J Sci Comput (2014) 58:627–647
DOI 10.1007/s10915-013-9746-4

Optimal Error Analysis of Galerkin FEMs for Nonlinear Joule Heating Equations

Huadong Gao

Received: 14 December 2012 / Revised: 28 April 2013 / Accepted: 22 June 2013 /
Published online: 10 July 2013

Abstract We study in this paper two linearized backward Euler schemes with Galerkin finite element approximations for the time-dependent nonlinear Joule heating equations. By introducing a time-discrete (elliptic) system as proposed in Li and Sun (Int J Numer Anal Model 10:622–633, 2013; SIAM J Numer Anal (to appear)), we split the error function as the temporal error function plus the spatial error function, and then we present unconditionally optimal error estimates of rth order Galerkin FEMs ($1 \leq r \leq 3$). Numerical results in two and three dimensional spaces are provided to confirm our theoretical analysis and show the unconditional stability (convergence) of the schemes.

Keywords Nonlinear parabolic system · Unconditional convergence · Optimal error estimate · Linearized semi-implicit scheme · Galerkin method

Mathematics Subject Classification 65N12 · 65N30 · 35K61

1 Introduction

In this paper, we focus on error estimates of linearized backward Euler Galerkin finite element methods for the time-dependent nonlinear Joule heating equations defined by

$$\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\phi|^2, \tag{1.1}$$

$$-\nabla \cdot (\sigma(u)\nabla\phi) = 0, \tag{1.2}$$

for $x \in \Omega$ and $t \in [0, T]$, where Ω is a bounded, smooth and convex domain in $\mathbb{R}^d$, $d = 2, 3$. The boundary and initial conditions are taken to be

$$u(x,t) = 0, \ \phi(x,t) = g(x,t), \quad \text{for } x \in \partial\Omega, \ t \in [0, T], \tag{1.3}$$

H. Gao (✉)
Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong
e-mail: mahdgao@163.com; hdgao2-c@my.city.edu.hk

$$u(x, 0) = u_0(x), \quad \text{for } x \in \Omega. \tag{1.4}$$

The above nonlinear system (1.1)–(1.4) describes the model of electric heating of a conducting body, where the first unknown u is the temperature and the second unknown ϕ is the electric potential with $\sigma(s)$ being the temperature-dependent electric conductivity satisfying

$$\kappa \le \sigma(s) \le K, \tag{1.5}$$

for some positive constants κ and K.

Theoretical analysis for the Joule heating system can be found in [3,5,7,16,32,33]. Among these works, existence and uniqueness of a C^α solution in three-dimensional space was proved in Yuan and Liu [33]. Based on this result, one can get higher regularity with suitable assumptions on the initial and boundary conditions. Numerical methods and analysis for the Joule heating problems can be found in [2,4,12,31,34,35]. For the two dimensional problem, optimal L^2-norm error estimates of linearized semi-implicit schemes with Galerkin and mixed FEMs were obtained in [31] and [34] under a weak time step condition, respectively. A linearized semi-implicit Euler scheme with a linear Galerkin FEM for the three dimensional model was presented in [12] and an optimal L^2-norm error estimate was obtained under the time step restriction $\tau = O(h^{\frac{1}{2}})$. A more general time discretization with higher-order finite element approximations was studied in [2]. An optimal L^2-norm error estimate was given under the conditions $\tau = O(h^{\frac{3}{2p}})$ and $r \ge 2$, where p is the order of discretization in time direction and r is the degree of piecewise polynomial approximations used.

In the consideration of practical computations, linearized (semi)-implicit schemes are more efficient since at each time step, the schemes only require solving a linear system. However, the time step restriction condition of linearized schemes arising from error analysis is always a crucial issue. We refer to [1,9–11,13,14,17,18,20,23,24,26,28–30] for works on some typical nonlinear parabolic problems. Because of difficulties in obtaining the L^∞ boundedness of the numerical solution, which is an essential condition for error analysis of nonlinear problems, most previous works require certain time step restriction conditions. There are some attempts to reduce the time step restriction conditions. Recently, a new approach was introduced by Li and Sun [19] (also see [21]) to get unconditional stability and optimal error analysis of a linearized backward Euler Galerkin FEM for the time-dependent Joule heating equations. The approach was based on a new splitting technique by a corresponding time-discrete system. With certain proved regularity of the solution of the time-discrete system, one can see

$$\|U_h^n - R_h U^n\|_{L^\infty} \le C h^{-d/2} \|U_h^n - R_h U^n\|_{L^2} \le C h^{-d/2} h^{r+1}.$$

where U_h^n is the FEM solution and R_h is the Ritz projection operator. Therefore, the boundedness of U_h^n in L^∞-norm can be obtained without time step restriction. With this new approach, optimal error estimates for a linear FEM was obtained almost unconditionally in [19] (i.e., the step sizes $h,\ \tau \le s_0$ for some small positive constant s_0). In this paper, we present two linearized schemes with Galerkin FEMs for the time-dependent nonlinear Joule heating system (1.1)–(1.4). The first scheme is semi-decoupled and at each time step, one has to solve Φ_h^{n+1} and U_h^{n+1} one by one. The second one is fully decoupled and at each time step Φ_h^{n+1} and U_h^{n+1} can be solved in parallel. We apply the Li-Sun error splitting method to analyze the Galerkin FEMs. The main difficulty is that error estimates for high-order Galerkin FEMs with the splitting method require rigorous analysis of higher regularity of the solution of the corresponding time-discrete system. For instance, we have to prove the uniform boundedness of the time-discrete solution in H^4-norm for a cubic FEM. As there is no numerical experiment in [19], we present numerical examples in two and three dimensional spaces in this paper. To demonstrate the unconditional stability, we take a fixed τ with several refined spatial meshes. In our numerical tests, the errors are proportional to the temporal error $O(\tau)$ as $h/\tau \to 0$, which show clearly that no time-step condition is needed and the schemes are unconditionally stable.

The rest of the paper is organized as follows. In Sect. 2, we present two linearized schemes with Galerkin finite element methods and our main results on error estimates. We split the error function as the temporal error function plus the spatial error function by introducing a corresponding time-discrete system. In Sect. 3, we provide a priori estimates for the temporal error and suitable regularity of the solution of the time-discrete system. In Sect. 4, we provide optimal spatial error estimates for the Galerkin finite element solutions in L^2 and H^1-norm unconditionally. Numerical examples for both two and three dimensional models are given in Sect. 5 to confirm our theoretical analysis.

2 Galerkin Methods and Main Results

Before presenting the schemes, we clarify some conventional notations. For integer $k \geq 0$ and $1 \leq p \leq \infty$, let $W^{k,p}(\Omega)$ be the Sobolev space with the norm

$$\|u\|_{W^{k,p}} = \begin{cases} \left(\sum_{|\beta|\leq k} \int_\Omega |D^\beta u|^p \, dx\right)^{\frac{1}{p}}, & \text{for } 1 \leq p < \infty, \\ \sum_{|\beta|\leq k} \text{ess sup}_\Omega |D^\beta u|, & \text{for } p = \infty, \end{cases}$$

where

$$D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \ldots \partial x_d^{\beta_d}},$$

for the multi-index $\beta = (\beta_1, \ldots, \beta_d)$, $\beta_1 \geq 0, \ \ldots, \ \beta_d \geq 0$, and $|\beta| = \beta_1 + \cdots + \beta_d$. When $p = 2$ we also note $H^k(\Omega) := W^{k,2}(\Omega)$.

For $t \in (0, T]$, the weak formulation of the system (1.1)–(1.2) with the boundary conditions (1.3) is defined by

$$(u_t, \xi_u) + (\nabla u, \nabla \xi_u) = (\sigma(u)|\nabla\phi|^2, \xi_u), \quad \forall \, \xi_u \in H_0^1(\Omega), \tag{2.1}$$

$$(\sigma(u)\nabla\phi, \nabla\xi_\phi) = 0, \quad \forall \, \xi_\phi \in H_0^1(\Omega). \tag{2.2}$$

Let $\mathcal{T}_h$ be a regular partition of Ω into triangles $T_j, \ j = 1, \ldots, M$ in $\mathbb{R}^2$ or tetrahedra in $\mathbb{R}^3$, and $h = \max_{1\leq j\leq M}\{\text{diam}\, T_j\}$ be the mesh size. For a triangle T_j with two nodes (or a tetrahedron with three nodes) on the boundary, we use $\widetilde{T}_j$ to denote the triangle with one curved edge (or a tetrahedron with one curved face) with the same nodes as T_j. For interior element, we simply set $\widetilde{T}_j$ as T_j itself. Following classical FEM theory [27,36], for a given partition of Ω, we define the finite element space

$$\widehat{S}_h = \{v_h \in C(\overline{\Omega}_h) : v_h|_{T_j} \text{is a polynomial of degree } r\},$$

$$\widehat{V}_h = \{v_h \in C(\overline{\Omega}_h) : v_h|_{T_j} \text{is a polynomial of degree } r \text{ and } v_h = 0 \text{on } \partial\Omega_h\},$$

we can see that $\widehat{S}_h$ is a subspace of $H^1(\Omega_h)$ and $\widehat{V}_h$ is a subspace of $H_0^1(\Omega_h)$. Let $G : \Omega_h \to \Omega$ be a mapping such that both G and G^{-1} are Lipschitz continuous and, for interior element G is the identity mapping, for T_j at the boundary, G maps T_j onto $\widetilde{T}_j$ smoothly. We define an operator $\mathcal{G} : L^2(\Omega_h) \to L^2(\Omega)$ by $\mathcal{G}v(x) = v(G^{-1}(x))$ for $x \in \Omega$. Defining

$$S_h = \{\mathcal{G}v_h : v_h \in \widehat{S}_h\},$$

it is easy to see that S_h is a finite element subspace of $H^1(\Omega)$. For any $v \in H^1(\Omega)$, We define $\Pi_h v = \mathcal{G}\widehat{\Pi}_h \mathcal{G}^{-1} v$, where $\widehat{\Pi}_h : C_0(\overline{\Omega}_h) \to \widehat{S}_h$ is the Lagrange interpolation operator of degree r, then $\forall\, v \in W^{r+1,p}(\Omega)$

$$\|v - \Pi_h v\|_{L^p} + h\|v - \Pi_h v\|_{W^{1,p}} \le Ch^{r+1}\|v\|_{W^{r+1,p}}, \quad \text{for } 1 \le p \le \infty. \tag{2.3}$$

We set

$$V_h = \{\mathcal{G}v_h : v_h \in \widehat{V}_h\},$$

and it is easy to verify that V_h is a finite element subspace of $H_0^1(\Omega)$. We define $R_h : H_0^1(\Omega) \to V_h$ to be a Ritz projection operator by

$$(\nabla(v - R_h v), \nabla w) = 0, \quad \forall\, w \in V_h.$$

By the standard theory of finite element methods to elliptic equations [6,27],

$$\|v - R_h v\|_{L^2} + h\|v - R_h v\|_{H^1} \le Ch^{r+1}\|v\|_{H^{r+1}}. \tag{2.4}$$

Moreover, let $\{t_n\}_{n=0}^N$ be a partition in time direction with $t_n = n\tau,\ T = N\tau$ and

$$u^n = u(x, t_n), \quad \phi^n = \phi(x, t_n).$$

For any sequence of functions $\{f^n\}_{n=0}^{N-1}$, we define

$$D_\tau f^{n+1} = \frac{f^{n+1} - f^n}{\tau}.$$

Now we introduce two linearized schemes to solve the time-dependent nonlinear Joule heating Eqs. (1.1)–(1.4).

