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A Stabilized Crank–Nicolson Mixed Finite Volume Element Formulation for the Non-stationary Incompressible Boussinesq Equations

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Abstract At first, a semi-discrete Crank–Nicolson (CN) formulation with respect to time for the non-stationary incompressible Boussinesq equations is presented. Then, a fully discrete stabilized CN mixed finite volume element (SCNMFVE) formulation based on two local Gauss integrals and parameter-free is established directly from the semi-discrete CN formulation with respect to time. Next, the error estimates for the fully discrete SCNMFVE solutions are derived by means of the standard CN mixed finite element method. Finally, some numerical experiments are presented illustrating that the numerical errors are consistent with theoretical results, the computing load for the fully discrete SCNMFVE formulation are far fewer than that for the stabilized mixed finite volume element (SMFVE) formulation with the first time accuracy, and its accumulation of truncation errors in the computational process is far lesser than that of the SMFVE formulation with the first time accuracy. Thus, the advantage of the fully discrete SCNMFVE formulation for the non-stationary incompressible Boussinesq equations is shown sufficiently.

Keywords Non-stationary incompressible Boussinesq equations · Stabilized Crank–Nicolson mixed finite volume element formulation · Local Gauss integrals and parameter-free · Error estimate · Numerical simulation

Mathematics Subject Classification 65M18 · 65N12

1 Introduction

The non-stationary incompressible Boussinesq equations are a nonlinear system of partial differential equations (PDEs) including the velocity vector field and the pressure field as

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well as the temperature field (see [17,18,23]), which are also known as the non-stationary conduction-convection problem and may be denoted by the following nonlinear system of PDEs.

Problem I Find $\boldsymbol{u} = (u_1, u_2),\ p$, and T such that, for $t_N > 0$,

$$\begin{cases} \boldsymbol{u}_t - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = T\boldsymbol{j}, & (x, y, t) \in \Omega \times (0, t_N), \\ \nabla \cdot \boldsymbol{u} = 0, & (x, y, t) \in \Omega \times (0, t_N), \\ T_t - \gamma_0^{-1} \Delta T + (\boldsymbol{u} \cdot \nabla) T = 0, & (x, y, t) \in \Omega \times (0, t_N), \\ \boldsymbol{u}(x, y, t) = \boldsymbol{u}_0(x, y, t),\ T(x, y, t) = \varphi(x, y, t), & (x, y, t) \in \partial\Omega \times (0, t_N), \\ \boldsymbol{u}(x, y, 0) = \boldsymbol{u}^0(x, y),\ T(x, y, 0) = \psi(x, y) & (x, y) \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded and connected domain, $\boldsymbol{u} = (u_1, u_2)$ represents the fluid velocity vector, p the pressure, T the temperature, t_N the total time, $\boldsymbol{j} = (0, 1)$ the unit vector, $\nu = \sqrt{Pr/Re}$, Re the Reynolds number, Pr the Prandtl number, $\gamma_0 = \sqrt{RePr}$, and $\boldsymbol{u}_0(x, y, t)$, $\boldsymbol{u}^0(x, y)$, $\varphi(x, y, t)$ and $\psi(x, y)$ are given functions. For the sake of convenience and without loss of generality, we may as well suppose that $\boldsymbol{u}_0(x, y, t) = \boldsymbol{u}^0(x, y) = \boldsymbol{0}$ and $\varphi(x, y, t) = 0$ in the following theoretical analysis.

In general, there are no analytical solutions for Problem I due to it being a system of nonlinear PDEs including the velocity vector, the pressure, and the temperature. One has to rely on numerical solutions (see, e.g., [9,17,18]). However, most of the existing papers use either the finite element (FE) method or finite difference (FD) schemes as discretization tools.

Compared to FD and FE methods, the finite volume element (FVE) method (see [6,13, 22]) is considered as most effective discretization approach for PDEs, since it is generally easier to implement and offer flexibility in handling complicated computing domains. More importantly, it can ensure local mass conservation and a highly desirable property in many applications. It is also referred to as a box method (see [3]) or a generalized difference method (see [14,15]). It has been widely used to find numerical solutions of various types of PDEs, for example, second order elliptic equations, parabolic equations, Stokes equations, and the Navier–Stokes equations (see, e.g., [2–4,6–8,11,13–15,22,24,25]).

A fully discrete FVE formulation without any stabilization (see [16]) and a fully discrete stabilized mixed FVE (SMFVE) formulation (see [17]) for the non-stationary incompressible Boussinesq equations are proposed, but they do only have the first-order time accuracy. Thus, in order to obtain sufficiently time accuracy, they need to refine time steps so that moving forward steps and the accumulation of truncation errors in the computational process could greatly increase. Therefore, in this study, we improve the methods in [16,17] and establish a fully discrete stabilized Crank–Nicolson (CN) mixed finite volume element (SCNMFVE) formulation based on two local Gauss integrals and parameter-free for the non-stationary incompressible Boussinesq equations. Thus, although the trial function spaces of velocity, temperature, and pressure of SCNMFVE formulation are the same as those in [16,17], the SCNMFVE solutions improve one-order time accuracy more than those in [16,17] such that it could alleviate the calculating load and the accumulation of truncation errors in the computational process. In addition, an optimizing reduced Petrov–Galerkin least squares mixed FE formulation based on residuals (see [19]) and a reduced mixed FE formulation without any stabilization (see [20]) have been established for the non-stationary incompressible Boussinesq equations, but they are completely different from the fully discrete SCNMFVE formulation since the FVE method is entirely different from and far more advantageous than the FE method as is mentioned above.

The remainder of this article is organized as follows. In Sect. 2, we first present the semi-discrete CN formulation with respect to time for the non-stationary incompressible Boussinesq equations. And then, we establish directly the fully discrete SCNMFVE formulation from the semi-discrete formulation with respect to time. Thus, we could avoid the discussion for semi–discrete SCNMFVE formulation with respect to spatial variables such that our theoretical analysis becomes simpler than the existing other methods (see, e.g., [11,12,17]). In Sect. 3, the error estimates for the fully discrete SCNMFVE solutions are derived by means of the standard CN mixed FE (CNMFE) method. In Sect. 4, some numerical experiments are presented illustrating that the numerical errors are consistent with theoretical results, the computing load of the fully discrete SCNMFVE formulation are far fewer than those of the SMFVE formulation with the first time accuracy, and the accumulation of its truncation errors in the computational process is far lesser than those of the SMFVE formulations with the first time accuracy. Thus, it is shown that the fully discrete SCNMFVE formulation for the non-stationary incompressible Boussinesq equations is far more advantageous than the SMFVE formulation in [17]. Section 5 provides main conclusions and some discussions.

2 Semi-Discrete Formulation About Time and Fully Discrete SCNMFVE Formulation

2.1 Semi-Discrete CN Formulation About Time

The Sobolev spaces along with their properties used in this context are standard (see [1]). Let $X = H_0^1(\Omega)^2$, $M = L_0^2(\Omega) = \{q \in L^2(\Omega);\ \int_\Omega q\mathrm{d}x\mathrm{d}y = 0\}$, and $W = H_0^1(\Omega)$. Thus, a mixed variational formulation for Problem I is written as follows.

Problem II Find $(\boldsymbol{u}, p, T) \in H^1(0, t_N; X)^2 \times L^2(0, t_N; M) \times H^1(0, t_N; W)$ such that, for almost all $t \in (0, t_N)$,

$$\begin{cases} (\boldsymbol{u}_t, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}) + a_1(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - b(p, \boldsymbol{v}) = (T\boldsymbol{j}, \boldsymbol{v}), & \forall \boldsymbol{v} \in X, \\ b(q, \boldsymbol{u}) = 0, & \forall q \in M, \\ (T_t, \phi) + d(T, \phi) + a_2(\boldsymbol{u}, T, \phi) = 0, & \forall \phi \in W, \\ \boldsymbol{u}(x, y, 0) = \boldsymbol{0}, T(x, y, 0) = \psi(x, y), & (x, y) \in \Omega, \end{cases} \tag{2}$$

where $(\cdot, \cdot)$ denotes inner product in $L^2(\Omega)^2$ or $L^2(\Omega)$ and

$$a(\boldsymbol{u}, \boldsymbol{v}) = \nu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \mathrm{d}x\mathrm{d}y, \forall \boldsymbol{u}, \boldsymbol{v} \in X, b(q, \boldsymbol{v}) = \int_\Omega q \ \mathrm{div}\boldsymbol{v} \mathrm{d}x\mathrm{d}y, \forall \boldsymbol{v} \in X, q \in M,$$

$$a_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \frac{1}{2} \int_\Omega [(\boldsymbol{u}\nabla\boldsymbol{v}) \cdot \boldsymbol{w} - (\boldsymbol{u}\nabla\boldsymbol{w}) \cdot \boldsymbol{v}] \,\mathrm{d}x\mathrm{d}y, \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in X,$$

$$a_2(\boldsymbol{u}, T, \phi) = \frac{1}{2} \int_\Omega [(\boldsymbol{u} \cdot \nabla T)\phi - (\boldsymbol{u} \cdot \nabla\phi)T] \,\mathrm{d}x\mathrm{d}y, \quad \forall \boldsymbol{u} \in X, \forall T, \phi \in W,$$

$$d(T, \phi) = \gamma_0^{-1} \int_\Omega \nabla T \cdot \nabla\phi \mathrm{d}x\mathrm{d}y, \ \forall T, \phi \in W.$$

The above trilinear forms $a_1(\cdot, \cdot, \cdot)$ and $a_2(\cdot, \cdot, \cdot)$ hold the following properties (see [16, 18]):

$$a_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -a_1(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}), \ \ a_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in X, \tag{3}$$

$$a_2(\boldsymbol{u}, T, \phi) = -a_2(\boldsymbol{u}, \phi, T), \ \ a_2(\boldsymbol{u}, \phi, \phi) = 0, \quad \forall \boldsymbol{u} \in X, \ \forall T, \phi \in W. \tag{4}$$

The above bilinear forms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$, and $b(\cdot,\cdot)$ have the following properties (also see [16,18]):

$$a(\boldsymbol{v},\boldsymbol{v}) \ge \nu|\boldsymbol{v}|_1^2,\ \ \forall \boldsymbol{v}\in X;\ \ |a(\boldsymbol{u},\boldsymbol{v})| \le \nu|\boldsymbol{u}|_1|\boldsymbol{v}|_1,\ \ \forall \boldsymbol{u},\ \boldsymbol{v}\in X, \tag{5}$$

$$d(\phi,\phi) \ge \gamma_0^{-1}|\phi|_1^2,\ \ \forall \phi\in W;\ \ |d(T,\phi)| \le \gamma_0^{-1}|T|_1|\phi|_1,\ \ \forall T,\ \phi\in W, \tag{6}$$

$$\sup_{\boldsymbol{v}\in X}\frac{b(q,\boldsymbol{v})}{|\boldsymbol{v}|_1} \ge \beta\|q\|_0,\ \ \forall q\in M, \tag{7}$$

where β is a constant. Define

$$N_0 = \sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in X}\frac{a_1(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})}{|\boldsymbol{u}|_1\cdot|\boldsymbol{v}|_1\cdot|\boldsymbol{w}|_1},\quad \tilde{N}_0 = \sup_{\boldsymbol{u}\in X,(T,\phi)\in W\times W}\frac{a_2(\boldsymbol{u},T,\phi)}{|\boldsymbol{u}|_1\cdot|T|_1\cdot|\phi|_1}. \tag{8}$$

The following result is classical (see Theorem 1.4.1 in [23] or Theorem 5.2 in [18]).

Theorem 2.1 *If $\psi\in L^2(\Omega)$, then the problem* II *has at least a solution which, in addition, is unique provided that $\|\psi\|_0^2 \le 2\nu^2 t_N/(2N_0 t_N^{-1}\exp(t_N)+\nu\gamma_0\tilde{N}_0^2)$, and there are the following prior estimates*

$$\|\boldsymbol{u}\|_0^2+\nu\|\nabla\boldsymbol{u}\|_{L^2(L^2)}^2 \le t_N^2\|\psi\|_0^2\exp(t_N),\quad \|T\|_0^2+\gamma_0^{-1}\|\nabla T\|_{L^2(L^2)}^2 \le \|\psi\|_0^2.$$

Let N be the positive integer, $k=t_N/N$ denote the time step increment, $t_n=nk$ $(0\le n\le N)$, and $(\boldsymbol{u}^n,p^n,T^n)$ be the semi–discrete approximation of $(\boldsymbol{u}(t),p,T)$ at $t_n=nk(n=0,1,\ldots,N)$ about time, $\bar{\boldsymbol{u}}^n=(\boldsymbol{u}^n+\boldsymbol{u}^{n-1})/2$, and $\bar{T}^n=(T^n+T^{n-1})/2$. If the differential quotients $\boldsymbol{u}_t$ and T_t in Problem II at time $t=t_n$ are, respectively, approximated by means of the backward difference quotients $\bar{\partial}_t\boldsymbol{u}^n=(\boldsymbol{u}^n-\boldsymbol{u}^{n-1})/k$ and $\bar{\partial}_tT^n=(T^n-T^{n-1})/k$. Thus, the semi–discrete CN scheme for Problem II with respect to time is written as follows.
Problem III Find $(\boldsymbol{u}^n,p^n,T^n)\in X\times M\times W$ $(1\le n\le N)$ such that

$$\begin{cases}(\boldsymbol{u}^n,\boldsymbol{v})+ka(\bar{\boldsymbol{u}}^n,\boldsymbol{v})+ka_1(\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{u}}^n,\boldsymbol{v})-kb(p^n,\boldsymbol{v}) & \\ \quad =k(\bar{T}^n\boldsymbol{j},\boldsymbol{v})+(\boldsymbol{u}^{n-1},\boldsymbol{v}), & \forall\boldsymbol{v}\in X,\\ b(q,\boldsymbol{u}^n)=0, & \forall q\in M,\\ (T^n,\phi)+kd(\bar{T}^n,\phi)+ka_2(\bar{\boldsymbol{u}}^n,\bar{T}^n,\phi)=(T^{n-1},\phi), & \forall\phi\in W,\\ \boldsymbol{u}^0=0,\quad T^0=\psi(x,y), & (x,y)\in\Omega.\end{cases} \tag{9}$$

There are the following results for Problem III.

