J Sci Comput (2015) 65:940–955
DOI 10.1007/s10915-015-9992-8

Error Estimates of a Pressure-Stabilized Characteristics Finite Element Scheme for the Oseen Equations

Hirofumi Notsu · Masahisa Tabata

Received: 20 July 2013 / Revised: 9 November 2014 / Accepted: 26 January 2015
Published online: 4 February 2015

Abstract Error estimates with the optimal convergence order are proved for a pressure-stabilized characteristics finite element scheme for the Oseen equations. The scheme is a combination of Lagrange–Galerkin finite element method and Brezzi–Pitkäranta's stabilization method. The scheme maintains the advantages of both methods; (i) It is robust for convection-dominated problems and the system of linear equations to be solved is symmetric. (ii) Since the P1 finite element is employed for both velocity and pressure, the number of degrees of freedom is much smaller than that of other typical elements for the equations, e.g., P2/P1. Therefore, the scheme is efficient especially for three-dimensional problems. The theoretical convergence order is recognized by two- and three-dimensional numerical results.

Keywords Error estimates · The finite element method · The method of characteristics · Pressure-stabilization · The Oseen equations

Mathematics Subject Classification 65M12 · 65M60 · 65M25 · 76D07

1 Introduction

The purpose of this paper is to prove the stability and convergence of a pressure-stabilized characteristics finite element scheme for the Oseen equations. The core part of this scheme consists of a characteristics (Lagrange–Galerkin) method and Brezzi–Pitkäranta's

H. Notsu (✉)
Waseda Institute for Advanced Study, Waseda University, 3-4-1, Ohkubo,
Shinjuku, Tokyo 169-8555, Japan
e-mail: h.notsu@aoni.waseda.jp

M. Tabata
Department of Mathematics, Waseda University, 3-4-1, Ohkubo,
Shinjuku, Tokyo 169-8555, Japan
e-mail: tabata@waseda.jp

stabilization method. A similar scheme has been proposed by us for the Navier–Stokes equations [21,23]. To the best of our knowledge it is the earliest work which combines the two methods, characteristics and stabilization. The stability of this scheme has its own interesting structure (Theorem 1), which is effectively used in the convergence proof not only for the Oseen equations here but also for the Navier–Stokes equations in a forthcoming paper [25].

Historically, in order to deal with convection-dominated problems, a lot of ideas have been proposed, e.g., upwind methods [1,4,8,15,18,20,33,35], characteristics(-based) methods [3,11,13,14,21–24,26–29,32] and so on. Our scheme belongs to the second group, and is the simplest one, i.e., the trajectory of fluid particle is discretized by the backward Euler method. It has common advantages of characteristics methods, robust stability with little numerical diffusion and symmetry of the resulting matrix of the system of linear equations.

For the purpose of reducing the number of degrees of freedom (DOF) we apply the stabilization method. We employ Brezzi–Pitkäranta's pressure-stabilization method [7], where the cheapest P1 element is used for both velocity and pressure. The number of DOF of this element is much smaller than that of typical *stable* finite elements, P2/P1 (Hood-Taylor) and P1-bubble/P1 (mini) [17]. As for other stabilized methods, e.g., pressure-stabilizing/Petrov–Galerkin (PSPG) and Galerkin-least-square (GLS) methods, cf. [4,15,16,35], the numbers of DOF become larger than that of P1/P1 element. Furthermore, the advantage of the symmetric matrix is lost when combined with the characteristics method, e.g., PSPG.

Thus, the scheme to be considered in this paper leads to a small-size symmetric resulting matrix, which can be solved by efficient iterative solvers for symmetric matrices, e.g., minimal residual method (MINRES), conjugate residual method (CR) and so on [2,30]. It is, therefore, efficient especially in three-dimensional computation. The scheme is proved to be essentially unconditionally stable (the required condition is only (6)) and convergent with optimal error estimates. A stabilized characteristics finite element scheme with an L^2-type pressure-stabilization is proposed in [19] for the Navier–Stokes equations and the convergence is proved under a strong stability condition on the discretization parameters. Since the stability condition is caused from the estimate of the incompressibility, it will be still required in the convergence proof for the Oseen equations if the proof is applied directly.

The paper is organized as follows. A pressure-stabilized characteristics finite element scheme for the Oseen equations is shown in Sect. 2. The main results on stability and error estimates are stated in Sect. 3 and they are proved in Sect. 4. The theoretical convergence order is recognized in two- and three-dimensional computations in Sect. 5. Finally conclusions are given in Sect. 6.

2 A Pressure-Stabilized Characteristics Finite Element Scheme

In this section we set the Oseen problems and state our pressure-stabilized characteristics finite element scheme.

Let Ω be a bounded domain in $\mathbb{R}^d (d = 2, 3)$, $\Gamma \equiv \partial\Omega$ be the boundary of Ω and T be a positive constant. Let m be a non-negative integer. We use the Sobolev spaces $W^{1,\infty}(\Omega)$, $W_0^{1,\infty}(\Omega)$, $H^m(\Omega)$ and $H_0^1(\Omega)$. For any normed space X with norm $\|\cdot\|_X$, we define function spaces $C^0([0, T]; X)$ and $H^m(0, T; X)$ consisting of X-valued functions in $C^0([0, T])$ and $H^m(0, T)$, respectively. We use the same notation $(\cdot, \cdot)$ to represent the $L^2(\Omega)$ inner product for scalar-, vector- and matrix-valued functions. $L_0^2(\Omega)$ is a function space defined by

$$L^2_0(\Omega) \equiv \left\{q \in L^2(\Omega);\ (q,1)=0\right\}.$$

We often omit $[0,T]$ and/or Ω if there is no confusion, e.g., $C^0(H^1)$ in place of $C^0([0,T];H^1(\Omega))$. The integer d is also often omitted from superscripts. For t_0 and $t_1 \in \mathbb{R}$ we introduce a function space

$$Z^m(t_0,t_1) \equiv \left\{v \in H^j\left(t_0,t_1;H^{m-j}(\Omega)^d\right);\ j=0,\ldots,m,\ \|v\|_{Z^m(t_0,t_1)} < \infty\right\},$$

where the norm $\|\cdot\|_{Z^m(t_0,t_1)}$ is defined by

$$\|v\|_{Z^m(t_0,t_1)} \equiv \left\{\sum_{j=0}^{m} \|v\|^2_{H^j(t_0,t_1;H^{m-j}(\Omega))}\right\}^{1/2}.$$

We denote $Z^m(0,T)$ by Z^m.

We consider an initial boundary value problem; find $(u,p):\ \Omega\times(0,T)\to\mathbb{R}^d\times\mathbb{R}$ such that

$$\frac{Du}{Dt} - \nabla\big(2\nu D(u)\big) + \nabla p + \lambda u = f \qquad \text{in } \Omega\times(0,T), \tag{1a}$$

$$\nabla\cdot u = 0 \qquad \text{in } \Omega\times(0,T), \tag{1b}$$

$$u = 0 \qquad \text{on } \Gamma\times(0,T), \tag{1c}$$

$$u = u^0 \qquad \text{in } \Omega,\ \text{at } t=0, \tag{1d}$$

where u is the velocity, p is the pressure, $f \in C^0([0,T];L^2(\Omega)^d)$ is a given external force, $u^0 \in H^1_0(\Omega)^d$ is a given initial velocity, $\lambda:\ \Omega\times(0,T)\to\mathbb{R}^{d\times d}$ is a given reaction rate, ν is a viscosity, $D(u)$ is the strain-rate tensor defined by

$$D_{ij}(u) \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (i,j=1,\ldots,d),$$

and D/Dt is a material derivative defined by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + w\cdot\nabla$$

for a given velocity $w:\ \Omega\times(0,T)\to\mathbb{R}^d$.

