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Legendre Spectral Projection Methods for Fredholm–Hammerstein Integral Equations

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Abstract In this paper, we consider the Legendre spectral Galerkin and Legendre spectral collocation methods to approximate the solution of Hammerstein integral equation. The convergence of the approximate solutions to the actual solution is discussed and the rates of convergence are obtained. We are able to obtain similar superconvergence rates for the iterated Legendre Galerkin solution for Hammerstein integral equations with smooth kernel as in the case of piecewise polynomial based Galerkin method.

Keywords Hammerstein integral equations · Smooth kernels · Spectral method · Galerkin method · Collocation method · Legendre polynomials · Superconvergence rates

1 Introduction

In this section, we consider the following Hammerstein integral equation

$$x(t) - \int_{-1}^{1} k(t,s)\psi(s,x(s))ds = f(t), \quad -1 \le t \le 1, \tag{1.1}$$

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where k, f and ψ are known functions and x is the unknown solution to be found in a Banach space $\mathbb{X}$. Hammerstein integral equations (1.1) arises as a reformulation of boundary value problems with certain nonlinear boundary conditions.

Several numerical methods are available in literature to solve nonlinear integral equations. Various spectral methods for solving different type of integral equations are present in literature (see [3,4,15–19,21,22]). The Galerkin, collocation, Petrov–Galerkin, degenerate kernel and Nystr$\ddot{o}$m methods are commonly used projection methods for finding numerical solutions of the equation of type (1.1) (see [2,6–12,14]). In [12,13] Kumar and Sloan discussed a new type of collocation method and established superconvergence results for the solution of Hammerstein integral equations. Some recent results on the numerical solutions of the Hammerstein equations can be found in [11].

In the case of piecewise polynomial based projection methods, we consider $-1 = t_0 < t_1 < \cdots < t_n = 1$, a partition of $[-1, 1]$ and let $h = \max\{t_{i+1} - t_i : 0 \leq t_i \leq n-1\}$ denote the norm of the partition. We assume that $h \to 0$, as $n \to \infty$. In this case the approximating subspaces $\mathbb{X}_n = S^{\nu}_{r,n}$, the space of all piecewise polynomials of order r (i.e., of degree $\leq r-1$) with break points at $t_1, t_2, \ldots, t_{n-1}$ and with ν continuous derivatives, $-1 \leq \nu \leq r-2$. Let $\mathcal{P}_n$ be either orthogonal or interpolatory bounded projections from $\mathbb{X}$ onto $\mathbb{X}_n$. Then in Galerkin or in collocation method, the Hammerstein integral equation (1.1) is approximated by

$$x_n - \mathcal{P}_n\mathcal{K}\psi(x_n) = \mathcal{P}_n f, \tag{1.2}$$

where $\mathcal{K}\psi(x_n)(t) = \int_{-1}^{1} k(t,s)\psi(s, x_n(s))\,\mathrm{d}s$. The iterated solution is defined by $\tilde{x}_n = f + \mathcal{K}\psi(x_n)$. Under some suitable conditions on the kernel k and the right hand side function f of the Eq. (1.1), it is known that the orders of convergence for Galerkin and collocation solutions are $\mathcal{O}(h^r)$ and for the iterated Galerkin and iterated collocation solutions are $\mathcal{O}(h^{2r})$ (see [9,10]). However, to get better accuracy in piecewise polynomial based projection methods, the number of partition points should be increased. Hence in such cases, one has to solve a large system of nonlinear equations, which is computationally very much expensive.

In this paper, we have applied Galerkin and collocation method to solve Eq. (1.1) using global polynomial basis functions. Use of global polynomials will imply smaller nonlinear systems, something which is highly desirable in practical computations. Hence we choose to use global polynomials rather than piecewise polynomial basis functions in this paper. In particular, we use Legendre polynomials, which can be generated recursively with ease and possess nice property of orthogonality. Further, these Legendre polynomials are less expensive computationally compared to piecewise polynomial basis functions. However, if $\mathcal{P}_n$ denotes either orthogonal or interpolatory projection from $\mathbb{X}$ into a subspace of global polynomials of degree $\leq n$, then $\|\mathcal{P}_n\|_\infty$ is unbounded. It is the purpose of this work to obtain similar convergence results for the approximate solutions in both L^2-norm and infinity norm using Legendre polynomial bases as in the case of piecewise polynomial bases.

We organize this paper as follows. In Sect. 2, we discuss the Legendre spectral Galerkin and Legendre spectral collocation methods to obtain convergence results. In Sect. 3, numerical results are given to illustrate the theoretical results. Throughout this paper, we assume that c is a generic constant.

2 Legendre Spectral Galerkin and Collocation Methods: Hammerstein Integral Equations with Smooth Kernel

In this section, we describe the Galerkin and collocation methods for solving Hammerstein integral equations using Legendre polynomial basis functions.

Let $\mathbb{X} = \mathcal{C}[-1, 1]$ and consider the following Hammerstein integral equation

$$x(t) - \int_{-1}^{1} k(t, s)\psi(s, x(s))\, ds = f(t), \quad -1 \leq t \leq 1, \tag{2.1}$$

where k, f and ψ are known functions and x is the unknown function to be determined. For a fixed $t \in [-1, 1]$, we denote $k_t(s) = k(t, s)$.

Throughout the paper, the following assumptions are made on f, $k(., .)$ and $\psi(., x(.))$:

(i) $f \in \mathcal{C}[-1, 1]$.

(ii) $\lim_{t \to t'} \|k(t, .) - k(t', .)\|_\infty = 0, \quad t, t' \in [-1, 1]$.

(iii) $M = \|k\|_\infty = \sup_{t,s \in [-1,1]} |k(t, s)| < \infty$.

(iv) The nonlinear function $\psi(s, x)$ is bounded and continuous over $[-1, 1] \times \mathbb{R}$. $\psi(s, x)$ is Lipschitz continuous in x, i.e., for any $x_1, x_2 \in \mathbb{R}$, $\exists\, c_1 > 0$ such that

$$|\psi(s, x_1) - \psi(s, x_2)| \leq c_1 |x_1 - x_2|, \ \forall s \in [-1, 1].$$

(v) The partial derivative $\psi^{(0,1)}(s, x(s))$ of ψ w.r.t the second variable exists and is Lipschitz continuous in x, i.e., for any $x_1, x_2 \in \mathbb{R}$, $\exists\, c_2 > 0$ such that

$$\left|\psi^{(0,1)}(s, x_1) - \psi^{(0,1)}(s, x_2)| \leq c_2 \right| x_1 - x_2|, \ \forall s \in [-1, 1].$$

From this, we have $\psi^{(0,1)}(., .) \in \mathcal{C}([-1, 1] \times \mathbb{R})$.

(vi) We assume that M and c_1 satisfy the condition that $2Mc_1 < 1$.

Note that under the above assumptions on f, k and ψ, for a sufficiently small number $h > 0$, we have

$$\begin{aligned}
|x(t+h) - x(t)| &= \left| f(t+h) + \int_{-1}^{1} k(t+h, s)\psi(s, x(s))\, ds - f(t) \right. \\
&\quad \left. - \int_{-1}^{1} k(t, s)\psi(s, x(s))\, ds \right| \\
&\leq |f(t+h) - f(t)| + \left| \int_{-1}^{1} [k(t+h, s) - k(t, s)]\psi(s, x(s))\, ds \right| \\
&\leq |f(t+h) - f(t)| + \sup_{s \in [-1,1]} |k(t+h, s) - k(t, s)| \int_{-1}^{1} |\psi(s, x(s))|\, ds \\
&\to 0 \ as\ h \to 0.
\end{aligned}$$

This implies $x \in \mathcal{C}[-1, 1]$.