The first linearized backward Euler Galerkin finite element method is to find $U_h^{n+1} \in V_h,\ \Phi_h^{n+1} \in S_h$ such that

$$\left(D_\tau U_h^{n+1},\, \xi_u\right) + \left(\nabla U_h^{n+1},\, \nabla\xi_u\right) = \left(\sigma(U_h^n)|\nabla\Phi_h^{n+1}|^2,\, \xi_u\right), \quad \forall\, \xi_u \in V_h, \tag{2.5}$$

$$\left(\sigma(U_h^n)\nabla\Phi_h^{n+1},\, \nabla\xi_\phi\right) = 0, \quad \forall\, \xi_\phi \in V_h, \tag{2.6}$$

with the initial and boundary conditions $U_h^0 = \Pi_h u^0$ and $\Phi_h^{n+1}|_{\partial\Omega} = \Pi_h g^{n+1}|_{\partial\Omega}$.

The second one is the fully decoupled linearized backward Euler Galerkin FEMs, which is to find $U_h^{n+1} \in V_h,\ \Phi_h^{n+1} \in S_h$ such that

$$\left(D_\tau U_h^{n+1},\, \xi_u\right) + \left(\nabla U_h^{n+1},\, \nabla\xi_u\right) = \left(\sigma(U_h^n)|\nabla\Phi_h^n|^2,\, \xi_u\right), \quad \forall\, \xi_u \in V_h\,, \tag{2.7}$$

$$\left(\sigma(U_h^n)\nabla\Phi_h^{n+1},\, \nabla\xi_\phi\right) = 0, \quad \forall\, \xi_\phi \in V_h\,, \tag{2.8}$$

with boundary conditions $\Phi_h^{n+1}|_{\partial\Omega} = \Pi_h g^{n+1}|_{\partial\Omega}$ and initial conditions $U_h^0 = \Pi_h u^0$ and Φ_h^0, which is the solution of

$$\left(\sigma(u^0)\nabla\Phi_h^0,\, \nabla\xi_\phi\right) = 0, \quad \forall\, \xi_\phi \in V_h,$$

with boundary condition $\Phi_h^0|_{\partial\Omega} = \Pi_h g^0|_{\partial\Omega}$.

The scheme (2.5)–(2.6) is semi-decoupled. At each time step, one has to solve (2.6) for Φ_h^{n+1} and then (2.5) for U_h^{n+1}. A similar semi-decoupled scheme was given in [19], where

Φ_h^0 was obtained by solving an elliptic PDE. The scheme (2.7)–(2.8) is fully decoupled. At each time step, one only needs to solve two systems of U_h^{n+1} and Φ_h^{n+1} in parallel.

In this paper, we only present error analysis for the linearized scheme (2.5)–(2.6). The analysis presented here can be easily extended to the second linearized scheme (2.7)–(2.8), which will be confirmed numerically in Sect. 5.

In the rest part of this paper, we assume that $\sigma(s) \in C^r(\mathbb{R})$ and the solution to the initial boundary value problem (1.1)–(1.4) exists and satisfies

$$\begin{cases} \|u\|_{L^\infty((0,T);H^{r+1})} + \|u_t\|_{L^2((0,T);H^{r^*})} + \|u_t\|_{L^\infty((0,T);H^1)} + \|u_{tt}\|_{L^2((0,T);H^1)} \le C, \\ \|\phi\|_{L^\infty((0,T);W^{r+1,4})} + \|\phi_{tt}\|_{L^2((0,T);H^1)} + \|g\|_{L^\infty((0,T);W^{r+1,4})} \le C. \end{cases} \tag{2.9}$$

where $r^* = \max(r, 2)$.

We present our main results on error estimates in the following theorem.

Theorem 2.1 *Suppose that the system (1.1)–(1.2) with the boundary conditions (1.3) and initial condition (1.4) has a unique solution (u, ϕ) satisfying (2.9). Then the finite element system (2.5)–(2.6) with $U_h^0 = \Pi u^0$ (for $r \le 3$) admits a unique solution $(U_h^{n+1}, \Phi_h^{n+1})$, and there exist two positive constants τ_0 and h_0 such that when $\tau < \tau_0$ and $h \le h_0$*

$$\max_{1\le n\le N} \|U_h^n - u^n\|_{L^2} + \max_{1\le n\le N} \|\Phi_h^n - \phi^n\|_{L^2} \le C_0(\tau + h^{r+1}), \tag{2.10}$$

and

$$\max_{1\le n\le N} \|U_h^n - u^n\|_{H^1} + \max_{1\le n\le N} \|\Phi_h^n - \phi^n\|_{H^1} \le C_0(\tau + h^r), \tag{2.11}$$

where C_0 is a positive constant, independent of n, h and τ.

For simplicity, through out this paper, we denote by C a generic positive constant and ϵ a generic small positive constant, which are independent of n, h, τ and C_0 in the above theorem.

For $n = 0, 1, \ldots, N-1$, we define the U^{n+1} and Φ^{n+1} to be the solutions of the following elliptic system (or time discrete parabolic equations)

$$D_\tau U^{n+1} - \Delta U^{n+1} = \sigma(U^n)|\nabla\Phi^{n+1}|^2, \tag{2.12}$$

$$-\nabla\cdot(\sigma(U^n)\nabla\Phi^{n+1}) = 0, \tag{2.13}$$

with $U^0 = u_0$ and boundary conditions

$$U^{n+1}(x) = 0, \quad \Phi^{n+1}(x) = g(x, t_{n+1}), \quad \text{for } x \in \partial\Omega. \tag{2.14}$$

In terms of the LS-splitting proposed in [19,20], the error functions under certain norm $\|\cdot\|$ can be written by

$$\|U_h^n - u^n\| \le \|e^n\| + \|e_h^n\| + \|U^n - R_h U^n\|,$$
$$\|\Phi_h^n - \phi^n\| \le \|\eta^n\| + \|\eta_h^n\| + \|\Phi^n - \Pi_h\Phi^n\|,$$

with

$$e^n = U^n - u^n, \quad e_h^n = U_h^n - R_h U^n,$$
$$\eta^n = \Phi^n - \phi^n, \quad \eta_h^n = \Phi_h^n - \Pi_h\Phi^n.$$

Here we can see both e^n and η^n have zero trace

$$e^n = \eta^n = 0, \quad \text{for } x \in \partial\Omega.$$

To prove our main results in Theorem 2.1, we will analyze the temporal error functions (e^n, η^n) in Sect. 3 and the spatial error functions (e_h^n, η_h^n) in Sect. 4, respectively. We present the Gagliardo–Nirenberg inequality and discrete Gronwall's inequality in the following two lemmas which will be frequently used in our proofs.

Lemma 2.1 *Gagliardo–Nirenberg inequality (see [25]): Let u be a function defined on Ω and $\partial^s u$ be any partial derivative of u of order s, then*

$$\|\partial^j u\|_{L^p} \le C\|\partial^m u\|_{L^r}^a \, \|u\|_{L^q}^{1-a} + C\|u\|_{L^q},$$

for $0 \le j < m$ and $\frac{j}{m} \le a \le 1$ with

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q},$$

except $1 < r < \infty$ and $m - j - \frac{n}{r}$ is a non-negative integer, in which case the above estimate holds only for $\frac{j}{m} \le a < 1$.

Lemma 2.2 *Discrete Gronwall's inequality [15]: Let τ, B and a_k, b_k, c_k, γ_k, for integers $k \ge 0$, be non-negative numbers such that*

$$a_n + \tau \sum_{k=0}^{n} b_k \le \tau \sum_{k=0}^{n} \gamma_k a_k + \tau \sum_{k=0}^{n} c_k + B, \quad \text{for} \quad n \ge 0,$$

suppose that $\tau\gamma_k < 1$, for all k, and set $\sigma_k = (1 - \tau\gamma_k)^{-1}$. Then

$$a_n + \tau \sum_{k=0}^{n} b_k \le \exp\left(\tau \sum_{k=0}^{n} \gamma_k \sigma_k\right)\left(\tau \sum_{k=0}^{n} c_k + B\right), \quad \text{for} \quad n \ge 0.$$

3 Temporal Error Estimates

Theorem 3.1 *Suppose that the time-dependent nonlinear Joule heating system (1.1)–(1.4) has a unique solution (u, ϕ) satisfying (2.9). Then the elliptic system (2.12)–(2.14) with $U^0 = u^0$ admits a unique solution (U^{n+1}, Φ^{n+1}) such that*

$$\max_{0\le n\le N} \|e^n\|_{H^1} + \max_{1\le n\le N} \|\eta^n\|_{H^1} \le C\tau, \tag{3.1}$$

and

$$\max_{0\le n\le N} \|U^n\|_{H^{r+1}} + \sum_{n=1}^{N} \|D_\tau U^n\|_{H^{r*}}^2 \,\tau \le C, \quad \max_{1\le n\le N} \|\Phi^n\|_{W^{r+1,4}} \le C. \tag{3.2}$$

