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Spectral Element Method for Mixed Inhomogeneous Boundary Value Problems of Fourth Order

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Abstract In this paper, we investigate spectral element method for fourth order problems with mixed inhomogeneous boundary conditions. Some results on the composite Legendre quasi-orthogonal approximation are established, which play important roles in spectral element method with non-uniform meshes and non-uniform approximation modes. As an example of applications, the spectral element scheme is provided for a model problem, with the convergence analysis. Numerical results demonstrate its spectral accuracy, and coincide with the analysis well. In particular, the suggested method is convenient for local mesh refinement and local mode increment, and so it works well even for the solutions changing rapidly, oscillating seriously, or behaving differently in different subdomains.

Keywords Spectral element method · Fourth order problems · Mixed inhomogeneous boundary conditions.

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1 Introduction

Spectral method possesses the high accuracy, and so it has been applied successfully to numerical simulations in science and engineering, see [2,6,12–15,26] and the references therein. Theoretically, the larger the modes used in spectral approximations, the smaller the numerical errors. However, it is not convenient to use very large modes in actual computations. Thus, it is reasonable to study spectral element method. Some authors developed pseudospectral element method, which is also called spectral element method oftentimes, see [7,24] and the references therein.

We considered second order problems mostly. But it is also important to deal with high order problems, see, e.g., [1,4,5,11,21,22]. Recently, Guo, Shen and Wang [17] proposed the generalized Jacobi orthogonal approximation, which provides a useful tool for spectral method of high order problems. Meanwhile, Guo, Sun and Zhang [18] developed the generalized Jacobi quasi-orthogonal approximation, which is applicable to spectral element method for high order problems in one-dimension. Guo and Jia [16] considered the Legendre quasi-orthogonal approximation in two dimensions with its applications to spectral element method for second order problems. Whereas, so far, there is few results on the spectral element method for mixed inhomogeneous boundary value problems of high order.

The aim of this paper is to develop spectral element method for fourth order problems with mixed inhomogeneous boundary conditions. Unlike many numerical approaches using certain interpolations, the spectral element method is based on certain orthogonal approximations. As we know, the orthogonal approximations possess the high accuracy, but do not match the boundary values of approximated functions usually. Thus, the main difficulties of designing spectral element schemes for high order problems are how to match numerical solutions and their derivatives on all interfaces of adjacent elements, and how to ensure their global spectral accuracy. On the other hand, the behaviors of solutions of considered problems might be very different on different parts of domains. For raising numerical accuracy and saving computational time, it seems better to adopt non-uniform meshes and non-uniform approximation modes. It also brings certain additional difficulties. In order to remedy the above troubles, we first introduce several new Legendre quasi-orthogonal approximations defined on a square, corresponding to various kinds of boundary conditions. Then, we propose the composite Legendre quasi-orthogonal approximation on the whole complex domain with rectangle partition, and derive its error estimate precisely, which serves as the mathematical foundation of spectral element method for mixed inhomogeneous boundary value problems of fourth order. Next, we design five kinds of proper base functions corresponding to the local approximations in the different elements, on the different edges of adjacent elements and the different vertices respectively, so that the numerical solutions and their derivatives keep the continuity. The above approximation and base functions are specially appropriate for spectral element method with non-uniform meshes and non-uniform modes. Moreover, since it approximates the solutions in the interiors, at the edges and the vertices of various elements independently, with different modes under a very weak restriction, the corresponding spectral method is very convenient for local mesh refinement and local mode increment. As an example of applications, we provide the spectral element scheme for a model problem of fourth order with mixed inhomogeneous boundary conditions, and prove its global spectral accuracy. The numerical results demonstrate its high effectiveness and coincide with the analysis well. In particular, such method works well even for the solutions changing rapidly, oscillating very seriously, or behaving differently in different parts of domain.

The next section is for preliminaries. In Sect. 3, we establish the basic result on the composite Legendre quasi-orthogonal approximation. In Sect. 4, we provide the spectral element scheme for a model problem with the convergence analysis. In Sect. 5, we present some numerical results. The final section is for concluding remarks.

2 Preliminaries

Let the interval $I = \{\, x \mid a < x < b\}$ and $h = b - a$. The shifted Jacobi weight function

$$\chi^{(\alpha,\beta)}(x) = (\frac{2}{h})^{\alpha+\beta}(b-x)^{\alpha}(x-a)^{\beta}.$$

For any integer $r \geq 0$, we define the weighted Sobolev spaces $H^r_{\chi^{(\alpha,\beta)}}(I)$ and its norm $\|v\|_{r,\chi^{(\alpha,\beta)},I}$ as usual. In particular, $H^0_{\chi^{(\alpha,\beta)}}(I) = L^2_{\chi^{(\alpha,\beta)}}(I)$ with the inner product $(u,v)_{\chi^{(\alpha,\beta)},I}$ and the norm $\|v\|_{\chi^{(\alpha,\beta)},I}$. We omit the subscript $\chi^{(\alpha,\beta)}$ in notations whenever $\alpha = \beta = 0$. The shifted Legendre polynomials $L_l(\frac{1}{h}(2x-b-a))$ form a complete $L^2(I)-$orthogonal system.

Let N be any positive integer. $\mathcal{P}_N(I)$ stands for the set of all algebraic polynomials of degree at most N. Throughout this paper, we denote by c a generic positive constant independent of any function and any mode N.

The shifted Legendre orthogonal projection $P_{N,I} : L^2(I) \to \mathcal{P}_N(I)$ is defined by

$$(P_{N,I}v - v, \phi)_I = 0, \qquad \forall \phi \in \mathcal{P}_N(I).$$

According to the basic result on the standard Legendre orthogonal approximation (see page 387 of [14], as well as [19]) coupled with a affine variable transformation, we know that if $v \in L^2(I),\ \partial_x^r v \in L^2_{\chi^{(r,r)}}(I)$, integers $r \geq 0$ and $r \leq N+1$, then

$$\|P_{N,I}v - v\|_I \leq c(\frac{h}{N})^r \|\partial_x^r v\|_{\chi^{(r,r)},I}. \tag{2.1}$$

Let

$$H^1_0(I) = \{v \in H^1(I) \mid v(a) = v(b) = 0\}, \qquad \mathcal{P}^{1,0}_N(I) = \mathcal{P}_N(I) \cap H^1_0(I).$$

The orthogonal projection $P^{1,0}_{N,I} : H^1_0(I) \to \mathcal{P}^{1,0}_N(I)$ is defined by

$$(\partial_x(P^{1,0}_{N,I}v - v), \partial_x\phi)_I = 0, \qquad \forall \phi \in \mathcal{P}^{1,0}_N(I).$$

By virtue of Theorem 3.4 of [19], we have that if $v \in H^1_0(I),\ \partial_x^r v \in L^2_{\chi^{(r-1,r-1)}}(I)$, integers $1 \leq r \leq N+1$ and $N \geq 1$, then

$$\|\partial_x^k(P^{1,0}_{N,I}v - v)\|_I \leq c(\frac{h}{N})^{r-k}\|\partial_x^r v\|_{\chi^{(r-1,r-1)},I}, \qquad k = 0, 1.$$

Next, we set

$$v_0(x) = v(x) - \frac{1}{b-a}(v(b)(x-a) + v(a)(b-x)),$$

and define the quasi-orthogonal projection

$${}_*P^1_{N,I}v(x) = P^{1,0}_{N,I}v_0(x) + \frac{1}{b-a}(v(b)(x-a) + v(a)(b-x)). \tag{2.2}$$

According to Lemma 2.4 of [20], we assert that if $v \in H^1(I)$, $\partial_x^r v \in L^2_{\chi^{(r-1,r-1)}}(I)$, integers $1 \le r \le N+1$ and $N \ge 1$, then

$$\|\partial_x^k({}_*P^1_{N,I}v - v)\|_I \le c(\frac{h}{N})^{r-k}\|\partial_x^r v\|_{\chi^{(r-1,r-1)},I}, \qquad k = 0, 1. \tag{2.3}$$

Furthermore, let

$$H_0^2(I) = \{v \in H^2(I) \mid v(a) = v(b) = \partial_x v(a) = \partial_x v(b) = 0\}, \quad \mathcal{P}_N^{2,0}(I) = \mathcal{P}_N(I) \cap H_0^2(I).$$

The orthogonal projection $P^{2,0}_{N,I} : H_0^2(I) \to \mathcal{P}_N^{2,0}(I)$ is defined by

$$(\partial_x^2(P^{2,0}_{N,I}v - v), \partial_x^2\phi)_I = 0, \qquad \forall \phi \in \mathcal{P}_N^{2,0}(I).$$

By virtue of Theorem 2.5 of [22], we conclude that if $v \in H_0^2(I)$, $\partial_x^r v \in L^2_{\chi^{(r-2,r-2)}}(I)$, integers $2 \le r \le N+1$ and $N \ge 2$, then

$$\|\partial_x^k(P^{2,0}_{N,I}v - v)\|_I \le c(\frac{h}{N})^{r-k}\|\partial_x^r v\|_{\chi^{(r-2,r-2)},I}, \qquad k = 0, 1, 2. \tag{2.4}$$

We now turn to the Legendre quasi-orthogonal projection in the space $H^2(I)$. For this purpose, let $z(x) = \frac{1}{h}(2x - b - a)$, and

$$\begin{aligned} f^-(x) &= \tfrac{1}{4}(z^3(x) - 3z(x) + 2), & f^+(x) &= \tfrac{1}{4}(-z^3(x) + 3z(x) + 2), \\ g^-(x) &= \tfrac{h}{8}(z^3(x) - z^2(x) - z(x) + 1), & g^+(x) &= \tfrac{h}{8}(z^3(x) + z^2(x) - z(x) - 1). \end{aligned} \tag{2.5}$$

It can be checked that

$$\begin{aligned} &f^-(b) = \partial_x f^-(a) = \partial_x f^-(b) = 0, && f^-(a) = 1, \\ &f^+(a) = \partial_x f^+(a) = \partial_x f^+(b) = 0, && f^+(b) = 1, \\ &g^-(a) = g^-(b) = \partial_x g^-(b) = 0, && \partial_x g^-(a) = 1, \\ &g^+(a) = g^+(b) = \partial_x g^+(a) = 0, && \partial_x g^+(b) = 1. \end{aligned} \tag{2.6}$$

We introduce the auxiliary function

$$v_{B,I}(x) = v(a)f^-(x) + v(b)f^+(x) + \partial_x v(a)g^-(x) + \partial_x v(b)g^+(x). \tag{2.7}$$

Furthermore, we set

$$v_0(x) = v(x) - v_{B,I}(x) \in H_0^2(I). \tag{2.8}$$

Then, we define the Legendre quasi-orthogonal projection ${}_*P^2_{N,I}v$ as

$${}_*P^2_{N,I}v(x) = P^{2,0}_{N,I}v_0(x) + v_{B,I}(x). \tag{2.9}$$

Obviously,

$$\partial_x^k {}_*P^2_{N,I}v(x) = \partial_x^k v(x), \qquad \text{for } x = a, b \text{ and } k = 0, 1. \tag{2.10}$$

Moveover, with the aid of Lemma 2.4 of [21], we know that if $v \in H^2(I)$, $\partial_x^r v \in L^2_{\chi^{(r-2,r-2)}}(I)$, integers $2 \le r \le N+1$ and $N \ge 2$, then

$$\|\partial_x^k({}_*P^2_{N,I}v - v)\|_I \le c(\frac{h}{N})^{r-k}\|\partial_x^r v\|_{\chi^{(r-2,r-2)},I}, \qquad k = 0, 1, 2. \tag{2.11}$$

In the numerical analysis of spectral method for mixed inhomogeneous boundary value problems of fourth order, we need other projections. For instance, let

$$\overline{H}^2(I) = \{v \in H^2(I) \mid v(a) = v(b) = \partial_x v(b) = 0\}, \qquad \overline{\mathcal{P}}_N(I) = \mathcal{P}_N(I) \cap \overline{H}^2(I). \tag{2.12}$$

We introduce the orthogonal projection $\overline{P}^2_{N,I} : \overline{H}^2(I) \to \overline{\mathcal{P}}_N(I)$, defined by

$$(\partial_x^2(\overline{P}^2_{N,I}v - v), \partial_x^2\phi)_I = 0, \qquad \forall \phi \in \overline{\mathcal{P}}_N(I).$$

Proposition 2.1 *If* $v \in \overline{H}^2(I)$, $\partial_x^r v \in L^2_{\chi^{(r-2,r-2)}}(I)$, *integers* $2 \le r \le N+1$ *and* $N \ge 2$, *then*

$$\|\partial_x^k(\overline{P}^2_{N,I}v - v)\|_I \le c(\frac{h}{N})^{r-k}\|\partial_x^r v\|_{\chi^{(r-2,r-2)},I}, \qquad k = 0, 1, 2. \tag{2.13}$$

Proof Let

$$\phi^*(x) = \int_x^b (\eta - x)P_{N-2,I}\partial_\eta^2 v(\eta)d\eta, \qquad \phi(x) = \phi^*(x) - \frac{1}{h^2}\phi^*(a)(b-x)^2.$$

A direct calculation shows $\phi(a) = \phi(b) = \partial_x\phi(b) = 0$. Thus $\phi \in \overline{P}_N(I)$ for $N \ge 2$. Consequently, we use projection theorem to deduce that

$$\|\partial_x^2(\overline{P}^2_{N,I}v - v)\|_I \le \|\partial_x^2(\phi - v)\|_I \le \|P_{N-2,I}\partial_x^2 v - \partial_x^2 v\|_I + \frac{2}{h^{\frac{3}{2}}}|\phi^*(a)|. \tag{2.14}$$

Due to $v \in \overline{H}^2(I)$, we derive that

$$|\phi^*(a)| = |\int_a^b (\eta - a)(P_{N-2,I}\partial_\eta^2 v(\eta) - \partial_\eta^2 v(\eta))d\eta| \le \frac{\sqrt{3}}{3}h^{\frac{3}{2}}\|P_{N-2,I}\partial_x^2 v - \partial_x^2 v\|_I.$$

Substituting the above inequality into (2.14), we use (2.1) to reach the result (2.13) with $k = 2$.

We next prove the result (2.13) with $k = 0$. Let $g \in L^2(I)$ and consider an auxiliary problem. It is to find $w \in \overline{H}^2(I)$ such that

$$(\partial_x^2 w, \partial_x^2 z)_I = (g, z)_I, \qquad \forall z \in \overline{H}^2(I). \tag{2.15}$$

Taking $z = w$ in the above equality and using the Poincaré inequality, we obtain $\|\partial_x^2 w\|_I \le c\|g\|_I$. Moreover, due to the property of elliptic equation, we have $\partial_x^4 w(x) = g(x)$ in the sense of distributions, and so $\|\partial_x^4 w\|_I \le c\|g\|_I$. By taking $z = \overline{P}^2_{N,I}v - v$ in (2.15) and using (2.13) with $k = 2$, we verify that

$$\begin{aligned}
|(g, \overline{P}^2_{N,I}v - v)_I| &= |(\partial_x^2 w, \partial_x^2(\overline{P}^2_{N,I}v - v))_I| \\
&= |(\partial_x^2(\overline{P}^2_{N,I}w - w), \partial_x^2(\overline{P}^2_{N,I}v - v))_I| \\
&\le \|(\partial_x^2(\overline{P}^2_{N,I}w - w)\|_I\|\partial_x^2(\overline{P}^2_{N,I}v - v)\|_I \\
&\le c(\tfrac{h}{N})^r\|\partial_x^4 w\|_{\chi^{(2,2)},I}\|\partial_x^r v\|_{\chi^{(r-2,r-2)},I} \\
&\le c(\tfrac{h}{N})^r\|g\|_I\|\partial_x^r v\|_{\chi^{(r-2,r-2)},I}.
\end{aligned}$$

Finally, we obtain from the above inequality that

$$\|\overline{P}^2_{N,I}v - v\|_I = \sup_{g\in L^2(I), g\neq 0}\frac{|(g, \overline{P}^2_{N,I}v - v)_I|}{\|g\|_I} \le c(\frac{h}{N})^r\|\partial_x^r v\|_{\chi^{(r-2,r-2)},I},$$

which is the result (2.13) with $k = 0$.