Remark 1 If w is replaced by u and $\lambda=0$, (1) becomes the Navier–Stokes problem. Here, we focus on the Oseen problem (1). The discussion on the Navier–Stokes problem will be presented in the forthcoming paper [25].

We impose assumptions for the given velocity w and reaction rate λ.

Hypothesis 1 *The function w satisfies $w \in C^0([0,T];W^{1,\infty}_0(\Omega)^d)$.*

Hypothesis 2 *The function λ satisfies $\lambda \in C^0([0,T];L^\infty(\Omega)^{d\times d})$.*

Let $V \equiv H^1_0(\Omega)^d$ and $Q \equiv L^2_0(\Omega)$ be function spaces. We define bilinear forms a on $V\times V$, b on $V\times Q$, $d(t)$ on $V\times V$ and $\mathscr{A}$ on $(V\times Q)\times(V\times Q)$ by

$$a(u,v) \equiv 2\nu\big(D(u),D(v)\big), \quad b(v,q) \equiv -(\nabla\cdot v,q), \quad d(t)(u,v) \equiv (\lambda(t)u,v),$$
$$\mathscr{A}((u,p),(v,q)) \equiv a(u,v) + b(v,p) + b(u,q),$$

respectively. Then, we can write the weak formulation of (1); find $(u,p):\ (0,T)\to V\times Q$ such that for $t\in(0,T)$

$$\left(\frac{Du}{Dt}(t), v\right) + \mathscr{A}((u, p)(t), (v, q)) + d(t)(u(t), v) = (f(t), v), \quad \forall (v, q) \in V \times Q, \quad (2)$$

with $u(0) = u^0$.

Let $X : (0, T) \to \mathbb{R}^d$ be a solution of the system of ordinary differential equations,

$$\frac{dX}{dt} = w(X, t). \quad (3)$$

Then, if u is smooth, it holds that

$$\frac{Du}{Dt}(X(t), t) = \frac{d}{dt}u\big(X(t), t\big).$$

Let Δt be a time increment, $N_T \equiv \lfloor T/\Delta t \rfloor$ be a total number of time steps, $t^n \equiv n\Delta t$ for $n = 0, \ldots, N_T$ and $X(\cdot; x, t^n)$ be the solution of (3) satisfying an initial condition $X(t^n) = x$. Then, we can consider a first order approximation of the material derivative at (x, t^n) $(n \geq 1)$ as follows.

$$\begin{aligned}\frac{Du}{Dt}(x, t^n) &= \frac{d}{dt}u\big(X(t; x, t^n), t\big)\Big|_{t=t^n} \\ &= \frac{u(X(t^n; x, t^n), t^n) - u(X(t^{n-1}; x, t^n), t^{n-1})}{\Delta t} + O(\Delta t) \\ &= \frac{u^n - u^{n-1} \circ X_1^n}{\Delta t}(x) + O(\Delta t), \end{aligned} \quad (4)$$

where $X_1^n(x)$ is a function defined by

$$X_1^n(x) \equiv x - w^n(x)\Delta t,$$

and we have used notations, $u^n \equiv u(\cdot, t^n)$ and

$$v \circ X_1^n(x) \equiv v(X_1^n(x)).$$

The point $X_1^n(x)$ is called an upwind point of x. The approximation (4) of Du/Dt is the basic idea to devise numerical schemes based on the method of characteristics. It has been combined with finite element and difference methods to lead to powerful numerical schemes for flow problems, cf. [11,22,24,27,29].

The next proposition gives a sufficient condition to guarantee all upwind points are in Ω.

Proposition 1 ([29, Proposition 1]) *Under Hypothesis 1 and the inequality*

$$\Delta t < \frac{1}{\|w\|_{C^0(W^{1,\infty}(\Omega))}}, \quad (5)$$

it holds that for any $n = 0, \ldots, N_T$

$$X_1^n(\Omega) = \Omega.$$

Here we fix Δt_0, which satisfies (5) and

$$\det\left(\frac{\partial X_1^n}{\partial x}\right) \geq \frac{1}{2}, \quad \forall n \in \{0, \ldots, N_T\}, \quad \forall \Delta t \in (0, \Delta t_0].$$

In the following we suppose that

$$\Delta t \in (0, \Delta t_0]. \quad (6)$$

For the sake of simplicity we assume Ω is a polygonal ($d = 2$) or polyhedral ($d = 3$) domain. Let $\mathscr{T}_h = \{K\}$ be a triangulation of $\bar{\Omega}$ ($= \bigcup_{K\in\mathscr{T}_h} K$), h_K be a diameter of $K \in \mathscr{T}_h$, and $h \equiv \max_{K\in\mathscr{T}_h} h_K$ be the maximum element size. We consider a regular family of subdivisions $\{\mathscr{T}_h\}_{h\downarrow 0}$ satisfying the inverse assumption [9], i.e., there exists a positive constant α_0 independent of h such that

$$\frac{h}{h_K} \le \alpha_0, \quad \forall K \in \mathscr{T}_h, \quad \forall h. \tag{7}$$

We define function spaces X_h, M_h, V_h and Q_h by

$$X_h \equiv \{v_h \in C^0(\bar{\Omega})^d;\ v_h|_K \in P_1(K)^d,\ \forall K \in \mathscr{T}_h\},$$
$$M_h \equiv \{q_h \in C^0(\bar{\Omega});\ q_h|_K \in P_1(K),\ \forall K \in \mathscr{T}_h\},$$

$V_h \equiv X_h \cap V$ and $Q_h \equiv M_h \cap Q$, respectively, where $P_1(K)$ is a polynomial space of linear functions on $K \in \mathscr{T}_h$. Let δ_0 be a positive constant and $(\cdot, \cdot)_K$ be the $L^2(K)^d$ inner product. We define bilinear forms $\mathscr{C}_h$ on $H^1(\Omega) \times H^1(\Omega)$ and $\mathscr{A}_h$ on $(V \times H^1(\Omega)) \times (V \times H^1(\Omega))$ by

$$\mathscr{C}_h(p, q) \equiv \delta_0 \sum_{K\in\mathscr{T}_h} h_K^2 (\nabla p, \nabla q)_K, \tag{8}$$
$$\mathscr{A}_h((u, p), (v, q)) \equiv a(u, v) + b(v, p) + b(u, q) - \mathscr{C}_h(p, q),$$

respectively. Let $f_h = \{f_h^n\}_{n=1}^{N_T} \subset L^2(\Omega)^d$ and $u_h^0 \in V_h$ be given. Our pressure-stabilized characteristics finite element scheme is to find $\{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that for $n = 1, \ldots, N_T$

$$\left(\frac{u_h^n - u_h^{n-1} \circ X_1^n}{\Delta t}, v_h\right) + \mathscr{A}_h((u_h^n, p_h^n), (v_h, q_h)) + d^n(u_h^{n-1}, v_h) = (f_h^n, v_h), \quad \forall (v_h, q_h) \in V_h \times Q_h, \tag{9}$$

where we have simply denoted $d(t^n)$ by d^n.