Proof We first prove the temporal error estimate (3.1) and then prove the uniform bound (3.2) for all the three cases $r = 1,\ 2$ and 3. It is clear that $e^0 = 0$. By (1.1)–(1.2) and (2.12)–(2.14), the temporal error functions (e^n, η^n) satisfy

$$\begin{aligned} D_\tau e^{n+1} - \Delta e^{n+1} &= (\sigma(U^n) - \sigma(u^n))|\nabla\phi^{n+1}|^2 \\ &\quad + \sigma(U^n)(\nabla\phi^{n+1} + \nabla\Phi^{n+1}) \cdot \nabla\eta^{n+1} - R_u^{n+1}, \end{aligned} \tag{3.3}$$

and

$$-\nabla \cdot (\sigma(U^n)\nabla\eta^{n+1}) = \nabla \cdot [(\sigma(u^n) - \sigma(U^n))\nabla\phi^{n+1}] - \nabla \cdot R_\phi^{n+1} \tag{3.4}$$

where R_u^{n+1} and R_ϕ^{n+1} are the truncation errors. With the regularity given in (2.9), we have

$$\|R_u^{n+1}\|_{H^1} \leq C\tau, \quad \|R_\phi^{n+1}\|_{H^1} \leq C\tau. \tag{3.5}$$

Using classical energy method as done in [19], we can derive that there exists a small positive constant τ_0 such that when $\tau < \tau_0$,

$$\max_{1\leq n\leq N} \|e^n\|_{L^2}^2 + \sum_{m=0}^{N-1} \|e^{m+1}\|_{H^1}^2 \tau + \max_{1\leq n\leq N} \|\eta^n\|_{H^1}\tau \leq C\tau^2. \tag{3.6}$$

It follows from (3.6) that, for $1 \leq n \leq N$

$$\|D_\tau e^n\|_{L^2}, \quad \|D_\tau \eta^n\|_{H^1}, \quad \|D_\tau U^n\|_{L^2}, \quad \|D_\tau \Phi^n\|_{H^1} \leq C, \quad \|e^n\|_{H^1} \leq C\tau^{1/2}. \tag{3.7}$$

To obtain the H^s-norm estimates, $s = 2,\ 3$ and 4, we need the following lemma and we refer to [8] for the details of the proof. □

Lemma 3.1 *Suppose that $\Omega \in \mathbb{R}^3$ be a bounded and smooth domain and $v \in H^k(\Omega)$ is a solution of*

$$-\Delta v = f, \quad x \in \Omega,$$

satisfying $v|_{\partial\Omega} = g$, where g can be extended to a function on Ω such that $g \in W^{k+1,p}(\Omega)$. Then

$$\|v\|_{W^{k+1,p}} \leq C\|f\|_{W^{k-1,p}} + C\|g\|_{W^{k+1,p}}, \quad \text{for } 2 \leq p < \infty.$$

We rewrite (3.4) by

$$\begin{aligned}-\Delta\eta^{n+1} = {} & \frac{1}{\sigma(U^n)}\left(\nabla\cdot[(\sigma(u^n) - \sigma(U^n))\nabla\phi^{n+1}] - \nabla\cdot R_\phi^{n+1}\right)\\ & + \frac{\sigma'(U^n)}{\sigma(U^n)}\nabla U^n \cdot \nabla\eta^{n+1}.\end{aligned}$$

Applying Lemma 3.1 to the above equation, we have

$$\begin{aligned}\|\eta^{n+1}\|_{H^2} &\leq C\|\nabla e^n \nabla\phi^{n+1}\|_{L^2} + C\|e^n\Delta\phi^{n+1}\|_{L^2} + C\|\nabla\cdot R_\phi^{n+1}\|_{L^2}\\ &\quad + C\|\nabla e^n\cdot\nabla\eta^{n+1}\|_{L^2} + C\|\nabla u^n\cdot\nabla\eta^{n+1}\|_{L^2}\\ &\leq \epsilon\|\nabla\eta^{n+1}\|_{L^6}^2 + \epsilon^{-1}C\|\nabla e^n\|_{L^3}^2 + C\|e^n\|_{H^1} + C\tau\\ &\leq \epsilon C\|\eta^{n+1}\|_{H^2}^2 + C\epsilon^{-1}\|e^n\|_{H^1}\|e^n\|_{H^2} + C\|e^n\|_{H^1} + C\tau,\end{aligned}$$

which with $C\epsilon \leq \frac{1}{2}$ reduces to

$$\|\eta^{n+1}\|_{H^2} \leq C\|e^n\|_{H^1}\|e^n\|_{H^2} + C\|e^n\|_{H^1} + C\tau. \tag{3.8}$$

Now we prove a primary estimate by mathematical induction

$$\|\eta^{n+1}\|_{H^2} \leq 1, \quad \text{for } 0 \leq n \leq N-1. \tag{3.9}$$

From (3.8), it is clear that $\|\eta^1\|_{H^2} \leq C\tau$, (3.9) holds for $n = 0$ if we require $C\tau \leq 1$. We assume that (3.9) holds for $n \leq k-1$.

By applying Lemma 3.1 to (3.3), with estimates (3.6), (3.7) and the above assumption (3.9), we can derive that

$$\begin{aligned}\|e^k\|_{H^2} &\le \|D_\tau e^k\|_{L^2} + \big\|(\sigma(U^{k-1}) - \sigma(u^{k-1}))|\nabla\phi^k|^2\big\|_{L^2}\\ &\quad+\big\|\sigma(U^{k-1})(\nabla\phi^k + \nabla\Phi^k)\cdot\nabla\eta^k\big\|_{L^2} + \|R_u^k\|_{L^2}\\ &\le C\|\nabla\eta^k\|_{L^4}^2 + C + C\tau\\ &\le C\|\eta^k\|_{H^2}^2 + C\\ &\le C.\end{aligned}$$

With (3.7), substituting the above estimate into (3.8) gives

$$\|\eta^{k+1}\|_{H^2} \le C\|e^k\|_{H^1} + C\tau \le C\tau^{1/2} + C\tau,$$

and therefore, $\|\eta^{k+1}\|_{H^2} \le 1$ when $C\tau^{1/2} + C\tau \le 1$.

Thus, we complete the induction and obtain

$$\|\eta^{n+1}\|_{H^2} \le C\|e^n\|_{H^1} + C\tau, \tag{3.10}$$

and

$$\|e^n\|_{H^2} \le C, \quad \|U^n\|_{H^2} \le C, \quad \|e^n\|_{L^\infty} \le C, \quad \|U^n\|_{L^\infty} \le C. \tag{3.11}$$

Again, we rewrite Eqs. (2.12) and (2.13) by

$$-\Delta U^{n+1} = \sigma(U^n)|\nabla\Phi^{n+1}|^2 - D_\tau U^{n+1}, \tag{3.12}$$

and

$$-\Delta\Phi^{n+1} = \frac{\sigma'(U^n)}{\sigma(U^n)}\nabla U^n\cdot\nabla\Phi^{n+1}. \tag{3.13}$$

Applying Lemma 3.1 to Eq. (3.13) gives

$$\begin{aligned}\|\Phi^{n+1}\|_{W^{2,4}} &\le C\Big\|\frac{\sigma'(U^n)}{\sigma(U^n)}\nabla U^n\cdot\nabla\Phi^{n+1}\Big\|_{L^4} + C\|g^{n+1}\|_{W^{2,4}}\\ &\le C\|\nabla U^n\|_{L^6}\,\|\nabla\Phi^{n+1}\|_{L^{12}} + C\\ &\le C\big\|\Phi^{n+1}\big\|_{H^1}^{\frac{5}{7}}\,\big\|\Phi^{n+1}\big\|_{W^{2,4}}^{\frac{2}{7}} + C\\ &\le \frac{2}{7}\|\Phi^{n+1}\|_{W^{2,4}} + C,\end{aligned} \tag{3.14}$$

where we have used the Gagliardo–Nirenberg inequality in Lemma 2.1. It follows that $\|\Phi^{n+1}\|_{W^{2,4}} \le C$.

With (3.11) and the above uniform bound for Φ^{n+1}, multiplying the Eq. (3.3) by $-\Delta e^{n+1}$ yields further

$$\begin{aligned}&D_\tau(\|e^{n+1}\|_{H^1}^2) + \|\Delta e^{n+1}\|_{L^2}^2\\ &\quad\le C\big\|(\sigma(U^n) - \sigma(u^n))|\nabla\phi^{n+1}|^2\big\|_{L^2}^2\\ &\quad+C\big\|\sigma(U^n)(\nabla\phi^{n+1} + \nabla\Phi^{n+1})\cdot\nabla\eta^{n+1}\big\|_{L^2}^2 + C\|R_u^{n+1}\|_{L^2}^2\\ &\le C\|\sigma(U^n)\|_{L^\infty}^2\,(\|\nabla\phi^{n+1}\|_{L^\infty}^2 + \|\nabla\Phi^{n+1}\|_{L^\infty}^2)\,\|\nabla\eta^{n+1}\|_{L^2}^2\\ &\quad+C\|e^n\|_{L^2}^2\|\nabla\phi^{n+1}\|_{L^\infty}^4 + C\tau^2\\ &\le C\tau^2.\end{aligned} \tag{3.15}$$

Thanks to Gronwall's inequality, there exists a small constant τ_0, such that when $\tau \le \tau_0$

$$\max_{0\le n\le N} \|e^n\|_{H^1}^2 + \sum_{m=0}^{N-1} \|e^{m+1}\|_{H^2}^2 \tau \le C\tau^2, \tag{3.16}$$

where we have noted the fact that $\|e^{n+1}\|_{H^2} \le C\|\Delta e^{n+1}\|_{L^2}$. The estimate (3.16) also implies that $\|D_\tau U^{n+1}\|_{H^1} \le C$. Substituting the above results into (3.10) gives

$$\max_{1\le n\le N} \|\eta^n\|_{H^2} \le C\tau. \tag{3.17}$$

Thus, we complete the proof of the temporal error estimate (3.1) by combining estimates (3.6) and (3.16).