The result (2.13) with $k = 1$ follows from space interpolation. □

Now, we consider the corresponding quasi-orthogonal projection. To do this, let $z(x)$ be the same as before, and set

$$\overline{v}_B(x) = \frac{1}{4}(z(x)-1)^2 v(a) + \frac{1}{4}(z(x)+1)(3-z(x))v(b) + \frac{h}{4}(z^2(x)-1)\partial_x v(b). \quad (2.16)$$

Let $\overline{v}_0(x) = v(x) - \overline{v}_B(x)$. Then, we define the quasi-orthogonal projection ${}_*\overline{P}^2_{N,I}v$ as

$${}_*\overline{P}^2_{N,I}v(x) = \overline{P}^2_{N,I}\overline{v}_0(x) + \overline{v}_B(x). \quad (2.17)$$

It is easy to show that

$${}_*\overline{P}^2_{N,I}v(a) = v(a), \qquad {}_*\overline{P}^2_{N,I}v(b) = v(b), \qquad \partial_x {}_*\overline{P}^2_{N,I}v(b) = \partial_x v(b). \quad (2.18)$$

Proposition 2.2 *If* $v \in H^2(I)$, $\partial_x^r v \in L^2_{\chi^{(r-2,r-2)}}(I)$, *integers* $2 \le r \le N+1$ *and* $N \ge 2$, *then*

$$\|\partial_x^k({}_*\overline{P}^2_{N,I}v - v)\|_I \le c(\frac{h}{N})^{r-k}\|\partial_x^r v\|_{\chi^{(r-2,r-2)},I}, \qquad k = 0, 1, 2. \quad (2.19)$$

Proof Clearly, ${}_*\overline{P}^2_{N,I}v(x) - v(x) = \overline{P}^2_{N,I}\overline{v}_0(x) - \overline{v}_0(x)$. Hence,

$$\begin{aligned}\|\partial_x^k({}_*\overline{P}^2_{N,I}v - v)\|_I &= \|\partial_x^k(\overline{P}^2_{N,I}\overline{v}_0 - \overline{v}_0)\|_I \\ &\le c(\tfrac{h}{N})^{r-k}(\|\partial_x^r v\|_{\chi^{(r-2,r-2)},I} + \|\partial_x^r \overline{v}_B\|_{\chi^{(r-2,r-2)},I}).\end{aligned} \quad (2.20)$$

If $r \ge 3$, then $\partial_x^r \overline{v}_B(x) = 0$. In this case, the desired result (2.19) follows immediately. On the other hand, by differentiating (2.16) and using integration by parts twice, we derive that

$$\partial_x^2 \overline{v}_B(x) = \frac{2}{h^2}(v(a) - v(b)) + \frac{2}{h}\partial_x v(b) = \frac{2}{h^2}\int_a^b (x-a)\partial_x^2 v(x)dx.$$

Accordingly, we use the Cauchy inequality to obtain

$$\|\partial_x^2 \overline{v}_B\|_I^2 \le \frac{4}{h^4}\int_a^b \left(\int_a^b (\eta - a)^2 d\eta \int_a^b (\partial_\eta^2 v(\eta))^2 d\eta\right) dx \le \frac{4}{3}\|\partial_x^2 v\|_I^2.$$

Substituting the above inequality into (2.20), we reach the result (2.19) with $r = 2$. □

Remark 2.1 For any function $v \in H^2(I)$ with $v(a) = v(b) = \partial_x v(a) = 0$, we could define the Legendre quasi-orthogonal projection ${}_*\overline{\overline{P}}^2_{N,I}v$ and derive its error estimate similarly.

3 Legendre Quasi-Orthogonal Approximation in Two Dimensions

In this section, we develop the Legendre quasi-orthogonal approximation in two dimensions, which plays an important role in the forthcoming discussions.

3.1 Legendre Quasi-Orthogonal Approximation in Rectangles

Let $I_i = \{x_i \mid a_i < x_i < b_i\}$ and $h_i = b_i - a_i$, $i = 1, 2$. For any positive integers N_i, we define the sets $\mathcal{P}_{N_i}(I_i)$ and $\mathcal{P}^{2,0}_{N_i}(I_i)$, and the orthogonal projections $P^{2,0}_{N_i,I_i}v$ from $H^2_0(I_i)$

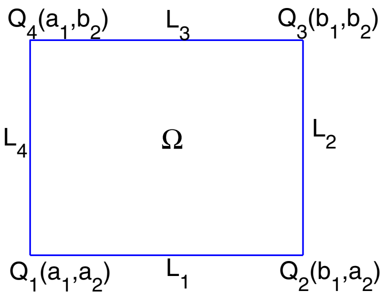

Fig. 1 Rectangle Ω.

onto $\mathcal{P}_{N_i}^{2,0}(I_i)$, in the same way as in Section 2. Moreover, we define the quasi-orthogonal projection ${}_*P_{N_i,I_i}^2 v$, like the definition (2.9).

Now, let $\Omega = \{\mathbf{x} = (x_1, x_2) \mid a_i < x_i < b_i,\ i = 1, 2\}$ with the boundary $\partial\Omega$. The four corners of Ω are denoted by $Q_1 = (a_1, a_2)$, $Q_2 = (b_1, a_2)$, $Q_3 = (b_1, b_2)$ and $Q_4 = (a_1, b_2)$, while the four edges of Ω are denoted by $L_1 = \overline{Q_1 Q_2}$, $L_2 = \overline{Q_2 Q_3}$, $L_3 = \overline{Q_3 Q_4}$ and $L_4 = \overline{Q_4 Q_1}$, respectively, see Figure 1. Let $\chi(\mathbf{x})$ be a certain weight function. For integer $r \geq 0$, we define the weighted Sobolev space $H_\chi^r(\Omega)$ as usual. In particular, $H_\chi^0(\Omega) = L^2(\Omega)$ with the inner product $(u, v)_{\chi,\Omega}$ and the norm $\|v\|_{\chi,\Omega}$. We omit the subscript χ whenever $\chi(\mathbf{x}) \equiv 1$.

Let $z_i(x_i) = \frac{1}{h_i}(2x_i - b_i - a_i)$, and

$$
\begin{aligned}
f_i^-(x_i) &= \tfrac{1}{4}(z_i^3(x_i) - 3z_i(x_i) + 2), & f_i^+(x_i) &= \tfrac{1}{4}(-z_i^3(x_i) + 3z_i(x_i) + 2),\\
g_i^-(x_i) &= \tfrac{h_i}{8}(z_i^3(x_i) - z_i^2(x_i) - z_i(x_i) + 1), & g_i^-(x_i) &= \tfrac{h_i}{8}(z_i^3(x_i) + z_i^2(x_i) - z_i(x_i) - 1).
\end{aligned} \tag{3.1}
$$

Due to (2.7), we have that

$$
\begin{aligned}
v_{B,I_1}(\mathbf{x}) = {}& v(a_1, x_2) f_1^-(x_1) + v(b_1, x_2) f_1^+(x_1) + \partial_{x_1} v(a_1, x_2) g_1^-(x_1)\\
&+ \partial_{x_1} v(b_1, x_2) g_1^+(x_1),
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
v_{B,I_2}(\mathbf{x}) = {}& v(x_1, a_2) f_2^-(x_2) + v(x_1, b_2) f_2^+(x_2) + \partial_{x_2} v(x_1, a_2) g_2^-(x_2)\\
&+ \partial_{x_2} v(x_1, b_2) g_2^+(x_2).
\end{aligned} \tag{3.3}
$$

Let

$$
\begin{aligned}
w_{B,\partial\Omega}(\mathbf{x}) = {}& v_{B,I_2}(a_1, x_2) f_1^-(x_1) + v_{B,I_2}(b_1, x_2) f_1^+(x_1)\\
&+ \partial_{x_1} v_{B,I_2}(a_1, x_2) g_1^-(x_1) + \partial_{x_1} v_{B,I_2}(b_1, x_2) g_1^+(x_1),
\end{aligned} \tag{3.4}
$$

or equivalently,

$$
\begin{aligned}
w_{B,\partial\Omega}(\mathbf{x}) = {}& v(a_1, a_2) f_1^-(x_1) f_2^-(x_2) + v(a_1, b_2) f_1^-(x_1) f_2^+(x_2)\\
&+ \partial_{x_2} v(a_1, a_2) f_1^-(x_1) g_2^-(x_2) + \partial_{x_2} v(a_1, b_2) f_1^-(x_1) g_2^+(x_2)\\
&+ v(b_1, a_2) f_1^+(x_1) f_2^-(x_2) + v(b_1, b_2) f_1^+(x_1) f_2^+(x_2)\\
&+ \partial_{x_2} v(b_1, a_2) f_1^+(x_1) g_2^-(x_2) + \partial_{x_2} v(b_1, b_2) f_1^+(x_1) g_2^+(x_2)\\
&+ \partial_{x_1} v(a_1, a_2) g_1^-(x_1) f_2^-(x_2) + \partial_{x_1} v(a_1, b_2) g_1^-(x_1) f_2^+(x_2)\\
&+ \partial_{x_1} \partial_{x_2} v(a_1, a_2) g_1^-(x_1) g_2^-(x_2) + \partial_{x_1} \partial_{x_2} v(a_1, b_2) g_1^-(x_1) g_2^+(x_2)\\
&+ \partial_{x_1} v(b_1, a_2) g_1^+(x_1) f_2^-(x_2) + \partial_{x_1} v(b_1, b_2) g_1^+(x_1) f_2^+(x_2)\\
&+ \partial_{x_1} \partial_{x_2} v(b_1, a_2) g_1^+(x_1) g_2^-(x_2) + \partial_{x_1} \partial_{x_2} v(b_1, b_2) g_1^+(x_1) g_2^+(x_2).
\end{aligned} \tag{3.5}
$$

The above function can be rewritten as the following equivalent form,

$$
\begin{aligned}
w_{B,\partial\Omega}(\mathbf{x}) = {}& v_{B,I_1}(x_1, a_2) f_2^-(x_2) + v_{B,I_1}(x_1, b_2) f_2^+(x_2)\\
&+ \partial_{x_2} v_{B,I_1}(x_1, a_2) g_2^-(x_2) + \partial_{x_2} v_{B,I_1}(x_1, b_2) g_2^+(x_2).
\end{aligned} \tag{3.6}
$$

Clearly, $w_{B,\partial\Omega}(\mathbf{x})$ is a polynomial of degree three for the variables x_1 and x_2. Further, we set

$$v_{B,\partial\Omega}(\mathbf{x}) = v_{B,I_1}(\mathbf{x}) + v_{B,I_2}(\mathbf{x}) - w_{B,\partial\Omega}(\mathbf{x}). \tag{3.7}$$

By using (3.4), (3.6), (2.6), (3.2) and (3.3), we have from (3.7) that

$$v_{B,\partial\Omega}(\mathbf{x}) = v(\mathbf{x}), \qquad \partial_n v_{B,\partial\Omega}(\mathbf{x}) = \partial_n v(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega. \tag{3.8}$$

Next, let $N_{B,\nu}$ be positive integers, and the pair $\mathbf{N}_B = (N_{B,1}, N_{B,2}, N_{B,3}, N_{B,4})$. We introduce the following projection corresponding the boundary $\partial\Omega$,

$$\begin{aligned}{}_*P^2_{\mathbf{N}_B,\partial\Omega} v_{B,\partial\Omega}(\mathbf{x}) = {} & {}_*P^2_{N_{B,1},I_1} v(x_1,a_2) f_2^-(x_2) + {}_*P^2_{N_{B,3},I_1} v(x_1,b_2) f_2^+(x_2) \\ & + {}_*P^2_{N_{B,1},I_1} \partial_{x_2} v(x_1,a_2) g_2^-(x_2) + {}_*P^2_{N_{B,3},I_1} \partial_{x_2} v(x_1,b_2) g_2^+(x_2) \\ & + {}_*P^2_{N_{B,4},I_2} v(a_1,x_2) f_1^-(x_1) + {}_*P^2_{N_{B,2},I_2} v(b_1,x_2) f_1^+(x_1) \\ & + {}_*P^2_{N_{B,4},I_2} \partial_{x_1} v(a_1,x_2) g_1^-(x_1) + {}_*P^2_{N_{B,2},I_2} \partial_{x_1} v(b_1,x_2) g_1^+(x_1) \\ & - w_{B,\partial\Omega}(\mathbf{x}). \end{aligned} \tag{3.9}$$

It can be checked that at the four corners $Q_\nu,\ 1 \le \nu \le 4$,

$$\begin{aligned} & {}_*P^2_{\mathbf{N}_B,\partial\Omega} v_{B,\partial\Omega}(\mathbf{x}) = v(\mathbf{x}), \\ & \partial_{x_i}({}_*P^2_{\mathbf{N}_B,\partial\Omega} v_{B,\partial\Omega}(\mathbf{x})) = \partial_{x_i} v(\mathbf{x}), \qquad i = 1, 2, \\ & \partial_{x_1}\partial_{x_2}({}_*P^2_{\mathbf{N}_B,\partial\Omega} v_{B,\partial\Omega}(\mathbf{x})) = \partial_{x_1}\partial_{x_2} v(\mathbf{x}). \end{aligned} \tag{3.10}$$

Now, let the pair $\mathbf{N} = (N_1, N_2)$. We define the Legendre quasi-orthogonal projection on the rectangle Ω as

$${}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega} v(\mathbf{x}) = P^{2,0}_{N_1,I_1} P^{2,0}_{N_2,I_2}(v - v_{B,\partial\Omega})(\mathbf{x}) + {}_*P^2_{\mathbf{N}_B,\partial\Omega} v_{B,\partial\Omega}(\mathbf{x}). \tag{3.11}$$

If $\mathbf{N}_B = (N_1, N_2, N_1, N_2)$, then we denote the approximation ${}_*P^2_{\mathbf{N}_B,\partial\Omega} v_{B,\partial\Omega}(\mathbf{x})$ by ${}_*P^2_{\mathbf{N},\partial\Omega} v_{B,\partial\Omega}(\mathbf{x})$ for simplicity. In this case, by using (3.2) and (3.3), we obtain from (3.9) that

$${}_*P^2_{\mathbf{N},\partial\Omega} v_{B,\partial\Omega}(\mathbf{x}) = {}_*P^2_{N_2,I_2} v_{B,I_1}(\mathbf{x}) + {}_*P^2_{N_1,I_1} v_{B,I_2}(\mathbf{x}) - w_{B,\partial\Omega}(\mathbf{x}). \tag{3.12}$$

In order to estimate the approximation error of ${}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega} v(\mathbf{x})$, we need some preparations.

Proposition 3.1 *We have*

$$P^{2,0}_{N_1,I_1} P^{2,0}_{N_2,I_2}(v - v_{B,\partial\Omega})(\mathbf{x}) = {}_*P^2_{N_1,I_1} {}_*P^2_{N_2,I_2} v(\mathbf{x}) - {}_*P^2_{\mathbf{N},\partial\Omega} v_{B,\partial\Omega}(\mathbf{x}). \tag{3.13}$$

Proof We use the definition (2.9) to calculate ${}_*P^2_{N_2,I_2}(v - v_{B_1,I_1})(\mathbf{x})$. Thanks to (3.2) and (3.6), the term $v_{B,I}(x)$ in (2.9) is now replaced by $v_{B,I_2}(\mathbf{x}) - w_{B,\partial\Omega}(\mathbf{x})$. Thus, we use (3.7) to deduce that

$$\begin{aligned}{}_*P^2_{N_2,I_2}(v - v_{B_1,I_1})(\mathbf{x}) &= P^{2,0}_{N_2,I_2}(v - v_{B,I_1} - v_{B,I_2} + w_{B,\partial\Omega})(\mathbf{x}) + v_{B,I_2}(\mathbf{x}) - w_{B,\partial\Omega}(\mathbf{x}) \\ &= P^{2,0}_{N_2,I_2}(v - v_{B,\partial\Omega})(\mathbf{x}) + v_{B,I_2}(\mathbf{x}) - w_{B,\partial\Omega}(\mathbf{x}). \end{aligned}$$

In other words,

$${}_*P^2_{N_2,I_2} v(\mathbf{x}) = P^{2,0}_{N_2,I_2}(v - v_{B,\partial\Omega})(\mathbf{x}) + {}_*P^2_{N_2,I_2} v_{B,I_1}(\mathbf{x}) + v_{B,I_2}(\mathbf{x}) - w_{B,\partial\Omega}(\mathbf{x}).$$

Furthermore,

$$\begin{aligned}{}_*P^2_{N_1,I_1} {}_*P^2_{N_2,I_2} v(\mathbf{x}) = {} & {}_*P^2_{N_1,I_1} P^{2,0}_{N_2,I_2}(v - v_{B,\partial\Omega})(\mathbf{x}) + {}_*P^2_{N_1,I_1} {}_*P^2_{N_2,I_2} v_{B,I_1}(\mathbf{x}) \\ & + {}_*P^2_{N_1,I_1} v_{B,I_2}(\mathbf{x}) - {}_*P^2_{N_1,I_1} w_{B,\partial\Omega}(\mathbf{x}). \end{aligned}$$

It can be checked that

$$
\begin{aligned}
&{}_*P^2_{N_1,I_1}P^{2,0}_{N_2,I_2}(v-v_{B,\partial\Omega})(\mathbf{x}) = P^{2,0}_{N_1,I_1}P^{2,0}_{N_2,I_2}(v-v_{B,\partial\Omega})(\mathbf{x}),\\
&{}_*P^2_{N_1,I_1}{}_*P^2_{N_2,I_2}v_{B,I_1}(\mathbf{x}) = {}_*P^2_{N_2,I_2}v_{B,I_1}(\mathbf{x}), \qquad {}_*P^2_{N_1,I_1}w_{B,\partial\Omega}(\mathbf{x}) = w_{B,\partial\Omega}(\mathbf{x}).
\end{aligned}
$$

Therefore, we use (3.12) to derive that

$$
\begin{aligned}
{}_*P^2_{N_1,I_1}{}_*P^2_{N_2,I_2}v(\mathbf{x}) &= P^{2,0}_{N_1,I_1}P^{2,0}_{N_2,I_2}(v-v_{B,\partial\Omega})(\mathbf{x}) + {}_*P^2_{N_2,I_2}v_{B,I_1}(\mathbf{x})\\
&\quad + {}_*P^2_{N_1,I_1}v_{B,I_2}(\mathbf{x}) - w_{B,\partial\Omega}(\mathbf{x})\\
&= P^{2,0}_{N_1,I_1}P^{2,0}_{N_2,I_2}(v-v_{B,\partial\Omega})(\mathbf{x}) + {}_*P^2_{\mathbf{N},\partial\Omega}v_{B,\partial\Omega}(\mathbf{x}).
\end{aligned}
$$

This ends the proof. □

For notational convenience, let $\mathbf{h}=(h_1,h_2)$ and $\mathbf{r}=(r_1,r_2)$ in the forthcoming discussions. Moreover,

$$
\chi_i^{(\alpha,\beta)}(x_i) = \left(\frac{2}{h_i}\right)^{\alpha+\beta}(b_i-x_i)^{\alpha}(x_i-a_i)^{\beta}, \qquad i=1,2.
$$

