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C^0-Nonconforming Triangular Prism Elements for the Three-Dimensional Fourth Order Elliptic Problem

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Abstract In this paper, using the bubble functions, we construct two C^0-nonconforming triangular prism elements for the fourth order elliptic problem in three dimensions. By the abstract convergence theorem in (Chen et al. in Numer. Math. (2012, accepted)), one element is proved to be of first order convergence and the other one is proved to be of second order convergence.

Keywords Fourth order elliptic problem · Bubble functions · C^0-nonconforming elements · Error estimates

1 Introduction

The construction of appropriate finite element spaces for the fourth order elliptic boundary value problem is an appealing subject. This problem has been well-studied in two dimensional spaces [4, 5, 8, 13, 16, 23, 27]. Recently some papers for three dimensional fourth order elliptic problem were presented [24–27]. In those papers, the domain is divided into triangles or rectangles in two dimensional spaces and tetrahedrons or cuboid in three dimensional spaces, respectively. However, there has been very little work devoted to triangular prism elements.

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A conforming finite element space for the fourth order elliptic problem consists of piecewise polynomials that are globally continuously differentiable (C^1). To meet this smoothness requirement, the degree of piecewise polynomials must be very high. In two dimensional case, Argyris element [8] with 5-degree polynomials, Bell element [8] with incomplete 5-degree polynomials are conforming triangular elements. Bogner-Fox-Schmit (BFS) element [8] with bicubic degree polynomials is a conforming rectangular element. To lower the polynomial degree, some macroelements were created on triangle grids, see e.g., [8, 15]. Recently, a macro type of biquadratic C^1 finite element was constructed on rectangle grids [10], which is a rectangular version of the C^1 Powell-Sabin element [15]. In three dimensional case, the situation becomes more complicated. A conforming tetrahedral element was first constructed in [28] using 9-degree polynomials, The number of degrees of freedom is 220. In [5], a three-dimensional conforming BFS element on cuboid mesh with tri-cubic degree polynomials and 64 degrees of freedom was developed. This element is of second order convergence. Macroelements have also been developed for the three-dimensional problems, see, e.g. [10].

To reduce the order of polynomials and degrees of freedom on each element, many lower order nonconforming elements have been constructed and used in practice. In two dimensions, there are many well-known nonconforming elements. Morley element [8, 9, 11–13, 18], Veubeke element [8, 23] are triangular elements and they are even not C^0-continuous. Zienkiewicz element [8, 13] is a C^0-triangular element, but this element is convergent only on some special meshes [16], because the mean values of normal derivatives on the boundary of the element are not continuous across the elements. The rectangle Morley element [27] is not C^0-continuous. Adini or ACM element [8, 13] is a C^0-rectangular element. The mean values of normal derivatives on the boundary of Adini element are not continuous across the elements, and its convergence depends on the special geometric property of the rectangular mesh.

In three-dimensional case, several nonconforming elements for the fourth order problems were developed in [24–27]. On the tetrahedral meshes, the three-dimensional Morley element was presented in [24]. The three-dimensional Zienkiewicz element, and a quasi-conforming element by modifying the three-dimensional Zienkiewicz element were presented in [25, 26]. On the cuboid meshes, the three-dimensional Morley-like element, the three-dimensional Adini element, and the three-dimensional BFS-like element were presented in [27]. All of these nonconforming elements are of first order convergence and they are the generalizations of the corresponding two-dimensional elements. Among them, three-dimensional Zienkiewicz element and three-dimensional BFS-like element are C^0-continuous, others are non-C^0-continuous. It should be pointed out that the above three-dimensional BFS-like element is different from the one in [5]. It is nonconforming and only of first order convergence, while the one in [5] is conforming and of second order convergence. We note that the degrees of freedom of these nonconforming elements are substantially smaller than those of known conforming elements. There are some other ways constructing elements. Quasi-conforming elements [22, 29], Generalized-conforming elements [14, 19] and Double set parameter elements [4, 6] are nonstandard elements. We do not describe them in detail here.

In [21] Stummel presented a sufficient and necessary condition for the convergence of nonconforming finite elements, named Generalized Patch-Test, but it is difficult to use in practice. In [17] Shi presented a sufficient condition, named F-E-M Test, which is easier to use in practice. For the fourth-order elliptic problem, to satisfy the strong F-E-M Test, the function values and the first-order derivatives of the shape functions should be continuous in the mean across the elements. In three-dimensional case it makes the order of element interpolation matrix very high. As a result, it is difficult to check the nonsingularity of this matrix.

For the columnar regions with complex base, the triangular prism elements have advantages than the tetrahedral and cuboid elements. In this paper, we construct two C^0-nonconforming elements for the fourth order elliptic problem. The idea is to divide the shape function space into two subspaces using bubble functions. One subspace is responsible for the C^0-continuity of the shape functions and getting the approximation error. Another one, which contains the bubble functions, is responsible for the continuity in the mean of the normal derivatives of the shape functions across the elements and getting the consistency error. The resulting element interpolation matrix is a block lower triangular matrix which greatly simplifies the proof of the nonsingularity of this matrix. In [7], an abstract convergence theorem was given, which builds a theoretical frame to prove the convergence of C^0-nonconforming elements for the fourth order elliptic problem. In this paper, by using this convergence theorem, for the two proposed C^0-nonconforming triangular prism elements, we can prove that one element is of first order convergence and the other one is of second order convergence.

