

A Hybridized Discontinuous Galerkin Method with Reduced Stabilization

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Abstract In this paper, we propose a hybridized discontinuous Galerkin (HDG) method with reduced stabilization for the Poisson equation. The reduce stabilization proposed here enables us to use piecewise polynomials of degree k and $k - 1$ for the approximations of element and inter-element unknowns, respectively, unlike the standard HDG methods. We provide the error estimates in the energy and L^2 norms under the chunkiness condition. In the case of $k = 1$, it can be shown that the proposed method is closely related to the Crouzeix–Raviart nonconforming finite element method. Numerical results are presented to verify the validity of the proposed method.

Keywords Hybridized discontinuous Galerkin methods · Error estimates · Reduced stabilization · Crouzeix–Raviart element

Mathematics Subject Classification 65N30

1 Introduction

In this paper, we propose a new hybridized discontinuous Galerkin (HDG) method with reduced stabilization. We consider the Poisson equation with homogeneous Dirichlet boundary condition as a model problem:

$$-\Delta u = f \quad \text{in } \Omega, \quad (1a)$$

$$u = 0 \quad \text{on } \partial\Omega. \quad (1b)$$

Here $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain, and $f \in L^2(\Omega)$ is a given function. For simplicity, we here deal with only the two-dimensional case, although the proposed method can be applied to the three-dimensional problems.

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The HDG methods are already applied to various problems and are still being developed. For second order elliptic problems, the HDG methods were introduced and analyzed by Cockburn et al. [7,8]. The embedded discontinuous Galerkin (EDG) methods, which is the continuous-approximation version of the HDG method, were analyzed in [9]. In these HDG methods, the numerical traces are hybridized in a mixed formulation. In [16], another approach of the HDG method was proposed. The hybridization presented in [16] is based on the *hybrid displacement method* proposed by Tong for linear elasticity problems [17]. The resulting scheme is almost equivalent to the IP-H [8].

In the formulations of HDG methods, element and hybrid unknowns are introduced. The element unknown can be eliminated by the hybrid unknown, which allows us to reduce the number of the globally coupled degrees of freedom. In all the standard HDG methods, we need to use two polynomials of equal degree for the approximations of the element and hybrid unknowns in order to achieve optimal convergences. The motivation of the reduced stabilization we propose is to use piecewise polynomials of degree k and $k - 1$ for approximations of the element and hybrid unknowns, respectively, which we call P_k - P_{k-1} approximation. In [4,5], reduced stabilization was introduced for the discontinuous Galerkin method (the RIP-method). In [14], Lehrenfeld first proposed a reduced HDG scheme by adjusting a numerical flux, which is equivalent to our proposed scheme. However, no error analysis was presented.

In [9], it is proved that the hybrid part of an EDG solution coincides with the trace of a finite element solution with the linear element. Analogously, the proposed method with P_1 - P_0 approximation is closely related to the Crouzeix–Raviart nonconforming finite element method [12]. We prove that our approximate solution coincides with the Crouzeix–Raviart approximation at the midpoints of edges.

We provide a priori error estimates under the chunkiness condition. The optimal error estimates in the energy norm are proved. In terms of the L^2 -errors, it is shown that the convergence rates are optimal when the scheme is symmetric. However, for the nonsymmetric schemes, we prove only the sub-optimal estimates in the L^2 norm due to the lack of adjoint consistency. We also provide the easy implementation by means of the Gaussian quadrature formula in the two-dimensional case. Unfortunately, such implementation is impossible in the three-dimensional case.

This paper is organized as follows. Sect. 2 is devoted to the preliminaries. In Sect. 3, we introduce reduced stabilization, and describe the proposed method. In Sect. 4, we provide the error estimates in the energy and L^2 norms under the chunkiness condition. In Sect. 5, numerical results are presented to verify the validity of our scheme. Finally, in Sect. 6, we end with a conclusion.

2 Preliminaries and Notation

2.1 Chunkiness Condition

Let $\{\mathcal{T}_h\}_h$ be a family of meshes of Ω . Each element $K \in \mathcal{T}_h$ is assumed to be a polygonal domain *star-shaped* with respect to a ball of which radius is ρ_K . Let $h_K = \text{diam}K$ and $h = \max_{K \in \mathcal{T}_h} h_K$. We assume that the boundary ∂K of $K \in \mathcal{T}_h$ is composed of m -faces and m is bounded by M from above independently of h . Let us denote $\mathcal{E}_h = \{e \subset \partial K : K \in \mathcal{T}_h\}$. In this paper, we assume that the family $\{\mathcal{T}_h\}_h$ satisfies the *chunkiness condition* [3,13]: there exists a positive constant γ_C independent of h such that

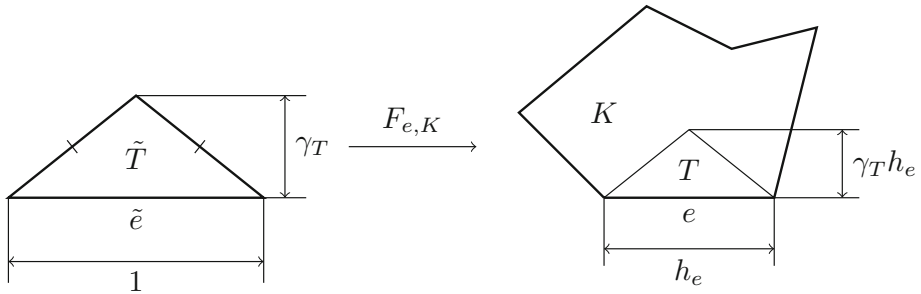


Fig. 1 Triangle condition

$$\frac{h_K}{\rho_K} \leq \gamma_C \quad \forall K \in \mathcal{T}_h. \tag{2}$$

From the chunkiness condition, a kind of cone condition follows [3, 13]. Let \tilde{T} be a reference triangle with the height of $\gamma_T > 0$. We assume that \tilde{T} is an isosceles triangle. Let \tilde{e} denote the base of \tilde{T} . For each $K \in \mathcal{T}_h$ and $e \subset \partial K$ (Fig. 1), let $F_{e,K}$ be an affine-linear mapping from \tilde{T} onto $T \subset K$ such that $F_{e,K}(\tilde{e}) = e$ and the height of T is equal to $\gamma_T h_e$. The constant γ_T depends only on the chunkiness parameter γ_C . Note that there exists a constant $\gamma_E \geq 1$ such that