We shall use the following notations,

$$
\begin{aligned}
A^{(0)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v) &= (\tfrac{h_1}{N_1})^{r_1}\|\partial_{x_1}^{r_1}v\|_{\chi_1^{(r_1-2,r_1-2)},\Omega} + (\tfrac{h_2}{N_2})^{r_2}\|\partial_{x_2}^{r_2}v\|_{\chi_2^{(r_2-2,r_2-2)},\Omega}\\
&\quad + \min\{(\tfrac{h_1}{N_1})^{2}(\tfrac{h_2}{N_2})^{r_2-2}\|\partial_{x_1}^{2}\partial_{x_2}^{r_2-2}v\|_{\chi_2^{(r_2-4,r_2-4)},\Omega},\ (\tfrac{h_1}{N_1})^{r_1-2}(\tfrac{h_2}{N_2})^{2}\|\partial_{x_1}^{r_1-2}\partial_{x_2}^{2}v\|_{\chi_1^{(r_1-4,r_1-4)},\Omega}\},\\
A^{(1)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v) &= \left(\frac{h_1}{N_1}\right)^{r_1-1}\|\partial_{x_1}^{r_1}v\|_{\chi_1^{(r_1-2,r_1-2)},\Omega} + \left(\frac{h_2}{N_2}\right)^{r_2-1}\|\partial_{x_1}\partial_{x_2}^{r_2-1}v\|_{\chi_2^{(r_2-3,r_2-3)},\Omega}\\
&\quad + \left(\frac{h_2}{N_2}\right)^{r_2-1}\|\partial_{x_2}^{r_2}v\|_{\chi_2^{(r_2-2,r_2-2)},\Omega} + \left(\frac{h_1}{N_1}\right)^{r_1-1}\|\partial_{x_1}^{r_1-1}\partial_{x_2}v\|_{\chi_1^{(r_1-3,r_1-3)},\Omega}\\
&\quad + \left(\frac{h_1}{N_1}\right)^{r_1-3}\left(\frac{h_2}{N_2}\right)^{2}\|\partial_{x_1}^{r_1-2}\partial_{x_2}^{2}v\|_{\chi_1^{(r_1-4,r_1-4)},\Omega} + \left(\frac{h_1}{N_1}\right)^{2}(\tfrac{h_2}{N_2})^{r_2-3}\|\partial_{x_1}^{2}\partial_{x_2}^{r_2-2}v\|_{\chi_2^{(r_2-4,r_2-4)},\Omega},\\
A^{(2)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v) &= \left(\frac{h_1}{N_1}\right)^{r_1-2}\|\partial_{x_1}^{r_1}v\|_{\chi_1^{(r_1-2,r_1-2)},\Omega} + \left(\frac{h_2}{N_2}\right)^{r_2-2}\|\partial_{x_1}^{2}\partial_{x_2}^{r_2-2}v\|_{\chi_2^{(r_2-4,r_2-4)},\Omega}\\
&\quad + \left(\frac{h_2}{N_2}\right)^{r_2-2}\|\partial_{x_2}^{r_2}v\|_{\chi_2^{(r_2-2,r_2-2)},\Omega} + \left(\frac{h_1}{N_1}\right)^{r_1-2}\|\partial_{x_1}^{r_1-2}\partial_{x_2}^{2}v\|_{\chi_1^{(r_1-4,r_1-4)},\Omega}\\
&\quad + \left(\frac{h_1}{N_1}\right)^{r_1-2}\|\partial_{x_1}^{r_1-1}\partial_{x_2}v\|_{\chi_1^{(r_1-3,r_1-3)},\Omega} + \left(\frac{h_2}{N_2}\right)^{r_2-2}\|\partial_{x_1}\partial_{x_2}^{r_2-1}v\|_{\chi_2^{(r_2-3,r_2-3)},\Omega}\\
&\quad + \min\{\left(\frac{h_1}{N_1}\right)\left(\frac{h_2}{N_2}\right)^{r_2-3}\|\partial_{x_1}^{2}\partial_{x_2}^{r_2-2}v\|_{\chi_2^{(r_2-4,r_2-4)},\Omega},\\
&\quad \times \left(\frac{h_1}{N_1}\right)^{r_1-3}\left(\frac{h_2}{N_2}\right)\|\partial_{x_1}^{r_1-2}\partial_{x_2}^{2}v\|_{\chi_1^{(r_1-4,r_1-4)},\Omega}\}.
\end{aligned}
$$

Remark 3.1 The weight functions appearing in the norms involved in the quantities $A^{(\mu)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v)$ are helpful for covering certain singularities of approximated functions at the corners of domain. If we ignore the weights and $r=r_1=r_2\ge 4$, then

$$
A^{(\mu)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v) \le \left(\frac{h_1}{N_1}+\frac{h_2}{N_2}\right)^{r-\mu}|v|_{H^r(\Omega)}, \qquad \mu=0,1,2.
$$

Proposition 3.2 *If integers* $4\le r_i\le N_i+1$ *for* $i=1,2$, *then*

$$
\|{}_*P^2_{N_2,I_2}{}_*P^2_{N_1,I_1}v - v\|_{H^{\mu}(\Omega)} \le cA^{(\mu)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v), \qquad \mu=0,1,2. \tag{3.14}
$$

Proof We have

$$
\|\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}({}_*P^2_{N_2,I_2}{}_*P^2_{N_1,I_1}v - v)\|_{\Omega} \le F_1(v)+F_2(v)+F_3(v), \tag{3.15}
$$

with

$$
\begin{aligned}
F_1(v) &= \|\partial_{x_1}^{k_1}({}_*P^2_{N_1,I_1}\partial_{x_2}^{k_2}({}_*P^2_{N_2,I_2}v - v) - \partial_{x_2}^{k_2}({}_*P^2_{N_2,I_2}v - v))\|_\Omega,\\
F_2(v) &= \|\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}({}_*P^2_{N_1,I_1}v - v)\|_\Omega, \qquad F_3(v) = \|\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}({}_*P^2_{N_2,I_2}v - v)\|_\Omega.
\end{aligned}
$$

Also, we have

$$
\|\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}({}_*P^2_{N_2,I_2}{}_*P^2_{N_1,I_1}v - v)\|_\Omega \le \widetilde{F}_1(v) + F_2(v) + F_3(v),
$$

with

$$
\widetilde{F}_1(v) = \|\partial_{x_2}^{k_2}({}_*P^2_{N_2,I_2}\partial_{x_1}^{k_1}({}_*P^2_{N_1,I_1}v - v) - \partial_{x_1}^{k_1}({}_*P^2_{N_1,I_1}v - v))\|_\Omega.
$$

We use (2.11) with $k = k_1$ and $r = 2$, and (2.11) with $k = k_2$ and $r = r_2 - 2$ successively, to derive that

$$
\begin{aligned}
F_1(v) &\le c(\tfrac{h_1}{N_1})^{2-k_1}\|\partial_{x_1}^2\partial_{x_2}^{k_2}({}_*P^2_{N_2,I_2}v - v)\|_\Omega\\
&\le c(\tfrac{h_1}{N_1})^{2-k_1}(\tfrac{h_2}{N_2})^{r_2-k_2-2}\|\partial_{x_1}^2\partial_{x_2}^{r_2-2}v\|_{\chi_2^{(r_2-4,r_2-4)},\Omega}.
\end{aligned}
$$

Similarly,

$$
\widetilde{F}_1(v) \le c(\frac{h_1}{N_1})^{r_1-k_1-2}(\frac{h_2}{N_2})^{2-k_2}\|\partial_{x_1}^{r_1-2}\partial_{x_2}^2 v\|_{\chi_1^{(r_1-4,r_1-4)},\Omega}.
$$

Next, we use (2.11) with $k = k_1$ and $r = r_1 - k_2$ to obtain

$$
F_2(v) \le c(\tfrac{h_1}{N_1})^{r_1-k_1-k_2}\|\partial_{x_1}^{r_1-k_2}\partial_{x_2}^{k_2}v\|_{\chi_1^{(r_1-k_2-2,r_1-k_2-2)},\Omega}.
$$

Also, thanks to (2.11) with $k = k_2$ and $r = r_2 - k_1$, we have

$$
F_3(v) \le c(\tfrac{h_2}{N_2})^{r_2-k_1-k_2}\|\partial_{x_1}^{k_1}\partial_{x_2}^{r_2-k_1}v\|_{\chi_2^{(r_2-k_1-2,r_2-k_1-2)},\Omega}.
$$

A combination of (3.15) and the previous estimates leads to

$$
\begin{aligned}
\|\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}({}_*P^2_{N_2,I_2}{}_*P^2_{N_1,I_1}v - v)\|_\Omega &\le c(\tfrac{h_1}{N_1})^{r_1-k_1-k_2}\|\partial_{x_1}^{r_1-k_2}\partial_{x_2}^{k_2}v\|_{\chi_1^{(r_1-k_2-2,r_1-k_2-2)},\Omega}\\
&\quad + c(\tfrac{h_2}{N_2})^{r_2-k_1-k_2}\|\partial_{x_1}^{k_1}\partial_{x_2}^{r_2-k_1}v\|_{\chi_2^{(r_2-k_1-2,r_2-k_1-2)},\Omega}\\
&\quad + c\min\{(\tfrac{h_1}{N_1})^{2-k_1}(\tfrac{h_2}{N_2})^{r_2-k_2-2}\|\partial_{x_1}^2\partial_{x_2}^{r_2-2}v\|_{\chi_2^{(r_2-4,r_2-4)},\Omega},\\
&\qquad (\tfrac{h_1}{N_1})^{r_1-k_1-2}(\tfrac{h_2}{N_2})^{2-k_2}\|\partial_{x_1}^{r_1-2}\partial_{x_2}^2 v\|_{\chi_1^{(r_1-4,r_1-4)},\Omega}\}.
\end{aligned}
\tag{3.16}
$$

Finally, a careful calculation with (3.16) leads to the desired result (3.14). □

Now, let $\mathbf{r}_B = (r_{B,1}, r_{B,2}, r_{B,3}, r_{B,4})$. We shall use the following notations,

$$\begin{aligned}
B^{(0)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v) &= h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}}\|\partial_{x_1}^{r_{B,1}} v(x_1,a_2)\|_{\chi_1^{(r_{B,1}-2,r_{B,1}-2)},I_1}\\
&+h_2^{\frac{3}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-1}\|\partial_{x_1}^{r_{B,1}-1}\partial_{x_2} v(x_1,a_2)\|_{\chi_1^{(r_{B,1}-3,r_{B,1}-3)},I_1}\\
&+h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}}\|\partial_{x_1}^{r_{B,3}} v(x_1,b_2)\|_{\chi_1^{(r_{B,3}-2,r_{B,3}-2)},I_1}\\
&+h_2^{\frac{3}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-1}\|\partial_{x_1}^{r_{B,3}-1}\partial_{x_2} v(x_1,b_2)\|_{\chi_1^{(r_{B,3}-3,r_{B,3}-3)},I_1}\\
&+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}}\|\partial_{x_2}^{r_{B,4}} v(a_1,x_2)\|_{\chi_2^{(r_{B,4}-2,r_{B,4}-2)},I_2}\\
&+h_1^{\frac{3}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}-1}\|\partial_{x_2}^{r_{B,4}-1}\partial_{x_1} v(a_1,x_2)\|_{\chi_2^{(r_{B,4}-3,r_{B,4}-3)},I_2}\\
&+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}}\|\partial_{x_2}^{r_{B,2}} v(b_1,x_2)\|_{\chi_2^{(r_{B,2}-2,r_{B,2}-2)},I_2}\\
&+h_1^{\frac{3}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-1}\|\partial_{x_2}^{r_{B,2}-1}\partial_{x_1} v(b_1,x_2)\|_{\chi_2^{(r_{B,2}-3,r_{B,2}-3)},I_2},
\end{aligned}$$

$$\begin{aligned}
B^{(1)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v) &= (h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-1}+h_2^{-\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}})\|\partial_{x_1}^{r_{B,1}} v(x_1,a_2)\|_{\chi_1^{(r_{B,1}-2,r_{B,1}-2)},I_1}\\
&+(h_2^{\frac{3}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-2}+h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-1})\|\partial_{x_1}^{r_{B,1}-1}\partial_{x_2} v(x_1,a_2)\|_{\chi_1^{(r_{B,1}-3,r_{B,1}-3)},I_1}\\
&+(h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-1}+h_2^{-\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}})\|\partial_{x_1}^{r_{B,3}} v(x_1,b_2)\|_{\chi_1^{(r_{B,3}-2,r_{B,3}-2)},I_1}\\
&+(h_2^{\frac{3}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-2}+h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-1})\|\partial_{x_1}^{r_{B,3}-1}\partial_{x_2} v(x_1,b_2)\|_{\chi_1^{(r_{B,3}-3,r_{B,3}-3)},I_1}\\
&+(h_1^{-\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}}+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}-1})\|\partial_{x_2}^{r_{B,4}} v(a_1,x_2)\|_{\chi_2^{(r_{B,4}-2,r_{B,4}-2)},I_2}\\
&+(h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}-1}+h_1^{\frac{3}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}-2})\|\partial_{x_2}^{r_{B,4}-1}\partial_{x_1} v(a_1,x_2)\|_{\chi_2^{(r_{B,4}-3,r_{B,4}-3)},I_2}\\
&+(h_1^{-\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}}+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-1})\|\partial_{x_2}^{r_{B,2}} v(b_1,x_2)\|_{\chi_2^{(r_{B,2}-2,r_{B,2}-2)},I_2}\\
&+(h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-1}+h_1^{\frac{3}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-2})\|\partial_{x_2}^{r_{B,2}-1}\partial_{x_1} v(b_1,x_2)\|_{\chi_2^{(r_{B,2}-3,r_{B,2}-3)},I_2},
\end{aligned}$$

$$\begin{aligned}
&B^{(2)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v)\\
&= (h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-2}+h_2^{-\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-1}+h_2^{-\frac{3}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}})\|\partial_{x_1}^{r_{B,1}} v(x_1,a_2)\|_{\chi_1^{(r_{B,1}-2,r_{B,1}-2)},I_1}\\
&\quad+(h_2^{\frac{3}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-3}+h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-2}+h_2^{-\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-1})\|\partial_{x_1}^{r_{B,1}-1}\partial_{x_2} v(x_1,a_2)\|_{\chi_1^{(r_{B,1}-3,r_{B,1}-3)},I_1}\\
&\quad+(h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-2}+h_2^{-\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-1}+h_2^{-\frac{3}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}})\|\partial_{x_1}^{r_{B,3}} v(x_1,b_2)\|_{\chi_1^{(r_{B,3}-2,r_{B,3}-2)},I_1}\\
&\quad+(h_2^{\frac{3}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-3}+h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-2}+h_2^{-\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-1})\|\partial_{x_1}^{r_{B,3}-1}\partial_{x_2} v(x_1,b_2)\|_{\chi_1^{(r_{B,3}-3,r_{B,3}-3)},I_1}\\
&\quad+(h_1^{-\frac{3}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}}+h_1^{-\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}-1}+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}-2})\|\partial_{x_2}^{r_{B,4}} v(a_1,x_2)\|_{\chi_2^{(r_{B,4}-2,r_{B,4}-2)},I_2}\\
&\quad+(h_1^{-\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}-1}+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}-2}+h_1^{\frac{3}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}-3})\|\partial_{x_2}^{r_{B,4}-1}\partial_{x_1} v(a_1,x_2)\|_{\chi_2^{(r_{B,4}-3,r_{B,4}-3)},I_2}\\
&\quad+(h_1^{-\frac{3}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}}+h_1^{-\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-1}+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-2})\|\partial_{x_2}^{r_{B,2}} v(b_1,x_2)\|_{\chi_2^{(r_{B,2}-2,r_{B,2}-2)},I_2}\\
&\quad+(h_1^{-\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-1}+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-2}+h_1^{\frac{3}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-3})\|\partial_{x_2}^{r_{B,2}-1}\partial_{x_1} v(b_1,x_2)\|_{\chi_2^{(r_{B,2}-3,r_{B,2}-3)},I_2}.
\end{aligned}$$

If $\mathbf{N}_B = (N_1, N_2, N_1, N_2)$, then we denote the quantity $B^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v)$ by $B^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N},\partial\Omega}(v)$, $\mu = 0, 1, 2$.