The rest of this paper is organized as follows. Section 2 gives the preliminaries. Section 3 gives the abstract convergence lemma. Section 4 gives detailed descriptions of two triangular prism elements. Section 5 shows the error estimates of the two elements. Some concluding remarks will be made at the end of the paper.

2 Preliminaries

We consider the following fourth order elliptic boundary value problem [8]:

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset R^3$ is a bounded convex domain with Lipschitz continuous boundary $\partial\Omega$, $f \in L^2(\Omega)$, $\mathbf{n} = (n_1, n_2, n_3)^T$ is the unit outer normal to $\partial\Omega$ and Δ is the standard Laplace operator.

$\forall u, v \in H_0^2(\Omega)$, define

$$a(u, v) = \int_\Omega \sum_{i,j=1}^{3} \partial_{ij} u \partial_{ij} v dx, \quad f(v) = \int_\Omega f v dx. \tag{2.2}$$

Here $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. The weak form of (2.1) is: Seek $u \in H_0^2(\Omega)$ satisfying

$$a(u, v) = f(v), \quad \forall v \in H_0^2(\Omega). \tag{2.3}$$

We adopt the standard notation $H^m(\Omega)$ for Sobolev space [1] on Ω with norm

$$\|v\|_{m,\Omega}^2 = \sum_{|\alpha| \le m} \left\| D^\alpha v \right\|_{0,\Omega}^2,$$

and semi-norm $|v|_{m,\Omega}^2 = \sum_{|\alpha|=m} \|D^\alpha v\|_{0,\Omega}^2$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is an index,

$$|\alpha| = \sum_{i=1}^{3} \alpha_i, \, D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}, \quad \|w\|_{0,\Omega}^2 = \int_\Omega w^2 dx.$$

We set

$$H_0^m(\Omega) = \left\{ v \in H^m(\Omega); \; \frac{\partial^j v}{\partial n^j} = 0, \; \text{on } \partial\Omega, \; 0 \le j \le m-1 \right\}.$$

The energy norm of (2.2) is defined by

$$|||v||| = a(v,v)^{\frac{1}{2}} = |v|_{2,\Omega}.$$

By Poincaré inequality [8], it is well known that $|\cdot|_{2,\Omega}$ is a norm on $H_0^2(\Omega)$ and is equivalent to $\|\cdot\|_{2,\Omega}$. Then (2.3) has a unique solution by Lax-Milgram Theorem [8].

Let $\mathcal{T}_h$ be a triangulation of Ω into triangular prisms with mesh size h, and $\{\mathcal{T}_h\}$ be a family of triangulations with $h \to 0$. Let (T, P_T, Σ_T) be a finite element where $T \in \mathcal{T}_h$ is a triangular prism, P_T the shape function space and Σ_T the vector of degrees of freedom, and let Σ_T be P_T-unisolvent [8]. Throughout the paper, we assume that $\{\mathcal{T}_h\}$ is regular and quasi-uniform, namely it satisfies that:

$$h_T/\rho_T \le \sigma_1, \qquad h_T/h_{T'} \le \sigma_2, \quad \forall T, T' \in \mathcal{T}_h, \; \forall h, \tag{2.4}$$

where h_T and ρ_T are the diameters of T and the largest ball contained in T, respectively, $\sigma_1 > 0, \sigma_2 > 0$ are constants independent of h. Let $F \subset \partial T$ be a face of T and $\mathcal{F}_h = \{F; F \subset \partial T, T \in \mathcal{T}_h\}$.

For each $\mathcal{T}_h$, the nonconforming finite element space V_h is a piecewise polynomial space and $V_h \not\subset H_0^2(\Omega)$. The discrete problem of (2.3) is: Find $u_h \in V_h$ satisfying

$$a_h(u_h, v_h) = f(v_h), \quad \forall v_h \in V_h, \tag{2.5}$$

where

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} \int_T \sum_{i,j=1}^{3} \partial_{ij} u_h \partial_{ij} v_h dx. \tag{2.6}$$

We introduce the following mesh-dependent energy norm $|||\cdot|||_h$:

$$|||v_h|||_h^2 = \sum_{T \in \mathcal{T}_h} |v_h|_{2,T}^2, \quad \forall v_h \in V_h. \tag{2.7}$$

3 An Abstract Convergence Lemma

For nonconforming elements, the basic mathematical theory has been established [3, 8, 13, 17, 21, 29] for different problems. In this section, we will give an abstract convergence lemma for the fourth order elliptic problem, which was proved in [7].

Suppose $F = T \cap T'$. We define

$$[w]|_F = w|_{T \cap F} - w|_{T' \cap F}; \qquad [w]|_F = w|_F, \quad \text{if } F \subset \partial\Omega.$$

The following result is well known as the Strang Lemma [2, 8, 20].

Lemma 3.1 *Assume that* $|||\cdot|||_h$ *is a norm on* V_h. *Let* u *and* u_h *be the solutions of* (2.3) *and* (2.5), *respectively. Then*

$$|||u-u_h|||_h \le C\left(\inf_{v_h\in V_h}|||u-v_h|||_h + \sup_{w_h\in V_h}\frac{|a_h(u,w_h)-f(w_h)|}{|||w_h|||_h}\right), \tag{3.1}$$

where $C>0$ *is a constant independent of* h.