Theorem 2.2 *Under the assumptions of* Theorem 2.1*, Problem III has a unique series of solutions $(\boldsymbol{u}^n,p^n,T^n)\in X\times M\times W(n=1,2,\ldots,N)$ such that*

$$\|\boldsymbol{u}^n\|_0^2+k\sqrt{\nu}\|\nabla\boldsymbol{u}^n\|_0 \le \sqrt{\nu^{-1}\gamma_0}\|\psi\|_0, \tag{10}$$

$$\|T^n\|_0+k\gamma_0^{-1/2}\|\nabla T^n\|_0 \le \max\{2,\sqrt{2}k\gamma_0^{-1/2}\}\|\psi\|_1,\quad k\|p^n\|_0\le C\|\psi\|_1, \tag{11}$$

where C used next is a constant independent of k, but dependent on ψ and known data. And if $(\boldsymbol{u},p,T)\in[H_0^1(\Omega)\cap H^2(\Omega)]^2\times[H^1(\Omega)\cap M]\times[H_0^1(\Omega)\cap H^2(\Omega)]$ is the exact solution for the problem I, we have the following error estimates

$$\|\boldsymbol{u}(t_n)-\boldsymbol{u}^n\|_0+k[\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}^n)\|_0+\|p(t_n)-p^n\|_0]\le Ck^2, \tag{12}$$

$$\|T(t_n)-T^n\|_0+k\|\nabla(T(t_n)-T^n)\|_0\le Ck^2,\ 1\le n\le N. \tag{13}$$

Proof First, we prove that Problem III has a unique series of solutions. For $1 \leq n \leq N$, we consider the following linearized auxiliary problem:

$$\begin{cases} \left(\boldsymbol{u}_m^n, \boldsymbol{v}\right) + ka\left(\bar{\boldsymbol{u}}_m^n, \boldsymbol{v}\right) + ka_1\left(\bar{\boldsymbol{u}}_{m-1}^n, \bar{\boldsymbol{u}}_m^n, \boldsymbol{v}\right) - kb\left(p_m^n, \boldsymbol{v}\right) & \\ \quad = k\left(\bar{T}_m^n \boldsymbol{j}, \boldsymbol{v}\right) + \left(\boldsymbol{u}_m^{n-1}, \boldsymbol{v}\right), & \forall \boldsymbol{v} \in X, m = 1, 2, \ldots, \\ b\left(q, \boldsymbol{u}_m^n\right) = 0, & \forall q \in M, m = 1, 2, \ldots, \\ \left(T_m^n, \phi\right) + kd\left(\bar{T}_m^n, \phi\right) + ka_2\left(\bar{\boldsymbol{u}}_{m-1}^n, \bar{T}_m^n, \phi\right) = \left(T_m^{n-1}, \phi\right), & \forall \phi \in W, m = 1, 2, \ldots, \\ \boldsymbol{u}_m^0 = 0, m = 0, 1, \ldots, \quad T_m^0 = \psi(x, y), m = 1, 2, \ldots, & (x, y) \in \Omega. \end{cases} \tag{14}$$

By taking $\boldsymbol{v} = \boldsymbol{u}_m^n + \boldsymbol{u}_m^{n-1}$, $q = p_m^n$, and $\phi = T_m^n + T_m^{n-1}$ in the system of equations (14), and by using (3) and (4), Hölder inequality, and Cauchy inequality, we obtain

$$\begin{aligned} 2\left(\|\boldsymbol{u}_m^n\|_0^2 - \|\boldsymbol{u}_m^{n-1}\|_0^2\right) + k\nu\|\nabla\left(\boldsymbol{u}_m^n + \boldsymbol{u}_m^{n-1}\right)\|_0^2 &= 2k\left(\bar{T}_m^n \boldsymbol{j}, \boldsymbol{u}_m^n + \boldsymbol{u}_m^{n-1}\right) \\ &\leq k\|T_m^n + T_m^{n-1}\|_{-1}\|\nabla\left(\boldsymbol{u}_m^n + \boldsymbol{u}_m^{n-1}\right)\|_0 \\ &\leq \frac{k}{2\nu}\|T_m^n + T_m^{n-1}\|_{-1}^2 + \frac{k\nu}{2}\|\nabla\left(\boldsymbol{u}_m^n + \boldsymbol{u}_m^{n-1}\right)\|_0^2 \end{aligned} \tag{15}$$

and

$$2\left(\|T_m^n\|_0^2 - \|T_m^{n-1}\|_0^2\right) + k\gamma_0^{-1}\|\nabla\left(T_m^n + T_m^{n-1}\right)\|_0^2 = 0. \tag{16}$$

Summing (15) and (16) from 1 to n and simplifying yield

$$\|\boldsymbol{u}_m^n\|_0^2 + k\nu\sum_{i=1}^n \|\nabla\left(\boldsymbol{u}_m^i + \boldsymbol{u}_m^{i-1}\right)\|_0^2 \leq \frac{k}{2\nu}\sum_{i=1}^n \|T_m^i + T_m^{i-1}\|_{-1}^2 \tag{17}$$

and

$$\|T_m^n\|_0^2 + k\gamma_0^{-1}\sum_{i=1}^n \|\nabla\left(T_m^i + T_m^{i-1}\right)\|_0^2 \leq 2\|\psi\|_0^2. \tag{18}$$

By using (18), from (17), we obtain

$$\|\boldsymbol{u}_m^n\|_0^2 + k\nu\sum_{i=1}^n \|\nabla\left(\boldsymbol{u}_m^i + \boldsymbol{u}_m^{i-1}\right)\|_0^2 \leq \nu^{-1}\gamma_0\|\psi\|_0^2. \tag{19}$$

By extracting square root for (18) and (19) and using $\left(\sum_{i=1}^n b_i^2\right)^{1/2} \geq \sum_{i=1}^n |b_i|/\sqrt{n}$ and $\|a+b\|_0 \geq \|a\|_0 - \|b\|_0$, we obtain

$$\|T_m^n\|_0 + k\gamma_0^{-1/2}\|\nabla T_m^n\|_0 \leq \max\{2, \sqrt{2}k\gamma_0^{-1/2}\}\|\psi\|_1, \tag{20}$$

$$\|\boldsymbol{u}_m^n\|_0^2 + k\sqrt{\nu}\|\nabla\boldsymbol{u}_m^n\|_0 \leq \sqrt{\nu^{-1}\gamma_0}\|\psi\|_0. \tag{21}$$

By using the first equation of (9), (7), (20), and (21), we easily obtain

$$\|p_m^n\|_0 \leq C\|\psi\|_1. \tag{22}$$

Thus, for $1 \leq n \leq N$, if $\psi = 0$, the system of linear equations (14) has only zero solution. Therefore, the system of linear equations (14) has a unique series of solutions $(\boldsymbol{u}_m^n, p_m^n, T_m^n) \in X \times M \times W$ $(m = 1, 2, \ldots)$. Because the spaces $X \times M \times W$ are weakly and sequentially compact Hilbert spaces, by fixed point theorem (see [21]), we can conclude that the sequence of solutions $(\boldsymbol{u}_m^n, p_m^n, T_m^n) \in X \times M \times W$ has a subsequence [without loss of generality, we still might denote by $(\boldsymbol{u}_m^n, p_m^n, T_m^n)$] that is uniquely and weakly convergent to $(\boldsymbol{u}^n, p^n, T^n) \in X \times M \times W$ for Problem III, i.e., Problem III has at least a series of solutions $(\boldsymbol{u}^n, p^n, T^n) \in X \times M \times W (n = 1, 2, \ldots, N)$. Using the same technique as the proof of the uniqueness of solution for Problem II (see Theorem 1.4.1 in [23] or Theorem 5.2 in [18]), we can prove that the series of solutions for Problem III is unique.

Second, we prove that (10) and (11) hold. By taking $\boldsymbol{v} = \boldsymbol{u}^n + \boldsymbol{u}^{n-1}$, $q = p^n$, and $\phi = T^n + T^{n-1}$ in Problem III, and by using (3) and (4), Hölder inequality, and Cauchy inequality, we obtain

$$\begin{aligned} 2\left(\|\boldsymbol{u}^n\|_0^2 - \|\boldsymbol{u}^{n-1}\|_0^2\right) + k\nu\|\nabla\left(\boldsymbol{u}^n + \boldsymbol{u}^{n-1}\right)\|_0^2 &= 2k\left(\bar{T}^n \boldsymbol{j}, \boldsymbol{u}^n + \boldsymbol{u}^{n-1}\right) \\ &\le k\|T^n + T^{n-1}\|_{-1}\|\nabla\left(\boldsymbol{u}^n + \boldsymbol{u}^{n-1}\right)\|_0 \\ &\le \frac{k}{2\nu}\|T^n + T^{n-1}\|_{-1}^2 + \frac{k\nu}{2}\|\nabla\left(\boldsymbol{u}^n + \boldsymbol{u}^{n-1}\right)\|_0^2 \end{aligned} \tag{23}$$

and

$$2\left(\|T^n\|_0^2 - \|T^{n-1}\|_0^2\right) + k\gamma_0^{-1}\|\nabla\left(T^n + T^{n-1}\right)\|_0^2 = 0. \tag{24}$$

Summing (23) and (24) from 1 to n and simplifying yield

$$\|\boldsymbol{u}^n\|_0^2 + k\nu\sum_{i=1}^{n}\|\nabla\left(\boldsymbol{u}^i + \boldsymbol{u}^{i-1}\right)\|_0^2 \le \frac{k}{2\nu}\sum_{i=1}^{n}\|T^i + T^{i-1}\|_{-1}^2 \tag{25}$$

and

$$\|T^n\|_0^2 + k\gamma_0^{-1}\sum_{i=1}^{n}\|\nabla\left(T^i + T^{i-1}\right)\|_0^2 \le 2\|\psi\|_0^2. \tag{26}$$

By using (26), from (25), we obtain

$$\|\boldsymbol{u}^n\|_0^2 + k\nu\sum_{i=1}^{n}\|\nabla\left(\boldsymbol{u}^i + \boldsymbol{u}^{i-1}\right)\|_0^2 \le \nu^{-1}\gamma_0\|\psi\|_0^2. \tag{27}$$

By extracting square root for (26) and (27) and using $\left(\sum_{i=1}^n b_i^2\right)^{1/2} \ge \sum_{i=1}^n |b_i|/\sqrt{n}$ and $\|a+b\|_0 \ge \|a\|_0 - \|b\|_0$, we obtain

$$\|T^n\|_0 + k\gamma_0^{-1/2}\|\nabla T^n\|_0 \le \max\left\{2, \sqrt{2}k\gamma_0^{-1/2}\right\}\|\psi\|_1, \tag{28}$$

$$\|\boldsymbol{u}^n\|_0^2 + k\sqrt{\nu}\|\nabla\boldsymbol{u}^n\|_0 \le \sqrt{\nu^{-1}\gamma_0}\|\psi\|_0. \tag{29}$$