Remark 3.2 We denote by $\partial_\tau v$ the tangential derivative on the boundary $\partial\Omega$. If we ignore the weights in the quantities $B^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v)$ and $r_B = r_{B,1} = r_{B,2} = r_{B,3} = r_{B,4} \geq 3,\ h = h_1 = h_2,\ N_B = N_{B,1} = N_{B,2} = N_{B,3} = N_{B,4}$, then

$$B^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v) \leq \frac{h^{r_B+\frac{1}{2}-\mu}}{N_B^{r_B-1-\mu}}(\|\partial_\tau^{r_B} v\|_{\partial\Omega} + \|\partial_\tau^{r_B-1}\partial_n v\|_{\partial\Omega}), \qquad \mu = 0, 1, 2.$$

Proposition 3.3 *If integers* $3 \leq r_{B,\nu} \leq N_{B,\nu} + 1$ *for* $1 \leq \nu \leq 4$, *then*

$$\|{}_*P^2_{\mathbf{N}_B,\partial\Omega} v_{B,\partial\Omega} - v_{B,\partial\Omega}\|_{H^\mu(\Omega)} \leq c B^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v), \qquad \mu = 0, 1, 2. \tag{3.17}$$

Proof By virtue of (3.9), (3.7), (3.2) and (3.3), we derive that

$$\begin{aligned}
&{}_*P^2_{\mathbf{N}_B,\partial\Omega} v_{B,\partial\Omega} - v_{B,\partial\Omega}\\
&\quad= f_2^-(x_2)({}_*P^2_{N_{B,1},I_1} v(x_1,a_2) - v(x_1,a_2)) + g_2^-(x_2)({}_*P^2_{N_{B,1},I_1}\partial_{x_2} v(x_1,a_2) - \partial_{x_2} v(x_1,a_2))\\
&\quad+ f_2^+(x_2)({}_*P^2_{N_{B,3},I_1} v(x_1,b_2) - v(x_1,b_2)) + g_2^+(x_2)({}_*P^2_{N_{B,3},I_1}\partial_{x_2} v(x_1,b_2) - \partial_{x_2} v(x_1,b_2))\\
&\quad+ f_1^-(x_1)({}_*P^2_{N_{B,4},I_2} v(a_1,x_2) - v(a_1,x_2)) + g_1^-(x_1)({}_*P^2_{N_{B,4},I_2}\partial_{x_1} v(a_1,x_2) - \partial_{x_1} v(a_1,x_2))\\
&\quad+ f_1^+(x_1)({}_*P^2_{N_{B,2},I_2} v(b_1,x_2) - v(b_1,x_2)) + g_1^+(x_1)({}_*P^2_{N_{B,2},I_2}\partial_{x_1} v(b_1,x_2) - \partial_{x_1} v(b_1,x_2)).
\end{aligned}$$

Moreover, thanks to (3.1), a careful calculation shows that for $i = 1, 2$,

$$\begin{aligned}
&\|f_i^-(x_i)\|_{I_i} = \|f_i^+(x_i)\|_{I_i} = \sqrt{\tfrac{13}{35}}h_i^{\frac{1}{2}}, && \|g_i^-(x_i)\|_{I_i} = \|g_i^+(x_i)\|_{I_i} = \sqrt{\tfrac{1}{105}}h_i^{\frac{3}{2}},\\
&\|\partial_{x_i} f_i^-(x_i)\|_{I_i} = \|\partial_{x_i} f_i^+(x_i)\|_{I_i} = \sqrt{\tfrac{6}{5}}h_i^{-\frac{1}{2}}, && \|\partial_{x_i} g_i^-(x_i)\|_{I_i} = \|\partial_{x_i} g_i^+(x_i)\|_{I_i} = \sqrt{\tfrac{2}{15}}h_i^{\frac{1}{2}},\\
&\|\partial^2_{x_i} f_i^-(x_i)\|_{I_i} = \|\partial^2_{x_i} f_i^+(x_i)\|_{I_i} = \sqrt{12}h_i^{-\frac{3}{2}}, && \|\partial^2_{x_i} g_i^-(x_i)\|_{I_i} = \|\partial^2_{x_i} g_i^+(x_i)\|_{I_i} = 2h_i^{-\frac{1}{2}}.
\end{aligned} \tag{3.18}$$

Therefore, we use (2.11) and (3.18) to reach the desired result (3.17). □

We now in a position to estimate $\|{}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega} v - v\|_{H^\mu(\Omega)}$.

Theorem 3.1 *If* $v \in H^{\frac{5}{2}+\delta}(\Omega)$ *with* $\delta > 0$, *integers* $4 \leq r_i \leq N_i + 1$ *for* $i = 1, 2$, *and* $3 \leq r_{B,\nu} \leq N_{B,\nu} + 1$ *for* $1 \leq \nu \leq 4$, *then*

$$\|{}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega} v - v\|_{H^\mu(\Omega)} \leq c(A^{(\mu)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v) + B^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v) + B^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N},\partial\Omega}(v)), \qquad \mu = 0, 1, 2, \tag{3.19}$$

provided that $A^{(\mu)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v)$, $B^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v)$ *and* $B^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N},\partial\Omega}(v)$ *are finite.*

Proof According to (3.11) and (3.13), we have

$$\begin{aligned}
\|{}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega} v - v\|_{H^\mu(\Omega)} &\leq \|{}_*P^2_{N_2,I_2}({}_*P^2_{N_1,I_1} v) - v\|_{H^\mu(\Omega)}\\
&\quad+ \|{}_*P^2_{\mathbf{N}_B,\partial\Omega} v_{B,\partial\Omega} - v_{B,\partial\Omega}\|_{H^\mu(\Omega)} + \|v_{B,\partial\Omega} - {}_*P^2_{\mathbf{N},\partial\Omega} v_{B,\partial\Omega}\|_{H^\mu(\Omega)}.
\end{aligned} \tag{3.20}$$

Like (3.17), we have

$$\|{}_*P^2_{\mathbf{N},\partial\Omega} v_{B,\partial\Omega} - v_{B,\partial\Omega}\|_{H^\mu(\Omega)} \leq c B^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N},\partial\Omega}(v), \tag{3.21}$$

Then, a combination of (3.14), (3.17), (3.20) and (3.21) leads to the desired result (3.19). □

Remark 3.3 As a special case, we may take $r = r_1 = r_2, r_B = r_{B,1} = r_{B,2} = r_{B,3} = r_{B,4}, h = h_1 = h_2, N = N_1 = N_2$, and $N_B = N_{B,1} = N_{B,2} = N_{B,3} = N_{B,4}$. If we ignore the weights appearing in all norms, then (3.19) implies that for $\mu = 0, 1, 2$,

$$\|{}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}v - v\|_{H^\mu(\Omega)} \le c(\frac{h}{N})^{r-\mu}|v|_{H^r(\Omega)} + c(\frac{h^{r_B+\frac{1}{2}-\mu}}{N^{r_B-1-\mu}} + \frac{h^{r_B+\frac{1}{2}-\mu}}{N_B^{r_B-1-\mu}})(\|\partial_\tau^{r_B}v\|_{\partial\Omega} + \|\partial_\tau^{r_B-1}\partial_n v\|_{\partial\Omega}).$$

Remark 3.4 If $\mathbf{N}_B = (N_1, N_2, N_1, N_2)$, then we use (3.11) and (3.13) to find that ${}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}v(\mathbf{x}) = {}_*P^2_{N_1,I_1}{}_*P^2_{N_2,I_2}v(\mathbf{x})$. Accordingly, we see from the proof of Theorem 3.1 that

$$\|{}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}v - v\|_{H^\mu(\Omega)} \le cA^{(\mu)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v), \qquad \mu = 0, 1, 2. \tag{3.22}$$

If we ignore the weights appearing in all norms, then (3.22) with $r = r_1 = r_2$ implies

$$\|{}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}v - v\|_{H^\mu(\Omega)} \le c(\frac{h_1}{N_1} + \frac{h_2}{N_2})^{r-\mu}|v|_{H^r(\Omega)}, \qquad \mu = 0, 1, 2.$$

But, in these cases, for keeping the continuity of ${}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}v(\mathbf{x})$ and $\partial_n{}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}v(\mathbf{x})$ at interfaces of adjacent elements, we should use the uniform $\mathbf{N}_B$. In other words, it is only appropriate for spectral element method with uniform mode. In opposite, the definition (3.11) allows us to use different modes at different interfaces of adjacent elements, and still keep the same continuity.

3.2 Other Quasi-Orthogonal Approximations in Rectangles

For solving mixed inhomogeneous boundary value problems, we need other kinds of Legendre quasi-orthogonal approximations. For fixedness, we assume that certain Neumann or Robin boundary conditions are imposed on the edge L_4. Let $\partial^*\Omega = L_1 \cup L_2 \cup L_3$, and

$$\overline{H}^2(\Omega) = \{v \in H^2(\Omega) \mid v = 0 \text{ on } \partial\Omega \text{ and } \partial_n v = 0 \text{ on } \partial^*\Omega\}, \qquad \overline{\mathcal{P}}_{\mathbf{N}}(\Omega) = \overline{H}^2(\Omega) \cap \mathcal{P}_{\mathbf{N}}(\Omega).$$

The meanings of $\mathcal{P}_{N_i}(I_i)$, $P^{2,0}_{N_i,I_i}v$ and ${}_*P^2_{N_i,I_i}v$ are the same as before. For any positive integers N_i, we define the sets $\overline{\mathcal{P}}_{N_i}(I_i)$ and the orthogonal projections $\overline{P}^2_{N_i,I_i}v$ from $\overline{H}^2(I_i)$ onto $\overline{\mathcal{P}}_{N_i}(I_i)$, in the same way as in Section 2. Moreover, we define the quasi-orthogonal ${}_*\overline{P}^2_{N_i,I_i}v$, like the definition (2.17).

Let $v_{B,I_2}(\mathbf{x})$ be the same as in (3.3), and

$$\overline{v}_{B,I_1}(\mathbf{x}) = v(a_1, x_2)f_1^-(x_1) + v(b_1, x_2)f_1^+(x_1) + \partial_{x_1}v(b_1, x_2)g_1^+(x_1). \tag{3.23}$$

Furthermore,

$$\overline{w}_{B,\partial\Omega}(\mathbf{x}) = v_{B,I_2}(a_1, x_2)f_1^-(x_1) + v_{B,I_2}(b_1, x_2)f_1^+(x_1) + \partial_{x_1}v_{B,I_2}(b_1, x_2)g_1^+(x_1), \tag{3.24}$$

or equivalently,

$$\begin{aligned}\overline{w}_{B,\partial\Omega}(\mathbf{x}) = {} & v(a_1, a_2)f_1^-(x_1)f_2^-(x_2) + v(a_1, b_2)f_1^-(x_1)f_2^+(x_2)\\ & +\partial_{x_2}v(a_1, a_2)f_1^-(x_1)g_2^-(x_2) + \partial_{x_2}v(a_1, b_2)f_1^-(x_1)g_2^+(x_2)\\ & +v(b_1, a_2)f_1^+(x_1)f_2^-(x_2) + v(b_1, b_2)f_1^+(x_1)f_2^+(x_2)\\ & +\partial_{x_2}v(b_1, a_2)f_1^+(x_1)g_2^-(x_2) + \partial_{x_2}v(b_1, b_2)f_1^+(x_1)g_2^+(x_2)\\ & +\partial_{x_1}v(b_1, a_2)g_1^+(x_1)f_2^-(x_2) + \partial_{x_1}v(b_1, b_2)g_1^+(x_1)f_2^+(x_2)\\ & +\partial_{x_1}\partial_{x_2}v(b_1, a_2)g_1^+(x_1)g_2^-(x_2) + \partial_{x_1}\partial_{x_2}v(b_1, b_2)g_1^+(x_1)g_2^+(x_2).\end{aligned} \tag{3.25}$$

The above function can be rewritten as

$$\overline{w}_{B,\partial\Omega}(\mathbf{x}) = \overline{v}_{B,I_1}(x_1,a_2)f_2^-(x_2) + \overline{v}_{B,I_1}(x_1,b_2)f_2^+(x_2) \\ +\partial_{x_2}\overline{v}_{B,I_1}(x_1,a_2)g_2^-(x_2) + \partial_{x_2}\overline{v}_{B,I_1}(x_1,b_2)g_2^+(x_2). \tag{3.26}$$

Clearly, $\overline{w}_{B,\partial\Omega}(\mathbf{x})$ is a polynomial of degree three for the variable x_1 and x_2. Further, we set

$$\overline{v}_{B,\partial\Omega}(\mathbf{x}) = \overline{v}_{B,I_1}(\mathbf{x}) + v_{B,I_2}(\mathbf{x}) - \overline{w}_{B,\partial\Omega}(\mathbf{x}). \tag{3.27}$$

By using (3.24), (3.26), (2.6), (3.23) and (3.3), we have from (3.27) that

$$\overline{v}_{B,\partial\Omega}(\mathbf{x}) = v(\mathbf{x}) \text{ for } \mathbf{x}\in\partial\Omega, \qquad \partial_n\overline{v}_{B,\partial\Omega}(\mathbf{x}) = \partial_n v(\mathbf{x}) \text{ for } \mathbf{x}\in\partial^*\Omega. \tag{3.28}$$

Now, we introduce the following projection corresponding the boundary $\partial\Omega$,

$$\begin{aligned} {}_*\overline{P}^2_{\mathbf{N}_B,\partial\Omega}\overline{v}_{B,\partial\Omega}(\mathbf{x}) = {}& {}_*\overline{P}^2_{N_{B,1},I_1}v(x_1,a_2)f_2^-(x_2) + {}_*\overline{P}^2_{N_{B,3},I_1}v(x_1,b_2)f_2^+(x_2) \\ &+ {}_*\overline{P}^2_{N_{B,1},I_1}\partial_{x_2}v(x_1,a_2)g_2^-(x_2) + {}_*\overline{P}^2_{N_{B,3},I_1}\partial_{x_2}v(x_1,b_2)g_2^+(x_2) \\ &+ {}_*P^2_{N_{B,4},I_2}v(a_1,x_2)f_1^-(x_1) + {}_*P^2_{N_{B,2},I_2}v(b_1,x_2)f_1^+(x_1) \\ &+ {}_*P^2_{N_{B,2},I_2}\partial_{x_1}v(b_1,x_2)g_1^+(x_1) - \overline{w}_{B,\partial\Omega}(\mathbf{x}). \end{aligned} \tag{3.29}$$

It can be checked that

$$\begin{aligned} &{}_*\overline{P}^2_{\mathbf{N}_B,\partial\Omega}\overline{v}_{B,\partial\Omega}(\mathbf{x}) = v(\mathbf{x}), \quad \partial_{x_2}({}_*\overline{P}^2_{\mathbf{N}_B,\partial\Omega}\overline{v}_{B,\partial\Omega}(\mathbf{x})) = \partial_{x_2}v(\mathbf{x}), \quad \text{at } Q_\nu,\ 1\le\nu\le 4, \\ &\partial_{x_1}({}_*\overline{P}^2_{\mathbf{N}_B,\partial\Omega}\overline{v}_{B,\partial\Omega}(\mathbf{x})) = \partial_{x_1}v(\mathbf{x}), \quad \partial_{x_1}\partial_{x_2}({}_*\overline{P}^2_{\mathbf{N}_B,\partial\Omega}\overline{v}_{B,\partial\Omega}(\mathbf{x})) = \partial_{x_1}\partial_{x_2}v(\mathbf{x}), \quad \text{at } Q_2,\ Q_3. \end{aligned} \tag{3.30}$$

Let the pair $\mathbf{N} = (N_1, N_2)$. Then, we define the Legendre quasi-orthogonal projection as

$$ {}_*\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}v(\mathbf{x}) = \overline{P}^2_{N_1,I_1}P^{2,0}_{N_2,I_2}(v-\overline{v}_{B,\partial\Omega})(\mathbf{x}) + {}_*\overline{P}^2_{\mathbf{N}_B,\partial\Omega}\overline{v}_{B,\partial\Omega}(\mathbf{x}). \tag{3.31}$$

If $\mathbf{N}_B = (N_1, N_2, N_1, N_2)$, then we denote the approximation ${}_*\overline{P}^2_{\mathbf{N}_B,\partial\Omega}\overline{v}_{B,\partial\Omega}(\mathbf{x})$ by ${}_*\overline{P}^2_{\mathbf{N},\partial\Omega}\overline{v}_{B,\partial\Omega}(\mathbf{x})$. In this case,

$$ {}_*\overline{P}^2_{\mathbf{N},\partial\Omega}\overline{v}_{B,\partial\Omega}(\mathbf{x}) = {}_*P^2_{N_2,I_2}\overline{v}_{B,I_1}(\mathbf{x}) + {}_*\overline{P}^2_{N_1,I_1}v_{B,I_2}(\mathbf{x}) - \overline{w}_{B,\partial\Omega}(\mathbf{x}). \tag{3.32}$$

In order to estimate the approximation error of ${}_*\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}v(\mathbf{x})$, we need some preparations. Firstly, an argument as the proof of Proposition 3.1, with (2.9), (3.23), (3.26), (3.27) and (3.32), leads to the following result.

Proposition 3.4 *We have*

$$\overline{P}^2_{N_1,I_1}P^{2,0}_{N_2,I_2}(v-\overline{v}_{B,\partial\Omega})(\mathbf{x}) = {}_*\overline{P}^2_{N_1,I_1}{}_*P^2_{N_2,I_2}v(\mathbf{x}) - {}_*\overline{P}^2_{\mathbf{N},\partial\Omega}\overline{v}_{B,\partial\Omega}(\mathbf{x}). \tag{3.33}$$

With the aid of (2.11) and (2.19), an argument as in the proof of Proposition 3.2 leads to the following result.

