CrossMark

A Sparse Grid Stochastic Collocation Method for Elliptic Interface Problems with Random Input

Qian Zhang[1] · Zhilin Li[1,2] · Zhiyue Zhang[1]

Abstract In this paper, numerical solutions of elliptic partial differential equations with both random input and interfaces are considered. The random coefficients are piecewise smooth in the physical space and moderately depend on a large number of random variables in the probability space. To relieve the *curse of dimensionality*, a sparse grid collocation algorithm based on the Smolyak construction is used. The numerical method consists of an immersed finite element discretization in the physical space and a Smolyak construction of the extreme of Chebyshev polynomials in the probability space, which leads to the solution of uncoupled deterministic problems as in the Monte Carlo method. Numerical experiments on two-dimensional domains are also presented. Convergence is verified and compared with the Monte Carlo simulations.

Keywords Sparse grids · Stochastic inputs · Interface · Immersed finite element · Smolyak construction

1 Introduction

In many applications, a simulation domain is often composed of several materials separated by curves or surfaces from each other, and this often leads to a so-called interface problem consisting of a boundary value problem of a partial differential equation coupled by jump conditions across the material interface required by pertinent physics. Elliptic interface problems have a variety of applications in many scientific and engineering disciplines, including fluid dynamics, materials science, and biological systems. An efficient solver for this type of interface problem is critical for numerical simulations. The immersed finite element method

✉ Zhiyue Zhang
zhangzhiyue@njnu.edu.cn

1   School of Mathematical Sciences, Jiangsu Key Laboratory for NSLSCS, Nanjing Normal University, Nanjing 210023, China

2   Center for Research in Scientific Computation and Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA

(IFEM) [17–20] has been proposed as a good choice. The IFEM is based on uniform meshes which have many advantages over the usual fitted meshes. The degree of freedom remains the same as the standard linear finite element method. When the discontinuity disappears, the IFEM becomes the standard FEM. The optimal accuracy is obtained by modifying the basis functions on interface elements. Moreover, one does not need to generate a new grid at each time step when a moving interface is involved [14].

On the other hand, in many cases, the information available for a given problem is far from complete and is in general very limited. This leads to the use of stochastic partial differential equations (SPDEs), whose input data, such as model coefficients, forcing terms, boundary conditions, or geometry that are known to some accuracy. There has been growing interest in designing efficient numerical methods for the solution of SPDEs. The methods for solving such problems include Monte Carlo sampling based methods [7,11,22,27], stochastic Galerkin methods [1,3,13,26,34,45], and stochastic collocation methods [2,4,29,35,38].

In real applications, one may encounter both of the two difficulties mentioned above, i.e., an interface problem together with randomness. However, to our knowledge, few papers in the literature deal with interface problems with random input on uniform meshes. In recent years it has become more and more important to model and simulate interface problems with stochastic input parameters. One simple approach to solve interface problems with stochastic input is the Monte Carlo method which has a rate of convergence that may be considered slow. In this paper, we aim at a direct and more effective method. We especially address the situation where the probability space has a moderate dimensionality, which means that the stochastic problem depends on a moderately large number of random variables. One intrusive approach, such as the gPC Galerkin method [37,39,40], is usually hard to implement, because it requires the modification of the deterministic code and this may be difficult, expensive, and time consuming for many complex computational problems. The stochastic collocation method as a non-intrusive method seems to be an ideal approach for computing the numerical solutions of the interface problems with random input discussed in this paper. Stochastic collocation method will not be affected by the complexity of the numerical schemes for the governing differential equations and only needs the execution of the deterministic algorithm as in the Monte Carlo method. In addition, the stochastic collocation method essentially preserves the fast convergence of the corresponding Galerkin method. For these reasons, stochastic collocation methods have become popular, and attracted much attention recently [4,15,24,25,28].

If the number of random variables is moderately large, one should rather consider sparse tensor product spaces as first proposed by Smolyak [33]. We present an isotropic sparse collocation algorithm based on the Smolyak construction for the elliptic interface problem whose discontinuous coefficients depend on a moderate number of random variables. A Galerkin method using bi-orthogonal polynomials is used to solve the stochastic elliptic interface problem in [43]. The proposed method which is equal to the full tensor product method decouples the equation and yields a number of uncoupled systems which motives us to adopt the so-called *sparse grid* collocation method [6] based on the Smolyak construction. There is a clear advantage of the isotropic Smolyak method with respect to the full tensor, and this justifies our claim that the use of the Smolyak approximation can reduce the curse of dimensionality.

This work claims that the sparse grid methods, which are "non-intrusive", are efficient for the elliptic interface problems with random input data. Due to the easy implementation, the "non-intrusive" methods have attracted much attention. The developments of them become active and produce many different techniques. There are recently many alternative "non-intrusive" approaches, for example, least-squares projection on polynomial spaces

[9,23,36,44], compressed sensing approach [10,32,41,42] and the least interpolation method [25].

The remainder of the paper is organized as follows. In Sect. 2 we formulate the mathematical problem and introduce some definitions and assumptions. We describe our numerical methods and error estimation in Sect. 3. In Sect. 4, we present several numerical examples. We conclude in the last section.

2 The problem description

Let $D \subset \mathbb{R}^2$ be a convex bounded polygonal domain which is separated into two subdomains D^+ and D^- by a smooth interface Γ, and $(\Omega, \mathcal{F}, P)$ be a complete probability space. Here Ω is the set of outcomes, $\mathcal{F} \in 2^{\Omega}$ is the σ-algebra of events, and $P : \mathcal{F} \to [0, 1]$ is a probability measure. Consider the stochastic elliptic interface problem: Find a random function, $u : \Omega \times D \to \mathbb{R}$, such that P-almost everywhere in Ω, or in other words, almost surely, the following equation holds:

$$-\nabla \cdot (\beta(\omega, \cdot)\nabla u(\omega, \cdot)) = f(\omega, \cdot) \qquad \text{in } D^+ \cup D^-, \qquad (2.1)$$

$$[u(\omega, \cdot)]_{\Gamma} = 0 \quad \left[\beta(\omega, \cdot)\frac{\partial u(\omega, \cdot)}{\partial \mathbf{n}}\right]_{\Gamma} = 0 \qquad \text{on } \Gamma, \qquad (2.2)$$

$$u = 0 \qquad \text{on } \partial D. \qquad (2.3)$$

Here $\mathbf{n}$ is the unit outward normal direction pointing to the domain D^+. The symbol $[\cdot]_{\Gamma}$ means the jump across the interface Γ, i.e., the value on D^+ minus the value on D^-. The coefficient $\beta(\omega, \mathbf{x}) : \Omega \times D \to \mathbb{R}$ is a piecewise random function, that is,

$$\beta(\omega, \mathbf{x}) = \begin{cases} \beta^-(\omega, \mathbf{x}) & \mathbf{x} \in D^-, \\ \beta^+(\omega, \mathbf{x}) & \mathbf{x} \in D^+. \end{cases} \qquad (2.4)$$

We assume that $\beta^-(\omega, \cdot)$ and $\beta^+(\omega, \cdot)$ are smooth enough in each sub-domain for any realization $\omega \in \Omega$. In addition, we shall make the following assumptions of the data:

1. The coefficient $\beta(\omega, \cdot)$ is uniformly bounded and coercive, i.e., there exist $\beta_{min}, \beta_{max} \in (0, \infty)$ such that $P\{\omega \in \Omega : \beta(\omega, \mathbf{x}) \in [\beta_{min}, \beta_{max}], \forall \mathbf{x} \in D\} = 1$;
2. For any $\omega \in \Omega$, the function $f(\omega, \cdot)$ belongs to the space $L^2(D)$ and is square integrable with respect to P in the sense of

$$\int_{\Omega} \|f(\omega, \cdot)\|_{L^2(D)}^2 dP(\omega) < \infty.$$

Next, we introduce some Hilbert spaces:

- $L_P^2(\Omega) \otimes H_0^1(D)$, equipped with the norm $\|v\|_{L_P^2(\Omega)\otimes H_0^1(D)}^2 = \int_D \mathbb{E}[|\nabla v|^2]d\mathbf{x}$, where $\mathbb{E}[\cdot]$ stands for the expectation.
- $\widetilde{H}^2(D) = \{v \in H_0^1(D) : v \in H^2(D^s), s = +, -\}$, equipped with the norm

$$\|v\|_{\widetilde{H}^2(D)}^2 = \|v\|_{H^2(D^+)}^2 + \|v\|_{H^2(D^-)}^2.$$

Multiplying the Eq. (2.1) by a function $v \in L_P^2(\Omega) \otimes H_0^1(D)$, integrating by parts and using the interface conditions (2.2), we obtain the weak form of the problem (2.1)–(2.3): Find $u \in L_P^2(\Omega) \otimes H_0^1(D)$ such that

$$\int_D \mathbb{E}[\beta \nabla u \cdot \nabla v] d\mathbf{x} = \int_D \mathbb{E}[fv] d\mathbf{x} \quad \text{for all } v \in L_P^2(\Omega) \otimes H_0^1(D). \tag{2.5}$$

A straightforward application of the Lax–Milgram theorem together with the assumptions made above allows one to state the well-posedness of the problem (2.5).