The first term of (3.1) is the approximation error and the second term of (3.1) is the consistency error.

For any $F\subset\partial T, \forall T\in\mathcal{T}_h$, let $\mathbf{n}=(n_1,n_2,n_3)^T$ be the unit outer normal to F and τ, s be two unit vectors orthogonal each other on F, then we have

$$\partial_j = \beta_{\tau j}\partial_\tau + \beta_{sj}\partial_s + \beta_{nj}\partial_n, \qquad \beta_{\tau j}^2+\beta_{sj}^2+\beta_{nj}^2 = 1, \quad 1\le j\le 3,$$

where

$$\partial_j = \frac{\partial}{\partial x_j}, \quad \partial_\tau = \frac{\partial}{\partial \tau}, \quad \partial_s = \frac{\partial}{\partial s}, \quad \partial_n = \frac{\partial}{\partial n}.$$

By Green formula [8],

$$\begin{aligned} a_h(u,w_h) &= \sum_{T\in\mathcal{T}_h}\int_T\sum_{i,j=1}^3 \partial_{ij}u\,\partial_{ij}w_h\,dx \\ &= \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^3\left\{\int_{\partial T}(\partial_{ij}u\,\partial_j w_h n_i - \partial_{iij}u\,w_h n_j)ds + \int_T \partial_{iijj}u\,w_h\,dx\right\} \\ &= \sum_{T\in\mathcal{T}_h}\int_{\partial T}\left\{\sum_{i,j=1}^3\partial_{ij}u(\beta_{\tau j}\partial_\tau w_h+\beta_{sj}\partial_s w_h+\beta_{nj}\partial_n w_h)n_i - \partial_n\Delta u\,w_h\right\}ds \\ &\quad + \sum_{T\in\mathcal{T}_h}\int_T\Delta^2 u\,w_h\,dx. \end{aligned}$$

Since $\Delta^2 u = f$, we have

$$\begin{aligned} a_h(u,w_h)-f(w_h) = \sum_{T\in\mathcal{T}_h}\int_{\partial T}\Bigg\{\sum_{i,j=1}^3\partial_{ij}u(\beta_{\tau j}\partial_\tau w_h \\ + \beta_{sj}\partial_s w_h + \beta_{nj}\partial_n w_h)n_i - \partial_n\Delta u\,w_h\Bigg\}ds. \end{aligned} \tag{3.2}$$

If $V_h\subset H_0^1(\Omega)$, then

$$\forall F\subset\partial T, \quad \forall T\in\mathcal{T}_h, \quad [w_h]|_F = [\partial_\tau w_h]|_F = [\partial_s w_h]|_F = 0.$$

We get

$$a_h(u,w_h)-f(w_h) = \sum_{T\in\mathcal{T}_h}\sum_{i,j=1}^3\beta_{nj}\sum_{F\subset\partial T}\int_F\partial_{ij}u\,\partial_n w_h n_i\,ds, \quad \forall w_h\in V_h. \tag{3.3}$$

Let I_k be the C^0 piecewise polynomial interpolation operator, such that I_k is affine equivalent and $I_k v = v$, on T, $\forall v \in P_k(T)$, $\forall T \in \mathcal{T}_h$, then it is well known that [2, 8]:

$$|v - I_k v|_{l,T} \le C h^{k+1-l} |v|_{k+1,T}, \quad 0 \le l \le k \ \forall v \in H^{k+1}(T). \tag{3.4}$$

The finite element interpolation operator on V_h is denoted by Π_h. Now we give the following abstract convergence lemma for C^0-nonconforming elements for the fourth order elliptic problem.

Lemma 3.2 [7] *Assume that* $\||\cdot\||_h$ *is a norm of* V_h. *Suppose that there is an integer* $m \ge 2$, *such that*

(H1) $V_h \subset H_0^1(\Omega)$,

(H2) $\||v - \Pi_h v\||_h \le C h^{m-1} |v|_{m+1,\Omega}, \quad \forall v \in H^{m+1}(\Omega)$,

(H3) $\int_F p[\partial_n w_h] ds = 0, \quad \forall p \in P_{m-2}(F),\ \forall F \in F_h,\ \forall w_h \in V_h$,

then

$$\||u - u_h\||_h \le C h^{m-1} |u|_{m+1,\Omega}. \tag{3.5}$$

Here u *and* u_h *are the solutions of* (2.2) *and* (2.5), *respectively,* $C > 0$ *is a constant independent of* h *and* Π_h *is the finite element interpolation operator on* V_h.

4 C^0-Nonconforming Triangular Prism Elements

Let $\hat{T}$ be the triangular prism element with nodes $\hat{a}_i$, $1 \le i \le 6$, the face of $\hat{T}$ opposite $\hat{a}_i\hat{a}_{i+3}$ is denoted by $\hat{F}_i$, $1 \le i \le 3$. Triangle $\hat{a}_1\hat{a}_2\hat{a}_3$ is denoted by $\hat{F}_4$ and triangle $\hat{a}_4\hat{a}_5\hat{a}_6$ is denoted by $\hat{F}_5$. The barycentric coordinates of this two triangles, named $\hat{\lambda}_i$, $1 \le i \le 3$, have the following properties [8]:

$$\hat{\lambda}_i \in P_1(\hat{F}_4), \qquad \hat{\lambda}_i(\hat{a}_j) = \delta_{ij}, \quad \sum_{i=1}^{3} \hat{\lambda}_i = 1, \quad \hat{\lambda}_i|_{\hat{F}_i} = 0,\ 1 \le i, j \le 3, \tag{4.1}$$

where $P_k(\hat{F})$ is the polynomial space of degree not greater than k on $\hat{F}$. Set $\hat{\lambda_4} = \hat{x}_3$, $\hat{\lambda}_5 = 1 - \hat{x}_3$. $\hat{l}_1 = \hat{a}_2\hat{a}_3$, $\hat{l}_2 = \hat{a}_1\hat{a}_3$, $\hat{l}_3 = \hat{a}_1\hat{a}_2$, $\hat{l}_4 = \hat{a}_5\hat{a}_6$, $\hat{l}_5 = \hat{a}_4\hat{a}_6$, $\hat{l}_6 = \hat{a}_4\hat{a}_5$, $\hat{l}_7 = \hat{a}_1\hat{a}_4$, $\hat{l}_8 = \hat{a}_2\hat{a}_5$, $\hat{l}_9 = \hat{a}_3\hat{a}_6$, and $\hat{b}_i$ be midpoint of $\hat{l}_i$, $1 \le i \le 9$.

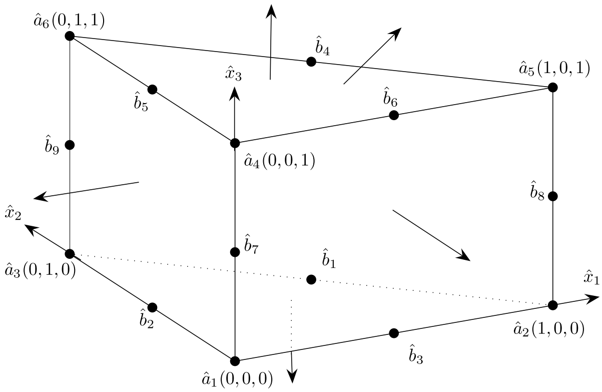
4.1 C^0TP1 Element

The shape function space of C^0TP1 element is taken as:

$$\hat{P}_{TP1} = \hat{P}_2^* \oplus b_{\hat{T}} Q_1, \tag{4.2}$$

where $\hat{P}_2^* = P_2(\hat{T}) \cup \{\hat{x}_1^2\hat{x}_3, \hat{x}_1\hat{x}_3^2, \hat{x}_2^2\hat{x}_3, \hat{x}_2\hat{x}_3^2, \hat{x}_1\hat{x}_2\hat{x}_3\}$, $b_{\hat{T}} = \hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3\hat{\lambda}_4\hat{\lambda}_5$ is the bubble function with $b_{\hat{T}}|_{\hat{F}_i} = 0,\ 1 \le i \le 5$, $Q_1 = \{\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4, \hat{\lambda}_4\hat{\lambda}_5\}$. Here $b_{\hat{T}} Q_1$ is the basis function

Fig. 1 Degrees of freedom of C^0TP1 element



set of the product of $b_{\hat{T}}$ and the set Q_1. It is easy to see that the dimension of $\hat{P}_{TP1}$ is 20. The degrees of freedom of C^0TP1 element are:

$$\hat{v}(\hat{a}_i), \quad 1 \le i \le 6; \qquad \hat{v}(\hat{b}_i), \quad 1 \le i \le 9; \qquad \int_{\hat{F}_i} \frac{\partial \hat{v}}{\partial \hat{n}} d\hat{s}, \quad 1 \le i \le 5. \tag{4.3}$$

See Fig. 1.

The corresponding interpolation operator $\hat{\Pi}_1 : H^3(\hat{T}) \to \hat{P}_{TP1}$ is defined by

$$\begin{cases} (\hat{v} - \hat{\Pi}_1\hat{v})(\hat{a}_i) = 0, & 1 \le i \le 6, \ (\hat{v} - \hat{\Pi}_1\hat{v})(\hat{b}_i) = 0, \ 1 \le i \le 9, \\ \displaystyle\int_{\hat{F}_i} \frac{\partial(\hat{v} - \hat{\Pi}_1\hat{v})}{\partial \hat{n}} d\hat{s} = 0, & 1 \le i \le 5. \end{cases} \tag{4.4}$$

Lemma 4.1 *The interpolation operator $\hat{\Pi}_1$ is well posed, namely, the degrees of freedom* (4.3) *are $\hat{P}_{TP1}$-unisolvent.*

Proof Because both the dimension of $\hat{P}_{TP1}$ and the number of degrees of freedom are 20, it is sufficient to show that if $\hat{v} \in \hat{P}_{TP1}$ and

$$\hat{v}(\hat{a}_i) = 0, \quad 1 \le i \le 6; \qquad \hat{v}(\hat{b}_i) = 0, \quad 1 \le i \le 9; \qquad \int_{\hat{F}_i} \frac{\partial \hat{v}}{\partial \hat{n}} d\hat{s} = 0, \quad 1 \le i \le 5,$$

then $v \equiv 0$.