Next, we prove the estimate (3.2) for $r = 1,\ 2,\ 3$. From (3.16) we can also derive that

$$\begin{aligned}\sum_{m=0}^{N-1} \|D_\tau U^{m+1}\|_{H^2}^2 \tau &\le \sum_{m=0}^{N-1} \left(\|D_\tau e^{m+1}\|_{H^2}^2 \tau + \|D_\tau u^{m+1}\|_{H^2}^2 \tau\right)\\ &\le C\tau^{-2} \sum_{m=0}^{N-1} \|e^{m+1}\|_{H^2}^2 \tau + C\\ &\le C.\end{aligned} \tag{3.18}$$

By combining (3.11), (3.14) and (3.18), we see that the uniform bound (3.2) holds for $r = 1$.

For the case $r = 2$, we apply Lemma 3.1 to the Eqs. (3.12) and (3.13) again to deduce

$$\begin{aligned}\|U^{n+1}\|_{H^3} &\le C\left\|\sigma(U^n)|\nabla\Phi^{n+1}|^2 - D_\tau U^{n+1}\right\|_{H^1}\\ &\le C\|\sigma(U^n)\|_{L^\infty} \|\nabla\Phi^{n+1}\|_{L^\infty} \|\Phi^{n+1}\|_{H^2}\\ &\quad + C\|\sigma(U^n)\|_{H^1} \|\nabla\Phi^{n+1}\|_{L^\infty}^2 + C\\ &\le C,\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\|\Phi^{n+1}\|_{W^{3,4}} &\le C\left\|\frac{\sigma'(U^n)}{\sigma(U^n)}\nabla U^n \cdot \nabla\Phi^{n+1}\right\|_{W^{1,4}} + C\|g\|_{W^{3,4}}\\ &\le \left\|\frac{\sigma'(U^n)}{\sigma(U^n)}\right\|_{W^{1,\infty}} \left(\|U^n\|_{W^{2,4}}\|\Phi^{n+1}\|_{W^{1,\infty}} + \|U^n\|_{W^{1,\infty}}\|\Phi^{n+1}\|_{W^{2,4}}\right) + C\\ &\le C\|U^n\|_{H^3}\|\Phi^{n+1}\|_{W^{1,\infty}} + \|U^n\|_{H^3}\|\Phi^{n+1}\|_{W^{2,4}}\\ &\le C,\end{aligned} \tag{3.20}$$

which imply the uniform bound (3.2) holds for $r = 2$.

Now we turn our proof to the uniform bound (3.2) for the case $r = 3$. We multiply (3.3) by $-D_\tau \Delta e^{n+1}$ to get

$$\begin{aligned}&D_\tau(\|\Delta e^{n+1}\|_{L^2}^2) + \|\nabla D_\tau e^{n+1}\|_{L^2}^2\\ &\le C\|\nabla\left((\sigma(U^n) - \sigma(u^n))|\nabla\phi^{n+1}|^2\right)\|_{L^2}^2\\ &\quad - \left(\sigma(U^n)(\nabla\phi^{n+1} + \nabla\Phi^{n+1}) \cdot \nabla\eta^{n+1},\ D_\tau\Delta e^{n+1}\right) + \|\nabla R_u^{n+1}\|_{L^2}^2\\ &\le -\left(\sigma(U^n)(\nabla\phi^{n+1} + \nabla\Phi^{n+1}) \cdot \nabla\eta^{n+1},\ D_\tau\Delta e^{n+1}\right) + C\|e^n\|_{H^1}^2 + C\tau^2,\end{aligned}$$

which shows further

$$
\begin{aligned}
&\|\Delta e^{n+1}\|_{L^2}^2 + \sum_{m=0}^{n} \tau \|\nabla D_\tau e^{m+1}\|_{L^2}^2 \\
&\le -\sum_{m=0}^{n} \tau \left(\sigma(U^m)(\nabla \phi^{m+1} + \nabla \Phi^{m+1}) \cdot \nabla \eta^{m+1},\ D_\tau \Delta e^{m+1}\right) + C\tau^2 \\
&= \sum_{m=1}^{n} \tau \left(D_\tau \left(\sigma(U^m)(\nabla \phi^{m+1} + \nabla \Phi^{m+1}) \cdot \nabla \eta^{m+1}\right),\ \Delta e^m\right) \\
&\quad - \left(\sigma(U^n)(\nabla \phi^{n+1} + \nabla \Phi^{n+1}) \cdot \nabla \eta^{n+1},\ \Delta e^{n+1}\right) + C\tau^2 \\
&\le \sum_{m=1}^{n} \tau \|D_\tau \left(\sigma(U^m)(\nabla \phi^{m+1} + \nabla \Phi^{m+1}) \cdot \nabla \eta^{m+1}\right)\|_{L^2}^2 \\
&\quad + \frac{1}{2}\|\Delta e^{n+1}\|_{L^2}^2 + \sum_{m=1}^{n} \tau \|\Delta e^m\|_{L^2}^2 + C\tau^2 \\
&\le C\sum_{m=1}^{n} \tau \|\nabla (D_\tau \eta^{m+1})\|_{L^2}^2 + \frac{1}{2}\|\Delta e^{n+1}\|_{L^2}^2 + \sum_{m=1}^{n} \tau \|\Delta e^m\|_{L^2}^2 + C\tau^2 , \qquad (3.21)
\end{aligned}
$$

where the summation by parts is used in the temporal direction. In order to estimate $\|\nabla(D_\tau \eta^{m+1})\|_{L^2}^2$, we take D_τ to both sides of the Eq. (3.4) and multiply the result with $D_\tau \eta^{n+1}$ to deduce that

$$
\begin{aligned}
\|D_\tau \eta^{n+1}\|_{H^1}^2 &\le C\|(D_\tau \sigma(U^n))\nabla \eta^{n+1}\|_{L^2}^2 + C\|D_\tau e^n\|_{L^2}^2 + C\tau^2 \\
&\le C\|D_\tau e^n\|_{L^2}^2 + C\tau^2 \\
&\le C\|\Delta e^n\|_{L^2}^2 + C\tau^2 , \qquad (3.22)
\end{aligned}
$$

where we have used the Eq. (3.3).

Substituting (3.22) into (3.21), with the help of Gronwall's inequality, we have

$$
\max_{0\le n\le N} \|e^n\|_{H^2}^2 + \sum_{m=0}^{N-1} \|\nabla D_\tau e^{m+1}\|_{L^2}^2 \tau \le C\tau^2, \qquad (3.23)
$$

when τ is less than certain $\tau_0 > 0$. It follows that

$$
\|D_\tau U^{n+1}\|_{H^2} \le C.
$$

Moreover, by applying Lemma 3.1 to (3.3), we can obtain

$$
\begin{aligned}
\|e^{n+1}\|_{H^3} &\le C\|D_\tau e^{n+1}\|_{H^1} + C\|(\sigma(U^n) - \sigma(u^n))|\nabla \phi^{n+1}|^2\|_{H^1} \\
&\quad + C\|\sigma(U^n)(\nabla \phi^{n+1} + \nabla \Phi^{n+1}) \cdot \nabla \eta^{n+1}\|_{H^1} + C\|R_u^{n+1}\|_{H^1} \\
&\le C\|D_\tau e^{n+1}\|_{H^1} + C\tau , \qquad (3.24)
\end{aligned}
$$

and therefore, by estimate (3.23), we have

$$\begin{aligned}\sum_{m=0}^{N-1}\|D_\tau U^{m+1}\|_{H^3}^2\tau &\le \sum_{m=0}^{N-1}(\|D_\tau e^{m+1}\|_{H^3}^2+\|D_\tau u^{m+1}\|_{H^3}^2)\tau\\ &\le \sum_{m=0}^{N-1}\left(C\tau^{-2}\|e^{m+1}\|_{H^3}^2\right)\tau+\sum_{m=0}^{N-1}\|D_\tau u^{m+1}\|_{H^3}^2\tau\\ &\le C\tau^{-2}\left(\sum_{m=0}^{N-1}(\|D_\tau e^{m+1}\|_{H^1}^2+\tau^2)\tau\right)+C\\ &\le C\,. \end{aligned}\tag{3.25}$$

Finally, we apply Lemma 3.1 to Eqs. (3.12) and (3.13) to get further

$$\begin{aligned}\|U^{n+1}\|_{H^4} &\le C\|\sigma(U^n)|\nabla\Phi^{n+1}|^2-D_\tau U^{n+1}\|_{H^2}\\ &\le C\|U^n\|_{H^2}C(\|\Phi^{n+1}\|_{W^{1,\infty}}^2+\|\Phi^{n+1}\|_{W^{1,\infty}}\|\Phi^{n+1}\|_{W^{2,\infty}})\\ &\quad +C\|U^n\|_{L^\infty}\|\Phi^{n+1}\|_{W^{1,\infty}}^2\|\Phi^{n+1}\|_{W^{3,4}}^2+C\\ &\le C,\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}\|\Phi^{n+1}\|_{W^{4,4}} &\le C\|\frac{\sigma'(U^n)}{\sigma(U^n)}\nabla U^n\cdot\nabla\Phi^{n+1}\|_{W^{2,4}}+C\|g\|_{W^{4,4}}\\ &\le C\|U^n\|_{W^{3,4}}\|\Phi^{n+1}\|_{W^{1,\infty}}\|U^n\|_{W^{1,\infty}}+C\|U^n\|_{W^{2,\infty}}\|\Phi^{n+1}\|_{W^{2,4}}\\ &\quad +C\|U^n\|_{W^{1,\infty}}\|\Phi^{n+1}\|_{W^{3,4}}+C\\ &\le C\,,\end{aligned}\tag{3.27}$$

where we have used estimates (3.19) and (3.20).

By combining (3.25), (3.26) and (3.27) , we have proved that (3.2) holds in the case $r=3$. Thus, we obtain the uniform boundedness of the solution to the elliptic system for all the three cases.