By using the first equation of (9), (7), (28), and (29), we easily obtain

$$\|p^n\|_0 \le C\|\psi\|_1. \tag{30}$$

Finally, we prove that the error estimates (12) and (13) hold. Put $\boldsymbol{e}^n = \boldsymbol{u}(t_n) - \boldsymbol{u}^n$, $\theta^n = T(t_n) - T^n$, and $\eta^n = p(t_n) - p^n$. Subtracting Problem III from Problem II taking $t = t_{n-\frac{1}{2}}$, $\boldsymbol{v} = \boldsymbol{e}^n + \boldsymbol{e}^{n-1}$, $\varphi = \theta^n + \theta^{n-1}$, and $q = \eta^n$, using Taylor's formula, we obtain that

$$\begin{aligned} \|\boldsymbol{e}^n\|_0^2 - \|\boldsymbol{e}^{n-1}\|_0^2 + \frac{k\nu}{2}\|\nabla\left(\boldsymbol{e}^n + \boldsymbol{e}^{n-1}\right)\|_0^2 &= k\left((\theta^n + \theta^{n-1})\boldsymbol{j}, \boldsymbol{e}^n + \boldsymbol{e}^{n-1}\right) \\ &\quad + \frac{k^3}{24}\left(\boldsymbol{u}_{ttt}(\xi_{1n}), \boldsymbol{e}^n + \boldsymbol{e}^{n-1}\right) \\ &\quad + \frac{k^3\nu}{4}\left(\nabla\boldsymbol{u}_{tt}(\xi_{2n}), \nabla(\boldsymbol{e}^n + \boldsymbol{e}^{n-1})\right) \\ &\quad - \frac{k^3}{4}\left(T_{tt}(\xi_{3n})\boldsymbol{j}, \boldsymbol{e}^n + \boldsymbol{e}^{n-1}\right) + \boldsymbol{\Phi}, \end{aligned} \tag{31}$$

$$\begin{aligned}&\|\theta^n\|_0^2-\|\theta^{n-1}\|_0^2+\frac{k}{2\gamma_0}\|\nabla(\theta^n+\theta^{n-1})\|_0^2\\&\quad=\frac{k^3}{24}\left(T_{ttt}(\zeta_{1n}),\theta^n+\theta^{n-1}\right)+\frac{k^3}{4\gamma_0}\left(\nabla T_{tt}(\zeta_{2n}),\nabla(\theta^n+\theta^{n-1})\right)+\Psi,\end{aligned}\tag{32}$$

where $\boldsymbol{\Phi}=ka_1(\boldsymbol{u}(t_{n-\frac{1}{2}}),\boldsymbol{u}(t_{n-\frac{1}{2}}),\boldsymbol{e}^n+\boldsymbol{e}^{n-1})-ka_1(\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{u}}^n,\boldsymbol{e}^n+\boldsymbol{e}^{n-1})$ and $\Psi=ka_2(\boldsymbol{u}(t_{n-\frac{1}{2}}),T(t_{n-\frac{1}{2}}),\theta^n+\theta^{n-1})-ka_2(\bar{\boldsymbol{u}}^n,\bar{T}^n,\theta^n+\theta^{n-1})$ ($t_{n-1}\leq\xi_{1n},\xi_{2n},\xi_{3n},\zeta_{1n},\zeta_{2n}\leq t_n$). By using Taylor's formula, Hölder inequality, and Cauchy inequality, there are $\xi_{in}\in[t_{n-1},t_n]$ ($i=4,5$) such that

$$\begin{aligned}\boldsymbol{\Phi}&=ka_1\left(\boldsymbol{u}(t_{n-\frac{1}{2}}),\boldsymbol{u}(t_{n-\frac{1}{2}}),\boldsymbol{e}^n+\boldsymbol{e}^{n-1}\right)-ka_1\left(\bar{\boldsymbol{u}}^n,\bar{\boldsymbol{u}}^n,\boldsymbol{e}^n+\boldsymbol{e}^{n-1}\right)\\&=ka_1\left(\boldsymbol{u}(t_{n-\frac{1}{2}})-\bar{\boldsymbol{u}}^n,\boldsymbol{u}(t_{n-\frac{1}{2}}),\boldsymbol{e}^n+\boldsymbol{e}^{n-1}\right)+ka_1\left(\bar{\boldsymbol{u}}^n,\boldsymbol{u}(t_{n-\frac{1}{2}})-\bar{\boldsymbol{u}}^n,\boldsymbol{e}^n+\boldsymbol{e}^{n-1}\right)\\&=\frac{k^3}{16}a_1\left(\boldsymbol{u}_{tt}(\xi_{4n}),\boldsymbol{u}(t_{n-\frac{1}{2}}),\boldsymbol{e}^n+\boldsymbol{e}^{n-1}\right)+\frac{k^3}{16}a_1\left(\bar{\boldsymbol{u}},\boldsymbol{u}_{tt}(\xi_{5n}),\boldsymbol{e}^n+\boldsymbol{e}^{n-1}\right)\\&\leq\frac{k\nu}{12}\|\nabla\left(\boldsymbol{e}^n+\boldsymbol{e}^{n-1}\right)\|_0^2+\frac{k^5N_0^2}{128\nu}\|\nabla\boldsymbol{u}(t)\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)}\\&\quad\times\left(\|\nabla\boldsymbol{u}(t)\|^2_{L^{2,\infty}(t_{n-1},t_n;L^2)}+\|\nabla\bar{\boldsymbol{u}}^n\|_0^2\right).\end{aligned}\tag{33}$$

By using Hölder inequality and Cauchy inequality, we have that

$$\begin{aligned}&\left|\frac{k^3}{24}\left(\boldsymbol{u}_{ttt}(\xi_{1n}),\boldsymbol{e}^n+\boldsymbol{e}^{n-1}\right)+\frac{k^3\nu}{4}\left(\nabla\boldsymbol{u}_{tt}(\xi_{2n}),\nabla(\boldsymbol{e}^n+\boldsymbol{e}^{n-1})\right)-\frac{k^3}{4}\left(T_{tt}(\xi_{3n})\boldsymbol{j},\boldsymbol{e}^n+\boldsymbol{e}^{n-1}\right)\right|\\&\quad\leq\frac{k\nu}{12}\|\nabla\left(\boldsymbol{e}^n+\boldsymbol{e}^{n-1}\right)\|_0^2+\frac{k^5}{64\nu}\|\boldsymbol{u}\|^2_{W^{3,\infty}(t_{n-1},t_n;H^{-1})}+\frac{9\nu k^5}{16}\|\nabla\boldsymbol{u}\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)}\\&\quad\quad+\frac{9k^5}{16\nu}\|T\|^2_{W^{3,\infty}(t_{n-1},t_n;H^{-1})}.\end{aligned}\tag{34}$$

Combining (33) and (34) with (31), using Hölder inequality and Cauchy inequality, and simplifying yield

$$\|\boldsymbol{e}^n\|_0^2-\|\boldsymbol{e}^{n-1}\|_0^2+\frac{k\nu}{4}\|\nabla\left(\boldsymbol{e}^n+\boldsymbol{e}^{n-1}\right)\|_0^2\leq\frac{3k}{\nu}\|\theta^n+\theta^{n-1}\|_{-1}^2+\tilde{C}^2k^5,\tag{35}$$

where $\tilde{C}^2=\frac{1}{64\nu}\|\boldsymbol{u}\|^2_{W^{3,\infty}(t_{n-1},t_n;H^{-1})}+\frac{9\nu}{16}\|\nabla\boldsymbol{u}\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)}+\frac{9}{16\nu}\|T\|^2_{W^{3,\infty}(t_{n-1},t_n;H^{-1})}+\frac{N_0^2}{128\nu}\|\nabla\boldsymbol{u}(t)\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)}\left(\|\nabla\boldsymbol{u}(t)\|^2_{L^{2,\infty}(t_{n-1},t_n;L^2)}+\|\nabla\bar{\boldsymbol{u}}^n\|_0^2\right)$.

By using the same techniques as proving (33), there are $\zeta_{in}\in[t_{n-1},t_n]$ ($i=3,4$) such that

$$\begin{aligned}\Psi&=ka_2\left(\boldsymbol{u}(t_{n-\frac{1}{2}}),T(t_{n-\frac{1}{2}}),\theta^n+\theta^{n-1}\right)-ka_2\left(\bar{\boldsymbol{u}}^n,\bar{T}^n,\theta^n+\theta^{n-1}\right)\\&\leq\frac{k}{8\gamma_0}\|\nabla\left(\theta^n+\theta^{n-1}\right)\|_0^2+\frac{k^5\tilde{N}_0^2\gamma_0}{64}\|\nabla T(t)\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)}\|\nabla\bar{\boldsymbol{u}}^n\|_0^2\\&\quad+\frac{k^5\tilde{N}_0^2\gamma_0}{64}\|\nabla\boldsymbol{u}(t)\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)}\|\nabla T(t)\|^2_{L^{2,\infty}(t_{n-1},t_n;L^2)}.\end{aligned}\tag{36}$$

By using Hölder inequality and Cauchy inequality, we have that

$$\left|\frac{k^3}{24}\left(T_{ttt}(\zeta_{1n}), \theta^n+\theta^{n-1}\right)+\frac{k^3}{4\gamma_0}\left(\nabla T_{tt}(\zeta_{2n}), \nabla(\theta^n+\theta^{n-1})\right)\right|$$
$$\leq \frac{k}{8\gamma_0}\|\nabla\left(\theta^n+\theta^{n-1}\right)\|_0^2+\frac{k^5\gamma_0}{144}\|T\|^2_{W^{3,\infty}(t_{n-1},t_n;H^{-1})}+\frac{k^5}{4}\|\nabla T\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)}. \quad (37)$$

Combining (36) and (37) with (32) and simplifying yield

$$\|\theta^n\|_0^2-\|\theta^{n-1}\|_0^2+\frac{k}{4\gamma_0}\|\nabla\left(\theta^n+\theta^{n-1}\right)\|_0^2 \leq \hat{C}^2k^5, \quad (38)$$

where $\hat{C}^2k^5 = \frac{\tilde{N}_0^2\gamma_0}{64}\|\nabla \boldsymbol{u}(t)\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)}\|\nabla T(t)\|^2_{L^{2,\infty}(t_{n-1},t_n;L^2)} + \frac{1}{4}\|T\|^2_{W^{2,\infty}(t_{n-1},t_n;H^1)} + \frac{\gamma_0}{144}\|T\|^2_{W^{3,\infty}(t_{n-1},t_n;H^{-1})} + \frac{\tilde{N}_0^2\gamma_0}{64}\|\nabla T(t)\|^2_{W^{2,\infty}(t_{n-1},t_n;L^2)}\|\nabla\bar{\boldsymbol{u}}^n\|_0^2$.

Summing (38) from 1 to n yields that

$$\|\theta^n\|_0^2+\frac{k}{4\gamma_0}\sum_{i=1}^{n}\|\nabla\left(\theta^i+\theta^{i-1}\right)\|_0^2 \leq \hat{C}^2nk^5. \quad (39)$$

By extracting square root for (39) and using $\left(\sum_{i=0}^n b_i^2\right)^{1/2} \geq \sum_{i=0}^n |b_i|/\sqrt{n}$ and $\|a+b\|_0 \geq \|a\|_0-\|b\|_0$, we obtain

$$\|\theta^n\|_0+\frac{k}{2\sqrt{\gamma_0}}\|\nabla\theta^n\|_0 \leq \hat{C}\sqrt{t_N}k^2, \quad (40)$$

which yields (13). By (35) and (40), we have

$$\|\boldsymbol{e}^n\|_0^2-\|\boldsymbol{e}^{n-1}\|_0^2+\frac{k\nu}{4}\|\nabla\left(\boldsymbol{e}^n+\boldsymbol{e}^{n-1}\right)\|_0^2 \leq \hat{C}_0^2k^5, \quad (41)$$

where $\hat{C}_0^2 = 12\gamma_0\hat{C}^2T_\infty\nu^{-1}k^3+\tilde{C}^2$. Summing (41) from 1 to n yields that

$$\|\boldsymbol{e}^n\|_0^2+\frac{k\nu}{4}\sum_{i=1}^{n}\|\nabla\left(\boldsymbol{e}^i+\boldsymbol{e}^{i-1}\right)\|_0^2 \leq \hat{C}_0^2nk^5, \quad (42)$$

By extracting square root for (42) and using $\left(\sum_{i=0}^n b_i^2\right)^{1/2} \geq \sum_{i=0}^n |b_i|/\sqrt{n}$ and $\|a+b\|_0 \geq \|a\|_0-\|b\|_0$, we obtain

$$\|\boldsymbol{e}^n\|_0+\frac{k\sqrt{\nu}}{2}\|\nabla\boldsymbol{e}^n\|_0 \leq \hat{C}_0^2\sqrt{t_N}k^2. \quad (43)$$

By using Taylor's formula, there are $\xi_{in} \in [t_{n-1}, t_n]$($i = 5, 6, 7, 8, 9, 10, 11$) such that

$$\begin{aligned}
b(p(t_n)-p^n, \boldsymbol{v}) = &\frac{1}{k}(\boldsymbol{e}^n-\boldsymbol{e}^{n-1}, \boldsymbol{v})+\frac{\nu}{2}\left(\nabla(\boldsymbol{e}^n+\boldsymbol{e}^{n-1}), \nabla\boldsymbol{v}\right)\\
&+\frac{1}{2}\Big[a_1\left(\boldsymbol{e}^n+\boldsymbol{e}^{n-1}, \boldsymbol{u}(t_{n-\frac{1}{2}}), \boldsymbol{v}\right)+a_1\left(\bar{\boldsymbol{u}}^n, \boldsymbol{e}^n+\boldsymbol{e}^{n-1}, \boldsymbol{v}\right)\\
&-\left((\theta^n+\theta^{n-1})\boldsymbol{j}, \boldsymbol{v}\right)\Big]\\
&-\frac{k^2}{48}\left(\boldsymbol{u}_{ttt}(\xi_{5n}), \boldsymbol{v}\right)-\frac{k^2}{48}\left(\boldsymbol{u}_{ttt}(\xi_{6n}), \boldsymbol{v}\right)-\frac{\nu k^2}{16}\left(\nabla\boldsymbol{u}_{tt}(\xi_{7n}), \nabla\boldsymbol{v}\right)\\
&-\frac{\nu k^2}{16}\left(\nabla\boldsymbol{u}_{tt}(\xi_{8n}), \nabla\boldsymbol{v}\right)+\frac{k}{2}b(\boldsymbol{v}, p_t(\xi_{9n}))-\frac{k^2}{16}a_1(\boldsymbol{u}_{tt}(\xi_{10n}), \boldsymbol{u}(t_{n-\frac{1}{2}}), \boldsymbol{v})
\end{aligned}$$

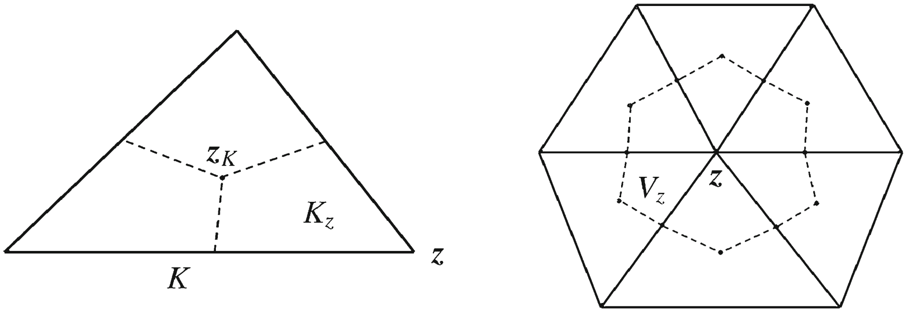

Fig. 1 *Left chart* is a triangle K partitioned into three sub-regions K_z. *Right chart* is a sample region with *dotted lines* indicating the corresponding control volume V_z

$$-\frac{k^2}{16}a_1(\bar{\boldsymbol{u}}^n, \boldsymbol{u}_{tt}(\xi_{11n}), \boldsymbol{v}) - \frac{k^2}{16}(T_{tt}(\xi_{3n})\boldsymbol{j}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in X. \tag{44}$$

Then, with (40), (43), (44), (7), Hölder inequality, and Cauchy inequality, we have that

$$\|p(t_n) - p^n\|_0 \leq \beta^{-1} \sup_{\boldsymbol{v}\in X} \frac{b\,(p(t_n) - p^n, \boldsymbol{v})}{\|\nabla v\|_0} \leq Ck. \tag{45}$$

Combining (45) and (43) yields (12), which completes the proof of Theorem 2.2. □

2.2 Fully Discrete SCNMFVE Formulation for Problem I

In order to establish the fully discrete SCNMFVE formulations for Problem II, it is necessary to introduce an FVE approximation for the spatial variables of Problem III (more details see [2,3,14,15]).