$$\frac{h_T}{h_e} \leq \gamma_E. \tag{3}$$

2.2 Function Spaces

We introduce the *piecewise* Sobolev spaces over \mathcal{T}_h , i.e., $H^k(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K \in H^k(K) \forall K \in \mathcal{T}_h\}$. The *skeleton* of \mathcal{T}_h is defined by $\Gamma_h = \bigcup_{e \in \mathcal{E}_h} e$. We use $L^2_D(\Gamma_h) = \{\hat{v} \in L^2(\Gamma_h) : \hat{v} = 0 \text{ on } \partial\Omega\}$ and $\mathbf{V} = H^2(\mathcal{T}_h) \times L^2_D(\Gamma_h)$ for the hybridized formulation of the continuous problem. Let us define $\mathbf{V}(h) = \{(v, v|_{\Gamma_h}) : v \in H^2(\Omega)\} \subset H^2(\Omega) \times H^{3/2}(\Gamma_h)$, where $v|_{\Gamma_h}$ stands for the trace of v_h on Γ_h . We use the following inner products:

$$(u, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \int_K u v dx, \quad \langle u, v \rangle_{\partial\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \int_{\partial K} u v ds$$

for $u, v \in L^2(\Omega)$ or $L^2_D(\Gamma_h)$.

2.3 Finite Element Spaces and Projections

Let $\mathcal{P}^k(\mathcal{T}_h)$ be the function space of element-wise polynomials of degree k over \mathcal{T}_h , and $\mathcal{P}^l(\mathcal{E}_h)$ be the space of edge-wise polynomials of degree l over \mathcal{E}_h , where k and l are nonnegative integers. Then we define $V_h^k = \mathcal{P}^k(\mathcal{T}_h)$ and $\hat{V}_h^l = \mathcal{P}^l(\mathcal{E}_h) \cap L^2_D(\Gamma_h)$. We employ $V_h^{k,l} = V_h^k \times \hat{V}_h^l$ as finite element spaces of \mathbf{V} . Let us denote by \mathbf{P}_k the L^2 -projection from $L^2(\Gamma_h)$ onto $\mathcal{P}^k(\mathcal{E}_h)$.

2.4 Mesh-Dependent Norms

Let $\|\cdot\|_m$ and $|\cdot|_m$ be the usual Sobolev norms and seminorms in the sense of [1], respectively. We introduce auxiliary mesh-dependent seminorms:

$$|v|_{1,h}^2 = \sum_{K \in \mathcal{T}_h} |v|_{1,K}^2 \quad \text{for } v \in H^1(\mathcal{T}_h), \tag{4}$$

$$|v|_{2,h}^2 = \sum_{K \in \mathcal{T}_h} h_K^2 |v|_{2,K}^2 \quad \text{for } v \in H^2(\mathcal{T}_h), \tag{5}$$

$$|\mathbf{v}|_j^2 = \sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K} \frac{1}{h_e} \|\mathbf{P}_{k-1}(\hat{v} - v)\|_{0,e}^2 \quad \text{for } \mathbf{v} = \{v, \hat{v}\} \in \mathbf{V}, \tag{6}$$

where h_e is the diameter of e . Note that $\mathbf{P}_{k-1}v$ in (6) is defined by $\mathbf{P}_{k-1}(\text{trace}(v|_K))$, which is well-defined, whereas v may be double-valued on ∂K . In our error analysis, we use the following energy norm:

$$\|\mathbf{v}\|^2 = |v|_{1,h}^2 + |v|_{2,h}^2 + |\mathbf{v}|_j^2 \quad \text{for } \mathbf{v} = \{v, \hat{v}\} \in \mathbf{V}.$$

2.5 Trace and Inverse Inequalities

We here state the trace and inverse inequalities without proofs. The constants appearing in the inequalities are independent of h , $K \in \mathcal{T}_h$ and $e \subset \partial K$ under the chunkiness condition.

Lemma 1 (Trace inequality) *Let $K \in \mathcal{T}_h$ and e be an edge of K . There exists a constant C independent of K , e and h such that*

$$\|v\|_{0,e} \leq Ch_e^{-1/2} (\|v\|_{0,K}^2 + h_K^2 |v|_{1,K}^2)^{1/2} \quad \forall v \in H^1(K). \tag{7}$$

Proof Refer to [13]. □

Lemma 2 (Inverse inequality) *Let $K \in \mathcal{T}_h$. There exists a constant C independent of K and h such that*

$$|v_h|_{1,K} \leq Ch_K^{-1} \|v_h\|_{0,K} \quad \forall v_h \in \mathcal{P}^k(K). \tag{8}$$

Proof Refer to [3]. □

We will use the lemma below to bound the terms of the complementary projection, $\mathbf{I} - \mathbf{P}_{k-1}$.