$\hat{v}|_{\hat{F}_i} \in P_2(\hat{F}_i)$, $i = 4, 5$, and $\hat{v} = 0$, at the 3 vertices of $\hat{F}_i$ and the middle points of 3 sides of $\hat{F}_i$, hence

$$\hat{v}|_{\hat{F}_i} = 0, \quad i = 4, 5. \tag{4.5}$$

$\hat{v}|_{\hat{F}_i} \in P_2(\hat{F}_i) \oplus \{\hat{x}_1^2\hat{x}_3, \hat{x}_1\hat{x}_3^2\}$ (or $\{\hat{x}_2^2\hat{x}_3, \hat{x}_2\hat{x}_3^2\}$), $i = 1, 2, 3$, which is the serendipity element space; and $\hat{v} = 0$, at the 4 vertices of $\hat{F}_i$ and the middle points of 4 sides of $\hat{F}_i$, hence

$$\hat{v}|_{\hat{F}_i} = 0, \quad i = 1, 2, 3. \tag{4.6}$$

By (4.2) $\hat{v}$ has the following expression

$$\hat{v} = b_{\hat{T}}\hat{p}, \quad \hat{p} = \sum_{i=1}^{4}\alpha_i\hat{\lambda}_i + \alpha_5\hat{\lambda}_4\hat{\lambda}_5.$$

Since $b_{\hat{T}}|_{\hat{F}_i} = 0$, $\hat{\lambda}_i|_{\hat{F}_i} = 0$, we have

$$\int_{\hat{F}_i}\frac{\partial\hat{v}}{\partial\hat{n}}d\hat{s} = \int_{\hat{F}_i}\frac{\partial b_{\hat{T}}}{\partial\hat{n}}\hat{p}d\hat{s} = \int_{\hat{F}_i}\frac{\partial\hat{\lambda}_i}{\partial\hat{n}}\left(\prod_{j=1 j\neq i}^{5}\hat{\lambda}_j\right)\hat{p}ds.$$

Set $\Lambda_i = \prod_{j=1 j\neq i}^{5}\hat{\lambda}_j$, $1\le i\le 5$, It is easy to see that $\frac{\partial\hat{\lambda}_i}{\partial\hat{n}}\neq 0$, $1\le i\le 5$. Then we can have

$$\int_{\hat{F}_i}\Lambda_i\hat{p}d\hat{s} = 0, \quad i = 1,2,3,4,5.$$

That is,

$$\begin{aligned}
\int_{\hat{F}_1}\Lambda_1\hat{p}d\hat{s} &= \int_{\hat{l}_1}d\hat{l}\int_0^1\hat{\lambda}_2\hat{\lambda}_3\hat{x}_3(1-\hat{x}_3)\big\{\alpha_2\hat{\lambda}_2+\alpha_3\hat{\lambda}_3+\alpha_4\hat{x}_3+\alpha_5\hat{x}_3(1-\hat{x}_3)\big\}\\
&= \frac{|\hat{l}_1|}{360}\{5\alpha_2+5\alpha_3+5\alpha_4+2\alpha_5\} = 0;\\
\int_{\hat{F}_2}\Lambda_2\hat{p}d\hat{s} &= \int_{\hat{l}_2}d\hat{l}\int_0^1\hat{\lambda}_1\hat{\lambda}_3\hat{x}_3(1-\hat{x}_3)\big\{\alpha_1\hat{\lambda}_1+\alpha_3\hat{\lambda}_3+\alpha_4\hat{x}_3+\alpha_5\hat{x}_3(1-\hat{x}_3)\big\}\\
&= \frac{|\hat{l}_2|}{360}\{5\alpha_1+5\alpha_3+5\alpha_4+2\alpha_5\} = 0;\\
\int_{\hat{F}_3}\Lambda_3\hat{p}d\hat{s} &= \int_{\hat{l}_3}d\hat{l}\int_0^1\hat{\lambda}_1\hat{\lambda}_2\hat{x}_3(1-\hat{x}_3)\big\{\alpha_1\hat{\lambda}_1+\alpha_3\hat{\lambda}_3+\alpha_4\hat{x}_3+\alpha_5\hat{x}_3(1-\hat{x}_3)\big\}\\
&= \frac{|\hat{l}_3|}{360}\{5\alpha_1+5\alpha_2+5\alpha_4+2\alpha_5\} = 0;\\
\int_{\hat{F}_4}\Lambda_4\hat{p}d\hat{s} &= \int_{\hat{F}_4}\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3\{\alpha_1\hat{\lambda}_1+\alpha_2\hat{\lambda}_2+\alpha_3\hat{\lambda}_3\}d\hat{s} = \frac{|\hat{F}_4|}{180}\{\alpha_1+\alpha_2+\alpha_3\} = 0;\\
\int_{\hat{F}_5}\Lambda_5\hat{p}d\hat{s} &= \int_{\hat{F}_5}\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3\{\alpha_1\hat{\lambda}_1+\alpha_2\hat{\lambda}_2+\alpha_3\hat{\lambda}_3+\alpha_4\}d\hat{s}\\
&= \frac{|\hat{F}_4|}{180}\{\alpha_1+\alpha_2+\alpha_3+3\alpha_4\} = 0.
\end{aligned}$$

The above linear systems can be expressed by

$$AX = 0,$$

where $X = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)^T$,

$$A = \begin{pmatrix} 0 & 5 & 5 & 5 & 2 \\ 5 & 0 & 5 & 5 & 2 \\ 5 & 5 & 0 & 5 & 2 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{pmatrix}.$$