We complete the proof of Theorem 3.1. □

4 Spatial Error Estimates

Theorem 4.1 *Suppose that the time-dependent nonlinear Joule heating system* (1.1)–(1.4) *has a unique solution* (u,ϕ) *satisfying* (2.9). *Then the fully-discrete system* (2.5)–(2.6) *with* $U_h^0=\Pi_h u^0$ *for* $r\le 3$ *admits a unique solution* (U_h^{n+1},Φ_h^{n+1})*, such that*

$$\max_{0\le n\le N}\|e_h^n\|_{L^2}+\max_{1\le n\le N}\|\eta_h^n\|_{L^2}\le Ch^{r+1}\,,\tag{4.1}$$

$$\max_{0\le n\le N}\|\nabla e_h^n\|_{L^2}+\max_{1\le n\le N}\|\nabla\eta_h^n\|_{L^2}\le Ch^{r}\,.\tag{4.2}$$

Proof The proof for linear FEM has been given in [19], here we only analyze the quadratic and cubic FEMs. Since the coefficient matrices for U_h^{n+1} and Φ_h^{n+1} are symmetric positive definite, it is clear that the FEM system (2.5)–(2.6) is uniquely solvable. By using the inverse inequality, it is easy to verify that the L^2-norm estimate (4.1) implies the H^1-norm estimate (4.2). Thus, we only need to prove (4.1). We first prove

$$\|e_h^n\|_{L^2}^2\le C_1h^{2r+2},\quad 0\le n\le N,\tag{4.3}$$

by mathematical induction, where C_1 is a positive constant independent of $n,\ h,\ \tau$ and the general constant C. As $u^0 = U^0$, from the Lagrange interpolation error estimate (2.3) and the Ritz projection error estimate (2.4), we can easily obtain

$$\|e_h^0\|^2 = \|\Pi_h u^0 - R_h u^0\|^2 \le C_2 h^{2r+2},$$

where C_2 is a positive constant independent of $\tau,\ h$ and n. Therefore, if we require $C_1 \ge C_2$, (4.3) holds for $n = 0$. We assume that (4.3) holds for $n \le k-1$. We need to find C_1, which is independent of $n,\ h,\ \tau$ and the general constant C, such that (4.3) also holds for $n \le k$.

With our assumption, by inverse inequality we have

$$\|e_h^n\|_{L^\infty} \le Ch^{-d/2}\|e_h^n\|_{L^2} \le CC_1 h^{r+1-d/2}.$$

It is clear that when $CC_1h^{r+1-d/2} \le 1$ we get $\|e_h^n\|_{L^\infty} \le 1$, which implies $\|U_h^n\|_{L^\infty} \le C$ for $n \le k-1$.

The weak formulation of the time-discrete elliptic system (2.12)–(2.14) is

$$\left(D_\tau U^{n+1},\ \xi_u\right) + \left(\nabla U^{n+1},\ \nabla\xi_u\right) = \left(\sigma(U^n)|\nabla\Phi^{n+1}|^2,\ \xi_u\right), \quad \forall\, \xi_u \in H_0^1\,, \tag{4.4}$$

$$\left(\sigma(U^n)\nabla\Phi^{n+1},\ \nabla\xi_\phi\right) = 0, \quad \forall\, \xi_\phi \in H_0^1\,. \tag{4.5}$$

Then, the spatial error functions (e_h^n, η_h^n) satisfy

$$\begin{aligned}
&\left(D_\tau e_h^{n+1},\ \xi_u\right) + \left(\nabla e_h^{n+1},\ \nabla\xi_u\right)\\
&= \left(D_\tau(U^{n+1} - R_hU^{n+1}),\ \xi_u\right) + \left((\sigma(U_h^n) - \sigma(U^n))|\nabla\Phi^{n+1}|^2,\ \xi_u\right)\\
&\quad +2\left((\sigma(U_h^n) - \sigma(U^n))\nabla\Phi^{n+1}\cdot\nabla(\Phi_h^{n+1} - \Phi^{n+1}),\ \xi_u\right)\\
&\quad +\left(\sigma(U_h^n)|\nabla(\Phi_h^{n+1} - \Phi^{n+1})|^2,\ \xi_u\right)\\
&\quad +2\left(\sigma(U^n)\nabla\Phi^{n+1}\cdot\nabla(\Phi_h^{n+1} - \Phi^{n+1}),\ \xi_u\right)\\
&:= \sum_{j=1}^{5} I_j^{n+1}(\xi_u), \quad \forall\, \xi_u \in V_h\,,
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
\left(\sigma(U^n)\nabla\eta_h^{n+1},\ \nabla\xi_\phi\right) &= \left((\sigma(U^n) - \sigma(U_h^n))\nabla\Phi_h^{n+1},\ \nabla\xi_\phi\right)\\
&\quad + \left(\sigma(U^n)\nabla(\Phi^{n+1} - \Pi_h\Phi^{n+1}),\ \nabla\xi_\phi\right)\,, \quad \forall\, \xi_\phi \in V_h\,.
\end{aligned} \tag{4.7}$$

Taking $\xi_u = e_h^{n+1}$ into (4.6), now we estimate the five residual terms of the right-hand side of (4.6). The first two terms are bounded by

$$\begin{aligned}
I_1^{n+1}(e_h^{n+1}) &\le \epsilon\|e_h^{n+1}\|_{H^1}^2 + \epsilon^{-1}C\|D_\tau U^{n+1} - R_hD_\tau U^{n+1}\|_{H^{-1}}^2\\
&\le \epsilon\|e_h^{n+1}\|_{H^1}^2 + \epsilon^{-1}C\|D_\tau U^{n+1}\|_{H^{r*}}^2 h^{2r+2}\,,
\end{aligned}$$

and

$$\begin{aligned}
I_2^{n+1}(e_h^{n+1}) &\le \|\sigma(U_h^n) - \sigma(U^n)\|_{L^2}\|\nabla\Phi^{n+1}\|_{L^\infty}^2\|e_h^{n+1}\|_{L^2}\\
&\le C\|e_h^{n+1}\|_{L^2}^2 + C\|e_h^n\|_{L^2}^2 + Ch^{2r+2},
\end{aligned}$$

where we have used the embedding inequality $\|\nabla\Phi^{n+1}\|_{L^\infty} \le C\|\Phi^{n+1}\|_{W^{2,4}}$ in Lemma 2.1 and noted $\|\Phi^{n+1}\|_{W^{2,4}} \le C$ and $\|U\|_{H^{r+1}} \le C$ which have been proved in Theorem 3.1.

By inverse inequality, for the third term we have

$$\begin{aligned} I_3^{n+1}(e_h^{n+1}) &\le 2\|\sigma(U_h^n)-\sigma(U^n)\|_{L^2}\|\nabla\Phi^{n+1}\|_{L^\infty}\|\nabla(\Phi_h^{n+1}-\Phi^{n+1})\|_{L^3}\|e_h^{n+1}\|_{L^6} \\ &\le \epsilon\|e_h^{n+1}\|_{H^1}^2+\epsilon^{-1}C(\|e_h^n\|_{L^2}^2+h^{2r+2})(h^{-d/3}\|\nabla\eta_h^{n+1}\|_{L^2}^2+h^{2r+2}) \\ &\le \epsilon\|e_h^{n+1}\|_{H^1}^2+\epsilon^{-1}C(\|e_h^n\|_{L^2}^2+h^{2r+2})(h^{-d/3}\|\nabla\eta_h^{n+1}\|_{L^2}^2+Ch^{2r+2})\,. \end{aligned}$$

Moreover, for the fourth term

$$\begin{aligned} I_4^{n+1}(e_h^{n+1}) &\le \|\sigma(U_h^n)\|_{L^\infty}\|\nabla(\Phi_h^{n+1}-\Phi^{n+1})\|_{L^2}\|\nabla(\Phi_h^{n+1}-\Phi^{n+1})\|_{L^3}\|e_h^{n+1}\|_{L^6} \\ &\le C\|e_h^{n+1}\|_{H^1}(h^{-d/6}\|\nabla\eta_h^{n+1}\|_{L^2}^2+h^r\|\nabla\eta_h^{n+1}\|_{L^2}+h^{2r}) \\ &\le \epsilon\|e_h^{n+1}\|_{H^1}^2+\epsilon^{-1}C(h^{-d/6}\|\nabla\eta_h^{n+1}\|_{L^2}^2+h^r\|\nabla\eta_h^{n+1}\|_{L^2}+h^{2r})^2. \end{aligned}$$

Finally, by integration by parts and noting the fact that $-\nabla\cdot(\sigma(U^n)\nabla\Phi^{n+1})=0$, we have

$$\begin{aligned} I_5^{n+1}(e_h^{n+1}) &= -2\left(\Phi_h^{n+1}-\Phi^{n+1},\nabla\cdot(\sigma(U^n)\nabla\Phi^{n+1}e_h^{n+1})\right) \\ &= -2\left(\Phi_h^{n+1}-\Phi^{n+1},\sigma(U^n)\nabla\Phi^{n+1}\cdot\nabla e_h^{n+1}\right) \\ &\le C\|\Phi_h^{n+1}-\Phi^{n+1}\|_{L^2}\|\sigma(U^n)\|_{L^\infty}\|\nabla\Phi^{n+1}\|_{L^\infty}\|\nabla e_h^{n+1}\|_{L^2} \\ &\le \epsilon\|e_h^{n+1}\|_{H^1}^2+\epsilon^{-1}C(\|\eta_h^{n+1}\|_{L^2}^2+h^{2r+2}). \end{aligned}$$

On the other hand, taking $\xi_\phi=\eta_h^{n+1}$ into the Eq. (4.7) gives

$$\begin{aligned} \|\nabla\eta_h^{n+1}\|_{L^2} &\le C\|(\sigma(U_h^n)-\sigma(U^n))\nabla\Phi_h^{n+1}\|_{L^2}+C\|\sigma(U^n)\nabla(\Phi^{n+1}-\Pi_h\Phi^{n+1})\|_{L^2} \\ &\le C\|U_h^n-U^n\|_{L^6}\|\nabla\eta_h^{n+1}\|_{L^3}+C\|U_h^n-U^n\|_{L^2}+Ch^r \\ &\le Ch^{-d/6}(\|e_h^n\|_{H^1}+h^r)\|\nabla\eta_h^{n+1}\|_{L^2}+C\|e_h^n\|_{L^2}+Ch^r\,. \end{aligned} \tag{4.8}$$

By the assumption of the induction that (4.3) holds for $n\le k-1$ and applying inverse inequality, we have

$$\|e_h^n\|_{L^2}\le C_1h^{r+1},\quad \|e_h^n\|_{H^1}\le CC_1h^r.$$

Thus, taking the above inequalities into (4.8) results in

$$\|\nabla\eta_h^{n+1}\|_{L^2}\le(CC_1h^{r+1-d/6}+Ch^{r-d/6})\|\nabla\eta_h^{n+1}\|_{L^2}+CC_1h^{r+1}+Ch^r,$$

and therefore, requiring $CC_1h^{r+1-d/6}+Ch^{r-d/6}\le 1/2$ and $C_1h\le 1$ yields

$$\|\nabla\eta_h^{n+1}\|_{L^2}\le Ch^r. \tag{4.9}$$

Moreover, we use Aubin–Nitsche technique [6] to estimate $\|\eta_h^n\|_{L^2}$. Rewriting (4.7) by

$$\left(\sigma(U^n)\nabla(\Phi^{n+1}-\Phi_h^{n+1}),\ \nabla\xi_\phi\right)+\left((\sigma(U^n)-\sigma(U_h^n))\nabla\Phi_h^{n+1},\quad \nabla\xi_\phi\right)=0,\quad \forall\,\xi_\phi\in V_h, \tag{4.10}$$

and defining ψ as the solution to the elliptic equation

$$-\nabla\cdot\left(\sigma(U^n)\nabla\psi\right)=\Phi^{n+1}-\Phi_h^{n+1}, \tag{4.11}$$

with Dirichlet boundary condition $\psi=0$ on $\partial\Omega$. With Lemma 3.1, it can be deduced that $\|\psi\|_{H^2}\le C\|\Phi^{n+1}-\Phi_h^{n+1}\|$.