Let $\mathcal{C}^r[-1, 1]$ denote the space of r-times continuously differentiable functions. For the rest of the paper we assume that the kernel $k(., .) \in \mathcal{C}^r([-1, 1] \times [-1, 1])$, the nonlinear function $\psi(., .) \in \mathcal{C}^r([-1, 1] \times \mathbb{R})$ and $f \in \mathcal{C}^r[-1, 1]$. Denote

$$(D^{i,j}k)(t, s) = \frac{\partial^{i+j}}{\partial t^i \partial s^j} k(t, s), \quad t, s \in [-1, 1],$$

and

$$\|k\|_{r,\infty} = max\left\{ \left\| D^{(i,j)}k \right\|_\infty : 0 \leq i \leq r, 0 \leq j \leq r \right\}.$$

Now for $j = 1, 2, \dots r$, we have from estimate (2.1) that

$$x^{(j)}(t) = f^{(j)}(t) + \int_{-1}^{1} \left\{ \frac{\partial^j}{\partial t^j} k(t,s) \right\} \psi(s, x(s)) ds.$$

Hence by our assumptions on f, k and ψ, it follows that $x \in \mathcal{C}^r[-1, 1]$. We write

$$\|x\|_{r,\infty} = \max \left\{ \left\| x^{(j)} \right\|_\infty : 0 \le j \le r \right\},$$

where $x^{(j)}$ denotes the j-th derivative of x.

Let

$$\mathcal{K}y(t) = \int_{-1}^{1} k(t,s) y(s) \,\mathrm{d}s, \quad t \in [-1, 1], \; y \in \mathbb{X}.$$

Note that, using Holder's inequality we have for any $y \in \mathbb{X}$,

$$\left\| \mathcal{K}y \right\|_\infty = \sup_{t \in [-1,1]} |\mathcal{K}y(t)| = \sup_{t \in [-1,1]} \left| \int_{-1}^{1} k(t,s) y(s) ds \right| \le \sup_{t,s \in [-1,1]} |k(t,s)| \int_{-1}^{1} |y(s)| ds$$
$$\le \sqrt{2} M \|y\|_{L^2}, \tag{2.2}$$

and

$$\left\| \mathcal{K}y \right\|_{L^2} \le \sqrt{2} \left\| \mathcal{K}y \right\|_\infty \le 2M \|y\|_{L^2}. \tag{2.3}$$

This implies

$$\|\mathcal{K}\|_{L^2} \le 2M. \tag{2.4}$$

We will use Kumar and Sloan [12] technique for finding the approximate solution of the Eq. (2.1). The projection method will now be applied to an equivalent equation for the function z defined by

$$z(t) := \psi(t, x(t)), \quad t \in [-1, 1]. \tag{2.5}$$

Note that, since $\psi(.,.) \in \mathcal{C}^r([-1, 1] \times \mathbb{R})$ and $x \in \mathcal{C}^r[-1, 1]$, using chain rule for higher derivatives it is easy to obtain that $z \in \mathcal{C}^r[-1, 1]$.

The desired exact solution x of (2.1) is obtained by the equation

$$x(t) = f(t) + \int_{-1}^{1} k(t,s) z(s) \,\mathrm{d}s, \quad t \in [-1, 1]. \tag{2.6}$$

For our convenience, we consider a nonlinear operator $\Psi : \mathbb{X} \to \mathbb{X}$ defined by

$$\Psi(x)(t) := \psi(t, x(t)). \tag{2.7}$$

Then the Eq. (2.1) will take the form

$$x = \mathcal{K}z + f, \tag{2.8}$$

and Eq. (2.5) becomes

$$z = \Psi(\mathcal{K}z + f). \tag{2.9}$$

Let $\mathcal{T}(u) := \Psi(\mathcal{K}u + f)$, $u \in \mathbb{X}$, then the Eq. (2.9) can be written as

$$z = \mathcal{T}z. \tag{2.10}$$

Theorem 2.1 *Let $\mathbb{X} = \mathcal{C}[-1, 1]$, $f \in \mathbb{X}$ and $k(.,.) \in \mathcal{C}([-1, 1] \times [-1, 1])$ with $M = \sup_{t,s\in[-1,1]} |k(t, s)| < \infty$. Let $\psi(s, y(s)) \in \mathcal{C}([-1, 1] \times \mathbb{R})$ satisfies the Lipschitz condition in the second variable, i.e.,*

$$\big|\psi(s, y_1) - \psi(s, y_2)\big| \leq c_1\big|y_1 - y_2\big|, \quad y_1, y_2 \in \mathbb{X},$$

with $2Mc_1 < 1$. Then the operator equation $z = \mathcal{T}z$ has a unique solution $z_0 \in \mathbb{X}$, i.e., we have $z_0 = \mathcal{T}z_0$.

Proof Let $z_1, z_2 \in \mathcal{C}[-1, 1]$. Using Lipschitz's continuity of $\psi(., x(.))$ and the estimate (2.2), we have

$$\begin{aligned}\big\|\mathcal{T}z_1 - \mathcal{T}z_2\big\|_\infty &= \big\|\Psi(\mathcal{K}z_1 + f) - \Psi(\mathcal{K}z_2 + f)\big\|_\infty \\ &\leq c_1\big\|\mathcal{K}(z_1 - z_2)\big\|_\infty \\ &\leq c_1 M\sqrt{2}\big\|z_1 - z_2\big\|_{L^2} \\ &\leq 2Mc_1\big\|z_1 - z_2\big\|_\infty. \end{aligned} \tag{2.11}$$

By assumption $2Mc_1 < 1$, hence $\mathcal{T}$ is a contraction mapping on $\mathbb{X}$. Since $\mathbb{X} = \mathcal{C}[-1, 1]$ with $\|.\|_\infty$ norm is a Banach space, $\mathcal{T}$ has a unique fixed point in $\mathbb{X}$, by Banach contraction theorem. We denote this unique solution as z_0. Hence the proof follows. □

Next we will apply Legendre Galerkin and Legendre collocation methods to the Eq. (2.9). To do this, we let $\mathbb{X}_n = \text{span}\{\phi_0, \phi_1, \phi_2, \ldots, \phi_n\}$ be the sequence of Legendre polynomial subspaces of $\mathbb{X}$ of degree $\leq n$, where $\{\phi_0, \phi_1, \phi_2, \ldots, \phi_n\}$ forms an orthonormal basis for $\mathbb{X}_n$. Here ϕ_i's are given by

$$\phi_i(s) = \sqrt{\frac{2i + 1}{2}} L_i(s), \quad i = 0, 1, \ldots, n, \tag{2.12}$$

where L_i's are the Legendre polynomials of degree $\leq i$. These Legendre polynomials can be generated by the following three-term recurrence relation

$$L_0(s) = 1, L_1(s) = s, \quad s \in [-1, 1], \tag{2.13}$$

and for $i = 1, 2, \ldots, n - 1$

$$(i + 1)L_{i+1}(s) = (2i + 1)sL_i(s) - iL_{i-1}(s), \quad s \in [-1, 1]. \tag{2.14}$$

Orthogonal projection operator: Let $\mathbb{X} = \mathcal{C}[-1, 1]$ and let the operator $\mathcal{P}_n^G : \mathbb{X} \to \mathbb{X}_n$ be the orthogonal projection defined by

$$\mathcal{P}_n^G x = \sum_{j=0}^{n} \langle x, \ \phi_j\rangle\phi_j, \quad x \in \mathbb{X}, \tag{2.15}$$

where $\langle x, \ \phi_j\rangle = \int_{-1}^{1} x(t)\phi_j(t)dt$.

We quote the following proposition and lemma which follows from (Canuto et al. [5], pp 283-287).

Proposition 2.1 *Let $\mathcal{P}_n^G : \mathbb{X} \to \mathbb{X}_n$ denote the orthogonal projection defined by (2.15). Then the projection $\mathcal{P}_n^G$ satisfies the following properties.*

(i) $\|\mathcal{P}_n^G u\|_{L^2} \leq p_1\|u\|_\infty$, *where p_1 is a constant independent of n.*

(ii) *There exists a constant $c > 0$ such that for any $n \in \mathbb{N}$ and $u \in \mathbb{X}$,*

$$\left\|\mathcal{P}_n^G u - u\right\|_{L^2} \leq c \inf_{\phi \in \mathbb{X}_n} \|u - \phi\|_{L^2} \to 0, \text{ as } n \to \infty. \tag{2.16}$$

Lemma 2.1 *Let $\mathcal{P}_n^G$ be the orthogonal projection defined by (2.15). Then for any $u \in \mathcal{C}^r[-1, 1]$, there hold*

$$\left\|u - \mathcal{P}_n^G u\right\|_{L^2} \leq cn^{-r}\left\|u^{(r)}\right\|_{L^2}, \tag{2.17}$$

$$\left\|u - \mathcal{P}_n^G u\right\|_\infty \leq cn^{\frac{3}{4}-r}\left\|u^{(r)}\right\|_{L^2}, \tag{2.18}$$

where c is a constant independent of n.