Lemma 3 *There exists a constant C independent of h such that, for all $v \in H^1(K)$,*

$$h_e^{-1} \|(\mathbf{I} - \mathbf{P}_{k-1})v\|_{0,e}^2 \leq C |v|_{1,K}^2. \tag{9}$$

Proof Let \tilde{T} be a reference triangle and \tilde{e} be the base of \tilde{T} as illustrated in Fig. 1. Let $\tilde{v} \in \mathcal{P}^k(\tilde{T})$ be arbitrarily fixed. We define a linear functional on $H^1(\tilde{T})$ by

$$G(\tilde{w}) = \langle (\mathbf{I} - \mathbf{P}_{k-1})\tilde{v}, \tilde{w} \rangle_{\tilde{e}}.$$

Note that the functional G vanishes on $\mathcal{P}^0(\tilde{T})$. By the Schwarz and trace inequalities, we have

$$\begin{aligned} |G(\tilde{w})| &\leq \|(\mathbf{I} - \mathbf{P}_{k-1})\tilde{v}\|_{0,\tilde{e}} \|\tilde{w}\|_{0,\tilde{e}} \\ &\leq \|(\mathbf{I} - \mathbf{P}_{k-1})\tilde{v}\|_{0,\tilde{e}} \cdot Ch_{\tilde{e}}^{-1/2} (\|\tilde{w}\|_{0,\tilde{T}}^2 + h_{\tilde{T}}^2 |\tilde{w}|_{1,\tilde{T}}^2)^{1/2} \\ &\leq C \|(\mathbf{I} - \mathbf{P}_{k-1})\tilde{v}\|_{0,\tilde{e}} \|\tilde{w}\|_{1,\tilde{T}}. \end{aligned}$$

By the Bramble–Hilbert lemma, we have

$$|G(\tilde{w})| \leq C \|(I - P_{k-1})\tilde{v}\|_{0,\tilde{e}} |\tilde{w}|_{1,\tilde{T}}.$$

Taking $\tilde{w} = \tilde{v}$ gives us

$$\|(I - P_{k-1})\tilde{v}\|_{0,\tilde{e}} \leq C |\tilde{v}|_{1,\tilde{T}}. \tag{10}$$

Let $F(\tilde{x}) = B\tilde{x} + \mathbf{d}$ be an affine mapping from \tilde{T} onto $T \subset K$ such that $F(\tilde{e}) = e$. Choosing $\tilde{v} = v \circ F$, we have

$$\|(I - P_{k-1})\tilde{v}\|_{0,\tilde{e}} = \frac{\text{meas}(\tilde{e})^{1/2}}{\text{meas}(e)^{1/2}} \|(I - P_{k-1})v\|_{0,e}, \tag{11}$$

where $\text{meas}(e)$ is the measure of e . From [6, Theorem 3.1.2.], it follows that

$$\begin{aligned} |\tilde{v}|_{1,\tilde{T}} &\leq C \|B\| |\det B|^{-1/2} |v|_{1,T} \\ &\leq C \frac{h_T}{2\rho_{\tilde{T}}} \frac{\text{meas}(\tilde{T})^{1/2}}{\text{meas}(T)^{1/2}} |v|_{1,T}, \end{aligned} \tag{12}$$

where $\rho_{\tilde{T}}$ is the radius of the inscribed ball of \tilde{T} . The measure of T is given by

$$\text{meas}(T) = \frac{\gamma_T h_e}{2} \text{meas}(e). \tag{13}$$

From (12), (13) and (3), we have

$$\begin{aligned} |\tilde{v}|_{1,\tilde{T}} &\leq C \frac{h_T}{h_e^{1/2} \text{meas}(e)^{1/2}} |v|_{1,T} \\ &\leq C \gamma_E \frac{h_e^{1/2}}{\text{meas}(e)^{1/2}} |v|_{1,T}. \end{aligned} \tag{14}$$

From (10), (11) and (14), it follows that

$$\|(I - P_{k-1})v\|_{0,e} \leq Ch_e^{1/2} |v|_{1,T} \leq Ch_e^{1/2} |v|_{1,K},$$

which completes the proof. □

2.6 Approximation Property

The approximation property in the energy norm follows from those of $\mathcal{P}^k(\mathcal{T}_h)$ and $\mathcal{P}^k(\mathcal{E}_h)$.

Lemma 4 (Approximation property) *Let $v \in H^{k+1}(\Omega)$ and $\mathbf{v} = \{v, v|_{\Gamma_h}\}$. We assume that the finite element space for the hybrid unknown is discontinuous. Then there exists a positive constant C independent of h such that*

$$\inf_{\mathbf{v}_h \in \mathbf{V}_h^{k,k-1}} \|\mathbf{v} - \mathbf{v}_h\| \leq Ch^k |v|_{k+1}. \tag{15}$$

Proof It is known that there exists $\mathbf{w}_h = \{w_h, \hat{w}_h\} \in \mathbf{V}_h^{k,k}$ such that

$$\|v - w_h\|_0 \leq Ch^{k+1} |v|_{k+1}, \tag{16}$$

$$|v - w_h|_{i,h} \leq Ch^{k+1-i} |v|_{k+1} \quad (i = 1, 2), \tag{17}$$

$$\left(\sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K} h_e^{-1} \|v - \hat{w}_h\|_{0,e}^2 \right)^{1/2} \leq Ch^k |v|_{k+1}. \tag{18}$$

Let us define $\mathbf{v}_h = \{w_h, \mathbf{P}_{k-1}\hat{w}_h\} \in \mathbf{V}_h^{k,k-1}$. Note that $|v - \mathbf{v}_h|_j = |v_h|_j = |w_h|_j$. By the trace inequality for $K \in \mathcal{T}_h$ and $e \subset \partial K$, we have

$$\begin{aligned} \|\mathbf{P}_{k-1}(\hat{w}_h - w_h)\|_{0,e} &\leq \|\hat{w}_h - w_h\|_{0,e} \\ &\leq \|\hat{w}_h - v\|_{0,e} + \|v - w_h\|_{0,e} \\ &\leq \|\hat{w}_h - v\|_{0,e} + Ch_e^{-1/2}(\|v - w_h\|_{0,K}^2 + h_K^2|v - w_h|_{1,K}^2)^{1/2}. \end{aligned}$$

From the above and (16)–(18), it follows that $\|\mathbf{v} - \mathbf{v}_h\| \leq Ch^k|v|_{k+1}$. □

Remark In the standard HDG methods, we can impose the continuity at nodes on the hybrid unknowns to reduce the number of degrees of freedom, which is so-called *continuous* approximation. However, the approximation property with respect to the energy norm does not hold for the continuous approximation since $\mathbf{P}_{k-1}v_h$ is not continuous at nodes in general even if v_h is continuous. Indeed, it is found by numerical experiments that the convergence rates in the energy and L^2 norms are sub-optimal for the continuous approximation.

