\phi_{m_3}, \psi_{p_3}) & a_h^\tau(\phi_{m_3}, \psi_{m_1}) & a_h^\tau(\phi_{m_3}, \psi_{m_2}) & a_h^\tau(\phi_{m_3}, \psi_{m_3}) \end{pmatrix},$$

where we have used the abbreviation $\psi_{p_i} = \psi_{V_{p_i}}$ and $\psi_{m_i} = \psi_{V_{m_i}}, i = 1, 2, 3$. By the definition of Π, on τ, there hold

$$\phi_{p_i}^* = \psi_{V_{p_i}} + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\left(\psi_{V_{m_{i+1}}} + \psi_{V_{m_{i+2}}}\right)$$

and

$$\phi_{m_i}^* = \frac{2}{\sqrt{3}}\psi_{V_{m_i}}.$$

Therefore, for all $v \in \mathcal{U}_h$ and all $j = 1, 2, 3$,

$$\begin{aligned} a^\tau(v, \phi_{p_j}) &= a_h^\tau\left(v, \phi_{p_j}^*\right) \\ &= a_h^\tau\left(v, \psi_{V_{p_j}}^*\right) + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\left(a_h^\tau\left(v, \psi_{V_{m_{j+1}}}^*\right) + a_h^\tau\left(v, \psi_{V_{m_{j+2}}}^*\right)\right), \\ a^\tau\left(v, \phi_{m_j}\right) &= a_h^\tau\left(v, \phi_{m_j}^*\right) = \frac{2}{\sqrt{3}}a_h^\tau\left(v, \psi_{V_{m_j}}\right). \end{aligned}$$

Namely, we have the relationship

$$A_\tau^v C_\tau = A_\tau^e$$

with the invertible matrix

$$C_\tau = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}-\frac{1}{\sqrt{3}} & \frac{1}{2}-\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 0 & 0 \\ \frac{1}{2}-\frac{1}{\sqrt{3}} & 0 & \frac{1}{2}-\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{3}} & 0 \\ \frac{1}{2}-\frac{1}{\sqrt{3}} & \frac{1}{2}-\frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}.$$

Consequently, the global stiffness matrix of the quadratic FVM can be obtained by multiplying its corresponding FEM stiffness matrix with a given low-triangular matrix C. Since C is invertible and the existence and uniqueness of the FE solution solutions are well-known, we immediately obtain the existence and uniqueness of the quadratic FV solution.

3.3 The Inf–Sup Condition

With the previous preparations, we are ready to show the following stability result.

Theorem 3.3 *The following inf–sup condition*

$$\inf_{v\in\mathcal{U}_h}\sup_{w\in\mathcal{U}_h}\frac{a_h(v,w^*)}{|v|_1|w|_1}\geq\frac{\alpha_0}{2} \tag{3.10}$$

holds for all $h>0$ if α is piecewise constant with respect to $\mathcal{T}_h$ and holds for sufficiently small $h>0$ if α is piecewise in $W^{1,\infty}$ with respect to $\mathcal{T}_h$.

Proof In the simple case that α is piecewise constant with respect to $\mathcal{T}_h$.

$$\sup_{w\in\mathcal{U}_h}\frac{a_h(v,w^*)}{|v|_1|w|_1}\geq\frac{a(v,v)}{|v|_1^2}\geq\alpha_0.$$

In the case α is only piecewise in $W^{1,\infty}$, we define the average of α in τ as

$$\alpha_\tau=\frac{1}{|\tau|}\int_\tau\alpha,\ \tau\in\mathcal{T}_h$$

and let

$$\bar{a}_h(v,w^*)=-\sum_{\tau\in\mathcal{T}_h}\left(\sum_{p\in\mathcal{N}_h}\int_{\partial V_p\cap\tau}\alpha_\tau\frac{\partial v}{\partial\mathbf{n}}\,w(p)ds-\sum_{m\in\mathcal{M}_h}\int_{\partial V_m\cap\tau}\alpha_\tau\frac{\partial u}{\partial\mathbf{n}}\,v(m)ds\right).$$