Interpoaltory projection operator: Let $\{\tau_0, \tau_1, \ldots, \tau_n\}$ be the zeros of the Legendre polynomial of degree $n + 1$ and define interpolatory projection $\mathcal{P}_n^C : \mathbb{X} \to \mathbb{X}_n$ by

$$\mathcal{P}_n^C u \in \mathbb{X}_n, \;\; \mathcal{P}_n^C u(\tau_i) = u(\tau_i), \; i = 0, 1, \ldots, n, \; u \in \mathbb{X}. \tag{2.19}$$

According to the analysis of (Canuto et al. [5]), $\mathcal{P}_n^C$ satisfies the following lemmas.

Lemma 2.2 *Let $\mathcal{P}_n^C : \mathbb{X} \to \mathbb{X}_n$ be the interpolatory projection defined by (2.19). Then there hold*

(i) $\left\|\mathcal{P}_n^C u\right\|_{L^2} \leq p_2\left\|u\right\|_\infty$, *where p_2 is a constant independent of n.*
(ii) There exists a constant $c > 0$ such that for any $n \in \mathbb{N}$ and $u \in \mathbb{X}$,

$$\left\|\mathcal{P}_n^C u - u\right\|_{L^2} \leq c \inf_{\phi \in \mathbb{X}_n} \left\|u - \phi\right\|_{L^2} \to 0, \text{ as } n \to \infty. \tag{2.20}$$

Lemma 2.3 *Let $\mathcal{P}_n^C : \mathbb{X} \to \mathbb{X}_n$ be the interpolatory projection defined by (2.19). Then for any $u \in \mathcal{C}^r[-1, 1]$, there exists a constant c independent of n such that*

$$\left\|u - \mathcal{P}_n^C u\right\|_{L^2} \leq cn^{-r}\left\|u^{(r)}\right\|_{L^2}, \tag{2.21}$$

$$\left\|u - \mathcal{P}_n^C u\right\|_\infty \leq cn^{\frac{1}{2}-r}\left\|u^{(r)}\right\|_{L^2}. \tag{2.22}$$

Throughout this paper, we assume that the projection operator $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ is either orthogonal projection $\mathcal{P}_n^G$ or interpolatory projection operator $\mathcal{P}_n^C$ defined as above. From the above discussed properties of $\mathcal{P}_n^G$ and $\mathcal{P}_n^C$, we have

$$\left\|\mathcal{P}_n u\right\|_{L^2} \leq p\left\|u\right\|_\infty, \; u \in \mathbb{X}, \tag{2.23}$$

where p is a constant independent of n. Also estimates (2.16) and (2.20) imply that

$$\left\|\mathcal{P}_n u - u\right\|_{L^2} \to 0, \text{ as } n \to \infty, \; \forall u \in \mathcal{C}[-1, 1]. \tag{2.24}$$

Further we have from Lemmas 2.1 and 2.3 that

$$\left\|u - \mathcal{P}_n u\right\|_{L^2} \leq cn^{-r}\left\|u^{(r)}\right\|_{L^2}, \tag{2.25}$$

$$\left\|u - \mathcal{P}_n u\right\|_\infty \leq cn^{\beta-r}\left\|u^{(r)}\right\|_{L^2}, \; 0 < \beta < 1, \text{ and } r = 0, 1, 2, \cdots \tag{2.26}$$

where c is a constant independent of n, $\beta = \frac{3}{4}$ for orthogonal projection operators and $\beta = \frac{1}{2}$ for interpolatory projections. Note that $\left\|\mathcal{P}_n u - u\right\|_\infty \nrightarrow 0$, as $n \to \infty$ for any $u \in \mathcal{C}[-1, 1]$.

The projection method for Eq. (2.9) is seeking an approximate solution $z_n \in \mathbb{X}_n$ such that

$$z_n = \mathcal{P}_n \Psi(\mathcal{K} z_n + f). \tag{2.27}$$

If $\mathcal{P}_n$ is chosen to be $\mathcal{P}_n^G$, the above scheme (2.27) leads to the Legendre Galerkin method, whereas if $\mathcal{P}_n$ is replaced by $\mathcal{P}_n^C$ we get the Legendre collocation method.

Let $\mathcal{T}_n$ be the operator defined by

$$\mathcal{T}_n(u) := \mathcal{P}_n\Psi(\mathcal{K}u + f),\ u \in \mathbb{X}. \tag{2.28}$$

Then the Eq. (2.27) can be written as

$$z_n = \mathcal{T}_n z_n. \tag{2.29}$$

Corresponding approximate solution x_n of x is given by

$$x_n = \mathcal{K}z_n + f. \tag{2.30}$$

In order to obtain more accurate approximation solution, we further consider the iterated projection method for (2.9). To this end, we define the iterated solution as

$$\tilde{z}_n = \Psi(\mathcal{K}z_n + f). \tag{2.31}$$

Applying $\mathcal{P}_n$ on both sides of the Eq. (2.31), we obtain

$$\mathcal{P}_n\tilde{z}_n = \mathcal{P}_n\Psi(\mathcal{K}z_n + f). \tag{2.32}$$

From Eqs. (2.27) and (2.32), it follows that $\mathcal{P}_n\tilde{z}_n = z_n$. Using this, we see that the iterated solution $\tilde{z}_n$ satisfies the following equation

$$\tilde{z}_n = \Psi(\mathcal{K}\mathcal{P}_n\tilde{z}_n + f). \tag{2.33}$$

Letting $\widetilde{\mathcal{T}}_n(u) := \Psi(\mathcal{K}\mathcal{P}_n u + f)$, $u \in \mathbb{X}$, the Eq. (2.33) can be written as $\tilde{z}_n = \widetilde{\mathcal{T}}_n\tilde{z}_n$. Corresponding approximate solution $\tilde{x}_n$ of x is given by

$$\tilde{x}_n = \mathcal{K}\tilde{z}_n + f. \tag{2.34}$$

We quote the following theorem from [20] which gives us the condition under which the solvability of one equation leads to the solvability of other equation.

Theorem 2.2 (Vainikko [20]) *Let $\widehat{\mathcal{F}}$ and $\widetilde{\mathcal{F}}$ be continuous operators over an open set Ω in a Banach space $\mathbb{X}$. Let the equation $x = \widetilde{\mathcal{F}}x$ has an isolated solution $\tilde{x}_0 \in \Omega$ and let the following conditions be satisfied.*

(a) *The operator $\widehat{\mathcal{F}}$ is Frechet differentiable in some neighborhood of the point $\tilde{x}_0$, while the linear operator $\mathcal{I} - \widehat{\mathcal{F}}'(\tilde{x}_0)$ is continuously invertible.*
(b) *Suppose that for some $\delta > 0$ and $0 < q < 1$, the following inequalities are valid (the number δ is assumed to be so small that the sphere $\|x - \tilde{x}_0\| \leq \delta$ is contained within Ω).*

$$\sup_{\|x-\tilde{x}_0\|\leq\delta} \left\|(\mathcal{I} - \widehat{\mathcal{F}}'(\tilde{x}_0))^{-1}(\widehat{\mathcal{F}}'(x) - \widehat{\mathcal{F}}'(\tilde{x}_0))\right\| \leq q, \tag{2.35}$$

$$\alpha = \left\|(\mathcal{I} - \widehat{\mathcal{F}}'(\tilde{x}_0))^{-1}(\widehat{\mathcal{F}}(\tilde{x}_0) - \widetilde{\mathcal{F}}(\tilde{x}_0))\right\| \leq \delta(1 - q). \tag{2.36}$$

Then the equation $x = \widehat{\mathcal{F}}x$ has a unique solution $\hat{x}_0$ in the sphere $\|x - \tilde{x}_0\| \leq \delta$. Moreover, the inequality

$$\frac{\alpha}{1+q} \leq \|\hat{x}_0 - \tilde{x}_0\| \leq \frac{\alpha}{1-q} \tag{2.37}$$

is valid.