It is easy to see that taking $\xi_\phi = \Pi_h \psi$ into (4.10) gives

$$\left(\sigma(U^n)\nabla(\Phi^{n+1} - \Phi_h^{n+1}),\ \nabla\Pi_h\psi\right) + \left((\sigma(U^n) - \sigma(U_h^n))\nabla\Phi_h^{n+1},\ \nabla\Pi_h\psi\right) = 0.$$

With the help of the estimate (4.9) and the above identity, by multiplying $(\Phi^{n+1} - \Phi_h^{n+1})$ at both sides of Eq. (4.11), we have

$$\begin{aligned}
\|\Phi^{n+1} - \Phi_h^{n+1}\|_{L^2}^2 &= \left(\sigma(U^n)\nabla(\Phi^{n+1} - \Phi_h^{n+1}),\ \nabla\psi\right) \\
&= \left(\sigma(U^n)\nabla(\Phi^{n+1} - \Phi_h^{n+1}),\ \nabla(\psi - \Pi_h\psi)\right) \\
&\quad - \left((\sigma(U^n) - \sigma(U_h^n))\nabla\Phi_h^{n+1},\ \nabla\Pi_h\psi\right) \\
&\le C\|\nabla(\Phi^{n+1} - \Phi_h^{n+1})\|_{L^2}\|\nabla(\psi - \Pi_h\psi)\|_{L^2} \\
&\quad + C\|U^n - U_h^n\|_{L^2}(\|\nabla\eta_h^{n+1}\|_{L^3} + \|\nabla\Pi_h\Phi^{n+1}\|_{L^3})\|\nabla\Pi_h\psi\|_{L^6} \\
&\le Ch\|\nabla(\Phi^{n+1} - \Phi_h^{n+1})\|_{L^2}\|\psi\|_{H^2} \\
&\quad + C\|U^n - U_h^n\|_{L^2}(h^{-d/6}\|\nabla\eta_h^{n+1}\|_{L^2} + C)\|\psi\|_{H^2} \\
&\le Ch^{r+1}\|\psi\|_{H^2} + CC_1(h^{2r+1-d/6} + h^{r+1})\|\psi\|_{H^2},
\end{aligned}$$

which in fact implies that when $C_1(h^{r+1-d/6} + h) \le 1$

$$\|\Phi^{n+1} - \Phi_h^{n+1}\|_{L^2} \le Ch^{r+1}. \tag{4.12}$$

Therefore, with (4.9) and (4.12) and estimates for I_j^{n+1}, $j = 1, \ldots, 5$, by taking $\xi_u = e_h^{n+1}$ into the spatial error Eq. (4.6), we can derive

$$\begin{aligned}
D_\tau\left(\|e_h^{n+1}\|_{L^2}^2\right) + \|\nabla e_h^{n+1}\|_{L^2}^2 &\le \epsilon\|e_h^{n+1}\|_{H^1}^2 + C\epsilon^{-1}\|e_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}\|e_h^n\|_{L^2}^2 \\
&\quad + C(\epsilon^{-1} + \|D_\tau U^{n+1}\|_{H^{r*}}^2)h^{2r+2}.
\end{aligned}$$

Thus we can choose a small positive number ϵ and use Gronwall's inequality with induction to obtain that there exists a $\tau_0 > 0$ such that when $\tau < \tau_0$

$$\begin{aligned}
\|e_h^{n+1}\|_{L^2}^2 + \tau\sum_{m=1}^{n+1}\|e_h^m\|_{H^1}^2 &\le \exp\left(\frac{TC}{1 - \tau C}\right)(CT + C_2)h^{2r+2} \\
&\le \exp(2TC)(CT + C_2)h^{2r+2},
\end{aligned}$$

where we have used $\sum_{n=1}^N \|D_\tau U^n\|_{H^{r*}}^2 \tau \le C$ and noted the homogeneous Dirichlet boundary condition. Thus, (4.3) holds for $n = k$ if we take $C_1 \ge \exp(2TC)(CT + C_2)$. We complete the induction.

With the above estimates, we have the following result directly from (4.12)

$$\|\Phi^n - \Phi_h^n\|_{L^2} \le Ch^{r+1}. \tag{4.13}$$

The proof of Theorem 4.1 is complete. □

We complete the proof of Theorem 2.1 by combining Theorem 3.1, Theorem 4.1, the interpolation error estimate (2.3) and the projection error estimate (2.4). □

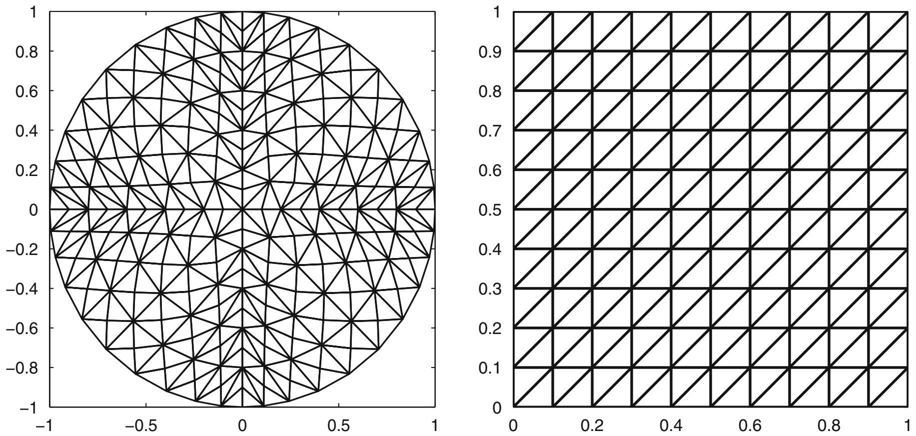

Fig. 1 The FEM meshes of the unit circle and the unit square with $M = 10$

5 Numerical Results

In this section, we provide some numerical examples to confirm our theoretical analysis. The computations are performed with free software FEniCS [22]. We set the final time $T = 1.0$ in all the computations.

Example 5.1 $(2d)$ We rewrite the system (1.1)–(1.2) by

$$\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\phi|^2 + f_1, \tag{5.1}$$

$$-\nabla \cdot (\sigma(u)\nabla\phi) = f_2, \tag{5.2}$$

and the electric conductivity σ takes the form

$$\sigma(u) = \frac{1}{1+u^2} + 1.$$

The functions $f_1,\ f_2$ and the initial and boundary conditions are determined correspondingly by the exact solution

$$u(x, y, t) = \exp(x + y - t), \quad \phi(x, y, t) = 1 + \sin(x + y + t).$$

Here we only present convergence rate results of the scheme (2.5)–(2.6), and it should be remarked that the fully decoupled scheme (2.7)–(2.8) has similar convergence results. We test the scheme (2.5)–(2.6) on two different domains, one is the unit circle with $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ and another is the unit square with $\Omega = (0, 1) \times (0, 1)$. A regular triangulation with M elements in the radial direction is made on the unit circle, and a uniform triangulation with $M + 1$ nodes in both horizontal and vertical directions is made on the unit square, see Fig. 1 for the case $M = 10$. Here we can see the mesh size $h = O(1/M)$. We solve Eqs. (5.1)–(5.2) by the two linearized backward Euler scheme (2.5)–(2.6) and the fully decoupled scheme (2.7)–(2.8), denoted by Scheme I and Scheme II, respectively.

To confirm our error estimates in Theorem 2.1, we choose $\tau = h^{r+1},\ r = 1,\ 2,\ 3$, for the linear, quadratic and cubic FE methods, respectively. Thus, from our theoretical analysis the L^2-norm errors are of scale $O(h^{r+1} + h^{r+1}) \sim O(h^{r+1})$ and the errors in H^1-norm are of scale $O(h^{r+1} + h^r) \sim O(h^r)$. We present the L^2 and H^1-norm errors of Scheme I in Table 1

Table 1 L^2 and H^1 errors of Scheme I for the unit circle (*Example 5.1. (2d)*)

	Linear ($\tau = h^2$)		Quadratic ($\tau = h^3$)		Cubic ($\tau = h^4$)	
	L^2	H^1	L^2	H^1	L^2	H^1
L^2 and H^1 errors of $U_h^N - u(\cdot, 1)$						
M=5	1.8169e−02	2.0747e−01	6.1606e−04	1.1769e−02	9.8082e−05	5.2944e−04
M=10	4.6111e−03	1.0088e−01	8.1802e−05	2.9602e−03	6.4353e−06	5.9366e−05
M=20	1.1563e−03	4.9836e−02	1.0727e−05	7.4498e−04	4.1246e−07	7.1520e−06
Order	1.99	1.03	2.92	1.99	3.95	3.10
L^2 and H^1 errors of $\Phi_h^N - \phi(\cdot, 1)$						
M=5	2.6713e−02	2.4261e−01	6.4313e−04	1.3113e−02	7.7874e−05	5.8134e−04
M=10	6.8652e−03	1.2175e−01	8.4243e−05	3.3166e−03	5.0630e−06	6.6763e−05
M=20	1.7398e−03	6.0891e−02	1.0900e−05	8.3713e−04	3.2026e−07	8.2091e−06
Order	1.97	1.00	2.94	1.98	3.96	3.07