Moreover, we have the following regularity result for the problem with respect to $\mathbf{x}$ (see [5]). The solution to (2.5) has realizations in the space $\widetilde{H}^2(D)$, i.e., for any $\omega \in \Omega$, $u(\omega, \cdot) \in \widetilde{H}^2(D)$ and $\|u(\omega, \cdot)\|_{\widetilde{H}^2(D)} \le C\|f(\omega, \cdot)\|_{L^2(D)}$.

2.1 Finite-dimensional noise assumption

In some problems the source of randomness can be characterized exactly by a finite number of independent random variables. However, in many applications, the random inputs are stochastic processes which are infinite-dimensional objects. For example, the coefficient $\beta^{\pm}(\omega, \mathbf{x})$ often represents an uncertain material property, e.g., conductivity, that is a stochastic process in space. So, to solve the problem (2.5) numerically, the first step is to reduce the infinite-dimensional probability space to a finite-dimensional space. The Karhunen–Loève expansion [12,13,31] is one of the most widely used techniques for dimension reduction.

Let the mean and the covariance of $\beta^{\pm}(\omega, \mathbf{x})$ be defined as

$$\beta_0^{\pm}(\mathbf{x}) = \int_{\Omega} \beta^{\pm}(\omega, \mathbf{x}) dP, \quad \forall \mathbf{x} \in D^{\pm}$$

and

$$Cov_{\beta^{\pm}}(\mathbf{x}, \mathbf{x}_1) = \int_{\Omega} \left(\beta^{\pm}(\omega, \mathbf{x}) - \beta_0^{\pm}(\mathbf{x}) \right) \left(\beta^{\pm}(\omega, \mathbf{x}_1) - \beta_0^{\pm}(\mathbf{x}_1) \right) dP,$$

respectively. Then the Karhunen–Loève (KL) expansion of $\beta^{\pm}(\omega, \mathbf{x})$ is

$$\beta^{\pm}(\omega, \mathbf{x}) = \beta_0^{\pm}(\mathbf{x}) + \sum_{n=1}^{\infty} \sqrt{\lambda_n^{\pm}} \beta_n^{\pm}(\mathbf{x}) y_n^{\pm}(\omega),$$

where $\beta_n^{\pm}$ are the orthogonal and normalized eigenfunctions and $\lambda_n^{\pm}$ are the corresponding eigenvalues of the following eigenvalue problem

$$\int_{D^{\pm}} Cov_{\beta^{\pm}}(\mathbf{x}, \mathbf{x}_1) \beta_n^{\pm}(\mathbf{x}_1) d\mathbf{x}_1 = \lambda_n^{\pm} \beta_n^{\pm}(\mathbf{x}).$$

The random variables $y_n^{\pm}(\omega)$ are mutually uncorrelated with zero mean value and unit variance and are defined by

$$y_n^{\pm}(\omega) = \frac{1}{\sqrt{\lambda_n^{\pm}}} \int_{D^{\pm}} \beta^{\pm}(\omega, \mathbf{x} - \beta_0^{\pm}(\mathbf{x}) \beta_n^{\pm}(\mathbf{x})) d\mathbf{x}, \quad n = 1, 2, \cdots.$$

It is shown in [13] that the Karhunen–Loève expansion is optimal among all possible representations of random processes in the sense of the mean-square error. The truncated Karhunen–Loève expansion reads

$$\beta_{N^{\pm}}^{\pm}(\omega, \mathbf{x}) = \beta_0^{\pm}(\mathbf{x}) + \sum_{n=1}^{N^{\pm}} \sqrt{\lambda_n^{\pm}} \beta_n^{\pm}(\mathbf{x}) y_n^{\pm}(\omega). \tag{2.6}$$

In practice, one has to choose the number of the truncated terms $N^{\pm}$ properly so that the influence of truncation error is relatively small and can be omitted.

In summary, the above descriptions motivate us to make the following assumption.

Assumption 2.1 (*Finite-dimensional noise*) The coefficients in the original equation have the form

$$\beta^{\pm}(\omega, \mathbf{x}) = \beta^{\pm}\left(y_1^{\pm}(\omega), y_2^{\pm}(\omega), \ldots, y_{N^{\pm}}^{\pm}(\omega), \mathbf{x}\right)$$

$$= \beta_0^{\pm}(\mathbf{x}) + \sum_{n=1}^{N^{\pm}} \sqrt{\lambda_n^{\pm}} \beta_n^{\pm}(\mathbf{x}) y_n^{\pm}(\omega) \text{ in } \Omega \times D^{\pm},$$

where $N^{\pm}$ are positive integers, $\{y_n^{\pm}\}_{n=1}^{N^{\pm}}$ are real-valued and independent random variables with mean value zero and unit variance. The function f has a similar form $f(\omega, \mathbf{x}) = f(y_1^f(\omega), \ldots, y_{Nf}^f(\omega), \mathbf{x})$. Define $\mathbf{y} = (y_1, \ldots, y_N) = (y_1^+, \ldots, y_{N^+}^+, y_1^-, \ldots, y_{N^-}^-, y_1^f, \ldots, y_{Nf}^f)$ with $N = N^+ + N^- + N^f$, we can rewrite $\beta^{\pm}(\omega, \mathbf{x}) = \beta^{\pm}(\mathbf{y}, \mathbf{x})$ and $f(\omega, \mathbf{x}) = f(\mathbf{y}, \mathbf{x})$.

For $n = 1, \ldots, N$, we assume that the image of y_n, i.e., the set $\Theta_n \equiv y_n(\Omega)$, is a bounded interval in $\mathbb{R}$, and the random variable y_n has a known density function $\rho_n : \Theta_n \to \mathbb{R}^+$. Since the random variables y_n are independent, the joint probability density of $\mathbf{y} = (y_1, \ldots, y_N)$ is $\rho(\mathbf{y}) = \prod_{n=1}^{N} \rho_n(y_n) : \Theta \to \mathbb{R}^+$, where $\Theta \equiv \prod_{n=1}^{N} \Theta_n \subset \mathbb{R}^N$.

Under the finite-dimensional assumption, the stochastic problem (2.1)–(2.3) now becomes a deterministic elliptic interface problem with N-dimensional parameter, i.e., find $u(\mathbf{y}, \mathbf{x}) : \Theta \times D \to \mathbb{R}$, for all $\mathbf{y} \in \Theta$, the following holds

$$\begin{cases} -\nabla \cdot (\beta(\mathbf{y}, \mathbf{x}) \nabla u(\mathbf{y}, \mathbf{x})) = f(\mathbf{y}, \mathbf{x}) & \mathbf{x} \in D^+ \cup D^-, \\ [u(\mathbf{y}, \mathbf{x})]_{\Gamma} = 0 \quad \left[\beta \dfrac{\partial u(\mathbf{y}, \mathbf{x})}{\partial \mathbf{n}}\right]_{\Gamma} = 0 & \mathbf{x} \in \Gamma, \\ u(\mathbf{y}, \mathbf{x}) = 0 & \mathbf{x} \in \partial D. \end{cases} \quad (2.7)$$

Note that here and later in this paper the gradient notation, ∇, means differentiation with respect to $\mathbf{x} \in D$. The stochastic variational formulation (2.5) has a deterministic equivalent: Find $u(\mathbf{y}, \mathbf{x}) \in L_{\rho}^2(\Theta) \otimes H_0^1(D)$ such that

$$\int_{\Theta \times D} \rho(\mathbf{y}) \beta(\mathbf{y}, \mathbf{x}) \nabla u(\mathbf{y}, \mathbf{x}) \cdot \nabla v(\mathbf{y}, \mathbf{x}) d\mathbf{y} d\mathbf{x}$$

$$= \int_{\Theta \times D} \rho(\mathbf{y}) f(\mathbf{y}, \mathbf{x}) v(\mathbf{y}, \mathbf{x}) d\mathbf{y} d\mathbf{x}, \quad \forall v \in L_{\rho}^2(\Theta) \otimes H_0^1(D). \quad (2.8)$$

3 Stochastic collocation methods

In this section, we introduce the stochastic collocation method for computing the statistical moments of the solution u to the problem (2.7). In the stochastic collocation method, the numerical solution is sought in a finite-dimensional space $\mathcal{P}_{\mathbf{p}}(\Theta) \otimes W_h(D)$ which can be regarded as an approximation of $L_{\rho}^2(\Theta) \otimes H_0^1(D)$.

