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Viscous Regularization for the Non-equilibrium Seven-Equation Two-Phase Flow Model

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Abstract In this paper, a viscous regularization is derived for the non-equilibrium seven-equation two-phase flow model (SEM). This regularization, based on an entropy condition, is an artificial viscosity stabilization technique that selects a weak solution satisfying an entropy-minimum principle. The viscous regularization ensures nonnegativity of the entropy residual, is consistent with the viscous regularization for Euler equations when one phase disappears, does not depend on the spatial discretization scheme chosen, and is compatible with the generalized Harten entropies. We investigate the behavior of the proposed viscous regularization for two important limit-cases. First, a Chapman–Enskog expansion is performed for the regularized SEM and we show that the five-equation flow model of Kapila is recovered with a well-scaled viscous regularization. Second, a low-Mach asymptotic limit of the regularized seven-equation flow model is carried out whereby the scaling of the non-dimensional numbers associated with the viscous terms is determined such that an incompressible two-phase flow model, with a properly scaled regularization, is recovered. Both limit-cases are illustrated with one-dimensional numerical results, including two-phase flow shock tube tests and steady-state two-phase flows in converging-diverging nozzles. A continuous finite element discretization is employed for all numerical simulations.

Keywords Seven-equation model · Two-phase flow model · Viscous regularization · Artificial viscosity method · Five-equation model of Kapila · Low-Mach asymptotics

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1 Introduction

Compressible two-phase fluid flows are found in numerous industrial applications. Their numerical solution is an ongoing area of research in modeling and simulation. A variety of two-phase models, with different levels of mechanical and thermodynamical non-equilibrium, has been developed, e.g., the five-equation model of Kapila [25,29,44], the more traditional six-equation models [45], and the fully non-equilibrium seven-equation model (SEM) [5,6,9,43]. These models are obtained by integrating the one-phase flow Hyperbolic Conservative System of Equations (HCSE), i.e., Euler equations, weighted by a characteristic or indicator function for each phase [6,17]. The resulting systems of two-phase flow equations contain non-conservative terms and relaxation terms that describe the interactions between phases, and are supplemented by equations for the volume fraction. These systems of equations are described as Hyperbolic Non-Conservative System of Equations (HNCSE) and are usually solved using discontinuous discretization schemes (e.g., finite volume and discontinuous Galerkin approaches). By assuming that the non-conservative system of equations is hyperbolic, an exact Riemann solver could be used but is often ruled out because of its complexity due to the number of equations involved. Instead, approximate Riemann solvers, a well-established approach for single-phase flows, are employed, see [3,41–43,49], while ensuring the correct low-Mach limit and a consistent discretization for the non-conservative terms [1,41].

In review of solutions to nonlinear, hyperbolic, initial boundary value problems, it is well known that, even with smooth initial data, the existence of a globally smooth solution may not occur because of the nonlinearity of the flux functions. For hyperbolic conservative systems of equations (HCSE), the concept of a weak solution was introduced by Lax to guarantee the existence of a global solution [31]; however, uniqueness of the solution is lost because the problem may allow infinitely many weak solutions. Thus, an additional condition, the"entropy condition," is usually imposed to ensure convergence to a unique solution, the "entropy solution". Although there are several different ways to define the entropy condition, it is generally hoped that they are all equivalent in the sense that they select the same entropy solution (this can be demonstrated for Burgers equations, see [18,35]). For numerical schemes, the entropy condition and solution are sought through the utilization of so-called conservative formulations of the governing laws, along with the addition of an appropriately specified physical or artificial viscosity, either added directly to the governing equations or implied by the discretization employed (truncation error). That is, a regularization is selected, or built, which is consistent with the entropy condition, thereby guaranteeing that the numerical computation captures the physically relevant solution. However, except for the discrete equation method (DEM) for two-phase flows, [6] and the references contained therein, which reverses the order of model construction and solution so that only conservative equations are used, for hyperbolic non-conservative systems of equations (HNCSE) such as the seven-equation two-phase flow model, the theory of Lax cannot be invoked to establish weak and entropy solutions in the sense of distributions. Instead, the work by Del Maso–LeFloch–Murat [12] must be used whereby the non-conservative products are defined as a bounded Borel measure by introducing a Lipschitz family of paths. Generally speaking, the main idea consists in specifying values for the solution at points of discontinuity so that the non-conservative products present in the governing laws are well-defined, hence allowing for the existence of a weak solution. For instance, the notion of weak and entropy solutions are extended to HNCSE in [32,33], leading to the derivation of a generalized Rankine–Hugoniot jump condition that is path-dependent. Furthermore, the notion of rarefaction and shocks waves has also been extended to HNCSE, see [34]. It can be shown that the classical Rankine–Hugoniot

jump condition for conservative systems [31] is recovered from the its generalization to non-conservative systems. Hence, the case of conservative systems of equations can be seen as a subset of non-conservative systems. For the five-equation two-phase model of Kapila et al. [29] the non-conservative nature of the volume fraction evolution equation is treated in a novel manner in [8,44] with a construction of Riemann or approximate Riemann solvers with an augmented system.

Another approach, followed in this paper, to establish the existence of a weak solution of non-conservative terms, consists in considering the vanishing viscosity solution obtained by adding a parabolic viscous regularization to the HNCSE. This approach was first studied by Bianchini et al. [10] and further generalized by Alouges et al. [2]. In their work, Bianchini et al. demonstrate that the solution to the vanishing viscosity system is unique as the vanishing viscosity tends to zero (Theorem 1, p. 229 in [10]). Moreover, Le Floch generalized the notion of entropy condition to non-conservative systems of equations in the space of bounded functions of bounded variation by introducing an entropy function and an associated entropy inequality to ensure convergence of the numerical solution to the entropy solution [32].

In this paper, we derive a viscous regularization for the non-equilibrium seven-equation two-phase flow model of [6] by using an entropy condition under the form of an inequality. The foundation for this work can be traced back to viscous regularizations for single-phase Euler and Navier-Stokes equations, notably [24] and the references therein. The proposed viscous regularization for the seven-equation model is consistent with the minimum entropy principle and Harten's generalized entropies. The minimum entropy principle states that the entropy of a fluid parcel is constant along its pathline and only increases upon crossing a shock wave, a viscous layer, or a thermal conduction region. Numerically computing shockwave solutions without proper stabilization leads to non physical spurious oscillations. When adding viscosity in the vicinity of the shockwave, one replaces the physical effect of the entropy production of the shockwave with the equivalent entropy increase of a viscous layer. If the viscous coefficient is not set appropriately the resulting entropy production will not be equivalent to that of the true shock and the numerical scheme may even create a train of oscillations in the solution, radiating away from the shock/viscous layer, and whose gradients will indeed attempt to produce an entropy production equivalent to that of the shock jump. One needs to properly set the viscosity coefficient along with the jump conditions for the given mesh size in order to produce a monotonic solution increase through the viscous layer with as little smearing as possible, but with an entropy production matching that of the true shock. Tadmor [46] gives an alternative statement (more useful for constructing numerical methods and proofs) of the minimum entropy principle by transforming the Lagrangian statement above into that of an Eulerian reference frame. Further discussion may also be found in Guermond and Popov [24].

In the work presented here, we also ensure that the proposed viscous regularization scales appropriately in two limits of both practical and theoretical importance: the multi-phase low-Mach regime and the reduced multiphase mechanical equilibrium model (treated as a Chapman–Enskog type expansion). First, a two-phase low-Mach asymptotic analysis is carried out to determine the conditions that need to be satisfied by the artificial dissipative terms in order to yield a well-scaled regularization in the low-Mach case; see [14] for the single-phase analog. Second, we investigate the behavior of the viscous regularization when the seven-equation two-phase flow model devolves to the five-equation model of Kapila [29] through a Chapman–Enskog expansion [15,25]. We show that the regularized seven-equation model yields a regularized five-equation model of Kapila.

One of the key aspects of the viscous regularization derived here is that it is agnostic of the spatial discretization scheme, unlike approximate Riemann solvers. Therefore, this viscous

regularization may be employed to stabilize numerical schemes based on either continuous or discontinuous spatial approximations. For examples of prior applications of the technique to the single-phase Euler equations, we refer the reader to [14,24] (for finite volume, continuous Galerkin, and spectral FEM discretizations) and to [50] (for discontinuous Galerkin discretizations).

The remainder of the paper is as follows. In Sect. 2, the seven-equation two-phase flow model (SEM) is recalled along with its main physical and mathematical properties. In Sect. 3, a viscous regularization is derived for the SEM and a Chapman–Enskog expansion of the regularized SEM is carried out to yield a regularized version of the five-equation two-phase flow model of Kapila. In Sect. 4, a low-Mach asymptotic study for the regularized SEM is performed and possible definitions for artificial viscosity coefficients are proposed in order to ensure well-scaled dissipative terms over a wide range of Mach numbers. Our theoretical approach is illustrated in Sect. 5 with one-dimensional numerical results. Finally, we conclude in Sect. 6.

2 The Seven-Equation Two-Phase Flow Model (SEM): Physical and Mathematical Properties

In this section, we recall the seven-equation two-phase flow model and discuss some of its main mathematical and physical properties.

2.1 Governing Equations

The seven-equation two-phase flow model employed in this paper is obtained by assuming that each phase satisfies the single-phase Euler equations (with phase-exchange terms) and by integrating the latter over a control volume after multiplication by a phasic characteristic function. The detailed derivation of the governing equations for a phase k in interaction with a phase j can be found in [6]. In the SEM, each phase obeys a mass, a momentum and an energy balance equation, supplemented by an equation for the volume fraction. The SEM equations are given in Eq. (1).

$$\frac{\partial \alpha_k A}{\partial t} + A u_{int} \cdot \nabla \alpha_k = A \mu_P (P_k - P_j) - S_{k\to j}, \tag{1a}$$

$$\frac{\partial (\alpha\rho)_k A}{\partial t} + \nabla \cdot [(\alpha\rho u)_k A] = -\Gamma_{k\to j}, \tag{1b}$$

$$\frac{\partial (\alpha\rho u)_k A}{\partial t} + \nabla \cdot \left[\alpha_k A (\rho u \otimes u + P\mathbb{I})_k\right] = P_{int} A \nabla \alpha_k + P_k \alpha_k \nabla A + A\lambda_u (u_j - u_k) - M_{k\to j}, \tag{1c}$$

$$\frac{\partial (\alpha\rho E)_k A}{\partial t} + \nabla \cdot \left[\alpha_k u_k A (\rho E + P)_k\right] = P_{\text{int}} A u_{int} \cdot \nabla \alpha_k - \bar{P}_{int} A \mu_P (P_k - P_j) + A\lambda_u \bar{u}_{int} \cdot (u_j - u_k) - E_{k\to j}, \tag{1d}$$

where α_k, ρ_k, u_k and E_k denote the volume fraction, the density, the velocity vector, and the total specific energy of phase k, respectively. The volume fraction, mass, momentum, and energy exchange terms from phase k to phase j are denoted by the symbols $S_{k\to j}$, Γkj, $M_{k\to j}$ and $E_{k\to j}$, respectively, and are set consistently with the second law of thermodynamics [5,39], thus only yielding entropy producing terms. They also obey to the following closure relations:

$$S_{1\to 2} + S_{2\to 1} = 0, \tag{2a}$$
$$\Gamma_{1\to 2} + \Gamma_{2\to 1} = 0, \tag{2b}$$
$$M_{1\to 2} + M_{2\to 1} = 0, \tag{2c}$$
$$E_{1\to 2} + E_{2\to 1} = 0. \tag{2d}$$

We have adopted a standard convention for vector and tensor operations: consider vector a with entries $(a_i)_{i=1,\ldots,\mathtt{dim}}$; $(a\otimes b)_{ij} = a_i b_j$; $\nabla a = \partial_{x_j} a_j$; $(\nabla a)_{ij} = \partial_{x_i} a_j$; for order-2 tensors $\mathbb{g}$, we have $(\nabla \mathbb{g})_j = \partial_{x_i} g_{ij}$, $(\mathbb{g}\cdot a)_i = g_{ij} a_j$, $\mathbb{g} : \mathbb{h} = g_{ij} h_{ij}$; summation is implied whenever an index is repeated. The phasic pressure P_k is computed from an equation of state that is assumed given as a function of the density ρ_k and the phasic internal energy $e_k = E_k - \frac{1}{2}u_k^2$. The A variable is geometric in nature and, in this work, is only spatially dependent. For example, in one-dimension A denotes the "cross-sectional flow area" of a channel or nozzle, while in two-dimensions A can denote a spatially variable "depth". In three-dimensions A could represent a spatially varying porosity. A is included for completeness of the presentation and is set to 1 in many applications. The interfacial pressure and velocity and their corresponding average values are denoted by P_{int}, u_{int}, $\bar{P}_{int}$ and $\bar{u}_{int}$, respectively; they are defined in Eq. (3).

$$P_{int} = \bar{P}_{int} + \frac{Z_k Z_j}{Z_k + Z_j} \frac{\nabla \alpha_k}{||\nabla \alpha_k||} \cdot (u_j - u_k), \tag{3a}$$
$$\bar{P}_{int} = \frac{Z_j P_k + Z_k P_j}{Z_k + Z_j}, \tag{3b}$$
$$u_{int} = \bar{u}_{int} + \frac{\nabla \alpha_k}{||\nabla \alpha_k||} \frac{P_j - P_k}{Z_k + Z_j}, \tag{3c}$$
$$\bar{u}_{int} = \frac{Z_k u_k + Z_j u_j}{Z_k + Z_j}. \tag{3d}$$

The interfacial variables P_{int} and u_{int} control the phase dynamics at the macro level because they specify velocity transport and forces acting upon volume fraction gradients, while the interfacial variables $\bar{P}_{int}$ and $\bar{u}_{int}$, which specify average transport velocity and pressure force, control the phase dynamics at the micro level. Note that the definitions of the interfacial variables P_{int} and u_{int} are not unique, see [5,7,19], for instance. The results presented in this paper apply to Eq. (1), regardless of the definitions for the interfacial variables as long as the entropy inequality given in Eq. (10) holds. Following [6], the pressure and velocity relaxation coefficients, μ_P and λ_u respectively, are functions of the acoustic impedances $Z_k = \rho_k c_k$ and the specific interfacial area A_{int}, as shown in Eq. (4).

$$\mu_P = \frac{A_{int}}{Z_k + Z_j}, \tag{4a}$$
$$\lambda_u = \frac{1}{2}\mu_P Z_k Z_j. \tag{4b}$$

The specific interfacial area (i.e., the interfacial surface area per unit volume of a two-phase mixture), A_{int}, is typically dependent upon flow regime conditions, is not unique, and can be provided as a correlation. In [6], A_{int} is chosen to be a function of the liquid volume fraction:

$$A_{int} = A_{int}^{max}\left[6.75\left(1-\alpha_{liq}\right)^2 \alpha_{liq}\right], \tag{5}$$

with $A_{int}^{max} = 5100\ \mathrm{m}^2/\mathrm{m}^3$. With this definition, the interfacial area is zero in the limits $\alpha_k = 0$ and $\alpha_k = 1$.

The set of equations satisfied by phase j are simply obtained by substituting k by j and j by k in Eq. (1), keeping the same definition of the interfacial variables and using Eq. (2). The equation for the volume fraction of phase j is simply replaced by the algebraic relation

$$\alpha_j = 1 - \alpha_k, \tag{6}$$

which reduces the number of partial differential equations from eight to seven and yields the seven-equation two-phase flow model (SEM).

2.2 Mathematical Properties and Entropy Equation for the SEM Without Viscous Regularization

Some properties of the seven-equation model are discussed next. A set of $5+2\texttt{dim}$ (with $\texttt{dim}$ the geometry's dimension) waves is present in the model: two acoustic waves per phase, one contact wave per phase per domain dimension, and one volume fraction wave propagating at the interfacial velocity u_{int}. These waves (eigenvalues of the Jacobian for the inviscid flux terms) are as follows for each phase k:

$$\begin{aligned}
&\lambda_{1,k} = u_k \cdot \bar{n} - c_k \\
&\lambda_{2,k} = u_k \cdot \bar{n} + c_k \\
&\lambda_{2+d,k} = u_k \cdot \bar{n} \text{ for } d = 1 \ldots \texttt{dim} \\
&\lambda_{3+\texttt{dim}} = u_{int} \cdot \bar{n},
\end{aligned} \tag{7}$$

where $\bar{n}$ is an unit vector pointing to a given direction. The eigenvalues given in Eq. (7) are unconditionally real (as long as the equation of state yields a real-valued sound speed) but not necessarily distinct. However, the seven-equation model admits a set of linearly independent eigenvectors under the *non-resonance condition* $(u_k - u_{int})^2 \neq c_k^2$ for each phase k (see proposition 2.1 in [11]), thus ensuring hyperbolicity of the model. Hyperbolicity is an extremely valuable property for the development of numerical methods since it is required of well-posed hyperbolic systems.

One may relax the seven-equation two-phase flow model to the ill-posed classical six-equation model, where a single pressure is used for both phases; this is accomplished by letting the pressure relaxation coefficient μ_P become very large, i.e., by letting it approach infinity. Note that as the pressure relaxation coefficient increases, so should the velocity relaxation coefficient λ_u; see Eq. (4). However, the six-equation model only relaxes the pressure parameter of the SEM and results in an ill-posed system of equations that can present unstable numerical solutions with sufficiently fine spatial resolution [6,28]. If one lets both the pressure and the velocity relaxation parameters tend to infinity, this further relaxes, or reduces, the seven-equation two-phase flow model to the hyperbolic and well-posed mechanical equilibrium five-equation model of Kapila [29].