Remark 2 (i) We can replace the third term by $d^n(u_h^n, v_h)$ and prove the stability and convergence of the scheme. The scheme, however, loses such an advantage of the Galerkin characteristics method that the resulting matrix is symmetric unless λ is symmetric. That is the reason why we consider scheme (9). (ii) Scheme (9) is equivalent to

$$\left(\frac{u_h^n}{\Delta t}, v_h\right) + a(u_h^n, v_h) + b(v_h, p_h^n) = \left(f_h^n + \frac{u_h^{n-1} \circ X_1^n}{\Delta t}, v_h\right) - d^n(u_h^{n-1}, v_h), \quad \forall v_h \in V_h,$$
$$b(u_h^n, q_h) - \mathscr{C}_h(p_h^n, q_h) = 0, \quad \forall q_h \in Q_h.$$

Selecting specific bases of V_h and Q_h and expanding u_h^n and p_h^n in terms of the associated basis functions, we get an algebraic system involving a symmetric matrix of the form

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}.$$

Here, A, B and C are sub-matrices derived from $\frac{1}{\Delta t}(u_h^n, v_h) + a(u_h^n, v_h)$, $b(u_h^n, q_h)$ and $\mathscr{C}_h(p_h^n, q_h)$, respectively, and the matrix is independent of the velocity w and current time t^n.

3 Main Results

In this section we present the main results of stability and error estimates, which are proved in Sect. 4.

We use c with or without subscript to represent the generic positive constant independent of the discretization parameters h and Δt, and it can take different values at different places. For $k \in \mathbb{N}$, $c(A_1, \ldots, A_k)$ means a positive constant depending on $A_1, \ldots, A_k$, which monotonically increases as each $A_i, i \in \{1, \ldots, k\}$, increases. Constants c_0, c_1, c_2 and c_* have particular meanings in this paper,

$$\begin{aligned} c_0 &= c(\|w\|_{C^0(L^\infty)}), \quad c_1 = c(\|\lambda\|_{C^0(L^\infty)}), \\ c_2 &= c(\|w\|_{C^0(W^{1,\infty})}, \|\lambda\|_{C^0(L^\infty)}, 1/\nu, \nu), \\ c_* &= c\big(\|w\|_{C^0(W^{1,\infty})}, \|\lambda\|_{C^0(L^\infty)}, 1/\nu, \nu, T\big). \end{aligned} \tag{10}$$

Constants c', c'_0 and c'_1, having the same meaning as c, c_0 and c_1, are used when they are distinguished from c, c_0 and c_1 near by, respectively. We use norms and seminorms, $\|\cdot\|_k \equiv \|\cdot\|_{H^k(\Omega)}$ $(k = 0, 1, 2)$, $\|\cdot\|_{V_h} \equiv \|\cdot\|_V \equiv \|\cdot\|_1$, $\|\cdot\|_{Q_h} \equiv \|\cdot\|_Q \equiv \|\cdot\|_0$,

$$\|u\|_{l^\infty(X)} \equiv \max_{n=0,\ldots,N_T} \|u^n\|_X, \qquad \|u\|_{l^2(X)} \equiv \left\{\Delta t \sum_{n=1}^{N_T} \|u^n\|_X^2\right\}^{1/2},$$

$$|q|_h \equiv \left\{\sum_{K\in\mathscr{T}_h} h_K^2 (\nabla q, \nabla q)_K\right\}^{1/2}, \qquad |p|_{l^\infty(|\cdot|_h)} \equiv \max_{n=0,\ldots,N_T} |p^n|_h,$$

$$|p|_{l^2(|\cdot|_h)} \equiv \left\{\Delta t \sum_{n=1}^{N_T} |p^n|_h^2\right\}^{1/2},$$

for $X = H^k(\Omega)$ and $H^k(\Omega)^d$ $(k = 0, 1)$. $\overline{D}_{\Delta t}$ is the backward difference operator defined by

$$\overline{D}_{\Delta t} a^n \equiv \frac{a^n - a^{n-1}}{\Delta t}.$$

Firstly we show the stability result.

Theorem 1 (Stability) *Suppose that Hypotheses 1 and 2 hold. Assume condition (6). Let $f_h = \{f_h^n\}_{n=1}^{N_T} \subset L^2(\Omega)^d$ and $u_h^0 \in V_h$ be given and (u_h, p_h) be the solution of (9). Suppose that there exists $p_h^0 \in Q_h$ such that*

$$b(u_h^0, q_h) - \mathscr{C}_h(p_h^0, q_h) = 0, \quad \forall q_h \in Q_h. \tag{11}$$

Then, there exists a constant c_* of (10) such that

$$\|u_h\|_{l^\infty(H^1)}, \ \ \|\overline{D}_{\Delta t} u_h\|_{l^2(L^2)}, \ \ |p_h|_{l^\infty(|\cdot|_h)}, \ \ \|p_h\|_{l^2(L^2)} \le c_*(\|u_h^0\|_1 + |p_h^0|_h + \|f_h\|_{l^2(L^2)}). \tag{12}$$

Remark 3 The relation (11) is satisfied if $(u_h^0, p_h^0) \in V_h \times Q_h$ is chosen as a Stokes projection of $(u^0, 0)$ (cf. Definition 1 below).

Secondary we give error estimates after preparing a (pressure-stabilized) Stokes projection using P1/P1-element and a hypothesis.

Definition 1 (*Stokes projection*) For $(u, p) \in V \times Q$ we define the Stokes projection $(\hat{u}_h, \hat{p}_h) \in V_h \times Q_h$ of (u, p) by

$$\mathscr{A}_h((\hat{u}_h, \hat{p}_h), (v_h, q_h)) = \mathscr{A}((u, p), (v_h, q_h)), \quad \forall (v_h, q_h) \in V_h \times Q_h. \tag{13}$$

Hypothesis 3 *The solution* (u, p) *of* (2) *satisfies* $u \in H^1(0, T; V \cap H^2(\Omega)^d) \cap Z^2$ *and* $p \in H^1(0, T; Q \cap H^1(\Omega))$.

Theorem 2 (Error estimate) *Suppose Hypotheses 1–3 hold. Assume condition* (6). *Suppose* $f_h = f$ *and that* u_h^0 *is the first component of the Stokes projection of* $(u^0, 0)$ *by* (13). *Let* (u_h, p_h) *be the solution of* (9). *Then, there exists a constant* c_* *of* (10) *such that*

$$\begin{aligned} &\|u_h - u\|_{l^\infty(H^1)}, \quad \left\| \overline{D}_{\Delta t} u_h - \frac{\partial u}{\partial t} \right\|_{l^2(L^2)}, \quad \|p_h - p\|_{l^2(L^2)} \\ &\qquad \le c_* (\Delta t \|u\|_{Z^2} + h \|(u, p)\|_{H^1(0,T;H^2 \times H^1)}). \end{aligned} \tag{14}$$

Remark 4 In the case of the inhomogeneous Dirichlet boundary condition $u = u_b$ in place of (1c), similar stability and convergence results are obtained if there exists a function $\tilde{u}_b \in H^1(H^2(\Omega)^d) \cap Z^2$ such that $\tilde{u}_b|_\Gamma = u_b$.

4 Proofs of Theorems 1 and 2

This section is devoted to the proofs of Theorems 1 and 2.

4.1 Preliminaries

We prepare six lemmas and a proposition to be used in the proofs. We omit the proofs of the first five lemmas only by referring to the related bibliography.