Here $W_h(D)$ is a finite element space, which contains piecewise polynomials defined on the triangulation $\mathcal{T}_h$ that has a maximum mesh-spacing parameter $h > 0$. For elliptic problems with smooth coefficients, the common choice is the standard linear finite element space. While for the interface problems the standard linear finite element can not achieve optimal accuracy, unless the triangulation is aligned with the interface. So we choose the immersed finite element space $S_h(D)$ which is discussed in Sect. 3.2 together with the immersed finite

element method. The semidiscrete approximation $u_h(\mathbf{y}, \cdot) : \Theta \to S_h(D)$ is obtained by the immersed finite element method.

In random spaces, $\mathcal{P}_{\mathbf{p}}(\Theta) \subset L^2_\rho(\Theta)$ is the span of tensor product polynomials with degree at most $\mathbf{p} = (p_1, \ldots, p_N)$, i.e., $\mathcal{P}_{\mathbf{p}}(\Theta) = \otimes_{n=1}^N \mathcal{P}_{p_n}(\Theta_n)$, with $\mathcal{P}_{p_n}(\Theta_n) = \text{span}\{y_n^k, k = 0, \ldots, p_n\}$, $n = 1, \ldots, N$. Hence the dimension of $\mathcal{P}_{\mathbf{p}}(\Theta)$ is $N_p = \prod_{n=1}^N (p_n + 1)$.

In the stochastic collocation method, we first evaluate approximation functions $u_h(\mathbf{y}_k, \cdot) \in S_h(D)$ to the solution of (2.7) on a suitable set of points $\mathbf{y}_k \in \Theta$ using the immersed finite element method (see Sect. 3.2). Then the fully discrete solution $u_{h,\mathbf{p}} \in C^0(\Theta; S_h(D))$ is a polynomial interpolation in the random space, i.e.,

$$u_{h,\mathbf{p}}(\mathbf{y}, \mathbf{x}) = \sum_k u_h(\mathbf{y}_k, \mathbf{x}) l_k^{\mathbf{p}}(\mathbf{y}), \tag{3.1}$$

where, for instance, the functions $l_k^{\mathbf{p}}$ can be taken as the Lagrange polynomials. Then the approximation of the expected value of u to the stochastic equation (2.1)–(2.3) can be evaluated as

$$\mathbf{E}[u] \approx \mathbf{E}[u_{h,\mathbf{p}}] = \sum_k u_h(\mathbf{y}_k, \mathbf{x}) \int_\Theta \rho(\mathbf{y}) l_k^{\mathbf{p}}(\mathbf{y}) d\mathbf{y}.$$

3.1 Smolyak approximation

First we assume $N = 1$, and let $\{y_1^i, \ldots, y_{m_i}^i\} \subset \Theta$ be a sequence of abscissas for Lagrange interpolation. Here the integer i means the level of approximation and m_i is the number of interpolation points used at level i. Then, the one-dimensional Lagrange interpolation is

$$\mathcal{U}^i(u) = \sum_{k=1}^{m_i} u(y_k^i) l_k^i,$$

where $l_k^i \in \mathcal{P}_{m_i-1}(\Theta)$ are the Lagrange polynomials of degree $m_i - 1$, i.e., $l_k^i(y) = \prod_{k=1, k \neq j}^{m_i} \frac{(y - y_k^i)}{(y_j^i - y_k^i)}$. In the multi-dimensional case, i.e., $N > 1$, the Lagrange interpolation based on the full tensor product is defined by

$$\mathcal{I}_{\mathbf{i}}^N(u)(\mathbf{y}) = (\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_N})(u)(\mathbf{y}) = \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_N=1}^{m_{i_N}} u(y_{j_1}^{i_1}, \ldots, y_{j_N}^{i_N})(l_{j_1}^{i_1} \otimes \cdots \otimes l_{j_N}^{i_N}). \tag{3.2}$$

Obviously, the number of the total interpolation points needed in the full tensor interpolation is $\eta = \prod_{n=1}^N m_{i_n}$. If we use the same points in each direction, i.e., $m_i = m$, $i = 1, \ldots, N$, then $\eta = m^N$. The exponential growth of interpolation nodes with respect to the dimension of random space is a drawback which is often called the *curse of dimensionality*. The sparse grid method proposed in [2,38], which greatly reduces the curse of dimensionality, is a good choice for solving the stochastic problem.

Now we briefly describe the isotropic Smolyak formulation which is a linear combination of low order tensor product formula (3.2). The Smolyak formula is then given by (see [29])

$$\mathcal{A}(w, N) = \sum_{w+1 \leq |\mathbf{i}| \leq w+N} (-1)^{w+N-|\mathbf{i}|} \binom{N-1}{w+N-|\mathbf{i}|} (\mathcal{U}^{i_1} \otimes \cdots \otimes \mathcal{U}^{i_N}), \tag{3.3}$$
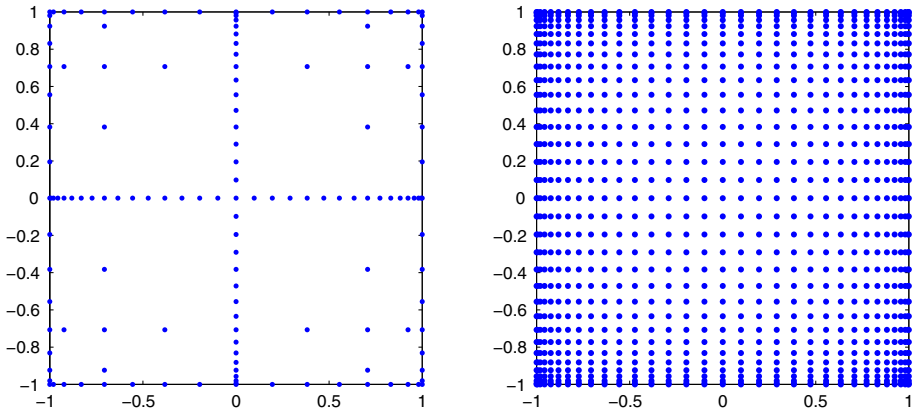
Fig. 1 Two-dimensional (N=2) interpolation nodes based on the extreme of Chebyshev polynomials (3.4). *Left* sparse grids $\mathcal{H}(w, N)$ with $w = 5$. Total number of points is 145. *Right* the tensor product grids based on the same one-dimensional nodes. Total number of nodes is 1089

where $\mathbf{i} \in \mathbb{N}_+^N$ and $|\mathbf{i}| = i_1 + \cdots + i_N$. The set of the sparse grids needed to compute $\mathcal{A}(w, N)(u)$ is

$$\mathcal{H}(w, N) = \sum_{w+1 \leq |\mathbf{i}| \leq w+N} (\vartheta^{i_1} \times \cdots \times \vartheta^{i_N}),$$

where $\vartheta^i = \{y_1^i, \ldots, y_{m_i}^i\}$ is the set of abscissas used by $\mathcal{U}^i$. In this paper we choose to use Clenshaw–Curtis abscissas which are the extreme of Chebyshev polynomials, that is, for any choice of $m_i > 1$,

$$y_j^i = -\cos\frac{\pi(j-1)}{m_i - 1}, \quad j = 1, \ldots, m_i. \tag{3.4}$$

In addition, we define $y_1^i = 0$ if $m_i = 1$, and choose $m_1 = 1$ and $m_i = 2^{i-1} + 1$ for $i > 1$.

Note that Clenshaw–Curtis abscissas are nested, so that the Smolyak approximation requires fewer interpolation points than the corresponding formula with non-nested points, for example, Gaussian abscissas. In Fig. 1, we show the sparse grids in $\mathcal{H}(w, N)$ with $w = 5$ and $N = 2$. For comparison, the full tensor product grids based on the same one-dimensional nodes are shown on the right of Fig. 1, and we observe that the sparse grid has significantly fewer nodes.

3.2 Immersed finite element methods

In this section we give a brief review of the immersed finite element space introduced in [20]. Since the triangulation $\mathcal{T}_h$ is independent of the interface Γ, there exist some elements that the interface passes through. We call these elements interface elements and the rest non-interface elements. The set of interface elements and non-interface elements are denoted by $\mathcal{T}_h^{int}$ and $\mathcal{T}_h^{non}$, respectively. For complicated interfaces, it may be necessary to refine the mesh to resolve the interface.