Firstly, let $\Im_h = \{K\}$ be a quasi-uniform triangulation of Ω with $h = \max h_K$, where h_K is the diameter of the triangle $K \in \Im_h$ (see [5,10,18]). In order to describe the SCNMFVE formulation, we introduce a dual partition $\Im_h^*$ based on $\Im_h$ whose elements are called the control volumes. We construct the control volume in the same way as in [2,3,14,15]. Let $z_K = (x_K, y_k)$ be the barycenter of $K \in \Im_h$. We connect z_K with line segments to the midpoints of the edges of K, thus partitioning K into three quadrilaterals $K_z(z = (x, y) \in Z_h(K)$, where $Z_h(K)$ are the vertices of K). Then with each vertex $z \in Z_h = \bigcup_{K\in\Im_h} Z_h(K)$ we associate a control volume V_z, which consists of the union of the sub-regions K_z, sharing the vertex z. Finally, we obtain a group of control volumes covering the domain Ω, which is called a barycenter-type dual partition $\Im_h^*$ of the triangulation $\Im_h$ (see Fig. 1). We denote the set of interior vertices of Z_h by Z_h°.

Since the FE triangulation $\Im_h$ is quasi-uniform, the dual partition $\Im_h^*$ is also quasi-uniform (see [2,3,5,10,14,15,18]). Moreover, the barycenter-type dual partition can lead to relatively simple calculations. To this end, the trial function spaces X_h, W_h, and M_h of the velocity, temperature, and pressure the velocity, the temperature and the pressure are, respectively, defined as follows:

$$X_h = \left\{\boldsymbol{v}_h \in X \cap C(\overline{\Omega})^2;\ \boldsymbol{v}_h|_K \in [\mathscr{P}_1(K)]^2, \forall K \in \Im_h\right\},$$
$$W_h = \left\{w_h \in W \cap C(\overline{\Omega});\ w_h|_K \in \mathscr{P}_1(K), \forall K \in \Im_h\right\},$$
$$M_h = \{q_h \in M;\ q_h|_K \in \mathscr{P}_1(K),\ \forall K \in \Im_h\},$$

where $\mathscr{P}_1(K)$ is the linear function space on K. Note that they are different from those in [16]. It is obvious that $X_h \subset X = H_0^1(\Omega)^2$ and $W_h \subset W = H_0^1(\Omega)$. For $(\boldsymbol{u}, T) \in X \times W$, let $(\Pi_h \boldsymbol{u}, \rho_h T)$ be the interpolation projection of $(\boldsymbol{u}, T)$ onto the trial function spaces $X_h \times W_h$. Then, due to the interpolation theory of Sobolev spaces (see [5,10,14,16,18]), we have the following error estimates

$$|\boldsymbol{u} - \Pi_h \boldsymbol{u}|_m \le Ch^{2-m}|\boldsymbol{u}|_2, \quad \forall \boldsymbol{u} \in H^2(\Omega)^2, m = 0, 1, \tag{46}$$

$$|T - \rho_h T|_m \le Ch^{2-m}|T|_2, \quad \forall T \in H^2(\Omega), m = 0, 1, \tag{47}$$

where C in this context indicates a positive constant which is possibly different at different occurrences, being independent of the spatial mesh size h and temporal mesh size k.

The test spaces $\tilde{X}_h$ and $\tilde{W}_h$ of the velocity and temperature are, respectively, chosen as follows:

$$\begin{aligned}
\tilde{X}_h = \big\{&\boldsymbol{v}_h \in L^2(\Omega)^2;\ \boldsymbol{v}_h|_{V_z} \in [\mathscr{P}_0(V_z)]^2 (V_z \cap \partial\Omega = \emptyset),\\
&\boldsymbol{v}_h|_{V_z} = \boldsymbol{0}(V_z \cap \partial\Omega \ne \emptyset), \forall V_z \in \Im_h^*\big\},\\
\tilde{W}_h = \big\{&w_h \in L^2(\Omega);\ w_h|_{V_z} \in \mathscr{P}_0(V_z)(V_z \cap \partial\Omega = \emptyset),\\
&w_h|_{V_z} = 0(V_z \cap \partial\Omega \ne \emptyset), \forall V_z \in \Im_h^*\big\},
\end{aligned} \tag{48}$$

where $\mathscr{P}_0(V_z)$ is the constant function space on V_z. In fact, they can be spanned by the following basis functions

$$\phi_z(x, y) = \begin{cases} 1, & (x, y) \in V_z, \\ 0, & \text{elsewhere}, \end{cases} \quad \forall \boldsymbol{z} \in Z_h^\circ. \tag{49}$$

For $(\boldsymbol{u}, w) \in X \times W$, let $(\Pi_h^* \boldsymbol{u}, \rho_h^* w)$ be the interpolation projection of $(\boldsymbol{u}, w)$ onto the test spaces $\tilde{X}_h \times \tilde{W}_h$, i.e.,

$$\left(\Pi_h^* \boldsymbol{u}, \rho_h^* w\right) = \sum_{z \in Z_h^\circ} (\boldsymbol{u}(z), w(z))\phi_z. \tag{50}$$

By the interpolation theory of Sobolev spaces (see [5,10,14,16,18,26]), we have

$$\|\boldsymbol{u} - \Pi_h^* \boldsymbol{u}\|_0 \le Ch|\boldsymbol{u}|_1; \quad \|w - \rho_h^* w\|_0 \le Ch|w|_1. \tag{51}$$

By using the same principle as mentioned in [16,17], the fully discrete SCNMFVE formulation for Problem II is written as follows.

Problem IV Find $(\boldsymbol{u}_h^n, p_h^n, T_h^n) \in X_h \times M_h \times W_h$ $(1 \le n \le N)$ such that

$$\begin{cases}
\left(\bar{\partial}_t \boldsymbol{u}_h^n, \Pi_h^* \boldsymbol{v}_h\right) + a_h\left(\bar{\boldsymbol{u}}_h^n, \Pi_h^* \boldsymbol{v}_h\right) + a_{1h}\left(\bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{u}}_h^n, \Pi_h^* \boldsymbol{v}_h\right) + b_h\left(p_h^n, \Pi_h^* \boldsymbol{v}_h\right) & \\
\quad = \left(\bar{T}_h^n \boldsymbol{j}, \Pi_h^* \boldsymbol{v}_h\right), & \forall \boldsymbol{v}_h \in X_h, \\
b\left(q_h, \boldsymbol{u}_h^n\right) + D_h\left(p_h^n, q_h\right) = 0, & \forall q_h \in M_h, \\
\left(\bar{\partial}_t T_h^n, \rho_h^* \varphi_h\right) + d_h\left(\bar{T}_h^n, \rho_h^* \varphi_h\right) + a_{2h}\left(\bar{\boldsymbol{u}}_h^n, \bar{T}_h^n, \rho_h^* \varphi_h\right) = 0, & \forall \varphi_h \in W_h, \\
\boldsymbol{u}_h^0 = \boldsymbol{0}, \quad T_h^0 = \rho_h \psi(x, y), & (x, y) \in \Omega,
\end{cases} \tag{52}$$

where

$$a_h\left(\boldsymbol{u}_h^n, \Pi_h^*\boldsymbol{v}_h\right) = -\nu \sum_{V_z\in\Im_h^*}\int_{\partial V_z}\left(\boldsymbol{v}_h(z)\nabla\boldsymbol{u}_h^n\right)\cdot\boldsymbol{n}\mathrm{d}s; \tag{53}$$

$$b_h\left(q_h, \Pi_h^*\boldsymbol{v}_h\right) = \sum_{V_z\in\Im_h^*}\boldsymbol{v}_h(z)\int_{\partial V_z} q_h\boldsymbol{n}\mathrm{d}s; \tag{54}$$

$$a_{1h}\left(\boldsymbol{u}_h^n, \boldsymbol{w}_h^n, \Pi_h^*\boldsymbol{v}_h\right) = \left(\left(\boldsymbol{u}_h^n\cdot\nabla\right)\boldsymbol{w}_h^n, \Pi_h^*\boldsymbol{v}_h\right) + \left(\left(\mathrm{div}\boldsymbol{u}_h^n\right)\boldsymbol{w}_h^n, \Pi_h^*\boldsymbol{v}_h\right)/2;$$

$$d_h\left(T_h^n, \rho_h^* w_h\right) = -\gamma_0^{-1}\sum_{V_z\in\Im_h^*} w_h(z)\int_{\partial V_z}\nabla T\cdot\boldsymbol{n}\mathrm{d}s; \tag{55}$$

$$a_{2h}\left(\boldsymbol{u}_h^n, T_h^n, \rho_h^*\varphi_h\right) = \left(\left(\boldsymbol{u}_h^n\cdot\nabla\right)T_h^n, \rho_h^*\varphi_h\right) + \left(\left(\mathrm{div}\boldsymbol{u}_h^n\right)T_h, \rho_h^*\varphi_h^n\right)/2; \tag{56}$$

$$D_h\left(p_h^n, q_h\right) = \varepsilon\sum_{K\in\Im_h}\left\{\int_{K,2} p_h^n q_h\mathrm{d}x\mathrm{d}y - \int_{K,1} p_h^n q_h\mathrm{d}x\mathrm{d}y\right\}, \tag{57}$$

here ε is a positive real number and $\int_{K,i} g(x,y)\mathrm{d}x\mathrm{d}y$ indicate an appropriate Gauss integral on K that is exact for polynomials of degree i $(i=1,2)$ and $g(x,y)=p_h q_h$ is a polynomial of degree not more than i $(i=1,2)$.

Thus, for all test functions $q_h\in M_h$, the trial function $p_h\in M_h$ must be piecewise constant when $i=1$. Consequently, we define the L^2—projection operator $\varrho_h : L^2(\Omega)\to\hat{W}_h$ such that $\forall p\in L^2(\Omega)$ satisfies

$$(p, q_h) = (\varrho_h p, q_h), \quad \forall q_h\in\hat{W}_h, \tag{58}$$

where $\hat{W}_h\subset L^2(\Omega)$ denotes the piecewise constant space associated with $\Im_h$. The projection operator ϱ_h has the following properties (see [5,18]).

$$\|\varrho_h p\|_0 \le C\|p\|_0, \quad \forall p\in L^2(\Omega), \tag{59}$$

$$\|p-\varrho_h p\|_0 \le Ch\|p\|_1, \quad \forall p\in H^1(\Omega). \tag{60}$$

Now, using the definition of ϱ_h, we can rewrite the bilinear form $D_h(\cdot,\cdot)$ as follows:

$$D_h(p_h, q_h) = \varepsilon(p_h-\varrho_h p_h, q_h) = \varepsilon(p_h-\varrho_h p_h, q_h-\varrho_h q_h). \tag{61}$$

3 Existence, Uniqueness, Stability, and Error Estimates for the SCNMFVE Solutions

In order to discuss the existence, the uniqueness, the stability, and the error estimates of the solutions for fully discrete SCNMFVE formulation with the second-order time accuracy or Problem IV, it is necessary to introduce some preliminary lemmas.

From [3,14,15,22] we have the following two lemmas.