Table 2 L^2 and H^1 errors of Scheme I for the unit square (*Example 5.1.* (2d))

	Linear ($\tau = h^2$)		Quadratic ($\tau = h^3$)		Cubic ($\tau = h^4$)	
	L^2	H^1	L^2	H^1	L^2	H^1
L^2 and H^1 errors of $U_h^N - u(\cdot, 1)$						
M=5	1.2792e−02	2.2304e−01	2.6174e−04	8.9883e−03	3.0465e−05	2.7032e−04
M=10	3.0923e−03	1.0833e−01	3.3569e−05	2.2344e−03	1.8984e−06	2.9916e−05
M=20	7.6450e−04	5.3636e−02	4.2697e−06	5.5831e−04	1.1988e−07	3.5972e−06
Order	2.03	1.03	2.97	2.00	3.99	3.12
L^2 and H^1 errors of $\Phi_h^N - \phi(\cdot, 1)$						
M=5	9.2838e−03	1.5860e−01	1.6634e−04	3.9396e−03	2.9810e−05	2.1941e−04
M=10	2.2941e−03	7.8388e−02	2.1810e−05	9.8571e−04	1.8648e−06	2.2564e−05
M=20	5.7123e−04	3.9040e−02	2.7859e−06	2.4629e−04	1.1657e−07	2.6397e−06
Order	2.01	1.01	2.95	2.00	4.00	3.19

for the unit circle and in Table 2 for the unit square, respectively. It is clear that for both unit circle and unit square the L^2-norm errors of u and ϕ are proportional to h^{r+1} and the H^1-norm errors are proportional to h^r, $r = 1, 2, 3$, which indicate the optimal convergence rates of the methods.

To show the unconditional convergence of the two schemes, we use the linear FE method to solve (5.1)–(5.2) with three different time steps $\tau = 0.01, 0.05, 0.25$ on gradually refined meshes with $M = 10i$, $i = 1, 2, \ldots, 6$ for both domains. The L^2-norm errors are given in Fig. 2 for Scheme I and in Fig. 3 for Scheme II, respectively. We should remark that the two schemes with linear FE approximations give L^2-norm errors of the scale $O(\tau + h^2)$. From Fig. 2 and (3), we can see that for a fixed τ, when refining the mesh gradually, the L^2-norm errors converge to a constant, i.e., the temporal error of the scale $O(\tau)$. It is easy to see that for both domains the two proposed schemes are unconditionally convergent (stable).

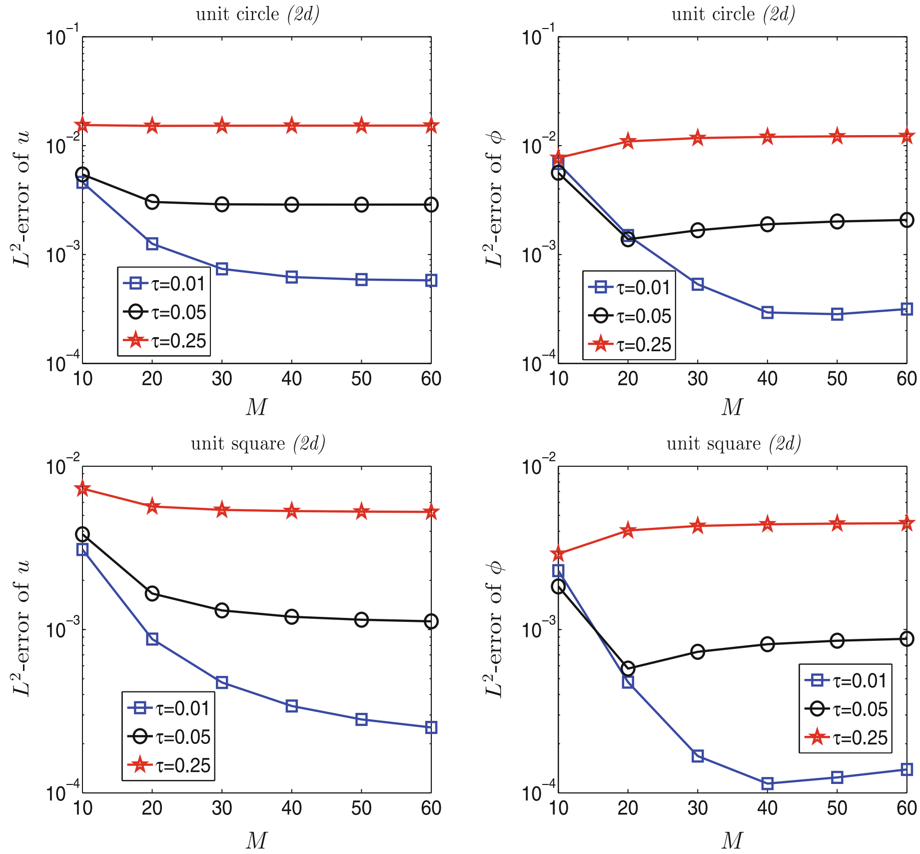

Fig. 2 L^2-norm errors of scheme I with linear FEM

Example 5.2 (3d) We consider Eqs. (5.1)–(5.2) in three-dimensional space with exact solution

$$u(x, y, z, t) = \exp(2x + y - z)(2t + \sin(t)),$$
$$\phi(x, y, z, t) = \sin(x - 2y)\cos(z)\exp(t),$$

where $\Omega = \{(x, y, z) : x^2 + y^2 + z^2 < 1\}$ is the unit ball. We solve the system by these two schemes with quadratic FE method. We take the time steps $\tau = 0.01, 0.05, 0.25$ for the scheme I and $\tau = 0.005, 0.01, 0.05$ for the scheme II. For the spatial discretizations, We use a regular tetrahedra partition with M elements in the radial direction (see Fig. 4 for the case $M = 10$). We refine the mesh gradually by taking $M = 5i$, $i = 1, 2, \ldots, 5$. Plots for L^2-norm errors against M are given in Fig. 5 for scheme (2.5)–(2.6) and in Fig. 6 for the fully decoupled scheme (2.7)–(2.8), respectively. From Theorem 2.1, the L^2-norm errors are of scale $O(\tau + h^3)$. From Figs. 5 and 6, we can see that if we fix τ and refine the mesh gradually, the L^2-norm errors will asymptotically converge to a constant.

This phenomenon also indicates the unconditional stability of the two schemes in three dimensional space. Previous error analysis for three-dimensional problems often requires a stronger time step restriction than for two-dimensional problems. Our numerical results

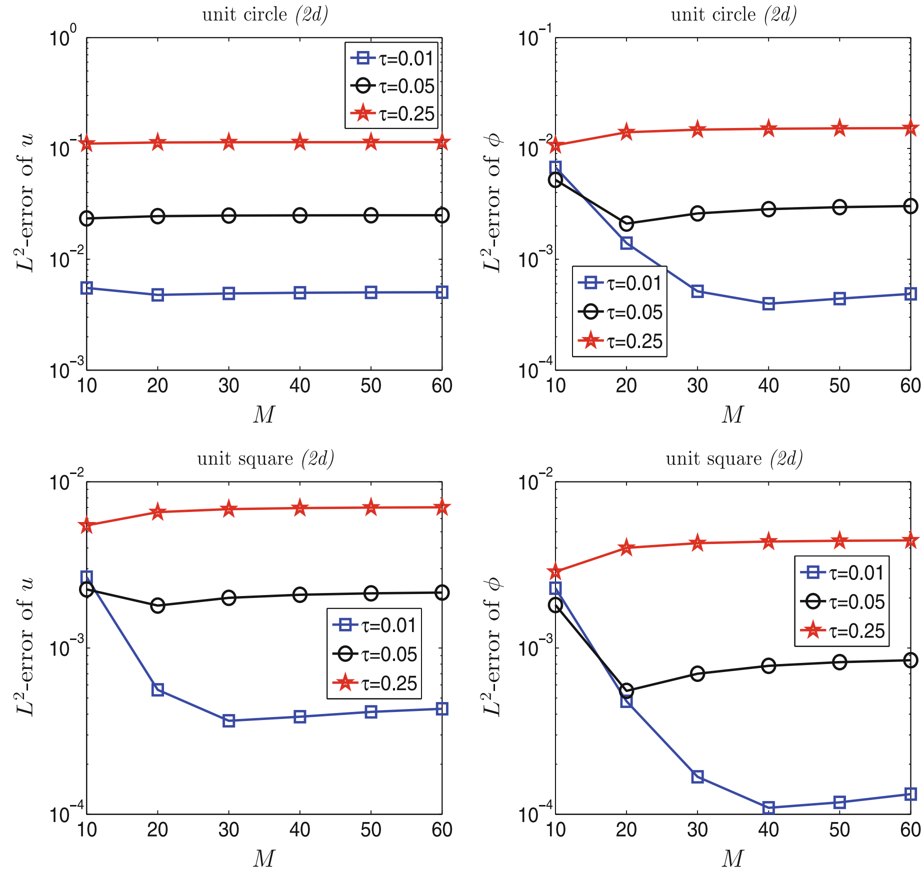

Fig. 3 L^2-norm errors of scheme II with linear FEM

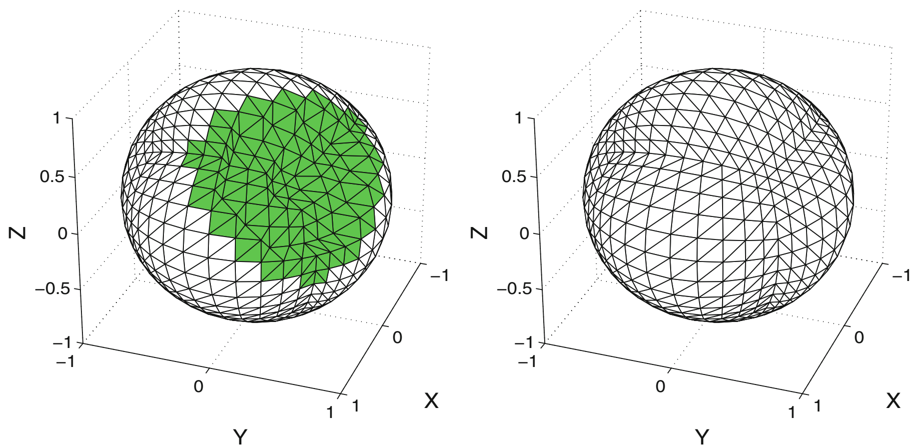

Fig. 4 The three dimensional mesh: inner structure and the surface of the partition with total 1,331 nodes and 6,000 elements ($M = 10$)