Lemma 1 (Discrete Gronwall's inequality, [34, Lemma 4.6]) *Let* a_1 *be a non-negative number,* Δt *be a positive number, and* $\{x^n\}_{n\ge 0}$, $\{y^n\}_{n\ge 1}$ *and* $\{b^n\}_{n\ge 1}$ *be non-negative sequences. Suppose*

$$\overline{D}_{\Delta t} x^n + y^n \le a_1 x^{n-1} + b^n, \quad \forall n \ge 1.$$

Then, it holds that

$$x^n + \Delta t \sum_{i=1}^n y^i \le \exp(a_1 n \Delta t) \Big(x^0 + \Delta t \sum_{i=1}^n b^i \Big), \quad \forall n \ge 1.$$

Lemma 2 (Korn's inequality, [12]) *Let* Ω *be a bounded domain with a Lipschitz-continuous boundary. Then, we have the following.*

(i) There exists a positive constant α_1 *such that*

$$(\|D(v)\|_0^2 + \|v\|_0^2)^{1/2} \ge \alpha_1 \|v\|_1, \quad \forall v \in H^1(\Omega)^d.$$

(ii) The norms $\|D(\cdot)\|_0$ *and* $\|\cdot\|_1$ *are equivalent in* $H_0^1(\Omega)^d$.

Lemma 3 ([5,9,10]) *There exist linear operators* $\Pi_h : H^2(\Omega) \to X_h$ *and* $\Pi_h^C : L_0^2(\Omega) \to Q_h$*, which satisfy*

$$\|\Pi_h v - v\|_1 \le \alpha_2 h \|v\|_2, \quad \forall v \in H^2(\Omega)^d, \tag{15a}$$

$$\|\Pi_h^C q - q\|_0 \le \alpha_3 h \|q\|_1, \quad \forall q \in L_0^2(\Omega) \cap H^1(\Omega), \tag{15b}$$

where α_2 *and* α_3 *are positive constants independent of* h.

Remark 5 Π_h is nothing but the Lagrange interpolation operator and Π_h^C is the Clément interpolation operator.

Lemma 4 ([16, eq. (3.6)]) *Let $\{\mathscr{T}_h\}_{h\downarrow 0}$ be a regular family of triangulations of $\bar{\Omega}$. Then, there exists a positive constant α_4 independent of h such that*

$$|q_h|_h \le \alpha_4 \|q_h\|_0, \quad \forall q_h \in Q_h. \tag{16}$$

Lemma 5 ([16, Lemma 3.2]) *There exists a positive constant $\alpha_5 = c(\nu)$ independent of h such that for any h*

$$\inf_{(u_h,p_h)\in V_h\times Q_h} \sup_{(v_h,q_h)\in V_h\times Q_h} \frac{\mathscr{A}_h((u_h, p_h), (v_h, q_h))}{\|(u_h, p_h)\|_{V\times Q}\|(v_h, q_h)\|_{V\times Q}} \ge \alpha_5. \tag{17}$$

Remark 6 The conventional inf-sup condition [17] requires a positive constant β^* independent of h such that

$$\inf_{q_h\in Q_h} \sup_{v_h\in V_h} \frac{b(v_h, q_h)}{\|v_h\|_1\|q_h\|_0} \ge \beta^*.$$

Although the condition does not hold true for the pair V_h and Q_h of the P1/P1 finite element spaces, $\mathscr{A}_h$ satisfies the stability inequality (17) for this pair.

The Stokes projection (13) has the following property, which is essentially proved in [6].

Proposition 2 *Suppose $(u, p) \in (V \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$. Then, there exists a positive constant $\alpha_6 = c(1/\nu, \nu)$ independent of h such that for any h the Stokes projection $(\hat{u}_h, \hat{p}_h)$ of (u, p) by (13) satisfies*

$$\|\hat{u}_h - u\|_1, \ \|\hat{p}_h - p\|_0, \ |\hat{p}_h - p|_h \le \alpha_6 h \|(u, p)\|_{H^2\times H^1}. \tag{18}$$

Proof Let Π_h and Π_h^{C} be the operators in Lemma 3. From (13) and Lemmas 3, 4 and 5 it holds that

$$\begin{aligned}
&\|(\hat{u}_h - \Pi_h u, \hat{p}_h - \Pi_h^{\mathrm{C}} p)\|_{V\times Q} \\
&\quad \le \frac{1}{\alpha_5} \sup_{(v_h,q_h)\in V_h\times Q_h} \frac{\mathscr{A}_h((\hat{u}_h - \Pi_h u, \hat{p}_h - \Pi_h^{\mathrm{C}} p), (v_h, q_h))}{\|(v_h, q_h)\|_{V\times Q}} \\
&\quad = \frac{1}{\alpha_5} \sup_{(v_h,q_h)\in V_h\times Q_h} \frac{\mathscr{A}_h((u - \Pi_h u, p - \Pi_h^{\mathrm{C}} p), (v_h, q_h)) + \mathscr{C}_h(p, q_h)}{\|(v_h, q_h)\|_{V\times Q}} \\
&\quad \le c(1/\nu, \nu)\big(\|(u - \Pi_h u, p - \Pi_h^{\mathrm{C}} p)\|_{V\times Q} + |p|_h\big) \\
&\quad \le c'(1/\nu, \nu) h \|(u, p)\|_{H^2\times H^1},
\end{aligned}$$

which implies (18). □

Lemma 6 *Assume Hypothesis 1 and condition (6). Then, for any $n \in \{0, \dots, N_T\}$ it holds that*

$$\|v - v \circ X_1^n\|_0 \le c_0 \Delta t \|v\|_1, \quad \forall v \in V. \tag{19}$$

Proof Let any $n \in \{0, \dots, N_T\}$ be fixed. Let $y(x, s) \equiv x - s w^n(x)\Delta t$ for $s \in [0, 1]$ and $J_s(x) \equiv \det(\partial y/\partial x)$ be the Jacobian. It holds that

$$J_s(x) \ge 1/2, \quad \forall s \in [0, 1],$$

$$v(x) - v \circ X_1^n(x) = \big[v\big(y(x, s)\big)\big]_{s=1}^{0} = \Delta t \int_0^1 \big[\{w^n(x) \cdot \nabla\} v\big]\big(y(x, s)\big) ds.$$

Changing the variable from x to y and using the above evaluation of the Jacobian, we have

$$\|v - v \circ X_1^n\|_0 \le c_0 \Delta t \int_0^1 \big\|\nabla v\big(y(\cdot, s)\big)\big\|_0 ds \le c_0\sqrt{2}\Delta t \int_0^1 \|\nabla v\|_0 ds \le c_0' \Delta t \|v\|_1,$$

which completes the proof. □

4.2 Proof of Theorem 1

From (9) with $v_h = 0 \in V_h$ and (11), it holds that for $n = 0, \ldots, N_T$

$$b\left(u_h^n, q_h\right) - \mathscr{C}_h\left(p_h^n, q_h\right) = 0, \quad \forall q_h \in Q_h,$$

which gives for $n = 1, \ldots, N_T$

$$b\left(\overline{D}_{\Delta t} u_h^n, q_h\right) - \mathscr{C}_h\left(\overline{D}_{\Delta t} p_h^n, q_h\right) = 0, \quad \forall q_h \in Q_h. \tag{20}$$

Substituting $(\overline{D}_{\Delta t} u_h^n, 0) \in V_h \times Q_h$ into (v_h, q_h) in (9) and using (20) with $q_h = -p_h^n$, we have for $n = 1, \ldots, N_T$

$$\Big(\frac{u_h^n - u_h^{n-1} \circ X_1^n}{\Delta t}, \overline{D}_{\Delta t} u_h^n\Big) + a\left(u_h^n, \overline{D}_{\Delta t} u_h^n\right) + \mathscr{C}_h\left(\overline{D}_{\Delta t} p_h^n, p_h^n\right) + d^n\left(u_h^{n-1}, \overline{D}_{\Delta t} u_h^n\right) = \left(f_h^n, \overline{D}_{\Delta t} u_h^n\right). \tag{21}$$