Lemma 3.1 *There hold the following results:*

$$a_h\left(\boldsymbol{u}_h, \Pi_h^*\boldsymbol{v}_h\right) = a(\boldsymbol{u}_h, \boldsymbol{v}_h), \quad a_{1h}\left(\boldsymbol{v}_h, \boldsymbol{u}_h, \Pi_h^*\boldsymbol{u}_h\right) = 0, \quad \forall\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h\in U_h,$$
$$d_h\left(T_h, \rho_h^*\phi_h\right) = d(T_h, \phi_h), \quad a_{2h}\left(\boldsymbol{u}_h, T_h, \rho_h^* T_h\right) = 0, \quad \forall T_h, \phi_h\in W_h, \forall\boldsymbol{u}_h\in X_h,$$
$$b_h\left(p_h, \Pi_h^*\boldsymbol{v}_h\right) = -b(p_h, \boldsymbol{v}_h), \quad \forall\boldsymbol{v}_h\in X_h, \forall p_h\in M_h.$$

Further, $a_h(\boldsymbol{u}_h, \Pi_h^* v_h)$ and $d_h(T_h, \rho_h^* w_h)$ are all symmetric, bounded, and positive definite, that is,

$$a_h\left(\boldsymbol{u}_h, \Pi_h^* \boldsymbol{v}_h\right) = a_h\left(\boldsymbol{v}_h, \Pi_h^* \boldsymbol{u}_h\right), \quad \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in U_h;$$
$$d_h\left(T_h, \rho_h^* w_h\right) = d_h\left(w_h, \rho_h^* T_h\right), \quad \forall T_h, w_h \in W_h,$$

and there exist a positive constants $h_0 \geq h > 0$ such that

$$a_h\left(\boldsymbol{u}_h, \Pi_h^* \boldsymbol{u}_h\right) = \nu |\boldsymbol{u}_h|_1^2, \quad |a_h\left(\boldsymbol{u}_h, \Pi_h^* \boldsymbol{v}_h\right)| \leq \nu \|\boldsymbol{u}_h\|_1 \|\boldsymbol{v}_h\|_1, \quad \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in X_h.$$
$$d_h\left(T_h, \rho_h^* T_h\right) \geq \gamma_0^{-1} |T_h|_1^2, \quad |d_h\left(T_h, \rho_h^* w_h\right)| \leq \tilde{C}_0 \|T_h\|_1 \|w_h\|_1, \quad \forall T_h, w_h \in W_h.$$

Lemma 3.2 *There holds the following statement:*

$$\left(\boldsymbol{u}_h, \Pi_h^* \boldsymbol{v}_h\right) = \left(\boldsymbol{v}_h, \Pi_h^* \boldsymbol{u}_h\right), \quad \forall \boldsymbol{u}_h, \boldsymbol{v}_h \in X_h.$$

For any $\boldsymbol{u} \in H^m(\Omega)^2$ $(m = 0, 1)$ and $\boldsymbol{v}_h \in X_h$,

$$|(\boldsymbol{u}, \boldsymbol{v}_h) - \left(\boldsymbol{u}, \Pi_h^* v_h\right)| \leq C h^{m+n} \|\boldsymbol{u}\|_m \|\boldsymbol{v}_h\|_n, \quad n = 0, 1.$$

Set $|||\boldsymbol{u}_h|||_0 = (\boldsymbol{u}_h, \Pi_h^* \boldsymbol{u}_h)^{1/2}$, then $|||\cdot|||_0$ is equivalent to $\|\cdot\|_0$ on X_h, namely there exist two positive constants C_1 and C_2 such that

$$C_1 \|\boldsymbol{u}_h\|_0 \leq |||\boldsymbol{u}_h|||_0 \leq C_2 \|\boldsymbol{u}_h\|_0, \quad \forall \boldsymbol{u}_h \in X_h. \tag{62}$$

Remark 1 For scalar functions, i.e., if $\boldsymbol{u}_h$ and $\boldsymbol{v}_h$ in X_h are, respectively, substituted with w_h and T_h in W_h, the results of Lemma 3.2 hold (see Theorem 3.2.1 and Lemma 5.1.5 in [14]).

The following discrete Gronwall Lemma (see [5,17,18]) is useful for the proofs of the existence, the uniqueness, the stability, and the error estimates of the solutions for Problem IV.

Lemma 3.3 *If $\{a_n\}$ and $\{b_n\}$ are two positive sequences, $\{c_n\}$ is a monotone positive sequence, and they satisfy $a_n + b_n \leq c_n + \bar{\lambda} \sum_{i=0}^{n-1} a_i (\bar{\lambda} > 0)$ and $a_0 + b_0 \leq c_0$, then $a_n + b_n \leq c_n \exp(n\bar{\lambda}) (n = 0, 1, 2, \ldots)$.*

There are the following results of the existence, the uniqueness, and the stability of the solutions for Problem IV.

Theorem 3.4 *Under the hypotheses of* Theorems 2.1 and 2.2*, there exists a unique series of solutions $(\boldsymbol{u}_h^n, p_h^n, T_h^n)(n = 1, 2, \ldots, N)$ for the fully discrete SCNMFVE formulation with the second-order time accuracy, i.e., Problem IV satisfying*

$$\|\boldsymbol{u}_h^n\|_0 + \|T_h^n\|_0 + k\|\nabla \boldsymbol{u}_h^n\|_0 + k\|\nabla T_h^n\|_0 + \sqrt{k}\|p_h^n\|_0 \leq C(\|\psi\|_0 + k\|\nabla\psi|\|_0), \tag{63}$$

which shows that the series of solutions of Problem IV is stable.

Proof First, we prove that Problem IV has a unique series of solutions. Because the finite dimensional subspaces $X_h \times M_h \times W_h$ are also weakly and sequentially compact Hilbert spaces, by using the same as the technique to prove that Problem III has a unique series of solutions and apply to fixed point theorem (see [21]) to Problem IV, we can prove that Problem VI has a unique sequence of solutions $(\boldsymbol{u}_h^n, p_h^n, T_h^n) \in X_h \times M_h \times W_h$.

And then, we prove that (63) holds. By taking $\boldsymbol{v}_h = \bar{\boldsymbol{u}}_h^n$ in the first equation of Problem IV and $q_h = p_h^n$ in the second equation of Problem IV and by using Lemmas 3.1 and 3.2, Hölder inequality, and Cauchy inequality, we obtain

$$\frac{1}{2}\left(\||\boldsymbol{u}_h^n|\|_0^2 - \||\boldsymbol{u}_h^{n-1}|\|_0^2\right) + k\nu\|\nabla\bar{\boldsymbol{u}}_h^n\|_0^2 + k\varepsilon\|p_h^n - \rho_h p_h^n\|_0^2 = k\left(\bar{T}_h^n \boldsymbol{j}, \Pi_h^*\bar{\boldsymbol{u}}_h^n\right) \le \frac{k}{2\nu}\|\bar{T}_h^n\|_{-1}^2 + \frac{k\nu}{2}\|\nabla\bar{\boldsymbol{u}}_h^n\|_0^2. \tag{64}$$

It follows from (64) that

$$\||\boldsymbol{u}_h^n|\|_0^2 - \||\boldsymbol{u}_h^{n-1}|\|_0^2 + 2k\nu\|\nabla\bar{\boldsymbol{u}}_h^n\|_0^2 + 2k\varepsilon\|p_h^n - \rho_h p_h^n\|_0^2 \le k\nu^{-1}\|\bar{T}_h^n\|_{-1}^2. \tag{65}$$

Summing (65) from 1 to n yields that

$$\||\boldsymbol{u}_h^n|\|_0^2 + \frac{k\nu}{2}\sum_{i=1}^{n}\|\nabla\left(\boldsymbol{u}_h^i + \boldsymbol{u}_h^{i-1}\right)\|_0^2 + 2k\varepsilon\|p_h^n - \rho_h p_h^n\|_0^2 \le k\nu^{-1}\sum_{i=1}^{n}\|\bar{T}_h^i\|_{-1}^2. \tag{66}$$

If $p_h^n \ne 0$, then it is easily see that $\|p_h^n\|_0 > \|\rho_h p_h^n\|_0$ from (2.51). Therefore, there exists a constant $\delta \in (0, 1)$ such that $\delta\|p_h^n\|_0 = \|\rho_h p_h^n\|_0$. By extracting square root for (66), using $\left(\sum_{i=1}^n b_i^2\right)^{1/2} \ge \sum_{i=1}^n |b_i|/\sqrt{n}$, $\|a+b\|_0 \ge \|a\|_0 - \|b\|_0$, and Lemma 3.2, and then, simplifying, we obtain

$$\|\boldsymbol{u}_h^n\|_0 + k\|\nabla\boldsymbol{u}_h^n\|_0 + \sqrt{k}\|p_h^n\|_0 \le C\left(k\sum_{i=1}^{n}\|\bar{T}_h^i\|_{-1}^2\right)^{1/2}. \tag{67}$$

By taking $\varphi_h = \bar{T}_h^n$ in the third equation of Problem IV and by using Lemmas 3.1 and 3.2, Hölder inequality, and Cauchy inequality, we obtain

$$\||T_h^n|\|_0^2 - \||T_h^{n-1}|\|_0^2 + \frac{k}{2\gamma_0}\|\nabla\left(T_h^n + T_h^{n-1}\right)\|_0^2 = 0. \tag{68}$$

Summing (68) from 1 to n yields that

$$\||T_h^n|\|_0^2 + \frac{k}{2\gamma_0}\sum_{i=1}^{n}\|\nabla\left(T_h^i + T_h^{i-1}\right)\|_0^2 = \||\psi|\|_0^2. \tag{69}$$

By extracting square root for (69) and using $\left(\sum_{i=0}^n b_i^2\right)^{1/2} \ge \sum_{n=0}^n |b_i|/\sqrt{n}$, $\|a+b\|_0 \ge \|a\|_0 - \|b\|_0$, and Lemma 3.2, we obtain

$$\|T_h^n\|_0 + k\|\nabla T_h^n\|_0 \le C(\|\psi\|_0 + k\|\nabla\psi|\|_0). \tag{70}$$

From (67) and (69) and by using Lemma 3.2, we obtain

$$\|\boldsymbol{u}_h^n\|_0 + k\|\nabla\boldsymbol{u}_h^n\|_0 + \sqrt{k}\|p_h^n\|_0 \le C\|\psi\|_0 \le C(\|\psi\|_0 + k\|\nabla\psi|\|_0). \tag{71}$$

Combining (70) with (71) yields (63). If $p_h^n = 0$, (63) is correct, which completes the proof of Theorem 3.4. □

Put

$$\begin{aligned}
\mathscr{A}\left((S_h\bar{\boldsymbol{u}}^n, Q_h p^n); (\boldsymbol{v}_h, q_h)\right) &= a(S_h\bar{\boldsymbol{u}}^n, \boldsymbol{v}_h) + a_1(S_h\bar{\boldsymbol{u}}^n, S_h\bar{\boldsymbol{u}}^n, \boldsymbol{v}_h)\\
&\quad - b(Q_h p^n, \boldsymbol{v}_h) + b(q_h, S_h\boldsymbol{u}^n),\\
\mathscr{A}\left((\bar{\boldsymbol{u}}^n, p^n); (\boldsymbol{v}_h, q_h)\right) &= a(\bar{\boldsymbol{u}}^n, \boldsymbol{v}_h) + a_1(\bar{\boldsymbol{u}}^n, \bar{\boldsymbol{u}}^n, \boldsymbol{v}_h) - b(p^n, \boldsymbol{v}_h) + b(q_h, \boldsymbol{u}^n).
\end{aligned} \tag{72}$$

By using the stabilized CN mixed FE (SCNMFE) methods (for example, see [12,17,18]) for the non-stationary Navier–Stokes equations, we obtain the following Lemma 3.5.

Lemma 3.5 *Let* $(S_h\boldsymbol{u}^n, Q_h p^n)$ *be the Navier–Stokes projection of the solutions* $(\boldsymbol{u}^n, p^n)$ *for Problem III on* $U_h \times M_h$*, that is, for the solutions* $(\boldsymbol{u}^n, p^n) \in U \times M$ *for Problem III, there exist* $(S_h\boldsymbol{u}^n, Q_h p^n)$ $(n = 1, 2, \ldots, N)$ *such that*

$$k\mathscr{A}\left((S_h\bar{\boldsymbol{u}}^n, Q_h p^n); (\boldsymbol{v}_h, q_h)\right) + (S_h\boldsymbol{u}^n - S_h\boldsymbol{u}^{n-1}, \boldsymbol{v}_h) + kD_h(Q_h p^n, q_h) \tag{73}$$
$$= k\mathscr{A}\left((\bar{\boldsymbol{u}}^n, p^n); (\boldsymbol{v}_h, q_h)\right) + (\boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \boldsymbol{v}_h), \forall(\boldsymbol{v}_h, q_h) \in U_h \times M_h, \quad n = 1, 2, \ldots, N,$$
$$S_h\boldsymbol{u}^0 = \Pi_h\boldsymbol{u}^0(x, y), \quad \boldsymbol{u}^0 = \boldsymbol{u}^0(x, y), \quad (x, y) \in \Omega. \tag{74}$$

Then, there hold

$$\|S_h\boldsymbol{u}^n\|_1 + \|Q_h p^n\|_0 \le C\left(\|\boldsymbol{u}^n\|_1 + \|p^n\|_0\right), \quad 1 \le n \le N. \tag{75}$$

If $k = O(h)$ *and the solution* $(\boldsymbol{u}^n, p^n) \in H^2(\Omega)^2 \times H^1(\Omega)$ $(n = 1, 2, \ldots, N)$ *for Problem III, then there hold the following error estimates*

$$\|\boldsymbol{u}^n - S_h\boldsymbol{u}^n\|_0 + k\|\nabla\left(\boldsymbol{u}^n - S_h\boldsymbol{u}^n\right)\|_0 + k\|p^n - Q_h p^n\|_0 \le Ch^2, \quad n = 1, 2, \ldots, N. \tag{76}$$

Remark 2 In fact, (73) and (74) are the system of error equations between standard SCN-MFE formulation and the semi-discrete CN formulation about time of the non-stationary Navier–Stokes equations, thus (75) and (76) are directly obtained from SCNMFE method (for example, see [12,17,18]).

By the FE methods (see, e.g., [5,10,18]) for elliptic equations, we have the following Lemma 3.6.