Next, we investigate the sign of the phasic and total entropy equations *without viscous regularization present*. The total entropy equation is simply obtained by summing over the phasic entropy equations. Because the exchange source terms are set consistently with the second law of thermodynamics [5,39], they only yield entropy producing terms (i.e., positive terms in the right-hand side of the total entropy equation) and are omitted here. Thus, we consider hereafter the SEM model with only pressure and velocity relaxation terms:

$$\frac{\partial \alpha_k A}{\partial t} + A u_{int} \cdot \nabla \alpha_k = A \mu_P (P_k - P_j), \tag{8a}$$

$$\frac{\partial \left(\alpha\rho\right)_k A}{\partial t} + \nabla \cdot [(\alpha\rho u)_k A] = 0, \tag{8b}$$

$$\frac{\partial\,(\alpha\rho u)_k\,A}{\partial t} + \nabla\cdot\left[\alpha_k A\,(\rho u\otimes u + P\mathbb{I})_k\right] = P_{int}A\nabla\alpha_k + P_k\alpha_k\nabla A + A\lambda_u(u_j - u_k), \quad (8c)$$

$$\frac{\partial\,(\alpha\rho E)_k\,A}{\partial t} + \nabla\cdot\left[\alpha_k u_k A\,(\rho E + P)_k\right] = P_{int}Au_{int}\cdot\nabla\alpha_k - \bar{P}_{int}A\mu_P(P_k - P_j) + A\lambda_u\bar{u}_{int}\cdot(u_j - u_k). \quad (8d)$$

An entropy equation can be derived for each phase k of system Eq. (8) and the sign of the entropy material derivative can be proved positive. The entropy function for a phase k is denoted by s_k and is a function of the phasic density ρ_k and phasic internal energy e_k. The full derivation is given in "Appendix 1" and only the final result is recalled here. The entropy of phase k satisfies the following equation:

$$(s_e)_k^{-1}\alpha_k\rho_k A\frac{Ds_k}{Dt} = \mu_P\frac{Z_k}{Z_k+Z_j}(P_j - P_k)^2 + \lambda_u\frac{Z_j}{Z_k+Z_j}(u_j - u_k)^2 + \|\nabla\alpha_k\|\frac{Z_k}{\left(Z_k+Z_j\right)^2}\left[Z_j(u_j - u_k) + \frac{\nabla\alpha_k}{\|\nabla\alpha_k\|}(P_k - P_j)\right]^2, \quad (9)$$

where $\frac{D(\cdot)}{Dt} = \partial_t(\cdot) + u_k\cdot\nabla(\cdot)$ is the material derivative. The right-hand side of Eq. (9) is unconditionally positive or zero since all terms are squared. Furthermore, the partial derivative of s_k with respect to the internal energy e_k, denoted by $(s_e)_k$, is shown to be equal to the inverse of the temperature of phase k, as in the case of the single-phase Euler equations [13,24], and thus is also a positive term. Equation (9) is valid for both phases $\{k, j\}$, ensuring positivity of the total entropy equation obtained by summation over the phases:

$$\sum_k (s_e)_k^{-1}\alpha_k\rho_k A\frac{Ds_k}{Dt} = \sum_k (s_e)_k^{-1}\alpha_k\rho_k A\,(\partial_t s_k + u_k\cdot\nabla s_k) \geq 0. \quad (10)$$

From a physical perspective, Eq. (10) states that the total entropy of the system increases as a function of time as long as the product $\alpha_k\rho_k$ remains positive. From a numerical perspective, the pressure and velocity relaxation terms add dissipation to the system of equations [see Eq. (9)].

Note that when one phase disappears, Eq. (10) degenerates to the single-phase entropy equation obtained for the single-phase Euler equations [6,13].

3 A Viscous Regularization for the Seven-Equation Two-Phase Flow Model

The objective of this section is to derive a viscous regularization for the SEM presented in Sect. 2. First, we present the methodology used, then derive the phasic entropy equation when accounting for the presence of dissipative terms, and employ the resulting phasic entropy equation to prove the minimum entropy principle. Finally, the scaling of the dissipative terms is investigated in the limit where the seven-equation two-phase flow model devolves, or reduces, to the five-equation two-phase flow model of Kapila [29] when considering large relaxation coefficients μ_P and λ_u [15].

3.1 Methodology

We wish to obtain a viscous regularization for the seven-equation two-phase flow model given in Eqs. (1) using the same methodology employed for the (single-phase) Euler equations

[14,24]. The method consists in adding dissipative terms to the system of equations under consideration and in deriving an entropy equation for the *regularized* system. By adequately choosing these artificial viscous fluxes, one can show that the sign of the entropy equation remains positive, resulting in an *entropy inequality* of the form $\frac{ds_k}{dt} \geq 0$. When considering HCSE (such as Euler equations), such an entropy inequality serves as the *entropy condition* and ensures convergence of the numerical solution to the physical or entropy solution [31]. However, the theory developed by Lax for HCSE [31] is not applicable for non-conservative systems (HNCSE), as reviewed in Sect. 1. Instead the theory developed by Del Maso–LeFloch–Murat is used to generalize the notions of weak solutions by introducing a path in order to define the non-conservative product at points of discontinuity [12]. Bianchini et al. extended the notion of vanishing viscosity solution to HNCSE (see Theorem 1. on page 229, [10]). Le Floch linked the notion of vanishing viscosity solution and the DLM theory to the notion of path (Theorem 3.2 on page 14, [33]). An entropy condition for HNCSE that is equivalent to Lax' admissibility criterion was introduced by LeFloch in [32] and examples of its application to gas dynamics and elastodynamics systems were provided in Sect. 4 of [32]. The generalization of the entropy condition to non-conservative systems (HNCSE) is of great importance for the seven-equation model, as it provides the theoretical foundations for a viscous regularization based on a minimum entropy principle. Namely, we will be able to demonstration that our derived viscous regularization for the SEM model (1) is consistent with an entropy inequality and (2) ensures convergence of the numerical solution to the physical or entropy solution. Furthermore, it is emphasized in [10,12,33] that the theory developed for HNCSE remains valid for HCSE: this remark is of importance since it will guarantee, from a theoretical perspective, that the viscous stabilization for the two-phase flow model will consistently devolve to the single-phase viscous stabilization when the SEM (a HNCSE) devolves to the Euler equations (a HCSE).

Because of the addition of the dissipative terms, the entropy equation obtained in Eq. (9) will be altered and will contain additional terms of yet unknown sign. By carefully choosing a definition for each of the dissipation terms, the sign of this new entropy equation can be determined (and kept positive). Derivation of the viscous regularization for the seven-equation model can be achieved by considering either the phasic entropy equation (Eq. 9) or the total entropy equation (Eq. 10). In the latter case, the minimum entropy principle can be verified for the whole two-phase flow system but may not ensure positivity of the entropy equation for each phase. However, positivity of the total entropy equation can also be achieved by requiring that the minimum entropy principle holds for each phase; this is the approach we have followed. This stronger requirement will also ensure consistency with the single-phase Euler equations when one of the phases disappears in the limit $\alpha_k \to 0$ or $\alpha_k \to 1$.

3.2 Entropy Equation for the SEM with Viscous Regularization

We start with the system of equations given in Eq. (8), where exchange terms have been omitted as explained in Sect. 2.2, and regularize this system by adding dissipation terms (viscous fluxes) to each equation, yielding:

$$\partial_t\left(\alpha_k A\right) + \underline{\underline{A u_{int} \cdot \nabla \alpha_k}} = \underline{\underline{A \mu_P \left(P_k - P_j\right)}} + \nabla \cdot l_k \tag{11a}$$

$$\partial_t\left(\alpha_k \rho_k A\right) + \nabla \cdot \left(\alpha_k \rho_k u_k A\right) = \nabla \cdot f_k \tag{11b}$$

$$\begin{aligned} &\partial_t\left(\alpha_k \rho_k u_k A\right) + \nabla \cdot \left[\alpha_k A \left(\rho_k u_k \otimes u_k + P_k \mathbb{I}\right)\right] \\ &\quad = \alpha_k P_k \nabla A + \underline{\underline{P_{int} A \nabla \alpha_k}} + \underline{\underline{A \lambda_u \left(u_j - u_k\right)}} + \nabla \cdot \mathbb{g}_k \end{aligned} \tag{11c}$$

$$\begin{aligned}&\partial_t\left(\alpha_k\rho_k E_k A\right)+\nabla\cdot\left[\alpha_k A u_k\left(\rho_k E_k+P_k\right)\right]\\&\quad=\underline{\underline{P_{int}Au_{int}\cdot\nabla\alpha_k}}-\underline{\underline{\mu_P A\bar{P}_{int}\left(P_k-P_j\right)}}\\&\quad\underline{\underline{+A\lambda_u\bar{u}_{int}\cdot\left(u_j-u_k\right)}}+\nabla\cdot\left(h_k+u\cdot\mathbb{g}_k\right)\end{aligned}\tag{11d}$$

where f_k, $\mathbb{g}_k$, h_k and l_k are phasic viscous terms, yet to be determined. The terms underlined in Eqs. (11) are part of the inviscid SEM model, see the unregularized system of equations (Eqs. (8)); these terms are entropy-producing and we refer the reader to "Appendix 1" for a detailed derivation of the entropy equation *without* viscous regularization present. In this section, we deal with the *regularized* system of equations for the SEM model and the above underlined terms will later be ignored in the derivation for brevity. The next step consists in deriving the entropy equation for the phase k, in a similar fashion as "Appendix 1" but with dissipative terms now present. The steps are as follows:

1. Derive the density and internal energy equations from Eqs. (11).
2. Assuming that the phasic entropy s_k is a function of density ρ_k and internal energy e_k, derive the entropy equation using the chain rule:

$$\frac{Ds_k}{Dt}=\left(s_\rho\right)_k\frac{D\rho_k}{Dt}+\left(s_e\right)_k\frac{De_k}{Dt}.\tag{12}$$

   The terms $(s_e)_k$ and $(s_\rho)_k$ denote the partial derivatives of s_k with respect to e_k and ρ_k, respectively.
3. Isolate the terms of interest and choose an appropriate expression for each of the viscous fluxes in order to ensure positivity of the entropy residual.

We first derive the density equation expressed in terms of the primitive variable ρ_k by combining Eqs. (11a) and (11b) to obtain:

$$\begin{aligned}&\alpha_k A\left[\partial_t\rho_k+u_k\cdot\nabla\rho_k\right]+\rho_k\alpha_k\nabla\cdot\left(Au_k\right)+\underline{\underline{A\rho_k\left(u_k-u_{int}\right)\cdot\nabla\alpha_k}}\\&\quad=-\underline{\underline{A\rho_k\mu_P\left(P_k-P_j\right)}}+\nabla\cdot f_k-\rho_k\nabla\cdot l_k.\end{aligned}\tag{13}$$

To derive an equation for the phasic internal energy, the phasic velocity equation is obtained by subtracting the density equation (multiplied by u_k) from the phasic momentum equation:

$$\begin{aligned}&\alpha_k\rho_k A\left[\partial_t u_k+\left(u_k\cdot\nabla\right)u_k\right]+\nabla\cdot\left(\alpha_k\rho_k AP_k\mathbb{I}\right)\\&\quad=\alpha_k P_k\nabla A+P_{int}A\nabla\alpha_k+A\lambda_u\left(u_j-u_k\right)+\nabla\cdot\mathbb{g}_k-u_k\nabla\cdot f_k\end{aligned}\tag{14}$$

After multiplying Eq. (14) by velocity u_k, the resulting phasic kinetic energy equation is subtracted from the phasic total energy equation to obtain the internal energy equation for phase k:

$$\begin{aligned}&\alpha_k\rho_k A\left[\partial_t e_k+u_k\cdot\nabla e_k\right]+\alpha_k P_k\nabla\cdot\left(Au_k\right)\\&\quad=\underline{\underline{P_{int}A\left(u_{int}-u_k\right)\cdot\nabla\alpha_k}}-\underline{\underline{\bar{P}_{int}A\mu_P\left(P_k-P_j\right)}}\\&\qquad\underline{\underline{+A\lambda_u\left(u_j-u_k\right)\cdot\left(\bar{u}_{int}-u_k\right)}}-\left(e_k-\tfrac{1}{2}\|u_k\|^2\right)\nabla\cdot f_k+\nabla\cdot h_k+\mathbb{g}_k:\nabla u_k.\end{aligned}\tag{15}$$

As stated earlier, the underlined terms in Eqs. (13) and (15), also present in the derivation of the entropy equation for the SEM *without* regularization, have been shown to yield entropy-producing terms (see "Appendix 1"). Because the focus of this Section is the verification of the minimum entropy principle for the SEM model when viscous regularization is added, we will omit the underlined terms for brevity in the remainder of the derivation. The phasic

entropy equation is now obtained by combining the density equation (Eq. 13) and the phasic internal energy equation (Eq. 15) through the chain rule given in Eq. (12) to yield:

$$\alpha_k \rho_k A \frac{Ds_k}{Dt} + \alpha_k \left(\rho_k^2 (s_\rho)_k + P_k (s_e)_k\right) \nabla\cdot(A u_k) = \left((\rho s_\rho)_k - (e s_e)_k\right) \nabla\cdot f_k - \rho_k^2 (s_\rho)_k \nabla\cdot l_k + (s_e)_k \left[\nabla\cdot h_k + \mathbb{g}_k : \nabla u_k + \tfrac{1}{2}\|u_k\|^2 \nabla\cdot f_k\right]. \tag{16}$$

The second law of thermodynamics for phase k is

$$T_k \mathrm{d}s_k = \mathrm{d}e_k - P_k \frac{\mathrm{d}\rho_k}{\rho_k^2}, \tag{17a}$$

which implies

$$(s_e)_k = T_k^{-1} \text{ and } (s_\rho)_k = -(s_e)_k \frac{P_k}{\rho_k^2}, \tag{17b}$$

that is,

$$\rho_k^2 (s_\rho)_k + P_k (s_e)_k = 0. \tag{17c}$$

Using Eqs. (17c), (16) can be rearranged as

$$\alpha_k \rho_k A \frac{Ds_k}{Dt} = \left((\rho s_\rho)_k - (e s_e)_k\right) \nabla\cdot f_k - \rho_k^2 (s_\rho)_k \nabla\cdot l_k + (s_e)_k \nabla\cdot\left(h_k + \tfrac{1}{2}\|u_k\|^2 f_k\right) + (s_e)_k \left(\mathbb{g}_k - f_k \otimes u_k\right) : \nabla u_k, \tag{18}$$

which is the phasic entropy equation obtained when the viscous regularization terms are included. In order to complete the proof of the minimum entropy principle, one needs to show that the phasic entropy residual $R_{e,k} := \frac{Ds_k}{Dt}$ is positive, which requires (1) positivity of the phasic product $\alpha_k \rho_k$ and (2) positivity of the right-handside of Eq. (18). Section 3.4 is devoted to proving positivity of the right-handside of Eq. (18) by choosing an adequate expression for the viscous fluxes. We also show that $\alpha_k \rho_k$ remains nonnegative under certain conditions. Then, in Sect. 3.5, the minimum entropy principle is proved by using results from Sect. 3.4.

3.3 Definitions of the Viscous Fluxes Based on the Entropy Condition

The right-hand side of Eq. (18) can be further simplified by introducing the viscous fluxes $\tilde{f}_k$ and $\tilde{h}_k$ and a viscous tensor $\mathbb{F}(u_k)$ as a function of f_k, $\mathbb{g}_k$, h_k and l_k as follows:

$$\tilde{f}_k = f_k - \rho_k l_k \tag{19a}$$

$$\alpha_k \rho_k A \mu_k \mathbb{F}(u_k) = \mathbb{g}_k - f_k \otimes u_k \tag{19b}$$

$$\tilde{h}_k = h_k + \tfrac{1}{2}\|u_k\|^2 f_k - (\rho e)_k l_k, \tag{19c}$$

where μ_k is a positive viscosity coefficient for phase k. The functional form of the dissipative terms given in Eqs. (19) are derived later in this section. Substituting the expressions of Eq. (19) into Eq. (18) yields:

$$\alpha_k \rho_k A \frac{Ds_k}{Dt} = \left((\rho s_\rho)_k - (e s_e)_k\right) \nabla\cdot \tilde{f}_k + (s_e)_k \nabla\cdot \tilde{h}_k + (s_e)_k \alpha_k \rho_k A \mu_k \mathbb{F}(u_k) : \nabla u_k + \rho_k l_k \cdot \nabla s_k, \tag{20}$$

or, after using the chain rule to combine partial derivatives,

$$\alpha_k \rho_k A \frac{Ds_k}{Dt} = \underbrace{\nabla\cdot\left[(s_e)_k \tilde{h}_k + \left((\rho s_\rho)_k - (e s_e)_k\right)\tilde{f}_k\right]}_{\mathcal{R}_0} - \underbrace{\left(\tilde{h}_k \cdot \nabla (s_e)_k + \tilde{f}_k \cdot \nabla\left[(\rho s_\rho)_k - (e s_e)_k\right]\right)}_{\mathcal{R}_1} + \underbrace{(s_e)_k\, \alpha_k \rho_k A \mu_k \mathbb{F}(u_k) : \nabla u_k}_{\mathcal{R}_2} + \underbrace{\rho_k l_k \cdot \nabla s_k}_{\mathcal{R}_3}. \tag{21}$$

We now split the right-hand-side of Eq. (21) into several residuals denoted by $\mathcal{R}_0$ through $\mathcal{R}_3$ and we analyze the sign of each of them separately.

The term $\mathcal{R}_3$ is a function of the gradient of the entropy. At the location of the minimum entropy, this gradient is zero; therefore, $\mathcal{R}_3$ has no effect on the minimum entropy principle. Thus, we observe that the minimum entropy principle will be verified independently of the definition of the dissipation term l_k used in the volume fraction equation, Eq. (11a); this is a noteworthy difference with respect to the derivation of the single-phase viscous regularization of [14,24]. We will later provide a possible definition for l_k.

Since $(s_e)_k := T_k^{-1}$ is defined as the inverse of the temperature and is thus positive, the sign of $\mathcal{R}_2$ is conditioned by the choice of the function $\mathbb{F}(u_k)$ so that its product with the tensor ∇u_k is positive. As in [14,24], $\mathbb{F}(u_k)$ is chosen to be proportional to the symmetric gradient of the velocity vector u_k (rate of deformation),

$$\mathbb{F}(u_k) = \nabla^s u_k. \tag{22}$$

With such a choice, the viscous regularization is also rotationally invariant.