3 Reduced HDG Method

3.1 The Standard HDG Scheme

To begin with, we present the standard HDG formulation: find $\mathbf{u}_h = \{u_h, \hat{u}_h\} \in \mathbf{V}_h^{k,k}$ such that

$$B_{\text{Std}}(\mathbf{u}_h, \mathbf{v}_h) = (f, v_h)_\Omega \quad \forall \mathbf{v}_h = \{v_h, \hat{v}_h\} \in \mathbf{V}_h^{k,k}, \tag{19}$$

where the bilinear form is defined by

$$\begin{aligned} B_{\text{Std}}(\mathbf{u}_h, \mathbf{v}_h) &= (\nabla u_h, \nabla v_h)_{\mathcal{T}_h} + \langle \mathbf{n} \cdot \nabla u_h, \hat{v}_h - v_h \rangle_{\partial \mathcal{T}_h} \\ &\quad + s \langle \mathbf{n} \cdot \nabla v_h, \hat{u}_h - u_h \rangle_{\partial \mathcal{T}_h} \\ &\quad + \langle \tau(\hat{u}_h - u_h), \hat{v}_h - v_h \rangle_{\partial \mathcal{T}_h}. \end{aligned} \tag{20}$$

Here s is a real number and τ is a stabilization parameter. The parameter τ takes a constant value τ_e/h_e on each edge e with $0 < \tau_0 \leq \tau_e \leq \tau_1$ for some τ_0, τ_1 . We refer to [16] for the details of the derivation.

3.2 Reduced HDG Schemes

Let us sketch the main idea of our method. The second term in the convectional scheme (20) can be rewritten as

$$\langle \mathbf{n} \cdot \nabla u_h, \hat{v}_h - v_h \rangle_{\partial \mathcal{T}_h} = \langle \mathbf{n} \cdot \nabla u_h, \mathbf{P}_{k-1}(\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h}$$

since $\mathbf{n} \cdot \nabla u_h \in \mathcal{P}^{k-1}(\mathcal{T}_h)$. The stabilization term is correspondingly decomposed into

$$\begin{aligned} \langle \tau(\hat{u}_h - u_h), (\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h} &= \langle \tau \mathbf{P}_{k-1}(\hat{u}_h - u_h), \mathbf{P}_{k-1}(\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h} \\ &\quad + \langle \tau(1 - \mathbf{P}_{k-1})(\hat{u}_h - u_h), (1 - \mathbf{P}_{k-1})(\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h} \end{aligned}$$

Our reduced stabilization is obtained by dropping the second term in the right-hand side. The proposed scheme reads: find $\mathbf{u}_h = \{u_h, \hat{u}_h\} \in \mathbf{V}_h^{k,k-1}$ such that

$$B_h(\mathbf{u}_h, \mathbf{v}_h) = (f, v_h)_\Omega \quad \forall \mathbf{v}_h = \{v_h, \hat{v}_h\} \in \mathbf{V}_h^{k,k-1}, \tag{21}$$

where the bilinear form is defined by

$$\begin{aligned}
 B_h(\mathbf{u}_h, \mathbf{v}_h) &= (\nabla u_h, \nabla v_h)_{\mathcal{T}_h} + \langle \mathbf{n} \cdot \nabla u_h, \hat{v}_h - v_h \rangle_{\partial \mathcal{T}_h} \\
 &\quad + s \langle \mathbf{n} \cdot \nabla v_h, \hat{u}_h - u_h \rangle_{\partial \mathcal{T}_h} \\
 &\quad + \langle \tau \mathbf{P}_{k-1}(\hat{u}_h - u_h), \mathbf{P}_{k-1}(\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h}.
 \end{aligned} \tag{22}$$

3.3 Local Conservativity

Let K be an element of \mathcal{T}_h , and let χ_K denote a characteristic function on K . Taking $\mathbf{v}_h = \{\chi_K, 0\}$ in (21), we find that our method as well as the other HDG methods satisfies the following local conservation property

$$- \int_{\partial K} \hat{\boldsymbol{\sigma}}(\mathbf{u}_h) \cdot \mathbf{n} ds = \int_K f dx, \tag{23}$$

where $\hat{\boldsymbol{\sigma}}$ is a numerical flux defined by

$$\hat{\boldsymbol{\sigma}}(\mathbf{u}_h) = \nabla u_h + \tau(\hat{u}_h - u_h)\mathbf{n}.$$

3.4 Implementation Using the Gaussian Quadrature Formula

In this section, we will show that the reduced stabilization term can be easily calculated by means of the Gaussian quadrature formula in the two-dimensional case. We can also avoid the calculation of the L^2 projections in the reduced stabilization term by using it.

For simplicity, we consider the case of the interval $I = [-1, 1]$. Let φ_m be the Legendre polynomial of order $m \geq 0$ on I . Let f be a smooth function on I . The k -point Gauss-Legendre quadrature rule on I is given by

$$\mathcal{G}_k[f] = \sum_{i=1}^k w_i f(a_i),$$

where $\{a_i, w_i\}_{i=1}^k$ are the quadrature points and weights. The standard stabilization term for P_k - P_k approximation can be exactly computed by using the $(k + 1)$ -point Gauss-Legendre quadrature rule. If we use the k -point quadrature rule instead of $(k + 1)$ -point one, then the reduced stabilization term is obtained.