By simple calculations, we get

$$\det A = 450 \neq 0.$$

Hence $X = 0$, namely, $\alpha_i = 0$, $1 \leq i \leq 5$. Then $\hat{v} = 0$. □

4.2 C^0TP2 Element

The shape function space of C^0TP2 element is taken as:

$$\hat{P}_{TP2} = \hat{P}_3^* \oplus b_{\hat{T}} Q_2, \tag{4.7}$$

where

$$\hat{P}_3^* = P_3(\hat{T}) \oplus \{\hat{x}_1^3\hat{x}_3, \hat{x}_1\hat{x}_3^3, \hat{x}_2^3\hat{x}_3, \hat{x}_2\hat{x}_3^3, \hat{x}_1^3\hat{x}_2\hat{x}_3, \hat{x}_1\hat{x}_2^3\hat{x}_3\},$$
$$Q_2 = \{\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_1\hat{\lambda}_2, \hat{\lambda}_2\hat{\lambda}_3, \hat{\lambda}_3\hat{\lambda}_1, \hat{\lambda}_1\hat{x}_3, \hat{\lambda}_2\hat{x}_3, \hat{\lambda}_3\hat{x}_3, \hat{\lambda}_1\hat{\lambda}_2\hat{x}_3, \hat{\lambda}_2\hat{\lambda}_3\hat{x}_3, \hat{\lambda}_3\hat{\lambda}_1\hat{x}_3, \hat{\lambda}_1^2\hat{\lambda}_2, \hat{\lambda}_2^2\hat{\lambda}_3, \hat{\lambda}_3^2\hat{\lambda}_1\}.$$

It is easy to see the dimension of $\hat{P}_{TP2}$ is 41. The degrees of freedom are:

$$\begin{aligned} &\hat{v}(\hat{a}_i), \quad \hat{v}_{\hat{x}_j}(\hat{a}_i), \quad 1 \leq i \leq 6,\ 1 \leq j \leq 3,\ \hat{v}(\hat{c}_j),\ j = 1, 2, \\ &\int_{\hat{F}_i} \frac{\partial \hat{v}}{\partial \hat{n}} \hat{q}\, d\hat{s}, \quad \hat{q} \in P_1(\hat{F}_i),\ 1 \leq i \leq 5, \end{aligned} \tag{4.8}$$

where $\hat{c}_1, \hat{c}_2$ are the central points of $\hat{F}_4$ and $\hat{F}_5$, respectively. See Fig. 2.

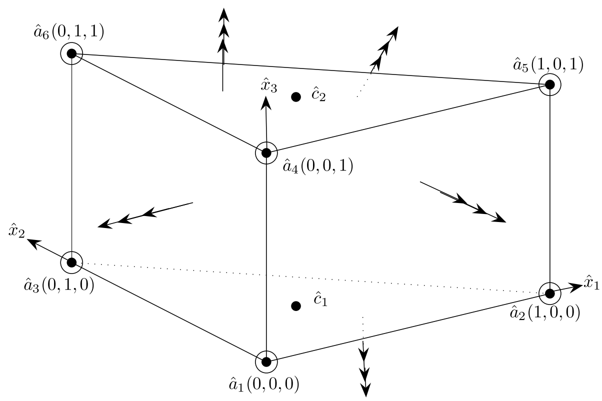


Fig. 2 Degrees of freedom of C^0TP2 element

The corresponding interpolation operator $\hat{\Pi}_2 : H^4(\hat{T}) \to \hat{P}_{TP2}$ is defined by

$$\begin{cases} (\hat{v} - \hat{\Pi}_2\hat{v})(\hat{a}_i) = 0, \ \dfrac{\partial(\hat{v} - \hat{\Pi}_2\hat{v})}{\partial \hat{x}_j}(\hat{a}_i) = 0, \quad 1 \le i \le 6, 1 \le j \le 3; \\ (\hat{v} - \hat{\Pi}_2\hat{v})(\hat{c}_j) = 0, \quad j = 1, 2; \qquad \displaystyle\int_{\hat{F}_i} \frac{\partial(\hat{v} - \hat{\Pi}_2\hat{v})}{\partial \hat{n}} \hat{p} d\hat{s} = 0, \quad \hat{p} \in P_1(\hat{F}_i), \ 1 \le i \le 5. \end{cases} \tag{4.9}$$

Lemma 4.2 *The interpolation operator $\hat{\Pi}_2$ is well posed, namely, the degrees of freedom* (4.8) *are $\hat{P}_{TP2}$-unisolvent.*

Proof It is easy to see that the number of degrees of freedom (4.8) is also 41, so it is sufficient to show that if $\hat{v} \in \hat{P}_{TP2}$ such that all the degrees of freedom of $\hat{v}$ are zero, then $\hat{v} \equiv 0$.