We now focus on the term denoted by $\mathcal{R}_1$, which is identical to the right-hand side of the single phase entropy equation for Euler equations (see "Appendix 2" in [14]). $\mathcal{R}_1$ is known to be positive when (i) assuming concavity of the entropy function s_k with respect to the internal energy e_k and the specific volume $1/\rho_k$ and (ii) when using the following definitions for the dissipative fluxes $\tilde{h}_k$ and $\tilde{f}_k$:

$$\tilde{f}_k = \alpha_k A \kappa_k \nabla \rho_k \tag{23a}$$

$$\tilde{h}_k = \alpha_k A \kappa_k \nabla (\rho e)_k\,, \tag{23b}$$

where κ_k is another positive viscosity coefficient. Finally, using Eqs. (23), the $\mathcal{R}_0$ can be recast as a function of the phasic entropy as follows:

$$\mathcal{R}_0 = \nabla\cdot(\alpha_k A \kappa_k \rho_k \nabla s_k)\,. \tag{24}$$

The entropy residual equation for phase k can now be written in its final form:

$$\alpha_k \rho_k A \frac{Ds_k}{Dt} - f_k \cdot \nabla s_k - \nabla\cdot(\alpha_k A \rho_k \kappa_k \nabla s_k) = -\alpha_k \rho_k A \kappa_k Q_k + (s_e)_k \alpha_k A \rho_k \mu_k \nabla^s u_k : \nabla u_k, \tag{25}$$

where Q_k is :

$$Q_k = \mathbf{X}_k^T \mathbb{\Sigma}_k \mathbf{X}_k$$

$$\text{with } \mathbf{X}_k = \begin{bmatrix} \nabla \rho_k \\ \nabla e_k \end{bmatrix} \text{ and } \mathbb{\Sigma}_k = \begin{bmatrix} \rho_k^{-2} \partial_{\rho_k}(\rho_k^2 \partial_{\rho_k} s_k) & \partial_{\rho_k, e_k} s_k \\ \partial_{\rho_k, e_k} s_k & \partial_{e_k, e_k} s_k \end{bmatrix}.$$

As with the single-phase Euler equations, one can demonstrate that $\mathbb{\Sigma}_k$ is a symmetric negative definite quadratic form when s_k is concave with respect to e_k and ρ_k^{-1} [14,24]. We have proved the following:

Lemma 1 *When the seven-equation model is regularized as shown in Eq. (11) and the corresponding entropy equation given in Eq. (25) holds, then the left-hand side of Eq. (25) is non-negative if and only if (i) the phasic entropy function s_k is concave with respect to e_k and ρ_k^{-1} and (ii) the following expressions for the viscous fluxes are employed:*

$$f_k = \alpha_k A \kappa_k \nabla \rho_k + \rho_k l_k \tag{26}$$

$$\mathcal{g}_k = \alpha_k A \mu_k \rho_k \nabla^s u_k + f_k \otimes u_k \tag{27}$$

$$h_k = \alpha_k A \kappa_k \nabla (\rho e)_k - \frac{\|u_k\|^2}{2} f_k + (\rho e)_k l_k \tag{28}$$

where κ_k and μ_k are positive viscous coefficients.

Equation (25) is constructed to satisfy the minimum entropy principle for the SEM *with* viscous regularization. At a location $x_{min}(t)$ where s_k reaches its minimum value at time t, the gradient, ∇s_k, and Laplacian, Δs_k, of the entropy are zero and positive at this particular location, respectively. Because the terms on the right-hand-side of Eq. (25) have been shown to be either positive or zero when the entropy reaches its spatial minimum, one concludes that $\alpha_k \rho_k \frac{Ds_k}{Dt} \geq 0$. For the minimum entropy principle to hold for each phase k, i.e., $\frac{Ds_k}{Dt} \geq 0$, positivity of the product $\alpha_k \rho_k$ is also required. To prove positivity of $\alpha_k \rho_k$, we need to derive an expression for the dissipative term l_k since it is present in the phasic continuity equation Eq. (11b); this is described in the next paragraph. Note that at an entropy extremum, the inequality $\alpha_k \rho_k \frac{Ds_k}{Dt} \geq 0$, holds *independently* of the definition of the dissipative term l_k utilized in the volume fraction equation.

3.4 Nonnegativity of the Phasic Density and the Phasic Volume Fraction

We now provide a possible expression for l_k by considering the volume fraction equation, Eq. (11a), by itself. It is an hyperbolic equation whose eigenvalue (speed) is u_{int}. An entropy relation can be derived for that equation alone (by multiplying it by α_k). Following the work of Guermond et al. for linear advection and Burgers' equations [22,23], it can be shown that a dissipative term ensuring selection of the entropy solution for the volume fraction equation is of the form $l_k = \beta_k A \nabla \alpha_k$, where β_k is a positive viscosity coefficient. The dissipative term is made proportional to the area A for consistency with the other definitions of the viscous terms. Now that an expression for the dissipative term l_k is known, we can show that (i) the phasic volume fraction α_k is bounded within the interval [0, 1] and (ii) the product $\alpha_k \rho_k$ remains nonnegative under certain conditions. The regularized volume fraction equation is as follows:

$$\partial_t (\alpha_k A) + u_{int} A \nabla \alpha_k = \nabla \cdot (A \beta_k \nabla \alpha_k) . \tag{29}$$

Let us assume that phasic volume fraction is initially bounded within the interval $[0, 1]$ and that a maximum is reached at a given point r_0 such as $\partial_d \alpha_k = 0$ and $\partial_{dd} \alpha_k \leq 0$ for $1 \leq d \leq \mathtt{dim}$. At the point where the maximum is reached, the regularized volume fraction equation becomes:

$$\partial_t \alpha_k = \beta_k \Delta \alpha_k \leq 0, \tag{30}$$

($\beta_k \geq 0$) which means that the volume fraction locally decreases as a function of time and thus remains bounded from above. Similarly, if a minimum is reached in r_0 (i.e., $\partial_d \alpha_k = 0$ and $\partial_{dd}\alpha_k \geq 0$), it can be shown that the volume fraction is also bounded from below. As a result, with the above regularization, the volume fraction should remain bounded in time within the given initial interval [0, 1], which yields the following lemma:

Lemma 2 *Assume the phasic volume fraction is regularized, Eq. (29), by a viscous flux denoted by l_k. The phasic volume fraction converges to a weak solution and is bounded if only if the viscous flux l_k is defined as*

$$l_k = \beta_k A \nabla \alpha_k,$$

where β_k is a positive viscous coefficient.

Remark 1 Note a viscous regularization of the phasic volume fraction equation is needed, as illustrated in the following example. Consider a domain with uniform initial phasic pressure, velocity, temperature and an initial step discontinuity in the volume fraction. Under such conditions and using the definitions given in Eq. (3), the interfacial pressure and velocity are equal to the phasic pressure and velocity, respectively. Also, each equation of Eq. (8) now devolves to the regularized volume fraction equation Eq. (29). Assuming that the phasic volume fraction equation is not stabilized by a viscous term, i.e., if $l_k = 0$, then this example case with the above initial condition would result in a numerical solution with spurious oscillations in the vicinity of the discontinuity.

Now that we have derived an expression for the dissipative term l_k, we consider the regularized phasic continuity equation to prove that the quantity $\alpha_k \rho_k$ remains nonnegative. Using the above definition of l_k, the regularized continuity equation is as follows:

$$\partial_t \left(\alpha_k \rho_k A\right) + \nabla \cdot \left(\alpha_k \rho_k u_k A\right) = \nabla \cdot \left[\kappa_k \alpha_k \nabla \rho_k + \beta_k \rho_k \nabla \alpha_k\right]. \tag{31}$$

In the particular case where $\beta_k = \kappa_k$, the dissipative terms in Eq. (31) become $\nabla \cdot (\kappa_k \nabla (\alpha_k \rho_k))$ and the resulting continuity equation is identical to the regularized continuity equation for single-phase when letting $\hat{\rho}_k = \alpha_k \rho_k$:

$$\partial_t \left(\hat{\rho}_k A\right) + \nabla \cdot \left(\hat{\rho}_k u_k A\right) = \nabla \cdot \left(\kappa_k \nabla \hat{\rho}_k\right). \tag{32}$$

Following Lemma 3.1 of [24] and using Eq. (32), a nonnegative density principle can be proved for the quantity $\hat{\rho}_k$. Thus, we conclude that the quantity $\alpha_k \rho_k$ remains nonnegative under the assumption $\beta_k = \kappa_k$. Note that unlike Lemma 3.2 of [24], the density $\hat{\rho}_k$ cannot be shown to be strictly positive because the phasic volume fraction α_k may locally be equal to zero.

3.5 Minimum Entropy Principle

In this section, the minimum entropy residual is proved for each phase. In the previous section, we have shown that (i) the phasic volume fraction α_k remains bounded within a given interval (ii) the quantity $\alpha_k \rho_k$ is nonnegative, and (iii) the right-hand side of Eq. (25) leads to entropy production (that is, the right-hand side is positive). As a result, the phasic entropy residual remains nonnegative, i.e. $\mathcal{R}_{e,k} \geq 0$ and that the minimum entropy principle holds for each phase and thus for the two-phase flow system as well:

Theorem 1 (Minimum entropy principle) *Assume that Lemmas 1 and 2 hold and that the product $\alpha_k \rho_k$ is non-negative. Assume that the solution of Eq. (11) is smooth, then the minimum entropy principle holds,*

$$s_k(x,t) \geq inf\, s_{k,0}(x) \text{ and } \sum_k s_k(x,t) \geq \sum_k inf\, s_{k,0}(x).$$

At this point, some remarks are in order. All of the dissipative terms have now been defined and are recalled here:

$$l_k = \beta_k A \nabla \alpha_k \tag{33a}$$
$$f_k = \alpha_k A \kappa_k \nabla \rho_k + \rho_k l_k \tag{33b}$$
$$\mathcal{G}_k = \alpha_k A \mu_k \rho_k \nabla^s u_k + f_k \otimes u_k \tag{33c}$$
$$h_k = \alpha_k A \kappa_k \nabla (\rho e)_k - \frac{\|u_k\|^2}{2} f_k + (\rho e)_k l_k \tag{33d}$$

1. The definition of the dissipative term l_k contains a viscosity coefficient β_k that is independent of the other viscosity coefficients, μ_k and κ_k. Its definition should account for the eigenvalue u_{int} and the entropy equation associated with the volume fraction equation.
2. The dissipative term f_k is a function of l_k. Thus, all of the other dissipative terms are also functions of l_k.
3. The partial derivatives $(s_e)_k$ and $(s_{\rho_k})_k$ can be computed using the definition provided in Eq. (17a) and are functions of the phasic thermodynamic variables: pressure, temperature and density.
4. All of the dissipative terms are chosen to be proportional to the void fraction α_k and the cross-sectional area A (except for l_k that is only proportional to A). Thus, when one of the phases disappears, the dissipative terms for that phase vanish, as expected for consistency. Similarly, when α_k goes to one, the *regularized* single-phase Euler equations with variable area are recovered.
5. By choosing $\beta_k = \mu_k = \kappa_k$ and $\mathbb{F}(u_k) = \nabla u_k$, the viscous flux expressions simplify to yield:

$$\partial_t (\alpha_k A) + A u_{int} \cdot \nabla \alpha_k = A \mu_P \left(P_k - P_j\right) + \nabla \cdot [A \kappa_k \nabla \alpha_k] \tag{34a}$$
$$\partial_t (\alpha_k \rho_k A) + \nabla \cdot (\alpha_k \rho_k u_k A) = \nabla \cdot \left[A \kappa_k \nabla (\alpha \rho)_k\right] \tag{34b}$$
$$\begin{aligned} &\partial_t (\alpha_k \rho_k u_k A) + \nabla \cdot [\alpha_k A (\rho_k u_k \otimes u_k + P_k \mathbb{I})] \\ &\quad = \alpha_k P_k \nabla A + P_{int} A \nabla \alpha_k + \nabla \cdot \left[A \kappa_k \nabla (\alpha \rho u)_k\right] \end{aligned} \tag{34c}$$
$$\begin{aligned} &\partial_t (\alpha_k \rho_k E_k A) + \nabla \cdot [\alpha_k A u_k (\rho_k E_k + P_k)] \\ &\quad = P_{int} A u_{int} \cdot \nabla \alpha_k - \mu_P \bar{P}_{int} \left(P_k - P_j\right) \\ &\qquad + A \lambda_u \bar{u}_{int} \cdot \left(u_j - u_k\right) + \nabla \cdot \left[A \kappa_k \nabla (\alpha \rho E)_k\right]. \end{aligned} \tag{34d}$$

   This particular choice of viscous regularization is analogous to the parabolic regularization for Euler equations [40]. Note that if one chooses $\mathbb{F}(u_k) = \nabla u_k$, the viscous regularization is no longer rotationally invariant.
6. Compatibility of the viscous regularization proposed in Eq. (33) with the generalized entropies identified in Harten et al. [27] is demonstrated in "Appendix 2".

At this point, we have derived a viscous regularization for the seven-equation two-phase flow model that ensures nonnegativity of the entropy residual, convergence of the numerical solution to the entropy solution when assuming concavity of the phasic entropy s_k, and consistency with the viscous regularization derived for Euler equations [14,24] in the limit

$\alpha_k \to 0, 1$. The viscous regularization involves a set of three positive viscosity coefficients for each phase, μ_k, κ_k, and β_k. The definition of these viscosity coefficients should be inferred from the scaled SEM in order to ensure well-scaled dissipative terms for a wide range of Mach numbers (subsonic, transonic and supersonic flows) and is the topic of Sect. 4. But first, we investigate, in the next paragraph, the effect of the viscous regularization in the case where the SEM devolves to the five-equation model of Kapila [29].

Remark 2 In Sect. 3.5, the phasic entropy residual has been shown to be nonnegative. However, when numerically computed, one can observe that the phasic entropy residual $\mathcal{R}_{e,k}$, exhibits some negative variation in the vicinity of a shock, as previously noted in [21]: this is due to the fact that the phasic entropy residual becomes a Dirac measure in the shock region and that this quantity is numerically sampled (numerical artifacts due to mesh/quadrature). We refer the reader to Sect. 4 of [21] for additional discussions. As a consequence, the absolute value of the entropy residual is used in computational results.

3.6 A Chapman–Enskog Expansion of the Regularized Seven-Equation Two-Phase Flow Model

The five-equation two-phase flow model of Kapila [29] is obtained from the non-regularized SEM by performing a Chapman–Enskog expansion. The steps of the derivation are well detailed in the literature and can be found in [15,25], for instance. The objective of this section is to perform a Chapman–Enskog expansion of the *regularized* SEM derived in Sect. 3.5 and to investigate the behavior of the dissipative terms: we wish to ensure that the dissipative terms remain well-scaled and can efficiently stabilize the resulting system of equations. Only the main results of the derivation are given here and have been obtained following a procedure similar to [15,25]. First, the pressure and velocity relaxation coefficients are scaled by a small coefficient ϵ that describes the strength of the perturbation: $\mu_P \to \frac{\mu_P}{\epsilon}$ and $\lambda_u \to \frac{\lambda_u}{\epsilon}$. Then, each variable is expanded in powers of ϵ, as shown, for instance, in Eq. (35).

$$P_k = P_{k,0} + P_{k,1}\varepsilon + P_{k,2}\varepsilon^2 + \dots \tag{35}$$

These expansions are inserted in the SEM equations Eqs. (11) with the dissipative fluxes obtained in Eqs. (33), yielding a hierarchy of equations for each power of ϵ. The dissipative terms derived in Sect. 3.5 do not depend the relaxation coefficients μ_P and λ_u and thus have the same scaling as the inviscid fluxes. From the leading-order momentum and energy equations (terms in powers of ϵ^{-1}), we determine that the leading-order phasic pressures and velocities are equal in each phase:

$$P_{k,0} = P_{j,0} = P_0 \quad \text{and} \quad u_{k,0} = u_{j,0} = u_0. \tag{36}$$

This also implies that

$$P_{int,0} = \bar{P}_{int,0} = P_0 \quad \text{and} \quad u_{int,0} = \bar{u}_{int,0} = u_0. \tag{37}$$

Using these results, the next-leading order equations (terms in ϵ^0) yield the regularized five-equation model of Kapila, which we recall in Eq. (38) where we have defined $\rho u = \sum_{i=k,j} \alpha_i \rho_i u_i$ and $\rho E = \sum_{i=k,j} \alpha_i \rho_i E_i$ as the mixture momentum and the mixture energy, respectively.

$$\frac{\partial \alpha_{k,0} A}{\partial t} + A u_0 \cdot \nabla \alpha_{k,0} = A K_k \nabla \cdot u_0 + \nabla \cdot (\beta_k \nabla \alpha_k)_0 \,, \tag{38a}$$

$$\frac{\partial \left(\alpha\rho\right)_{k,0} A}{\partial t} + \nabla \cdot \left[(\alpha\rho)_{k,0} u_0 A \right] = \nabla \cdot f_{k,0}, \tag{38b}$$

$$\frac{\partial\left(\alpha\rho\right)_{j,0}A}{\partial t}+\nabla\cdot\left[\left(\alpha\rho\right)_{j,0}u_0A\right]=\nabla\cdot f_{j,0}, \tag{38c}$$

$$\frac{\partial\left(\rho u\right)_0A}{\partial t}+\nabla\cdot\left[A\left(\rho u\otimes u+P\mathbb{I}\right)_0\right]=P_0\nabla A+\sum_{i=k,j}\nabla\cdot\mathcal{G}_{i,0}, \tag{38d}$$

$$\frac{\partial\left(\rho E\right)_0A}{\partial t}+\nabla\cdot\left[uA\left(\rho E+P\right)_0\right]=\sum_{i=k,j}\nabla\cdot\left(h_{i,0}+u_0\cdot\mathcal{G}_{i,0}\right). \tag{38e}$$

The mixture pressure is defined as $P=\sum_{i=k,j}\alpha_iP_i$ and is a function of the phasic pressures. The notation $(fg)_0$ means that we only keep the 0th order terms in the product fg. The function denoted by K_k in Eq. (38a) is given in Eq. (39) and is derived from the Chapman–Enskog expansion.

$$K_k=\alpha_k\alpha_j\frac{\rho_jc_j^2-\rho_kc_k^2}{\alpha_j\rho_kc_k^2+\alpha_k\rho_jc_j^2} \tag{39}$$

From the leading-order equations, we conclude that the regularized seven-equation model yield a suitably regularized five-equation model of Kapila derived via a Chapman–Enskog expansion: to leading order, the phasic pressure and velocity remain equal in each phase. In addition, we observe that the dissipative terms of the viscous regularization scale appropriately in the five-equation model limit, see Eqs. (38).

4 The Scaled Seven-Equation Two-Phase Flow Model with Viscous Regularization

In the previous section, we have presented a viscous regularization for the seven-equation two-phase flow model. However, two-phase fluids may be found in various flow regimes, from extremely low-Mach subsonic situations to supersonic cases. In this section, we write the non-dimensionalized version of the SEM to carry out a low-Mach asymptotic analysis. In order to recover the low-Mach incompressible equations, some requirements on the scaling of the viscous fluxes will be presented. We then propose an all-Mach scaling in order to obtain an appropriately scaled viscous regularization for many flow regimes (subsonic, transonic and supersonic flows).