Next we discuss the existence of approximate and iterated approximate solutions and their error bounds. To do this, we first recall the following definition of ν-convergence and a lemma from [1].

Definition 2.1 Let $\mathbb{X}$ be Banach space and $\mathbb{BL}(\mathbb{X})$ be space of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$. Let $\mathcal{K}_n,\ \mathcal{K} \in \mathbb{BL}(\mathbb{X})$. We say $\mathcal{K}_n$ is ν-convergent to $\mathcal{K}$ if

$$\|\mathcal{K}_n\| \le c < \infty,\ \|(\mathcal{K}_n - \mathcal{K})\mathcal{K}\| \to 0,\ \|(\mathcal{K}_n - \mathcal{K})\mathcal{K}_n\| \to 0,\ as\ n \to \infty.$$

Lemma 2.4 (Ahues et al. [1]) *Let $\mathbb{X}$ be a Banach space and $\mathcal{K}$, $\mathcal{K}_n$ be bounded linear operators on $\mathbb{X}$. If $\|\mathcal{K}_n - \mathcal{K}\| \to 0$, as $n \to \infty$ or $\mathcal{K}_n$ is ν-convergent to $\mathcal{K}$ and $(\mathcal{I} - \mathcal{K})^{-1}$ exists, then $(\mathcal{I} - \mathcal{K}_n)^{-1}$ exists and uniformly bounded on $\mathbb{X}$, for sufficiently large n.*

Lemma 2.5 *Let $z_0 \in \mathcal{C}^r[-1, 1]$, then the following hold*

$$\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty = \sup_{t\in[-1,1]} | < k_t(.), (\mathcal{I} - \mathcal{P}_n)z_0 > | \le M\sqrt{2}\|(\mathcal{I} - \mathcal{P}_n)z_0\|_{L^2}.$$

In particular we have $\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty \to 0,\ as\ n \to \infty.$

Proof Using Cauchy-Schwarz inequality and the estimate (2.25), we have

$$\begin{aligned}
\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\|_\infty &= \sup_{t\in[-1,1]} |\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0(t)| \\
&= \sup_{t\in[-1,1]} \left|\int_{-1}^{1} k(t,s)(\mathcal{I} - \mathcal{P}_n)z_0(s)ds\right| \qquad (2.38)\\
&= \sup_{t\in[-1,1]} | < k_t(.), (\mathcal{I} - \mathcal{P}_n)z_0 > | \\
&\le \sup_{t\in[-1,1]} \|k_t(.)\|_{L^2}\|(\mathcal{I} - \mathcal{P}_n)z_0\|_{L^2} \\
&\le \sqrt{2}M\|(\mathcal{I} - \mathcal{P}_n)z_0\|_{L^2} \qquad (2.39)\\
&\le c\sqrt{2}Mn^{-r}\|z_0^{(r)}\|_{L^2} \to 0,\ as\ n \to \infty. \qquad (2.40)
\end{aligned}$$

Hence the proof follows. □

Lemma 2.6 *Let $\mathcal{T}'(z_0)$ and $\widetilde{\mathcal{T}}_n'(z_0)$ be the Frechet derivatives of $\mathcal{T}(z)$ and $\widetilde{\mathcal{T}}_n(z)$, respectively at z_0. Then*

$$\begin{aligned}
&\|(\mathcal{I} - \mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)\|_{L^2} \to 0,\ as\ n \to \infty, \\
&\|(\mathcal{I} - \mathcal{P}_n)\mathcal{T}'(z_0)\|_{L^2} \to 0,\ as\ n \to \infty.
\end{aligned}$$

Proof We have $\widetilde{\mathcal{T}}_n'(z_0) = \Psi'(\mathcal{K}\mathcal{P}_n z_0 + f)\mathcal{K}\mathcal{P}_n$.

Now using the Lipschitz's continuity of $\psi^{(0,1)}(., x(.))$, Lemma 2.5 and boundedness of $\|\Psi'(\mathcal{K}z_0 + f)\|_\infty$, we have

$$\begin{aligned}
\|\Psi'(\mathcal{K}\mathcal{P}_n z_0 + f)\|_\infty &\le \|\Psi'(\mathcal{K}\mathcal{P}_n z_0 + f) - \Psi'(\mathcal{K}z_0 + f)\|_\infty + \|\Psi'(\mathcal{K}z_0 + f)\|_\infty \\
&\le c_2\|\mathcal{K}(\mathcal{P}_n - \mathcal{I})z_0\|_\infty + \|\Psi'(\mathcal{K}z_0 + f)\|_\infty \le B < \infty, \qquad (2.41)
\end{aligned}$$

where B is a constant independent of n.

This implies

$$\|\Psi'(\mathcal{K}\mathcal{P}_n z_0 + f)\|_{L^2} \le \sqrt{2}\|\Psi'(\mathcal{K}\mathcal{P}_n z_0 + f)\|_\infty \le \sqrt{2}B < \infty. \qquad (2.42)$$

Next, Let $\bar{B} := \{x \in \mathbb{X} : \|x\|_{L^2} \le 1\}$ be the closed unit ball in $\mathbb{X}$. We have $\widetilde{\mathcal{T}}_n'(z_0) = \Psi'(\mathcal{K}\mathcal{P}_n z_0 + f)\mathcal{K}\mathcal{P}_n$. Since $\{\mathcal{K}\mathcal{P}_n\}$ is a sequence of compact operators and $\Psi'(\mathcal{K}\mathcal{P}_n z_0 + f)$ is uniformly bounded, $\widetilde{\mathcal{T}}_n'(z_0)$ are compact operators. Thus $S = \{\widetilde{\mathcal{T}}_n'(z_0)x : x \in \bar{B}, n \in N\}$ is relatively compact set. Using estimate (2.24), we can conclude

$$\begin{aligned}\left\|(\mathcal{I}-\mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)\right\|_{L^2} &= \sup\left\{\left\|(\mathcal{I}-\mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)x\right\|_{L^2} : x \in \bar{B}\right\}\\ &= \sup\{\left\|(\mathcal{I}-\mathcal{P}_n)y\right\|_{L^2} : y \in S\} \to 0,\ n \to \infty. \end{aligned} \tag{2.43}$$

Similarly, since $\Psi'(\mathcal{K}z_0 + f)$ is bounded and $\mathcal{K}$ is compact, $\mathcal{T}'(z_0) = \Psi'(\mathcal{K}z_0 + f)\mathcal{K}$ is also compact and we have

$$\left\|(\mathcal{I}-\mathcal{P}_n)\mathcal{T}'(z_0)\right\|_{L^2} \to 0,\ as\ n \to \infty.$$

This completes the proof. □

Theorem 2.3 *Let $z_0 \in \mathcal{C}^r[-1,1]$ be an isolated solution of the Eq. (2.9). Assume that 1 is not an eigenvalue of the linear operator $\Psi'(\mathcal{K}z_0 + f)\mathcal{K}$, where $\Psi'(\mathcal{K}z_0 + f)\mathcal{K}$ denotes the Frechet derivative of $\Psi(\mathcal{K}z + f)$ at z_0. Let $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ be either orthogonal or interpolatory projection operator defined by (2.15) and (2.19) respectively. Then the Eq. (2.27) has a unique solution $z_n \in B(z_0,\delta) = \{z : \|z - z_0\|_{L^2} < \delta\}$ for some $\delta > 0$ and for sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n such that*

$$\frac{\alpha_n}{1+q} \le \left\|z_n - z_0\right\|_{L^2} \le \frac{\alpha_n}{1-q},$$

where $\alpha_n = \left\|(\mathcal{I} - \mathcal{T}_n{}'(z_0))^{-1}(\mathcal{T}_n(z_0) - \mathcal{T}(z_0))\right\|_{L^2}$. Further, we obtain

$$\left\|z_n - z_0\right\|_{L^2} \le c\left\|(\mathcal{P}_n - \mathcal{I})z_0\right\|_{L^2} = \mathcal{O}(n^{-r}),$$

where c is a constant independent of n.