Proposition 3.5 *If integers $4\le r_i\le N_i+1$ for $i=1,2$, then*

$$\|{}_*P^2_{N_2,I_2}{}_*\overline{P}^2_{N_1,I_1}v - v\|_{H^\mu(\Omega)} \le cA^{(\mu)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v), \qquad \mu = 0,1,2. \tag{3.34}$$

For notational convenience, we introduce the following quantities,

$$
\begin{aligned}
\overline{B}^{(0)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v) = {} & h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}}\|\partial_{x_1}^{r_{B,1}}v(x_1,a_2)\|_{\chi_1^{(r_{B,1}-2,r_{B,1}-2)},I_1}\\
&+h_2^{\frac{3}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-1}\|\partial_{x_1}^{r_{B,1}-1}\partial_{x_2}v(x_1,a_2)\|_{\chi_1^{(r_{B,1}-3,r_{B,1}-3)},I_1}\\
&+h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}}\|\partial_{x_1}^{r_{B,3}}v(x_1,b_2)\|_{\chi_1^{(r_{B,3}-2,r_{B,3}-2)},I_1}\\
&+h_2^{\frac{3}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-1}\|\partial_{x_1}^{r_{B,3}-1}\partial_{x_2}v(x_1,b_2)\|_{\chi_1^{(r_{B,3}-3,r_{B,3}-3)},I_1}\\
&+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}}\|\partial_{x_2}^{r_{B,4}}v(a_1,x_2)\|_{\chi_2^{(r_{B,4}-2,r_{B,4}-2)},I_2}\\
&+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}}\|\partial_{x_2}^{r_{B,2}}v(b_1,x_2)\|_{\chi_2^{(r_{B,2}-2,r_{B,2}-2)},I_2}\\
&+h_1^{\frac{3}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-1}\|\partial_{x_2}^{r_{B,2}-1}\partial_{x_1}v(b_1,x_2)\|_{\chi_2^{(r_{B,2}-3,r_{B,2}-3)},I_2},\\
\overline{B}^{(1)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v) = {} & (h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-1}+h_2^{-\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}})\|\partial_{x_1}^{r_{B,1}}v(x_1,a_2)\|_{\chi_1^{(r_{B,1}-2,r_{B,1}-2)},I_1}\\
&+(h_2^{\frac{3}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-2}+h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-1})\|\partial_{x_1}^{r_{B,1}-1}\partial_{x_2}v(x_1,a_2)\|_{\chi_1^{(r_{B,1}-3,r_{B,1}-3)},I_1}\\
&+(h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-1}+h_2^{-\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}})\|\partial_{x_1}^{r_{B,3}}v(x_1,b_2)\|_{\chi_1^{(r_{B,3}-2,r_{B,3}-2)},I_1}\\
&+(h_2^{\frac{3}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-2}+h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-1})\|\partial_{x_1}^{r_{B,3}-1}\partial_{x_2}v(x_1,b_2)\|_{\chi_1^{(r_{B,3}-3,r_{B,3}-3)},I_1}\\
&+(h_1^{-\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}}+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}-1})\|\partial_{x_2}^{r_{B,4}}v(a_1,x_2)\|_{\chi_2^{(r_{B,4}-2,r_{B,4}-2)},I_2}\\
&+(h_1^{-\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}}+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-1})\|\partial_{x_2}^{r_{B,2}}v(b_1,x_2)\|_{\chi_2^{(r_{B,2}-2,r_{B,2}-2)},I_2}\\
&+(h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-1}+h_1^{\frac{3}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-2})\|\partial_{x_2}^{r_{B,2}-1}\partial_{x_1}v(b_1,x_2)\|_{\chi_2^{(r_{B,2}-3,r_{B,2}-3)},I_2},
\end{aligned}
$$

$$
\begin{aligned}
&\overline{B}^{(2)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v)\\
&= (h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-2}+h_2^{-\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-1}+h_2^{-\frac{3}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}})\|\partial_{x_1}^{r_{B,1}}v(x_1,a_2)\|_{\chi_1^{(r_{B,1}-2,r_{B,1}-2)},I_1}\\
&\quad+(h_2^{\frac{3}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-3}+h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-2}+h_2^{-\frac{1}{2}}(\tfrac{h_1}{N_{B,1}})^{r_{B,1}-1})\|\partial_{x_1}^{r_{B,1}-1}\partial_{x_2}v(x_1,a_2)\|_{\chi_1^{(r_{B,1}-3,r_{B,1}-3)},I_1}\\
&\quad+(h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-2}+h_2^{-\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-1}+h_2^{-\frac{3}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}})\|\partial_{x_1}^{r_{B,3}}v(x_1,b_2)\|_{\chi_1^{(r_{B,3}-2,r_{B,3}-2)},I_1}\\
&\quad+(h_2^{\frac{3}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-3}+h_2^{\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-2}+h_2^{-\frac{1}{2}}(\tfrac{h_1}{N_{B,3}})^{r_{B,3}-1})\|\partial_{x_1}^{r_{B,3}-1}\partial_{x_2}v(x_1,b_2)\|_{\chi_1^{(r_{B,3}-3,r_{B,3}-3)},I_1}\\
&\quad+(h_1^{-\frac{3}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}}+h_1^{-\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}-1}+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,4}})^{r_{B,4}-2})\|\partial_{x_2}^{r_{B,4}}v(a_1,x_2)\|_{\chi_2^{(r_{B,4}-2,r_{B,4}-2)},I_2}\\
&\quad+(h_1^{-\frac{3}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}}+h_1^{-\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-1}+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-2})\|\partial_{x_2}^{r_{B,2}}v(b_1,x_2)\|_{\chi_2^{(r_{B,2}-2,r_{B,2}-2)},I_2}\\
&\quad+(h_1^{-\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-1}+h_1^{\frac{1}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-2}+h_1^{\frac{3}{2}}(\tfrac{h_2}{N_{B,2}})^{r_{B,2}-3})\|\partial_{x_2}^{r_{B,2}-1}\partial_{x_1}v(b_1,x_2)\|_{\chi_2^{(r_{B,2}-3,r_{B,2}-3)},I_2}.
\end{aligned}
$$

If $\mathbf{N}_B=(N_1,N_2,N_1,N_2)$, then we denote the quantity $\overline{B}^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v)$ by $\overline{B}^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N},\partial\Omega}(v)$, $\mu=0,1,2$.

Remark 3.5 If we ignore the weights in the quantities $\overline{B}^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v)$ and $r_B=r_{B,1}=r_{B,2}=r_{B,3}=r_{B,4}\geq 3$, $h=h_1=h_2$, $N_B=N_{B,1}=N_{B,2}=N_{B,3}=N_{B,4}$, then

$$
\overline{B}^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v)\leq \frac{h^{r_B+\frac{1}{2}-\mu}}{N_B^{r_B-1-\mu}}(\|\partial_\tau^{r_B}v\|_{\partial\Omega}+\|\partial_\tau^{r_B-1}\partial_n v\|_{\partial^*\Omega}),\qquad \mu=0,1,2.
$$

With the aid of (3.29), (3.27), (3.23), (3.3), (2.11), (2.19) and (3.18), we follow the same line as the proof of Proposition 3.3 to derive the following result.

Proposition 3.6 *If integers* $3 \le r_{B,\nu} \le N_{B,\nu} + 1$ *for* $1 \le \nu \le 4$, *then*

$$\|_*\overline{P}^2_{\mathbf{N}_B,\partial\Omega}\overline{v}_{B,\partial\Omega} - \overline{v}_{B,\partial\Omega}\|_{H^\mu(\Omega)} \le c\overline{B}^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v), \qquad \mu = 0, 1, 2. \tag{3.35}$$

We are now in position to estimate $\|_*\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}v - v\|_{H^\mu(\Omega)}$. By virtue of (3.31), (3.33), (3.35), (3.34) and an argument similar to the proof of Theorem 3.1, we can prove the following result.

Theorem 3.2 *If* $v \in H^{\frac{5}{2}+\delta}(\Omega)$ *with* $\delta > 0$, *integers* $4 \le r_i \le N_i + 1$ *for* $i = 1, 2$, *and* $3 \le r_{B,\nu} \le N_{B,\nu} + 1$ *for* $1 \le \nu \le 4$, *then*

$$\|_*\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}v - v\|_{H^\mu(\Omega)} \le c(A^{(\mu)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v) + \overline{B}^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v) + \overline{B}^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N},\partial\Omega}(v)), \qquad \mu = 0, 1, 2, \tag{3.36}$$

provided that $A^{(\mu)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v)$, $\overline{B}^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N}_B,\partial\Omega}(v)$ *and* $\overline{B}^{(\mu)}_{\mathbf{r}_B,\mathbf{h},\mathbf{N},\partial\Omega}(v)$ *are finite.*

Remark 3.6 As a special case, we may take $r = r_1 = r_2, r_B = r_{B,1} = r_{B,2} = r_{B,3} = r_{B,4}, h = h_1 = h_2, N = N_1 = N_2$, and $N_B = N_{B,1} = N_{B,2} = N_{B,3} = N_{B,4}$. If we ignore the weights appearing in all norms, then (3.36) implies that for $\mu = 0, 1, 2$,

$$\begin{aligned}\|_*\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}v - v\|_{H^\mu(\Omega)} &\le c(\frac{h}{N})^{r-\mu}|v|_{H^r(\Omega)} + c(\frac{h^{r_B+\frac{1}{2}-\mu}}{N^{r_B-1-\mu}} \\ &+ \frac{h^{r_B+\frac{1}{2}-\mu}}{N_B^{r_B-1-\mu}})(\|\partial_\tau^{r_B}v\|_{\partial\Omega} + \|\partial_\tau^{r_B-1}\partial_n v\|_{\partial^*\Omega}).\end{aligned}$$

Remark 3.7 If $\mathbf{N}_B = (N_1, N_2, N_1, N_2)$, then we use (3.31) and (3.33) to find that ${}_*\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}v(\mathbf{x}) = {}_*\overline{P}^2_{N_1,I_1}{}_*P^2_{N_2,I_2}v(\mathbf{x})$. Accordingly,

$$\|_*\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}v - v\|_{H^\mu(\Omega)} \le cA^{(\mu)}_{\mathbf{r},\mathbf{h},\mathbf{N},\Omega}(v), \qquad \mu = 0, 1, 2. \tag{3.37}$$

If we ignore the weights appearing in all norms, then (3.37) with $r = r_1 = r_2$ implies

$$\|_*\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}v - v\|_{H^\mu(\Omega)} \le c(\frac{h_1}{N_1} + \frac{h_2}{N_2})^{r-\mu}|v|_{H^r(\Omega)}, \qquad \mu = 0, 1, 2.$$

Remark 3.8 If $\partial_n v$ vanishes on other parts of the boundary, we could define the related Legendre quasi-orthogonal projection in the same way, and derive the error estimate similar to (3.36) and (3.37).

3.3 Composite Legendre Quasi-Orthogonal Approximation

We now turn to the composite Legendre quasi-orthogonal approximation on certain complex domains, which serves as the mathematical foundation of spectral element method for mixed inhomogeneous boundary value problems of fourth order.

Let Ω be a polygon with the boundary $\partial\Omega = \partial^*\Omega \cup \partial^{**}\Omega$ and $\partial^*\Omega \cap \partial^{**}\Omega = \emptyset$. We may impose Neumann or Robin boundary conditions on $\partial^{**}\Omega$. In this paper, we suppose that the domain Ω could be divided into rectangle subdomains Ω_k, namely,

$$\Omega_k = \{\mathbf{x} = (x_1, x_2) \mid a_{k,i} < x_i < b_{k,i},\ i = 1, 2\}, \qquad 1 \le k \le M,$$

with the boundary $\partial\Omega_k$, the edges $L_{k,\nu}$ and the vertices $Q_{k,\nu}$ $1 \le \nu \le 4$. Besides, $\partial^*\Omega_k = \partial\Omega_k \cap \partial^*\Omega$ and $\partial^{**}\Omega_k = \partial\Omega_k \cap \partial^{**}\Omega$.

We assume that the partition of Ω satisfies the following hypotheses,

(H$_1$). $\overline{\Omega} = \cup_{k=1}^{M} \overline{\Omega}_k$ and $\Omega_{k_1} \cap \Omega_{k_2} = \emptyset$ if $k_1 \ne k_2$,
(H$_2$). each vertex of Ω_k is also one of vertices of adjacent rectangles,
(H$_3$). if $\partial^{**}\Omega_k \ne \emptyset$, then Ω_k has at most two adjacent edges belonging to $\partial^{**}\Omega$,
(H$_4$). if $L_{k,\nu} \subseteq \partial^*\Omega$, then $L_{k,\nu} \nsubseteq \partial^{**}\Omega$.

Let $I_{k,i} = \{x_i \mid a_{k,i} < x_i < b_{k,i}\}$, and

$$h_{k,i} = b_{k,i} - a_{k,i}, \qquad \mathbf{h}_k = (h_{k,1}, h_{k,2}), \qquad \mathbf{N}_k = (N_{k,1}, N_{k,2}),$$
$$\mathbf{N}_{B,k} = (N_{B,k,1}, N_{B,k,2}, N_{B,k,3}, N_{B,k,4}), \qquad \mathbf{r}_{B,k} = (r_{B,k,1}, r_{B,k,2}, r_{B,k,3}, r_{B,k,4}).$$

If L_{k_1,ν_1} and L_{k_2,ν_2} are the same segment, say $L_{k_1,1} = L_{k_2,3}$, then we take $N_{B,k_1,1} = N_{B,k_2,3}$. The local weight function

$$\chi_{k,i}^{(\alpha,\beta)}(x_i) = (\frac{2}{h_{k,i}})^{\alpha+\beta}(b_{k,i} - x_i)^{\alpha}(x_i - a_{k,i})^{\beta}, \qquad i = 1, 2.$$

The functions $f_{k,i}^-(x_i)$, $f_{k,i}^+(x_i)$, $g_{k,i}^-(x_i)$ and $g_{k,i}^+(x_i)$ are defined in the same way as in (3.1). We also define the local quantities $A^{(\mu)}_{\mathbf{r}_k,\mathbf{h}_k,\mathbf{N}_k,\Omega_k}(v)$, $B^{(\mu)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega_k}(v)$, $B^{(\mu)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_k,\partial\Omega_k}(v)$, $\overline{B}^{(\mu)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega_k}(v)$ and $\overline{B}^{(\mu)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_k,\partial\Omega_k}(v)$ as in Subsections 3.1 and 3.2. Furthermore, let

$$\widetilde{B}^{(\mu)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega_k}(v) = \begin{cases} B^{(\mu)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega_k}(v), & \text{if } \partial^{**}\Omega_k = \emptyset, \\ \overline{B}^{(\mu)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega_k}(v), & \text{if } \partial^{**}\Omega_k \ne \emptyset. \end{cases}$$

If $\mathbf{N}_{B,k} = (N_{k,1}, N_{k,2}, N_{k,1}, N_{k,2})$, then we denote $\widetilde{B}^{(\mu)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega_k}(v)$ by $\widetilde{B}^{(\mu)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_k,\partial\Omega_k}(v)$.

Now, let

$$\mathbf{N} = (N_{1,1}, N_{1,2}, \cdots, N_{M,1}, N_{M,2}),$$
$$\mathbf{N}_B = (N_{B,1,1}, N_{B,1,2}, N_{B,1,3}, N_{B,1,4}, \cdots, N_{B,M,1}, N_{B,M,2}, N_{B,M,3}, N_{B,M,4}).$$

We define the composite Legendre quasi-orthogonal projection ${}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}v$ on the whole domain Ω by

$${}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}v|_{\Omega_k} = {}_*P^2_{\mathbf{N}_k,\mathbf{N}_{B,k},\Omega_k}v, \qquad 1 \le k \le M, \tag{3.38}$$

where the local projections ${}_*P^2_{\mathbf{N}_k,\mathbf{N}_{B,k},\Omega_k}v$ are constructed in such a way that

A. if $\partial^{**}\Omega_k = \emptyset$, then ${}_*P^2_{\mathbf{N}_k,\mathbf{N}_{B,k},\Omega_k}v$ is given by (3.11).
B. if $\partial^{**}\Omega_k \ne \emptyset$, say $\partial^{**}\Omega_k = L_{k,1}$, then ${}_*P^2_{\mathbf{N}_k,\mathbf{N}_{B,k},\Omega_k}v$ is given by (3.31).
C. if $\partial^{**}\Omega_k \ne \emptyset$, say $\partial^{**}\Omega_k = L_{k,j} \cup L_{k,j+1}$, (with $L_{k,5} = L_{k,1}$), then ${}_*P^2_{\mathbf{N}_k,\mathbf{N}_{B,k},\Omega_k}v$ is similar to the definition (3.31).

By using Theorems 3.1 and 3.2, we verify that if integers $4 \le r_{k,i} \le N_{k,i} + 1$ and $3 \le r_{B,k} \le N_{B,k,\nu} + 1$ for $1 \le k \le M$, $i = 1, 2$ and $1 \le \nu \le 4$, then

$$\begin{aligned} &\|{}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}v - v\|_{H^\mu(\Omega)} \\ &\le c\sum_{k=1}^{M}(A^{(\mu)}_{\mathbf{r}_k,\mathbf{h}_k,\mathbf{N}_k,\Omega_k}(v) + \widetilde{B}^{(\mu)}_{r_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega_k}(v) + \widetilde{B}^{(\mu)}_{r_{B,k},\mathbf{h}_k,\mathbf{N}_k,\partial\Omega_k}(v)), \qquad \mu = 0, 1, 2, \end{aligned} \tag{3.39}$$

provided that the quantities involved at the right side of the above inequality are finite.

Remark 3.9 As a special case, we may take $r_k = r_{k,1} = r_{k,2}, r_{B,k} = r_{B,k,1} = r_{B,k,2} = r_{B,k,3} = r_{B,k,4}, N_k = N_{k,1} = N_{k,2}, h = h_{k,1} = h_{k,2}$, and $N_B = N_{B,k,1} = N_{B,k,2} = N_{B,k,3} = N_{B,k,4}$, for $1 \le k \le M$. If we ignore the weights appearing in all norms, then (3.39), together with Remarks 3.3 and 3.6, leads to

$$\|{}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}v - v\|_{H^\mu(\Omega)} \le c\sum_{k=1}^{M}(\frac{h}{N_k})^{r_k-\mu}|v|_{H^{r_k}(\Omega_k)}$$
$$+c\sum_{k=1}^{M}(\frac{h^{r_{B,k}+\frac{1}{2}-\mu}}{N_k^{r_{B,k}-1-\mu}} + \frac{h^{r_{B,k}+\frac{1}{2}-\mu}}{N_B^{r_{B,k}-1-\mu}})(\|\partial_\tau^{r_{B,k}}v\|_{\partial\Omega_k} + \|\partial_\tau^{r_{B,k}-1}\partial_n v\|_{\partial\Omega_k}), \quad \mu = 0, 1, 2.$$

If $r_k = r_{k,1} = r_{k,2}, N_{k,1} = N_{B,k,1} = N_{B,k,3}$ and $N_{k,2} = N_{B,k,2} = N_{B,k,4}$ for $1 \le k \le M$, then (3.39), together with Remarks 3.4 and 3.7, leads to

$$\|{}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}v - v\|_{H^\mu(\Omega)} \le c\sum_{k=1}^{M}(\frac{h_{k,1}}{N_{k,1}} + \frac{h_{k,2}}{N_{k,2}})^{r_k-\mu}|v|_{H^{r_k}(\Omega_k)}, \qquad \mu = 0, 1, 2.$$

4 Spectral Element Method for Fourth Order Problems

In this section, we propose the spectral element method for mixed inhomogeneous boundary value problems of fourth order, which occur in many practical cases, such as laminated composite plates, free vibration of anisotropic plates under general edge conditions and so on, see [3,10,23,25] and the references therein.

