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A Weak Galerkin Finite Element Scheme for the Biharmonic Equations by Using Polynomials of Reduced Order

Ran Zhang · Qilong Zhai

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Abstract A new weak Galerkin (WG) finite element method for solving the biharmonic equation in two or three dimensional spaces by using polynomials of reduced order is introduced and analyzed. The WG method is on the use of weak functions and their weak derivatives defined as distributions. Weak functions and weak derivatives can be approximated by polynomials with various degrees. Different combination of polynomial spaces leads to different WG finite element methods, which makes WG methods highly flexible and efficient in practical computation. This paper explores the possibility of optimal combination of polynomial spaces that minimize the number of unknowns in the numerical scheme, yet without compromising the accuracy of the numerical approximation. Error estimates of optimal order are established for the corresponding WG approximations in both a discrete H^2 norm and the standard L^2 norm. In addition, the paper also presents some numerical experiments to demonstrate the power of the WG method. The numerical results show a great promise of the robustness, reliability, flexibility and accuracy of the WG method.

Keywords Weak Galerkin finite element methods · Weak Laplacian · Biharmonic equation · Polyhedral meshes

Mathematics Subject Classification Primary: 65N30, 65N15, 65N12, 74N20 · Secondary: 35B45, 35J50, 35J35

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R. Zhang (✉) · Q. Zhai
Department of Mathematics, Jilin University, Changchun, China
e-mail: zhangran@mail.jlu.edu.cn

Q. Zhai
e-mail: diql@mails.jlu.edu.cn

1 Introduction

This paper will concern with approximating the solution u of the biharmonic equation

$$\Delta^2 u = f, \quad \text{in } \Omega, \tag{1.1}$$

with clamped boundary conditions

$$u = g, \quad \text{on } \partial\Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial \mathbf{n}} = \phi, \quad \text{on } \partial\Omega, \tag{1.3}$$

where Δ is the Laplacian operator, Ω is a bounded polygonal or polyhedral domain in $\mathbb{R}^d$ for $d = 2, 3$ and $\mathbf{n}$ denotes the outward unit normal vector along $\partial\Omega$. We assume that f, g, ϕ are given, sufficiently smooth functions.

This problem mainly arises in fluid dynamics where the stream functions u of incompressible flows are sought and elasticity theory, in which the deflection of a thin plate of the clamped plate bending problem is sought [26,34,36].

Due to the significance of the biharmonic problem, a large number of methods for discretizing (1.1)–(1.3) have been proposed. These methods include dealing with the biharmonic operator directly, such as discretizing (1.1)–(1.3) on a uniform grid using a 13-point or 25-point direct approximation of the fourth order differential operator [9,24]; mixed methods, that is, splitting the biharmonic equation into two coupled Poisson equations [1,4–7,12,15,17–20,25,27]. Also there are some other approaches to the biharmonic problems, like the conformal mapping methods [11,35], integral equations [29], orthogonal spline collocation method [8] and the fast multipole methods [23], etc.

Among these methods, finite element methods are one of the most widely used technique, which is based on variational formulations of the equations considered. In fact, the biharmonic equation is also one of the most important applicable problems of the finite element methods, cf. [2,13,14,16,22,41]. The Galerkin methods, discretizing the corresponding variational form of (1.1) is given by seeking $u \in H^2(\Omega)$ satisfying

$$u|_{\partial\Omega} = g, \qquad \frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = \phi$$

such that

$$(\Delta u, \Delta v) = (f, v), \quad \forall v \in H_0^2(\Omega), \tag{1.4}$$

where $H_0^2(\Omega)$ is the subspace of $H^2(\Omega)$ consisting of functions with vanishing value and normal derivative on $\partial\Omega$.

Standard finite element methods for solving (1.1)–(1.3) based on the variational form (1.4) with conforming finite element require rather sophisticated finite elements such as the 21-degrees-of-freedom of Argyris (see [3]) or nonconforming elements of Hermite type. Since the complexity in the construction for the finite element with high continuous elements, H^2 conforming element are seldom used in practice for the biharmonic problem. To avoid using of C^1-elements, besides the mixed methods, an alternative approach, nonconforming and discontinuous Galerkin finite element methods have been developed for solving the biharmonic equation over the last several decades. Morley element [28] is a well known nonconforming element for the biharmonic equation for its simplicity. A C^0 interior penalty method was developed in [10,21]. In [30], a hp-version interior penalty discontinuous Galerkin method was presented for the biharmonic equation.

Recently a new class of finite element methods, called weak Galerkin(WG) finite element methods were developed for the biharmonic equation for its highly flexible and robust properties. The WG method refers to a numerical scheme for partial differential equations in which differential operators are approximated by weak forms as distributions over a set of generalized functions. This thought was first proposed in [38] for a model second order elliptic problem, and this method was further developed in [31,39,40]. In [32], a weak Galerkin method for the biharmonic equation was derived by using discontinuous functions of piecewise polynomials on general partitions of polygons or polyhedra of arbitrary shape. After that, in order to reduce the number of unknowns, a C^0 WG method [33] was proposed and analyzed. However, due to the continuity limitation, the C^0 WG scheme only works for the traditional finite partitions, while not arbitrary polygonal or polyhedral girds as allowed in [32].

In order to realize the aim that reducing the unknown numbers and suit for general partitions of polygons or polyhedra of arbitrary shape at the same time, in this paper we construct a reduction WG scheme based on the use of a discrete weak Laplacian plus a new stabilization that is also parameter free. The goal of this paper is to specify all the details for the reduction WG method for the biharmonic equations and present the numerical analysis by presenting a mathematical convergence theory.

An outline of the paper is as follows. In the remainder of the introduction we shall introduce some preliminaries and notations for Sobolev spaces. In Sect. 2 is devoted to the definitions of weak functions and weak derivatives. The WG finite element schemes for the biharmonic Eqs. (1.1)–(1.3) are presented in Sect. 3. In Sect. 4, we establish an optimal order error estimates for the WG finite element approximation in an H^2 equivalent discrete norm. In Sect. 5, we shall drive an error estimate for the WG finite element method in the standard L^2 norm. Section 6 contains the numerical results of the WG method. The theoretical results are illustrated by these numerical examples. Finally, we present some technical estimates for quantities related to the local L^2 projections into various finite element spaces and some approximation properties which are useful in the convergence analysis in "Appendix".

Now let us define some notations. Let D be any open bounded domain with Lipschitz continuous boundary in $\mathbb{R}^d$, $d = 2, 3$. We use the standard definition for the Sobloev space $H^s(D)$ and their associated inner products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$, and seminorms $|\cdot|_{s,D}$ for any $s \geq 0$.

The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively. When $D = \Omega$, we shall drop the subscript D in the norm and in the inner product notation.

The space $H(\text{div}; D)$ is defined as the set of vector-valued functions on D which, together with their divergence, are square integrable; i.e.,

$$H(\text{div}; D) = \left\{\mathbf{v} : \mathbf{v} \in [L^2(D)]^d, \nabla\cdot\mathbf{v} \in L^2(D)\right\}.$$

The norm in $H(\text{div}; D)$ is defined by

$$\|\mathbf{v}\|_{H(\text{div};D)} = \left(\|\mathbf{v}\|_D^2 + \|\nabla\cdot\mathbf{v}\|_D^2\right)^{\frac{1}{2}}.$$

2 Weak Laplacain and Discrete Weak Laplacian

For the biharmonic equation (1.1), the underlying differential operator is the Laplacian Δ. Thus, we shall first introduce a weak version for the Laplacian operator defined on a class of discontinuous functions as distributions [32].

Let K be any polygonal or polyhedral domain with boundary ∂K. A weak function on the region K refers to a function $v = \{v_0, v_b, \mathbf{v}_g\}$ such that $v_0 \in L^2(K)$, $v_b \in L^2(\partial K)$, and $\mathbf{v}_g \cdot \mathbf{n} \in L^2(\partial K)$, where $\mathbf{n}$ is the outward unit normal vector along ∂K. Denote by $\mathcal{W}(K)$ the space of all weak functions on K, that is,

$$\mathcal{W}(K) = \left\{ v = \{v_0, v_b, \mathbf{v}_g\} : v_0 \in L^2(K), v_b, \mathbf{v}_g \cdot \mathbf{n} \in L^2(\partial K) \right\}. \tag{2.1}$$

Recall that, for any $v \in \mathcal{W}(K)$, the weak Laplacian of $v = \{v_0, v_b, \mathbf{v}_g\}$ is defined as a linear functional $\Delta_w v$ in the dual space of $H^2(K)$ whose action on each $\varphi \in H^2(K)$ is given by

$$(\Delta_w v, \varphi)_K = (v_0, \Delta\varphi)_K - \langle v_b, \nabla\varphi \cdot \mathbf{n}\rangle_{\partial K} + \langle \mathbf{v}_g \cdot \mathbf{n}, \varphi\rangle_{\partial K}, \tag{2.2}$$

where $(\cdot, \cdot)_K$ stands for the L^2-inner product in $L^2(K)$ and $\langle \cdot, \cdot \rangle_{\partial K}$ is the inner product in $L^2(\partial K)$.

The Sobolev space $H^2(K)$ can be embedded into the space $\mathcal{W}(K)$ by an inclusion map $i_{\mathcal{W}} : H^2(K) \to \mathcal{W}(K)$ defined as follows

$$i_{\mathcal{W}}(\phi) = \{\phi|_K, \phi|_{\partial K}, (\nabla\phi \cdot \mathbf{n})\mathbf{n}|_{\partial K}\}, \quad \phi \in H^2(K).$$

With the help of the inclusion map $i_{\mathcal{W}}$, the Sobolev space $H^2(K)$ can be viewed as a subspace of $\mathcal{W}(K)$ by identifying each $\phi \in H^2(K)$ with $i_{\mathcal{W}}(\phi)$.

Analogously, a weak function $v = \{v_0, v_b, \mathbf{v}_g\} \in \mathcal{W}(K)$ is said to be in $H^2(K)$ if it can be identified with a function $\phi \in H^2(K)$ through the above inclusion map. Here the first components v_0 can be seen as the value of v in the interior and the second component v_b represents the value of v on ∂K. Denote $\nabla v \cdot \mathbf{n}$ by v_n, then the third component $\mathbf{v}_g$ represents $(\nabla v \cdot \mathbf{n})\mathbf{n}|_{\partial K} = v_n \mathbf{n}$. Obviously, $\mathbf{v}_g \cdot \mathbf{n} = \nabla v \cdot \mathbf{n}$. Note that if $v \notin H^2(K)$, then v_b and $\mathbf{v}_g$ may not necessarily be related to the trace of v_0 and $(\nabla v_0 \cdot \mathbf{n})\mathbf{n}$ on ∂K, respectively.