Proof Using Lemma 2.6, we have

$$\begin{aligned}\left\|\mathcal{T}_n{}'(z_0) - \mathcal{T}'(z_0)\right\|_{L^2} &= \left\|\mathcal{P}_n\Psi'(\mathcal{K}z_0 + f)\mathcal{K} - \Psi'(\mathcal{K}z_0 + f)\mathcal{K}\right\|_{L^2}\\ &= \left\|(\mathcal{P}_n - \mathcal{I})\Psi'(\mathcal{K}z_0 + f)\mathcal{K}\right\|_{L^2}\\ &= \left\|(\mathcal{P}_n - \mathcal{I})\mathcal{T}'(z_0)\right\|_{L^2} \to 0,\ as\ n \to \infty.\end{aligned}$$

Since we assume that 1 is not an eigen value of $\mathcal{T}'(z_0)$, $(\mathcal{I} - \mathcal{T}'(z_0))$ is invertible. Hence by applying Lemma 2.4, we have $(\mathcal{I} - \mathcal{T}_n'(z_0))^{-1}$ exists and uniformly bounded on $\mathbb{X}$, for some sufficiently large n, i.e., there exists some $A_1 > 0$ such that $\left\|(\mathcal{I} - \mathcal{T}_n'(z_0))^{-1}\right\|_{L^2} \le A_1 < \infty$.

Now from estimates (2.2) and (2.23), we have for any $z \in B(z_0,\delta)$,

$$\begin{aligned}\left\|[\mathcal{T}_n{}'(z_0) - \mathcal{T}_n{}'(z)]v\right\|_{L^2} &= \left\|[\mathcal{P}_n\Psi'(\mathcal{K}z_0 + f)\mathcal{K} - \mathcal{P}_n\Psi'(\mathcal{K}z + f)\mathcal{K}]v\right\|_{L^2}\\ &\le p\left\|[\Psi'(\mathcal{K}z_0 + f) - \Psi'(\mathcal{K}z + f)]\mathcal{K}v\right\|_\infty\\ &\le p\left\|\Psi'(\mathcal{K}z_0 + f) - \Psi'(\mathcal{K}z + f)\right\|_\infty\left\|\mathcal{K}v\right\|_\infty\\ &\le \sqrt{2}pM\left\|\Psi'(\mathcal{K}z_0 + f) - \Psi'(\mathcal{K}z + f)\right\|_\infty\left\|v\right\|_{L^2}.\end{aligned} \tag{2.44}$$

Taking use of the Lipschtiz's continuity of $\psi^{(0,1)}(.,x(.))$ and estimate (2.2), we have

$$\begin{aligned}\left\|\Psi'(\mathcal{K}z_0 + f) - \Psi'(\mathcal{K}z + f)\right\|_\infty &\le c_2\left\|\mathcal{K}(z_0 - z)\right\|_\infty\\ &\le \sqrt{2}c_2M\left\|z_0 - z\right\|_{L^2} \le \sqrt{2}Mc_2\delta.\end{aligned} \tag{2.45}$$

Using the estimate (2.45) in (2.44), we obtain

$$\left\|[\mathcal{T}_n{}'(z_0) - \mathcal{T}_n{}'(z)]v\right\|_{L^2} \leq 2pM^2c_2\delta\left\|v\right\|_{L^2}.$$

Thus we have

$$\sup_{\left\|z-z_0\right\|_{L^2}\leq\delta} \left\|(\mathcal{I} - \mathcal{T}_n{}'(z_0))^{-1}(\mathcal{T}_n{}'(z_0) - \mathcal{T}_n{}'(z))\right\|_{L^2} \leq 2A_1pM^2c_2\delta \leq q\ (say).$$

Here we choose δ in such a way that, $0 < q < 1$. This proves the Eq. (2.35) of Theorem 2.2.

Taking use of (2.25), we have

$$\begin{aligned}\alpha_n &= \left\|(\mathcal{I} - \mathcal{T}_n{}'(z_0))^{-1}(\mathcal{T}_n(z_0) - \mathcal{T}(z_0))\right\|_{L^2}\\ &\leq A_1\left\|\mathcal{T}_n(z_0) - \mathcal{T}(z_0)\right\|_{L^2}\\ &= A_1\left\|\mathcal{P}_n\Psi(\mathcal{K}z_0 + f) - \Psi(\mathcal{K}z_0 + f)\right\|_{L^2}\\ &= A_1\left\|(\mathcal{P}_n - \mathcal{I})\Psi(\mathcal{K}z_0 + f)\right\|_{L^2}\\ &= A_1\left\|(\mathcal{P}_n - \mathcal{I})z_0\right\|_{L^2} \to 0,\ as\ n \to \infty.\end{aligned}$$

By choosing n large enough such that $\alpha_n \leq \delta(1-q)$, the Eq. (2.36) of Theorem 2.2 is satisfied. Hence by applying Theorem 2.2, we obtain

$$\frac{\alpha_n}{1+q} \leq \left\|z_n - z_0\right\|_{L^2} \leq \frac{\alpha_n}{1-q},$$

and

$$\left\|z_n - z_0\right\|_{L^2} \leq \frac{\alpha_n}{1-q} \leq c\left\|(\mathcal{P}_n - \mathcal{I})z_0\right\|_{L^2}.$$

Hence from estimate (2.25), we have

$$\left\|z_n - z_0\right\|_{L^2} = \mathcal{O}(n^{-r}).$$

This completes the proof. □

Next we discuss the existence and convergence of the iterated approximate solution $\tilde{z}_n$ to z_0.

Theorem 2.4 *$\widetilde{\mathcal{T}}_n'(z_0)$ is ν-convergent to $\mathcal{T}'(z_0)$ in both infinity norm and L^2-norm.*

Proof Consider

$$\begin{aligned}\left|\widetilde{\mathcal{T}}_n'(z_0)z(t)\right| &= \left|\Psi'(\mathcal{K}\mathcal{P}_nz_0 + f)\mathcal{K}\mathcal{P}_nz(t)\right|\\ &\leq \left|\Psi'(\mathcal{K}\mathcal{P}_nz_0 + f) - \Psi'(\mathcal{K}z_0 + f)||\mathcal{K}\mathcal{P}_nz(t)\right|\\ &\quad+\left|\Psi'(\mathcal{K}z_0 + f)||\mathcal{K}\mathcal{P}_nz(t)\right|.\end{aligned} \tag{2.46}$$

Now using the Lipschtiz's continuity of $\psi^{(0,1)}(.,x(.))$ and Lemma 2.5, we have

$$\begin{aligned}\left\|\Psi'(\mathcal{K}\mathcal{P}_nz_0 + f) - \Psi'(\mathcal{K}z_0 + f)\right\|_\infty &\leq c_2\left\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\right\|_\infty\\ &\to 0,\ as\ n \to \infty.\end{aligned} \tag{2.47}$$

Using estimate (2.2) and (2.23), we have

$$\left\|\mathcal{K}\mathcal{P}_nz\right\|_\infty \leq \sqrt{2}M\left\|\mathcal{P}_nz\right\|_{L^2} \leq \sqrt{2}Mp\left\|z\right\|_\infty, \tag{2.48}$$

which implies

$$\|\mathcal{K}\mathcal{P}_n\|_\infty \leq \sqrt{2}Mp. \tag{2.49}$$

Now combining the estimates (2.46), (2.47) and (2.49), we obtain

$$\|\widetilde{\mathcal{T}}_n'(z_0)\|_\infty \leq \sqrt{2}Mp(c_2\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty + \|\Psi'(\mathcal{K}z_0+f)\|_\infty) < \infty.$$

This shows that $\|\widetilde{\mathcal{T}}_n'(z_0)\|_\infty$ is uniformly bounded.