We evaluate each term in (21) as follows.

$$\begin{aligned}
\Big(\frac{u_h^n - u_h^{n-1} \circ X_1^n}{\Delta t}, \overline{D}_{\Delta t} u_h^n\Big) &= \Big(\overline{D}_{\Delta t} u_h^n + \frac{1}{\Delta t}(u_h^{n-1} - u_h^{n-1} \circ X_1^n), \overline{D}_{\Delta t} u_h^n\Big) \\
&= \|\overline{D}_{\Delta t} u_h^n\|_0^2 + \frac{1}{\Delta t}\big(u_h^{n-1} - u_h^{n-1} \circ X_1^n, \overline{D}_{\Delta t} u_h^n\big) \\
&\ge \|\overline{D}_{\Delta t} u_h^n\|_0^2 - \frac{1}{\Delta t}\|u_h^{n-1} - u_h^{n-1} \circ X_1^n\|_0 \|\overline{D}_{\Delta t} u_h^n\|_0 \\
&\ge \|\overline{D}_{\Delta t} u_h^n\|_0^2 - c_0 \|u_h^{n-1}\|_1 \|\overline{D}_{\Delta t} u_h^n\|_0 \quad \text{(by (19))} \\
&\ge \|\overline{D}_{\Delta t} u_h^n\|_0^2 - \Big(c_0^2 \|u_h^{n-1}\|_1^2 + \frac{1}{4}\|\overline{D}_{\Delta t} u_h^n\|_0^2\Big) \\
&\ge \frac{3}{4}\|\overline{D}_{\Delta t} u_h^n\|_0^2 - c_0' \|D\big(u_h^{n-1}\big)\|_0^2, \qquad \text{(22a)} \\
a(u_h^n, \overline{D}_{\Delta t} u_h^n) &= \overline{D}_{\Delta t}\Big(\frac{1}{2} a(u_h^n, u_h^n)\Big) + \frac{\Delta t}{2} a(\overline{D}_{\Delta t} u_h^n, \overline{D}_{\Delta t} u_h^n) \\
&= \overline{D}_{\Delta t}\big(\nu \|D(u_h^n)\|_0^2\big) + \nu \Delta t \|D\big(\overline{D}_{\Delta t} u_h^n\big)\|_0^2 \\
&\ge \overline{D}_{\Delta t}\big(\nu \|D(u_h^n)\|_0^2\big), \qquad \text{(22b)} \\
\mathscr{C}_h\left(\overline{D}_{\Delta t} p_h^n, p_h^n\right) &= \overline{D}_{\Delta t}\Big(\frac{1}{2}\mathscr{C}_h(p_h^n, p_h^n)\Big) + \frac{\Delta t}{2}\mathscr{C}_h\left(\overline{D}_{\Delta t} p_h^n, \overline{D}_{\Delta t} p_h^n\right) \\
&= \overline{D}_{\Delta t}\Big(\frac{\delta_0}{2}|p_h^n|_h^2\Big) + \frac{\delta_0 \Delta t}{2}|\overline{D}_{\Delta t} p_h^n|_h^2 \\
&\ge \overline{D}_{\Delta t}\Big(\frac{\delta_0}{2}|p_h^n|_h^2\Big), \qquad \text{(22c)} \\
-d^n(u_h^{n-1}, \overline{D}_{\Delta t} u_h^n) &= -(\lambda^n u_h^{n-1}, \overline{D}_{\Delta t} u_h^n) \\
&\le c_1 \|u_h^{n-1}\|_0^2 + \frac{1}{8}\|\overline{D}_{\Delta t} u_h^n\|_0^2
\end{aligned}$$

$$\leq c_1' \| D(u_h^{n-1}) \|_0^2 + \frac{1}{8} \| \overline{D}_{\Delta t} u_h^n \|_0^2, \tag{22d}$$

$$\big(f_h^n, \overline{D}_{\Delta t} u_h^n \big) \leq 2 \| f_h^n \|_0^2 + \frac{1}{8} \| \overline{D}_{\Delta t} u_h^n \|_0^2, \tag{22e}$$

where Lemma 2 has been used for (22a) and (22d). Combining (22) with (21), we have for $n = 1, \ldots, N_T$

$$\overline{D}_{\Delta t} \Big(\nu \| D(u_h^n) \|_0^2 + \frac{\delta_0}{2} |p_h^n|_h^2 \Big) + \frac{1}{2} \| \overline{D}_{\Delta t} u_h^n \|_0^2 \leq (c_0' + c_1') \| D(u_h^{n-1}) \|_0^2 + 2 \| f_h^n \|_0^2. \tag{23}$$

Hence, the first three inequalities of (12) are obtained by applying Lemma 1 to (23).

Now we prove the last inequality of (12). From Lemmas 4, 5 and 6 it holds that

$$\begin{aligned}
\| p_h^n \|_0 &\leq \| (u_h^n, p_h^n) \|_{V \times Q} \leq \frac{1}{\alpha_5} \sup_{(v_h, q_h) \in V_h \times Q_h} \frac{\mathscr{A}_h((u_h^n, p_h^n), (v_h, q_h))}{\| (v_h, q_h) \|_{V \times Q}} \\
&= \frac{1}{\alpha_5} \sup_{(v_h, q_h) \in V_h \times Q_h} \frac{(f_h^n, v_h) - \frac{1}{\Delta t}(u_h^n - u_h^{n-1} \circ X_1^n, v_h) - d^n(u_h^{n-1}, v_h)}{\| (v_h, q_h) \|_{V \times Q}} \\
&\leq \frac{1}{\alpha_5} \Big\{ \| f_h^n \|_0 + \| \overline{D}_{\Delta t} u_h^n \|_0 + \frac{1}{\Delta t} \| u_h^{n-1} - u_h^{n-1} \circ X_1^n \|_0 + \| \lambda^n u_h^{n-1} \|_0 \Big\} \\
&\leq \frac{1}{\alpha_5} \big\{ \| f_h^n \|_0 + \| \overline{D}_{\Delta t} u_h^n \|_0 + (c_0 + c_1) \| u_h^{n-1} \|_1 \big\},
\end{aligned}$$

which yields the last inequality of (12) by the first and second inequalities of (12).