When employing artificial viscosity techniques, one needs to ensure that sufficient artificial viscosity is present in the shock and discontinuity regions to prevent spurious oscillations from forming in the numerical solution while little to no dissipation is added when the solution is smooth. A low-Mach asymptotic limit needs to be performed on the *regularized* SEM system of equations in order to properly scale the viscosity coefficients and to recover the incompressible asymptotic equations [26,47,48]. The purpose of this section is to derive the scaled SEM equations and investigate the scaling of the dissipative terms. First, the scaled SEM are derived. Then, two limit cases will be considered to determine appropriate scaling for the viscosity coefficients so that the dissipative terms remain well-scaled for:

1. (a) the isentropic low-Mach limit where the seven-equation model devolves to an incompressible system of equations, and
2. (b) the non-isentropic limit where shocks can occur.

Because each phase can experience different flow regimes, e.g., supersonic gas and subsonic liquid, we elect to keep distinct the three viscosity coefficients for each phase, μ_k, κ_k, and

β_k. The study is performed for the seven-equation model using the Stiffened Gas Equations of State (SGEOS) [37] given in Eq. (40).

$$P_k = (\gamma_k - 1)\,\rho_k(e_k - q_k) - \gamma_k P_{k,\infty} \tag{40}$$

Note that the ideal gas equations of state can be recovered by letting $P_{k,\infty} = q_k = 0$.

4.1 Derivation of the Non-dimensionalized Seven-Equation Two-Phase Flow Model

We consider the case where the relaxation coefficients, representing the micro scale or local phase interaction effects, and the gradient of volume fraction $\nabla\alpha_k$, representing the macro scale phase interaction effects, are eliminated (i.e., set to zero); that is, the two phases do not interact and the volume fraction of each phase remains constant over time [see Eq. (34a)]. Thus, the seven-equation model degenerates to two sets of Euler equations with a pseudo cross-sectional area $\alpha_k A$. In the remainder of this section, we assume that the volume fraction of each phase is non-zero. The first step in the study of the two limit cases (a) and (b) is to re-write each system of equations using non-dimensionalized variables. To do so, the following variables are introduced for each phase k:

$$\begin{aligned}
&\rho_k^* = \frac{\rho_k}{\rho_{k,\infty}},\ u_k^* = \frac{u_k}{u_{k,\infty}},\ P_k^* = \frac{P_k}{\rho_{k,\infty}c_{k,\infty}^2},\ e_k^* = \frac{e_k}{c_{k,\infty}^2},\ E_k^* = \frac{E_k}{c_{k,\infty}^2},\\
&\alpha_k^* = \frac{\alpha_k}{\alpha_{k,\infty}},\ x^* = \frac{x}{L_\infty},\ t_k^* = \frac{t_k}{L_\infty/u_{k,\infty}},\ \mu_k^* = \frac{\mu_k}{\mu_{k,\infty}},\ \kappa_k^* = \frac{\kappa_k}{\kappa_{k,\infty}},
\end{aligned} \tag{41}$$

where the subscript ∞ denotes the far-field or stagnation quantities and the superscript $*$ stands for the non-dimensional variables. The far-field reference quantities are chosen such that the dimensionless flow quantities are of order 1. The stagnation quantities for the pressure and velocity interfacial variables will be specified for each case. The reference phasic Mach number is given by

$$M_{k,\infty} = \frac{u_{k,\infty}}{c_{k,\infty}}. \tag{42}$$

With the scaling introduced in Eq. (41), the scaled equations for phase k with viscous regularization are as follows (the volume fraction equation is omitted because the volume fraction is assumed constant for each phase):

$$\begin{aligned}
&\partial_{t^*}(\alpha_k\rho_k A)^* + \nabla^*\cdot(\alpha_k\rho_k u_k A)^*\\
&\quad = \frac{1}{\mathrm{P\acute{e}}_{k,\infty}^{\kappa}}\nabla^*\cdot(A\kappa_k\alpha_k\nabla\rho_k)^* + \frac{1}{\mathrm{P\acute{e}}_{k,\infty}^{\beta}}\nabla^*\cdot(A\beta_k\rho_k\nabla\alpha_k)^*
\end{aligned} \tag{43a}$$

$$\begin{aligned}
&\partial_{t^*}(\alpha_k\rho_k u_k A)^* + \nabla^*\cdot[\alpha_k A(\rho_k u_k\otimes u_k)]^* + \frac{1}{M_{k,\infty}^2}\nabla^*\left(\alpha_k A P_k^*\right)\\
&\quad = \frac{1}{M_{k,\infty}^2}\alpha_k^* P_k^*\nabla^* A^* + \frac{1}{\mathrm{Re}_{k,\infty}}\nabla^*\cdot\left(\alpha_k A\mu_k\rho_k\nabla^s u_k\right)^*\\
&\qquad + \frac{1}{\mathrm{P\acute{e}}_{k,\infty}^{\kappa}}\nabla^*\cdot(\alpha_k A\kappa_k\nabla\rho_k\otimes u_k)^* + \frac{1}{\mathrm{P\acute{e}}_{k,\infty}^{\beta}}\nabla\cdot(A\beta_k\rho_k\nabla\alpha_k\otimes u_k)^*
\end{aligned} \tag{43b}$$

$$\partial_{t^*}\left(\alpha_k^* A\rho_k E_k\right)^* + \nabla^*\cdot\left[\alpha_k^* A u_k^*(\rho_k E_k)^*\right] + \nabla^*\cdot\left(\alpha_k^* A u_k P_k\right)^*$$

$$= \frac{1}{\text{Pé}^{\kappa}_{k,\infty}} \nabla^* \cdot (\alpha_k A \kappa_k \nabla (\rho_k e_k))^* + \frac{M^2_{k,\infty}}{\text{Pé}^{\kappa}_{k,\infty}} \nabla^* \cdot \left(A \alpha_k \kappa_k \frac{||u_k||^2}{2} \nabla \rho \right)^*$$
$$+ \frac{M^2_{k,\infty}}{\text{Re}_{k,\infty}} \nabla^* \cdot \left(\alpha_k A \mu_k \rho_k u_k : \nabla^s u_k \right)^* + \frac{1}{\text{Pé}^{\beta}_{k,\infty}} \nabla^* \cdot (\rho_k e_k A \beta_k \nabla \alpha_k)^* \quad (43c)$$

where the phasic numerical Reynolds number ($\text{Re}_{k,\infty}$) represents the ratio of fluid inertia force to viscous forces and the Péclet numbers ($\text{Pé}^{\kappa}_{k,\infty}$ and $\text{Pé}^{\beta}_{k,\infty}$) represent the ratio of advection rate to diffusion rate. These numbers are defined as:

$$\text{Re}_{k,\infty} = \frac{u_{k,\infty} L_\infty}{\mu_{k,\infty}}, \text{Pé}^{\kappa}_{k,\infty} = \frac{u_{k,\infty} L_\infty}{\kappa_{k,\infty}} \text{ and } \text{Pé}^{\beta}_{k,\infty} = \frac{u_{k,\infty} L_\infty}{\beta_{k,\infty}}. \quad (44)$$

The numerical Reynolds and Péclet numbers are obviously related to the viscosity coefficients $\mu_{k,\infty}$, $\kappa_{k,\infty}$ and $\beta_{k,\infty}$. Thus, once a scaling (in terms of powers of $M_{k,\infty}$) is obtained for $\text{Re}_{k,\infty}$, $\text{Pé}^{\kappa}_{k,\infty}$, and $\text{Pé}^{\beta}_{k,\infty}$ in the two limit cases (a) and (b) given above, it will impose a condition upon the definition of the phasic viscosity coefficients μ_k, κ_k, and β_k. For brevity, the superscripts * are omitted in the remainder of this section.

4.2 Scaling of $\text{Re}_{k,\infty}$, $\text{Pé}^{\kappa}_{k,\infty}$ and $\text{Pé}^{\beta}_{k,\infty}$ in the Low-Mach Asymptotic Regime (Case a)

In the low-Mach isentropic limit, the seven-equation model converges to an incompressible system of equations that is characterized, for each phase, by pressure fluctuations of order M_k^2 and a divergence-free constraint on the velocity, $\nabla \cdot (\alpha_k A u_k) = 0$. When adding dissipative terms as a viscous regularization of the flow equations, the main properties of the low-Mach asymptotic limit must be preserved. In the low-Mach limit, the two phases are assumed not to interact and, thus, the volume fraction of each phase remains constant. We begin by expanding each variable in powers of the Mach number. As an example, the expansion for the pressure is given by:

$$P_k(r,t) = P_{k,0}(r,t) + P_{k,1}(r,t) M_{k,\infty} + P_{k,2}(r,t) M^2_{k,\infty} + \ldots \quad (45)$$

By studying the resulting momentum equations for various powers of M_∞, we note that the leading- and first-order pressure terms, $P_{k,0}$ and $P_{k,1}$, are spatially constant if and only if $\text{Re}_{k,\infty} = \text{Pé}^{\kappa}_{k,\infty} = \text{Pé}^{\beta}_{k,\infty} = \mathcal{O}(1)$ (i.e., scale as 1). In this case, remembering that $\nabla \alpha_k = 0$ and $\alpha_k \neq 0$, we have at order $M^{-2}_{k,\infty}$:

$$A \nabla (\alpha_k P_k)_0 = 0 \Longrightarrow \nabla P_{k,0} = 0 \quad (46a)$$

and at order $M^{-1}_{k,\infty}$

$$A \nabla (\alpha_k P_k)_1 = 0 \Longrightarrow \nabla P_{k,1} = 0. \quad (46b)$$

From Eqs. (46) we infer that the leading- and first-order pressure terms are spatially independent which ensures that the pressure variations are of order Mach number squared, as expected in the low-Mach asymptotic limit. Using the scaling $\text{Re}_{k,\infty} = \text{Pé}^{\kappa}_{k,\infty} = \text{Pé}^{\beta}_{k,\infty} = 1$, the second-order momentum equations and the leading-order expressions for the continuity and energy equations are:

$$\partial_t (\alpha_k A \rho_k)_0 + \nabla \cdot (\alpha_k A \rho_k u_k)_0 = \nabla \cdot (\alpha_k A \kappa_k \nabla \rho_k)_0 + \nabla \cdot (A \beta_k \nabla \alpha_k)_0 \quad (47a)$$
$$\partial_t (\alpha_k A \rho_k u_k)_0 + \nabla \cdot (\alpha_k A \rho_k u_k \otimes u_k)_0 + A \nabla (\alpha_k P_k)_2$$

$$= \nabla\cdot\left[\alpha_k A\left(\mu_k\rho_k\nabla^s u_k + \kappa_k u_k\otimes\nabla\rho_k\right)\right]_0 + \nabla\cdot(A\beta_k\rho u\nabla\alpha_k)_0 \tag{47b}$$

$$\partial_t(\alpha_k A\rho_k E_k) + \nabla\cdot(\alpha_k A u_k\rho_k E_k)_0 + \nabla\cdot\ (\alpha_k A u_k P_k)_0$$
$$= \nabla\cdot[\alpha_k A\kappa_k\nabla\left(\rho_k e_k\right)]_0 + \nabla\cdot[A\rho_k e_k\beta_k\nabla\alpha_k]_0 \tag{47c}$$

where the notation $(fg)_0$ means that we only keep the 0th-order terms in the product fg. Recall that Eqs. (47) are written with the assumptions of Sect. 4.1, that is, that the micro and macro scale interaction effects are neglected. The set of equations given in Eq. (47) is similar to the single-phase Euler equations with variable area when interpreting $\alpha_k A$ as a pseudo-area [14]. The leading-order of the Stiffened Gas Equation of State (Eq. (40)) is also given by

$$P_{k,0} = (\gamma_k - 1)\rho_{k,0}(E_{k,0} - q_k) - \gamma P_{k,\infty} = (\gamma_k - 1)\rho_{k,0}(e_{k,0} - q_k) - \gamma_k P_{k,\infty}. \tag{48}$$

Using Eq. (48), the energy equation can be recast as a function of the leading-order pressure, P_0, as follows:

$$A\alpha_{k,0}\left[\partial_t P_k + u_k\cdot\nabla P_k\right]_0 + (\gamma_k - 1)\nabla\cdot\ [\alpha_k A u_k P_k]_0$$
$$+\left(P_{k,0} + \gamma_k P_{k\infty}\right)\nabla\cdot(\alpha_k A u_k)_0 = \nabla\cdot(\alpha_k A\kappa_k\nabla P_k)_0 + \nabla\cdot(A P_k\beta_k\nabla\alpha_k)_0\,. \tag{49}$$

From Eq. (46a), we infer that P_0 is spatially constant. Thus, Eq. (49) becomes

$$\frac{\alpha_{k,0}A}{\gamma_k\left(P_{k,0} + P_{k,\infty}\right)}\frac{dP_{k,0}}{dt} = -\nabla\cdot(\alpha_k A u_k)_0 \tag{50}$$

and, at steady state, we have the divergence-free constraint on velocity

$$\nabla\cdot(\alpha_k A u_k)_0 = 0. \tag{51}$$

That is, the leading-order of the product of velocity and area is divergence-free, which reduces to the standard divergence-free velocity field for a constant area A and volume fraction α_k, i.e., $\nabla\cdot u_{k,0} = 0$. Finally, recall that Eq. (50) was written for the stiffened gas law; to retrieve standard result for the ideal gas law, simply set $P_{k,\infty} = 0$. The same reasoning can be applied to the leading-order of the continuity equation (Eq. 47a) to show that the material derivative of the density variable is stabilized with appropriately scaled dissipative terms (the resulting regularization does not depend on the Mach number):

$$\partial_t\left(\alpha_k A\rho\right)_0 + \nabla\cdot(\alpha_k A\rho_k u_k)_0 = \nabla\cdot[\alpha_k A\kappa_k\nabla\rho + A\beta_k\rho_k\nabla\alpha_k]_0\,. \tag{52}$$

Therefore, we conclude with the following lemma:

Lemma 3 (Low-Mach asymptotic limit) *Assume the seven-equation model is regularized, Eq. (8), with the viscous fluxes defined in Eq. (33). Then, the regularized seven-equation model yields the correct incompressible limit in low-Mach situations if only if the phasic numerical numbers* $Re_{k,\infty}$, $Pr^{\kappa}_{k,\infty}$ *and* $Pr^{\beta}_{k,\infty}$ *scale as one.*

4.3 Scaling of $Re_{k,\infty}$, $Pé^{\kappa}_{k,\infty}$ and $Pé^{\beta}_{k,\infty}$ for Non-isentropic Flows (Case b)

Next, we consider the non-isentropic case. Recall that even subsonic flows can present shocks (for instance, a step initial condition in the pressure will trigger a shock formation, independently of the Mach number). The non-dimensional form of the seven-equation model given in Eq. (43) provides some insight on the dominant terms as a function of the Mach number. This is particularly obvious in the momentum equation, Eq. (43b), where the gradient of pressure is scaled by $1/M^2_{k,\infty}$. However, in the non-isentropic case, we can no longer expect

$\frac{\nabla(\alpha_k P_k)}{M^2_{k,\infty}} = \nabla\,(\alpha_k P_k)_2$ at the leading order and, therefore, the pressure gradient term may need to be stabilized by some dissipative terms scaling as $1/M^2_{k,\infty}$ so as to prevent spurious oscillations from forming in the numerical solution. By inspecting the dissipative terms present in the momentum equation, we note that by imposing that one of the dissipative terms scales as $1/M^2_{k,\infty}$ will lead to a total of eight different options (a scaling of either 1 or $M^2_{k,\infty}$ for each number, $\mathrm{Re}_{k,\infty}$, $\mathrm{Pé}^{\kappa}_{k,\infty}$, and $\mathrm{Pé}^{\beta}_{k,\infty}$). Three of these options are discussed next; the five other options are omitted for brevity and we leave it to the reader to verify that they can indeed be ruled out by following the same reasoning given below. The three options analyzed here are:

$$
\begin{aligned}
&(i)\ \mathrm{Re}_{k,\infty} = 1,\ \mathrm{Pé}^{\kappa}_{k,\infty} = M^2_{k,\infty} \text{ and } \mathrm{Pé}^{\beta}_{k,\infty} = 1,\\
&(ii)\ \mathrm{Re}_{k,\infty} = 1,\ \mathrm{Pé}^{\kappa}_{k,\infty} = 1 \text{ and } \mathrm{Pé}^{\beta}_{k,\infty} = M^2_{k,\infty} \text{ or}\\
&(iii)\ \mathrm{Re}_{k,\infty} = M^2_{k,\infty},\ \mathrm{Pé}^{\kappa}_{k,\infty} = 1 \text{ and } \mathrm{Pé}^{\beta}_{k,\infty} = 1.
\end{aligned}
$$

Any of these choices will also affect the stabilization of the volume fraction, continuity, and energy equations. For instance, using Péclet numbers equal to $M^2_{k,\infty}$ may effectively stabilize the volume fraction and continuity equations in the shock region but this may also add an excessive amount of dissipation for subsonic flows at the location of the contact wave. Such a behavior may not be suitable for accuracy purpose, making options (i) and (ii) inappropriate. The same reasoning, left to the reader, can be carried out for the energy equation (Eq. 43c) and results in the same conclusion. The remaining choice, option (iii), has the proper scaling: in this case, only the dissipation terms involving $\nabla^{s,*}u^*_k$ scale as $1/M^2_{k,\infty}$ because $\mathrm{Re}_{k,\infty} = M^2_{k,\infty}$, leaving the regularization of the volume fraction and continuity equations unaffected because $\mathrm{Pé}^{\beta}_{k,\infty} = \mathrm{Pé}^{\kappa}_{k,\infty} = 1$.