Lemma 5 *Let \mathbf{P}_{k-1} denote the L^2 -projection from $L^2(I)$ onto $\mathcal{P}^{k-1}(I)$. Then we have, for all $\hat{u}_h, \hat{v}_h \in \mathcal{P}^k(I)$,*

$$\mathcal{G}_k[\hat{u}_h \hat{v}_h] = \int_I \mathbf{P}_{k-1} \hat{u}_h \mathbf{P}_{k-1} \hat{v}_h ds. \tag{24}$$

Proof We can write $\hat{u}_h = \sum_{j=1}^k u_j \varphi_j$ and $\hat{v}_h = \sum_{j=1}^k v_j \varphi_j$. Note that the Legendre polynomial φ_k vanishes at the quadrature points, i.e., $\varphi_k(a_i) = 0$ for $1 \leq i \leq k$, and that \mathcal{G}_k is exact for polynomials of degree $\leq 2k - 1$. Then we have

$$\begin{aligned}
 \mathcal{G}_k[\hat{u}_h \hat{v}_h] &= \sum_{i=1}^k \left[w_i \sum_{j=1}^k u_j \varphi_j(a_i) \cdot \sum_{j=1}^k v_j \varphi_j(a_i) \right] \\
 &= \sum_{i=1}^k \left[w_i \sum_{j=1}^{k-1} u_j \varphi_j(a_i) \cdot \sum_{j=1}^{k-1} v_j \varphi_j(a_i) \right] \\
 &= \sum_{i=1}^k [w_i \mathbf{P}_{k-1} u_h(a_i) \mathbf{P}_{k-1} v_h(a_i)] \\
 &= \mathcal{G}_k[\mathbf{P}_{k-1} \hat{u}_h \mathbf{P}_{k-1} \hat{v}_h] \\
 &= \int_I \mathbf{P}_{k-1} \hat{u}_h \mathbf{P}_{k-1} \hat{v}_h ds,
 \end{aligned}$$

which completes the proof. □

Remark In the three-dimensional case, the efficient implementaion using the Gaussian cubature formula is impossible since there exists almost no cubature formula such that the nodes are the common zeros of orthogonal polynomials (see, for example, [10, 15]). Even for a triangle, it is known that there does not exist such a cubature formula of degree ≥ 3 , see [11]. Only in the case of $k = 1$, our method can be easily implemented by the barycentric rule.

3.5 Relation with the Crouzeix–Raviart Nonconforming Finite Element Method

In [9], it is proved that the numerical trace of the EDG method coincides with the approximate solution given by the conforming finite element method on a skeleton Γ_h . In this section, we reveal the relation between the Crouzeix–Raviart nonconforming finite element and our symmetric scheme($s = 1$) with P_1 – P_0 triangular elements. The meshes considered here are assumed to be triangular. Let Π_h denote the Crouzeix–Raviart interpolation operator with respect to a mesh \mathcal{T}_h . For $\hat{u} \in L^2(\Gamma_h)$, the interpolation $\Pi_h \hat{u} \in \mathcal{P}^1(\mathcal{T}_h)$ is given by

$$\int_{\Gamma_h} (\Pi_h \hat{u}) \hat{v}_h ds = \int_{\Gamma_h} \hat{u} \hat{v}_h ds \quad \forall \hat{v}_h \in \mathcal{P}^0(\mathcal{T}_h).$$

Theorem 1 *Let $\mathbf{u}_h = \{u_h, \hat{u}_h\} \in \mathbf{V}_h^{1,0}$ be the approximate solution provided by (21) with $s = 1$ and u_{CR} be the Crouzeix–Raviart approximation. Then we have*

$$\Pi_h \hat{u}_h = u_{CR}. \tag{25}$$

In particular, we have for all $e \in \mathcal{E}_h$,

$$\int_e \hat{u}_h ds = \int_e u_{CR} ds. \tag{26}$$

Proof By the definition of the Crouzeix–Raviart interpolation, we have

$$\langle \mathbf{n} \cdot \nabla u_h, \Pi_h \hat{v}_h - \hat{v}_h \rangle_{\partial \mathcal{T}_h} = 0.$$

and

$$\langle \tau \mathbf{P}_{k-1}(\hat{u}_h - u_h), \mathbf{P}_{k-1}(\Pi_h \hat{v}_h - \hat{v}_h) \rangle_{\partial \mathcal{T}_h} = 0. \tag{27}$$

Taking $\mathbf{v}_h = \{\Pi_h \hat{v}_h, \hat{v}_h\} \in \mathbf{V}_h^{1,0}$ in (21) yields

$$\begin{aligned}
 B_h(\mathbf{u}_h, \mathbf{v}_h) &= (\nabla u_h, \nabla(\Pi_h \hat{v}_h))_{\mathcal{T}_h} + s \langle \mathbf{n} \cdot \nabla(\Pi_h \hat{v}_h), \hat{u}_h - u_h \rangle_{\partial \mathcal{T}_h} \\
 &= (1 - s) (\nabla u_h, \nabla(\Pi_h \hat{v}_h))_{\mathcal{T}_h} + s (\nabla(\Pi_h \hat{u}_h), \nabla(\Pi_h \hat{v}_h))_{\mathcal{T}_h}
 \end{aligned} \tag{28}$$

When $s = 1$, the resulting equation for \hat{u}_h reads

$$(\nabla(\Pi_h \hat{u}_h), \nabla(\Pi_h \hat{v}_h))_{\mathcal{T}_h} = (f, \Pi_h \hat{v}_h)_{\Omega} \quad \forall \hat{v}_h \in \mathcal{P}^0(\mathcal{E}_h). \tag{29}$$

The solution of the equation above is uniquely determined to be u_{CR} . Hence we have $\Pi_h \hat{u}_h = u_{CR}$. □

Remark For higher-order polynomials, (28) does not hold since the Laplacian of u_h does not vanish. In the case of a polygonal element, the reduced stabilization term does not vanish, that is, (27) does not hold. Therefore we might not find a discrete equation in terms of only the hybrid unknown, like (29), in general cases.

4 Error Analysis

First, we prove the consistency, boundedness and coercivity of the bilinear form of our method.