4.3 Proof of Theorem 2

Let $(\hat{u}_h, \hat{p}_h)(t) \in V_h \times Q_h$ be the Stokes projection of $(u, p)(t) \in H^2(\Omega)^d \times H^1(\Omega)$ by (13) and set

$$e_h^n \equiv u_h^n - \hat{u}_h^n, \quad \epsilon_h^n \equiv p_h^n - \hat{p}_h^n, \quad \eta(t) \equiv (u - \hat{u}_h)(t).$$

From (2), (9), (13) and identity

$$e_h^n = \eta^n - u^n + u_h^n,$$

it holds that for any $(v_h, q_h) \in V_h \times Q_h$

$$\begin{aligned}
&\Big(\frac{e_h^n - e_h^{n-1} \circ X_1^n}{\Delta t}, v_h \Big) + a(e_h^n, v_h) + b(v_h, \epsilon_h^n) + b(e_h^n, q_h) - \mathscr{C}_h(\epsilon_h^n, q_h) + d^n(e_h^{n-1}, v_h) \\
&= \Big(\frac{\eta^n - \eta^{n-1} \circ X_1^n}{\Delta t} - \frac{u^n - u^{n-1} \circ X_1^n}{\Delta t}, v_h \Big) + \Big(\frac{Du^n}{Dt}, v_h \Big) + d^n(\eta^{n-1} + u^n - u^{n-1}, v_h) \\
&= \Big(\frac{\eta^n - \eta^{n-1} \circ X_1^n}{\Delta t} + \Big(\frac{Du^n}{Dt} - \frac{u^n - u^{n-1} \circ X_1^n}{\Delta t} \Big) + \lambda^n (\eta^{n-1} + u^n - u^{n-1}), v_h \Big) \\
&= (\tilde{f}_h^n, v_h),
\end{aligned} \tag{24}$$

where

$$\tilde{f}_h^n \equiv \frac{\eta^n - \eta^{n-1} \circ X_1^n}{\Delta t} + \Big(\frac{Du^n}{Dt} - \frac{u^n - u^{n-1} \circ X_1^n}{\Delta t} \Big) + \lambda^n (\eta^{n-1} + u^n - u^{n-1}).$$

We evaluate $\|\tilde{f}_h\|_{l^2(L^2)}$. It holds that

$$\|\tilde{f}_h^n\|_0 \le \left\|\frac{\eta^n - \eta^{n-1}\circ X_1^n}{\Delta t}\right\|_0 + \left\|\frac{Du^n}{Dt} - \frac{u^n - u^{n-1}\circ X_1^n}{\Delta t}\right\|_0 + \|\lambda^n(\eta^{n-1} + u^n - u^{n-1})\|_0$$
$$\equiv I_1^n + I_2^n + I_3^n.$$

Let $y(x,s) \equiv x - (1-s)w^n(x)\Delta t$ and $t(s) \equiv t^{n-1} + s\Delta t$ $(s \in [0,1])$. From Proposition 2 the terms I_i^n $(i = 1, 2, 3)$ are evaluated as

$$\begin{aligned}
I_1^n &\le \|\overline{D}_{\Delta t}\eta^n\|_0 + \frac{1}{\Delta t}\|\eta^{n-1} - \eta^{n-1}\circ X_1^n\|_0\\
&\le \frac{1}{\sqrt{\Delta t}}\|\eta\|_{H^1(t^{n-1},t^n;L^2)} + c_0\|\eta^{n-1}\|_1 \quad \text{(by (19))}\\
&\le \alpha_6 h\Big(\frac{1}{\sqrt{\Delta t}}\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)} + c_0\|(u,p)^{n-1}\|_{H^2\times H^1}\Big),\\
I_2^n &= \left\|\Big\{\Big(\frac{\partial}{\partial t} + w^n(\cdot)\cdot\nabla\Big)u\Big\}(\cdot,t^n) - \int_0^1 \Big\{\Big(\frac{\partial}{\partial t} + w^n(\cdot)\cdot\nabla\Big)u\Big\}\big(y(\cdot,s),t(s)\big)ds\right\|_0\\
&\le \Delta t\int_0^1 ds\int_s^1 \left\|\Big\{\Big(\frac{\partial}{\partial t} + w^n(\cdot)\cdot\nabla\Big)^2 u\Big\}\big(y(\cdot,s_1),t(s_1)\big)\right\|_0 ds_1\\
&\le c_0\sqrt{\Delta t}\|u\|_{Z^2(t^{n-1},t^n)} \quad (\text{by } \det(\partial y(\cdot,s)/\partial x) \ge 1/2,\ \forall s \in [0,1]),\\
I_3^n &\le c_1(\|\eta^{n-1}\|_0 + \|u^n - u^{n-1}\|_0)\\
&\le c_1\big(\alpha_6 h\|(u,p)^{n-1}\|_{H^2\times H^1} + \sqrt{\Delta t}\|u\|_{H^1(t^{n-1},t^n;L^2)}\big),
\end{aligned}$$

which imply

$$\|\tilde{f}_h\|_{l^2(L^2)} \le c_2(\Delta t\|u\|_{Z^2} + h\|(u,p)\|_{H^1(H^2\times H^1)}). \tag{25}$$

From the definitions of (u_h^0, p_h^0) and $(\hat{u}_h^0, \hat{p}_h^0)$ it holds that for any $q_h \in Q_h$

$$\begin{aligned}
b(e_h^0, q_h) - \mathscr{C}_h(\epsilon_h^0, q_h) &= b(u_h^0, q_h) - \mathscr{C}_h(p_h^0, q_h) - \big\{b(\hat{u}_h^0, q_h) - \mathscr{C}_h(\hat{p}_h^0, q_h)\big\}\\
&= b(u^0, q_h) - b(u^0, q_h) = 0.
\end{aligned}$$

Applying Theorem 1 to (24), we obtain

$$\begin{aligned}
&\|e_h\|_{l^\infty(H^1)},\quad \|\overline{D}_{\Delta t}e_h\|_{l^2(L^2)},\quad \|\epsilon_h\|_{l^2(L^2)}\\
&\quad\le c_*\big(\|e_h^0\|_1 + |\epsilon_h^0|_h + \Delta t\|u\|_{Z^2} + h\|(u,p)\|_{H^1(H^2\times H^1)}\big)
\end{aligned} \tag{26}$$

from (25). Since (u_h^0, p_h^0) and $(\hat{u}_h^0, \hat{p}_h^0)$ are the Stokes projections of $(u^0, 0)$ and (u^0, p^0) by (13), respectively, it holds that

$$\|e_h^0\|_1 = \|u_h^0 - \hat{u}_h^0\|_1 \le \|u_h^0 - u^0\|_1 + \|u^0 - \hat{u}_h^0\|_1 \le c\alpha_6 h\|(u^0, p^0)\|_{H^2\times H^1}, \tag{27a}$$

$$|\epsilon_h^0|_h = |p_h^0 - \hat{p}_h^0|_h \le |p_h^0 - 0|_h + |\hat{p}_h^0 - p^0|_h + |p^0|_h \le c\alpha_6 h\|(u^0, p^0)\|_{H^2\times H^1}. \tag{27b}$$

Combining (27) with (26), we obtain (14). □

5 Numerical Results

In this section two- and three-dimensional test problems are computed by scheme (9). Theorems 1 and 2 are valid for any positive number δ_0 in (8). Here we concentrate on recognizing the theoretical convergence order numerically and simply fix $\delta_0 = 1$.

Quadrature formulae of degree five for $d = 2$ (seven points) and 3 (fifteen points) [31] are employed for computation of the integral

$$\int_K u_h^{n-1} \circ X_1^n(x) v_h(x)\, dx$$

appearing in scheme (9).

In the following examples, N is the division number of each side of the domain (square or cube) and we (re)define $h \equiv 1/N$. The time increment Δt is set to be $\Delta t = h$ in order to keep the same orders of both Δt and h, since the error estimates in Theorem 2 are of $O(\Delta t + h)$.

In scheme (9) the initial function u_h^0 is chosen as the first component of the Stokes projection of $(u^0, 0)$ by (13). Let (u_h, p_h) be the solution of scheme (9) and (u, p) be each analytical solution set in the examples. We define Err by

$$Err \equiv \frac{\|u_h - \Pi_h u\|_{l^2(H^1)} + \|p_h - \Pi_h p\|_{l^2(L^2)}}{\|\Pi_h u\|_{l^2(H^1)} + \|\Pi_h p\|_{l^2(L^2)}} \tag{28}$$

as the relative error between (u_h, p_h) and (u, p), where the same symbol Π_h has been used as its scalar version, i.e., $\Pi_h : H^2(\Omega) \to M_h$. The system of linear equations in the coefficients of u_h^n and p_h^n of scheme (9) is solved by MINRES.