Next Consider

$$\begin{aligned}
|(\widetilde{\mathcal{T}}_n'(z_0)-\mathcal{T}'(z_0))\widetilde{\mathcal{T}}_n'(z_0)z(t)| &= \left|\{\Psi'(\mathcal{K}\mathcal{P}_nz_0+f)\mathcal{K}\mathcal{P}_n - \Psi'(\mathcal{K}z_0+f)\mathcal{K}\}\widetilde{\mathcal{T}}_n'(z_0)z(t)\right| \\
&\leq \left|\Psi'(\mathcal{K}\mathcal{P}_nz_0+f)(\mathcal{K}\mathcal{P}_n-\mathcal{K})\widetilde{\mathcal{T}}_n'(z_0)z(t)\right| \\
&\quad + \left|\{\Psi'(\mathcal{K}\mathcal{P}_nz_0+f)-\Psi'(\mathcal{K}z_0+f)\}\mathcal{K}\widetilde{\mathcal{T}}_n'(z_0)z(t)\right|.
\end{aligned} \tag{2.50}$$

Now for the first term in the above estimate (2.50), using Lemma 2.5 we have

$$\begin{aligned}
\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)z\|_\infty &= \sup_{t\in[-1,1]}\left|\int_{-1}^{1} k(t,s)(\mathcal{I}-\mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)z(s)ds\right| \\
&\leq \sqrt{2}M\|(\mathcal{I}-\mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)\|_{L^2}\|z\|_\infty.
\end{aligned} \tag{2.51}$$

For the second term of the estimate (2.50) using estimates (2.2), (2.47), we have

$$\begin{aligned}
&\|\{\Psi'(\mathcal{K}\mathcal{P}_nz_0+f)-\Psi'(\mathcal{K}z_0+f)\}\mathcal{K}\widetilde{\mathcal{T}}_n'(z_0)z\|_\infty \\
&\leq \|\Psi'(\mathcal{K}\mathcal{P}_nz_0+f)-\Psi'(\mathcal{K}z_0+f)\|_\infty\|\mathcal{K}\widetilde{\mathcal{T}}_n'(z_0)z\|_\infty \\
&\leq c_2\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)z_0\|_\infty\|\mathcal{K}\widetilde{\mathcal{T}}_n'(z_0)z\|_\infty \\
&\leq 2\sqrt{2}c_2M^2\|(\mathcal{I}-\mathcal{P}_n)z_0\|_{L^2}\|\widetilde{\mathcal{T}}_n'(z_0)\|_\infty\|z\|_\infty.
\end{aligned} \tag{2.52}$$

Now combining estimates (2.41), (2.50), (2.51) and (2.52), we see that

$$\begin{aligned}
&\|(\widetilde{\mathcal{T}}_n'(z_0)-\mathcal{T}'(z_0))\widetilde{\mathcal{T}}_n'(z_0)z\|_\infty \\
&\quad\leq \left\{\sqrt{2}MB\|(\mathcal{I}-\mathcal{P}_n)\widetilde{\mathcal{T}}_n'(z_0)\|_{L^2} + 2\sqrt{2}c_2M^2\|(\mathcal{I}-\mathcal{P}_n)z_0\|_{L^2}\|\widetilde{\mathcal{T}}_n'(z_0)\|_\infty\right\}\|z\|_\infty.
\end{aligned}$$

Hence using Lemma 2.6, estimate (2.25) and the uniform boundedness of $\|\widetilde{\mathcal{T}}_n'(z_0)\|_\infty$, we obtain

$$\|(\widetilde{\mathcal{T}}_n'(z_0)-\mathcal{T}'(z_0))\widetilde{\mathcal{T}}_n'(z_0)\|_\infty \to 0, \ as\ n \to \infty.$$

Following the similar steps we can prove that

$$\|(\widetilde{\mathcal{T}}_n'(z_0)-\mathcal{T}'(z_0))\mathcal{T}'(z_0)\|_\infty \to 0, \ as\ n \to \infty.$$

This shows that $\widetilde{\mathcal{T}}_n'(z_0)$ is ν-convergent to $\mathcal{T}'(z_0)$ in infinity norm.

On similar lines, we can show that $\widetilde{\mathcal{T}}_n'(z_0)$ is ν-convergent to $\mathcal{T}'(z_0)$ in L^2- norm. This completes the proof. □

Hence by applying the Lemma 2.4 and Theorem 2.4, we obtain the following theorem.

Theorem 2.5 *Let $z_0 \in \mathcal{C}^r[-1,1]$ be an isolated solution of the Eq.* (2.9). *Assume that* 1 *is not an eigenvalue of $\Psi'(\mathcal{K}z_0 + f)\mathcal{K}$. Then for sufficiently large n, the operator $\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0)$ is invertible on $\mathcal{C}[-1,1]$ and there exist constants $L,\ L_1 > 0$ independent of n such that $\left\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}\right\|_\infty \le L$ and $\left\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}\right\|_{L^2} \le L_1$.*

Theorem 2.6 *Let $z_0 \in \mathcal{C}^r[-1,1]$ be an isolated solution of the Eq.* (2.9). *Let $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ be either orthogonal or interpolatory projection operator defined by* (2.15) *and* (2.19) *respectively. Assume that* 1 *is not an eigenvalue of $\Psi'(\mathcal{K}z_0 + f)\mathcal{K}$, then for sufficiently large n, the iterated solution $\tilde{z}_n$ defined by* (2.33) *is the unique solution in the sphere $B(z_0, \delta) = \{z : \left\|z - z_0\right\|_\infty < \delta\}$. Moreover, there exists a constant $0 < q < 1$, independent of n such that*

$$\frac{\beta_n}{1+q} \le \left\|\tilde{z}_n - z_0\right\|_\infty \le \frac{\beta_n}{1-q},$$

where

$$\beta_n = \left\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}(\widetilde{\mathcal{T}}_n(z_0) - \mathcal{T}(z_0))\right\|_\infty.$$

Proof From Theorem 2.5, we can say, there exists a constant $L > 0$ such that $\left\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}\right\|_\infty \le L$, for sufficiently large value of n.

Consider for any $z \in B(z_0, \delta)$,

$$\begin{aligned} \left\|[\widetilde{\mathcal{T}}_n'(z) - \widetilde{\mathcal{T}}_n'(z_0)]v\right\|_\infty &= \left\|[\{\Psi'(\mathcal{K}\mathcal{P}_n z + f) - \Psi'(\mathcal{K}\mathcal{P}_n z_0 + f)\}\mathcal{K}\mathcal{P}_n]v\right\|_\infty \\ &\le \left\|\Psi'(\mathcal{K}\mathcal{P}_n z + f) - \Psi'(\mathcal{K}\mathcal{P}_n z_0 + f)\right\|_\infty \left\|\mathcal{K}\mathcal{P}_n v\right\|_\infty. \end{aligned} \tag{2.53}$$

Using Cauchy-Schawrz inequality and estimate (2.48), we have

$$\begin{aligned} \left\|\Psi'(\mathcal{K}\mathcal{P}_n z + f) - \Psi'(\mathcal{K}\mathcal{P}_n z_0 + f)\right\|_\infty &\le c_2\left\|\mathcal{K}\mathcal{P}_n(z_0 - z)\right\|_\infty \\ &\le \sqrt{2}Mc_2 p\left\|z - z_0\right\|_\infty \\ &\le \sqrt{2}Mc_2 p\delta. \end{aligned} \tag{2.54}$$

Combining estimates (2.48), (2.53), (2.54), we obtain

$$\left\|[\widetilde{\mathcal{T}}_n'(z) - \widetilde{\mathcal{T}}_n'(z_0)]v\right\|_\infty \le 2M^2 c_2 p^2 \delta \left\|v\right\|_\infty. \tag{2.55}$$

This implies

$$\sup_{\left\|z - z_0\right\|_\infty \le \delta} \left\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}(\widetilde{\mathcal{T}}_n'(z) - \widetilde{\mathcal{T}}_n'(z_0))\right\|_\infty \le 2LM^2 c_2 p^2 \delta \le q\ (say).$$

We choose δ in such a way that $0 < q < 1$. Hence this proves the Eq. (2.35) of Theorem 2.2.