As in Sect. 3.3, we suppose that the domain Ω is a union of several rectangles, with the boundary $\partial\Omega = \partial^*\Omega \cup \partial^{**}\Omega$, $\partial^*\Omega \cap \partial^{**}\Omega = \emptyset$.

Let d and β be nonnegative constants. We consider the following model problem,

$$\begin{cases} \Delta^2 U(\mathbf{x}) + dU(\mathbf{x}) = F(\mathbf{x}), & \text{in } \Omega, \\ \Delta U(\mathbf{x}) + \beta\partial_n U(\mathbf{x}) = G_2(\mathbf{x}), & \text{on } \partial^{**}\Omega, \\ \partial_n U(\mathbf{x}) = G_1(\mathbf{x}), & \text{on } \partial^*\Omega, \\ U(\mathbf{x}) = G_0(\mathbf{x}), & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

If $\partial^*\Omega = \partial\Omega$, then the above problem is a Dirichlet boundary value problem. Otherwise, it is a mixed inhomogeneous boundary value problem. In this case, if $\partial^{**}\Omega = \partial\Omega$ and $d = \beta = 0$, then we require an additional condition for ensuring the existence of solution. In fact, for any $z, w \in H^2(\Omega)$,

$$\iint\limits_{\Omega} z(\mathbf{x})\Delta w(\mathbf{x})dx_1dx_2 + \iint\limits_{\Omega}(\partial_{x_1}z(\mathbf{x})\partial_{x_1}w(\mathbf{x}) + \partial_{x_2}z(\mathbf{x})\partial_{x_2}w(\mathbf{x}))dx_1dx_2$$
$$= \int\limits_{\partial\Omega} z(\mathbf{x})\partial_n w(\mathbf{x})ds. \tag{4.2}$$

Therefore, by integrating the first equation of (4.1) and using (4.2) with $w = \Delta U$ and $z = 1$, we obtain

$$\iint\limits_{\Omega} F(\mathbf{x})dx_1dx_2 = \int\limits_{\partial\Omega}\partial_n G_2(\mathbf{x})ds. \tag{4.3}$$

Since the considered domain Ω is a union of several rectangles, we find that if $\beta = 0$ and $U(\mathbf{x}) = G_0(\mathbf{x}) \equiv 0$ on $\partial\Omega$, then the boundary condition $\Delta U(\mathbf{x}) = G_2(\mathbf{x})$ on $\partial^{**}\Omega$, implies

$\partial_n^2 U(\mathbf{x}) = G_2(\mathbf{x})$ on $\partial^{**}\Omega$. This is similar to the simply supported boundary condition in elastic mechanics.

We next derive the weak formulation of problem (4.1). As we know, for any $w, z \in H^2(\Omega)$,

$$\int\int_\Omega z(\mathbf{x})\Delta^2 w(\mathbf{x})dx_1dx_2 = \int\int_\Omega \Delta w(\mathbf{x})\Delta z(\mathbf{x})dx_1dx_2 + \int_{\partial\Omega}(z(\mathbf{x})\partial_n(\Delta w(\mathbf{x})) - \Delta w(\mathbf{x})\partial_n z(\mathbf{x}))ds. \tag{4.4}$$

Now, let

$$V(\Omega) = \{v \in H^2(\Omega) \mid v = G_0(\mathbf{x}) \text{ on } \partial\Omega, \text{ and } \partial_n v = G_1(\mathbf{x}) \text{ on } \partial^*\Omega\},$$
$$\overline{V}(\Omega) = \{v \in H^2(\Omega) \mid v = 0 \text{ on } \partial\Omega, \text{ and } \partial_n v = 0 \text{ on } \partial^*\Omega\}.$$

We introduce the bilinear form

$$a_{d,\beta}(u,v) = (\Delta u, \Delta v)_\Omega + d(u,v)_\Omega + \beta\int_{\partial^{**}\Omega}\partial_n u(\mathbf{x})\partial_n v(\mathbf{x})ds, \qquad \forall u, v \in H^2(\Omega).$$

By multiplying the first equation of (4.1) by $v \in \overline{V}(\Omega)$ and integrating the resulting equality by parts, we use (4.4) to obtain the weak form of (4.1). It is to find $U \in V(\Omega)$ such that

$$a_{d,\beta}(U,v) = (F,v)_\Omega + \int_{\partial^{**}\Omega} G_2(\mathbf{x})\partial_n v(\mathbf{x})ds, \qquad \forall v \in \overline{V}(\Omega). \tag{4.5}$$

If $F \in L^2(\Omega)$, $G_0 \in H^{\frac{3}{2}}(\partial\Omega)$, $G_1 \in H^{\frac{1}{2}}(\partial^*\Omega)$ and $G_2 \in H^{-\frac{1}{2}}(\partial^{**}\Omega)$, then the above problem admits a unique solution.

For solving (4.5) numerically, we first consider an auxiliary problem. We divide the domain Ω into rectangles Ω_k, $1 \le k \le M$. We also use the same notations as before, such as $\partial\Omega_k$, $\partial^*\Omega_k$, $\partial^{**}\Omega_k$, $I_{k,i}$, $h_{k,i}$ and $L_{k,\nu}$. Hereafter $L_{k,\nu}$ is the $\nu'th$ edge of $\partial\Omega_k$ as before. Furthermore, ${}_*P^m_{N_{B,k,\nu},I_{k,i}}v$, $m = 1, 2$, stand for the quasi-orthogonal projections on the edge $L_{k,\nu} \subset \partial\Omega$, which are similar to the definitions (2.2) and (2.9) respectively.

We now define the following operators,

$${}_*\widetilde{P}^1_{\mathbf{N}_B,\partial^*\Omega}v(\mathbf{x}) = \begin{cases} {}_*P^1_{N_{B,k,\nu},I_{k,1}}v(\mathbf{x}), & \text{if } \mathbf{x} \in L_{k,\nu} \subset \partial^*\Omega,\ \nu = 1,3, \\ {}_*P^1_{N_{B,k,\nu},I_{k,2}}v(\mathbf{x}), & \text{if } \mathbf{x} \in L_{k,\nu} \subset \partial^*\Omega,\ \nu = 2,4, \end{cases}$$

$${}_*\widetilde{P}^2_{\mathbf{N}_B,\partial\Omega}v(\mathbf{x}) = \begin{cases} {}_*P^2_{N_{B,k,\nu},I_{k,1}}v(\mathbf{x}), & \text{if } \mathbf{x} \in L_{k,\nu} \subset \partial\Omega,\ \nu = 1,3, \\ {}_*P^2_{N_{B,k,\nu},I_{k,2}}v(\mathbf{x}), & \text{if } \mathbf{x} \in L_{k,\nu} \subset \partial\Omega,\ \nu = 2,4. \end{cases}$$

Moreover,

$$V^*(\Omega) = \{v \in H^2(\Omega) \mid v = {}_*\widetilde{P}^2_{\mathbf{N}_B,\partial\Omega}G_0(\mathbf{x}) \text{ on } \partial\Omega, \text{ and } \partial_n v = {}_*\widetilde{P}^1_{\mathbf{N}_B,\partial^*\Omega}G_1(\mathbf{x}) \text{ on } \partial^*\Omega\}.$$

The auxiliary problem is to find $W \in V^*(\Omega)$ such that

$$a_{d,\beta}(W,v) = (F,v)_\Omega + \int_{\partial^{**}\Omega} G_2(\mathbf{x})\partial_n v(\mathbf{x})ds, \qquad \forall v \in \overline{V}(\Omega), \tag{4.6}$$

or equivalently,

$$\begin{cases} \Delta^2 W(\mathbf{x}) + dW(\mathbf{x}) = F(\mathbf{x}), & \text{in } \Omega, \\ \Delta W(\mathbf{x}) + \beta\partial_n W(\mathbf{x}) = G_2(\mathbf{x}), & \text{on } \partial^{**}\Omega, \\ \partial_n W(\mathbf{x}) = {}_*\widetilde{P}^1_{\mathbf{N}_B,\partial^*\Omega}G_1(\mathbf{x}), & \text{on } \partial^*\Omega, \\ W(\mathbf{x}) = {}_*\widetilde{P}^2_{\mathbf{N}_B,\partial\Omega}G_0(\mathbf{x}), & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

We have from (4.1) and (4.7) that

$$\begin{cases} \Delta^2(U(\mathbf{x}) - W(\mathbf{x})) + d(U(\mathbf{x}) - W(\mathbf{x})) = 0, & \text{in } \Omega, \\ \Delta(U(\mathbf{x}) - W(\mathbf{x})) + \beta\partial_n(U(\mathbf{x}) - W(\mathbf{x})) = 0, & \text{on } \partial^{**}\Omega, \\ \partial_n(U(\mathbf{x}) - W(\mathbf{x})) = G_1(\mathbf{x}) - {}_*\widetilde{P}^1_{\mathbf{N}_B,\partial^*\Omega} G_1(\mathbf{x}), & \text{on } \partial^*\Omega, \\ (U(\mathbf{x}) - W(\mathbf{x})) = G_0(\mathbf{x}) - {}_*\widetilde{P}^2_{\mathbf{N}_B,\partial\Omega} G_0(\mathbf{x}), & \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

According to the property of elliptic equation, we derive that

$$\begin{aligned} \|U - W\|_{H^2(\Omega)} &\le c(\|G_0 - {}_*\widetilde{P}^2_{\mathbf{N}_B,\partial\Omega} G_0\|_{H^{\frac{3}{2}}(\partial\Omega)} + \|G_1 - {}_*\widetilde{P}^1_{\mathbf{N}_B,\partial^*\Omega} G_1\|_{H^{\frac{1}{2}}(\partial^*\Omega)}) \\ &\le c(\|G_0 - {}_*\widetilde{P}^2_{\mathbf{N}_B,\partial\Omega} G_0\|^{\frac{1}{2}}_{H^1(\partial\Omega)} \|G_0 - {}_*\widetilde{P}^2_{\mathbf{N}_B,\partial\Omega} G_0\|^{\frac{1}{2}}_{H^2(\partial\Omega)} \\ &\quad + \|G_1 - {}_*\widetilde{P}^1_{\mathbf{N}_B,\partial^*\Omega} G_1\|^{\frac{1}{2}}_{L^2(\partial^*\Omega)} \|G_1 - {}_*\widetilde{P}^1_{\mathbf{N}_B,\partial^*\Omega} G_1\|^{\frac{1}{2}}_{H^1(\partial^*\Omega)}). \end{aligned} \tag{4.9}$$

For simplicity of statements, we introduce the following quantities,

$$\begin{aligned} K^{(1)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega_k}(v) &= \sum_{\substack{L_{k,\nu}\subset\partial\Omega_k \\ \nu=1,3}} (\frac{h_{k,1}}{N_{B,k,\nu}})^{r_{B,k,\nu}-\frac{3}{2}} \|\partial_{x_1}^{r_{B,k,\nu}} v\|_{\chi_{k,1}^{(r_{B,k,\nu}-2,r_{B,k,\nu}-2)},I_{k,1}} \\ &\quad + \sum_{\substack{L_{k,\nu}\subset\partial\Omega_k \\ \nu=2,4}} (\frac{h_{k,2}}{N_{B,k,\nu}})^{r_{B,k,\nu}-\frac{3}{2}} \|\partial_{x_2}^{r_{B,k,\nu}} v\|_{\chi_{k,2}^{(r_{B,k,\nu}-2,r_{B,k,\nu}-2)},I_{k,2}}, \\ K^{(2)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial^*\Omega_k}(v) &= \sum_{\substack{L_{k,\nu}\subset\partial^*\Omega_k \\ \nu=1,3}} (\frac{h_{k,1}}{N_{B,k,\nu}})^{r_{B,k,\nu}-\frac{1}{2}} \|\partial_{x_1}^{r_{B,k,\nu}} v\|_{\chi_{k,1}^{(r_{B,k,\nu}-2,r_{B,k,\nu}-2)},I_{k,1}} \\ &\quad + \sum_{\substack{L_{k,\nu}\subset\partial^*\Omega_k \\ \nu=2,4}} (\frac{h_{k,2}}{N_{B,k,\nu}})^{r_{B,k,\nu}-\frac{1}{2}} \|\partial_{x_2}^{r_{B,k,\nu}} v\|_{\chi_{k,2}^{(r_{B,k,\nu}-2,r_{B,k,\nu}-2)},I_{k,2}}. \end{aligned}$$

Then, with the aid of (2.3), (2.11), the inequality (4.9) implies

$$\|U - W\|_{H^2(\Omega)} \le c \sum_{k=1}^{M} (K^{(1)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega_k}(G_0) + K^{(2)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial^*\Omega_k}(G_1)). \tag{4.10}$$

Remark 4.1 As a special case, we may take $r_{B,k} = r_{B,k,1} = r_{B,k,2} = r_{B,k,3} = r_{B,k,4}$, $h = h_{k,1} = h_{k,2}$, and $N_B = N_{B,k,1} = N_{B,k,2} = N_{B,k,3} = N_{B,k,4}$ for $1 \le k \le M$. If we ignore the weights appearing in all norms, then (4.10) implies

$$\|U - W\|_{H^2(\Omega)} \le c \sum_{k=1}^{M} (\frac{h}{N_B})^{r_{B,k}-\frac{3}{2}} \|\partial_\tau^{r_{B,k}} G_0\|_{\partial\Omega_k} + c \sum_{k=1}^{M} (\frac{h}{N_B})^{r_{B,k}-\frac{1}{2}} \|\partial_\tau^{r_{B,k}} G_1\|_{\partial^*\Omega_k}.$$

If $r = r_{k,1} = r_{k,2}$, $N_{k,1} = N_{B,k,1} = N_{B,k,3}$ and $N_{k,2} = N_{B,k,2} = N_{B,k,4}$ for $1 \le k \le M$, then we have

$$\begin{aligned} \|U - W\|_{H^2(\Omega)} &\le c \sum_{k=1}^{M} (\frac{h_{k,1}}{N_{k,1}} + \frac{h_{k,2}}{N_{k,2}})^{r_{B,k}-\frac{3}{2}} \|\partial_\tau^{r_{B,k}} G_0\|_{\partial\Omega_k} \\ &\quad + c \sum_{k=1}^{M} (\frac{h_{k,1}}{N_{k,1}} + \frac{h_{k,2}}{N_{k,2}})^{r_{B,k}-\frac{1}{2}} \|\partial_\tau^{r_{B,k}} G_1\|_{\partial^*\Omega_k}. \end{aligned}$$

We are going to design the spectral element scheme with non-uniform meshes and non-uniform modes. We need five kinds of base functions. To do this, let $L_l(x)$ be the standard Legendre polynomials, and

$$\begin{aligned}
\phi_l^{(0)}(x) &= \tfrac{1}{\sqrt{2(2l+3)^2(2l+5)}}(L_l(x) - \tfrac{2(2l+5)}{2l+7}L_{l+2}(x) + \tfrac{2l+3}{2l+7}L_{l+4}(x)),\\
\phi_l^{(1)}(x) &= -\tfrac{\sqrt{2}}{2(l+2)(2l+3)}(L_l(x) - \tfrac{2l+3}{2l+5}L_{l+1}(x) - L_{l+2}(x) + \tfrac{2l+3}{2l+5}L_{l+3}(x)),\\
\phi_l^{(2)}(x) &= -\tfrac{\sqrt{2}}{2(l+2)(2l+3)}(L_l(x) + \tfrac{2l+3}{2l+5}L_{l+1}(x) - L_{l+2}(x) - \tfrac{2l+3}{2l+5}L_{l+3}(x)), \qquad l \geq 0.
\end{aligned}$$

It can be checked that

$$\phi_l^{(\mu)}(\pm 1) = \partial_x \phi_l^{(0)}(\pm 1) = \partial_x \phi_l^{(1)}(1) = \partial_x \phi_l^{(2)}(-1) = 0, \quad \|\partial_x^2 \phi_l^{(\mu)}\|_{L^2(-1,1)} = 1, \quad \mu = 0, 1, 2.$$

We also use the notations $f_{k,i}^-(x_i)$, $f_{k,i}^+(x_i)$, $g_{k,i}^-(x_i)$ and $g_{k,i}^+(x_i)$ as in the last section.