For $v \in H^2(K)$, from integration by parts we have

$$\begin{aligned}(\Delta_w v, \varphi)_K &= (v, \Delta\varphi)_K - \langle v, \nabla\varphi \cdot \mathbf{n}\rangle_{\partial K} + \langle \nabla v \cdot \mathbf{n}, \varphi\rangle_{\partial K} \\ &= (v_0, \Delta\varphi)_K - \langle v_b, \nabla\varphi \cdot \mathbf{n}\rangle_{\partial K} + \langle \mathbf{v}_g \cdot \mathbf{n}, \varphi\rangle_{\partial K}.\end{aligned}$$

Thus the weak Laplacian is identical with the strong Laplacian, i.e.,

$$\Delta_w i_{\mathcal{W}}(v) = \Delta v$$

for smooth functions in $H^2(K)$.

For numerical implementation purpose, we define a discrete version of the weak Laplacain operator by approximating Δ_w in polynomial subspaces of the dual of $H^2(K)$. To this end, for any non-negative integer $r \geq 0$, let $P_r(K)$ be the set of polynomials on K with degree no more than r.

Definition 2.1 ([32]) A discrete weak Laplacian operator, denoted by $\Delta_{w,r,K}$, is defined as the unique polynomial $\Delta_{w,r,K} v \in P_r(K)$ satisfying

$$(\Delta_{w,r,K} v, \varphi)_K = (v_0, \Delta\varphi)_K - \langle v_b, \nabla\varphi \cdot \mathbf{n}\rangle_{\partial K} + \langle \mathbf{v}_n \cdot \mathbf{n}, \varphi\rangle_{\partial K}, \quad \forall \varphi \in P_r(K). \tag{2.3}$$

From the integration by parts, we have

$$(v_0, \Delta\varphi)_K = (\Delta v_0, \varphi)_K + \langle v_0, \nabla\varphi \cdot \mathbf{n}\rangle_{\partial K} - \langle \nabla v_0 \cdot \mathbf{n}, \varphi\rangle_{\partial K}.$$

Substituting the above identity into (2.3) yields

$$(\Delta_{w,r,K} v, \varphi)_K - (\Delta v_0, \varphi)_K = \langle v_0 - v_b, \nabla\varphi \cdot \mathbf{n}\rangle_{\partial K} - \langle (\nabla v_0 - \mathbf{v}_g) \cdot \mathbf{n}, \varphi\rangle_{\partial K}, \tag{2.4}$$

for all $\varphi \in P_r(K)$.

3 Weak Galerkin Finite Element Scheme

Let $\mathcal{T}_h$ be a partition of the domain Ω into polygons in 2D or polyhedra in 3D. Assume that $\mathcal{T}_h$ is shape regular in the sense as defined in [39]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ be the set of all interior edges or flat faces.

Since v_n represents $\nabla v \cdot \mathbf{n}$, then v_n is naturally dependent on $\mathbf{n}$. To ensure a single valued function v_n on $e \in \mathcal{E}_h$, we introduce a set of normal directions on $\mathcal{E}_h$ as follows

$$\mathcal{N}_h = \{\mathbf{n}_e : \mathbf{n}_e \text{ is unit and normal to } e,\ \ e \in \mathcal{E}_h\}. \tag{3.1}$$

For any given integer $k \geq 2$, $T \in \mathcal{T}_h$, denote by $\mathcal{W}_k(T)$ the discrete weak function space given by

$$\mathcal{W}_k(T) = \{\{v_0, v_b, v_n\mathbf{n}_e\} : v_0 \in P_k(T), v_b, v_n \in P_{k-1}(e), e \subset \partial T\}. \tag{3.2}$$

By patching $\mathcal{W}_k(T)$ over all the elements $T \in \mathcal{T}_h$ through a common value on the interface $\mathcal{E}_h^0$, we arrive at a weak finite element space V_h given by

$$V_h = \left\{\{v_0, v_b, v_n\mathbf{n}_e\} : \{v_0, v_b, v_n\mathbf{n}_e\}\big|_T \in \mathcal{W}_k(T), \quad \forall T \in \mathcal{T}_h\right\}.$$

Denote by V_h^0 the subspace of V_h constituting discrete weak functions with vanishing traces; i.e.,

$$V_h^0 = \{\{v_0, v_b, v_n\mathbf{n}_e\} : \{v_0, v_b, v_n\mathbf{n}_e\} \in V_h, v_b|_e = 0, v_n|_e = 0, \quad e \in \partial T \cap \partial\Omega\}.$$

Denote by Λ_h the trace of V_h on $\partial\Omega$ from the component v_b. It is obvious that Λ_h consists of piecewise polynomials of degree $k-1$. Similarly, denote by Υ_h the trace of V_h from the component of v_n as piecewise polynomials of degree $k-1$. Denote by $\Delta_{w,k-2}$ the discrete weak Laplacian operator on the finite element space V_h computed by using (2.3) on each element T for $k \geq 2$, that is,

$$(\Delta_{w,k-2}v)|_T = \Delta_{w,k-2,T}(v|_T) \quad \forall v \in V_h. \tag{3.3}$$

For simplicity, we shall drop the subscript $k-2$ in the notation $\Delta_{w,k-2}$ for the discrete weak Laplacian operator. We also introduce the following notation

$$(\Delta_w v, \Delta_w w)_h = \sum_{T \in \mathcal{T}_h} (\Delta_w v, \Delta_w w)_T.$$

For each element $T \in \mathcal{T}_h$, denote by Q_0 the L^2 projection onto $P_k(T)$, $k \geq 2$. For each edge/face $e \subset \partial T$, denote by Q_b the L^2 projection onto $P_{k-1}(e)$. Now for any $u \in H^2(\Omega)$, we shall combine these two projections together to define a projection into the finite element space V_h such that on the element T

$$Q_h u = \{Q_0 u, Q_b u, (Q_b(\nabla u \cdot \mathbf{n}_e))\mathbf{n}_e\}.$$

Theorem 3.1 *Let $\mathbb{Q}_h$ be the local L^2 projection onto P_{k-2}. Then the following commutative diagram holds true on each element $T \in \mathcal{T}_h$:*

$$\Delta_w Q_h u = \mathbb{Q}_h \Delta u, \qquad \forall u \in H^2(T). \tag{3.4}$$

Proof For any $\phi \in P_{k-2}(T)$, from the definition of the discrete weak Laplacian and the L^2 projection

$$\begin{aligned}(\Delta_w Q_h u, \phi)_T &= (Q_0 u, \Delta\phi)_T - \langle Q_b u, \nabla\phi\cdot\mathbf{n}\rangle_{\partial T} + \langle Q_b(\nabla u\cdot\mathbf{n}_e)\mathbf{n}_e\cdot\mathbf{n}, \phi\rangle_{\partial T}\\ &= (u, \Delta\phi)_T - \langle u, \nabla\phi\cdot\mathbf{n}\rangle_{\partial T} + \langle\nabla u\cdot\mathbf{n}, \phi\rangle_{\partial T}\\ &= (\Delta u, \phi)_T = (\mathbb{Q}_h\Delta u, \phi),\end{aligned}$$

which implies (3.4). □

The commutative property (3.4) indicates that the discrete weak Laplacian of the L^2 projection of u is a good approximation of the Laplacian of u in the classical sense. This is a good property of the discrete weak Laplacian in application to algorithm and analysis.

For any $u_h = \{u_0, u_b, u_n\mathbf{n}_e\}$ and $v = \{v_0, v_b, v_n\mathbf{n}_e\}$ in V_h, we introduce a bilinear form as follows

$$\begin{aligned}s(u_h, v) = &\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle\nabla u_0\cdot\mathbf{n}_e - u_n, \nabla v_0\cdot\mathbf{n}_e - v_n\rangle_{\partial T}\\ &+\sum_{T\in\mathcal{T}_h} h_T^{-3}\langle Q_b u_0 - u_b, Q_b v_0 - v_b\rangle_{\partial T}.\end{aligned}$$

Weak Galerkin Algorithm 1 *Find $u_h = \{u_0, u_b, u_n\boldsymbol{n}_e\} \in V_h$ satisfying $u_b = Q_b g$ and $u_n = Q_b\phi$ on $\partial\Omega$ and the following equation:*

$$(\Delta_w u_h, \Delta_w v)_h + s(u_h, v) = (f, v_0),\ \forall v = \{v_0, v_b, v_n\boldsymbol{n}_e\} \in V_h^0. \tag{3.5}$$

Lemma 3.2 *For any $v \in V_h^0$, let $|\!|\!| v |\!|\!|$ be given by*

$$|\!|\!| v |\!|\!|^2 = (\Delta_w v, \Delta_w v)_h + s(v, v). \tag{3.6}$$

Then, $|\!|\!| \cdot |\!|\!|$ defines a norm in the linear space V_h^0.

Proof For simplicity, we shall only prove the positivity property for $|\!|\!| \cdot |\!|\!|$. Assume that $|\!|\!| v |\!|\!| = 0$ for some $v \in V_h^0$. It follows that $\Delta_w v = 0$ on each element T, $Q_b v_0 = v_b$ and $\nabla v_0\cdot\mathbf{n}_e = v_n$ on each edge ∂T. We claim that $\Delta v_0 = 0$ holds true locally on each element T. To this end, for any $\varphi \in P_{k-2}(T)$ we use $\Delta_w v = 0$ and the identify (2.4) to obtain

$$\begin{aligned}0 &= (\Delta_w v, \varphi)_T\\ &= (\Delta v_0, \varphi)_T + \langle Q_b v_0 - v_b, \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle v_n\mathbf{n}_e\cdot\mathbf{n} - \nabla v_0\cdot\mathbf{n}, \varphi\rangle_{\partial T}\\ &= (\Delta v_0, \varphi)_T,\end{aligned} \tag{3.7}$$

where we have used the fact that $Q_b v_0 - v_b = 0$ and

$$v_n\mathbf{n}_e\cdot\mathbf{n} - \nabla v_0\cdot\mathbf{n} = \pm(v_n - \nabla v_0\cdot\mathbf{n}_e) = 0$$

in the last equality. The identity (3.7) implies that $\Delta v_0 = 0$ holds true locally on each element T.