Now using the Lipschtiz's continuity of $\psi(., x(.))$ and Lemma 2.5, we have

$$\begin{aligned} \left\|\widetilde{\mathcal{T}}_n(z_0) - \mathcal{T}(z_0)\right\|_\infty &\le \left\|\Psi(\mathcal{K}\mathcal{P}_n z_0 + f) - \Psi(\mathcal{K}z_0 + f)\right\|_\infty \\ &\le c_1\left\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\right\|_\infty \to 0,\ as\ n \to \infty. \end{aligned} \tag{2.56}$$

Hence

$$\begin{aligned} \beta_n &= \left\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}(\widetilde{\mathcal{T}}_n(z_0) - \mathcal{T}(z_0))\right\|_\infty \\ &\le Lc_1\left\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)z_0\right\|_\infty \to 0,\ as\ n \to \infty. \end{aligned}$$

Choose n large enough such that $\beta_n \leq \delta(1-q)$. Then the Eq. (2.36) of Theorem 2.2 is satisfied. Thus by applying Theorem 2.2, we obtain

$$\frac{\beta_n}{1+q} \leq \left\|\tilde{z}_n - z_0\right\|_\infty \leq \frac{\beta_n}{1-q}$$

where

$$\beta_n = \left\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}(\widetilde{\mathcal{T}}_n(z_0) - \mathcal{T}(z_0))\right\|_\infty.$$

This completes the proof. □

Theorem 2.7 *Let $z_0 \in \mathcal{C}^r[-1,1]$ be an isolated solution of the Eq. (2.9). Let $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ be either orthogonal or interpolatory projection operator defined by (2.15) and (2.19) respectively. Assume that 1 is not an eigenvalue of $\Psi'(\mathcal{K}z_0 + f)\mathcal{K}$, then for sufficiently large n, the iterated solution $\tilde{z}_n$ defined by (2.33) is the unique solution in the sphere $B(z_0, \delta) = \{z : \left\|z - z_0\right\|_{L^2} < \delta\}$. Moreover, there exists a constant $0 < q < 1$, independent of n such that*

$$\frac{\beta_n}{1+q} \leq \left\|\tilde{z}_n - z_0\right\|_{L^2} \leq \frac{\beta_n}{1-q},$$

where

$$\beta_n = \left\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}(\widetilde{\mathcal{T}}_n(z_0) - \mathcal{T}(z_0))\right\|_{L^2}.$$

Proof Using the similar steps as in the proof of Theorem 2.6, the proof of the above theorem can be easily done. □

Theorem 2.8 *Let $z_0 \in \mathcal{C}[-1,1]$ be an isolated solution of the Eq. (2.9). Let $\tilde{z}_n$ defined by the iterated scheme (2.33). Then the following hold*

$$\left\|\tilde{z}_n - z_0\right\|_\infty \leq c \sup_{t\in[-1,1]} |< k_t, (\mathcal{I} - \mathcal{P}_n)z_0 >|, \tag{2.57}$$

and

$$\left\|\tilde{z}_n - z_0\right\|_{L^2} \leq c \sup_{t\in[-1,1]} |< k_t, (\mathcal{I} - \mathcal{P}_n)z_0 >|, \tag{2.58}$$

where c is a constant independent of n.

Proof It follows from Theorem 2.6 that

$$\frac{\beta_n}{1+q} \leq \left\|\tilde{z}_n - z_0\right\|_\infty \leq \frac{\beta_n}{1-q},$$

where

$$\beta_n = \left\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}(\widetilde{\mathcal{T}}_n(z_0) - \mathcal{T}(z_0))\right\|_\infty.$$

Hence from Theorem 2.5, estimates (2.39), (2.56), we have

$$\begin{aligned}\left\|\tilde{z}_n - z_0\right\|_\infty \leq \frac{\beta_n}{1-q} &\leq c\left\|(\mathcal{I} - \widetilde{\mathcal{T}}_n'(z_0))^{-1}(\widetilde{\mathcal{T}}_n(z_0) - \mathcal{T}(z_0))\right\|_\infty \\ &\leq cL\left\|\Psi(\mathcal{K}\mathcal{P}_n z_0 + f) - \Psi(\mathcal{K}z_0 + f)\right\|_\infty \\ &\leq cLc_1\left\|\mathcal{K}(\mathcal{P}_n - \mathcal{I})z_0\right\|_\infty \\ &\leq c \sup_{t\in[-1,1]} |< k_t(.), (\mathcal{I} - \mathcal{P}_n)z_0 >|.\end{aligned}$$

This proves the estimate (2.57).

Similarly for L^2- norm we can show that

$$\|\tilde{z}_n - z_0\|_{L^2} \leq \sqrt{2}\|\tilde{z}_n - z_0\|_\infty \leq c \sup_{t\in[-1,1]} | < k_t, (\mathcal{I} - \mathcal{P}_n)z_0 > |,$$

where c is a constant independent of n.

This completes the proof. □

Theorem 2.9 *Let $x_0 \in \mathcal{C}^r[-1, 1]$ be an isolated solution of the Eq. (2.1) and x_n be the Legendre Galerkin or Legendre collocation approximations of x_0. Then there hold*

$$\|x_0 - x_n\|_{L^2} = \mathcal{O}(n^{-r}),$$
$$\|x_0 - x_n\|_\infty = \mathcal{O}(n^{-r}).$$

Proof Using estimates (2.2), (2.8), (2.30) and Theorem 2.3, we have

$$\|x_0 - x_n\|_\infty = \|\mathcal{K}(z_0 - z_n)\|_\infty \leq \sqrt{2}M\|z_0 - z_n\|_{L^2} = \mathcal{O}(n^{-r}),$$

and

$$\|x_0 - x_n\|_{L^2} \leq \sqrt{2}\|x_0 - x_n\|_\infty = \mathcal{O}(n^{-r}).$$

Hence the proof follows. □

Now we discuss the convergence rates for the iterated approximate solutions. To distinguish between the iterated Legendre Galerkin method and iterated Legendre collocation method, we set the following notations. In case of iterated Legendre Galerkin method we denote $\tilde{z}_n = \tilde{z}_n^G$ and $\tilde{x}_n = \tilde{x}_n^G$, and for iterated Legendre collocation method we write $\tilde{z}_n = \tilde{z}_n^C$ and $\tilde{x}_n = \tilde{x}_n^C$.

Theorem 2.10 *Let $x_0 \in \mathcal{C}^r[-1, 1]$ be an isolated solution of the Eq. (2.1) and $\tilde{x}_n^G$ be the iterated Legendre Galerkin approximation of x_0. Then the following superconvergence rates hold*

$$\|x_0 - \tilde{x}_n^G\|_{L^2} = \mathcal{O}(n^{-2r}),$$
$$\|x_0 - \tilde{x}_n^G\|_\infty = \mathcal{O}(n^{-2r}).$$

Proof From Theorem 2.8, we have

$$\|\tilde{z}_n^G - z_0\|_\infty \leq c \sup_{t\in[-1,1]} \left| < k_t(.), (\mathcal{I} - \mathcal{P}_n^G)z_0(.) > \right|. \tag{2.59}$$

Using the orthogonality of the projection operators $\mathcal{P}_n^G$, Cauchy-Schwarz inequality and estimate (2.17) of Lemma 2.1, we obtain

$$\begin{aligned} \left| < k_t(.), (\mathcal{I} - \mathcal{P}_n^G)z_0(.) > \right| &= | < (\mathcal{I} - \mathcal{P}_n^G)k_t(.), (\mathcal{I} - \mathcal{P}_n^G)z_0(.) > | \\ &\leq \|(\mathcal{I} - \mathcal{P}_n^G)k_t(.)\|_{L^2}\|z_0 - \mathcal{P}_n^G z_0\|_{L^2} \\ &\leq cn^{-2r}\|z_0^{(r)}\|_{L^2}\|(k_t(.))^{(r)}\|_{L^2} \\ &\leq cn^{-2r}\|z_0^{(r)}\|_{L^2}\|k\|_{r,\infty}. \end{aligned} \tag{2.60}$$

Hence from (2.59) and (2.60), we have

$$\|\tilde{z}_n^G - z_0\|_\infty \leq cn^{-2r}\|z_0^{(r)}\|_{L^2}\|k\|_{r,\infty} = O(n^{-2r}), \tag{2.61}$$

and

$$\|\tilde{z}_n^G - z_0\|_{L^2} \leq \sqrt{2}\|\tilde{z}_n^G - z_0\|_\infty = \mathcal{O}(n^{-2r}). \tag{2.62}$$