Lemma 4 (Non-isentropic flows) *Assume the seven-equation model is regularized,* Eq. (8)*, with the viscous fluxes defined in* Eq. (33)*. Then, the regularized seven-equation model is efficiently stabilized in all Mach situations when experiencing non-isentropic flows if only if the phasic numerical numbers $Re_{k,\infty}$, $Pr^{\kappa}_{k,\infty}$ and $Pr^{\beta}_{k,\infty}$ scale as follows:*

$$
Re_{k,\infty} = M^2_{k,\infty},\ \mathrm{Pé}^{\kappa}_{k,\infty} = 1 \text{ and } \mathrm{Pé}^{\beta}_{k,\infty} = 1 \tag{53}
$$

Remark 3 In the above, the Péclet number $\mathrm{Pé}^{\kappa}_{k,\infty}$ was set to one to avoid adding an excessive amount of dissipation in contact waves (presence of a $1/\mathrm{Pé}^{\kappa}_{k,\infty}$ term in Eq. (43a)). However, if one can distinguish contact waves from shock / rarefaction waves (in a numerical scheme, for instance), then there is the possibility of having the local Péclet number $\mathrm{Pé}^{\kappa}_{k,\infty}$ set to one in contact waves and to $M^2_{k,\infty}$ in shock waves. As such, this option would allow stabilizing shock waves ($\mathrm{Pé}^{\kappa}_{k,\infty}$ scales as $M^2_{k,\infty}$) and would not be over-dissipative in the contact region ($\mathrm{Pé}^{\kappa}_{k,\infty}$ scales as 1):

$$
\begin{aligned}
&\mathrm{Re}_{k,\infty} = M^2_{k,\infty},\\
&\mathrm{Pé}^{\kappa}_{k,\infty} = \begin{cases} 1 & \text{in the contact region} \\ M^2_{k,\infty} & \text{in the shock region} \end{cases},\\
&\mathrm{Pé}^{\beta}_{k,\infty} = 1.
\end{aligned}
$$

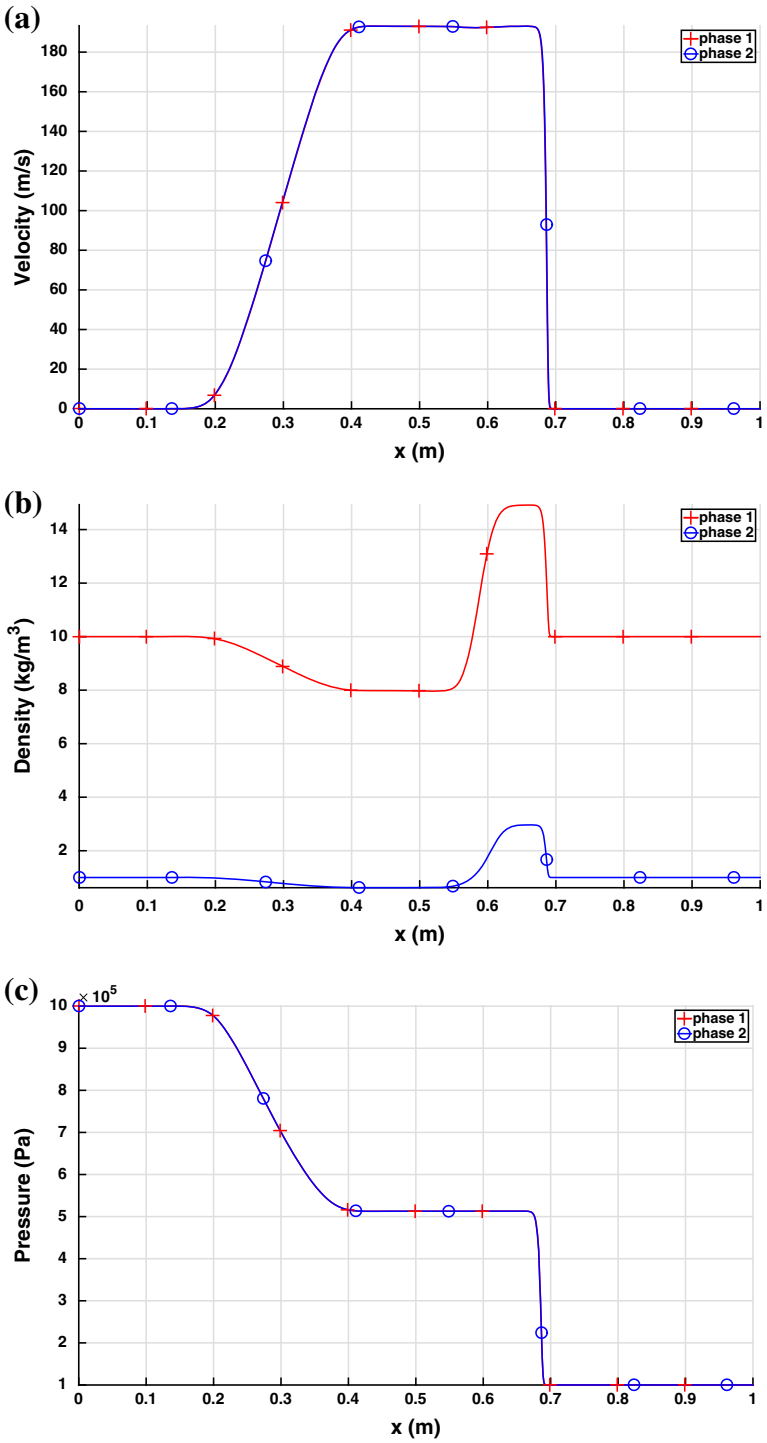

Fig. 1 Two-phase flow shock tube results using a LLF viscosity, ($t = 473\ \mu s$). **a** Velocity, **b** density, **c** pressure, **d** volume fraction

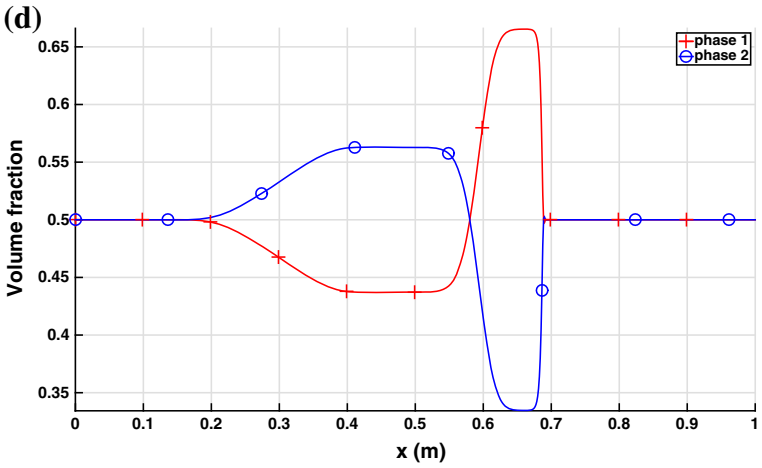

Fig. 1 continued

5 Numerical Illustrations

We recall that the seven-equation model for two-phase flow with viscous regularization is given in Eq. (11), with definitions for the artificial viscous fluxes from Eq. (33). When the pressure and velocity relaxation parameters are large, Eq. (11) become equivalent to the regularized five-equation model given by Eq. (38) and for which the phasic pressures and velocities are equal.

In our numerical simulations, the seven-equation model given is discretized using a *continuous* Galerkin finite element method. This is in contrast with the typical discretizations schemes adopted for two-phase flow computations and HNCSE in general. The reasons are two-fold. First, we wish to illustrate that the proposed viscous regularization is effective and agnostic of the discretization scheme and thus have opted for a less traditional discretization technique for HNCSE; note that when employing continuous finite elements, interfacial numerical fluxes need not be defined, which simplifies the implementation. Second, several open-source, parallel, continuous finite element libraries are available (e.g., MOOSE [20] which was employed here), thus possibly enabling the dissemination of multi-phase flow simulations using such tools. For temporal discretization, we use BDF2 [4], a second-order accurate backward differencing scheme. A Jacobian-free Newton Krylov (JFNK) solver is employed to solve the nonlinear system of discrete equations at the end of each time step. An approximate full Jacobian matrix is used as a preconditioner and is computed by finite difference (this is one of the options available in MOOSE and is reasonably efficient for 1-D simulations).

We illustrate the ability of the proposed viscous regularization to stabilize the seven-equation two-phase flow equations using three test cases. The first test is a shock tube configuration where we show that the viscous regularization prevents undershoots and overshoots near shocks and discontinuities. The second set of tests aims at illustrating the low-Mach asymptotic limit study performed in Sect. 4 by investigating the steady-state solution of a two-phase flow in a converging-diverging nozzle. Finally, in the third series of tests, we consider a shock tube with nearly pure phases on both side of the initial membrane (almost pure liquid to the left of it, almost pure vapor to the right). In all tests, the phasic pressure is computed using the Stiffened Gas equation of state [37]:

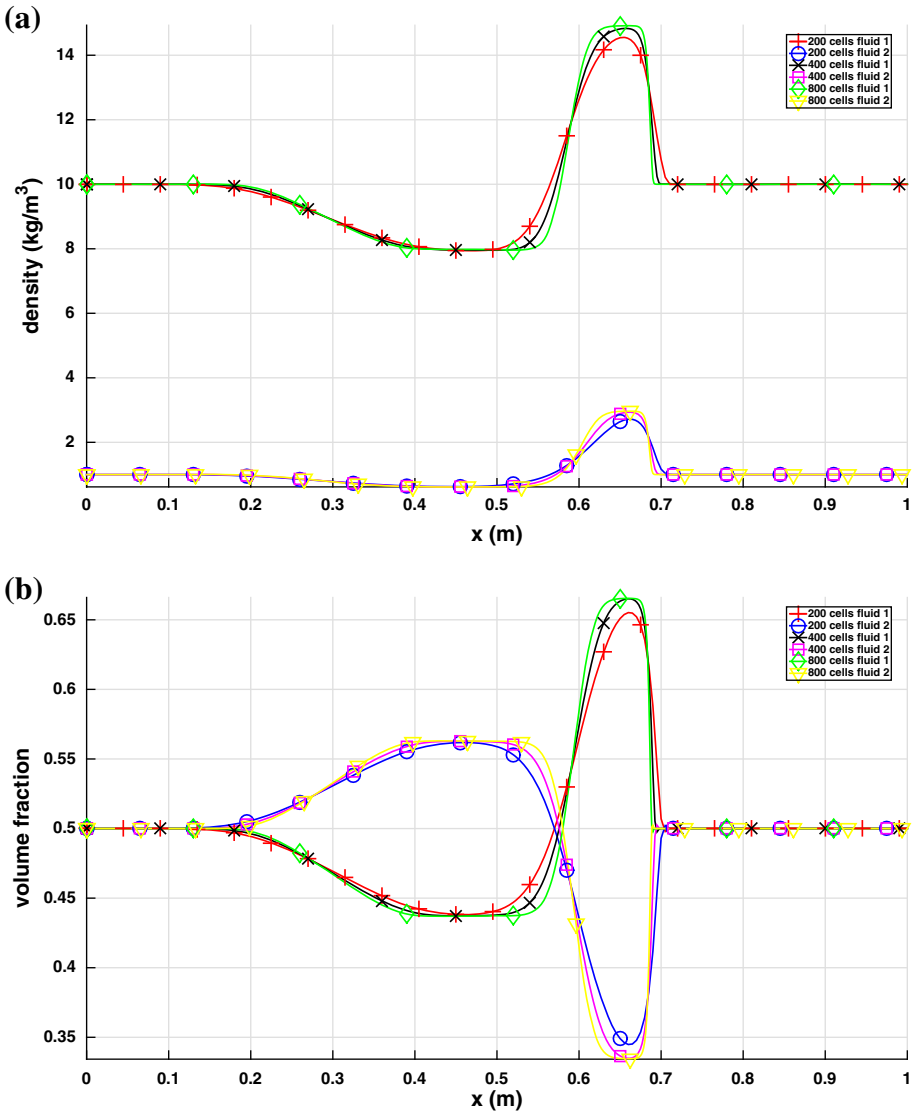

Fig. 2 Two-phase flow shock tube results using a LLF viscosity, mesh refinement study ($t = 473\ \mu s$). **a** Density, **b** volume fraction

$$P_k = (\gamma_k - 1)\rho_k(e_k - q_k) - \gamma_k P_{\infty,k} \quad \text{and} \quad \rho_k = \frac{P_k + P_{\infty,k}}{C_{v,k}(\gamma_k - 1)T_k}, \tag{54}$$

where the parameters γ_k, q_k and $P_{\infty,k}$ are fluid-dependent and will be specified for each test. Note that the Ideal gas law is recovered by setting $q_k = P_{\infty,k} = 0$ in Eq. (54).

5.1 Shock Tube Test Case

In this example, we set the relaxation coefficients μ_P and λ_u to large values; the pressures and velocities in each phase should become identical, as expected by performing a Chapman–

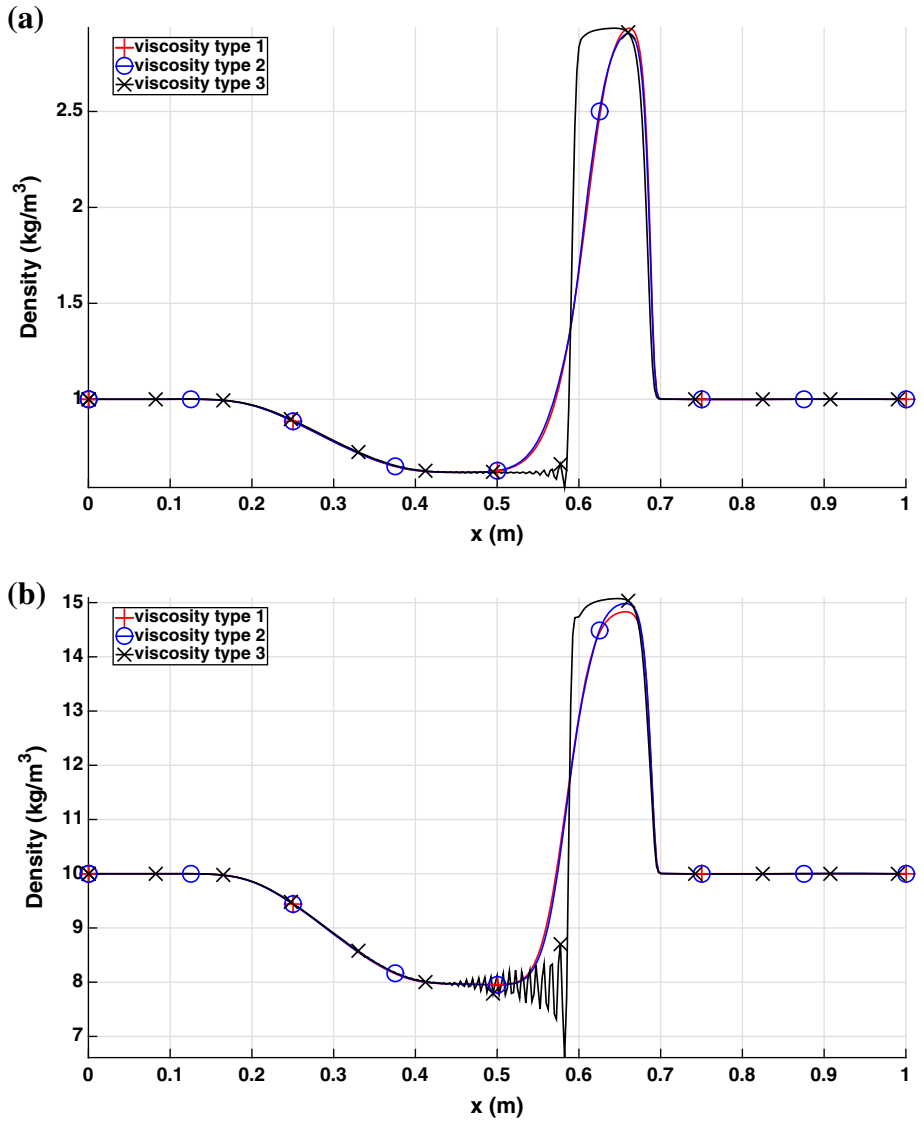

Fig. 3 Shock tube results: liquid and vapor densities computed using three different viscosity types at $t = 473\ \mu s$. **a** Liquid density, **b** vapor density

Enskog expansion. We consider two fluids, denoted by the subscripts 1 and 2, and employ the ideal gas law with the following parameters $\gamma_1 = 3$ and $\gamma_2 = 1.4$. The numerical illustration consists of a 1-D shock tube of length 1 meshed with 400 cells and containing two-phase mixtures separated by a membrane placed at $x = 0.5$. The initial left/right values of the pressures are $(P_{k,left} = 10^5)$ and $(P_{k,right} = 10^4)$ (for $k = 1, 2$). The initial density and volume fraction are uniform and set to $\rho_1 = 10$, $\rho_2 = 1$ and $\alpha = 0.5$, respectively. Both fluids are initially at rest. The relaxation coefficients μ_P and λ_u are computed from Eq. (4a) and Eq. (4b), respectively, along with Eq. (5) and $A_{int}^{max} = 4 \times 10^3\ m^{-1}$. At $t = 0$, the membrane is suddenly removed and the simulation is ran until $t = 473\ \mu s$ with a CFL of one. In Sect. 5.1.1,

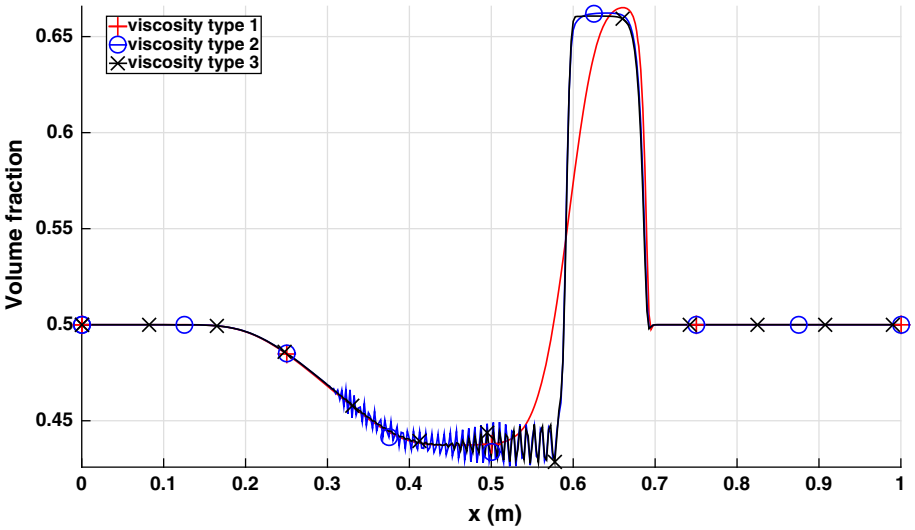

Fig. 4 Liquid volume fraction obtained using three different viscosity types at $t = 473\ \mu$s

Table 1 Stiffened gas equation of state parameters for steam and liquid water

Fluid	γ_k	$C_{v,k}$ (J kg^{-1} K^{-1})	$P_{\infty,k}$ (Pa)	q_k (J kg^{-1})
Liquid	2.35	1816	10^9	-1167×10^3
Vapor	1.43	1040	0	2030×10^3

numerical results are presented when defining the phasic viscosity coefficients proportional to the local maximum eigenvalues which we refer to as Lax-Friedrichs-like scheme. Then, a grid-convergence study is performed in Sect. 5.1.2 using the same definitions for the phasic viscosity coefficients as in Sect. 5.1.1. Then in Sect. 5.1.3, the influence of the dissipative terms present in the phasic volume and continuity equations onto the numerical solutions is investigated as an illustration of Remark 1.