Since $\hat{v}|_{\hat{F}_i} \in P_3(\hat{F}_i), i = 4, 5$, and $\hat{v} = 0$, at the 3 vertices and the barycenter of $\hat{F}_i$ and two one-order derivatives on 3 vertices of $\hat{F}_i$ are zero, hence

$$\hat{v}|_{\hat{F}_i} = 0, \quad i = 4, 5. \tag{4.10}$$

$\hat{v}|_{\hat{F}_i} \in P_3(\hat{F}_i) \oplus \{\hat{x}_1^3\hat{x}_3, \hat{x}_1\hat{x}_3^3\}$ (or $\{\hat{x}_2^3\hat{x}_3, \hat{x}_2\hat{x}_3^3\}$), $i = 1, 2, 3$, which is the Adini or ACM element space; and the function values as well as two first order derivatives are zero on 4 vertices of $\hat{F}_i$, hence

$$\hat{v}|_{\hat{F}_i} = 0, \quad i = 1, 2, 3. \tag{4.11}$$

By (4.7), $\hat{v}$ has the following expression

$$\hat{v} = b_{\hat{T}}\hat{p},$$

where

$$\begin{aligned} \hat{p} = {} & \alpha_1\hat{\lambda}_1 + \alpha_2\hat{\lambda}_2 + \alpha_3\hat{\lambda}_3 + \alpha_4\hat{\lambda}_1\hat{\lambda}_2 + \alpha_5\hat{\lambda}_2\hat{\lambda}_3 + \alpha_6\hat{\lambda}_3\hat{\lambda}_1 + \alpha_7\hat{\lambda}_1\hat{x}_3 + \alpha_8\hat{\lambda}_2\hat{x}_3 \\ & + \alpha_9\hat{\lambda}_3\hat{x}_3 + \alpha_{10}\hat{\lambda}_1\hat{\lambda}_2\hat{x}_3 + \alpha_{11}\hat{\lambda}_2\hat{\lambda}_3\hat{x}_3 + \alpha_{12}\hat{\lambda}_3\hat{\lambda}_1\hat{x}_3 + \alpha_{13}\hat{\lambda}_1^2\hat{\lambda}_2 + \alpha_{14}\hat{\lambda}_2^2\hat{\lambda}_3 \\ & + \alpha_{15}\hat{\lambda}_3^2\hat{\lambda}_1. \end{aligned}$$

By a similar argument used in Lemma 4.1, since $b_{\hat{T}}|_{\hat{F}_i} = 0, \hat{\lambda}_i|_{\hat{F}_i} = 0$, we have

$$\int_{\hat{F}_i} \frac{\partial \hat{v}}{\partial \hat{n}} \hat{q} d\hat{s} = 0 \quad \Longleftrightarrow \quad \int_{\hat{F}_i} \Lambda_i \hat{p}\hat{q} d\hat{s} = 0, \quad 1 \le i \le 5.$$

The above linear systems can be expressed by

$$AX = 0,$$

where $X = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{15})^T$,

$$A = \begin{pmatrix}
0 & 3 & 2 & 0 & 1 & 0 & 0 & \frac{3}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 & \frac{4}{7} & 0 \\
0 & 2 & 3 & 0 & 1 & 0 & 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{3}{7} & 0 \\
0 & \frac{5}{2} & \frac{5}{2} & 0 & 1 & 0 & 0 & \frac{3}{2} & \frac{3}{2} & 0 & \frac{3}{5} & 0 & 0 & \frac{1}{2} & 0 \\
3 & 0 & 2 & 0 & 0 & 1 & \frac{3}{2} & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{3}{7} \\
2 & 0 & 3 & 0 & 0 & 1 & 1 & 0 & \frac{3}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{4}{7} \\
\frac{5}{2} & 0 & \frac{5}{2} & 0 & 0 & 1 & \frac{3}{2} & 0 & \frac{3}{2} & 0 & 0 & \frac{3}{5} & 0 & 0 & \frac{1}{2} \\
3 & 2 & 0 & 1 & 0 & 0 & \frac{3}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 & \frac{4}{7} & 0 & 0 \\
2 & 3 & 0 & 1 & 0 & 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{3}{7} & 0 & 0 \\
\frac{5}{2} & \frac{5}{2} & 0 & 1 & 0 & 0 & \frac{3}{2} & \frac{3}{2} & 0 & \frac{3}{5} & 0 & 0 & \frac{1}{2} & 0 & 0 \\
2 & \frac{4}{3} & \frac{4}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{9} & \frac{1}{9} & \frac{1}{6} \\
\frac{4}{3} & 2 & \frac{4}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{2}{9} & \frac{1}{9} \\
\frac{4}{3} & \frac{4}{3} & 2 & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & \frac{1}{6} & \frac{2}{9} \\
2 & \frac{4}{3} & \frac{4}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 2 & \frac{4}{3} & \frac{4}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & \frac{2}{9} & \frac{1}{9} & \frac{1}{6} \\
\frac{4}{3} & 2 & \frac{4}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{4}{3} & 2 & \frac{4}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} & \frac{2}{9} & \frac{1}{9} \\
\frac{4}{3} & \frac{4}{3} & 2 & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{4}{3} & \frac{4}{3} & 2 & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{9} & \frac{1}{6} & \frac{2}{9}
\end{pmatrix}$$

By simple computations, we get

$$\det A = \frac{-1}{2^5 \times 3^2 \times 5^3 \times 7^3} \neq 0.$$

Hence $X = 0$, namely, $\alpha_i = 0,\ 1 \le i \le 12$. Then $v \equiv 0$. □

5 Convergence Analysis

In this section, we will give the convergence analysis of the elements given in Sect. 4.1 for the fourth order elliptic problem (2.1).