Next, we claim that $\nabla v_0 = 0$ also holds true locally on each element T. For this purpose, for any $\phi \in P_k(T)$, we utilize the Gauss formula to obtain

$$(\nabla v_0, \nabla\phi)_T = -(\Delta v_0, \phi)_T + \langle\nabla v_0\cdot\mathbf{n}, \phi\rangle_{\partial T} = \langle\nabla v_0\cdot\mathbf{n}, \phi\rangle_{\partial T}. \tag{3.8}$$

By letting $\phi = v_0$ on each element T and summing over all T we obtain

$$\sum_{T\in\mathcal{T}_h}(\nabla v_0, \nabla v_0)_T = \sum_{T\in\mathcal{T}_h}\langle\nabla v_0\cdot\mathbf{n}, v_0\rangle_{\partial T}. \tag{3.9}$$

For two elements $T_1, T_2 \in \mathcal{T}_h$, which share $e \in \mathcal{E}_h \setminus \partial\Omega$ as a common edge, denote v_0^1, v_0^2 the values of v in the interior of T_1, T_2, respectively. It follows from $Q_b v_0^1 = Q_b v_0^2 = v_b$ on edge e and the fact $\nabla v_0 \cdot \mathbf{n}_e = v_n \in P_{k-1}(e)$ that

$$\left\langle \nabla v_0^1 \cdot \mathbf{n}_{T_1}, v_0^1 \right\rangle_e + \left\langle \nabla v_0^2 \cdot \mathbf{n}_{T_2}, v_0^2 \right\rangle_e = \pm \left\langle v_n, v_0^1 - v_0^2 \right\rangle_e = \pm \left\langle v_n, Q_b v_0^1 - Q_b v_0^2 \right\rangle_e = 0,$$

where $\mathbf{n}_{T_1}, \mathbf{n}_{T_2}$ denote the outward unit normal vectors on e according to elements T_1, T_2, respectively. This, together with $\nabla v_0 \cdot \mathbf{n} = v_n = 0$ on the boundary edge $e \in \mathcal{E}_h \cap \partial\Omega$ implies

$$\sum_{T \in \mathcal{T}_h} \langle \nabla v_0 \cdot \mathbf{n}, v_0 \rangle_{\partial T} = 0.$$

It follows from Eq. (3.9) that $\|\nabla v_0\|_T = 0$ on each element T. Thus, $v_0 = const$ locally on each element and is then continuous across each interior edge e as

$$v_0|_e = Q_b v_0 = v_b.$$

The boundary condition of $v_b = 0$ then implies that $v \equiv 0$ on Ω, which completes the proof. □

Lemma 3.3 *The weak Galerkin finite element scheme (3.5) has a unique solution.*

Proof Assume $u_h^{(1)}$ and $u_h^{(2)}$ are two solutions of the WG finite element scheme (3.5). It is obvious that the difference $\rho_h = u_h^{(1)} - u_h^{(2)}$ is a finite element function in V_h^0 satisfying

$$(\Delta_w \rho_h, \Delta_w v)_h + s(\rho_h, v) = 0, \ \forall v \in V_h^0. \tag{3.10}$$

By letting $v = \rho_h$ in above Eq. (3.10) we obtain the following indentity

$$(\Delta_w \rho_h, \Delta_w \rho_h)_h + s(\rho_h, \rho_h) = 0.$$

It follows from Lemma 3.2 that $\rho_h \equiv 0$, which shows that $u_h^{(1)} = u_h^{(2)}$. This completes the proof. □

4 An Error Estimate

The goal of this section is to establish an error estimate for the WG-FEM solution u_h arising from (3.5).

First of all, let us derive an error equation for the WG finite element solution obtained from (3.5). This error equation is critical in convergence analysis.

Lemma 4.1 *Let u and $u_h \in V_h$ be the solution of (1.1)–(1.3) and (3.5), respectively. Denote by*

$$e_h = Q_h u - u_h$$

the error function between the L^2 projection of u and its weak Galerkin finite element solution. Then the error function e_h satisfies the following equation

$$(\Delta_\omega e_h, \Delta_\omega v)_h + s(e_h, v) = \ell_u(v) \tag{4.1}$$

for all $v \in V_h^0$. *Here*

$$\ell_u(v) = \sum_{T\in\mathcal{T}_h} \langle \Delta u - \mathbb{Q}_h \Delta u, \nabla v_0 \cdot \mathbf{n} - v_n \mathbf{n}_e \cdot \mathbf{n}\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u)\cdot \mathbf{n}, v_0 - v_b\rangle_{\partial T} + s(Q_h u, v). \tag{4.2}$$

Proof Using (2.4) with $\varphi = \Delta_\omega Q_h u = \mathbb{Q}_h \Delta u$ we obtain

$$\begin{aligned}(\Delta_\omega Q_h u, \Delta_\omega v)_T &= (\Delta v_0, \mathbb{Q}_h \Delta u)_T + \langle v_0 - v_b, \nabla(\mathbb{Q}_h \Delta u)\cdot \mathbf{n}\rangle_{\partial T} - \langle (\nabla v_0 - v_n \mathbf{n}_e)\cdot \mathbf{n}, \mathbb{Q}_h \Delta u\rangle_{\partial T} \\ &= (\Delta u, \Delta v_0)_T + \langle v_0 - v_b, \nabla(\mathbb{Q}_h \Delta u)\cdot \mathbf{n}\rangle_{\partial T} - \langle (\nabla v_0 - v_n \mathbf{n}_e)\cdot \mathbf{n}, \mathbb{Q}_h \Delta u\rangle_{\partial T},\end{aligned}$$

which implies that

$$(\Delta u, \Delta v_0)_T = (\Delta_\omega Q_h u, \Delta_\omega v)_T - \langle v_0 - v_b, \nabla(\mathbb{Q}_h \Delta u)\cdot \mathbf{n}\rangle_{\partial T} + \langle (\nabla v_0 - v_n \mathbf{n}_e)\cdot \mathbf{n}, \mathbb{Q}_h \Delta u\rangle_{\partial T}. \tag{4.3}$$

Next, it follows from the integration by parts that

$$(\Delta u, \Delta v_0)_T = (\Delta^2 u, v_0)_T + \langle \Delta u, \nabla v_0 \cdot \mathbf{n}\rangle_{\partial T} - \langle \nabla(\Delta u)\cdot \mathbf{n}, v_0\rangle_{\partial T}.$$

By summing over all T and then using the identity $(\Delta^2 u, v_0) = (f, v_0)$ we arrive at

$$\sum_{T\in\mathcal{T}_h} (\Delta u, \Delta v_0)_T = (f, v_0) + \sum_{T\in\mathcal{T}_h} \langle \Delta u, \nabla v_0 \cdot \mathbf{n} - v_n \mathbf{n}_e \cdot \mathbf{n}\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u)\cdot \mathbf{n}, v_0 - v_b\rangle_{\partial T},$$

where we have used the fact that v_n and v_b vanish on the boundary of the domain. Combining the above equation with (4.3) yields

$$(\Delta_\omega Q_h u, \Delta_\omega v)_h = (f, v_0) + \sum_{T\in\mathcal{T}_h} \langle \Delta u - \mathbb{Q}_h \Delta u, (\nabla v_0 - v_n \mathbf{n}_e)\cdot \mathbf{n}\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u)\cdot \mathbf{n}, v_0 - v_b\rangle_{\partial T}.$$

Adding $s(Q_h u, v)$ to both sides of the above equation gives

$$\begin{aligned}(\Delta_\omega Q_h u, \Delta_\omega v)_h + s(Q_h u, v) &= (f, v_0) + \sum_{T\in\mathcal{T}_h} \langle \Delta u - \mathbb{Q}_h \Delta u, (\nabla v_0 - v_n \mathbf{n}_e)\cdot \mathbf{n}\rangle_{\partial T} \\ &\quad - \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u)\cdot \mathbf{n}, v_0 - v_b\rangle_{\partial T} + s(Q_h u, v).\end{aligned} \tag{4.4}$$

Subtracting (3.5) from (4.4) leads to the following error equation

$$(\Delta_\omega e_h, \Delta_\omega v)_h + s(e_h, v) = \sum_{T\in\mathcal{T}_h} \langle \Delta u - \mathbb{Q}_h \Delta u, (\nabla v_0 - v_n \mathbf{n}_e)\cdot \mathbf{n}\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u)\cdot \mathbf{n}, v_0 - v_b\rangle_{\partial T} + s(Q_h u, v)$$

for all $v \in V_h^0$. This completes the derivation of (4.1). □

The following Theorem presents an optimal order error estimate for the error function e_h in the trip-bar norm. We believe this tripe-bar norm provides a discrete analogue of the usual H^2-norm.

Theorem 4.2 *Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from (3.5) with finite element functions of order $k \geq 2$. Assume that the exact solution of (1.1)–(1.3) is sufficiently regular such that $u \in H^{k+2}(\Omega)$. Then, there exists a constant C such that*

$$|||u_h - Q_h u||| \leq Ch^{k-1} \|u\|_{k+2}. \tag{4.5}$$

The above estimate is of optimal order in terms of the meshsize h, but not in the regularity assumption on the exact solution of the biharmonic equation.

Proof By letting $v = e_h$ in the error Eq. (4.1), we have

$$|||e_h|||^2 = \ell(e_h), \tag{4.6}$$

where

$$\begin{aligned} \ell(e_h) = & \sum_{T\in\mathcal{T}_h} \langle \Delta u - \mathbb{Q}_h \Delta u, (\nabla e_0 - e_n \mathbf{n}_e) \cdot \mathbf{n}\rangle_{\partial T} \\ & - \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u) \cdot \mathbf{n}, e_0 - e_b\rangle_{\partial T} \\ & + \sum_{T\in\mathcal{T}_h} h_T^{-1} \langle \nabla Q_0 u \cdot \mathbf{n}_e - Q_b(\nabla u \cdot \mathbf{n}_e), \nabla e_0 \cdot \mathbf{n}_e - e_n\rangle_{\partial T} \\ & + \sum_{T\in\mathcal{T}_h} h_T^{-3} \langle Q_b Q_0 u - Q_b u, Q_b e_0 - e_b\rangle_{\partial T}. \end{aligned} \tag{4.7}$$

The rest of the proof shall estimate each of the terms on the right-hand side of (4.7). For the first term, we use the Cauchy–Schwarz inequality and the estimates (7.5) and (7.6) in Lemma 7.4 (see "Appendix") with $m = k$ to obtain

$$\begin{aligned} & \left| \sum_{T\in\mathcal{T}_h} \langle \Delta u - \mathbb{Q}_h \Delta u, (\nabla e_0 - e_n \mathbf{n}_e) \cdot \mathbf{n}\rangle_{\partial T} \right| \\ & \leq \left(\sum_{T\in\mathcal{T}_h} h_T \|\Delta u - \mathbb{Q}_h \Delta u\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|\nabla e_0 \cdot \mathbf{n}_e - e_n\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ & \leq Ch^{k-1} \|u\|_{k+1} |||e_h|||. \end{aligned} \tag{4.8}$$

For the second term, using Lemmas 7.4, 7.6 and 7.9 we obtain

$$\begin{aligned} & \left| \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u) \cdot \mathbf{n}, e_0 - e_b\rangle_{\partial T} \right| \\ & \leq \left| \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u) \cdot \mathbf{n}, Q_b e_0 - e_b\rangle_{\partial T} \right| \\ & \quad + \left| \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u) \cdot \mathbf{n}, e_0 - Q_b e_0\rangle_{\partial T} \right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{T\in\mathcal{T}_h} \langle \nabla(\Delta u - \mathbb{Q}_h \Delta u)\cdot \mathbf{n}, Q_b e_0 - e_b\rangle_{\partial T} \right| \\
&\quad + \left| \sum_{T\in\mathcal{T}_h} \langle (\nabla(\Delta u) - Q_b(\nabla(\Delta u)))\cdot \mathbf{n}, e_0 - Q_b e_0\rangle_{\partial T} \right| \\
&\le \left(\sum_{T\in\mathcal{T}_h} h_T^3 \|\nabla(\Delta u - \mathbb{Q}_h\Delta u)\|_{\partial T}^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{T\in\mathcal{T}_h} h_T^{-3} \|Q_b e_0 - e_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{T\in\mathcal{T}_h} \|\nabla(\Delta u) - Q_b(\nabla(\Delta u))\|_{\partial T}^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{T\in\mathcal{T}_h} \|e_0 - Q_b e_0\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\le Ch^{k-1} \|u\|_{k+2} |||e_h|||, \qquad (4.9)
\end{aligned}
$$

where the H^{k+2}-norm of u is used because the estimate in Lemma 7.9 is not optimal in terms of the mesh parameter h.