5.1.1 Numerical Solution with a Lax-Friedrichs-Like Viscosity

In order to test the proposed viscous regularization, we need to define the phasic viscosity coefficients β_k, μ_k, and κ_k that appear in Eq. (33). Following Guermond et al. (Section 2.2 in [24]), we employ $\mu_k = \kappa_k = \beta_k = \frac{h}{2}(||u_k|| + c_k)$ where h is the grid size and $||u_k|| + c_k$ is the largest eigenvalue for phase k. With this choice of viscosity, the numerical scheme is equivalent to a local Lax-Friedrichs (LLF) scheme which is known to be over-dissipative. The numerical results are given in Fig. 1.

The viscous numerical solutions do not display any instability in the vicinity of the shock region ($x \simeq 0.73$ at $t = 473\ \mu s$). In Fig. 1b, the contact wave ($x \simeq 0.6$) is smeared because of the over-dissipative nature of the viscosity coefficients. This example shows that the viscous regularization developed in this paper can efficiently stabilize the numerical solution. We observe that the phasic pressures and velocities are indeed identical for large values of the relaxation parameters.

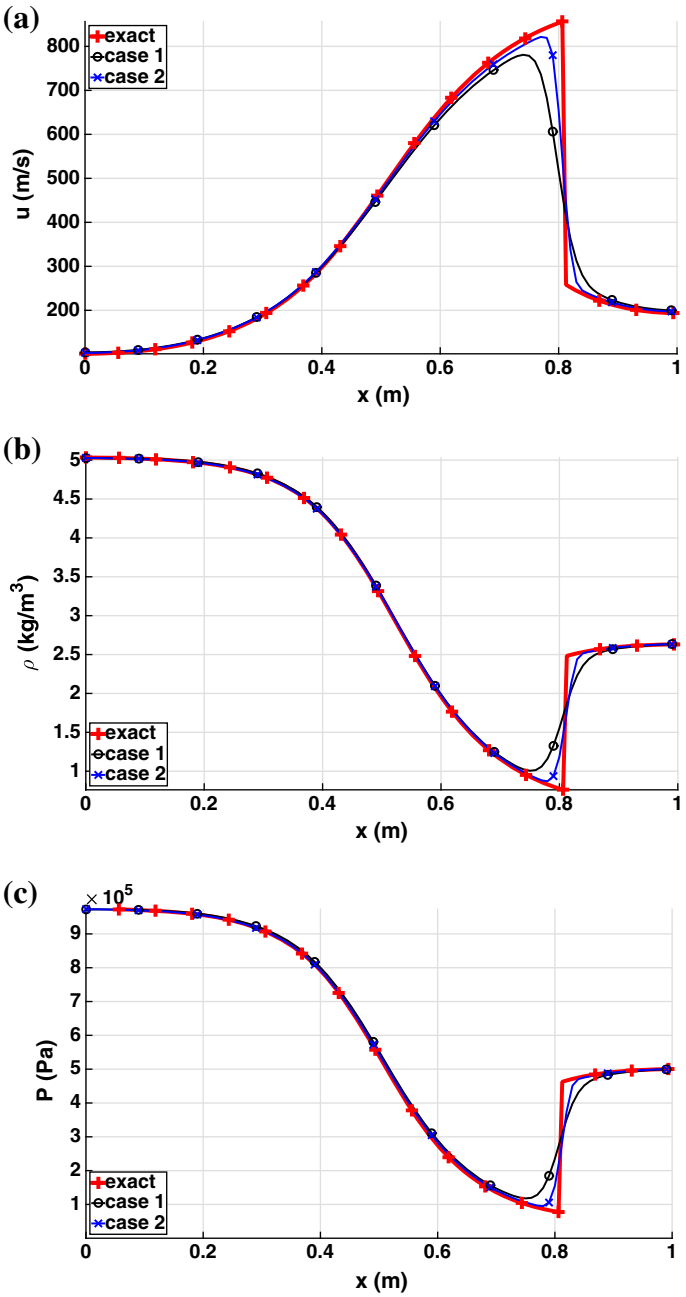

Fig. 5 Numerical and analytical steady-state vapor-phase solutions for the divergent-convergent two-phase flow nozzle problem. **a** Velocity. **b** Density. **c** Pressure. **d** Mach number

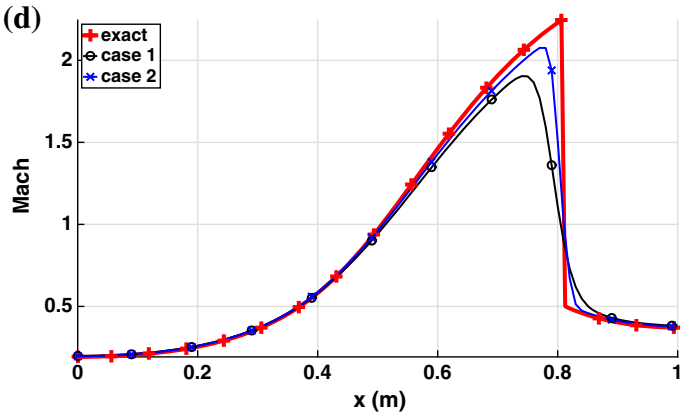

Fig. 5 continued

5.1.2 A Grid Convergence Study

The proposed regularization is grid-dependent, i.e., the phasic viscosity coefficients are set proportional to the grid size h. Thus, it is of interest to investigate the behavior of the numerical solution as the mesh is refined, especially in the vicinity of the contact (around $x = 0.6$) and shock (around $x = 0.7$) waves where the use of a LLF viscosity led to excessive smearing as shown in the density and the volume fraction profiles in Fig. 1b, d, respectively. Thus, we propose to run the same test as before using three different mesh resolutions: 200, 400 and 800 cells. The numerical solutions of the phasic densities and the phasic volume fractions obtained are displayed in Fig. 2a, b, respectively. From the phasic density profiles in Fig. 2a, it is observed that the contact and the shock waves are better resolved as the mesh is refined, as expected, indicating that the numerical solution converges, at least formally, to a weak solution. Similar observations can be drawn for the phasic volume fraction profiles, as shown in Fig. 2b, and were also observed for the other phasic variables, i.e., pressure and velocity (not shown).

5.1.3 Numerical Solutions with an Incomplete Viscous Regularization

In this test, we investigate the influence of regularizing some or all of the equations on the numerical solution, as an illustration of Remark 1. We use again the previous shock test case and select the LLF viscosity and consider the following 3 cases:

1. all viscosity coefficients non zero, i.e., the local Lax-Friedrichs scheme already shown in Fig. 1 (viscosity type 1),
2. the definitions for μ_k and κ_k are unchanged but β_k is set to zero (viscosity type 2). In doing so, we assess the effect of the viscous stabilization on the volume fraction, and
3. the viscosity coefficients κ_k and β_k are set to zero while μ_k is unchanged (viscosity type 3). In doing so, no stabilization is present in both the volume fraction and continuity equations.

For each case, the density and volume fraction profiles are presented in Figs. 3 and 4. In cases (1) and (2), the numerical solutions do not display any instability in the density profiles. Case (3) displays oscillations, which is expected since the continuity equations are not regularized (similarly to the results obtained in the single-phase case, [24]).

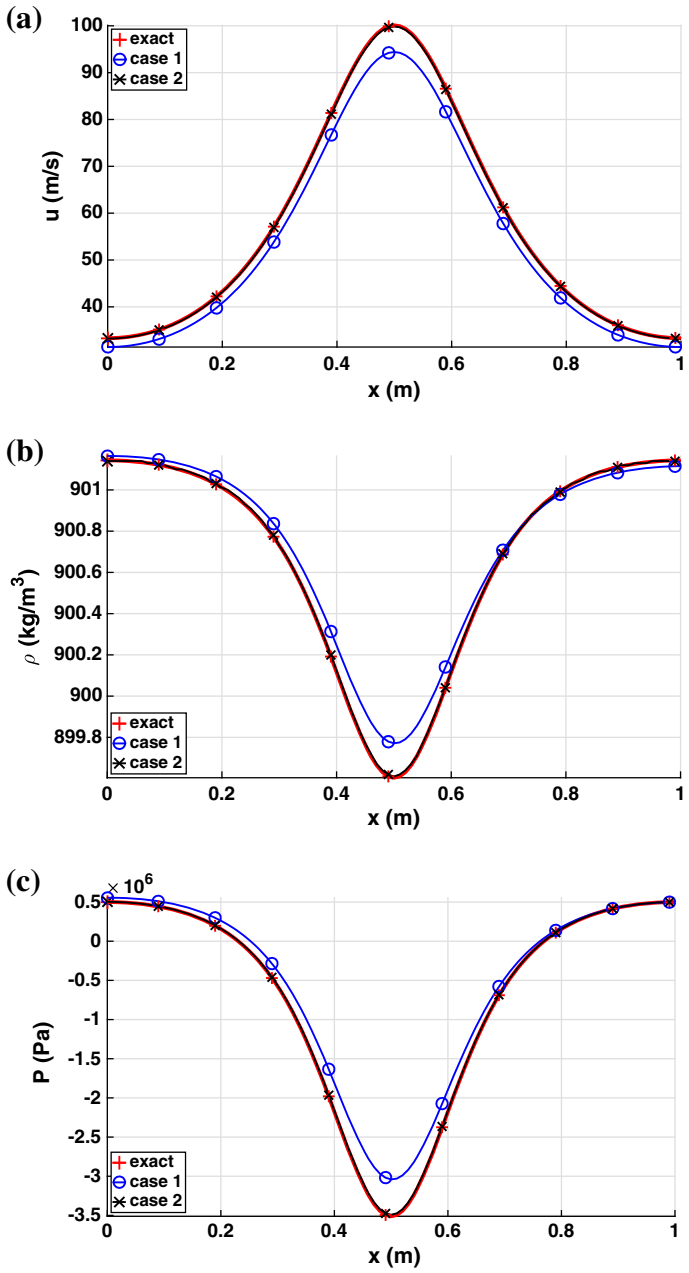

Fig. 6 Numerical and analytical steady-state liquid-phase solutions for the divergent-convergent two-phase flow nozzle problem. **a** Velocity, **b** density, **c** pressure. **d** Mach number

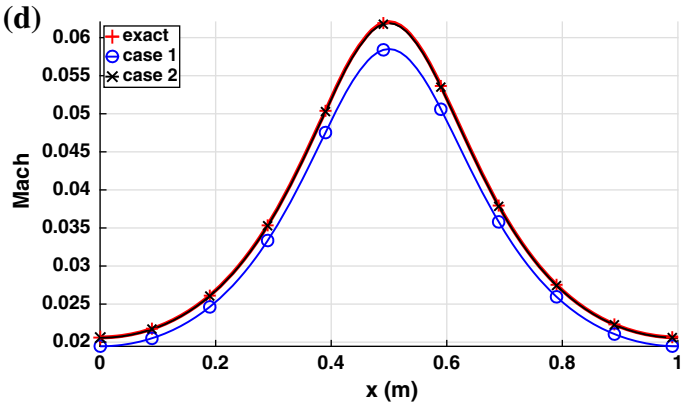

Fig. 6 continued

When inspecting the volume fraction profiles near the contact region (see Fig. 4), one notices that as soon as $\beta_k = 0$ (cases (2) and (3)), the simulations exhibit spurious oscillations in the contact region. In case (3), the continuity and the volume fraction equations are both no longer regularized ($\kappa_k = \beta_k = 0$); this leads to the formation of instabilities in the pre-contact region, the presence of an undershoot in the density profile and an overshoot in the volume fraction profile in the post-contact region as shown in Figs. 3 and 4, respectively.

5.2 Multi-Mach Mixture Flow Test Case

To illustrate the low-Mach asymptotic study presented in Sect. 4, a 1-D converging-diverging nozzle problem is considered for a two-phase flow where the relaxation coefficients μ_P and λ_u are set to zero (independent phases). The variable area expression for the nozzle is given by $A(x) = 1 + \frac{1}{2}\cos(2\pi x/L)$ with length $L = 1$ m. At the inlet, the stagnation pressure and temperature are set to $P_0 = 1$ MPa and $T_0 = 453\ K$, respectively. At the outlet, only the static pressure is specified: $P_s = 0.5$ MPa. Initially, the two phases are at rest; their temperatures are uniform and equal to their stagnation temperatures; their pressures linearly decrease from the stagnation pressure inlet value to the static pressure outlet value. The volume fraction α_k is set to 0.5. The stiffened gas equation of state, Eq. (54), is used to model the liquid and vapor with the parameters provided in Table 1. This test case is of interest for multiple reasons: (a) a steady state is reached, (b) an analytical solution is available [6,36], (c) due to the different compressibilities between the phases, the vapor phase will become supersonic and will develop a steady shock while the liquid phase will simultaneously be low-Mach flow ($M_{\text{liquid}} \approx 10^{-2}$). We illustrate the low-Mach asymptotic study of Sect. 4 with two cases. In case (1), the nozzle problem is run using the same phasic viscosity coefficients that were used for the previous shock tube test case (Sect. 5.1), i.e., $\mu_k = \kappa_k = \beta_k = \frac{h}{2}(||u_k|| + c_k)$ (note that the dissipative term in the volume fraction is not active because the volume fraction remains constant here). Using these definitions for the viscosity coefficients in the the expressions for the Reynolds and Péclet numbers, Eq. (44), we obtain: $\text{Re}_k = \text{Pé}_k^{\kappa} = \text{Pé}_k^{\beta} = 2M_k/(1 + M_k)$. Such a scaling will efficiently stabilize the vapor flow that becomes supersonic in the nozzle ($M_k \sim 1$). On the other hand, the liquid phase will reach a low-Mach steady-state flow and the above definitions of the viscosity coefficients are ill-scaled because they lead to $\text{Re}_k = \text{Pé}_k^{\kappa} = \text{Pé}_k^{\beta} \propto M_k$ while the

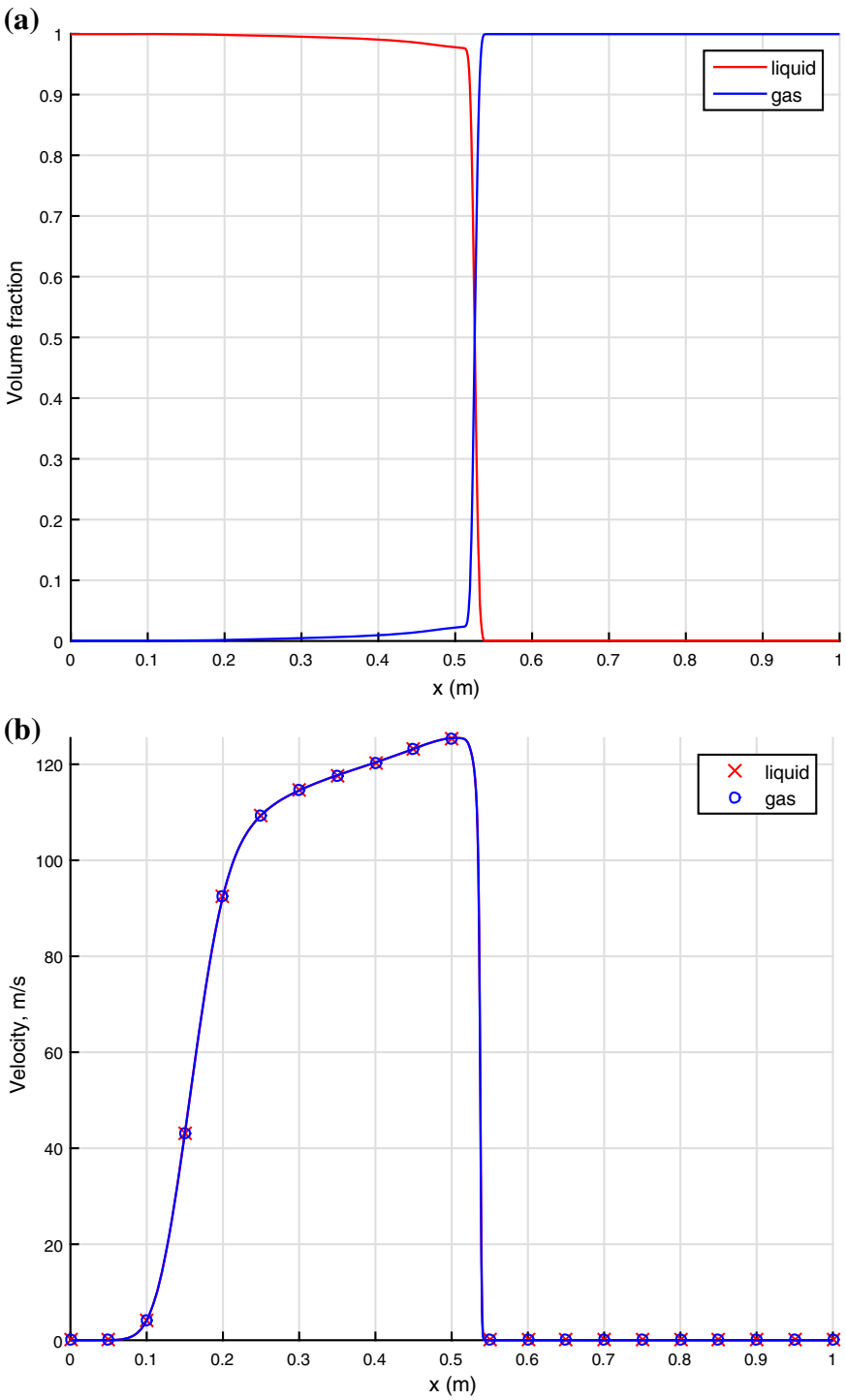

Fig. 7 Shock tube with nearly pure phases, 1000 cells, at time 2×10^{-4} s. **a** Volume fractions, **b** velocities, **c** pressures. **d** Densities