Lemma 3.6 *Let* $R_h : W \to W_h$ *be a generalized Ritz projection, i.e., for given* $\boldsymbol{u}_h^n \in X_h$, $T^{n-1} \in W$*, and* $T_h^{n-1} \in W_h$*, and for any* $T^n \in W$ $(n = 1, 2, \ldots, N)$*, there exist* $R_hT^n \in W_h$ $(n = 1, 2, \ldots, N)$ *such that*

$$\left(R_hT^n, w_h\right) + kd\left(R_h\bar{T}^n, w_h\right) + ka_2\left(\bar{\boldsymbol{u}}_h^n, R_h\bar{T}^n, w_h\right) - \left(R_hT^{n-1}, w_h\right) = \left(T^n, w_h\right)$$
$$+ kd\left(\bar{T}^n, w_h\right) + ka_2\left(\bar{\boldsymbol{u}}^n, \bar{T}^n, w_h\right) - \left(T^{n-1}, w_h\right), \quad \forall w_h \in W_h, \quad n = 1, 2, \ldots, N. \tag{77}$$

If $(\boldsymbol{u}^n, p^n, T^n)$ $(n = 1, 2, \ldots, N)$ *are the solutions of Problem III and* $T^n \in H^2(\Omega) \cap W$*, we have the following inequalities*

$$\|R_hT^n\|_0 + k\|\nabla R_hT^n\|_0 \le C\|\nabla T^n\|_0, \quad n = 0, 1, 2, \ldots, N, \tag{78}$$
$$\|R_hT^n - T^n\|_0 + k\|\nabla\left(R_hT^n - T^n\right)|_0 \le Ch^2\|\psi\|_2, \quad n = 0, 1, 2, \ldots, N. \tag{79}$$

Theorem 3.7 *Let* $(\boldsymbol{u}, p, T)$ *be the solution for Problem II and* $(\boldsymbol{u}_h^n, p_h^n, T_h^n)$ *the series of solutions of fully discrete SCMNFVE formulation with the second-order time accuracy (that is, Problem IV). Then, under the hypotheses of* Theorems 2.2 and 3.4*, if* $p_h^0 = p^0 = 0$ *(or* $p_h^0 = Q_h p^0$*),* $h = O(k)$*,* $N_0\nu^{-1}\|\nabla\bar{\boldsymbol{u}}_h^n\|_0 \le 1/4$*, and* $\psi \in H^1(\Omega)$*, we have the following error estimates*

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n\|_0 + \|T(t_n) - T_h^n\|_0 + k[\|p(t_n) - p_h^n\|_0 + \|\nabla(\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n)\|_0$$
$$+ \|\nabla\left(T(t_n) - T_h^n\right)\|_0] \le Ch^2\|\psi\|_1, \quad n = 1, 2, \ldots, N. \tag{80}$$

Proof By subtracting Problem IV from Problem III taking $\boldsymbol{v} = \boldsymbol{v}_h$, $q = q_h$, and $\varphi = \varphi_h$, and by using Lemmas 3.1 and 3.2, we obtain the system of error equations:

$$\begin{cases} \left(\boldsymbol{u}^n - \boldsymbol{u}_h^n, \boldsymbol{v}_h\right) + \left(\boldsymbol{u}_h^n - \Pi_h^* \boldsymbol{u}_h^n, \boldsymbol{v}_h - \Pi_h^* \boldsymbol{v}_h\right) + ka\left(\bar{\boldsymbol{u}}^n - \bar{\boldsymbol{u}}_h^n, \boldsymbol{v}_h\right) \\ \quad + ka_1\left(\bar{\boldsymbol{u}}^n, \bar{\boldsymbol{u}}^n, \boldsymbol{v}_h\right) - ka_{1h}\left(\bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{u}}_h^n, \Pi^* \boldsymbol{v}_h\right) - kb\left(p^n - p_h^n, \boldsymbol{v}_h\right) \\ \quad = k\left((\bar{T}^n - \bar{T}_h^n)\boldsymbol{j}, \boldsymbol{v}_h\right) - k\left(\bar{T}_h^n \boldsymbol{j}, \Pi_h^* \boldsymbol{v}_h - \boldsymbol{v}_h\right) + \left(\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h\right) \\ \qquad - \left(\boldsymbol{u}_h^{n-1}, \Pi_h^* \boldsymbol{v}_h - \boldsymbol{v}_h\right), \quad \forall \boldsymbol{v}_h \in X_h, \quad n = 1, 2, \ldots, N, \\ b\left(q_h, \boldsymbol{u}^n - \boldsymbol{u}_h^n\right) - \varepsilon\left(p_h^n - \varrho_h p_h^n, q_h - \varrho_h q_h\right) = 0, \forall q_h \in M_h, \quad n = 1, 2, \ldots, N, \\ \left(T^n - T_h^n, \varphi_h\right) - \left(T_h^n, \rho_h^* \varphi_h - \varphi_h\right) + kd\left(\bar{T}^n - \bar{T}_h^n, \varphi_h\right) + ka_2\left(\bar{\boldsymbol{u}}^n, \bar{T}^n, \varphi_h\right) \\ \quad - ka_{2h}\left(\bar{\boldsymbol{u}}_h^n, \bar{T}_h^n, \rho_h^* \varphi_h\right) = \left(T^{n-1} - T_h^{n-1}, \varphi_h\right) \\ \quad - \left(T_h^{n-1}, \rho_h^* \varphi_h - \varphi_h\right), \forall \varphi_h \in W_h, \quad n = 1, 2, \ldots, N, \\ \boldsymbol{u}^0 - \boldsymbol{u}_h^0 = \boldsymbol{0}, \quad T^0 - T_h^0 = \psi(x, y) - \rho_h \psi(x, y), \quad (x, y) \in \Omega. \end{cases} \tag{81}$$

Let $\zeta^n = Q_h p^n - p_h^n$, $\boldsymbol{E}^n = S_h \boldsymbol{u}^n - \boldsymbol{u}_h^n$, and $\bar{\boldsymbol{E}}^n = S_h \bar{\boldsymbol{u}}^n - \bar{\boldsymbol{u}}_h^n$. By using (73) and the system of error equations (81), we obtain

$$\begin{aligned} \frac{1}{2}\|\boldsymbol{E}^n\|_0^2 + k\nu|\bar{\boldsymbol{E}}^n|_1^2 &= \left(S_h \boldsymbol{u}^n - \boldsymbol{u}^n, \bar{\boldsymbol{E}}^n\right) + ka\left(S_h \bar{\boldsymbol{u}}^n - \bar{\boldsymbol{u}}^n, \bar{\boldsymbol{E}}^n\right) \\ &\quad + \left(\boldsymbol{u}^n - \boldsymbol{u}_h^n, \bar{\boldsymbol{E}}^n\right) + ka\left(\bar{\boldsymbol{u}}^n - \bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{E}}^n\right) - \frac{1}{2}\left(\boldsymbol{E}^{n-1}, \boldsymbol{E}^n\right) \\ &= \left(S_h \boldsymbol{u}^{n-1} - \boldsymbol{u}^{n-1}, \bar{\boldsymbol{E}}^n\right) + kb\left(Q_h p^n - p^n, \bar{\boldsymbol{E}}^n\right) + ka_1\left(\bar{\boldsymbol{u}}^n, \bar{\boldsymbol{u}}^n, \bar{\boldsymbol{E}}^n\right) \\ &\quad - ka_1\left(S_h \bar{\boldsymbol{u}}^n, S_h \bar{\boldsymbol{u}}^n, \bar{\boldsymbol{E}}^n\right) - ka_1\left(\bar{\boldsymbol{u}}^n, \bar{\boldsymbol{u}}^n, \bar{\boldsymbol{E}}^n\right) + ka_{1h}\left(\bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{u}}_h^n, \Pi_h^* \bar{\boldsymbol{E}}^n\right) \\ &\quad + kb\left(p^n - p_h^n, \bar{\boldsymbol{E}}^n\right) - \left(\boldsymbol{u}_h^n - \Pi_h^* \boldsymbol{u}_h^n, \bar{\boldsymbol{E}}^n - \Pi_h^* \bar{\boldsymbol{E}}^n\right) \\ &\quad + \left(\boldsymbol{u}_h^{n-1} - \Pi_h^* \boldsymbol{u}_h^{n-1}, \bar{\boldsymbol{E}}^n - \Pi_h^* \bar{\boldsymbol{E}}^n\right) - \frac{1}{2}(S_h \boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1}, \boldsymbol{E}^n) \\ &\quad + \left(\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1}, \bar{\boldsymbol{E}}^n\right) + k\left((\bar{T}^n - \bar{T}_h^n)\boldsymbol{j}, \bar{\boldsymbol{E}}^n\right) \\ &\quad - k\left(\bar{T}_h^n \boldsymbol{j}, \Pi_h^* \bar{\boldsymbol{E}}^n - \bar{\boldsymbol{E}}^n\right) \\ &= k\left((\bar{T}^n - \bar{T}_h^n)\boldsymbol{j}, \bar{\boldsymbol{E}}^n\right) - k\left(\bar{T}_h^n \boldsymbol{j}, \Pi_h^* \bar{\boldsymbol{E}}^n - \bar{\boldsymbol{E}}^n\right) - ka_1\left(\bar{\boldsymbol{E}}_h^n, \bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{E}}^n\right) \\ &\quad + ka_{1h}\left(\bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{E}}^n - \Pi_h^* \bar{\boldsymbol{E}}^n\right) + \frac{1}{2}\left(\boldsymbol{E}^{n-1}, \boldsymbol{E}^{n-1}\right) + kb\left(\zeta^n, \bar{\boldsymbol{E}}^n\right) \\ &\quad - \left(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1} - \Pi_h^*\left(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\right), \bar{\boldsymbol{E}}^n - \Pi_h^* \bar{\boldsymbol{E}}^n\right). \end{aligned} \tag{82}$$

By using Lemma 3.2, Hölder inequality, and Cauchy inequality, we have

$$\begin{aligned} &|k\left((\bar{T}^n - \bar{T}_h^n)\boldsymbol{j}, \bar{\boldsymbol{E}}^n\right) - k\left(T_h^n \boldsymbol{j}, \Pi_h^* \bar{\boldsymbol{E}}^n - \bar{\boldsymbol{E}}^n\right)| \\ &\leq Ck\|\bar{T}^n - \bar{T}_h^n\|_0 \|\bar{\boldsymbol{E}}^n\|_0 + Ckh^2 \|\nabla \bar{T}_h^n\|_0 \|\nabla \bar{\boldsymbol{E}}_n\|_0 \\ &\leq Ck\|\bar{T}^n - \bar{T}_h^n\|_0^2 + Ck\|\bar{\boldsymbol{E}}^n\|_0^2 + Ckh^4 + \frac{\nu k}{8}\|\nabla \bar{\boldsymbol{E}}^n\|_0^2, \end{aligned} \tag{83}$$

If $k = O(h)$, by using inverse error estimate and Taylor's formula, we obtain

$$\begin{aligned} &|\left(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1} - \Pi_h^*\left(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\right), \bar{\boldsymbol{E}}^n - \Pi_h^* \bar{\boldsymbol{E}}^n\right)| \leq Ch^2 \|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\|_1 |\bar{\boldsymbol{E}}^n|_1 \\ &\leq Ch^3\left(\|\nabla \boldsymbol{E}^n\|_0^2 + \|\nabla\left(S_h \boldsymbol{u}^n - S_h \boldsymbol{u}^{n-1}\right)\|_0^2 + \|\nabla \boldsymbol{E}^{n-1}\|_0^2\right) + \frac{k\nu}{8}\|\nabla \bar{\boldsymbol{E}}^n\|_0^2 \end{aligned}$$

$$\leq Ch\|\boldsymbol{E}^n\|_0^2 + Ck^2h^3 + Ch\|\boldsymbol{E}^{n-1}\|_0^2 + \frac{k\nu}{8}\|\nabla\bar{\boldsymbol{E}}^n\|_0^2. \tag{84}$$

□

Noting that $b(q_h, S_h\boldsymbol{u}^n - \boldsymbol{u}^n) = -k\varepsilon(Q_h p^n - \rho_h(Q_h p^n), q_h - \rho_h q_h)$, by the properties of operator ϱ_h and the second equation of (81), we have

$$\begin{aligned} b(\zeta^n, \bar{\boldsymbol{E}}^n) &= b(\zeta^n, S_h\bar{\boldsymbol{u}}^n - \bar{\boldsymbol{u}}^n) + b\left(\zeta^n, \bar{\boldsymbol{u}}^n - \bar{\boldsymbol{u}}_h^n\right) \\ &= -\frac{\varepsilon}{2}(\zeta^n - \varrho_h\zeta^n, \zeta^n - \varrho_h\zeta^n) - \frac{\varepsilon}{2}(\zeta^{n-1} - \varrho_h\zeta^{n-1}, \zeta^n - \varrho_h(\zeta^n) \\ &\leq -\frac{\varepsilon}{4}\|\zeta^n - \varrho_h\zeta^n\|_0^2 + \frac{\varepsilon}{4}\|\zeta^{n-1} - \varrho_h\zeta^{n-1}\|_0^2. \end{aligned} \tag{85}$$

If $N_0\nu^{-1}\|\nabla\bar{\boldsymbol{u}}_h^n\|_0 \leq 1/4$ ($n = 1, 2, \ldots, N$), by Lemma 3.2, (3), and (8), we obtain

$$k|a_{1h}\left(\bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{E}}^n - \Pi_h^*\bar{\boldsymbol{E}}^n\right) - a_1\left(\bar{\boldsymbol{E}}^n, \bar{\boldsymbol{u}}_h^n, \bar{\boldsymbol{E}}^n\right)| \leq Ckh^4 + \frac{k\nu}{4}\|\nabla\bar{\boldsymbol{E}}^n\|_0^2. \tag{86}$$