The third and fourth terms can be estimated by using the Cauchy–Schwarz inequality and the estimates (7.7) and (7.8) in Lemma 7.4 as follows

$$
\left| \sum_{T\in\mathcal{T}_h} h_T^{-1} \langle \nabla Q_0 u\cdot \mathbf{n}_e - Q_b(\nabla u\cdot \mathbf{n}_e), \nabla e_0\cdot \mathbf{n}_e - e_n\rangle_{\partial T} \right| \le Ch^{k-1}\|u\|_{k+1} |||e_h||| \qquad (4.10)
$$

and

$$
\left| \sum_{T\in\mathcal{T}_h} h_T^{-3} \langle Q_b Q_0 u - Q_b u, Q_b e_0 - e_b\rangle_{\partial T} \right| \le Ch^{k-1}\|u\|_{k+1} |||e_h|||. \qquad (4.11)
$$

Substituting (4.8)–(4.11) into (4.6) gives

$$
|||e_h|||^2 \le Ch^{k-1} \|u\|_{k+2} |||e_h|||,
$$

which implies (4.5) and hence completes the proof. □

5 Error Estimates in L^2

In this section, we shall establish some error estimates for all three components of the error function e_h in the standard L^2 norm.

First of all, let us derive an error estimate for the first component of the error function e_h by applying the usual duality argument in the finite element analysis. To this end, we consider the problem of seeking φ such that

$$
\begin{aligned}
\Delta^2 \varphi &= e_0, \quad \text{in } \Omega, \\
\varphi &= 0, \quad \text{on } \partial\Omega, \\
\frac{\partial \varphi}{\partial \mathbf{n}} &= 0, \quad \text{on } \partial\Omega.
\end{aligned}
\qquad (5.1)
$$

Assume that the dual problem has the H^4 regularity property in the sense that the solution function $\varphi \in H^4$ and there exists a constant C such that

$$\|\varphi\|_4 \leq C\|e_0\|. \tag{5.2}$$

Theorem 5.1 *Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from (3.5) with finite element functions of order $k \geq 2$. Let $k_0 = \min\{3, k\}$. Assume that the exact solution of (1.1)–(1.3) is sufficiently regular such that $u \in H^{k+2}(\Omega)$ and the dual problem (5.1) has the H^4 regularity. Then, there exists a constant C such that*

$$\|u_0 - Q_0 u\| \leq Ch^{k+k_0-2}\|u\|_{k+1}, \tag{5.3}$$

which means we have a sub-optimal order of convergence for $k = 2$ and optimal order of convergence for $k \geq 3$.

Proof Testing (5.1) by error function e_0 and then using the integration by parts gives

$$\begin{aligned}
\|e_0\|^2 &= (\Delta^2\varphi, e_0)\\
&= \sum_{T\in\mathcal{T}_h} (\Delta\varphi, \Delta e_0)_T + \sum_{T\in\mathcal{T}_h} \langle\nabla(\Delta\varphi)\cdot\mathbf{n}, e_0\rangle_{\partial T} - \sum_{T\in\mathcal{T}_h} \langle\Delta\varphi, \nabla e_0\cdot\mathbf{n}\rangle_{\partial T}\\
&= \sum_{T\in\mathcal{T}_h} (\Delta\varphi, \Delta e_0)_T + \sum_{T\in\mathcal{T}_h} \langle\nabla(\Delta\varphi)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T}\\
&\quad - \sum_{T\in\mathcal{T}_h} \langle\Delta\varphi, (\nabla e_0 - e_n\mathbf{n}_e)\cdot\mathbf{n}\rangle_{\partial T},
\end{aligned}$$

where we have used the fact that e_n and e_b vanishes on the boundary of the domain Ω. By letting $u = \varphi$ and $v_0 = e_h$ in (4.3), we can rewrite the above equation as follows

$$\begin{aligned}
\|e_0\|^2 &= (\Delta_w Q_h\varphi, \Delta_w e_h)_h + \sum_{T\in\mathcal{T}_h} \langle(\nabla(\Delta\varphi) - \nabla(\mathbb{Q}_h\Delta\varphi)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T}\\
&\quad - \sum_{T\in\mathcal{T}_h} \langle\Delta\varphi - \mathbb{Q}_h\Delta\varphi, (\nabla e_0 - e_n\mathbf{n}_e)\cdot\mathbf{n}\rangle_{\partial T}.
\end{aligned}$$

Next, by letting $v = Q_h\varphi$, from the error equation (4.1), we have

$$\begin{aligned}
(\Delta_w Q_h\varphi, \Delta_w e_h)_h &= \sum_{T\in\mathcal{T}_h} \langle(\Delta u - \mathbb{Q}_h\Delta u, (\nabla Q_0\varphi)\cdot\mathbf{n} - Q_b(\nabla\varphi\cdot\mathbf{n}_e)\mathbf{n}_e\cdot\mathbf{n}\rangle_{\partial T}\\
&\quad - \sum_{T\in\mathcal{T}_h} \langle\nabla(\Delta u - \mathbb{Q}_h\Delta u)\cdot\mathbf{n}, Q_0\varphi - Q_b\varphi\rangle_{\partial T}\\
&\quad - s(e_h, Q_h\varphi) + s(Q_h u, Q_h\varphi).
\end{aligned}$$

Combining the two equations above gives

$$\begin{aligned}
\|e_0\|^2 &= \sum_{T\in\mathcal{T}_h} \langle(\nabla(\Delta\varphi) - \nabla(\mathbb{Q}_h\Delta\varphi)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T}\\
&\quad - \sum_{T\in\mathcal{T}_h} \langle\Delta\varphi - \mathbb{Q}_h\Delta\varphi, (\nabla e_0\cdot\mathbf{n}_e - e_n)\cdot\mathbf{n}\rangle_{\partial T}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{T\in\mathcal{T}_h}\langle(\Delta u-\mathbb{Q}_h\Delta u,(\nabla Q_0\varphi)\cdot\mathbf{n}-Q_b(\nabla\varphi\cdot\mathbf{n}_e)\mathbf{n}_e\cdot\mathbf{n}\rangle_{\partial T}\\
&-\sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta u-\mathbb{Q}_h\Delta u)\cdot\mathbf{n},Q_0\varphi-Q_b\varphi\rangle_{\partial T}\\
&-s(e_h,Q_h\varphi)+s(Q_hu,Q_h\varphi).
\end{aligned}\tag{5.4}
$$

From the Cauchy–Schwarz inequality and Lemma 7.4, we can estimate the six terms on the right-hand side of the identity above as follows.

For the first term, it follows from Lemmas 7.4, 7.9 and the fact $k_0=\min\{k,3\}\le 3$ that

$$
\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\langle(\nabla(\Delta\varphi)-\nabla(\mathbb{Q}_h\Delta\varphi))\cdot\mathbf{n},e_0-e_b\rangle_{\partial T}\right|\\
&\le\left(\sum_{T\in\mathcal{T}_h}h_T^3\|\nabla(\Delta\varphi)-\nabla(\mathbb{Q}_h\Delta\varphi)\|_{\partial T}^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h}h_T^{-3}\|Q_be_0-e_b\|_{\partial T}^2\right)^{\frac12}\\
&\quad+\left(\sum_{T\in\mathcal{T}_h}\|\nabla(\Delta\varphi)-Q_b\nabla(\Delta\varphi)\|_{\partial T}^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h}\|e_0-Q_be_0\|_{\partial T}^2\right)^{\frac12}\\
&\le\left(\sum_{T\in\mathcal{T}_h}h_T^3\|\nabla(\Delta\varphi)-\nabla(\mathbb{Q}_h\Delta\varphi)\|_{\partial T}^2\right)^{\frac12}|\!|\!|e_h|\!|\!|\\
&\quad+C\lambda\left(\sum_{T\in\mathcal{T}_h}\|\nabla(\Delta\varphi)-Q_b\nabla(\Delta\varphi)\|_{\partial T}^2\right)^{\frac12}\cdot h^{-\frac12}\|e_0\|\\
&\quad+C\left(\sum_{T\in\mathcal{T}_h}\|\nabla(\Delta\varphi)-Q_b\nabla(\Delta\varphi)\|_{\partial T}^2\right)^{\frac12}\cdot h^{\frac32}|\!|\!|e_h|\!|\!|\\
&\le Ch^{k_0-1}(\|\varphi\|_{k_0+1}+h\delta_{k_0,2}\|\varphi\|_4)|\!|\!|e_h|\!|\!|+C\lambda h^{\frac12}\|\varphi\|_4\cdot h^{-\frac12}\|e_0\|\\
&\quad+Ch^{k_0-\frac52}(\|\varphi\|_{k_0+1}+h\delta_{k_0,2}\|\varphi\|_4)\cdot h^{\frac32}|\!|\!|e_h|\!|\!|\\
&\le Ch^{k_0-1}\|\varphi\|_4|\!|\!|e_h|\!|\!|+C\lambda\|\varphi\|_4\|e_0\|.
\end{aligned}\tag{5.5}
$$

For the second term, it follows from (7.3) with $m=k_0$ that

$$
\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\langle\Delta\varphi-\mathbb{Q}_h\Delta\varphi,(\nabla e_0\cdot\mathbf{n}_e-e_n)\cdot\mathbf{n}\rangle_{\partial T}\right|\\
&\le\left(\sum_{T\in\mathcal{T}_h}h_T\|\Delta\varphi-\mathbb{Q}_h\Delta\varphi\|_{\partial T}^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|\nabla e_0\cdot\mathbf{n}_e-e_n\|_{\partial T}^2\right)^{\frac12}\\
&\le Ch^{k_0-1}\|\varphi\|_{k_0+1}|\!|\!|e_h|\!|\!|\le Ch^{k_0-1}\|\varphi\|_4|\!|\!|e_h|\!|\!|.
\end{aligned}\tag{5.6}
$$

As to the third term, it follows from Cauchy–Schwarz inequality and Lemma 7.4 that

$$\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\langle \Delta u-\mathbb{Q}_h\Delta u, (\nabla Q_0\varphi)\cdot\mathbf{n}-Q_b(\nabla\varphi\cdot\mathbf{n}_e)\mathbf{n}_e\cdot\mathbf{n}\rangle_{\partial T}\right|\\
&\le\left(\sum_{T\in\mathcal{T}_h}h_T\|\Delta u-\mathbb{Q}_h\Delta u\|_{\partial T}^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|(\nabla Q_0\varphi)\cdot\mathbf{n}-Q_b(\nabla\varphi\cdot\mathbf{n}_e)\|_{\partial T}^2\right)^{\frac12}\\
&\le Ch^{k-1}\|u\|_{k+1}h^{k_0-1}\|\varphi\|_{k_0+1}\le Ch^{k+k_0-2}\|u\|_{k+1}\|\varphi\|_4.
\end{aligned}\tag{5.7}$$