On non-interface elements, we use the standard linear basis functions. On interface elements, the linear functions are broken along the interface to satisfy the jump conditions across the interface in some sense. We take a typical interface element $\triangle ABC$ whose geometric configuration is given in Fig. 2 as a demonstration. The line segment $\overline{SE}$ divides T into two

Fig. 2 A typical interface element and a neighboring element
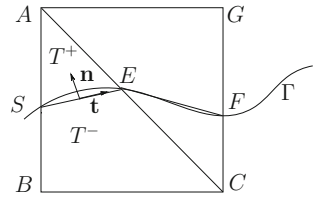
parts T^+ and T^-. Let **n** and **t** be the unit normal and tangential directions of the line segment $\overline{SE}$, respectively. We construct the following piecewise linear function on this element,

$$\phi(\mathbf{x}) = \begin{cases} \phi^+ = a^+ + b^+ x_1 + c^+ x_2, & \mathbf{x} = (x_1, x_2) \in T^+, \\ \phi^- = a^- + b^- x_1 + c^- x_2, & \mathbf{x} = (x_1, x_2) \in T^-. \end{cases} \quad (3.5)$$

The coefficients are chosen such that

$$\phi(A) = V_1, \ \phi(B) = V_2, \ \phi(C) = V_3, \quad (3.6)$$

$$\phi^+(S) = \phi^-(S), \ \phi^+(E) = \phi^-(E), \ \beta^+ \frac{\partial \phi^+}{\partial \mathbf{n}} = \beta^- \frac{\partial \phi^-}{\partial \mathbf{n}}, \quad (3.7)$$

where V_i, $i = 1, 2, 3$ are the nodal variables. Intuitively, there are six unknowns in (3.5) and six restrictions in (3.6)–(3.7). The piecewise linear function is uniquely determined by V_i, $i = 1, 2, 3$ (see [20]). Now we define the immersed finite element space $S_h(D)$ as the set of all piecewise linear functions that satisfy

$$\begin{cases} \phi|_T \text{ is the linear function if } T \in \mathcal{T}_h^{non}, \\ \phi|_T \text{ is the piecewise linear function defined in (3.5)–(3.7) if } T \in \mathcal{T}_h^{int}, \\ \phi \text{ is continuous at all nodal points}, \\ \phi(\mathbf{x}_b) = 0 \text{ if } \mathbf{x}_b \text{ is a nodal point on } \partial D. \end{cases}$$

The immersed finite element space S_h is a modification to the standard piecewise linear conforming finite element space when the coefficient β is discontinuous. The two spaces are the same when $\beta^+ = \beta^-$. For any $\mathbf{y} \in \Theta$, the immersed finite element approximation of (2.7) reads: Find $u_h(\mathbf{y}, \cdot) \in S_h(D)$ such that

$$a_h(u_h(\mathbf{y}, \mathbf{x}), v(\mathbf{x})) = \int_D f(\mathbf{y}, \mathbf{x}) v(\mathbf{x}) d\mathbf{x}, \quad \forall v(\mathbf{x}) \in S_h(D), \quad (3.8)$$

where the bilinear form $a_h(\cdot, \cdot)$ is defined by

$$a_h(w, v) = \sum_{T \in \mathcal{T}_h} \int_T \beta \nabla w \cdot \nabla v d\mathbf{x}, \quad \forall w, v \in S_h(D). \quad (3.9)$$

It has been proven in [8, Theorem 5.1] that the immersed finite element method has the optimal convergence order in L^2-norm, that is,

$$\|u(\mathbf{y}, \cdot) - u_h(\mathbf{y}, \cdot)\|_{L^2(D)} \leq Ch^2 \|u(\mathbf{y}, \cdot)\|_{\widetilde{H}^2(D)}, \quad \forall \mathbf{y} \in \Theta. \quad (3.10)$$

We note that $v_h(x)$ in S_h is likely discontinuous across the adjacent edges of two interface elements. That is why S_h is called a non-conforming finite element space in [20]. We can see this from the diagram in Fig. 2. If $\phi(A)$, $\phi(B)$, $\phi(C)$ and $\phi(G)$ are given, we can obtain $\phi|_{\triangle ABC}$ and $\phi|_{\triangle ACG}$ independently so that $\phi|_{\triangle ACG}(E) \neq \phi_{\triangle ACG}(E)$. The set of these edges where the function of $S_h(D)$ is discontinuous is denoted by $\mathcal{E}_h^{int}$. To cancel the non-conforming error caused by the discontinuity on $\mathcal{E}_h^{int}$, two correction terms are added to the

bilinear form and the resulting method is outperform than original immersed finite element method (see [16,21]).

3.3 Error analysis

We recall that u is the solution of the original stochastic problem (2.1)–(2.3), u_h is the semi-discrete approximation obtained by the IFEM and $\mathcal{A}(w, N)u_h$ is the fully discrete numerical solution. The error to be considered can be split as

$$
\|u - \mathcal{A}(w, N)u_h\|_{L^2_\rho(\Theta)\otimes L^2(D)} \leq \|u - u_h\|_{L^2_\rho(\Theta)\otimes L^2(D)}
$$
$$
+ \|u_h - \mathcal{A}(w, N)u_h\|_{L^2_\rho(\Theta)\otimes L^2(D)}. \tag{3.11}
$$

The first term is nothing but the approximation error in physical spaces, i.e., the error of the IFEM. By (3.10), we have

$$
\|u - u_h\|_{L^2_\rho(\Theta)\otimes L^2(D)} = \left(\int_\Theta \int_D \rho |u(\mathbf{y}, \mathbf{x}) - u_h(\mathbf{y}, \mathbf{x})|^2 d\mathbf{x} d\mathbf{y} \right)^{1/2}
$$
$$
\leq Ch^2 \|u\|_{L^2_\rho(\Theta)\otimes \widetilde{H}^2(D)}. \tag{3.12}
$$

The second term is the Smolyak approximation error. To estimate the approximation error, we first give the following lemma [29, Theorem 3.10].

Lemma 3.1 *Let $\Theta^* = \prod_{j=1, j\neq n}^N \Theta_j$ and y_n^* be an arbitrary element of Θ^*. For each $y_n \in \Theta_n$, assume that there exists τ_n such that $u(y_n, y_n^*, \mathbf{x})$ as a function of y_n admits an analytic extension $u(z, y_n^*, \mathbf{x})$, $z \in \mathbb{C}$, in the region of the complex plane*

$$
\sigma(\Theta_n; \tau_n) = \{z \in \mathbb{C}, dist(z, \Theta_n) \leq \tau_n\}. \tag{3.13}
$$

Also define the parameter

$$
\sigma = \frac{1}{2} \min_{n=1,\dots,N} \log \left(\frac{2\tau_n}{|\Theta_n|} + \sqrt{1 + \frac{4\tau_n^2}{|\Theta_n|^2}} \right). \tag{3.14}
$$

Then the isotropic Smolyak formula (3.3) based on Clenshaw–Curtis abscissas satisfies

$$
\|u - \mathcal{A}(w, N)(u)\|_{L^\infty(\Theta^N; W_h(D))} \leq C(\sigma, N)\eta^{-\mu_1} \text{ with } \mu_1 = \frac{\sigma}{1 + \log(2N)}, \quad (3.15)
$$

where $\eta = |\mathcal{H}(w, N)|$ is the number of collocation points, and the constant $C(\sigma, N)$ only depends on σ and N.

This lemma indicates that under some regularity assumptions the Smolyak formula based on Clenshaw–Curtis abscissas has at least algebraic convergence with respect to the number of collocation points.

However u_h is required to satisfy the regularity assumption made in the above lemma. It has been proved in [2, Lemma 3.2] that the problem satisfies the regularity assumption with $0 < \tau_n < 1/(2/\gamma_n)$ if the following holds:

$$
\left\| \frac{\partial_{y_n}^k \beta(\mathbf{y}, \cdot)}{\beta(\mathbf{y}, \cdot)} \right\|_{L^\infty(D)} \leq \gamma_n^k k! \quad \frac{\|\partial_{y_n}^k f(\mathbf{y}, \cdot)\|_{L^2(D)}}{1 + \|f(\mathbf{y}, \cdot)\|_{L^2(D)}} \leq \gamma_n^k k!. \tag{3.16}
$$

Under the assumption 2.1, we have

$$\frac{\partial_{y_n}^k \beta^+(\mathbf{y}, \mathbf{x})}{\beta(\mathbf{y}, \mathbf{x})} \leq \begin{cases} \sqrt{\lambda_n} \beta_n^+(\mathbf{x})/\beta_{min} & \text{if } k = 1 \\ 0 & \text{if } k > 1 \text{ or } n > N^+ \end{cases} \tag{3.17}$$

and similar results for β^- and f. Thus (3.16) is satisfied if we take $\gamma_n = \sqrt{\lambda_n} \|\beta_n^+\|_{L^\infty(D^+)}$ $/\beta_{min}$ for $n = 1, \ldots, N^+$. For the case of $n > N^+$, the constant γ_n can be chosen similarly. Note that the regularity results are valid also for the semidiscrete solution u_h.