Then the validity of (3.10) for piecewise constant coefficient yields

$$\inf_{v\in\mathcal{U}_h}\sup_{w\in\mathcal{U}_h}\frac{\bar{a}_h(v,w^*)}{|v|_1|w|_1}\geq\alpha_0.$$

On the other hand, since

$$|\alpha(x)-\alpha_\tau|\lesssim h|\alpha|_{1,\infty},$$

we have (see e.g. [25])

$$\left|\bar{a}_h(v,w^*)-a_h(v,w^*)\right|\lesssim h|v|_1|w|_1.$$

Thus for sufficiently small h,

$$a_h(v, v^*) \geq a_h(v, v^*) - h|v|_1^2 \geq \frac{\alpha_0}{2}|v|_1^2$$

which implies the inf–sup condition (3.10) for all piecewise $W^{1,\infty}$ coefficient α. □

Remark 3.4 The theorem tells us that just as the quadratic FEM, the stability of our quadratic FVM is independent of the shape of the mesh, i.e. the minimal angle of the triangles in the mesh. Noticing that the high order FV schemes over triangular meshes in the literature is not unconditional stable (see e.g. [6,10,17,18,25], The quadratic FV scheme in the paper might be the first unconditional stable high order FV scheme.

Remark 3.5 Combining the stability in this subsection and the existence and uniqueness in the previous subsection, our quadratic FV scheme is well-posed.

3.4 H^1 Error Estimates

The error estimate under H^1 norm of the finite volume solution can be obtained by the following routine work.

Theorem 3.6 *If α is piecewise $W^{1,\infty}$, then for sufficiently small h*

$$|u - u_h|_1 \lesssim h^2\|u\|_3 \tag{3.11}$$

holds with the hidden constant independent of the mesh size h.

Proof By the inf–sup condition (3.10), for all $v_h \in \mathcal{U}_h$, there holds

$$|u_h - v_h|_1 \lesssim \sup_{w_h \in \mathcal{U}_h} \frac{a_h\left(u_h - v_h, w_h^*\right)}{|w_h|_1} = \sup_{w_h \in \mathcal{U}_h} \frac{a_h\left(u - v_h, w_h^*\right)}{|w_h|_1}.$$

On the other hand, the following continuity of $a_h(\cdot, \cdot)$ has been shown in [25] :

$$a_h(v, w_h^*) \lesssim (|v|_1 + h|v|_2)|w_h|_1, \forall v \in H^1(\Omega), w_h \in \mathcal{U}_h.$$

Then by the triangle inequality, we have the following Céa type's inequality

$$|u - u_h|_1 \lesssim |u - v_h|_1 + h|u - v_h|_2, \forall v_h \in \mathcal{U}_h.$$

Choosing $v_h = u_I$, the interpolation of u in $\mathcal{U}_h$, we obtain the estimate (3.11). □

4 Concluding Remarks

The stability analysis for high order schemes is a challenging task in the mathematical theory of the finite volume method. The paper [26] set up a framework for the stability analysis of the finite volume schemes over quadrilateral meshes by using some novel ideas. This works attempt to shed some light on the stability analysis of the high order finite volume schemes over triangular meshes. The novelty here is a new one-to-one mapping from the trial space to the test space. With this mapping, an identity between the FV and FE bilinear forms has been set up. This identity can be regarded as a natural extension of the corresponding identity of the linear FV scheme established in [1,25]. Note that unlike the somehow artificial analysis in [17,18,25], the analysis in this paper might fit the aesthetical standard of some rigorous mathematicians.

Our ongoing project includes the extension of the approach in this paper to other high order finite volume schemes. Especially, we will investigate the stability and convergence properties for any order finite volume schemes over triangular or tetrahedron meshes. It is expected that by delicate designation, any order finite volume schemes over triangular or tetrahedron meshes have optimal convergence orders both under H^1 and L^2 norms.

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