For the forth term, by using Lemma 7.3, we have

$$\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\langle\nabla(\Delta u-\mathbb{Q}_h\Delta u)\cdot\mathbf{n}, Q_0\varphi-Q_b\varphi\rangle_{\partial T}\right|\\
&\le\left(\sum_{T\in\mathcal{T}_h}h_T^3\|\nabla(\Delta u-\mathbb{Q}_h\Delta u)\|_{\partial T}^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h}h_T^{-3}\|Q_0\varphi-\varphi\|_{\partial T}^2\right)^{\frac12}\\
&\le Ch^{k-1}(\|u\|_{k+1}+h\delta_{k,2}\|u\|_4)h^{t_0-1}\|\varphi\|_{k_0+1}\\
&\le Ch^{k-1}(\|u\|_{k+k_0-2}+h\delta_{k,2}\|u\|_4)\|\varphi\|_4.
\end{aligned}\tag{5.8}$$

As to the fifth term, we also use the Cauchy–Schwarz inequality and Lemma 7.4 to obtain

$$\begin{aligned}
|s(e_h, Q_h\varphi)|&\le\left|\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle\nabla e_0\cdot\mathbf{n}_e-e_n, \nabla Q_0\varphi\cdot\mathbf{n}_e-Q_b(\nabla\varphi\cdot\mathbf{n}_e)\rangle_{\partial T}\right|\\
&\quad+\left|\sum_{T\in\mathcal{T}_h}h_T^{-3}\langle Q_be_0-e_b, Q_bQ_0\varphi-Q_b\varphi\rangle_{\partial T}\right|\\
&\le Ch^{k_0-1}\|\varphi\|_4|\!|\!|e_h|\!|\!|.
\end{aligned}\tag{5.9}$$

The last term can be estimated as follows

$$\begin{aligned}
|s(Q_hu, Q_h\varphi)|&\le\left|\sum_{T\in\mathcal{T}_h}h_T^{-1}\langle(\nabla Q_0u\cdot\mathbf{n}_e-Q_b(\nabla u\cdot\mathbf{n}_e), (\nabla Q_0\varphi\cdot\mathbf{n}_e-Q_b(\nabla\varphi\cdot\mathbf{n}_e)\rangle_{\partial T}\right|\\
&\quad+\left|\sum_{T\in\mathcal{T}_h}h_T^{-3}\langle Q_bQ_0u-Q_bu, Q_bQ_0\varphi-Q_b\varphi\rangle_{\partial T}\right|\\
&\le Ch^{k-1}\|u\|_{k+1}h^{k_0-1}\|\varphi\|_{k_0+1}\\
&\le Ch^{k+k_0-2}\|u\|_{k+1}\|\varphi\|_4.
\end{aligned}\tag{5.10}$$

Substituting all the six estimates into (5.4) we obtain

$$\begin{aligned}
\|e_0\|^2\le\ &Ch^{k+k_0-2}(\|u\|_{k+1}+h\delta_{k,2}\|u\|_4)\|\varphi\|_4\\
&+Ch^{k_0-1}\|\varphi\|_4|\!|\!|e_h|\!|\!|+C\lambda\|\varphi\|_4\|e_0\|.
\end{aligned}$$

Using the regularity estimate (5.2) and choosing constant λ such that $C\lambda\|\varphi\|_4 < \frac{1}{2}\|e_0\|$, we arrive at

$$\begin{aligned}\|e_0\| &\le Ch^{k_0-1}|\!|\!|e_h|\!|\!| + Ch^{k+k_0-2}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4)\\ &\le Ch^{k+k_0-2}\|u\|_{k+2}.\end{aligned}$$

Together with the H^2 error estimate (4.5) we have the desired L^2 error estimate (5.3). □

In order to study the error estimates on edges, we shall introduce the edge-based L^2 norm here. To keep the consistency of order, the edge-based L^2 norm is different from the standard L^2 norm.

Definition 5.2 For any function v defined on the edges $\mathcal{E}_h$,

$$\|v\|_{\mathcal{E}_h}^2 = \sum_{e\in\mathcal{E}_h} h_e\|v\|_{L^2(e)}^2,$$

where h_e is the measure of edge $e \in \mathcal{E}_h$.

Next, we shall derive the estimates for the second and third components of the error function e_h.

Theorem 5.3 *Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from (3.5) with finite element functions of order $k \ge 2$. Let $k_0 = \min\{k, 3\}$. Assume that the exact solution of (1.1)–(1.3) is sufficiently regular such that $u \in H^{k+2}(\omega)$ and the dual problem (5.1) has the H^4 regularity property. Then, there exists a constant C such that*

$$\|u_b - Q_b u\|_{\mathcal{E}_h} \le Ch^{k+k_0-2}\|u\|_{k+2}, \tag{5.11}$$

$$\|u_n - Q_b(\nabla u_0\cdot\mathbf{n}_e)\|_{\mathcal{E}_h} \le Ch^{k+k_0-3}\|u\|_{k+2}. \tag{5.12}$$

Proof It is obvious that

$$\|e_b\|_{L^2(e)}^2 \le 2(\|Q_b e_0\|_{L^2(e)}^2 + \|Q_b e_0 - e_b\|_{L^2(e)}^2).$$

Summing over all edges, we have

$$\begin{aligned}\|u_b - Q_b u\|_{\mathcal{E}_h}^2 &= \sum_{e\in\mathcal{E}_h} h_e\|u_b - Q_b u\|_{L^2(e)}^2\\ &\le 2\left(\sum_{e\in\mathcal{E}_h} h_e\|Q_b e_0\|_{L^2(e)}^2 + \sum_{e\in\mathcal{E}_h} h_e\|Q_b e_0 - e_b\|_{L^2(e)}^2\right)\\ &\le C\left(\sum_{T\in\mathcal{T}_h} h_T\|Q_b e_0\|_{L^2(\partial T)}^2 + \sum_{T\in\mathcal{T}_h} h_T\|Q_b e_0 - e_b\|_{L^2(\partial T)}^2\right).\end{aligned} \tag{5.13}$$

We shall discuss the two terms separately. For the first part, by applying the trace inequality (7.1), the inverse inequality (7.2) and the error estimate for e_0 in Theorem 5.1, we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} h_T\|Q_b e_0\|_{L^2(\partial T)}^2 &\le \sum_{T\in\mathcal{T}_h} h_T\|e_0\|_{L^2(\partial T)}^2\\ &\le C\sum_{T\in\mathcal{T}_h}\left(\|e_0\|_{L^2(T)}^2 + h_T^2\|\nabla e_0\|_{L^2(T)}^2\right)\end{aligned}$$

$$\leq C \sum_{T\in\mathcal{T}_h} \|e_0\|^2_{L^2(T)}$$
$$\leq Ch^{2k+2k_0-4}\|u\|^2_{k+2}. \tag{5.14}$$

For the second part, we use the trip-bar norm to handle the second part.

$$\sum_{T\in\mathcal{T}_h} h_T\|Q_b e_0 - e_b\|^2_{L^2(\partial T)} \leq h^4 \sum_{T\in\mathcal{T}_h} h_T^{-3}\|Q_b e_0 - e_b\|^2_{L^2(\partial T)} \leq h^4 |||e_h|||^2 \tag{5.15}$$
$$\leq Ch^{2k+2k_0-4}\|u\|^2_{k+2}.$$

Combining the above two estimates gives the desired error estimate (5.11).

Similarly, we establish the error estimates for e_n.

$$\begin{aligned}
\|e_n\|^2_{\mathcal{E}_h} &= \sum_{e\in\mathcal{E}_h} h_e\|e_n\|^2_{L^2(e)} \\
&\leq C\left(\sum_{T\in\mathcal{T}_h} h_T\|\nabla e_0\cdot\mathbf{n}_e\|_{\partial T} + \sum_{T\in\mathcal{T}_h} h_T\|\nabla e_0\cdot\mathbf{n}_e - e_n\|_{\partial T}\right) \\
&\leq C\left(\sum_{T\in\mathcal{T}_h} h_T\|\nabla e_0\|_{\partial T} + h^2\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\nabla e_0\cdot\mathbf{n}_e - e_n\|_{\partial T}\right) \\
&\leq C\left(\sum_{T\in\mathcal{T}_h}\|\nabla e_0\|_T + h^2|||e_h|||\right) \\
&\leq C\left(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|e_0\|_T + h^2|||e_h|||\right) \\
&\leq C\left(h^{2k+2k_0-6} + h^{2k}\right)\|u\|^2_{k+2}.
\end{aligned} \tag{5.16}$$

Thus, we have

$$\|e_n\|_{\mathcal{E}_h} \leq Ch^{k+k_0-3}\|u\|_{k+2},$$

which completes the proof. □

6 Numerical Results

In this section, we would like to report some numerical results for the weak Galerkin finite element method proposed and analyzed in previous sections. Here we use the following finite element space

$$\tilde{V}_h = \{v = \{v_0, v_b, v_n\mathbf{n}_e\}, v_0 \in P_2(T), v_b, v_n \in P_1(e), T\in\mathcal{T}_h, e\subset\mathcal{E}_h\}.$$

For any given $v = \{v_0, v_b, v_n\mathbf{n}_e\} \in \tilde{V}_h$ and $\varphi \in P_0(T)$, we compute the discrete weak Laplacian $\Delta_w v$ on each element T as a function in $P_0(T)$ as follows

$$(\Delta_w v, \varphi)_T = (v_0, \Delta\varphi)_T - \langle v_b, \nabla\varphi\cdot\mathbf{n}\rangle_{\partial T} + \langle v_n\mathbf{n}_e\cdot\mathbf{n}, \varphi\rangle_{\partial T},$$

which could be simplified as

$$(\Delta_w v, \varphi)_T = \langle v_n\mathbf{n}_e\cdot\mathbf{n}, \varphi\rangle_{\partial T}.$$

Table 1 Errors and orders of Example 6.1 in H^2 and L^2 with $k = 2$

h	$\|\|\|u_h - Q_h u\|\|\|$	Order	$\|u_0 - Q_0 u\|$	Order
3.74355e−01	3.69061e−01		4.29897e−02	
1.91955e−01	1.89785e−01	9.59493e−01	1.11418e−02	1.94801
9.56362e−02	1.01110e−01	9.08440e−01	2.97175e−03	1.90660
4.78382e−02	5.57946e−02	8.57728e−01	8.08649e−04	1.87773
2.20971e−02	3.00721e−02	8.91700e−01	2.14457e−04	1.91483
1.10485e−02	1.55286e−02	9.53498e−01	5.49264e−05	1.96512