Using Lemma 3.1, the second term now can be estimated as

$$\|u_h - \mathcal{A}(w, N)u_h\|_{L_\rho^2(\Theta) \otimes L^2(D)} \leq C\|u_h - \mathcal{A}(w, N)u_h\|_{L_\rho^\infty(\Theta) \otimes L^2(D)} \leq C(\sigma, N)\eta^{-\mu_1}. \tag{3.18}$$

Substituting (3.12) and (3.18) into (3.11), we therefore get the following convergence result.

Theorem 3.2 *Under the assumption 2.1, it holds that*

$$\|u - \mathcal{A}(w, N)u_h\|_{L_\rho^2(\Theta) \otimes L^2(D)} \leq Ch^2\|u\|_{L_\rho^2(\Theta) \otimes \widetilde{H}^2(D)} + C(\sigma, N)\eta^{-\sigma/(1+\log(N))}, \tag{3.19}$$

where σ is defined in (3.14) and the constants C and $C(\sigma, N)$ are independent of h and η.

Using the theorem, the error in the expected value of u is easily estimated, i.e.,

$$\|\mathbb{E}[u] - \mathbb{E}[\mathcal{A}(w, N)u_h]\|_{L^2(D)} \leq \|u - \mathcal{A}(w, N)u_h\|_{L_\rho^2(\Theta) \otimes L^2(D)}$$
$$\leq Ch^2\|u\|_{L_\rho^2(\Theta) \otimes \widetilde{H}^2(D)} + C(\sigma, N)\eta^{-\sigma/(1+\log(N))}. \tag{3.20}$$

4 Numerical examples

In this section, we present some numerical examples to show the performance of the sparse grid stochastic collocation method. A comparison of the efficiency of the proposed sparse grid stochastic collocation method with both the Monte Carlo method and the sparse grid stochastic collocation method based on standard linear finite element methods will be given. For simplicity, the problems are defined in the rectangular domain $D = [-1, 1] \times [-1, 1]$ which is partitioned into $2N_h^2$ right triangles with mesh size h. We consider a deterministic right-hand function f and construct the random coefficient as

$$\beta(\mathbf{y}, \mathbf{x}) = \begin{cases} \beta^-(\mathbf{y}, \mathbf{x}) = \beta_0^-(\mathbf{x})\left(1 + 0.5 * \sum_{n=1}^{M} \frac{1}{n^2} y_n\right), & \mathbf{x} \in D^-, \\ \beta^+(\mathbf{y}, \mathbf{x}) = \beta_0^+(\mathbf{x})\left(1 + 0.5 * \sum_{n=M+1}^{2M} \frac{1}{n^2} y_n\right), & \mathbf{x} \in D^+, \end{cases} \tag{4.1}$$
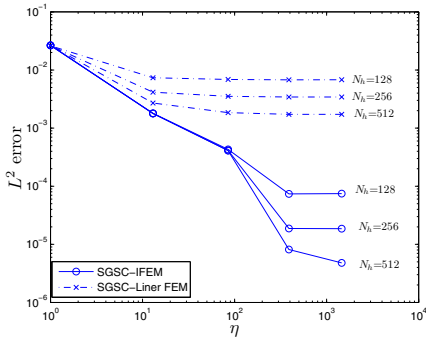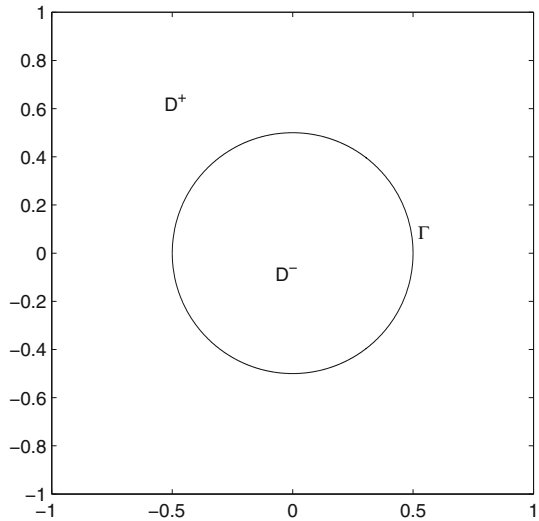
with $N = 2M$ the dimension of random space, and $y_n \in [-1, 1]$, $n = 1, \ldots, N$, are independent uniformly distributed random variables.

In all examples, we compute the $L^2(D)$ error to the expected value, i.e.,
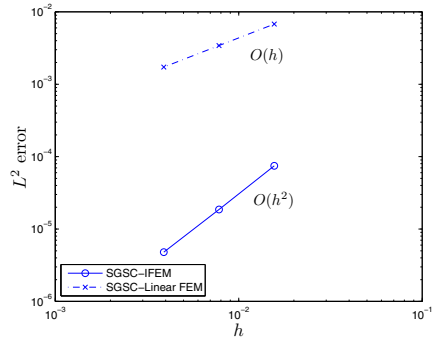
$$\text{Error} = \|\mathbb{E}[u] - \mathbb{E}[\mathcal{A}(w, N)u_h]\|_{L^2(D)}, \tag{4.2}$$

where the expected value of exact solution is approximated as $\mathbb{E}[u] \approx \mathbb{E}[\mathcal{A}(\widetilde{w}, N)u_h]$ with a larger $\widetilde{w}$.

Fig. 3 The domain and the interface of Example 1

(a) L^2 error versus the number of collocation points η

(b) L^2 error versus the mesh-spacing parameter h with $w = 4$

Fig. 4 A comparison between the sparse grid stochastic collocation (SGSC) method based on the IFEM and the standard linear FEM for solving Example 1 with $N=6$. **a** L^2 error versus the number of collocation points η. **b** L^2 error versus the mesh-spacing parameter h with $w = 4$

Example 1 The interface Γ is a circle centered at the origin with radius $r_0 = 0.5$, as shown in Fig. 3. The true solution is

$$
u = \begin{cases} \dfrac{r^3}{\beta^-} & \text{in } D^-, \\[3mm] \dfrac{r^3}{\beta^+} + \left(\dfrac{1}{\beta^-} - \dfrac{1}{\beta^+}\right) r_0^3 & \text{in } D^+. \end{cases} \tag{4.3}
$$

where $r = \sqrt{x_1^2 + x_2^2}$. In this example, we choose $\beta_0^+ = 100$ and $\beta_0^- = 1$ in (4.1).

First a low-dimensional random inputs with $N = 6$ is tested. We use the proposed sparse grid stochastic collocation method based on immersed finite element method (SGSC-IFEM) and linear finite element method (SGSC-Linear FEM) respectively. Figure 4 shows the L^2
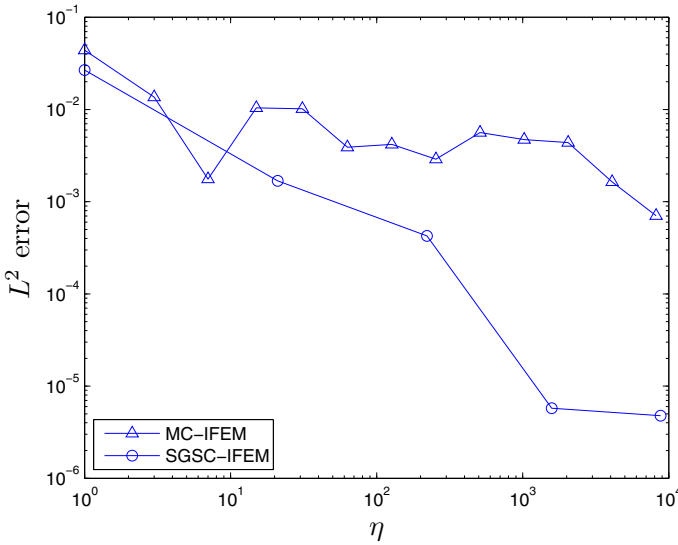
Fig. 5 A comparison between the sparse grid stochastic collocation method and the Monte Carlo approach for solving Example 1 with $N=10$ and $N_h = 512$

Fig. 6 The exact expectation of Example 1 with N=10
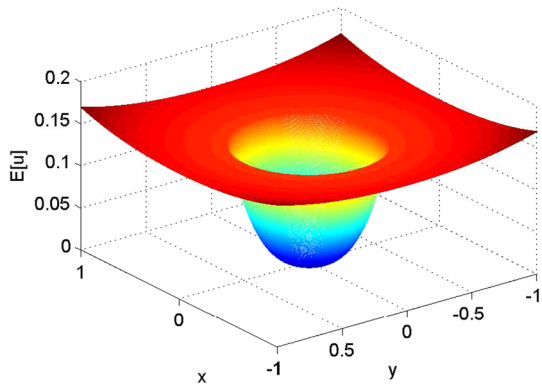
error in expected value of the solution versus the number of collocation points η and the mesh size h for different numerical methods. We plot the errors at different resolutions, such as $N_h = 128$, $N_h = 256$ and $N_h = 512$. From Fig. 4a we can see that when we select a fixed value of N_h, the error decays fast first and then stagnates as the number of collocation points increases. The error stagnates because the approximation error in the physical spaces becomes predominant. If we wish the error is reduced further, then we need to increase N_h. From Fig. 4b we can see that when we fix the "level" $w = 4$, the order of errors in L^2 norm is $O(h^2)$ for the IFEM, but only $O(h)$ for the standard linear FEM.