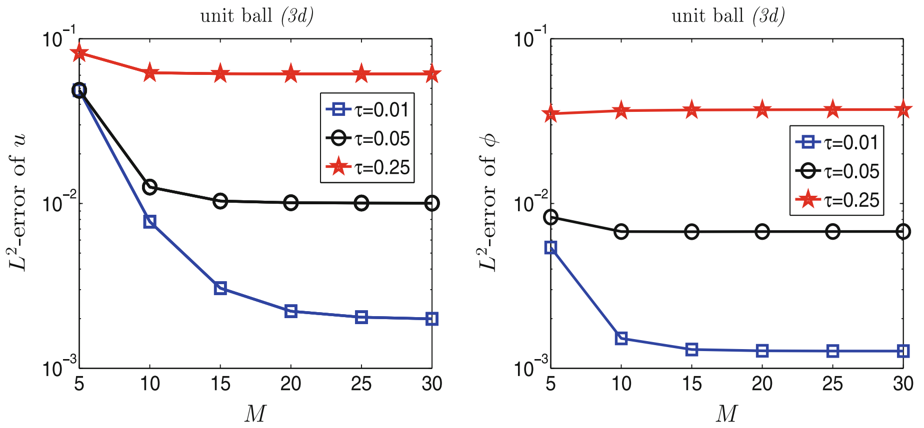

Fig. 5 L^2-norm errors of scheme I with quadratic FEM on a unit ball

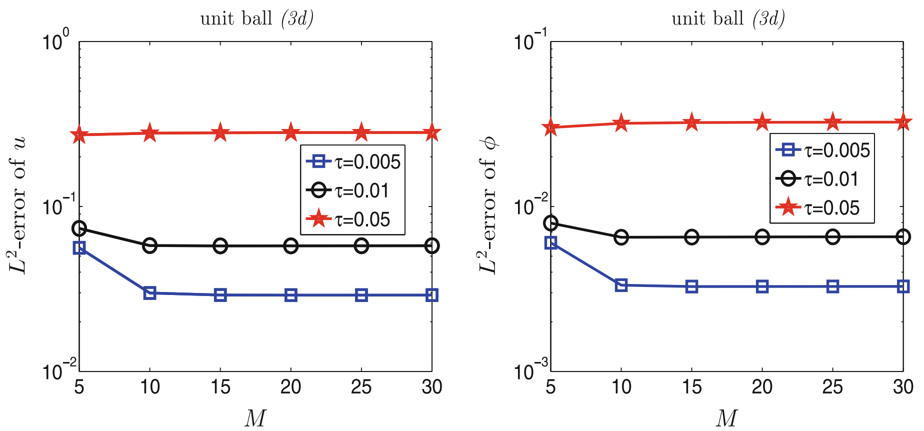

Fig. 6 L^2-norm errors of scheme II with quadratic FEM on a unit ball

for both two and three dimensional problems show clearly that no time step condition is needed.

6 Conclusions

We have presented two linearized backward Euler schemes for the nonlinear Joule heating equations in two and three dimensional spaces and provided unconditionally optimal error estimates for the r-order Galerkin FEMs ($1 \leq r \leq 3$) in both L^2 and H^1 norms. Numerical results for both two and three dimensional problems confirm our theoretical analysis and show clearly the unconditional stability of the two schemes. The technique presented in this paper can be applied to analyze higher order finite element methods for other nonlinear parabolic equations.

Acknowledgments The author would like to thank Professor Weiwei Sun for valuable suggestions and many constructive discussions. The work of the author was supported in part by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CityU 102712).

References

1. Achdou, Y., Guermond, J.L.: Convergence analysis of a finite element projection/Lagrange–Galerkin method for the incompressible Navier–Stokes equations. SIAM J. Numer. Anal. **37**, 799–826 (2000)
2. Akrivis, G., Larsson, S.: Linearly implicit finite element methods for the time dependent Joule heating problem. BIT **45**, 429–442 (2005)
3. Allegretto, W., Xie, H.: Existence of solutions for the time dependent thermistor equation. IMA. J. Appl. Math. **48**, 271–281 (1992)
4. Allegretto, W., Yan, N.: A posteriori error analysis for FEM of thermistor problems. Int. J. Numer. Anal. Model. **3**, 413–436 (2006)
5. Allegretto, W., Lin, Y., Ma, S.: Existence and long time behavior of solutions to obstacle thermistor equations. Discrete Contin. Dyn. Syst. Ser. A **8**, 757–780 (2002)
6. Brenner, S., Scott, L.: The Mathematical Theory of Finite Element Methods. Springer, New York (2002)
7. Cimatti, G.: Existence of weak solutions for the nonstationary problem of the joule heating of a conductor. Ann. Mat. Pura Appl. **162**, 33–42 (1992)
8. Chen, Y.Z., Wu, L.C.: Second order elliptic equations and elliptic systems, Translations of Mathematical Monographs 174, AMS, USA (1998)
9. Chen, Z., Hoffmann, K.-H.: Numerical studies of a non-stationary Ginzburg–Landau model for superconductivity. Adv. Math. Sci. Appl. **5**, 363–389 (1995)
10. Deng, Z., Ma, H.: Optimal error estimates of the Fourier spectral method for a class of nonlocal, nonlinear dispersive wave equations. Appl. Numer. Math. **59**, 988–1010 (2009)
11. Douglas Jr, J., Ewing, R., Wheeler, M.F.: A time-discretization procedure for a mixed finite element approximation of miscible displacement in porous media. RAIRO Anal. Numer. **17**, 249–265 (1983)
12. Elliott, C.M., Larsson, S.: A finite element model for the time-dependent joule heating problem. Math. Comput. **64**, 1433–1453 (1995)
13. Ewing, R.E., Wheeler, M.F.: Galerkin methods for miscible displacement problems in porous media. SIAM J. Numer. Anal. **17**, 351–365 (1980)
14. He, Y.: The Euler implicit/explicit scheme for the 2D time-dependent Navier–Stokes equations with smooth or non-smooth initial data. Math. Comput. **77**, 2097–2124 (2008)
15. Heywood, J.G., Rannacher, R.: Finite element approximation of the nonstationary Navier–Stokes problem IV: error analysis for second-order time discretization. SIAM J. Numer. Anal. **27**, 353–384 (1990)
16. Holst, M.J., Larson, M.G., Malqvist, A., Axel, R.: Convergence analysis of finite element approximations of the Joule heating problem in three spatial dimensions. BIT **50**, 781–795 (2010)
17. Hou, Y., Li, B., Sun, W.: Error analysis of splitting Galerkin methods for heat and sweat transport in textile materials. SIAM J. Numer. Anal. **51**, 88–111 (2013)
18. Kellogg, B., Liu, B.: The analysis of a finite element method for the Navier–Stokes equations with compressibility. Numer. Math. **87**, 153–170 (2000)
19. Li, B., Sun, W.: Error analysis of linearized semi-implicit Galerkin finite element methods for nonlinear parabolic equations. Int. J. Numer. Anal. Model. **10**, 622–633 (2013)
20. Li, B., Sun, W.: Unconditional convergence and optimal error estimates of a Galerkin-mixed FEM for incompressible miscible flow in porous media. SIAM J. Numer. Anal. **51**, 1959–1977 (2013)
21. Li, B.: Mathematical modeling, analysis and computation for some complex and nonlinear flow problems. PhD Thesis, City University of Hong Kong, Hong Kong (2012)
22. Logg, A., Mardal, K.-A., Wells, G.N., et al.: Automated Solution of Differential Equations by the Finite Element Method. Springer, Berlin (2012). doi:10.1007/978-3-642-23099-8
23. Ma, H., Sun, W.: Optimal error estimates of the Legendre–Petrov–Galerkin method for the Korteweg–de Vries equation. SIAM J. Numer. Anal. **39**, 1380–1394 (2001)
24. Mu, M., Huang, Y.: An alternating Crank–Nicolson method for decoupling the Ginzburg–Landau equations. SIAM J. Numer. Anal. **35**, 1740–1761 (1998)
25. Nirenberg, L.: An extended interpolation inequality. Ann. Scuola Norm. Sup. Pisa (3) **20**, 733–737 (1966)
26. Sun, W., Sun, Z.: Finite difference methods for a nonlinear and strongly coupled heat and moisture transport system in textile materials. Numer. Math. **120**, 153–187 (2012)
27. Thomee, V.: Galerkin Finite Element Methods for Parabolic Problems. Springer, Berlin (2006)
28. Tourigny, Y.: Optimal H^1 estimates for two time-discrete Galerkin approximations of a nonlinear Schrödinger equation. IMA J. Numer. Anal. **11**, 509–523 (1991)
29. Wang, K., Wang, H.: An optimal-order error estimate to ELLAM schemes for transient advection–diffusion equations on unstructured meshes. SIAM J. Numer. Anal. **48**, 681–707 (2010)
30. Wu, H., Ma, H., Li, H.: Optimal error estimates of the Chebyshev–Legendre spectral method for solving the generalized Burgers equation. SIAM J. Numer. Anal. **41**, 659–672 (2003)
31. Yue, X.: Numerical analysis of nonstationary thermistor problem. J. Comput. Math. **12**, 213–223 (1994)

32. Yuan, G.: Regularity of solutions of the thermistor problem. Appl. Anal. **53**, 149–155 (1994)
33. Yuan, G., Liu, Z.: Existence and uniqueness of the C^α solution for the thermistor problem with mixed boundary value. SIAM J. Math. Anal. **25**, 1157–1166 (1994)
34. Zhao, W.: Convergence analysis of finite element method for the nonstationary thermistor problem. Shandong Daxue Xuebao **29**, 361–367 (1994)
35. Zhou, S., Westbrook, D.R.: Numerical solutions of the thermistor equations. J. Comput. Appl. Math. **79**, 101–118 (1997)
36. Zlámal, M.: Curved elements in the finite element method. I. SIAM J. Numer. Anal. **10**, 229–240 (1973)