Table 2 Errors and orders of Example 6.1 in L^2 and L^∞ for e_b with $k = 2$

h	$\|Q_b u - u_b\|_{\mathcal{E}_h}$	Order	$\|Q_b u - u_b\|_\infty$	Order
3.74355e−01	1.21967e−01		1.18101e−01	
1.91955e−01	3.12884e−02	1.91858	3.27686e−02	1.84964
9.56362e−02	8.39049e−03	1.89880	8.84728e−03	1.88901
4.78382e−02	2.28623e−03	1.87578	2.39957e−03	1.88246
2.20971e−02	6.06514e−04	1.91436	6.33868e−04	1.92052
1.10485e−02	1.55351e−04	1.96501	1.62044e−04	1.96780

The error for the weak Galerkin solution is measured in six norms defined as follows:

$$\|\|\|e_h\|\|\|^2 = \sum_{T\in\mathcal{T}_h}\left(\int_T |\Delta_w v_h|^2 dT + h_T^{-1}\int_{\partial T} |(\nabla v_0)\cdot\mathbf{n}_e - v_n|^2 ds + h_T^{-3}\int_{\partial T}(Q_b v_0 - v_b)^2 ds\right) \quad \text{(A discrete } H^2 \text{ norm)}$$

$$\|Q_0 v - v_0\|^2 = \sum_{T\in\mathcal{T}_h}\int_T |Q_0 v - v_0|^2 dT \quad \text{(Element based } L^2 \text{ norm)}$$

$$\|Q_b v - v_b\|_{\mathcal{E}_h}^2 = \sum_{e\in\mathcal{E}_h} h_e \int_e |Q_b v - v_b|^2 ds \quad \text{(Edge based } L^2 \text{ norm for } v_b)$$

$$\|Q_b v - v_n\|_{\mathcal{E}_h}^2 = \sum_{e\in\mathcal{E}_h} h_e \int_e |Q_b v - v_n|^2 ds \quad \text{(Edge based } L^2 \text{ norm for } v_n)$$

$$\|Q_b v - v_b\|_\infty = \max_{e\in\mathcal{E}_h}\{|Q_b v - v_b|\} \quad \text{(Edge based } L^\infty \text{ norm for } v_b)$$

$$\|Q_b v - v_n\|_\infty = \max_{e\in\mathcal{E}_h}\{|Q_b(\nabla u_0\cdot\mathbf{n}_e) - v_n|\} \quad \text{(Edge based } L^\infty \text{ norm for } v_n)$$

Example 6.1 Consider the biharmonic problem (1.1)–(1.3) in the square domain $\Omega = (0, 1)^2$. It has the analytic solution $u(x) = x^2(1-x)^2 y^2(1-y)^2$, and the right hand side function f in (1.1) is computed to match the exact solution. The mesh size is denoted by $h = 1/n$. Table 1 shows that the convergence rates for the WG-FEM solution in the H^2 and L^2 norms are of order $O(h)$ and $O(h^2)$ when $k = 2$, respectively.

Table 2 shows that the errors and orders of Example 6.1 in L^2 and L^∞ for e_b. The numerical results are in consistency with theory for these two cases.

Table 3 shows that the errors and orders of Example 6.1 in L^2 and L^∞ for e_n. The numerical results are in consistency with theory for these two cases.

Table 3 Errors and orders of Example 6.1 in L^2 and L^∞ for e_n with $k = 2$

h	$\|Q_b(\nabla u \cdot \mathbf{n}_e) - u_n\|_{\mathcal{E}_h}$	Order	$\|Q_b(\nabla u \cdot \mathbf{n}_e) - u_n\|_\infty$	Order
3.74355e−01	1.18286e−01		5.28497e−02	
1.91858e+00	3.12884e−02	1.91858	1.51029e−02	1.80707
9.56362e−02	8.39049e−03	1.89880	7.33970e−03	1.04103
4.78382e−02	2.28623e−03	1.87578	3.41617e−03	1.10334
2.20971e−02	6.06514e−04	1.91436	1.18287e−03	1.53009
1.10485e−02	1.55351e−04	1.96501	3.30602e−04	1.83912

Table 4 Errors and orders of example 6.1 in H^2 and L^2 with $k = 3$

h	$\|\|\|u_h - Q_h u\|\|\|$	Order	$\|u_0 - Q_0 u\|$	Order
3.74355e−01	1.17819e−01		4.56114e−03	
1.91955e−01	3.56257e−02	1.72558	4.16403e−04	3.45334
9.56362e−02	1.00915e−02	1.81977	3.55158e−05	3.55145
4.78382e−02	2.56977e−03	1.97343	2.30985e−06	3.94259
2.20971e−02	6.44317e−04	1.99580	1.44990e−07	3.99378
1.10485e−02	1.61222e−04	1.99873	9.07702e−09	3.99759

Table 5 Errors and orders of example 6.1 in L^2 and L^∞ for e_b with $k = 3$

h	$\|Q_b u - u_b\|_{\mathcal{E}_h}$	Order	$\|Q_b u - u_b\|_\infty$	Order
3.74355e−01	8.34847e−03		1.15414e−02	
1.91955e−01	8.06272e−04	3.37217	1.08014e−03	3.41753
9.56362e−02	7.89345e−05	3.35254	9.02080e−05	3.58181
4.78382e−02	5.19889e−06	3.92438	5.93961e−06	3.92481
2.20971e−02	3.26604e−07	3.99259	3.72799e−07	3.99390
1.10485e−02	2.04554e−08	3.99699	2.33003e−08	3.99998

Table 6 Errors and orders of example 6.1 in L^2 and L^∞ for e_n with $k = 3$

h	$\|Q_b(\nabla u \cdot \mathbf{n}_e) - u_n\|_{\mathcal{E}_h}$	Order	$\|Q_b(\nabla u \cdot \mathbf{n}_e) - u_n\|_\infty$	Order
3.74355e−01	5.23031e−02		1.15371e−01	
1.91858e+00	8.83906e−03	2.56493	1.96390e−02	2.55449
9.56362e−02	1.50030e−03	2.55865	3.59916e−03	2.44799
4.78382e−02	1.89000e−04	2.98878	4.60320e−04	2.96695
2.20971e−02	2.33468e−05	3.01709	5.56932e−05	3.04707
1.10485e−02	2.89988e−06	3.00916	6.86324e−06	3.02054

Table 7 Errors and orders of Example 6.2 in H^2 and L^2 with $k = 2$

h	$\|\|\|u_h - Q_h u\|\|\|$	Order	$\|u_0 - Q_0 u\|$	Order
3.74355e−01	3.51847e+01		4.18608e+00	
1.91955e−01	1.79831e+01	9.68306e−01	1.06553e+00	1.97403
9.56362e−02	9.36621e+00	9.41104e−01	2.74735e−01	1.95546
4.78382e−02	4.90899e+00	9.32039e−01	7.07013e−02	1.95823
2.20971e−02	2.51557e+00	9.64541e−01	1.79112e−02	1.98087
1.10485e−02	1.26858e+00	9.87671e−01	4.49750e−03	1.99367

Table 8 Errors and orders of Example 6.2 in L^2 and L^∞ for e_b with $k = 2$

h	$\|Q_b u - u_b\|_{\mathcal{E}_h}$	Order	$\|Q_b u - u_b\|_\infty$	Order
3.74355e−01	1.15398e+01		1.10028e+01	
1.91955e−01	2.99335e+00	1.94679	2.97577e+00	1.88654
9.56362e−02	7.75705e−01	1.94818	7.77671e−01	1.93603
4.78382e−02	1.99884e−01	1.95635	2.00300e−01	1.95700
2.20971e−02	5.06547e−02	1.98039	5.07058e−02	1.98194
1.10485e−02	1.27205e−02	1.99354	1.27268e−02	1.99428

Table 9 Errors and orders of Example 6.2 in L^2 and L^∞ for e_n with $k = 2$

h	$\|Q_b(\nabla u \cdot \mathbf{n}_e) - u_n\|_{\mathcal{E}_h}$	Order	$\|Q_b(\nabla u \cdot \mathbf{n}_e) - u_n\|_\infty$	Order
3.74355e−01	1.15398e+01		4.02986e+00	
1.91955e−01	2.99335e+00	1.94679	1.26437e+00	1.67231
9.56362e−02	7.75705e−01	1.94818	4.40635e−01	1.52076
4.78382e−02	1.99884e−01	1.95635	1.74400e−01	1.33718
2.20971e−02	5.06547e−02	1.98039	5.22660e−02	1.73846
1.10485e−02	1.27205e−02	1.99354	1.37655e−02	1.92482

Table 10 Errors and orders of example 6.2 in H^2 and L^2 with $k = 3$

h	$\|\|\|u_h - Q_h u\|\|\|$	Order	$\|u_0 - Q_0 u\|$	Order
3.74355e−01	9.17084e+00		3.37369e−01	
1.91955e−01	2.46720e+00	1.89418	2.77383e−02	3.60438
9.56362e−02	6.52418e−01	1.91900	2.14578e−03	3.69231
4.78382e−02	1.65736e−01	1.97691	1.36946e−04	3.96982
2.20971e−02	4.16442e−02	1.99270	8.50154e−06	4.00974
1.10485e−02	1.04302e−02	1.99734	5.29568e−07	4.00484

Table 11 Errors and orders of example 6.2 in L^2 and L^∞ for e_b with $k = 3$

h	$\|Q_b u - u_b\|_{\mathcal{E}_h}$	Order	$\|Q_b u - u_b\|_\infty$	Order
3.74355e−01	5.34596e−01		7.40358e−01	
1.91955e−01	4.42882e−02	3.59346	6.53790e−02	3.50132
9.56362e−02	3.79823e−03	3.54353	5.37615e−03	3.60418
4.78382e−02	2.44771e−04	3.95582	3.54304e−04	3.92351
2.20971e−02	1.51601e−05	4.01308	2.24297e−05	3.98151
1.10485e−02	9.43450e−07	4.00619	1.40631e−06	3.99543

Table 12 Errors and orders of example 6.2 in L^2 and L^∞ for e_n with $k = 3$

h	$\|Q_b(\nabla u \cdot \mathbf{n}_e) - u_n\|_{\mathcal{E}_h}$	Order	$\|Q_b(\nabla u \cdot \mathbf{n}_e) - u_n\|_\infty$	Order
3.74355e−01	3.44921e+00		8.19181e+00	
1.91858e+00	5.38035e−01	2.68049	1.42296e+00	2.52529
9.56362e−02	7.93752e−02	2.76094	2.26764e−01	2.64963
4.78382e−02	9.99040e−03	2.99007	3.26867e−02	2.79442
2.20971e−02	1.23394e−03	3.01727	4.33160e−03	2.91573
1.10485e−02	1.52755e−04	3.01398	5.50588e−04	2.97585

In Tables 4, 5 and 6 we investigate the same problem for $k = 3$. Table 4 shows that the convergence rates for the WG-FEM solution in the H^2 and L^2 norms are of order $O(h^2)$ and $O(h^4)$. Tables 5 and 6 show the errors and orders in L^2 and L^∞ for e_b and e_n, which are also consistent with theoretical conclusions.