In practical stochastic collocation computing, the PDE solver is usually a "black-box" to provide the function evaluations at the sample points. In the following experiments, we use IFEM for both sparse grid stochastic collocation method and Monte Carlo method with $N_h = 512$, assuming that the approximation error in the "black-box" is small enough to investigate the convergence properties of the isotropic Smolyak method.
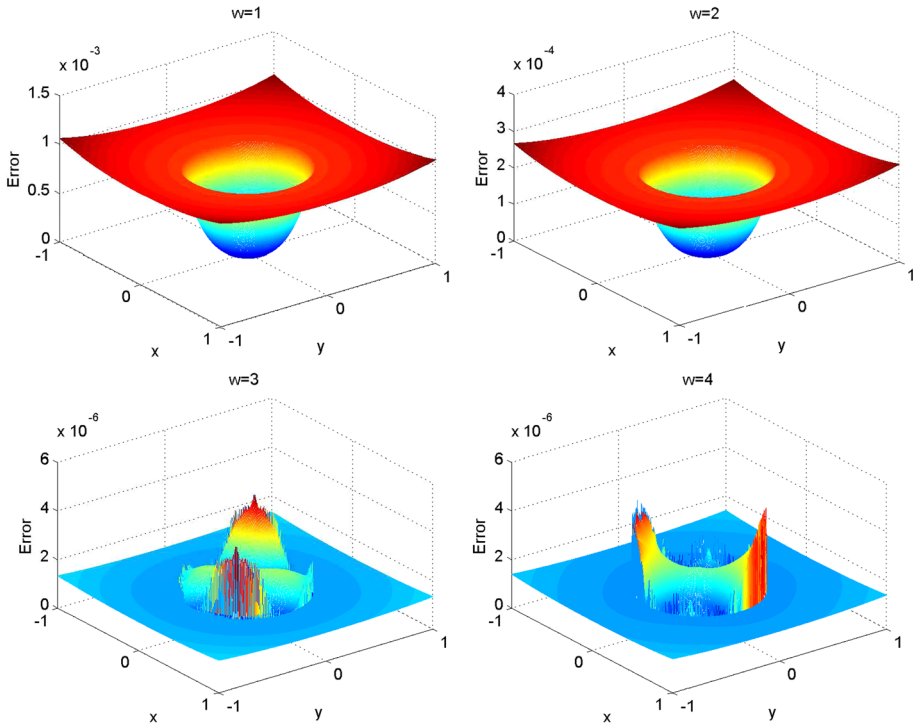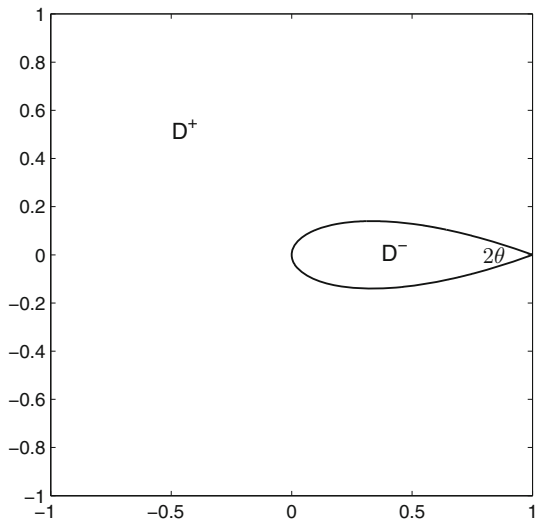
Fig. 7 The error distribution of the expected value of Example 1 with N=10 and w=1, 2, 3, 4

Fig. 8 The domain and the interface of Example 2. The interface has a corner of angle 2θ at $(1, 0)$

Now we choose a moderately high-dimensional random space with $N = 10$. Figure 5 shows the error convergence with respect to the number of grids for the sparse grid stochastic collocation method and Monte Carlo method. We can see that the sparse grid stochastic collocation method converges quickly, and it converges faster than the Monte Carlo finite
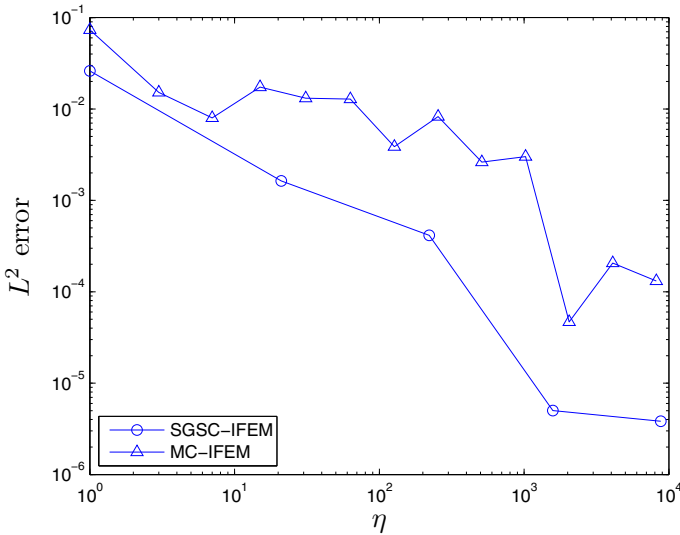
Fig. 9 A comparison between the SGSC-IFEM and the MC-IFEM for solving Example 2 with N=10

Fig. 10 The expected value of Example 2 with N=10
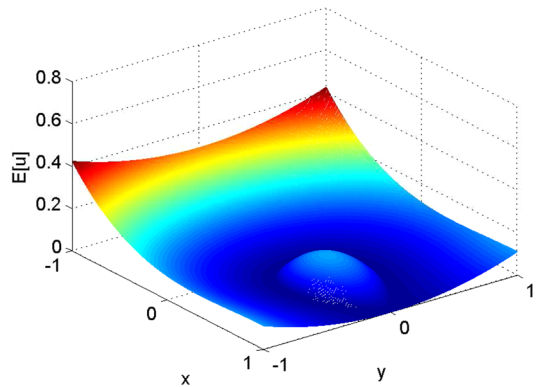
element method. In Fig. 7, we plot the error distribution of the expected value of numerical solution with different "level". From these figures we can see that the numerical error is small although very low "level" is used. In addition, we observe that the shape of the error distribution for $w = 1$ and 2 is almost the same as that of exact expectation (see Fig. 6), but changes completely when $w = 3$ and 4. The reason is that when w is small, such as $w = 1$ and 2, the error of the isotropic Smolyak approximation in the random space is larger than the approximation error in the physical space. As we increase w, the approximation error in the physical space becomes larger than that from the Smolyak approximation.