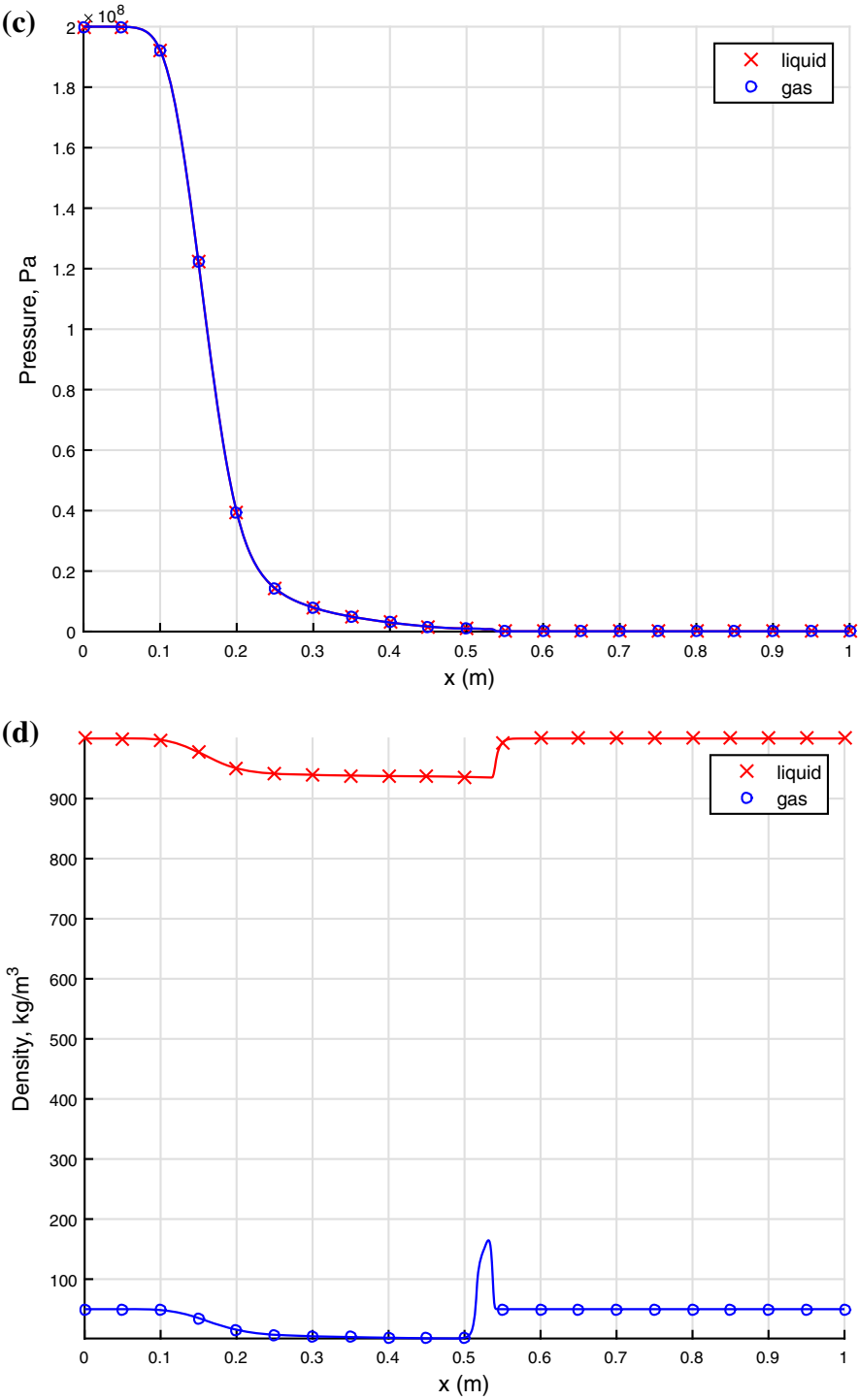

Fig. 7 continued

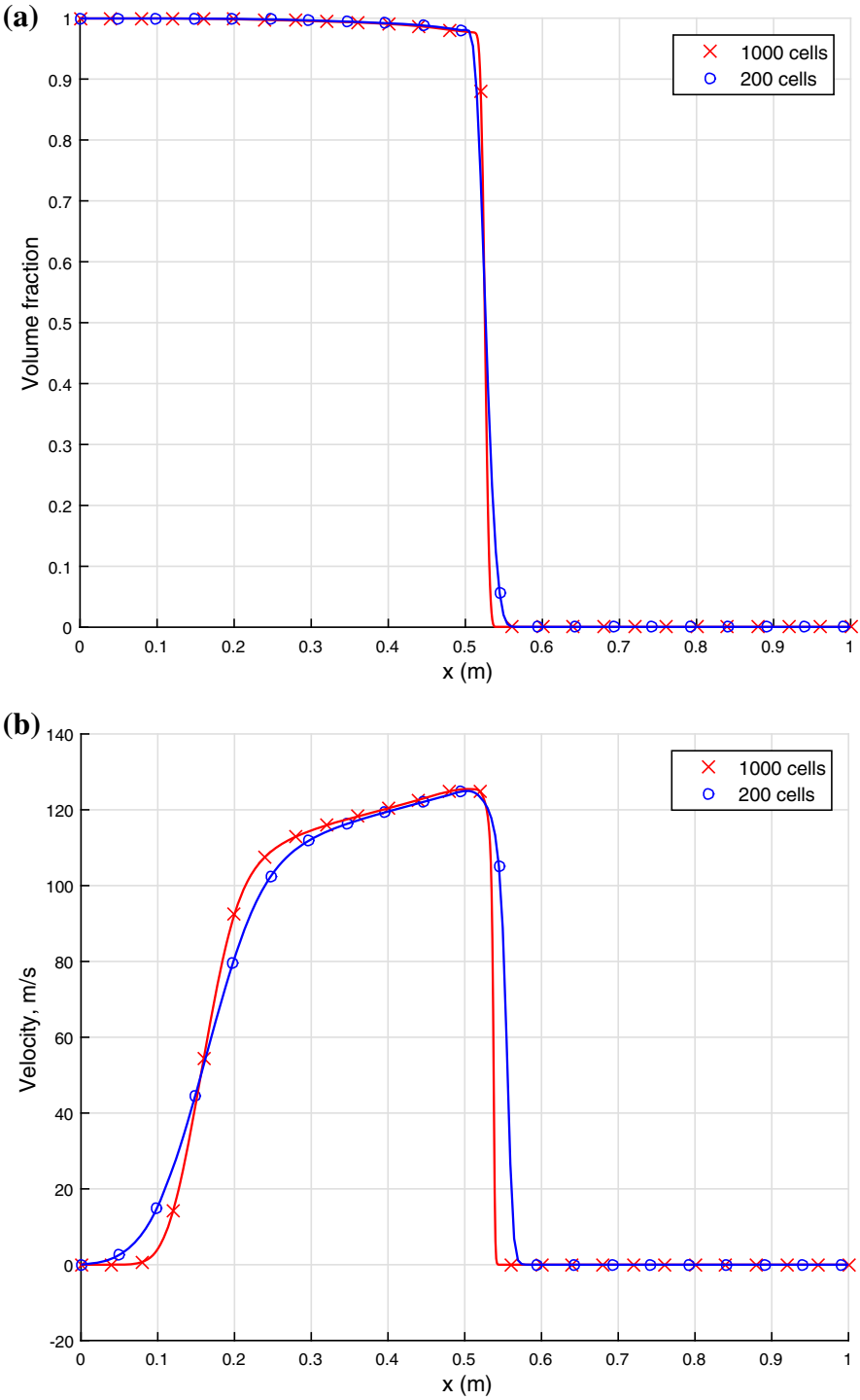

Fig. 8 Shock tube with nearly pure phases, grid convergence using 200 and 1000 cells. **a** Volume fractions, **b** velocities, **c** pressures. **d** Liquid density. **e** Vapor density

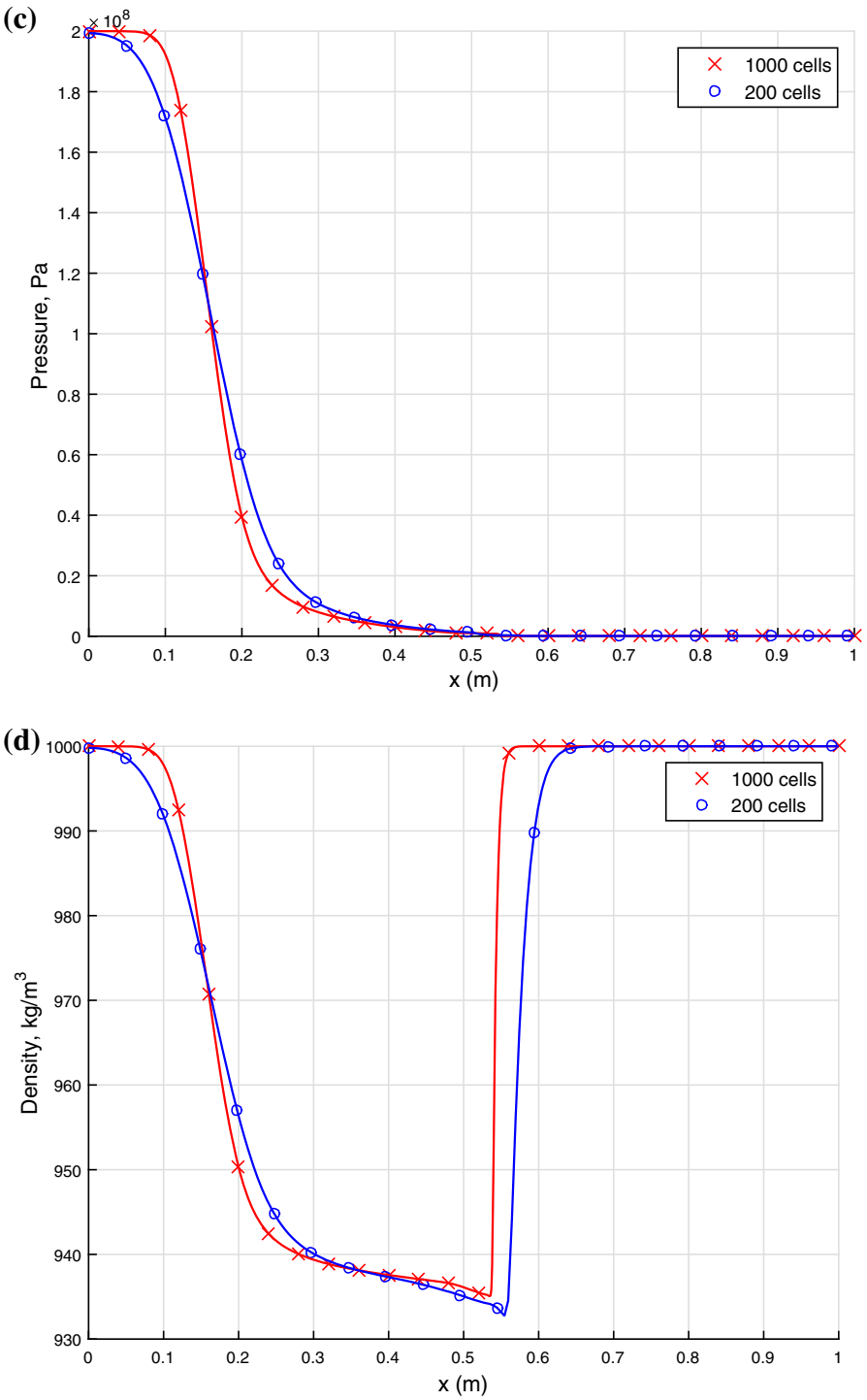

Fig. 8 continued

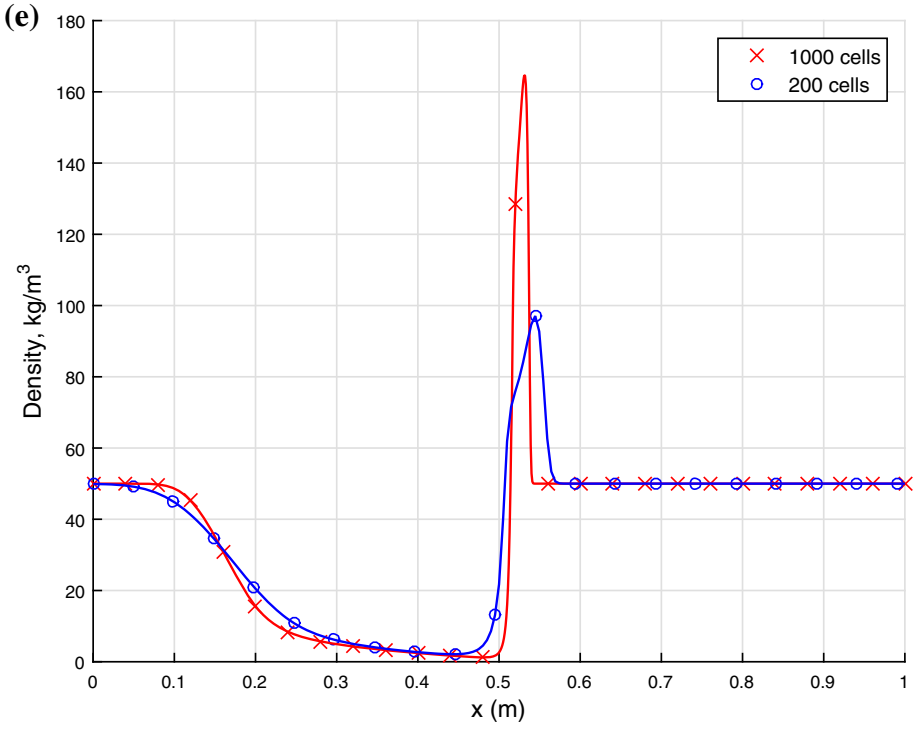

Fig. 8 continued

low-Mach asymptotic analysis of Sect. 4.2 requires that these non-dimensionalized numbers scale as 1 in the low-Mach regime (i.e., the local Lax-Friedrichs viscosities are too large in the low-Mach regime). Hence, the correct steady-state solution should not be obtained for the liquid phase using these viscosity definitions. In case (2), we propose a fix to the ill-scaling effects observed in the liquid phase by having non-dimensionalized numbers scaling as 1 for low-Mach flows. Hence, the following definitions of the phasic viscosity are used: $\mu_k = \kappa_k = \beta_k = M_k \times \frac{h}{2}(||u_k|| + c_k)$. Now, in the low-Mach regime, we have Re_k, $\mathrm{P\acute{e}}_k^{\kappa}$, and $= \mathrm{P\acute{e}}_k^{\beta}$ scaling as 1. Note that in the supersonic case, the viscous stabilization of the scheme is not altered since $M_k \simeq \mathcal{O}(1)$.

The steady-state numerical solutions for liquid and vapor phases are presented in Figs. 5 and 6, respectively, for the above cases (1) and (2).

In Fig. 5, the numerical solutions obtained for the vapor phase in cases (1) and (2) are almost identical, as expected, because the Mach number is of order 1 (see Mach plot in Fig. 5d). The numerical and exact solutions match well, except in the vicinity of the shock where smoothing occurs due to the over-dissipative nature of the local Lax-Friedrichs viscosity. As the mesh is refined, it is observed (not shown) that the numerical solution converges to the exact solution.

The steady-state numerical solution of the liquid phase is shown in Fig. 6. In case (1) (LLF viscosity without a low-Mach fix), the numerical solution does not match the exact solution; this illustrates the ill-scaling effect of the dissipative terms for low-Mach flows (see Mach plot in Fig. 6d). The numerical solution obtained in case (2) (LLF viscosity with

low-Mach fix, i.e., properly scaled LLF viscosity) converges to the exact solution and the correct low-Mach asymptotic limit is recovered.

5.3 Shock Tube with Nearly Pure Phases

Finally, we conclude with a liquid-gas shock tube where the two phases are initially in their nearly pure state ($\alpha_l = \varepsilon = 5 \times 10^{-4}$ for $x < 0.5$ and $\alpha_l = 1 - \varepsilon$ for $x > 0.5$). The initial conditions are: $P_{\text{left}} = 2 \times 10^8$ Pa, $P_{\text{right}} = 10^5$ Pa; the densities are uniform, $\rho_l = 1000$ kg/m^3, $\rho_g = 50$ kg/m^3. We use the stiffened gas equation of state with the following parameters: $\gamma_g = 1.4$, $\gamma_l = 4.4$, $P_{l,\infty} = 6 \times 10^8$ Pa, $P_{g,\infty} = 0$ Pa , $q_g = q_l = 0$ J/kg. The pressure and velocity relaxation coefficients, μ_P and λ_u are large (we use $A_{int}^{max} = 10^6$ in Eq. (5)). We employ 1000 mesh cells. The results are shown in Fig. 7 at time 200×10^{-6} s.

To show mesh convergence of the results, the same computation is now made on a mesh involving only 200 cells. These results are shown in Fig. 8, where it is clear that the method is spatially converging.

6 Conclusions

We have derived a viscous regularization for the hyperbolic seven-equation two-phase flow Model. The regularization ensures nonnegativity of the phasic entropy residual, convergence of the numerical solution to an entropy solution for concave phasic entropy functions s_k, and is consistent with the viscous regularization derived for Euler equations when one of the phases disappears. We have also demonstrated that the proposed viscous regularization is compatible with the generalized Harten entropies that were initially derived for Euler equations.

The viscous regularization for the SEM equations involves a set of two positive viscosity coefficients for each phase, μ_k and κ_k, and one for the volume fraction equation, β_k. Using non-dimensionalized equations, we have shown that the scaling of these viscosities is related to the numerical Reynolds and Péclet numbers, Re_k, Pé_k^{μ}, and Pé_k^{κ}. Adequate scaling of these numbers has been devised for two important limit-cases: the low-Mach asymptotic limit and for non-isentropic flows. In the low-Mach regime, we show that the incompressible equations are recovered when all of the non-dimensional numbers scale as one. The study of the non-isentropic case shows that the scaling of the non-dimensionalized numbers is Mach-number dependent to ensure well-scaled dissipative terms in the vicinity of the shock. Based on these results, a proper scaling of the viscosity coefficients can be derived for all Mach numbers, as was the case for single-phase flows (see [14]). We have also shown that the regularized SEM equations yields a regularized version of the five-equation flow model of Kapila by means of a Chapman–Enskog expansion.