Combining (83–86) with (82) yields that

$$\begin{aligned} &\|\boldsymbol{E}^n\|_0^2 + k\nu\|\nabla\bar{\boldsymbol{E}}^n\|_0^2 + \frac{k\varepsilon}{2}\|\zeta^n - \varrho_h\zeta^n\|_0^2 - \frac{k\varepsilon}{2}\|\zeta^{n-1} - \varrho_h\zeta^{n-1}\|_0^2 \\ &\quad \leq Ckh^4 + Ck^2h^3 + \|E^{n-1}\|_0^2 + Ch\|\boldsymbol{E}^{n-1}\|_0^2 + Ch\|\boldsymbol{E}^n\|_0^2 + Ck\|\bar{T}^n - \bar{T}_h^n\|_0^2. \end{aligned} \tag{87}$$

By summing (87) from 1 to n, if h is sufficiently small such that $Ch \leq 1/2$ in (87) and $p_h^0 = p^0 = 0$ (or $p_h^0 = Q_h p^0$), we obtain that

$$\begin{aligned} &\|\boldsymbol{E}_n\|_0^2 + 2k\nu\sum_{i=1}^n\|\bar{\boldsymbol{E}}_i\|_0^2 + k\varepsilon\|\zeta^n - \varrho_h\zeta^n\|_0^2 \\ &\quad \leq Cnkh^4 + Ck\sum_{i=1}^n\|\bar{T}^i - \bar{T}_h^i\|_0^2 + Ck\sum_{i=0}^{n-1}\|\boldsymbol{E}_i\|_0^2. \end{aligned} \tag{88}$$

Applying Gronwall Lemma 3.3 to (88) yields that

$$\begin{aligned} &\|\boldsymbol{E}_n\|_0^2 + k\sum_{i=1}^n\|\nabla\bar{\boldsymbol{E}}_i\|_0^2 + k\|\zeta^n - \varrho_h\zeta^n\|_0^2 \\ &\quad \leq C\left(h^4 + k\sum_{i=1}^n\|\bar{T}^i - \bar{T}_h^i\|_0^2\right)\exp(Ckn). \end{aligned} \tag{89}$$

By extracting square root for (89) and using $\left(\sum_{i=0}^n b_i^2\right)^{1/2} \geq \sum_{i=0}^n |b_i|/\sqrt{n}$, $\|a+b\|_0 \geq \|a\|_0 - \|b\|_0$, we obtain

$$\|\boldsymbol{E}_n\|_0 + k\|\nabla\bar{\boldsymbol{E}}_n\|_0 + k(\|\zeta^n\|_0 - \|\varrho_h\zeta^n\|_0) \leq C\left(h^4 + k\sum_{i=1}^n\|\bar{T}^i - \bar{T}_h^i\|_0^2\right)^{1/2}. \tag{90}$$

If $\zeta^n \neq 0$, then $\|\zeta^n\|_0 > \|\varrho_h\zeta^n\|_0$. Thus, there is a constant $\omega \in (0, 1)$ such that $\omega\|\zeta^n\|_0 = \|\varrho_h\zeta^n\|_0$. Then, by using triangle inequality, (90), and Lemma 3.4, we obtain

$$\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_0 + k[\|\nabla(\boldsymbol{u}^n - \boldsymbol{u}_h^n)|_0 + \|p^n - p_h^n\|_0] \leq C\left(h^4 + k\sum_{i=1}^n\|\bar{T}^i - \bar{T}_h^i\|_0^2\right)^{1/2}. \tag{91}$$

Let $e_n = R_h T^n - T_h^n$. On the one hand, by using the system of error equations (71), and Lemma 3.6, we obtain that

$$\begin{aligned}
\frac{1}{2}\|e_n\|_0^2 + k\gamma_0^{-1}\|\nabla\bar{e}_n\|_0^2 &= (e_n, \bar{e}_n) + kd(\bar{e}_n, \bar{e}_n) - \frac{1}{2}(e_n, e_{n-1})\\
&= \left[\left(R_h T^n - T^n, \bar{e}_n\right) + kd\left(R_h \bar{T}^n - \bar{T}^n, \bar{e}_n\right)\right]\\
&\quad + \left[\left(T^n - T_h^n, \bar{e}_n\right) + kd\left(\bar{T}^n - \bar{T}_h^n, \bar{e}_n\right)\right] - \frac{1}{2}(e_n, e_{n-1})\\
&= \left[\left(R_h T^{n-1} - T^{n-1}, \bar{e}_n\right) + ka_2\left(\bar{\boldsymbol{u}}^n, \bar{T}^n, \bar{e}_n\right) - ka_2\left(\bar{\boldsymbol{u}}_h^n, R_h\bar{T}^n, \bar{e}_n\right)\right]\\
&\quad + \left[\left(T_h^n, \rho_h^*\bar{e}_n - \bar{e}_n\right) + ka_{2h}\left(\bar{\boldsymbol{u}}_h^n, \bar{T}_h^n, \rho_h^*\bar{e}_n\right) - ka_2\left(\bar{\boldsymbol{u}}^n, \bar{T}^n, \bar{e}_n\right)\right.\\
&\quad \left. + \left(T^{n-1} - T_h^{n-1}, \bar{e}_n\right) - \left(T_h^{n-1}, \rho_h^*\bar{e}_n - \bar{e}_n\right)\right] - \frac{1}{2}(e_{n-1}, e_n)\\
&= \frac{1}{2}(e_{n-1}, e_n) + \left(T_h^n - T_h^{n-1}, \rho_h^*\bar{e}_n - \bar{e}_n\right)\\
&\quad + ka_{2h}\left(\bar{\boldsymbol{u}}_h^n, \bar{T}_h^n, \rho_h^*\bar{e}_n\right) - ka_2\left(\bar{\boldsymbol{u}}_h^n, R_h\bar{T}^n, \bar{e}_n\right). \qquad (92)
\end{aligned}$$

By using Lemma 3.2, Hölder inequality, and Cauchy inequality, we obtain

$$\begin{aligned}
(T_h^n - T_h^{n-1}, \rho_h^*\bar{e}_n - \bar{e}_n) &\le Ch(\|e_n\|_0 + \|R_h T^n - T^n\|_0 + h\|T^n - T^{n-1}\|_1\\
&\quad + \|T^{n-1} - R_h T^{n-1}\|_0 + \|e_{n-1}\|_0)\|\nabla\bar{e}_n\|_0\\
&\le Ch\left(h^4 + k^2h^2 + \|e_n\|_0^2 + \|e_{n-1}\|_0^2\right) + \frac{k}{4\gamma_0}\|\nabla\bar{e}_n\|_0^2, \qquad (93)
\end{aligned}$$

$$\frac{1}{2}(e_{n-1}, e_n) \le \frac{1}{2}\|e_{n-1}\|_0\|e_n\|_0 \le \frac{1}{4}\|e_{n-1}\|_0^2 + \frac{1}{4}\|e_n\|_0^2. \qquad (94)$$

If $N_0\nu^{-1}\|\nabla\bar{\boldsymbol{u}}_h^n\|_0 \le 1/4$ $(n = 1, 2, \ldots, N)$, by using Lemmas 3.1 and 3.2, (4), Hölder inequality, and Cauchy inequality, we obtain

$$ka_{2h}(\bar{\boldsymbol{u}}_h^n, \bar{T}_h^n, \rho_h^*\bar{e}_n) - ka_2(\bar{\boldsymbol{u}}_h^n, R_h\bar{T}^n, \bar{e}_n) \le \frac{k}{4\gamma_0}\|\nabla\bar{e}_n\|_0^2 + Ckh^4. \qquad (95)$$

If $k = O(h)$, by combining (93–95) with (92), we obtain

$$\|e_n\|_0^2 + k\|\nabla\bar{e}_n\|_0^2 \le Ck\left(h^4 + \|e_n\|_0^2 + \|e_{n-1}\|_0^2\right) + \|e_{n-1}\|_0^2. \qquad (96)$$

Summing (96) from 1 to n and using Lemma 3.6 and (27) yield that

$$\begin{aligned}
&\|e_n\|_0^2 + k\gamma_0^{-1}\sum_{i=1}^{n}\|\nabla\bar{e}_i\|_0^2 \le Cnkh^4 + \|e_0\|_0^2 + Ck\sum_{i=1}^{n}\|e_i\|_0^2\\
&\le Ch^4 + Ck\sum_{i=1}^{n}\|e_i\|_0^2 + C\|R_h\psi - \psi\|_0^2 + C\|\psi - \rho_h\psi\|_0^2 \le Ch^4 + Ck\sum_{i=1}^{n}\|e_i\|_0^2. \qquad (97)
\end{aligned}$$

If k is sufficiently small such that $Ck \le 1/2$ in (97), we obtain

$$\|e_n\|_0^2 + k\gamma_0^{-1}\sum_{i=1}^{n}\|\nabla\bar{e}_i\|_0^2 \le Ch^4 + Ck\sum_{i=0}^{n-1}\|e_i\|_0^2. \qquad (98)$$

Applying Gronwall Lemma 3.3 to (98) yields that

$$\|e_n\|_0^2 + k\gamma_0^{-1}\sum_{i=1}^{n}\|\nabla\bar{e}_i\|_0^2 \le Ch^4\exp(Cnk) \le Ch^4. \qquad (99)$$

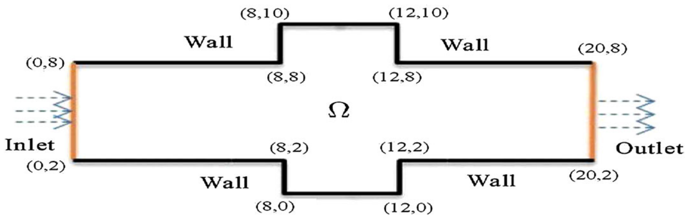

Fig. 2 The computational field and boundary conditions of flow

By extracting square root for (99) and using $\left(\sum_{i=1}^{n} b_i^2\right)^{1/2} \geq \sum_{i=1}^{n} |b_i|/\sqrt{n}$ and $\|a + b\|_0 \geq \|a\|_0 - \|b\|_0$, we obtain

$$\|e_n\|_0 + k\|\nabla e_n\|_0 \leq Ch^2. \tag{100}$$

By using triangle inequality, (100), and Lemma 3.6, we obtain

$$\|T^n - T_h^n\|_0 + k\|\nabla \left(T^n - T_h^n\right)\|_0 \leq Ch^2. \tag{101}$$

Combining (91) with (101) yields that

$$\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_0 + k[\|\nabla \left(\boldsymbol{u}^n - \boldsymbol{u}_h^n\right)|_0 + \|p^n - p_h^n\|_0] \leq Ch^2. \tag{102}$$

Combining (101) and (102) with Theorem 2.2 yields (80). If $\zeta^n = 0$, (80) is also correct, which completes the proof of Theorem 3.7.

Remark 3 It is known from Theorem 3.4 and its proof that, if $\|\psi\|_1$ is sufficiently small, then the conditions $N_0\nu^{-1}\|\nabla \bar{\boldsymbol{u}}_h^n\|_0 \leq 1/4$ $(n = 1, 2, \ldots, N)$ in Theorem 3.7 hold.

4 Some Numerical Experiments

In this section, some numerical experiments are used to show that the advantage of the fully discrete SCNMFVE formulation for the non-stationary incompressible Boussinesq equations.

Computational field $\bar{\Omega}$ consists of the channel of width to 6 and length to 20 and two same rectangular cavities at the bottom and top of the channel. Its two rectangular cavities all are width to 2 and length to 4 (see Fig. 2). It is first divided into some small squares with side length $\triangle x = \triangle y = 0.01$, and then each square is linked with diagonal in the same direction divided into two triangles, which constitutes triangularizations $\Im_h$ with $h = \sqrt{2} \times 10^{-2}$. The dual decomposition $\Im_h^*$ is taken as barycenter form, namely the barycenter of the right triangle $K \in \Im_h$ is taken as the node of the dual decomposition. Take $Re = 1000$, $Pr = 7$, and $\varepsilon = 1$. Except inflow of left boundary with a velocity of $\boldsymbol{u} = (0.1(y - 4.5)(5.5 - y), 0)^T$ $(4.5 \leq y \leq 5.5)$ and outflow of right boundary with velocity of $\boldsymbol{u} = (u_1, u_2)^T$ satisfying $u_2 = 0$ and $u_1(x, y, t) = u_1(19, y, t)$ $(19 \leq x \leq 20, 2 \leq y \leq 8, 0 \leq t \leq t_N)$, all initial and boundary value conditions are taken as $\mathbf{0}$. Time step increment is taken as $\Delta t = 0.01$.