Define $P_{TP1} = \{v = \hat{v} \circ G_T^{-1}; \forall \hat{v} \in \hat{P}_{TP1}\}$, $P_{TP2} = \{v = \hat{v} \circ G_T^{-1}; \forall \hat{v} \in \hat{P}_{TP2}\}$. Then $\Pi_1 : H^3(T) \to P_{TP1}$ and $\Pi_2 : H^4(T) \to P_{TP2}$ are the interpolation operators of C^0TP1 and C^0TP2 elements, respectively. The corresponding finite element spaces of C^0TP1 and C^0TP2 elements are defined by

$$V_{h1} = \{v_h|_T \in P_{TP1},\ \forall T \in \mathcal{T}_h,\ \text{the degrees of freedom (4.3)} \\ \text{are continuous across the elements and are zeros on } \partial\Omega\}, \tag{5.1}$$

$$V_{h2} = \{v_h|_T \in P_{TP2},\ \forall T \in \mathcal{T}_h,\ \text{the degrees of freedom (4.8)} \\ \text{are continuous across the elements and are zeros on } \partial\Omega\}. \tag{5.2}$$

The finite element interpolation operators Π_{h1}: $H^3(\Omega) \to V_{h1}$ and $\Pi_{h2} : H^4(\Omega) \to V_{h2}$ are defined by

$$\Pi_{h1}|_T = \Pi_1, \qquad \Pi_{h2}|_T = \Pi_2, \quad \forall T \in \mathcal{T}_h.$$

Under $x = G_T(\hat{x})$, let $\hat{a}_i \leftrightarrow a_i, 1 \le i \le 6$; $\hat{F}_i \leftrightarrow F_i,\ 1 \le i \le 5$; $\hat{l}_i \leftrightarrow l_i,\ \hat{b}_i \leftrightarrow b_i,\ 1 \le i \le 9$; $\hat{P}_{TP1} \leftrightarrow P_{TP1},\ \hat{P}_{TP2} \leftrightarrow P_{TP2}$; $\hat{v}(\hat{x}) = v(x)$.

The discrete variational problems to solve (2.3) are: Find $u_{h1} \in V_{h1}$ such that

$$a_h(u_{h1}, v_h) = f(v_h), \quad \forall v_h \in V_{h1}. \tag{5.3}$$

Find $u_{h2} \in V_{h2}$ such that

$$a_h(u_{h2}, v_h) = f(v_h), \quad \forall v_h \in V_{h2}. \tag{5.4}$$

It is easy to check that $|||\cdot|||_h$ is a norm of V_{h1} and V_{h2}, respectively, so (5.3) and (5.4) are unisolvent by Lax-Milgram Theorem [8].

For getting the error estimates for C^0TP1 and C^0TP2 elements, it is only needed to check (H1) (H2) (H3) of Lemma 3.2. By (4.5), (4.6), (4.10) and (4.11), it is easy to prove that

$$V_{h1} \subset H_0^1(\Omega), \quad V_{h2} \subset H_0^1(\Omega). \tag{5.5}$$

Because $P_2(T) \subset P_{TP1}$, $P_3(T) \subset P_{TP2}$, by the well-known interpolation theorem [2, 8], we have

$$\begin{cases} |||u - \Pi_{h1}u|||_h \le Ch|u|_{3,\Omega}, \\ |||u - \Pi_{h2}u|||_h \le Ch^2|u|_{4,\Omega}. \end{cases} \tag{5.6}$$

Here $u \in H_0^2(\Omega)$ is the solution of (2.3) with the additional regularity $u \in H^r(\Omega)$, where $r = 3$ for the C^0TP1 element and $r = 4$ for the C^0TP2 element. By the last sets of the degrees of freedom (4.3) and (4.8), we obtain that

$$\begin{cases} \displaystyle\int_F \left[\frac{\partial w_h}{\partial n}\right] ds = 0, \quad \forall F \in \mathcal{F}_h,\ \forall w_h \in V_{h1}, \\ \displaystyle\int_F p\left[\frac{\partial w_h}{\partial n}\right] ds = 0, \quad \forall F \in \mathcal{F}_h,\ \forall p \in P_1(F),\ \forall w_h \in V_{h2}. \end{cases} \tag{5.7}$$

By (5.5), (5.6), (5.7), we know that (H1) (H2) and (H3) are satisfied for C^0TP1 with $m = 2$ and for C^0TP2 with $m = 3$. Then by Lemma 3.2, we obtain the following convergence theorem for C^0TP1 and C^0TP2 elements.

Theorem 5.1 *Suppose that the mesh $\mathcal{T}_h$, into triangular prisms for C^0TP1 and C^0TP2 elements is regular in the sense of* (2.4), *u is the solution of* (2.3) *with the additional regularity $u \in H^r(\Omega)$, and u_{h1} and u_{h2} are the solutions of* (5.3) *and* (5.4), *respectively, then*

$$\begin{cases} |||u - u_{h1}|||_h \le Ch|u|_{3,\Omega}, \\ |||u - u_{h2}|||_h \le Ch^2|u|_{4,\Omega}, \end{cases} \tag{5.8}$$

where $r = 3$ for the C^0TP1 element and $r = 4$ for the C^0TP2 element.

6 Conclusion

In this paper, we present a method to construct the C^0-nonconforming elements for the fourth order elliptic problem by using the bubble functions. It makes the element interpolation matrices being block lower triangular and it is easy to choose the matched shape function spaces and degrees of freedom. By this method, we construct two C^0-nonconforming triangular prism elements to solve the fourth-order three-dimensional elliptic problem. One element is of first order convergence and the other one is of second order convergence. In the future work, we will carry out some numerical experiments to verify our theoretical results.

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