Example 6.2 Consider the biharmonic problem (1.1)–(1.3) in the square domain $\Omega = (0, 1)^2$. It has the analytic solution $u(x) = \sin(\pi x)\sin(\pi y)$, and the right hand side function f in (1.1) is computed accordingly.

The numerical results are presented in Tables 7, 8, 9, 10, 11 and 12 which confirm the theory developed in previous sections.

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7 Appendix: L^2 Projection and Some Technical Results

In this section, we shall present some technical results for the L^2 projection operators with respect to the finite element space V_h. These results are useful for the error estimates of the WG finite element method.

Lemma 7.1 *([39])* (Trace Inequality) *Let $\mathcal{T}_h$ be a partition of the domain Ω into polygons in 2D or polyhedra in 3D. Assume that the partition $\mathcal{T}_h$ satisfies the assumptions (A1), (A2), and (A3) as specified in [39]. Then, there exists a constant C such that for any $T \in \mathcal{T}_h$ and edge/face $e \in \partial T$, we have*

$$\|\theta\|_e^p \le C h_T^{-1}(\|\theta\|_T^p + h_T^p \|\nabla\theta\|_T^p), \tag{7.1}$$

where $\theta \in H^1(T)$ is any function.

Lemma 7.2 *([39]) (Inverse Inequality) Let $\mathcal{T}_h$ be a partition of the domain Ω into polygons or polyhedra. Assume that $\mathcal{T}_h$ satisfies all the assumptions (A1)–(A4) as specified in [39]. Then, there exists a constant $C(n)$ such that*

$$\|\nabla\varphi\|_T \le C(n) h_T^{-1}\|\varphi\|_T, \quad \forall T \in \mathcal{T}_h \tag{7.2}$$

for any piecewise polynomial φ of degree n on $\mathcal{T}_h$.

7.1 Approximation Properties

The following lemma provides some approximation properties for the projection operators Q_h and $\mathbb{Q}_h$.

Lemma 7.3 *([32]) Let $\mathcal{T}_h$ be a finite element partition of Ω satisfying the shape regularity assumptions. Then, for any $0 \le s \le 2$ and $2 \le m \le k$ we have*

$$\textstyle\sum_{T\in\mathcal{T}_h} h_T^{2s}\|u - Q_0 u\|_{s,T}^2 \le C h^{2(m+1)}\|u\|_{m+1}^2, \tag{7.3}$$

$$\textstyle\sum_{T\in\mathcal{T}_h} h_T^{2s}\|\Delta u - \mathbb{Q}_h \Delta u\|_{s,T}^2 \le C h^{2(m-1)}\|u\|_{m+1}^2. \tag{7.4}$$

Lemma 7.4 *Let $2 \le m \le k, \omega \in H^{m+2}(\Omega)$. There exists a constant C such that the following estimates hold true:*

$$\left(\sum_{T\in\mathcal{T}_h} h_T\|\Delta\omega - \mathbb{Q}_h\Delta\omega\|_{\partial T}^2\right)^{\frac{1}{2}} \le C h^{m-1}\|\omega\|_{m+1}, \tag{7.5}$$

$$\left(\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla(\Delta\omega - \mathbb{Q}_h\Delta\omega)\|_{\partial T}^2\right)^{\frac{1}{2}} \le C h^{m-1}(\|\omega\|_{m+1} + h\delta_{m,2}\|\omega\|_4), \tag{7.6}$$

$$\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\nabla(Q_0\omega)\cdot \boldsymbol{n}_e - Q_b(\nabla\omega\cdot\boldsymbol{n}_e)\|_{\partial T}^2\right)^{\frac{1}{2}} \le C h^{m-1}\|\omega\|_{m+1}, \tag{7.7}$$

$$\left(\sum_{T\in\mathcal{T}_h} h_T^{-3}\|Q_b Q_0\omega - Q_b\omega\|_{\partial T}^2\right)^{\frac{1}{2}} \le C h^{m-1}\|\omega\|_{m+1}, \tag{7.8}$$

$$\left(\sum_{T\in\mathcal{T}_h} \|\nabla(\Delta\omega) - Q_b(\nabla(\Delta\omega))\|_{\partial T}^2\right)^{\frac{1}{2}} \le C h^{m-\frac{3}{2}}\|\omega\|_{m+2}. \tag{7.9}$$

Here $\delta_{i,j}$ is the usual Kronecker's delta with value 1 when $i = j$ and value 0 otherwise.

Proof To derive (7.5), we use the trace inequality (7.1) and the estimate (7.4) to obtain

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h} h_T\|\Delta\omega-\mathbb{Q}_h\Delta\omega\|_{\partial T}^2\\&\quad\le C\sum_{T\in\mathcal{T}_h}\left(\|\Delta\omega-\mathbb{Q}_h\Delta\omega\|_T^2+h_T^2\|\nabla\left(\Delta\omega-\mathbb{Q}_h\Delta\omega\right)\|_T^2\right)\\&\quad\le Ch^{2m-2}\|\omega\|_{m+1}^2.\end{aligned}$$

As to (7.6), we use the trace inequality (7.1) and the estimate (7.4) to obtain

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h} h_T^3\|\nabla\left(\Delta\omega-\mathbb{Q}_h\Delta\omega\right)\|_{\partial T}^2\\&\quad\le C\sum_{T\in\mathcal{T}_h}\left(h_T^2\|\nabla(\Delta\omega-\mathbb{Q}_h\Delta\omega)\|_T^2+h_T^4\|\nabla^2(\Delta\omega-\mathbb{Q}_h\Delta\omega)\|_T^2\right)\\&\quad\le Ch^{2m-2}\left(\|\omega\|_{m+1}^2+h^2\delta_{m,2}\|\omega\|_4^2\right).\end{aligned}$$

As to (7.7), we have from the definition of Q_b, the trace inequality (7.1), and the estimate (7.3) that

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\nabla(Q_0\omega)\cdot\mathbf{n}_e-Q_b(\nabla\omega\cdot\mathbf{n}_e)\|_{\partial T}^2\\&\quad\le\sum_{T\in\mathcal{T}_h} h_T^{-1}\|(\nabla Q_0\omega-\nabla\omega)\cdot\mathbf{n}_e\|_{\partial T}^2\\&\quad\le C\sum_{T\in\mathcal{T}_h}\left(h_T^{-2}\|\nabla Q_0\omega-\nabla\omega\|_T^2+\|\nabla Q_0\omega-\nabla\omega\|_{1,T}^2\right)\\&\quad\le Ch^{2m-2}\|\omega\|_{m+1}^2.\end{aligned}$$

Notice that Q_b is a linear bounded operator, we use the definition of Q_b and the trace inequality (7.1) to obtain

$$\begin{aligned}&\sum_{T\in T_h} h_T^{-3}\|Q_bQ_0\omega-Q_b\omega\|_{\partial T}^2\\&\quad\le\sum_{T\in T_h}\left(h_T^{-4}\|Q_0\omega-\omega\|_T^2+h_T^{-2}\|\nabla(Q_0\omega-\omega)\|_T^2\right)\\&\quad\le Ch^{2m-2}\|\omega\|_{m+1}^2.\end{aligned}$$

To derive (7.9), we use the trace inequality (7.1) and the estimate (7.4) to obtain

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h}\|\nabla(\Delta\omega)-Q_b(\nabla(\Delta\omega))\|_{\partial T}^2\\&\quad\le C\sum_{T\in\mathcal{T}_h}\left(h_T^{-1}\|\nabla(\Delta\omega)-Q_b(\nabla(\Delta\omega))\|_T^2+h_T\|\nabla(\nabla(\Delta\omega)-Q_b(\nabla(\Delta\omega)))\|_T^2\right)\\&\quad\le Ch^{2m-3}\|\omega\|_{m+2}^2.\end{aligned}$$

This completes the proof of (7.9), and hence the lemma. □

7.2 Technical Inequalities

The goal here is to present some technical estimates useful for deriving error estimates for the WG finite element scheme (3.5).

Lemma 7.5 *There exists a constant C such that, for any $v = \{v_0, v_b, v_n\boldsymbol{n}_e\} \in V_h$, the following holds true*

$$\sum_{T\in\mathcal{T}_h} \|\Delta v_0\|_T^2 \le C|\!|\!|v|\!|\!|^2. \tag{7.10}$$

Proof From the identity (2.4) with $\phi = \Delta v_0$ we have

$$\|\Delta v_0\|_T^2 = (\Delta_w v, \Delta v_0)_T - \langle Q_b v_0 - v_b, \nabla(\Delta v_0)\cdot\mathbf{n}\rangle_{\partial T} + \langle(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}, \Delta v_0\rangle_{\partial T}.$$

Thus, using the Cauchy–Schwarz inequality, trace inequality, and the inverse inequality we obtain

$$\begin{aligned}
\|\Delta v_0\|_T^2 &\le \|\Delta_w v\|_T\|\Delta v_0\|_T + \|Q_b v_0 - v_b\|_{\partial T}\|\nabla(\Delta v_0)\cdot\mathbf{n}\|_{\partial T}\\
&\quad + \|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_{\partial T}\|\Delta v_0\|_{\partial T}\\
&\le C(\|\Delta_w v\|_T\|\Delta v_0\|_T + h_T^{-\frac{1}{2}}\|Q_b v_0 - v_b\|_{\partial T}\|\nabla(\Delta v_0)\cdot\mathbf{n}\|_T\\
&\quad + h_T^{-\frac{1}{2}}\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_{\partial T}\|\Delta v_0\|_T)\\
&\le C(\|\Delta_w v\|_T\|\Delta v_0\|_T + h_T^{-\frac{3}{2}}\|Q_b v_0 - v_b\|_{\partial T}\|\Delta v_0\|_T\\
&\quad + h_T^{-\frac{1}{2}}\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_{\partial T}\|\Delta v_0\|_T).
\end{aligned}$$

Hence,

$$\|\Delta v_0\|_T^2 \le C(\|\Delta_w v\|_T^2 + h_T^{-3}\|Q_b v_0 - v_b\|_{\partial T}^2 + h_T^{-1}\|(\nabla v_0 - v_n\mathbf{n}_e)\cdot\mathbf{n}\|_{\partial T}^2),$$

which verifies the inequality (7.10). □

Lemma 7.6 *([37], Lemma 10.4) There exists a constant C such that, for any $v \in V_h^0$, we have the following Poincaré inequality:*

$$\|v_0\|^2 \le C\left(\sum_{T\in\mathcal{T}_h}\|\nabla v_0\|_T^2 + h^{-1}\sum_{T\in\mathcal{T}_h}\|Q_b v_0 - v_b\|_{\partial T}^2\right). \tag{7.11}$$

The following lemma provides an estimate for the term $\sum_{T\in\mathcal{T}_h}\|\nabla v_0\|_T^2$. Note that v_0 is a piecewise polynomial of degree $k \ge 2$. Thus, Lemma 7.7 is concerned only with piecewise polynomials; no boundary condition is necessary.