The first kind of base functions correspond to the subdomains Ω_k, $1 \leq k \leq M$. If $\partial^{**}\Omega_k = \emptyset$, then we define the base functions as follows,

$$\psi_{\Omega_k,l,l'}(\mathbf{x}) = \begin{cases} \phi_l^{(0)}(z_{k,1}(x_1))\phi_{l'}^{(0)}(z_{k,2}(x_2)), & \mathbf{x} \in \Omega_k, \\ 0, & \text{otherwise}. \end{cases}$$

This function and its normal derivative vanish at the edges of Ω_k. If $\partial^{**}\Omega_k = L_{k,1}$, on which we impose Neumann or Robin boundary conditions, then we define the base function

$$\psi_{\Omega_k,l,l'}(\mathbf{x}) = \begin{cases} \phi_l^{(0)}(z_{k,1}(x_1))\phi_{l'}^{(1)}(z_{k,2}(x_2)), & \mathbf{x} \in \Omega_k, \\ 0, & \text{otherwise}. \end{cases}$$

This function vanish at all edges of Ω_k. Moreover, its normal derivative vanish at the edges $L_{k,\nu}$, $\nu = 2, 3, 4$. If $\partial^{**}\Omega_k = L_{k,1} \cup L_{k,2}$, then the corresponding base function

$$\psi_{\Omega_k,l,l'}(\mathbf{x}) = \begin{cases} \phi_l^{(2)}(z_{k,1}(x_1))\phi_{l'}^{(1)}(z_{k,2}(x_2)), & \mathbf{x} \in \Omega_k, \\ 0, & \text{otherwise}. \end{cases}$$

Similarly, we can define the base functions for other kinds of Ω_k.

We now number the edges of all subdomains as E_s, $1 \leq s \leq S$. The second kind of base functions correspond to the common edges of adjacent subdomains Ω_{k_1} and Ω_{k_2}. For instance, if $E_s = L_{k_1,2} = L_{k_2,4}$, then we define the corresponding base functions as follows,

$$\psi_{E_s,l}^{(0)}(\mathbf{x}) = \begin{cases} f_{k_1,1}^+(x_1)\phi_l^{(0)}(z_{k_1,2}(x_2)), & \mathbf{x} \in \Omega_{k_1}, \\ f_{k_2,1}^-(x_1)\phi_l^{(0)}(z_{k_2,2}(x_2)), & \mathbf{x} \in \Omega_{k_2}, \\ 0, & \text{otherwise}. \end{cases}$$
$$\psi_{E_s,l}^{(1)}(\mathbf{x}) = \begin{cases} g_{k_1,1}^+(x_1)\phi_l^{(0)}(z_{k_1,2}(x_2)), & \mathbf{x} \in \Omega_{k_1}, \\ g_{k_2,1}^-(x_1)\phi_l^{(0)}(z_{k_2,2}(x_2)), & \mathbf{x} \in \Omega_{k_2}, \\ 0, & \text{otherwise}. \end{cases}$$

They ensure the continuity of numerical solution and its normal derivative at the common edges of adjacent subdomains.

The third kind of base functions correspond to the edges lying on the boundary $\partial\Omega$. For instance, if $E_s = L_{k,1} \subset \partial^*\Omega$, then the corresponding base functions are as follows,

$$\psi^{(0)}_{E_s,l}(\mathbf{x}) = \begin{cases} \phi^{(0)}_l(z_{k,1}(x_1)) f^-_{k,2}(x_2), & \mathbf{x} \in \Omega_k, \\ 0, & \text{otherwise.} \end{cases}$$
$$\psi^{(1)}_{E_s,l}(\mathbf{x}) = \begin{cases} \phi^{(0)}_l(z_{k,1}(x_1)) g^-_{k,2}(x_2), & \mathbf{x} \in \Omega_k, \\ 0, & \text{otherwise.} \end{cases}$$

If $E_s = L_{k,1} \subset \partial^{**}\Omega$, then the corresponding base functions are as follows,

$$\psi^{(0)}_{E_s,l}(\mathbf{x}) = \begin{cases} \phi^{(0)}_l(z_{k,1}(x_1)) f^-_{k,2}(x_2), & \mathbf{x} \in \Omega_k, \\ 0, & \text{otherwise.} \end{cases} \qquad \psi^{(1)}_{E_s,l}(\mathbf{x}) = 0.$$

They also ensure the continuity of numerical solution and its normal derivative at the common edges of adjacent subdomains.

We next number the vertices of all subdomains as $V_j,\ 1 \le j \le J$. The fourth kind of base functions correspond to the common vertices of four adjacent subdomains $\Omega_{k_1}, \Omega_{k_2}, \Omega_{k_3}$ and Ω_{k_4}. If $V_j = Q_{k_1,3} = Q_{k_2,4} = Q_{k_3,1} = Q_{k_4,2}$, then we define the corresponding base functions as follows,

$$\psi^{(0)}_{V_j}(\mathbf{x}) = \begin{cases} f^+_{k_1,1}(x_1) f^+_{k_1,2}(x_2), & \text{in } \Omega_{k_1}, \\ f^-_{k_2,1}(x_1) f^+_{k_2,2}(x_2), & \text{in } \Omega_{k_2}, \\ f^-_{k_3,1}(x_1) f^-_{k_3,2}(x_2), & \text{in } \Omega_{k_3}, \\ f^+_{k_4,1}(x_1) f^-_{k_4,2}(x_2), & \text{in } \Omega_{k_4}, \\ 0, & \text{otherwise.} \end{cases} \qquad \psi^{(1)}_{V_j}(\mathbf{x}) = \begin{cases} g^+_{k_1,1}(x_1) f^+_{k_1,2}(x_2), & \text{in } \Omega_{k_1}, \\ g^-_{k_2,1}(x_1) f^+_{k_2,2}(x_2), & \text{in } \Omega_{k_2}, \\ g^-_{k_3,1}(x_1) f^-_{k_3,2}(x_2), & \text{in } \Omega_{k_3}, \\ g^+_{k_4,1}(x_1) f^-_{k_4,2}(x_2), & \text{in } \Omega_{k_4}, \\ 0, & \text{otherwise.} \end{cases}$$
$$\psi^{(2)}_{V_j}(\mathbf{x}) = \begin{cases} f^+_{k_1,1}(x_1) g^+_{k_1,2}(x_2), & \text{in } \Omega_{k_1}, \\ f^-_{k_2,1}(x_1) g^+_{k_2,2}(x_2), & \text{in } \Omega_{k_2}, \\ f^-_{k_3,1}(x_1) g^-_{k_3,2}(x_2), & \text{in } \Omega_{k_3}, \\ f^+_{k_4,1}(x_1) g^-_{k_4,2}(x_2), & \text{in } \Omega_{k_4}, \\ 0, & \text{otherwise.} \end{cases} \qquad \psi^{(3)}_{V_j}(\mathbf{x}) = \begin{cases} g^+_{k_1,1}(x_1) g^+_{k_1,2}(x_2), & \text{in } \Omega_{k_1}, \\ g^-_{k_2,1}(x_1) g^+_{k_2,2}(x_2), & \text{in } \Omega_{k_2}, \\ g^-_{k_3,1}(x_1) g^-_{k_3,2}(x_2), & \text{in } \Omega_{k_3}, \\ g^+_{k_4,1}(x_1) g^-_{k_4,2}(x_2), & \text{in } \Omega_{k_4}, \\ 0, & \text{otherwise.} \end{cases}$$

They ensure the continuity of numerical solution and its derivatives of first order.

The fifth kind of base functions correspond to the vertices lying on $\partial\Omega$. For instance, if $V_j = L_{k_1,1} \cap L_{k_2,1} \in \partial^*\Omega$, then we define the corresponding base functions as follows,

$$\psi^{(0)}_{V_j}(\mathbf{x}) = \begin{cases} f^+_{k_1,1}(x_1) f^-_{k_1,2}(x_2), & \text{in } \Omega_{k_1}, \\ f^-_{k_2,1}(x_1) f^-_{k_2,2}(x_2), & \text{in } \Omega_{k_2}, \\ 0, & \text{otherwise.} \end{cases} \qquad \psi^{(1)}_{V_j}(\mathbf{x}) = \begin{cases} g^+_{k_1,1}(x_1) f^-_{k_1,2}(x_2), & \text{in } \Omega_{k_1}, \\ g^-_{k_2,1}(x_1) f^-_{k_2,2}(x_2), & \text{in } \Omega_{k_2}, \\ 0, & \text{otherwise.} \end{cases}$$
$$\psi^{(2)}_{V_j}(\mathbf{x}) = \begin{cases} f^+_{k_1,1}(x_1) g^-_{k_1,2}(x_2), & \text{in } \Omega_{k_1}, \\ f^-_{k_2,1}(x_1) g^-_{k_2,2}(x_2), & \text{in } \Omega_{k_2}, \\ 0, & \text{otherwise.} \end{cases} \qquad \psi^{(3)}_{V_j}(\mathbf{x}) = \begin{cases} g^+_{k_1,1}(x_1) g^-_{k_1,2}(x_2), & \text{in } \Omega_{k_1}, \\ g^-_{k_2,1}(x_1) g^-_{k_2,2}(x_2), & \text{in } \Omega_{k_2}, \\ 0, & \text{otherwise.} \end{cases}$$

Further, we can define the base functions corresponding the corners of the domain Ω. For instance, if the corner $V_j = Q_{k,1} \in \partial^*\Omega$, then the base functions are as follows,

$$\psi^{(0)}_{V_j}(\mathbf{x}) = \begin{cases} f^-_{k,1}(x_1) f^-_{k,2}(x_2), & \text{in } \Omega_k, \\ 0, & \text{otherwise.} \end{cases} \quad \psi^{(1)}_{V_j}(\mathbf{x}) = \begin{cases} g^-_{k,1}(x_1) f^-_{k,2}(x_2), & \text{in } \Omega_k, \\ 0, & \text{otherwise.} \end{cases}$$
$$\psi^{(2)}_{V_j}(\mathbf{x}) = \begin{cases} f^-_{k,1}(x_1) g^-_{k,2}(x_2), & \text{in } \Omega_k, \\ 0, & \text{otherwise.} \end{cases} \quad \psi^{(3)}_{V_j}(\mathbf{x}) = \begin{cases} g^-_{k,1}(x_1) g^-_{k,2}(x_2), & \text{in } \Omega_k, \\ 0, & \text{otherwise.} \end{cases}$$

Now, let $\mathcal{P}_{\mathbf{N},\mathbf{N}_B}(\Omega)$ be the finite-dimensional set spanned by the following base functions,

$$\begin{array}{ll} \psi_{\Omega_k,l,l'}(\mathbf{x}), & 0 \le l \le N_{k,1}-4, \quad 0 \le l' \le N_{k,2}-4, \quad 1 \le k \le M, \\ \psi^{(\mu)}_{E_s,l}(\mathbf{x}), & 0 \le l \le N_s, \quad \mu = 0,1, \quad 1 \le s \le S, \\ \psi^{(\mu)}_{V_j}(\mathbf{x}), & 0 \le \mu \le 3, \quad 1 \le j \le J. \end{array}$$

In addition, if $E_s = L_{k_1,1} = L_{k_2,3}$, then $N_s = N_{B,k_1,1} = N_{B,k_2,3}$, etc.. Furthermore,

$$V_{\mathbf{N},\mathbf{N}_B}(\Omega) = \mathcal{P}_{\mathbf{N},\mathbf{N}_B}(\Omega) \cap V^*(\Omega), \qquad \overline{V}_{\mathbf{N},\mathbf{N}_B}(\Omega) = \mathcal{P}_{\mathbf{N},\mathbf{N}_B}(\Omega) \cap \overline{V}(\Omega).$$

The spectral scheme for (4.6) is to find $w_{\mathbf{N},\mathbf{N}_B} \in V_{\mathbf{N},\mathbf{N}_B}(\Omega)$ such that

$$a_{d,\beta}(w_{\mathbf{N},\mathbf{N}_B}, \phi) = (F,\phi)_\Omega + \int_{\partial^{**}\Omega} G_2(\mathbf{x})\partial_n\phi(\mathbf{x})ds, \quad \forall \phi \in \overline{V}_{\mathbf{N},\mathbf{N}_B}(\Omega). \tag{4.11}$$

We now deal with the convergence of (4.11). To do this, we introduce the operator $\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega} : V^*(\Omega) \to V_{\mathbf{N},\mathbf{N}_B}(\Omega)$, defined by

$$a_{d,\beta}(\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}v - v, \phi) = 0, \qquad \forall \phi \in \overline{V}_{\mathbf{N},\mathbf{N}_B}(\Omega). \tag{4.12}$$

The following two propositions play important roles in the error estimates.

Proposition 4.1 *For any $v \in V^*(\Omega)$ and $w \in V_{\mathbf{N},\mathbf{N}_B}(\Omega)$, we have*

$$a_{d,\beta}(v - \overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}v, v - \overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}v) \le a_{d,\beta}(v-w, v-w). \tag{4.13}$$

Proposition 4.2 *If Ω is a union of rectangles and $v \in \overline{V}(\Omega)$, then $\|\Delta v\|_\Omega = |v|_{H^2(\Omega)}$.*

Now, we use (4.6) and (4.12) to obtain

$$a_{d,\beta}(\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}W, \phi) = (F,\phi)_\Omega + \int_{\partial^{**}\Omega} G_2(\mathbf{x})\partial_n\phi(\mathbf{x})ds, \qquad \forall \phi \in \overline{V}_{\mathbf{N},\mathbf{N}_B}(\Omega).$$

Subtracting the above equation from (4.11) yields

$$a_{d,\beta}(w_{\mathbf{N},\mathbf{N}_B} - \overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}W, \phi) = 0, \qquad \forall \phi \in \overline{V}_{\mathbf{N},\mathbf{N}_B}(\Omega). \tag{4.14}$$

This implies $w_{\mathbf{N},\mathbf{N}_B} = \overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}W$.

Clearly, $w_{\mathbf{N},\mathbf{N}_B}(\mathbf{x}) - W(\mathbf{x}) = 0$ on $\partial\Omega$, and $\partial_n(w_{\mathbf{N},\mathbf{N}_B}(\mathbf{x}) - W(\mathbf{x})) = 0$ on $\partial^*\Omega$. Thus, by Proposition 4.2,

$$\|\Delta(w_{\mathbf{N},\mathbf{N}_B} - W)\|_\Omega = |w_{\mathbf{N},\mathbf{N}_B} - W|_{H^2(\Omega)}. \tag{4.15}$$

Besides, ${}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}v \in V_{\mathbf{N},\mathbf{N}_B}(\Omega)$. Therefore, we use (4.15), (4.13) with $v = W$ and $w = {}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}U$, (3.39) with $v = U$, and (4.10) successively, to derive that if $d \ge 0$, $4 \le r_{k,i} \le N_{k,i}+1$ and $3 \le r_{B,k} \le N_{B,k,\nu}+1$ for $1 \le k \le M$, $i = 1,2$ and $1 \le \nu \le 4$, then

$$\begin{aligned} |w_{\mathbf{N},\mathbf{N}_B} - U|^2_{H^2(\Omega)} &\le 2|w_{\mathbf{N},\mathbf{N}_B} - W|^2_{H^2(\Omega)} + 2|U-W|^2_{H^2(\Omega)} \\ &\le 2a_{d,\beta}(w_{\mathbf{N},\mathbf{N}_B} - W, w_{\mathbf{N},\mathbf{N}_B} - W) + 2|U-W|^2_{H^2(\Omega)} \\ &= 2a_{d,\beta}(\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}W - W, \overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}W - W) + 2|U-W|^2_{H^2(\Omega)} \\ &\le 2a_{d,\beta}({}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}U - W, {}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}U - W) + 2|U-W|^2_{H^2(\Omega)} \\ &\le 4a_{d,\beta}({}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}U - U, {}_*P^2_{\mathbf{N},\mathbf{N}_B,\Omega}U - U) + 4a_{d,\beta}(W-U, W-U) + 2|U-W|^2_{H^2(\Omega)}, \end{aligned}$$

whence

$$|w_{\mathbf{N},\mathbf{N}_B}-U|_{H^2(\Omega)}\le c\sum_{k=1}^{M}(A^{(2)}_{\mathbf{r}_k,\mathbf{h}_k,\mathbf{N}_k,\Omega_k}(U)+\widetilde{B}^{(2)}_{r_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega_k}(U)+\widetilde{B}^{(2)}_{r_{B,k},\mathbf{h}_k,\mathbf{N}_k,\partial\Omega_k}(U)$$
$$+K^{(1)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega}(G_0)+K^{(2)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial^*\Omega}(G_1)). \tag{4.16}$$

If, in addition, $d>0$, then

$$\begin{aligned}\|w_{\mathbf{N},\mathbf{N}_B}-U\|^2_{H^2(\Omega)}&\le 2\|w_{\mathbf{N},\mathbf{N}_B}-W\|^2_{H^2(\Omega)}+2\|U-W\|^2_{H^2(\Omega)}\\&\le 2a_{d,\beta}(w_{\mathbf{N},\mathbf{N}_B}-W,w_{\mathbf{N},\mathbf{N}_B}-W)+2\|U-W\|^2_{H^2(\Omega)},\end{aligned}$$

whence

$$\|w_{\mathbf{N},\mathbf{N}_B}-U\|_{H^2(\Omega)}\le c\sum_{k=1}^{M}(A^{(2)}_{\mathbf{r}_k,\mathbf{h}_k,\mathbf{N}_k,\Omega_k}(U)+\widetilde{B}^{(2)}_{r_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega_k}(U)+\widetilde{B}^{(2)}_{r_{B,k},\mathbf{h}_k,\mathbf{N}_k,\partial\Omega_k}(U)$$
$$+K^{(1)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial\Omega}(G_0)+K^{(2)}_{\mathbf{r}_{B,k},\mathbf{h}_k,\mathbf{N}_{B,k},\partial^*\Omega}(G_1)). \tag{4.17}$$