Lemma 6 (Consistency) *Let u be the exact solution of (1a)(1b), and $\mathbf{u} = \{u, u|_{\Gamma_h}\}$. Then we have*

$$B_h(\mathbf{u}, \mathbf{v}_h) = (f, v_h)_{\Omega} \quad \forall \mathbf{v}_h \in \mathbf{V}_h^{k,k}. \tag{30}$$

Proof Since $\hat{u} - u = 0$ on Γ_h and the normal derivative of u is single-valued, we have

$$\begin{aligned} B_h(\mathbf{u}, \mathbf{v}_h) &= (\nabla u, \nabla v_h)_{\mathcal{T}_h} - \langle \mathbf{n} \cdot \nabla u, v_h \rangle_{\partial \mathcal{T}_h} \\ &= (-\Delta u, v_h)_{\mathcal{T}_h} \\ &= (f, v_h)_{\Omega}. \end{aligned} \tag{31}$$

□

Lemma 7 (Boundedness) *There exists a constant C_b independent of h such that*

$$|B_h(\mathbf{w}, \mathbf{v})| \leq C_b \|\mathbf{w}\| \|\mathbf{v}\| \quad \forall \mathbf{w}, \mathbf{v} \in \mathbf{V}(h) + \mathbf{V}_h^{k,k-1}. \tag{32}$$

Proof Let $\mathbf{w} = \{w, \hat{w}\} = \{\bar{w} + w_h, \bar{w}|_{\Gamma_h} + \hat{w}_h\}$ and $\mathbf{v} = \{v, \hat{v}\} = \{\bar{v} + v_h, \bar{v}|_{\Gamma_h} + \hat{v}_h\}$, where $\bar{w}, \bar{v} \in H^2(\Omega)$ and $\{v_h, \hat{v}_h\}, \{w_h, \hat{w}_h\} \in \mathbf{V}_h^{k,k-1}$. We estimate each term in the bilinear form separately. By the Schwarz inequality, we have

$$|(\nabla w, \nabla v)_K| \leq \|\nabla w\|_{0,K} \|\nabla v\|_{0,K}. \tag{33}$$

To bound the second term in the bilinear form, we decompose it as

$$\begin{aligned} \langle \mathbf{n} \cdot \nabla w, \hat{v} - v \rangle_{\partial \mathcal{T}_h} &= \langle \mathbf{n} \cdot \nabla(\bar{w} + w_h), (\bar{v} + \hat{v}_h) - (\bar{v} + v_h) \rangle_{\partial \mathcal{T}_h} \\ &= \langle \mathbf{n} \cdot \nabla \bar{w}, \hat{v}_h - v_h \rangle_{\partial \mathcal{T}_h} + \langle \mathbf{n} \cdot \nabla w_h, \hat{v}_h - v_h \rangle_{\partial \mathcal{T}_h}. \end{aligned} \tag{34}$$

Since $\langle \mathbf{n} \cdot \nabla \bar{w}, z \rangle_{\partial \mathcal{T}_h} = 0$ for any single-valued function z , we have

$$\begin{aligned} \langle \mathbf{n} \cdot \nabla \bar{w}, \hat{v}_h - v_h \rangle_{\partial \mathcal{T}_h} &= \langle \mathbf{n} \cdot \nabla \bar{w}, \mathbf{P}_{k-1}(\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h} + \langle \mathbf{n} \cdot \nabla \bar{w}, (1 - \mathbf{P}_{k-1})(\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h} \\ &= \langle \mathbf{n} \cdot \nabla \bar{w}, \mathbf{P}_{k-1}(\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h} + \langle \mathbf{n} \cdot \nabla \bar{w}, (1 - \mathbf{P}_{k-1})(\bar{v} - v_h) \rangle_{\partial \mathcal{T}_h}. \end{aligned} \tag{35}$$

Similarly, noting that $\langle \mathbf{n} \cdot \nabla w_h, (1 - \mathbf{P}_{k-1})z \rangle_{\partial \mathcal{T}_h} = 0$ for any single-valued function z , we have

$$\begin{aligned} & \langle \mathbf{n} \cdot \nabla w_h, \hat{v}_h - v_h \rangle_{\partial \mathcal{T}_h} \\ &= \langle \mathbf{n} \cdot \nabla w_h, \mathbf{P}_{k-1}(\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h} + \langle \mathbf{n} \cdot \nabla w_h, (1 - \mathbf{P}_{k-1})(\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h} \\ &= \langle \mathbf{n} \cdot \nabla w_h, \mathbf{P}_{k-1}(\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h} + \langle \mathbf{n} \cdot \nabla w_h, (1 - \mathbf{P}_{k-1})(\bar{v} - v_h) \rangle_{\partial \mathcal{T}_h}. \end{aligned} \tag{36}$$

From (34), (35) and (36), it follows that

$$\langle \mathbf{n} \cdot \nabla w, \hat{v} - v \rangle_{\partial \mathcal{T}_h} = \langle \mathbf{n} \cdot \nabla w, \mathbf{P}_{k-1}(\hat{v}_h - v_h) + (1 - \mathbf{P}_{k-1})(\bar{v} - v_h) \rangle_{\partial \mathcal{T}_h}. \tag{37}$$

By the trace inequality and Lemma 3, we get

$$\begin{aligned} |\langle \mathbf{n} \cdot \nabla w, \hat{v} - v \rangle_{\partial \mathcal{T}_h}| &\leq C(|w|_{1,h}^2 + h^2|w|_{2,h}^2)^{1/2}(|\mathbf{v}|_j^2 + |\bar{v} - v_h|_{1,h}^2)^{1/2} \\ &\leq C \|\mathbf{w}\| \|\mathbf{v}\|. \end{aligned} \tag{38}$$

In the same manner, the third term in the bilinear form can be bounded. The stabilization term is bounded as

$$\begin{aligned} |\langle \tau \mathbf{P}_{k-1}(\hat{w} - w), \mathbf{P}_{k-1}(\hat{v} - v) \rangle_{\partial \mathcal{T}_h}| &= |\langle \tau \mathbf{P}_{k-1}(\hat{w}_h - w_h), \mathbf{P}_{k-1}(\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h}| \\ &\leq \tau_1 |\mathbf{w}|_j |\mathbf{v}|_j. \end{aligned} \tag{39}$$

Combining (33), (38) and (39), we obtain

$$|B_h(\mathbf{w}, \mathbf{v})| \leq C_b \|\mathbf{w}\| \|\mathbf{v}\|, \tag{40}$$

where the constant C_b depends on the constants of the trace inequality and τ_1 , but is independent of h . The proof is completed. \square

Lemma 8 (Coercivity) *Assume that τ_0 is sufficiently large. Then there exists a constant $C_c > 0$ independent of h such that*

$$B_h(\mathbf{v}_h, \mathbf{v}_h) \geq C_c \|\mathbf{v}_h\|^2 \quad \forall \mathbf{v}_h \in \mathbf{V}_h^{k,k-1}. \tag{41}$$

When $s = -1$, it holds for any $\tau > 0$.