Firstly, the numerical solutions of the velocity, temperature, and pressure obtained by the SCNMFVE formulation Problem IV with the second-order time accuracy at $t = 5000$ are depicted graphically at the bottom charts in Figs. 3, 4, and 5, respectively. If we find the solution at $t = 5000$ by means of the SMFVE formulation with the first-order time accuracy in [17], in order to obtain the same accuracy solution as that of the SCNMFVE formulation, the time step for the SMFVE formulation must be taken as $k = 0.0001$. Thus, the numerical

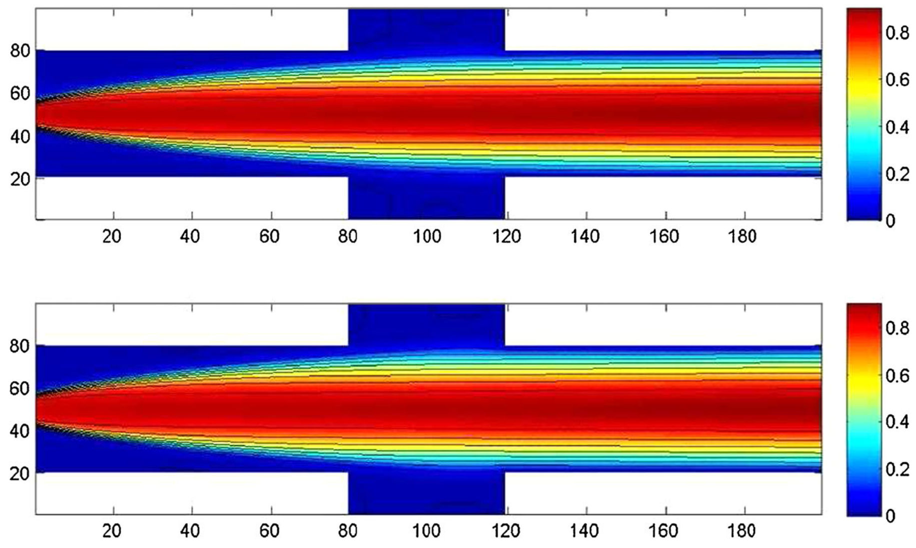

Fig. 3 *Top chart* is the SMFVE solution of velocity $\boldsymbol{u}$ and *bottom chart* is the SCNMFVE solution of velocity $\boldsymbol{u}$ when $Re = 1000$ and $Pr = 7$ at time $t = 5000$

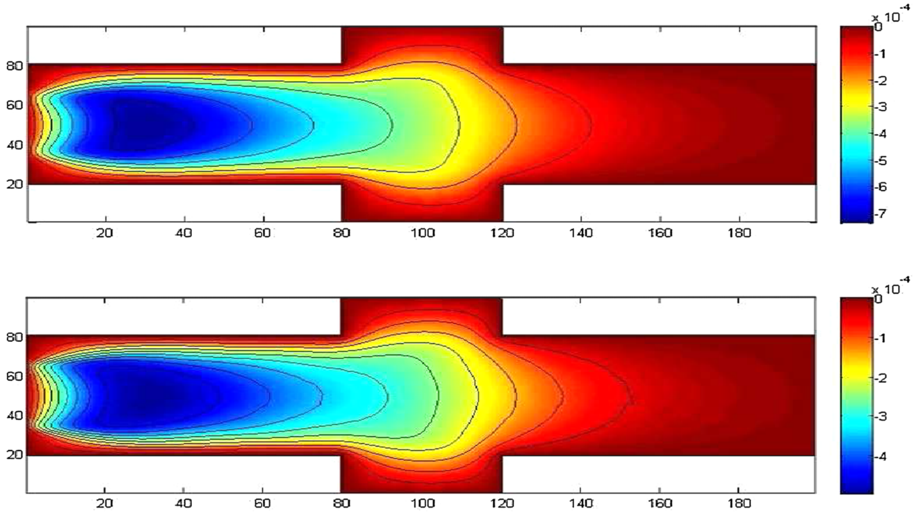

Fig. 4 *Top chart* is the SMFVE solution of temperature T and *bottom chart* is the SCNMFVE solution of temperature T when $Re = 1000$ and $Pr = 7$ at time $t = 5000$

solutions of the velocity, temperature, and pressure obtained by the SMFVE formulation at $t = 5000$ need implement 5×10^7 steps, which are 100 times for implementing steps 5×10^5 of SCNMFVE formulation such that it increases greatly the truncation error accumulation in computational process. The numerical solutions of the velocity, temperature, and pressure obtained by the SMFVE formulation at $t = 5000$ are depicted graphically at the top charts in Figs. 3, 4, and 5, respectively. Comparing every two charts in Figs. 3, 4, and 5 shows that the solutions obtained by the SCNMFVE formulation are better than the SMFVE solutions.

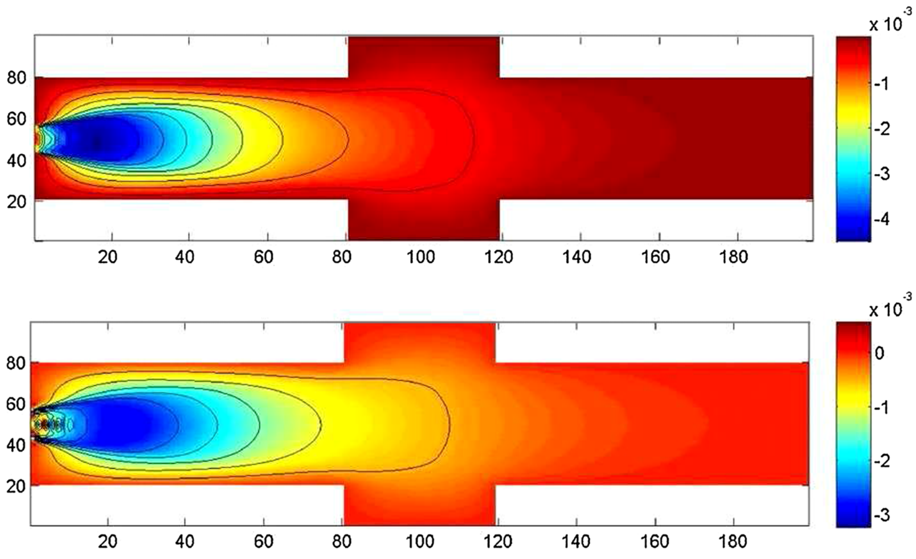

Fig. 5 *Top chart* is the SMFVE solution of pressure p and *bottom chart* is the SCNMFVE solution of pressure p when $Re = 1000$ and $Pr = 7$ at time $t = 5000$

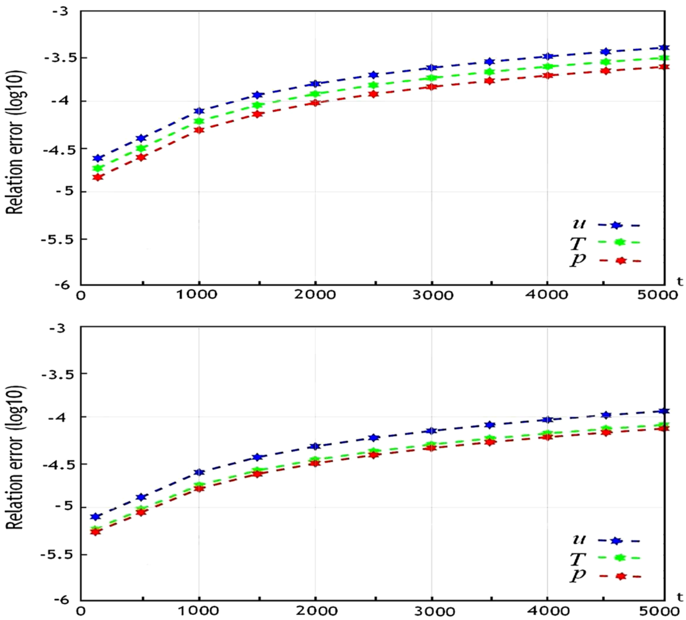

Fig. 6 When $Re = 10^3$ and $Pr = 0.71$, the *top chart* and the *bottom chart* are the relative errors (log 10) of the SMFVE solutions and the SCNFVE solution of the velocity $\boldsymbol{u}$, the temperature T, and the pressure p at the time $t \in (0, 5000]$, respectively

Especially, the the SCNMFVE solutions of the temperature and pressure are far better than the SMFVE solutions.

The curves of the top and bottom charts in Fig. 6 are the relative errors (log 10) of the SMFVE solutions and the SCNMFVE solutions at time $t \in (0, 5000]$ with respect to

L^2-norm, respectively. Due to the truncation error accumulation in computational process, the errors of numerical solutions appear increase (see Fig. 6), but the truncation error accumulation for the SCNMFVE formulation Problem IV is far smaller than that for the SMFVE formulation and the relative errors (which illustrate that the numerical errors are consistent with theoretical results, since they does not exceed 2×10^{-4}) of SCNMFVE numerical solutions are far smaller than those of the SMFVE solutions (also see Fig. 6). Moreover, it is shown that the fully discrete SCNMFVE formulation for the non-stationary incompressible Boussinesq equations is far more advantageous than the SMFVE formulation in [17].

5 Conclusions and Discussions

In this study, we have established the semi-discrete CN formulation with respect to time for the non-stationary incompressible Boussinesq equations and have built the fully discrete SCNMFVE formulation based on two local Gauss integrals and parameter-free directly from the semi-discrete CN formulation with respect to time. Thus, we have avoided the discussion for semi-discrete SCNMFVE formulation with respect to spatial variables such that our theoretical analysis becomes simpler than the existing other methods (for example, see [11,16, 17]). We have also provided the error estimates for the fully discrete SCNMFVE solutions and have used some numerical experiments to illustrate that the numerical errors were consistent with theoretical results, the computing load for the fully discrete SCNMFVE formulation was far fewer than that for the SMFVE formulation with the first time accuracy, and its accumulation of truncation errors in the computational process was far lesser than that of the SMFVE formulation with the first time accuracy. Thus, we have shown the advantage of the fully discrete SCNMFVE formulation for the non-stationary incompressible Boussinesq equations, i.e., the fully discrete SCNMFVE formulation not only has the second-order time accuracy, but it also satisfies the discrete B-B inequality. Thereby, it is different from existing other methods (for example, see [11,16,17]).

References

1. Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
2. An, J., Sun, P., Luo, Z.D., Huang, X.M.: A stabilized fully discrete finite volume element formulation for non-stationary Stokes equations. Math. Numer. Sin. **33**(2), 213–224 (2011)
3. Bank, R.E., Rose, D.J.: Some error estimates for the box methods. SIAM J. Numer. Anal. **24**(4), 777–787 (1987)
4. Blanc, P., Eymerd, R., Herbin, R.: A error estimate for finite volume methods for the Stokes equations on equilateral triangular meshes. Numer. Methods Partial Diff. Equ. **20**, 907–918 (2004)
5. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer-Verlag, New York (1991)
6. Cai, Z., McCormick, S.: On the accuracy of the finite volume element method for diffusion equations on composite grid. SIAM J. Numer. Anal. **27**(3), 636–655 (1990)
7. Chatzipantelidis, P., Lazarrov, R.D., Thomée, V.: Error estimates for a finite volume element method for parabolic equations in convex in polygonal domains. Numer. Methods Partial Diff. Equ. **20**, 650–674 (2004)
8. Chou, S.H., Kwak, D.Y.: A covolume method based on rotated bilinears for the generalized Stokes problem. SIAM J. Numer. Anal. **35**, 494–507 (1998)
9. Christon, M.A., Gresho, P.M., Sutton, S.B.: Computational predictability of time-dependent natural convection flows in enclosures (including a benchmark solution). MIT Spec. Issue Therm. Convect. Int. J. Numer. Methods Fluids **40**(8), 953–980 (2002). special issue
10. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)

11. He, G.L., He, Y.N., Chen, Z.X.: A penalty finite volume method for the transient Navier–Stokes equations. Appl. Numer. Math. **58**, 1583–1613 (2008)
12. He, Y., Lin, Y., Sun, W.: Stabilized finite element method for the non-stationary Navier–Stokes problem. Discrete Continuous Dyn. Syst. B **6**(1), 41–68 (2006)
13. Jones, W.P., Menziest, K.P.: Analysis of the cell-centred finite volume method for the diffusion equation. J. Comput. Phys. **165**, 45–68 (2000)
14. Li, R.H., Chen, Z.Y., Wu, W.: Generalized difference methods for differential equations—numerical analysis of finite volume methods. In: Monographs and Textbooks in Pure and Applied Mathematics, vol. 226. Marcel Dekker Inc., New York (2000)
15. Li, Y., Li, R.H.: Generalized difference methods on arbitrary quadrilateral networks. J. Comput. Math. **17**, 653–672 (1999)
16. Li, H., Luo, Z.D., Sun, P., An, J.: A finite volume element formulation and error analysis for the non-stationary conduction–convection problem. J. Math. Anal. Appl. **396**, 864–879 (2012)
17. Luo, Z.D., Li, H., Sun, P.: A fully discrete stabilized mixed finite volume element formulation for the non-stationary conduction–convection problem. J. Math. Anal. Appl. **44**(1), 71–85 (2013)
18. Luo, Z.D.: The Bases and Applications of Mixed Finite Element Methods. Chinese Science Press, Beijing (2006). in Chinese
19. Luo, Z.D., Chen, J., Navon, I.M., Zhu, J.: An optimizing reduced PLSMFE formulation for non-stationary conduction–convection problems. Int. J. Numer. Meth. Fluids **60**, 409–436 (2009)
20. Luo, Z.D., Xie, Z.H., Chen, J.: A reduced MFE formulation based on POD for the non-stationary conduction–convection problems. Acta Math. Sci. **31**(5), 765–1785 (2011)
21. Rudin, W.: Functional and Analysis, 2nd edn. McGraw-Hill Companies Inc., New York (1973)
22. Süli, E.: Convergence of finite volume schemes for Poissons equation on nonuniform meshes. SIAM J. Numer. Anal. **28**(5), 1419–1430 (1991)
23. Wu, J.H.: The 2D incompressible Boussinesq equations. Peking University Summer School Lecture Notes, Beijing, July 23–August 3 (2012)
24. Yang, M., Song, H.L.: A postprocessing finite volume method for time-dependent Stokes equations. Appl. Numer. Math. **59**, 1922–1932 (2009)
25. Ye, X.: On the relationship between finite volume and finite element methods applied to the Stokes equations. Numer. Methods Partial Diff. Equ. **17**, 440–453 (2001)
26. Zlámal, M.: Finite element methods for nonlinear parabolic equations. RAIRO Analyse numer. **11**(1), 93–107 (1977)