Lemma 7.7 *Let φ be any piecewise polynomial of degree $k \ge 2$ on each element T. Denote by $\nabla_h\varphi$ and $\Delta_h\varphi$ the gradient and Laplacian of φ taken on each element. Then, for any $\varepsilon > 0$, there exists a constant C such that*

$$\begin{aligned}
\|\nabla_h\varphi\|^2 &\le \varepsilon\|\varphi\|^2 + C\varepsilon^{-1}\|\Delta_h\varphi\|^2\\
&\quad + C\varepsilon^{-1}h^{-1}\left(\sum_{e\in\mathcal{E}_h}\int_e\left(\frac{\partial\varphi_L}{\partial\boldsymbol{n}_L} + \frac{\partial\varphi_R}{\partial\boldsymbol{n}_R}\right)^2 ds\right)\\
&\quad + Ch^{-1}\left(\sum_{e\in\mathcal{E}_h}\int_e(Q_b\varphi_R - Q_b\varphi_L)^2 ds\right).
\end{aligned}\tag{7.12}$$

Here φ_L is the trace of φ on e as seen from the "left" or the opposite direction of $\mathbf{n}_e$. If e is a boundary edge, then the trace from the outside of Ω is defined as zero.

Proof On each element T, we have

$$\int_T |\nabla\varphi|^2 dT = -\int_T \varphi\Delta\varphi\, dT + \int_{\partial T} \frac{\partial\varphi}{\partial\mathbf{n}}\,\varphi\, ds$$
$$= -\int_T \varphi\Delta\varphi dT + \int_{\partial T} \frac{\partial\varphi}{\partial\mathbf{n}}\, Q_b\varphi\, ds.$$

Summing over all $T \in \mathcal{T}_h$, we have

$$\|\nabla_h\varphi\|^2 = -\int_\Omega \varphi\Delta_h\varphi dT + \sum_{T\in\mathcal{T}_h}\int_{\partial T} \frac{\partial\varphi}{\partial\mathbf{n}}\, Q_b\varphi\, ds. \tag{7.13}$$

Using the identity $a_L b_L + a_R b_R = (a_L + a_R)b_L + a_R(b_R - b_L)$ we obtain

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T} \frac{\partial\varphi}{\partial\mathbf{n}}\, Q_b\varphi\, ds = \sum_{e\in\mathcal{E}_h}\int_e \left(\frac{\partial\varphi_L}{\partial\mathbf{n}_L}\, Q_b\varphi_L + \frac{\partial\varphi_R}{\partial\mathbf{n}_R}\, Q_b\varphi_R\right) ds$$
$$= \sum_{e\in\mathcal{E}_h}\int_e \left(\frac{\partial\varphi_L}{\partial\mathbf{n}_L} + \frac{\partial\varphi_R}{\partial\mathbf{n}_R}\right) Q_b\varphi_L ds$$
$$+ \sum_{e\in\mathcal{E}_h}\int_e \frac{\partial\varphi_R}{\partial\mathbf{n}_R}\,(Q_b\varphi_R - Q_b\varphi_L) ds.$$

Thus, from the Cauchy–Schwarz inequality we have

$$\left|\sum_{T\in\mathcal{T}_h}\int_{\partial T} \frac{\partial\varphi}{\partial\mathbf{n}}\, Q_b\varphi\, ds\right| \le \left(\sum_{e\in\mathcal{E}_h}\int_e \left(\frac{\partial\varphi_L}{\partial\mathbf{n}_L} + \frac{\partial\varphi_R}{\partial\mathbf{n}_R}\right)^2 ds\right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}_h}\int_e |Q_b\varphi_L|^2\, ds\right)^{\frac{1}{2}}$$
$$+ \left(\sum_{e\in\mathcal{E}_h}\int_e \left|\frac{\partial\varphi_R}{\partial\mathbf{n}_R}\right|^2 ds\right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}_h}\int_e (Q_b\varphi_R - Q_b\varphi_L)^2 ds\right)^{\frac{1}{2}}. \tag{7.14}$$

Next, we use the trace inequality (7.1) and the inverse inequality (7.2) to obtain

$$\int_e |Q_b\varphi_L|^2\, ds \le \int_e |\varphi_L|^2\, ds$$
$$\le C\left[h^{-1}\int_T \varphi^2 dT + h\int_T |\nabla\varphi|^2 dT\right]$$
$$\le Ch^{-1}\int_T \varphi^2 dT, \tag{7.15}$$

and

$$\int_e \left|\frac{\partial\varphi_R}{\partial\mathbf{n}_R}\right|^2 ds \le C\left[h^{-1}\int_T |\nabla\varphi|^2 dT + h\int_T |\nabla^2\varphi|^2 dT\right]$$
$$\le Ch^{-1}\int_T |\nabla\varphi|^2 dT. \tag{7.16}$$

Substituting (7.15) and (7.16) into (7.14) yields

$$\left|\sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial\varphi}{\partial\mathbf{n}}\,Q_b\varphi\,ds\right| \le Ch^{-\frac{1}{2}}\|\varphi\|\left(\sum_{e\in\mathcal{E}_h}\int_e\left(\frac{\partial\varphi_L}{\partial\mathbf{n}_L}+\frac{\partial\varphi_R}{\partial\mathbf{n}_R}\right)^2 ds\right)^{\frac{1}{2}} + Ch^{-\frac{1}{2}}\|\nabla_h\varphi\|\left(\sum_{e\in\mathcal{E}_h}\int_e(Q_b\varphi_R-Q_b\varphi_L)^2 ds\right)^{\frac{1}{2}}. \quad (7.17)$$

Substituting (7.17) into (7.13) gives

$$\|\nabla_h\varphi\|^2 \le \|\Delta_h\varphi\|\,\|\varphi\| + Ch^{-\frac{1}{2}}\|\varphi\|\left(\sum_{e\in\mathcal{E}_h}\int_e\left(\frac{\partial\varphi_L}{\partial\mathbf{n}_L}+\frac{\partial\varphi_R}{\partial\mathbf{n}_R}\right)^2 ds\right)^{\frac{1}{2}} + Ch^{-\frac{1}{2}}\|\nabla_h\varphi\|\left(\sum_{e\in\mathcal{E}_h}\int_e(Q_b\varphi_R-Q_b\varphi_L)^2 ds\right)^{\frac{1}{2}},$$

which, through an use of Young's inequality, implies the desired estimate (7.12). This completes the proof. □

Lemma 7.8 *There exists a constant C such that for any $v=\{v_0,v_b,v_n\mathbf{n}_e\}\in V_h^0$ the following Poincaré type inequality holds true*

$$\|\nabla_h v_0\| \le C|||v|||. \quad (7.18)$$

In addition, we have the following estimate

$$\|\nabla_h v_0\| \le \lambda h^{-1}\|v\| + Ch|||v|||, \quad (7.19)$$

where λ is a positive constant.

Proof The first component v_0 is a piecewise polynomial of degree $k\ge 2$. Using the estimate (7.12) in Lemma 7.7 we have

$$\|\nabla_h v_0\|^2 \le \varepsilon\|v\|^2 + C\varepsilon^{-1}\|\Delta_h v_0\|^2 + C\varepsilon^{-1}h^{-1}\left(\sum_{e\in\mathcal{E}_h}\int_e\left(\frac{\partial v_{0L}}{\partial\mathbf{n}_L}+\frac{\partial v_{0R}}{\partial\mathbf{n}_R}\right)^2 ds\right) + Ch^{-1}\left(\sum_{e\in\mathcal{E}_h}\int_e(Q_b v_{0R}-Q_b v_{0L})^2 ds\right). \quad (7.20)$$

By inserting $v_n\mathbf{n}_e\cdot\mathbf{n}$ in each integrand we obtain

$$\sum_{e\in\mathcal{E}_h}\int_e\left(\frac{\partial v_{0L}}{\partial\mathbf{n}_L}+\frac{\partial v_{0R}}{\partial\mathbf{n}_R}\right)^2 ds \le C\sum_{T\in\mathcal{T}_h}\|\nabla v_0\cdot\mathbf{n}_e - v_n\|_{\partial T}^2.$$

Similarly, by inserting v_b

$$\sum_{e\in\mathcal{E}_h}\int_e(Q_b v_{0R}-Q_b v_{0L})^2 ds \le C\sum_{T\in\mathcal{T}_h}\|Q_b v_0 - v_b\|_{\partial T}^2.$$

Substituting the above two inequalities into (7.20) yields

$$\|\nabla_h v_0\|^2 \le \varepsilon\|v\|^2 + C\varepsilon^{-1}\|\Delta_h v_0\|^2 + Ch^{-1}\sum_{T\in\mathcal{T}_h}\|Q_b v_0 - v_b\|_{\partial T}^2 + C\varepsilon^{-1}h^{-1}\sum_{T\in\mathcal{T}_h}\|\nabla v_0\cdot\mathbf{n}_e - v_n\|_{\partial T}^2. \quad (7.21)$$

Using the Poincaré inequality (7.11) and the estimate (7.10) we arrive at

$$\|\nabla_h v_0\|^2 \le \varepsilon C\|\nabla_h v\|^2 + C\varepsilon^{-1}|||v|||^2,$$

which leads to the inequality (7.18) for sufficiently small ε.

Finally, by setting $\varepsilon = \lambda h^{-2}$ in (7.21) we arrive at

$$\|\nabla_h v_0\|^2 \le \lambda h^{-2}\|v\|^2 + Ch^2|||v|||^2,$$

where λ is a positive constant. This verifies the inequality (7.19), and hence completes the proof of the lemma. □

Lemma 7.9 *There exists a constant C such that for any $v = \{v_0, v_b, v_n\mathbf{n}_e\} \in V_h^0$ one has*

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}(v_0 - Q_b v_0)^2 ds \le Ch|||v|||^2 \quad (7.22)$$

and

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}(v_0 - Q_b v_0)^2 ds \le C\lambda h^{-1}\|v\|^2 + Ch^3|||v|||^2. \quad (7.23)$$

Proof From the trace inequality (7.1) and the inverse inequality (7.2), we have

$$\int_{\partial T}(v_0 - Q_b v_0)^2 ds \le Ch\int_T|\nabla v_0|^2 dT.$$

Summing over all $T \in \mathcal{T}_h$ yields

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}(v_0 - Q_b v_0)^2 ds \le Ch\sum_{T\in\mathcal{T}_h}\int_T|\nabla v_0|^2 dT, \quad (7.24)$$

which, combined with (7.18) and (7.19), completes the proof of the lemma. □

Remark 7.1 The estimate (7.22) in Lemma 7.9 is sufficient for us to derive an optimal order error estimate for the WG finite element solution arising from (3.5). But the estimate (7.22) is sub-optimal in terms of the mesh parameter h. We conjecture that the following inequality holds true

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}(v_0 - Q_b v_0)^2 ds \le Ch^3\,|||v|||^2. \quad (7.25)$$

However, with the current mathematical approach, we are unable to verify the validity of (7.25). This estimate is then left to interested readers or researchers as an open problem.

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