Proof Letting $\mathbf{u}_h = \mathbf{v}_h$ in (21), we have

$$B_h(\mathbf{v}_h, \mathbf{v}_h) \geq |v_h|_{1,h}^2 - |1 - s| |\langle \mathbf{n} \cdot \nabla v_h, \hat{v}_h - v_h \rangle_{\partial \mathcal{T}_h}| + \tau_0 |v_h|_j^2. \tag{42}$$

Note that

$$\langle \mathbf{n} \cdot \nabla v_h, \hat{v}_h - v_h \rangle_{\partial \mathcal{T}_h} = \langle \mathbf{n} \cdot \nabla v_h, \mathbf{P}_{k-1}(\hat{v}_h - v_h) \rangle_{\partial \mathcal{T}_h}. \tag{43}$$

By the trace inequality and Young’s inequality, it follows that, for any $\varepsilon > 0$,

$$B_h(\mathbf{v}_h, \mathbf{v}_h) \geq (1 - C\varepsilon)|v_h|_{1,h}^2 + (\tau_0 - \varepsilon^{-1})|v_h|_j^2. \tag{44}$$

If $\tau_0 > C + 1$, then we can take $\varepsilon = (\tau_0^{-1} + C^{-1})/2$. Therefore we obtain

$$\begin{aligned} B_h(\mathbf{v}_h, \mathbf{v}_h) &\geq \frac{1}{2}(|v_h|_{1,h}^2 + |v_h|_j^2) \\ &\geq C \|\mathbf{v}_h\|^2, \end{aligned} \tag{45}$$

where we have used the inverse inequality. If $s = -1$, then the second term in the right-hand side in (42) vanishes. From this, we see that (45) holds for any $\tau > 0$ when $s = -1$. \square

Next, we prove the error estimates with respect to the energy norm.

Theorem 2 (Quasi-best approximation) *Let u be the exact solution of (1a)(1b) and $\mathbf{u} := \{u, u|_{\Gamma_h}\} \in V$. Let $\mathbf{u}_h \in \mathbf{V}_h^{k,k-1}$ be an approximate solution provided by our method (21). Then we have*

$$\|\mathbf{u} - \mathbf{u}_h\| \leq C \inf_{\mathbf{v}_h \in \mathbf{V}_h^{k,k-1}} \|\mathbf{u} - \mathbf{v}_h\|, \tag{46}$$

where C is a positive constant independent of h .

Proof Let $\mathbf{v}_h \in \mathbf{V}_h^{k,k-1}$ be arbitrary. By the coercivity, consistency and boundedness, we have

$$\begin{aligned} C_c \|\mathbf{u}_h - \mathbf{v}_h\|^2 &\leq B_h(\mathbf{u}_h - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) \\ &= B_h(\mathbf{u} - \mathbf{v}_h, \mathbf{u}_h - \mathbf{v}_h) \\ &\leq C_b \|\mathbf{u} - \mathbf{v}_h\| \|\mathbf{u}_h - \mathbf{v}_h\|, \end{aligned} \tag{47}$$

from which it follows that

$$\|\mathbf{u}_h - \mathbf{v}_h\| \leq \frac{C_b}{C_c} \|\mathbf{u} - \mathbf{v}_h\|. \tag{48}$$

By the triangle inequality, we have

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\| &\leq \|\mathbf{u} - \mathbf{v}_h\| + \|\mathbf{v}_h - \mathbf{u}_h\| \\ &\leq \left(1 + \frac{C_b}{C_c}\right) \|\mathbf{u} - \mathbf{v}_h\|, \end{aligned}$$

which implies (46). □

By the approximation property, the optimal-order error estimate in the energy norm follows immediately.

Theorem 3 *Let the notation be the same in Theorem 2. If $u \in H^{k+1}(\Omega)$, then we have*

$$\|\mathbf{u} - \mathbf{u}_h\| \leq Ch^k |u|_{k+1}.$$

Finally, we prove the L^2 -error estimates.

Theorem 4 (L^2 -error estimates) *Let the notation be the same as in Theorem 2. If $u \in H^{k+1}(\Omega)$, then we have*

$$\|u - u_h\|_0 \leq Ch^{k+1} |u|_{k+1} \text{ for } s = 1, \tag{49}$$

$$\|u - u_h\|_0 \leq Ch^k |u|_{k+1} \text{ for } s \neq 1, \tag{50}$$

where C is a positive constant independent of h .

Proof First, we prove (49). We can use Aubin-Nitsche’s trick for $s = 1$. Let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the exact solution of the equation $-\Delta\psi = u - u_h$, and define $\boldsymbol{\psi} = \{\psi, \psi|_{\Gamma_h}\}$. For any $\boldsymbol{\psi}_h \in \mathbf{V}_h^{k,k-1}$, by the consistency and boundedness, we have

$$\begin{aligned} \|u - u_h\|_0^2 &= B_h(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\psi}) \\ &= B_h(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\psi} - \boldsymbol{\psi}_h) \\ &\leq C_b \|\mathbf{u} - \mathbf{u}_h\| \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|. \end{aligned} \tag{51}$$

Note that there exists $\psi_h \in V_h^{k,k-1}$ such that

$$\|\psi - \psi_h\| \leq Ch|\psi|_2 \leq Ch\|u - u_h\|_0.$$

By Lemma 3, we obtain (49). For the proof of the case $s \neq 1$, we show the following inequality in a similar manner presented in [2].