Remark 4.2 If we ignore the weight functions appearing in the right sides of (4.16), and $r_k=r_{k,1}=r_{k,2}, r_{B,k}=r_{B,k,1}=r_{B,k,2}=r_{B,k,3}=r_{B,k,4}, h=h_{k,1}=h_{k,2}, N_k=N_{k,1}=N_{k,2}$, $N_B=N_{B,k,1}=N_{B,k,2}=N_{B,k,3}=N_{B,k,4}$ for $1\le k\le M$, then with the help of Remarks 3.9 and 4.1, we assert that for $d\ge 0$,

$$\begin{aligned}|w_{\mathbf{N},\mathbf{N}_B}-U|^2_{H^2(\Omega)}\le{}& c\sum_{k=1}^{M}(\frac{h}{N_k})^{r_k-2}|U|_{H^{r_k}(\Omega_k)}\\&+c\sum_{k=1}^{M}(\frac{h^{r_{B,k}-\frac{3}{2}}}{N_k^{r_{B,k}-3}}+\frac{h^{r_{B,k}-\frac{3}{2}}}{N_B^{r_{B,k}-3}})(\|\partial_\tau^{r_{B,k}}U\|_{\partial\Omega_k}+\|\partial_\tau^{r_{B,k}-1}\partial_nU\|_{\partial\Omega_k})\\&+c\sum_{k=1}^{M}(\frac{h}{N_B})^{r_{B,k}-\frac{3}{2}}\|\partial_\tau^{r_{B,k}}G_0\|_{\partial\Omega_k}+c\sum_{k=1}^{M}(\frac{h}{N_B})^{r_{B,k}-\frac{1}{2}}\|\partial_\tau^{r_{B,k}}G_1\|_{\partial^*\Omega_k}.\end{aligned} \tag{4.18}$$

If $r_k=r_{k,1}=r_{k,2}, N_{k,1}=N_{B,k,1}=N_{B,k,3}$ and $N_{k,2}=N_{B,k,2}=N_{B,k,4}$ for $1\le k\le M$, then we have form (4.16) that for $d\ge 0$,

$$\begin{aligned}|w_{\mathbf{N},\mathbf{N}_B}-U|^2_{H^2(\Omega)}\le{}& c\sum_{k=1}^{M}(\frac{h_{k,1}}{N_{k,1}}+\frac{h_{k,2}}{N_{k,2}})^{r_k-2}|U|_{H^{r_k}(\Omega_k)}\\+c\sum_{k=1}^{M}(\frac{h_{k,1}}{N_{k,1}}+\frac{h_{k,1}}{N_{k,1}})^{r_{B,k}-\frac{3}{2}}&\|\partial_\tau^{r_{B,k}}G_0\|_{\partial\Omega_k}+c\sum_{k=1}^{M}(\frac{h_{k,1}}{N_{k,1}}+\frac{h_{k,1}}{N_{k,1}})^{r_{B,k}-\frac{1}{2}}\|\partial_\tau^{r_{B,k}}G_1\|_{\partial^*\Omega_k}.\end{aligned} \tag{4.19}$$

If, in addition, $d>0$, then the above two estimates are also valid for $||w_{\mathbf{N},\mathbf{N}_B}-U||^2_{H^2(\Omega)}$.

Remark 4.3 In the norms involved at the right sides of the error estimates (4.16) and (4.17), there exist some weights tending to zero as the points go to the corners of elements. It is useful for covering the singularities of solutions at the corners of the boundary of considered domain.

Remark 4.4 Since we are allowed to use different mesh sizes and different approximation modes in different elements, at different common edges of adjacent elements and in different directions independently, with a weak restriction, the proposed scheme is very convenient for local mesh refinements and local mode increments.

Remark 4.5 In order to compare the above results with the corresponding results of finite element method, we take the same mesh size and the same approximation mode for all elements and ignore all weights appearing in the error estimates, then we obtain from (4.16) with trace theorem that

$$\begin{aligned}|w_{\mathbf{N},\mathbf{N}_B} - U|^2_{H^2(\Omega)} &\le c(\tfrac{h}{N})^{r-2}(|U|_{H^r(\Omega)} + \|\partial_\tau^{r-\frac{1}{2}} G_0\|_{\partial\Omega} + \|\partial_\tau^{r-\frac{3}{2}} G_1\|_{\partial^*\Omega})\\ &\le c(\tfrac{h}{N})^{r-2}|U|_{H^r(\Omega)}.\end{aligned}$$

For fixed mode N, we have

$$|w_{\mathbf{N},\mathbf{N}_B} - U|^2_{H^2(\Omega)} \le ch^{r-2}|U|_{H^r(\Omega)}.$$

If, in addition, $d \ge 0$, $\partial^*\Omega = \partial\Omega$ or $d > 0$, $\partial^{**}\Omega \ne \emptyset$, then by embedding theorem,

$$||w_{\mathbf{N},\mathbf{N}_B} - U||^2_{H^2(\Omega)} \le ch^{r-2}|U|_{H^r(\Omega)}.$$

In fact, the above error estimate with $N = 3$ and $r = 4$ is just the same as the corresponding result of high order finite element method for problem (4.1) with the rectangle Ω, $d = 0$, $\partial^*\Omega = \partial\Omega$ and $G_0(x, y) = G_1(x, y) \equiv 0$, given by Chen [8], which could be regarded as pseudospectral element method. We also refer to the similar results with $r = 5$, see page 358 of [8] and Zlamal [27]. Whereas, our new method is a spectral element method and so different from the methods of [8,27]. Moreover, the results (4.16) and (4.17) are valid for all $4 \le r \le N + 1$.

Remark 4.6 We could use an duality argument to derive the optimal estimate of $\|w_{\mathbf{N},\mathbf{N}_B} - U\|_{L^2(\Omega)}$. For simplicity of statements, we focus on the case with $G_0(x, y) = G_1(x, y) \equiv 0$. Accordingly, $V(\Omega) = V^*(\Omega) = \overline{V}(\Omega)$, $V_{\mathbf{N},\mathbf{N}_B}(\Omega) = \overline{V}_{\mathbf{N},\mathbf{N}_B}(\Omega)$ and $W = U$. Let $g \in L^2(\Omega)$. We consider an auxiliary problem. It is to find $\eta \in V(\Omega)$ such that

$$a_{d,\beta}(\eta, z) = (g, z)_\Omega, \qquad \forall z \in V(\Omega). \tag{4.20}$$

By taking $z = \eta$ in (4.20), we use the Poincaré inequality to assert that $||\eta||_{H^2(\Omega)} \le ||g||_\Omega$. Furthermore, by taking $z = w_{\mathbf{N},\mathbf{N}_B} - U$ in (4.20), we obtain

$$a_{d,\beta}(\eta, w_{\mathbf{N},\mathbf{N}_B} - U) = (g, w_{\mathbf{N},\mathbf{N}_B} - U)_\Omega.$$

It was shown in (4.14) that $w_{\mathbf{N},\mathbf{N}_B} = \overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}U \in \overline{V}_{\mathbf{N},\mathbf{N}_B}(\Omega)$. Thereby, we use (4.12) with $v = U$ and $\phi = \overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}\eta$ to deduce that

$$a_{d,\beta}(\overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}\eta, w_{\mathbf{N},\mathbf{N}_B} - U) = 0.$$

A combination of the above two equalities leads to

$$a_{d,\beta}(\eta - \overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}\eta, w_{\mathbf{N},\mathbf{N}_B} - U) = (g, w_{\mathbf{N},\mathbf{N}_B} - U)_\Omega.$$

If $r_k = r_{k,1} = r_{k,2}$, $N_{k,1} = N_{B,k,1} = N_{B,k,3}$ and $N_{k,2} = N_{B,k,2} = N_{B,k,4}$ for $1 \le k \le M$, then we have from (4.19) that

$$\begin{aligned}|(g, w_{\mathbf{N},\mathbf{N}_B} - U)_\Omega| &\le c\|w - \overline{P}^2_{\mathbf{N},\mathbf{N}_B,\Omega}w\|_{H^2(\Omega)}\|w_{\mathbf{N},\mathbf{N}_B} - U\|_{H^2(\Omega)}\\ &\le c\sum_{k=1}^{M}(\frac{h_{k,1}}{N_{k,1}} + \frac{h_{k,2}}{N_{k,2}})^{r_k}|U|_{H^{r_k}(\Omega_k)}|\eta|_{H^4(\Omega_k)}.\end{aligned}$$

If, in addition, $\partial\Omega = \partial^*\Omega$ or $d > 0$, then by virtue of the property of elliptic equation, there exists $\zeta_\Omega > 0$, such that $\|\eta\|_{H^4(\Omega)} \le \zeta_\Omega \|g\|_{L^2(\Omega)}$ (cf. [9]). Consequently, we verify that for $r \ge 4$,

$$\|w_{\mathbf{N},\mathbf{N}_B} - U\|_{L^2(\Omega)} = \sup_{g\in L^2(\Omega), g\neq 0} \frac{|(g, w_{\mathbf{N},\mathbf{N}_B} - U)_\Omega|}{\|g\|_{L^2(\Omega)}} \le c\zeta_\Omega \sum_{k=1}^{M} (\frac{h_{k,1}}{N_{k,1}} + \frac{h_{k,2}}{N_{k,2}})^{r_k} |U|_{H^{r_k}(\Omega_k)}.$$

5 Numerical Results

In this section, we present some numerical results. In actual computations, we expand the numerical solution of (4.11) as

$$\begin{aligned} w_{\mathbf{N},\mathbf{N}_B}(\mathbf{x}) = & \sum_{k=1}^{M} \sum_{l'=0}^{N_{k,2}-4} \sum_{l=0}^{N_{k,1}-4} a_{k,l,l'} \psi_{\Omega_k,l,l'}(\mathbf{x}) + \sum_{s=1}^{S} \sum_{\mu=0}^{1} \sum_{l=0}^{N_s} b_{s,\mu,l} \psi^{(\mu)}_{E_s,l}(\mathbf{x}) \\ & + \sum_{j=1}^{J} \sum_{\mu=0}^{3} c_{j,\mu} \psi^{(\mu)}_{V_j}(\mathbf{x}). \end{aligned} \tag{5.1}$$

In addition, if the edges E_s or the vertices V_j are located on the boundary $\partial\Omega$, then the corresponding coefficients $b_{s,\mu,l}$ and $c_{j,\mu}$ are determined by the boundary conditions $w_{\mathbf{N},\mathbf{N}_B}(\mathbf{x})(\mathbf{x}) = {}_*\widetilde{P}^2_{\mathbf{N}_B,\partial\Omega} G_0(\mathbf{x})$ on $\partial\Omega$, and $\partial_n w_{\mathbf{N},\mathbf{N}_B}(\mathbf{x}) = {}_*\widetilde{P}^1_{\mathbf{N}_B,\partial^*\Omega} G_1(\mathbf{x})$ on $\partial^*\Omega$. By substituting (5.1) into (4.11), we obtain a system of algebra equations with the unknown coefficients $a_{k,l,l'}$, $b_{s,\mu,l}$ and $c_{j,\mu}$.

Now, let $\xi_{N,l}$ $(0 \le l \le N)$ be the zeros of the Legendre polynomial $L_{N+1}(\xi)$. Meanwhile, $\omega_{N,l}$ $(0 \le l \le N)$ stand for the corresponding Christoffel numbers of the Legendre-Gauss interpolation. Furthermore, $x^{(k,i)}_{N,l} = \frac{1}{2}(h_{k,i}\xi_{N,l} + b_{k,i} - a_{k,i})$, $i = 1, 2$. The errors of numerical solutions are measured by the discrete average errors

$$E_{ave,N} = (\sum_{k=1}^{M} \sum_{l'=0}^{N_{k,2}-4} \sum_{l=0}^{N_{k,1}-4} (U(x^{(k,1)}_{N,l}, x^{(k,2)}_{N,l'}) - w_{\mathbf{N},\mathbf{N}_B}(x^{(k,1)}_{N,l}, x^{(k,2)}_{N,l'}))^2 \omega_{N,l}\omega_{N,l'})^{\frac{1}{2}},$$

and the discrete maximum errors

$$E_{max,N} = \max_{1\le k\le M} \max_{0\le l'\le N_{k,2}-4} \max_{0\le l\le N_{k,1}-4} |U(x^{(k,1)}_{N,l}, x^{(k,2)}_{N,l'}) - w_{\mathbf{N},\mathbf{N}_B}(x^{(k,1)}_{N,l}, x^{(k,2)}_{N,l'})|.$$

In actual computation, we take the concave domain $\Omega = \Omega_A \cup \Omega_B$ with $\Omega_A = \{\, \mathbf{x} \mid -1 < x_1 \le 0.5, \; -1 < x_2 < 1 \,\}$ and $\Omega_B = \{\, \mathbf{x} \mid 0.5 < x_1 < 1, \; -0.5 < x_2 < 0.5 \,\}$.

We use (4.11) to solve problem (4.6) with $d = \beta = 0$ and the test function

$$U(x_1, x_2) = \sqrt{x_1^2 + x_2^2 + 1}\cos(\nu(x_1 + x_2)), \tag{5.2}$$

which oscillates seriously for large ν. We first use the uniform partition of domain Ω, with the mesh size $h_{k,1} = h_{k,2} = h$ for $1 \le k \le M$. In Table 1, we list the discrete errors $E_{ave,N}$ and $E_{max,N}$, with $\nu = 20$. Clearly, the numerical errors decay rapidly as N increases and h decreases. This coincides the analysis well. In Table 2, we list the discrete errors $E_{ave,N}$ and $E_{max,N}$, with $N = 15$ and $\nu = 40$. The spectral scheme (4.11) still works well even for the solutions oscillating seriously.

Table 1 The numerical errors with test function (5.2) and $\nu = 20$

	$E_{ave,N}$			$E_{max,N}$		
	$N=5$	$N=10$	$N=15$	$N=5$	$N=10$	$N=15$
$h=\frac{1}{2}$	1.3351e+01	4.2276e-02	1.6884e-05	2.3936e+00	4.9040e-03	1.2057e-06
$h=\frac{1}{4}$	5.0218e-01	4.6031e-05	3.2041e-10	1.0403e-01	4.4983e-06	2.9794e-11
$h=\frac{1}{8}$	9.6899e-03	2.4627e-08	2.7816e-12	1.9551e-03	2.8065e-09	2.9095e-12

Table 2 The numerical errors with test function (5.2) and $\nu = 40$

	$E_{ave,N}$	$E_{max,N}$
$h=\frac{1}{2}$	3.0247e-01	2.1915e-02
$h=\frac{1}{4}$	1.7291e-05	1.4119e-06
$h=\frac{1}{8}$	3.4901e-10	3.1821e-11

Table 3 The errors $E_{max,N}$ with test function (5.3) and $\nu = 10$

	$h_*=h$	$h_*=\frac{1}{2}h$	$h_*=\frac{1}{3}h$
$h=\frac{1}{2}$	1.1622e-03	2.8923e-06	8.4407e-08
$h=\frac{1}{4}$	2.8938e-06	5.4187e-09	3.3538e-11
$h=\frac{1}{8}$	5.4022e-09	2.3724e-12	3.9587e-12

Table 4 The errors $E_{max,N}$ with test function (5.3) and $\nu = 100$, $h_* = \frac{1}{3}h$

	$N=5$	$N=10$	$N=15$
$h=\frac{1}{16}$	1.0058e+00	4.8533e-03	2.6921e-06
$h=\frac{1}{32}$	8.0521e-02	3.4986e-06	1.2350e-08
$h=\frac{1}{48}$	5.2516e-03	1.1684e-07	1.0888e-09

We next consider problem (4.6) with $d = \beta = 0$ and the test function

$$U(x_1, x_2) = \sqrt{x_1^2 + x_2^2 + 1}\cos\left(\frac{\nu}{\sqrt{1.2 - x_1}}\right), \tag{5.3}$$

which oscillates very seriously as $x_1 \to 1$. In this case, for raising numerical accuracy and saving work, we use non-uniform mesh sizes. More precisely, we take the mesh size $h_{k,2} = h$ in the x_2-direction, while we take $h_{k,1} = h$ if $\Omega_k \subseteq \Omega_A$, and $h_{k,1} = h_* = \frac{1}{n}h$ if $\Omega_k \subset \Omega_B$. In Table 3, we list the discrete errors $E_{max,N}$, with $\nu = 10$ and $N = 15$. As predicted in the analysis, the spectral scheme (4.11) with non-uniform mesh sizes provides very accurate numerical results, even for the solutions oscillating seriously and locally. In fact, this is one of advantages of scheme (4.11).

Finally, we consider problem (4.6) with $d = \beta = 0$ and the test function (5.3) with very big $\nu = 100$. In this case, the solution (5.3) oscillates extremely, and even more seriously as $x_1 \to 1$. We take $h_{k,1} = h$ if $\Omega_k \subset \Omega_A$, and $h_{k,1} = h_* = \frac{1}{3}h$ if $\Omega_k \subset \Omega_B$. In Table 4, we list the discrete errors $E_{max,N}$ with different h and N. We find that the numerical error delays very fast as h decreases and N increases. It shows again that the proposed spectral scheme

(4.11) is specially appropriate for the solutions oscillating seriously. Indeed, this is another advantage of the spectral element method.

6 Concluding Remarks

In this paper, we provided the spectral element method for fourth order problems with mixed inhomogeneous boundary conditions. Since we designed the specific base functions properly, it is very appropriate for non-uniform meshes and non-uniform modes on different elements, at different common edges of adjacent elements and in different directions. Moreover, it is convenient for local mesh refinement and local mode increment. Therefore, this new approach works well for numerical simulations of problems whose solutions oscillate seriously and behave in differently ways on different parts of domain. As an example of applications, we constructed the spectral element scheme for a model problem and proved its global spectral accuracy. The numerical results demonstrated its high effectiveness. As the mathematical foundation of such new spectral element method, we proposed the composite Legendre quasi-orthogonal approximation, and established the basic approximation results. This approximation not only keeps the continuity of approximated functions and their derivatives of first order at all common edges of adjacent elements, but also possesses the global spectral accuracy on the whole complex domain.

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