Example 1 (2D) Let $d = 2$. In problem (1) we set $\Omega = (0, 1)^2$, $T = 1$, four values of ν,

$$\nu = 10^{-k}, \quad k = 1, \ldots, 4,$$

and functions w and λ,

$$w(x,t) = \Big(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\Big)(x,t), \quad \lambda(x,t) = \begin{pmatrix} \sin\pi(x_1+t) & \sin\pi(x_2+t) \\ \cos\pi(x_2+t) & \sin\pi(x_1+x_2+t) \end{pmatrix},$$

where ψ is a function defined by

$$\psi(x,t) = \frac{\sqrt{3}}{2\pi} \sin^2(\pi x_1) \sin^2(\pi x_2) \sin\{\pi(x_1+x_2+t)\}.$$

The functions f and u^0 are given so that the exact solution is

$$(u, p)(x,t) = \big(w(x,t), \sin\{\pi(x_1 + 2x_2 + t)\}\big).$$

The solution is normalized so that $\|u\|_{C^0(L^\infty)} = \|p\|_{C^0(L^\infty)} = 1$.

We set $N = 16, 32, 64, 128$ and 256. The left of Fig. 1 shows a sample mesh ($N = 16$). The right of Fig. 1 exhibits the graphs of Err versus h $(= \Delta t)$ in logarithmic scale. We can see that Err is almost of first order in h and Δt for all ν.

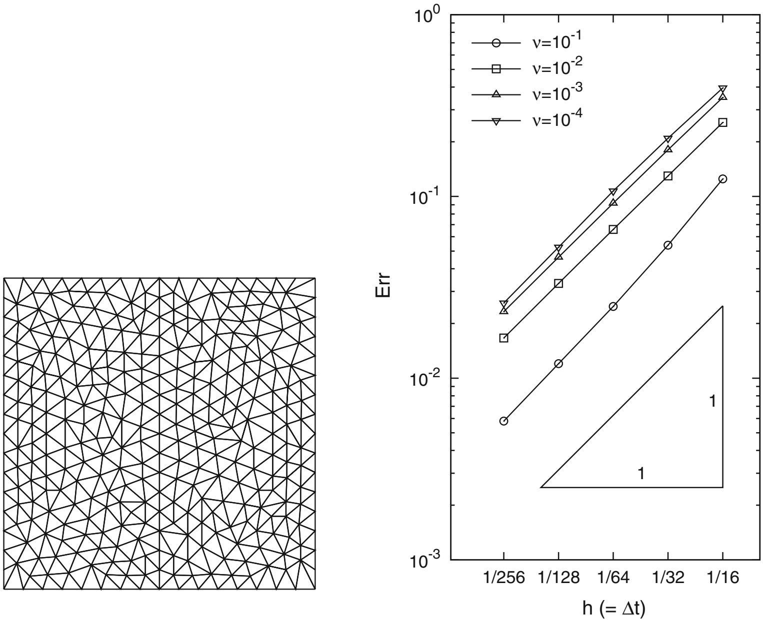

Fig. 1 A sample mesh (*left*, $N = 16$) and graphs of Err versus h (*right*) for Example 1

Example 2 (3D) Let $d = 3$. In problem (1) we set $\Omega = (0, 1)^3$ and functions w and λ,

$$w(x, t) = \mathrm{rot}\, \Psi(x, t),$$
$$\lambda(x, t) = \begin{pmatrix} \sin \pi(x_1 + t) & \sin \pi(x_2 + t) & \sin \pi(x_3 + t) \\ \cos \pi(x_2 + t) & \sin \pi(x_1 + x_2 + t) & \sin \pi(x_2 + x_3 + t) \\ \cos \pi(x_3 + t) & \cos \pi(x_2 + x_3 + t) & \sin \pi(x_1 + x_2 + x_3 + t) \end{pmatrix},$$

where Ψ is a function defined by

$$\Psi_1(x, t) \equiv \frac{8\sqrt{3}}{27\pi} \sin(\pi x_1) \sin^2(\pi x_2) \sin^2(\pi x_3) \sin\{\pi(x_2 + x_3 + t)\},$$
$$\Psi_2(x, t) \equiv \frac{8\sqrt{3}}{27\pi} \sin^2(\pi x_1) \sin(\pi x_2) \sin^2(\pi x_3) \sin\{\pi(x_3 + x_1 + t)\},$$
$$\Psi_3(x, t) \equiv \frac{8\sqrt{3}}{27\pi} \sin^2(\pi x_1) \sin^2(\pi x_2) \sin(\pi x_3) \sin\{\pi(x_1 + x_2 + t)\}.$$

Constants T and ν are the same as those in Example 1. The functions f and u^0 are given so that the exact solution is

$$(u, p)(x, t) = \big(w(x, t), \sin\{\pi(x_1 + 2x_2 + x_3 + t)\}\big).$$

The solution is normalized so that $\|u\|_{C^0(L^\infty)} = \|p\|_{C^0(L^\infty)} = 1$.

We set $N = 8$, 16, 32 and 64. The left of Fig. 2 shows a sample mesh ($N = 8$). The right of Fig. 2 exhibits graphs of Err versus h $(= \Delta t)$ in logarithmic scale. We can see that Err is of better order than first one for $\nu = 10^{-1}$ and is almost of first order for $\nu = 10^{-k}$ $(k = 2, 3, 4)$ in h and Δt.

These results for both Examples 1 and 2 are consistent with Theorem 2.

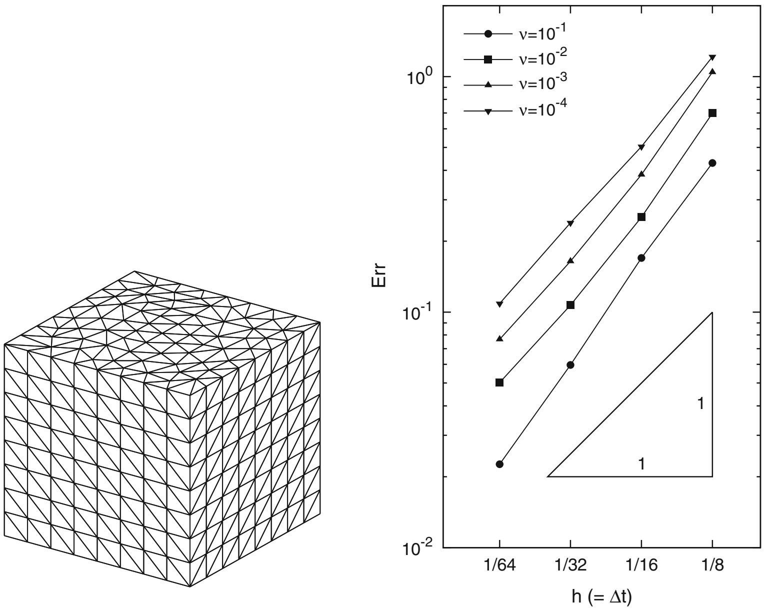

Fig. 2 A sample mesh (*left*, $N = 8$) and graphs of Err versus h (*right*) for Example 2

6 Conclusions

A combined finite element scheme with a characteristics method and Brezzi–Pitkäranta's stabilization method for the Oseen equations has been studied. Stability (Theorem 1) and convergence (Theorem 2) results with the optimal error estimates for the velocity and the pressure have been proved. The scheme has the advantages of both of the characteristics method and Brezzi–Pitkäranta's stabilization method, i.e., robustness for convection-dominated problems, symmetry of the resulting matrix and the small number of degrees of freedom. In order to construct the initial approximate velocity we have also introduced a stabilized Stokes projection, which works well in the analysis without any loss of convergence order. The theoretical convergence order has been recognized in two- and three-dimensional test problems in Examples 1 and 2, respectively. To devise general higher-order stabilized characteristics schemes is a future work. A corresponding stabilized characteristics scheme for the Navier–Stokes equations will be studied in a forthcoming paper [25].