$$\|w - w_h\|_0 \leq C\|\mathbf{w} - \mathbf{w}_h\| \quad \forall \mathbf{w} = \{w, w|_{\Gamma_h}\} \in \mathbf{V}(h), \mathbf{w}_h \in \mathbf{V}_h^{k,k-1}. \tag{52}$$

Let $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of $-\Delta\varphi = w - w_h$ and define $\boldsymbol{\varphi} = \{\varphi, \varphi|_{\Gamma_h}\}$. Then we have

$$\begin{aligned} \|w - w_h\|_0^2 &= (\nabla\varphi, \nabla(w - w_h))_{\mathcal{T}_h} - \langle \mathbf{n} \cdot \nabla\varphi, \hat{w}_h - w_h \rangle_{\partial\mathcal{T}_h} \\ &\leq C\|\varphi\|_{2,\Omega} \|\mathbf{w} - \mathbf{w}_h\|. \end{aligned}$$

Since $\|\varphi\|_{2,\Omega} \leq C\|w - w_h\|_0$, we have (52). From Theorem 3, (50) follows immediately. \square

5 Numerical Results

5.1 Discontinuous Approximation

We consider the following test problem:

$$-\Delta u = 2\pi^2 \sin(\pi x) \sin(\pi y) \quad \text{in } \Omega, \tag{53}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{54}$$

where the domain Ω is the unit square and the source function is chosen so that the exact solution is $u(x, y) = \sin(\pi x) \sin(\pi y)$. We employed unstructured triangular meshes and P_k - P_{k-1} discontinuous approximation for $1 \leq k \leq 3$. The schemes are all symmetric. Tables 1 and 2 display the convergence histories of the reduced and standard HDG schemes, respectively. The mesh size is given by $h \approx 0.1 \times 2^{-(l-1)}$. It can be observed that the convergence rates of the piecewise H^1 -error and L^2 -error are optimal in all the cases, which agrees with our theoretical results. We also see that the absolute errors of the reduced HDG method is approximately as same as those of the standard HDG method. It suggests that the complementary projection part of the hybrid quantity, namely $(I - P_{k-1})\hat{u}_h$, actually does not contribute to accuracy.

5.2 Continuous Approximation

As mentioned in Sect 2.6, the approximation property does not hold for the continuous approximations. As a result, the convergence order in the energy and L^2 norms may not be optimal. We carried out numerical experiments to observe the convergence rate. The same test problem as in the previous is considered. We computed the approximate solutions by the reduced HDG method with the P_2 - P_1 and P_3 - P_2 continuous approximations. The same meshes as in the previous were used. The results are shown at Table 3. We observe that the convergence rates in the piecewise H^1 and L^2 norms are sub-optimal, which indicates the reduced stabilization is not suitable for the continuous approximations.

Table 1 Convergence history of the reduced HDG methods with discontinuous approximations

	l	$\ u - u_h\ $		$\ \nabla u - \nabla u_h\ $	
		Error	Order	Error	Order
P1P0	1	6.7399E-03	–	2.7656E-01	–
	2	1.5971E-03	2.48	1.3137E-01	1.28
	3	3.9742E-04	2.05	6.7522E-02	0.98
	4	9.7854E-05	2.01	3.2294E-02	1.06
P2P1	1	1.2851E-04	–	2.1856E-02	–
	2	1.3557E-05	3.88	4.8848E-03	2.58
	3	1.5594E-06	3.18	1.1857E-03	2.08
	4	1.8789E-07	3.03	2.8859E-04	2.03
P3P2	1	5.7044E-06	–	1.1525E-03	–
	2	2.7034E-07	5.25	1.1943E-04	3.91
	3	1.8682E-08	3.93	1.6192E-05	2.94
	4	9.7700E-10	4.23	1.7701E-06	3.17

Table 2 Convergence history of the standard HDG methods with discontinuous approximations

	l	$\ u - u_h\ $		$\ \nabla u - \nabla u_h\ $	
		Error	Order	Error	Order
P1P1	1	4.5794E-03	–	2.5585E-01	–
	2	1.0083E-03	2.61	1.2030E-01	1.30
	3	2.7100E-04	1.93	6.2282E-02	0.97
	4	6.0270E-05	2.15	2.9429E-02	1.07
P2P2	1	1.2353E-04	–	2.2591E-02	–
	2	1.2762E-05	3.91	5.0321E-03	2.59
	3	1.5147E-06	3.14	1.2195E-03	2.09
	4	1.8201E-07	3.04	2.9701E-04	2.02
P3P3	1	5.8623E-06	–	1.1410E-03	–
	2	2.7867E-07	5.25	1.1733E-04	3.92
	3	1.9385E-08	3.92	1.5902E-05	2.94
	4	1.0046E-09	4.24	1.7365E-06	3.17

Table 3 Convergence history of the reduced HDG methods with continuous approximations

	l	$\ u - u_h\ $		$\ \nabla u - \nabla u_h\ $	
		Error	Order	Error	Order
P2P1	1	6.6188E-03	–	2.3451E-01	–
	2	1.4856E-03	2.58	1.1206E-01	1.27
	3	3.7717E-04	2.02	5.6537E-02	1.01
	4	8.9346E-05	2.06	2.7466E-02	1.04
P3P2	1	1.5101E-04	–	1.4131E-02	–
	2	1.5681E-05	3.90	3.1251E-03	2.60
	3	1.8543E-06	3.14	7.6405E-04	2.07
	4	2.2391E-07	3.03	1.8754E-04	2.01

6 Conclusion

We proposed a new hybridized discontinuous Galerkin method with reduced stabilization. We devised an efficient implementation of our method by means of the Gaussian quadrature formula in the two-dimensional case. The error estimates in the energy and L^2 norms were proved under the chunkiness condition. It was also shown that our method with P_1 – P_0 approximation is closely related to the Crouzeix–Raviart nonconforming finite element method. Numerical results confirmed the validity of the proposed schemes.

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