Using estimates (2.2), (2.8), (2.34) and (2.62), we have

$$\|x_0 - \tilde{x}_n^G\|_\infty = \|\mathcal{K}(z_0 - \tilde{z}_n^G)\|_\infty \leq \sqrt{2}M\|\tilde{z}_n^G - z_0\|_{L^2} = \mathcal{O}(n^{-2r}),$$

and

$$\|x_0 - \tilde{x}_n^G\|_{L^2} \leq \sqrt{2}\|x_0 - \tilde{x}_n^G\|_\infty = \mathcal{O}(n^{-2r}).$$

Hence the proof follows. □

Theorem 2.11 *Let $x_0 \in \mathcal{C}^r[-1,1]$ be an isolated solution of the Eq. (2.1) and $\tilde{x}_n^C$ be the iterated Legendre collocation approximation of x_0. Then the following hold*

$$\|x_0 - \tilde{x}_n^C\|_{L^2} = \mathcal{O}(n^{-r}),$$
$$\|x_0 - \tilde{x}_n^C\|_\infty = \mathcal{O}(n^{-r}).$$

Proof Using Theorem 2.8, estimate (2.21) of Lemma 2.3 and Cauchy-Schwarz inequality, we have for the interpolatory projection operator $\mathcal{P}_n^C$

$$\begin{aligned}\|\tilde{z}_n^C - z_0\|_\infty &\leq c \sup_{t\in[-1,1]} |< k_t(.), (\mathcal{I} - \mathcal{P}_n^C)z_0(.) >| \\ &\leq c \sup_{t\in[-1,1]} \|k_t(.)\|_{L^2}\|z_0 - \mathcal{P}_n^C z_0\|_{L^2} \\ &\leq c\sqrt{2}M\|z_0 - \mathcal{P}_n^C z_0\|_{L^2} \\ &\leq \sqrt{2}Mcn^{-r}\|z_0^{(r)}\|_{L^2} = O(n^{-r}), \end{aligned} \tag{2.63}$$

and

$$\|\tilde{z}_n^C - z_0\|_{L^2} \leq \sqrt{2}\|\tilde{z}_n^C - z_0\|_\infty = \mathcal{O}(n^{-r}). \tag{2.64}$$

Using estimates (2.2), (2.8), (2.34) and (2.64), we have

$$\|x_0 - \tilde{x}_n^C\|_\infty = \|\mathcal{K}(z_0 - \tilde{z}_n^C)\|_\infty \leq \sqrt{2}M\|\tilde{z}_n^C - z_0\|_{L^2} = \mathcal{O}(n^{-r}),$$

and

$$\|x_0 - \tilde{x}_n^C\|_{L^2} \leq \sqrt{2}\|x_0 - \tilde{x}_n^C\|_\infty = \mathcal{O}(n^{-r}).$$

Hence the proof follows. □

Remark From Theorems 2.9, 2.10, and 2.11 we observe that the Legendre Galerkin and Legendre collocation solutions have same order of convergence, $O(n^{-r})$ both in L^2-norm and infinity norm. The iterated Legendre Galerkin solution converges with the order $O(n^{-2r})$ in both L^2-norm and infinity norm whereas the iterated Legendre collocation solution converges with the order $O(n^{-r})$ in both L^2-norm and in infinity norm.

3 Numerical Example

In this section we present the numerical results. For that we take the Legendre polynomials as the basis functions of $\mathbb{X}_n$ from the three-term recurrence relation

$$\phi_0(s) = 1, \phi_1(s) = s, \quad s \in [-1, 1],$$

and

$$(i+1)\phi_{i+1}(s) = (2i+1)s\phi_i(s) - i\phi_{i-1}(s), \quad s \in [-1,1], \quad i = 1,2,\ldots,n-1. \quad (3.1)$$

We present the errors of the approximation solutions and the iterated approximation solutions under the Legendre Galerkin and Legendre collocation methods in both L^2-norm and infinity norm in Tables 1–4. We use n to represent the highest degree of the Legendre polynomials employed in the computation. The numerical algorithm was run on a PC with Intel Pentium 1.83 GHz CPU, 512MB RAM, and the programs were compiled by using Matlab.

Example 3.1 We consider the following Hammerstein integral equation

$$x(t) - \int_{-1}^{1} k(t,s)\psi(s,x(s))ds = f(t), \quad -1 \le t \le 1, \quad (3.2)$$

with the kernel function $k(t,s) = \left(\frac{3\sqrt{2}\pi}{16}\right)\cos\left(\frac{\pi|s-t|}{4}\right)$, $\psi(s,x(s)) = [x(s)]^2$ and the function $f(t) = \left(\frac{-1}{4}\right)\cos\left(\frac{\pi t}{4}\right)$ where the exact solution is given by $x(t) = \cos\left(\frac{\pi t}{4}\right)$.

Example 3.2 We consider the following Hammerstein integral equation

$$x(t) - \int_{-1}^{1} k(t,s)\psi(s,x(s))ds = f(t), -1 \le t \le 1 \quad (3.3)$$

Table 1 Legendre Galerkin method

n	$\|x - x_n^G\|_{L^2}$	$\|x - x_n^G\|_\infty$	$\|x - \tilde{x}_n^G\|_{L^2}$	$\|x - \tilde{x}_n^G\|_\infty$
2	0.16612096e−02	0.34084469e−02	0.27140926e−04	0.21215376e−04
4	0.20439952e−02	0.22166056e−02	0.44270028e−04	0.35440976e−04
5	0.86759925e−05	0.21021360e−04	0.31719179e−08	0.24794071e−08
7	0.24104371e−07	0.64562011e−07	0.99307177e−13	0.78714812e−13
8	0.41715829e−10	0.11321033e−09	0.21675239e−14	0.31086244e−14

Table 2 Legendre collocation method

n	$\|x - x_n^C\|_{L^2}$	$\|x - x_n^C\|_\infty$	$\|x - \tilde{x}_n^C\|_{L^2}$	$\|x - \tilde{x}_n^C\|_\infty$
2	0.23728226e−02	0.63183241e−02	0.50686226e−03	0.39620083e−03
4	0.20524588e−02	0.22438193e−02	0.44452727e−04	0.35587527e−04
5	0.86760263e−05	0.20971481e−04	0.36041468e−08	0.28172709e−08
7	0.24104407e−07	0.64465292e−07	0.11634294e−12	0.91926466e−13
8	0.58748349e−10	0.22963709e−09	0.13749494e−14	0.27755576e−14

Table 3 Legendre Galerkin method

n	$\|x - x_n^G\|_{L^2}$	$\|x - x_n^G\|_\infty$	$\|x - \tilde{x}_n^G\|_{L^2}$	$\|x - \tilde{x}_n^G\|_\infty$
2	0.89991060e−01	0.102514167204	0.80983908e−02	0.47581867e−02
4	0.58544673e−02	0.23433077e−03	0.20570498e−03	0.23405378e−03
5	0.19729463e−03	0.78969100e−04	0.69322253e−05	0.78875757e−05
6	0.82051733e−04	0.33521203e−05	0.83117109e−07	0.32803189e−07
7	0.29426261e−05	0.32842007e−05	0.86590496e−09	0.33481582e−10

Table 4 Legendre collocation method

n	$\|x - x_n^C\|_{L^2}$	$\|x - x_n^C\|_\infty$	$\|x - \tilde{x}_n^C\|_{L^2}$	$\|x - \tilde{x}_n^C\|_\infty$
2	0.65655572e−01	0.60995024e−01	0.53667135e−02	0.61135425e−02
4	0.30616615e−02	0.28443301e−03	0.62491114e−04	0.29159128e−03
5	0.22815333e−03	0.30485943e−04	0.71788910e−05	0.98738609e−05
6	0.34864807e−04	0.33492602e−05	0.81036081e−06	0.10174882e−07
7	0.36051726e−05	0.32389936e−05	0.86647503e−09	0.80064485e−09

with the kernel function $k(t, s) = e^{-2s} \sin(t)$, $\psi(s, x(s)) = [x(s)]^2$ and the function $f(t) = t^2$ where the exact solution is given by $x(t) = t^2 + 1.95778sin(t)$.

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