Example 2 The interface is the zero level set of the function is $\varphi(x_1, x_2) = -x_2^2 + ((x_1 - 1)\tan\theta)^2 x_1$, where θ is a parameter. The interface has a corner of angle 2θ at $(1, 0)$ as shown in Fig. 8. The exact solution is chosen as $u = \varphi(x_1, x_2)/\beta$. It is easy to verify that the solution indeed satisfies the PDE and the jump conditions using the fact of $\mathbf{n} = \nabla\varphi/|\nabla\varphi|$. In this case, we choose $\beta_0^- = 1$ and $\beta_0^+ = 10$ in (4.1). The dimension of random space is set to be $N = 10$.
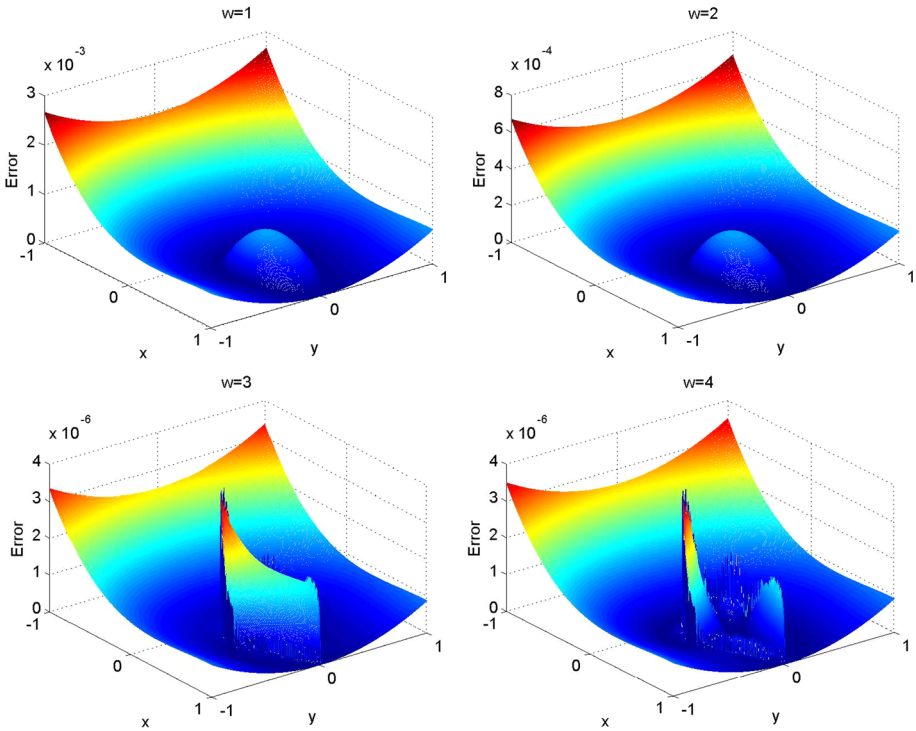
Fig. 11 The error distribution of the expected value of Example 2 with $N=10$ and $w=1, 2, 3, 4$
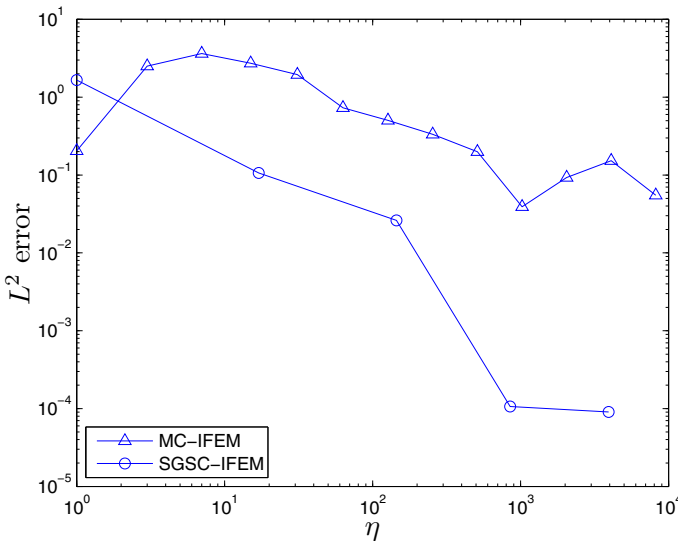

Fig. 12 A comparison between the SGSC-IFEM and the MC-IFEM for solving Example 3 with $N=8$

A comparison between the sparse gird collocation method and the Monte Carlo method is given in Fig. 9. It can be concluded again that the sparse grid collocation method is accurate and performs better than the Monte Carlo method. We plot the expected value of the exact

Fig. 13 The expected value of Example 3 with $N=8$
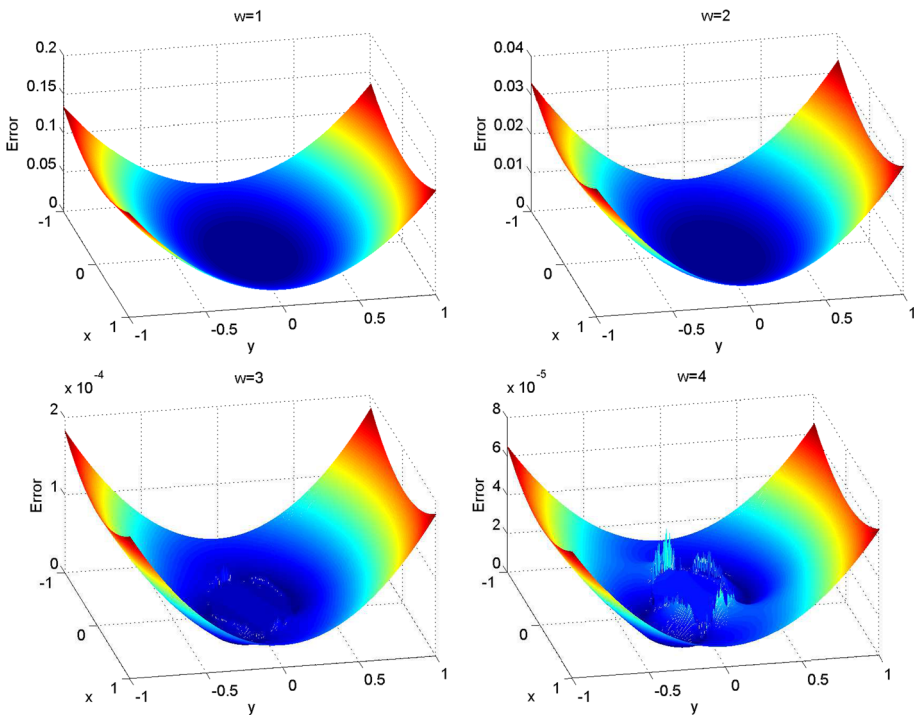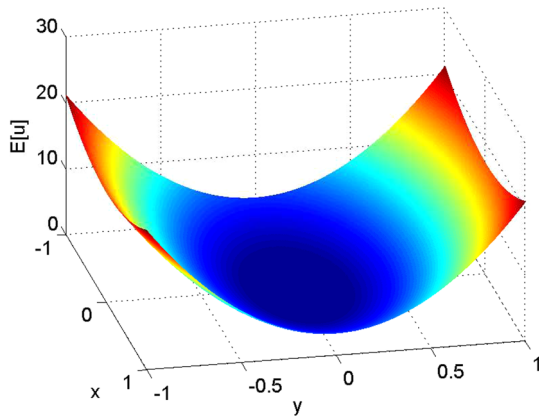

Fig. 14 The error distribution of the expected value of Example 3 with $N=8$ and $w=1, 2, 3, 4$

solution in Fig. 10. The corresponding error distributions of expected value from "level" 1 to "level" 4 are displayed in Fig. 11. From these figures we observe the similar phenomena as Example 1. The shape of the error distribution for $w = 1$ and 2 is almost the same as that of exact expectation (see Fig. 10), but changes completely when $w = 3$ and 4.

Example 3 We consider the case where β_0^+ and β_0^- in (4.1) is not a piecewise constant,

$$\begin{cases} \beta_0^+ = 1 & \text{in } D^+, \\ \beta_0^- = 10 + 5(x_1^2 - x_1 x_2 + x_2^2) & \text{in } D^-. \end{cases} \tag{4.4}$$

The interface is the zero level set of $\varphi(x_1, x_2) = x_1^2/(0.5^2) + x_2^2/(0.25)^2 - 1$. The exact solution is chosen as $u = \varphi(x_1, x_2)/\beta$. And we set $N = 8$.

A comparison between the sparse gird collocation method and the Monte Carlo method is given in Fig. 12. It can be concluded again that the sparse grid collocation method is accurate and performs better than the Monte Carlo method.

The expected value of the exact solution is presented in Fig. 13. The corresponding error distributions of expected value from "level" 1 to "level" 4 are displayed in Fig. 14. We can see that the shape of the error distribution is almost the same as that of exact expectation (see Fig. 13) when $w = 1$ and 2, changes a little when $w = 3$, and changes completely when $w = 4$. We can conclude that when w is small, such as $w = 1$ and 2, the error of the isotropic Smolyak method is dominant. As we increase w, the approximation error in the physical space becomes larger than that from the Smolyak approximation.

5 Conclusion and future work

We have presented a stochastic collocation method for the numerical solution of elliptic partial differential equations with both random inputs and interfaces. To relieve the *curse of dimensionality*, we use the sparse grid collocation method based on the isotropic Smolyak construction instead of using the full tensor product construction. In the error analysis, we divide the error into two parts and provide the error estimates respectively. Numerical examples have shown that the sparse grid collocation method preserves a high level of accuracy and it is a valid alternative to the more traditional Monte Carlo method. For the sparse grid stochastic collocation method, the deterministic PDE solver is usually considered as a "black-box" solver which is the case for interface problems. In order to apply the sparse isotropic Smolyak approximation, we must guarantee the error in the "black-box" small enough. The IFEM is a good choice as verified by our numerical experiments. The accuracy of the IFEM, in fact, reduces the total computational cost.