Acknowledgments The authors are thankful to anonymous referees for their valuable comments. This work was supported by JSPS (the Japan Society for the Promotion of Science) under the Japanese-German Graduate Externship (Mathematical Fluid Dynamics) and by Waseda University under Project research, Spectral analysis and its application to the stability theory of the Navier–Stokes equations of Research Institute for Science and Engineering. The authors are indebted to JSPS also for Grant-in-Aid for Young Scientists (B), No. 26800091 to the first author and for Grants-in-Aid for Scientific Research (C), No. 25400212 and (S), No. 24224004 to the second author.

References

1. Baba, K., Tabata, M.: On a conservative upwind finite element scheme for convective diffusion equations. RAIRO Anal. Numer. **15**, 3–25 (1981)

2. Barrett, R., Berry, M., Chan, T.F., Demmel, J., Donato, J., Dongarra, J., Eijkhout, V., Pozo, R., Romine, C., van der Vorst, H.: Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods. SIAM, Philadelphia (1994)
3. Boukir, K., Maday, Y., Métivet, B., Razafindrakoto, E.: A high-order characteristics/finite element method for the incompressible Navier-Stokes equations. Int. J. Numer. Methods Fluids **25**, 1421–1454 (1997)
4. Braack, M., Burman, E., John, V., Lube, G.: Stabilized finite element methods for the generalized Oseen problem. Comput. Methods Appl. Mech. Eng. **196**, 853–866 (2007)
5. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods. Springer, New York (2002)
6. Brezzi, F., Douglas Jr, J.: Stabilized mixed methods for the Stokes problem. Numer. Math. **53**, 225–235 (1988)
7. Brezzi, F., Pitkäranta, J.: On the stabilization of finite element approximations of the Stokes equations. In: Hackbusch, W. (ed.) Efficient Solutions of Elliptic Systems, pp. 11–19. Vieweg, Wiesbaden (1984)
8. Brooks, A.N., Hughes, T.J.R.: Streamline upwind/Petrov–Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. Comput. Methods Appl. Mech. Eng. **32**, 199–259 (1982)
9. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
10. Clément, P.: Approximation by finite element functions using local regularization. RAIRO Anal. Numer. **9**, 77–84 (1975)
11. Douglas Jr, J., Russell, T.F.: Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures. SIAM J. Numer. Anal. **19**, 871–885 (1982)
12. Duvaut, G., Lions, J.L.: Inequalities in Mechanics and Physics. Springer, Berlin (1976)
13. Ewing, R.E., Russell, T.F.: Multistep Galerkin methods along characteristics for convection–diffusion problems. In: Vichnevetsky, R., Stepleman, R.S. (eds.) Advances in Computer Methods for Partial Differential Equations. IV, pp. 28–36. IMACS, New Brunswick (1981)
14. Ewing, R.E., Russell, T.F., Wheeler, M.F.: Simulation of miscible displacement using mixed methods and a modified method of characteristics. In: Proceedings of the Seventh Reservoir Simulation Symposium, Society of Petroleum Engineers of AIME. pp. 71–81 (1983)
15. Franca, L.P., Frey, S.L.: Stabilized finite element methods: II. The incompressible Navier–Stokes equations. Comput. Methods Appl. Mech. Eng. **99**, 209–233 (1992)
16. Franca, L.P., Stenberg, R.: Error analysis of some Galerkin least squares methods for the elasticity equations. SIAM J. Numer. Anal. **28**, 1680–1697 (1991)
17. Girault, V., Raviart, P.-A.: Finite Element Methods for Navier–Stokes Equations, Theory and Algorithms. Springer, Berlin (1986)
18. Hughes, T.J.R., Franca, L.P., Mallet, M.: A new finite element formulation for computational fluid dynamics: VI. Convergence analysis of the generalized SUPG formulation for linear time-dependent multidimensional advective-diffusive systems. Comput. Methods Appl. Mech. Eng. **63**, 97–112 (1987)
19. Jia, H., Li, K., Liu, S.: Characteristic stabilized finite element method for the transient Navier–Stokes equations. Comput. Methods Appl. Mech. Eng. **199**, 2996–3004 (2010)
20. Johnson, C.: Numerical Solution of Partial Differential Equations by the Finite Element Method. Cambridge University Press, Cambridge (1987)
21. Notsu, H.: Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme. Transactions of the Japan Society for Computational. Eng. Sci. **2008**, 20080032 (2008)
22. Notsu, H., Rui, H., Tabata, M.: Development and L^2-analysis of a single-step characteristics finite difference scheme of second order in time for convection-diffusion problems. J. Algorithms Comput. Technol. **7**, 343–380 (2013)
23. Notsu, H., Tabata, M.: A combined finite element scheme with a pressure stabilization and a characteristic-curve method for the Navier–Stokes equations. Trans. Jpn Soc. Ind. Appl. Math. **18**, 427–445 (2008). (in Japanese)
24. Notsu, H., Tabata, M.: A single-step characteristic-curve finite element scheme of second order in time for the incompressible Navier–Stokes equations. J. Sci. Comput. **38**, 1–14 (2009)
25. Notsu, H., Tabata, M.: Error Estimates of a Pressure-Stabilized Characteristics Finite Element Scheme for the Navier–Stokes Equations. Waseda University, WIAS Discussion Paper, No. 2013-002 (2013). http://www.waseda.jp/wias/achievement/dp/data/dp2013002
26. Pironneau, O.: On the transport-diffusion algorithm and its applications to the Navier–Stokes equations. Numer. Math. **38**, 309–332 (1982)
27. Pironneau, O.: Finite Element Methods for Fluids. Wiley, Chichester (1989)

28. Pironneau, O., Tabata, M.: Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type. Int. J. Numer. Methods Fluids **64**, 1240–1253 (2010)
29. Rui, H., Tabata, M.: A second order characteristic finite element scheme for convection–diffusion problems. Numer. Math. **92**, 161–177 (2002)
30. Saad, Y.: Iterative Methods for Sparse Linear Systems. SIAM, Philadelphia (2003)
31. Stroud, A.H.: Approximate Calculation of Multiple Integrals. Prentice-Hall, Englewood Cliffs (1971)
32. Süli, E.: Convergence and nonlinear stability of the Lagrange–Galerkin method for the Navier–Stokes equations. Numer. Math. **53**, 459–483 (1988)
33. Tabata, M.: A finite element approximation corresponding to the upwind finite differencing. Mem. Numer. Math. **4**, 47–63 (1977)
34. Tabata, M., Tagami, D.: Error estimates for finite element approximations of drag and lift in nonstationary Navier–Stokes flows. Jpn. J. Ind. Appl. Math. **17**, 371–389 (2000)
35. Tezduyar, T.E., Mittal, S., Ray, S.E., Shih, R.: Incompressible flow computations with stabilized bilinear and linear equal-order-interpolation velocity–pressure elements. Comput. Methods Appl. Mech. Eng. **95**, 221–242 (1992)