The ability of the proposed regularization to stabilize the SEM equations was numerically demonstrated using a shock tube problem, with the definitions of the artificial viscosity coefficients borrowed from the local Lax-Friedrichs scheme. We have also performed a grid-convergence and analyzed the effect of removing stabilization from the volume fraction and the continuity equations. The scaling effect of the viscous regularization on the numerical solution in the low-Mach asymptotic limit was numerically illustrated using a 1-D converging-diverging nozzle. The proposed viscous regularization is agnostic of the spatial discretization scheme and all of our numerical examples employed a *continuous* finite element method. Shock-tube simulations with nearly pure phases have been successfully performed.

As an extension of this work, the two-phase flow viscous regularization presented here should be utilized and tested with high-order, less dissipative, artificial viscosities, such as the entropy viscosity method (e.g., see [23] for single-phase supersonic flows and [14] for single-phase subsonic and transonic flows). We also note that the proposed regularization can also be employed using definitions of viscosity coefficients traditionally used in single-phase flows, e.g., the Lapidus viscosity [16,30] and pressure-based viscosity, for two-phase flows [38]. We intend on reporting on these findings in a subsequent publication.

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Appendix 1: Entropy Equation for the Multi-dimensional Seven Equation Model Without Viscous Regularization

This appendix provides the steps that lead to the derivation of the phasic entropy equation of the seven-equation two-phase flow Model [6]. For the purpose of this appendix, two phases are considered with no interphase mass or heat transfer and denoted by the indexes j and k. In the seven-equation two-phase flow Model, each phase obeys to the following set of equations (Eq. (55)):

$$\partial_t\left(\alpha_k A\right) + A u_{int}\cdot\nabla\alpha_k = A\mu_P\left(P_k - P_j\right) \tag{55a}$$

$$\partial_t\left(\alpha_k\rho_k A\right) + \nabla\cdot\left(\alpha_k\rho_k u_k A\right) = 0 \tag{55b}$$

$$\begin{aligned}&\partial_t\left(\alpha_k\rho_k u_k A\right) + \nabla\cdot\left[\alpha_k A\left(\rho_k u_k\otimes u_k + P_k\mathbb{I}\right)\right]\\ &\quad = \alpha_k P_k\nabla A + P_{int}A\nabla\alpha_k + A\lambda_u\left(u_j - u_k\right)\end{aligned} \tag{55c}$$

$$\begin{aligned}&\partial_t\left(\alpha_k\rho_k E_k A\right) + \nabla\cdot\left[\alpha_k A u_k\left(\rho_k E_k + P_k\right)\right]\\ &\quad = P_{int}A u_{int}\cdot\nabla\alpha_k - A\mu_P\bar{P}_{int}\left(P_k - P_j\right)\\ &\qquad + \bar{u}_{int}A\lambda_u\left(u_j - u_k\right)\end{aligned} \tag{55d}$$

where ρ_k, u_k, E_k and P_k denote the density, velocity, specific total energy, and pressure of phase k, respectively. μ_P and λ_u and the pressure and velocity relaxation parameters, respectively. We recall that we assume that the cross section A is only function of space: $\partial_t A = 0$ (a value of $A \neq 1$ is mostly of practical important for 1D nozzle problems). Variables with subscript int correspond to the interfacial variables; their definitions are given in Eq. (56).

$$\begin{cases} P_{int} = \bar{P}_{int} - \frac{\nabla\alpha_k}{||\nabla\alpha_k||}\frac{Z_k Z_j}{Z_k+Z_j}\left(u_k - u_j\right)\\ \bar{P}_{int} = \frac{Z_k P_j + Z_j P_k}{Z_k+Z_j}\\ u_{int} = \bar{u}_{int} - \frac{\nabla\alpha_k}{||\nabla\alpha_k||}\frac{P_k - P_j}{Z_k+Z_j}\\ \bar{u}_{int} = \frac{Z_k u_k + Z_j u_j}{Z_k+Z_j}\end{cases} \tag{56}$$

where $Z_k = \rho_k c_k$ and $Z_j = \rho_j c_j$ are the impedances of phases k and j, respectively. The speed of sound is denoted by the symbol c.

The first step in proving the entropy minimum principle for Eq. (55) consists of recasting these equations using the primitive variables $(\alpha_k, \rho_k, u_k, e_k)$, where e_k is the specific internal energy of phase k. We introduce the material derivative $\frac{D(\cdot)}{Dt} = \partial_t(\cdot) + u_k \cdot \nabla(\cdot)$ for simplicity.

The continuity equation can be expressed as follows:

$$\alpha_k A \frac{D\rho_k}{Dt} + \rho_k A \mu_P \left(P_k - P_j\right) + \rho_k A \left(u_k - u_{int}\right) \cdot \nabla \alpha_k + \rho_k \alpha_k \nabla \cdot (A u_k) = 0. \quad (57)$$

The momentum and continuity equations are combined to yield an equation for the velocity:

$$\alpha_k \rho_k A \frac{Du_k}{Dt} + \nabla \left(\alpha_k A P_k\right) = \alpha_k P_k \nabla A + P_{int} A \nabla \alpha_k + A \lambda_u \left(u_j - u_k\right). \quad (58)$$

A kinetic energy equation is obtained by taking the vector scalar product of the previous results with u_k to yield:

$$\alpha_k \rho_k A \frac{D\left(u_k^2/2\right)}{Dt} + u_k \nabla \left(\alpha_k A P_k\right) = u_k \Big(\alpha_k P_k \nabla A + P_{int} A \nabla \alpha_k + A \lambda_u \left(u_j - u_k\right) \Big). \quad (59)$$

The internal energy equation is obtained by subtracting the above kinetic energy equation from the total energy equation:

$$\begin{aligned} \alpha_k \rho_k A \frac{De_k}{Dt} + \alpha_k P_k \nabla \cdot (A u_k) = {} & P_{int} A \left(u_{int} - u_k\right) \cdot \nabla \alpha_k - \bar{P}_{int} A \mu_P \left(P_k - P_j\right) \\ & + A \lambda_u \left(u_j - u_k\right) \cdot \left(\bar{u}_{int} - u_k\right). \end{aligned} \quad (60)$$

In the next step, we assume the existence of a phasic entropy s_k that is function of the density ρ_k and the internal energy e_k. Using the chain rule,

$$\frac{Ds_k}{Dt} = (s_\rho)_k \frac{D\rho_k}{Dt} + (s_e)_k \frac{De_k}{Dt}, \quad (61)$$

we combine the density and internal energy equations $(\rho_k (s_\rho)_k \times Eq.\,(57) + (s_e)_k \times Eq.\,(60))$ to obtain the following entropy equation:

$$\begin{aligned} & \alpha_k \rho_k A \frac{Ds_k}{Dt} + \underbrace{\alpha_k \left(P_k (s_e)_k + \rho_k^2 (s_\rho)_k\right) \nabla \cdot (A u_k)}_{\text{(a)}} \\ & = (s_e)_k A \left[P_{int} (u_{int} - u_k) \cdot \nabla \alpha_k - \bar{P}_{int} A \mu_P (P_k - P_j) + A \lambda_u (\bar{u}_{int} - u_k) \cdot (u_j - u_k) \right] \\ & \quad - \rho_k^2 (s_\rho)_k \left[\mu_P A (P_k - P_j) + A (u_k - u_{int}) \cdot \nabla \alpha_k \right] \end{aligned} \quad (62)$$

where $(s_e)_k$ and $(s_\rho)_k$ denote the partial derivatives of entropy s_k with respect to the internal energy e_k and the density ρ_k, respectively. The term denoted by (a) on the left-hand side of Eq. (62) can be set to zero by invoking the Gibbs relation from the second law of thermodynamics:

$$T_k ds_k = de_k - \frac{P_k}{\rho_k^2} d\rho_k \text{ with } (s_e)_k = \frac{1}{T_k} \text{ and } (s_\rho)_k = -\frac{P_k}{\rho_k^2} (s_e)_k \quad (63)$$

which yields

$$P_k (s_e)_k + \rho_k^2 (s_\rho)_k = 0. \quad (64)$$

Finally, Eq. (62) is as follows:

$$((s_e)_k)^{-1} \alpha_k \rho_k \frac{Ds_k}{Dt} = \underbrace{\left[P_{int} (u_{int} - u_k) + P_k (u_k - u_{int})\right] \cdot \nabla \alpha_k}_{\text{(b)}} + \underbrace{\mu_P (P_k - P_j)(P_k - \bar{P}_{int})}_{\text{(c)}}$$

$$+ \underbrace{\lambda_u(u_j - u_k)\cdot(\bar{u}_{int} - u_k)}_{\text{(d)}} \tag{65}$$

The right-hand side of Eq. (65) has been split into three terms, (b), (c), and (d); next we analyze each of these terms separately. The terms (c) and (d) can be easily recast by using the definitions of $\bar{u}_{int}$ and $\bar{P}_{int}$ given in Eq. (56):

$$\begin{aligned}
\mu_P(P_k - P_j)(P_k - \bar{P}_{int}) &= \mu_P \frac{Z_k}{Z_k + Z_j}(P_j - P_k)^2,\\
\lambda_u(u_j - u_k)\cdot(\bar{u}_{int} - u_k) &= \lambda_u \frac{Z_j}{Z_k + Z_j}(u_j - u_k)^2.
\end{aligned} \tag{66}$$

By definition, μ_P, λ_u, and Z_k are all positive. Thus, the above terms (c) and (d) are unconditionally positive.

We now inspect term (b). Once again, we use the definitions of P_{int} and u_{int} and the following relations:

$$\begin{aligned}
u_{int} - u_k &= \frac{Z_j}{Z_k + Z_j}(u_j - u_k) - \frac{\nabla\alpha_k}{\|\nabla\alpha_k\|}\frac{Pk - P_j}{Z_k + Z_j},\\
P_{int} - P_k &= \frac{Z_k}{Z_k + Z_j}(P_j - P_k) - \frac{\nabla\alpha_k}{\|\nabla\alpha_k\|}\frac{Z_k Z_j}{Z_k + Z_j}(u_k - u_j).
\end{aligned}$$

Then, term (b) becomes:

$$\begin{aligned}
&[P_{int}(u_{int} - u_k) + P_k(u_k - u_{int})]\cdot\nabla\alpha_k = (P_{int} - P_k)(u_{int} - u_k)\cdot\nabla\alpha_k\\
&= \frac{Z_k}{(Z_k + Z_j)^2}\nabla\alpha_k\cdot\left[Z_j(u_j - u_k)(P_j - P_k) + \frac{\nabla\alpha_k}{\|\nabla\alpha_k\|}Z_j^2(u_j - u_k)^2\right.\\
&\left. + \frac{\nabla\alpha_k}{\|\nabla\alpha_k\|}(P_k - P_j)^2 + \frac{\nabla\alpha_k\cdot\nabla\alpha_k}{\|\nabla\alpha_k\|^2}(P_k - P_j)Z_j(u_k - u_j)\right]
\end{aligned} \tag{67}$$

The above equation is factorized by $\|\nabla\alpha_k\|$ and then recast under a quadratic form using $\frac{\nabla\alpha_k\cdot\nabla\alpha_k}{\|\nabla\alpha_k\|^2} = 1$. This yields:

$$\begin{aligned}
&[(u_{int} - u_k)P_{int} + (u_k - u_{int})P_k]\nabla\alpha_k\\
&= \|\nabla\alpha_k\|\frac{Z_k}{(Z_k + Z_j)^2}\left[Z_j(u_j - u_k) + \frac{\nabla\alpha_k}{\|\nabla\alpha_k\|}(P_k - P_j)\right]^2
\end{aligned} \tag{68}$$

Thus, using Eq. (65), Eq. (66), Eq. (67) and Eq. (68), the entropy equation obtained in [6] holds and is recalled here for convenience:

$$\begin{aligned}
(s_e)_k^{-1}\alpha_k\rho_k A\frac{Ds_k}{Dt} &= \mu_P\frac{Z_k}{Z_k + Z_j}(P_j - P_k)^2 + \lambda_u\frac{Z_j}{Z_k + Z_j}(u_j - u_k)^2\\
&+ \|\nabla\alpha_k\|\frac{Z_k}{(Z_k + Z_j)^2}\left[Z_j(u_j - u_k) + \frac{\nabla\alpha_k}{\|\nabla\alpha_k\|}(P_k - P_j)\right]^2.
\end{aligned}$$

Appendix 2: Compatibility of the Viscous Regularization for the Seven-Equation Two-Phase Model with the Generalized Harten Entropies

We investigate in this appendix whether the viscous regularization of the seven-equation two-phase model derived in Sect. 3 is compatible with some or all generalized entropies

identified in Harten et al. [27]. Considering the single-phase Euler equations, Harten et al. [27] demonstrated that a function $\rho\mathscr{H}(s)$ is called a generalized entropy and is strictly concave if $\mathscr{H}$ is twice differentiable and

$$\mathscr{H}'(s) \geq 0, \quad \mathscr{H}'(s)c_p^{-1} - \mathscr{H}'' \geq 0, \ \forall\,(\rho, e) \in \mathbb{R}_+^2, \tag{69}$$

where $c_p\,(\rho, e) = T\,\partial_T s\,(\rho, e)$ is the specific heat at constant pressure (T is a function of e and ρ through the equation of state). Because the seven-equation two-phase model was initially derived by assuming that each phase obeys the single-phase Euler equation, we want to investigate whether the above property still holds when considering the seven-equation model with viscous regularization included. To do so, we consider a phasic generalized entropy, $\mathscr{H}_k(s_k)$ and a phasic specific heat at constant pressure, $c_{p,k}\,(\rho_k, e_k) = T_k\partial_{T_k} s_k\,(\rho_k, T_k)$ characterized by Eq. (69). The objective is to find an entropy inequality verified by $\rho_k\mathscr{H}_k(s_k)$.

We start from the entropy inequality verified by s_k,

$$\alpha_k\rho_k A\frac{Ds_k}{Dt} = f_k\cdot\nabla s_k + \nabla\cdot(\alpha_k A\rho_k\kappa_k\nabla s_k) - \alpha_k\rho_k A\kappa_k Q_k + (s_e)_k\alpha_k A\rho_k\mu_k\nabla^s u_k : \nabla u_k. \tag{70}$$

Eq. (70) is multiplied by $\mathscr{H}_k'(s_k)$ to yield:

$$\begin{aligned}\alpha_k\rho_k A\frac{D\mathscr{H}_k(s_k)}{Dt} &= \nabla\cdot(\alpha_k A\rho_k\kappa_k\nabla\mathscr{H}_k(s_k)) - \mathscr{H}_k''(s_k)\alpha_k A\kappa_k\rho_k\|\nabla s_k\|^2\\ &\quad + \mathscr{H}_k'(s_k)\,f_k\cdot\nabla s_k - \mathscr{H}_k'(s_k)\alpha_k\rho_k A\kappa_k Q_k\\ &\quad + \mathscr{H}_k'(s_k)(s_e)_k\alpha_k A\rho_k\mu_k\nabla^s u_k : \nabla u_k\end{aligned} \tag{71}$$

Let us now multiply the continuity equation of phase k by $\mathscr{H}_k(s_k)$ and add the result to the above equation to obtain:

$$\begin{aligned}&\partial_t\,(\alpha_k\rho_k A\mathscr{H}_k(s_k)) + \nabla\cdot(\alpha_k\rho_k u_k A\mathscr{H}_k(s_k))\\ &\quad - \nabla\cdot[\alpha_k A\rho_k\kappa_k\nabla\mathscr{H}_k(s_k) + \alpha_k A\kappa_k\mathscr{H}_k(s_k)\nabla\rho_k\ + A\kappa_k\rho_k\mathscr{H}_k(s_k)\nabla\alpha_k]\\ &= \underbrace{-\mathscr{H}_k''(s_k)\alpha_k A\kappa_k\rho_k\|\nabla s_k\|^2 - \mathscr{H}_k'(s_k)\alpha_k A\kappa_k\rho_k Q_k}_{\mathbb{T}_0}\\ &\quad + \underbrace{\mathscr{H}_k'(s_k)(s_e)_k\alpha_k A\rho_k\mu_k\nabla^s u_k : \nabla u_k}_{\mathbb{T}_1}.\end{aligned} \tag{72}$$

As in Sect. 3, the left-hand side of Eq. (72) is split into two residuals denoted by $\mathbb{T}_0$ and $\mathbb{T}_1$ in order to study the sign of each of them. Obviously the sign of $\mathbb{T}_1$ is positive since it is assumed that $\mathscr{H}_k'(s_k) \geq 0$. To investigate the sign of $\mathbb{T}_0$, we use Eq. (69) to get:

$$-\mathbb{T}_0 \leq \mathscr{H}_k'(s_k)\alpha_k A\kappa_k\rho_k\left(c_{p,k}^{-1}\|\nabla s_k\|^2 + Q_k\right). \tag{73}$$

The right-hand side of Eq. (73) is a quadratic form that was already defined in Appendix 5 of [24] and can be recast in the matrix form $X_k^t\mathbb{S}X_k$ where $\mathbb{S}$ is a 2×2 matrix and the vector X_k was previously defined in Sect. 3. In [24], matrix $\mathbb{S}$ is shown to be negative semi-definite which allows us to conclude that $\mathbb{T}_0$ is unconditionally positive using Eq. (73). Then, knowing the sign of the two residuals $\mathbb{T}_0$ and $\mathbb{T}_1$, we conclude that:

$$\begin{aligned}&\partial_t\,(\alpha_k\rho_k A\mathscr{H}_k(s_k)) + \nabla\cdot(\alpha_k\rho_k u_k A\mathscr{H}_k(s_k))\\ &\quad - \nabla\cdot[\alpha_k A\rho_k\kappa_k\nabla\mathscr{H}_k(s_k) + \alpha_k A\kappa_k\mathscr{H}_k(s_k)\nabla\rho_k\ + A\kappa_k\rho_k\mathscr{H}_k(s_k)\nabla\alpha_k] \geq 0.\end{aligned}$$

Subsequently, we conclude that an entropy inequality is satisfied for all generalized entropies $\rho_k \mathscr{H}_k(s_k)$ when using the viscous regularization derived in Sect. 3 for the seven-equation two-phase model. Note that the above inequality holds as well for the total entropy of the system (i.e., summation over the phasic entropy statements).

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