In this paper, we only considered the case that the coefficient of the PDE has random input. In real applications, it is also important to consider the interface itself as a random curve owing to the lack of information for the media (see [30]). Whether our current work can be extended to the random interfaces is under investigation.

References

1. Acharjee, S., Zabaras, N.: Uncertainty propagation in finite deformations—a spectral stochastic Lagrangian approach. Comput. Methods Appl. Mech. Eng. **195**(19), 2289–2312 (2006)
2. Babuška, I., Nobile, F., Tempone, R.: A stochastic collocation method for elliptic partial differential equations with random input data. SIAM Rev. **52**(2), 317–355 (2010)
3. Babuška, I., Tempone, R., Zouraris, G.E.: Galerkin finite element approximations of stochastic elliptic partial differential equations. SIAM J. Numer. Anal. **42**(2), 800–825 (2004)

4. Bieri, M.: A sparse composite collocation finite element method for elliptic SPDEs. SIAM J. Numer. Anal. **49**(6), 2277–2301 (2011)

5. Bramble, J.H., King, J.T.: A finite element method for interface problems in domains with smooth boundaries and interfaces. Adv. Comput. Math. **6**(1), 109–138 (1996)

6. Bungartz, H., Griebel, M.: Sparse grids. Acta Numer. **13**, 147–269 (2004)

7. Cao, Y., Hussaini, M.Y., Zang, T.A.: An efficient Monte Carlo method for optimal control problems with uncertainty. Comput. Optim. Appl. **26**(3), 219–230 (2003)

8. Chou, S., Kwak, D.Y., Wee, K.T.: Optimal convergence analysis of an immersed interface finite element method. Adv. Comput. Math. **33**(2), 149–168 (2010)

9. Cohen, A., Davenport, M.A., Leviatan, D.: On the stability and accuracy of least squares approximations. Found. Comput. Math. **13**(5), 819–834 (2013)

10. Doostan, A., Owhadi, H.: A non-adapted sparse approximation of PDEs with stochastic inputs. J. Comput. Phys. **230**(8), 3015–3034 (2011)

11. Fishman, G.: Monte Carlo: Concepts, Algorithm, and Applications. Springer, Berlin (1996)

12. Frauenfelder, P., Schwab, C., Todor, R.A.: Finite elements for elliptic problems with stochastic coefficients. Comput. Methods Appl. Mech. Eng. **194**(2), 205–228 (2005)

13. Ghanem, R .G., Spanos, P .D.: Stochastic finite elements: a spectral approach, vol. 41. Springer, Berlin (1991)

14. He, X., Lin, T., Lin, Y., Zhang, X.: Immersed finite element methods for parabolic equations with moving interface. Numer. Methods Partial Differ. Equ. **29**(2), 619–646 (2013)

15. Jakeman, J.D., Narayan, A., Xiu, D.: Minimal multi-element stochastic collocation for uncertainty quantification of discontinuous functions. J. Comput. Phys. **242**, 790–808 (2013)

16. Ji, H., Chen, J., Li, Z.: A symmetric and consistent immersed finite element method for interface problems. J. Sci. Comput. **61**, 533–557 (2014)

17. Kwak, D.Y., Wee, K.T., Chang, K.S.: An analysis of a broken P_1-nonconforming finite element method for interface problems. SIAM J. Numer. Anal. **48**(6), 2117–2134 (2010)

18. Li, Z., Ito, K.: The Immersed Interface Method: Numerical Solutions of PDEs Involving Interfaces and Irregular Domains, vol. 33. SIAM, Philadelphia, PA (2006)

19. Li, Z., Lin, T., Lin, Y., Rogers, R.C.: An immersed finite element space and its approximation capability. Numer. Methods Partial Differ. Equ. **20**(3), 338–367 (2004)

20. Li, Z., Lin, T., Wu, X.: New Cartesian grid methods for interface problems using the finite element formulation. Numer. Math. **96**(1), 61–98 (2003)

21. Lin, T., Lin, Y., Zhang, X.: Partially penalized immersed finite element methods for elliptic interface problems. SIAM J. Numer. Anal. **53**(2), 1121–1144 (2015)

22. Martin, J., Wilcox, L.C., Burstedde, C., Ghattas, O.: A stochastic Newton MCMC method for large-scale statistical inverse problems with application to seismic inversion. SIAM J. Sci. Comput. **34**(3), A1460–A1487 (2012)

23. Migliorati, G., Nobile, F., Schwerin, E., Tempone, R.: Approximation of quantities of interest in stochastic PDEs by the random discrete L^2 projection on polynomial spaces. SIAM J. Sci. Comput. **35**(3), A1440–A1460 (2013)

24. Motamed, M., Nobile, F., Tempone, R.: A stochastic collocation method for the second order wave equation with a discontinuous random speed. Numer. Math. **123**(3), 493–536 (2013)

25. Narayan, A., Xiu, D.: Stochastic collocation methods on unstructured grids in high dimensions via interpolation. SIAM J. Sci. Comput. **34**(3), A1729–A1752 (2012)

26. Narayanan, V.A.B., Zabaras, N.: Stochastic inverse heat conduction using a spectral approach. Int. J. Numer. Methods Eng **60**(9), 1569–1593 (2004)

27. Niederreiter, H.: Random Number Generation and Quasi-Monte Carlo Methods, vol. 63. SIAM, Philadelphia, PA (1992)

28. Nobile, F., Tempone, R., Webster, C.G.: An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data. SIAM J. Numer. Anal. **46**(5), 2411–2442 (2008)

29. Nobile, F., Tempone, R., Webster, C.G.: A sparse grid stochastic collocation method for partial differential equations with random input data. SIAM J. Numer. Anal. **46**(5), 2309–2345 (2008)

30. Nouy, A., Clement, A., Schoefs, F., Moës, N.: An extended stochastic finite element method for solving stochastic partial differential equations on random domains. Comput. Methods Appl. Mech. Eng. **197**(51), 4663–4682 (2008)

31. Phoon, K.K., Huang, S.P., Quek, S.T.: Implementation of Karhunen–Loeve expansion for simulation using a wavelet-Galerkin scheme. Probab. Eng. Mech. **17**(3), 293–303 (2002)

32. Rauhut, H., Ward, R.: Sparse Legendre expansions via l1-minimization. J. Approx. Theory **164**(5), 517–533 (2012)

33. Smolyak, S.: Quadrature and interpolation formulas for tensor products of certain classes of functions. Sov. Math. Dokl. **4**, 240–243 (1963)
34. Stefanou, G.: The stochastic finite element method: past, present and future. Comput. Methods Appl. Mech. Eng. **198**(9), 1031–1051 (2009)
35. Tang, T., Zhou, T.: Convergence analysis for stochastic collocation methods to scalar hyperbolic equations with a random wave speed. Commun. Comput. Phys **8**(1), 226–248 (2010)
36. Tang, T., Zhou, T.: On discrete least square projection in unbounded domain with random evaluations and its application to parametric uncertainty quantification. SIAM J. Sci. Comput. **36**, A2272–A2295 (2014)
37. Xiu, D.: Numerical Methods for Stochastic Computations: A Spectral Method Approach. Princeton University Press, Princeton (2010)
38. Xiu, D., Hesthaven, J.S.: High-order collocation methods for differential equations with random inputs. SIAM J. Sci. Comput. **27**(3), 1118–1139 (2005)
39. Xiu, D., Karniadakis, G.: Modeling uncertainty in steady state diffusion problems via generalized polynomial chaos. Comput. Methods Appl. Mech. Eng. **191**(43), 4927–4948 (2002)
40. Xiu, D., Karniadakis, G.: The Wiener–Askey polynomial chaos for stochastic differential equations. SIAM J. Sci. Comput. **24**(2), 619–644 (2002)
41. Xu, Z., Zhou, T.: On sparse interpolation and the design of deterministic interpolation points. SIAM J. Sci. Comput. **36**, A1752–A1769 (2014)
42. Yan, L., Guo, L., Xiu, D.: Stochastic collocation algorithms using l-1minimization. Int. J. Uncertain. Quantif. **2**(3), 279–293 (2012)
43. Zhou, T.: Stochastic Galerkin methods for elliptic interface problems with random input. J. Comput. Appl. Math. **236**(5), 782–792 (2011)
44. Zhou, T., Narayan, A., Xu, Z.: Multivariate discrete least-squares approximations with a new type of collocation grid. SIAM J. Sci. Comput. **36**, A2401–A2422 (2014)
45. Zhou, T., Tang, T.: Galerkin methods for stochastic hyperbolic problems using bi-orthogonal polynomials. J. Sci. Comput. **51**(2), 274–292 (2012)