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Fourier Type Error Analysis of the Direct Discontinuous Galerkin Method and Its Variations for Diffusion Equations

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Abstract In this paper we present Fourier type error analysis on the recent four discontinuous Galerkin methods for diffusion equations, namely the direct discontinuous Galerkin (DDG) method (Liu and Yan in SIAM J. Numer. Anal. 47(1):475–698, 2009); the DDG method with interface corrections (Liu and Yan in Commun. Comput. Phys. 8(3):541–564, 2010); and the DDG method with symmetric structure (Vidden and Yan in SIAM J. Numer. Anal., 2011); and a DG method with nonsymmetric structure (Yan, A discontinuous Galerkin method for nonlinear diffusion problems with nonsymmetric structure, 2011). The Fourier type L^2 error analysis demonstrates the optimal convergence of the four DG methods with suitable numerical fluxes. The theoretical predicted errors agree well with the numerical results.

Keywords Discontinuous Galerkin method · Diffusion equation · Stability · Consistency · Convergence · Supraconvergence

1 Introduction

The focus of this paper is to perform Fourier type error analysis on four discontinuous Galerkin methods for diffusion problems. We have (1) the direct discontinuous Galerkin (DDG) method [16]; (2) the DDG method with interface corrections [17]; (3) a variation of the DDG method with symmetric structure [26]; and (4) a discontinuous Galerkin method

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with nonsymmetric structure [30]. In this paper, for the simplicity of presentation we present the four discontinuous Galerkin methods for the diffusion equations on the simple one dimensional linear heat equation

$$U_t - U_{xx} = 0, \quad \text{for } x \in [0, 2\pi] \tag{1.1}$$

with suitable given boundary conditions and initial condition $U(x, 0) = \sin(\alpha x)$. We would like to point out, however, that the four discontinuous Galerkin methods are actually designed and can be analyzed for much more general multidimensional nonlinear convection diffusion equations, see, e.g. [17]. The points we would like to make in this paper can be represented well by the simple case (1.1).

The discontinuous Galerkin (DG) method is a finite element method using a completely discontinuous piecewise function space for the numerical solution and the test function. One main feature of the DG method is the flexibility afforded by local approximation spaces combined with the suitable design of numerical fluxes crossing cell interfaces. The application to hyperbolic problems has been quite successful since it was originally introduced by Reed and Hill [19] in 1973 for neutron transport equations. A major development of the DG method for nonlinear hyperbolic conservation laws is carried out by Cockburn, Shu, and collaborators. We refer to [10, 11, 13] for reviews and further references.

However, the application of the DG method to diffusion problems has been a challenging task because of the subtle difficulty in defining appropriate numerical fluxes for diffusion terms, see e.g. [21]. There have been several DG methods suggested in literature to solve the problem. One class is the *interior penalty* (IP) methods, dates back to 1982 by Arnold in [1] (also by Baker in [3] and Wheeler in [27]), the Baumann and Oden method [5, 18], the NIPG method [20] and the IIPG method [14]. Another class is the local discontinuous Galerkin (LDG) method introduced in [12] by Cockburn and Shu (originally studied by Bassi and Rebay in [4] for compressible Navier-Stokes equations). Here we refer to the unified analysis paper [2] in 2002 for the review of different DG methods for diffusion equation. More recent works for diffusion with DG methods are those by Van Leer and Nomura in [24], Gassner et al. in [15], Cheng and Shu in [7], and Brenner et al. in [6].

Recently in [16] we introduce a discontinuous Galerkin method to solve diffusion equations. The scheme is based on the direct weak formulation, thus we name it the direct discontinuous Galerkin (DDG) method. We propose a general numerical flux formula for the solution derivative $\widehat{u_x}$ at the cell interface, which involves the average $\overline{u_x}$ and even order derivative jumps $[\partial_x^{2m} u]$ $(m = 0, 1, \ldots, [\frac{k}{2}])$ across the cell interfaces. We then introduce a concept of admissibility to identify suitable numerical fluxes to further obtain stability and energy norm error estimate. However, numerical experiments in [16] show that measured in L^2 norm the scheme accuracy is sensitive to the coefficients chosen in the numerical flux formula, and for higher order p^k $(k \geq 4)$ polynomial approximations it is difficult to identify suitable coefficients in the numerical flux formula to obtain optimal $(k+1)$th order of accuracy.

In [17], we introduce a refined version of the DDG method by adding extra interface correction terms such that a simpler numerical flux formula can be used and numerically optimal accuracy under L^2 norm can be achieved for all p^k polynomials. The interface terms are added due to the observation that the test function is also discontinuous at the cell interface and the derivative of the test function does contribute to the interface flux when higher order approximations are used. Another feature of the refined DDG method is that the scheme is not sensitive to the coefficients in the numerical flux formula. Numerical tests show a large class of admissible numerical fluxes can lead to optimal order of accuracy.

More recently in [26], we further explore the DDG method by introducing a numerical flux for the test function derivative $\widetilde{v_x}$ and adding more interface terms in the scheme formulation, and obtain a DDG scheme with symmetric structure. The symmetric structure helps us prove the optimal $(k+1)$th order $L^2(L^2)$ error estimate with any p^k polynomial approximations. Detailed admissibility analysis confirms that there exists a large class of admissible numerical fluxes, basically any coefficients pair that falls in a parabolic region is admissible (see [26] or details in [25]).

When adding the test function derivative $\widetilde{v_x}$ interface terms in a negative sense, we end up with a discontinuous Galerkin method with nonsymmetric structure in [30]. This nonsymmetric DG method is closely related to the Baumann-Oden method [5] and the NIPG method [20]. Essentially the scheme degenerates to the Baumann-Oden or the NIPG scheme if the second derivative jump term $[u_{xx}]$ is not included in the numerical flux formula for $\widehat{u_x}$ (same as $[v_{xx}]$ in $\widetilde{v_x}$). On the other hand, numerically we observe our nonsymmetric scheme can lead to optimal $(k+1)$th order convergence with any p^k polynomial approximations. As we know, only sub-optimal kth order of accuracy can be obtained for the Baumann-Oden method and the NIPG method with p^k ($k = even$) polynomial approximations. In a word, the optimal accuracy is recovered with our nonsymmetric scheme [30].

Notice that it is difficult to obtain L^2 error analysis for the DDG method [16], the DDG method with interface corrections [17], and the nonsymmetric DG method [30]. On the other hand, Fourier analysis is a new technique to study the stability and error estimates for the discontinuous Galerkin method and other related schemes, especially in some cases where standard finite element technique can not be applied. Numerically we observe the DDG method [16] is sensitive to the coefficients in the numerical flux formula to obtain optimal convergence with even-th order polynomial approximations, thus in this paper we focus on the p^2 quadratic case and use the Fourier analysis technique to analyze the error behavior of the four DG methods. The Fourier analysis does have several advantages over the standard finite element techniques. It can be used to analyze some of the "bad" schemes [31]; it can be used for stability analysis for some of the non-standard methods such as the spectral volume (SV) method [32], which belongs to the class of Petrov-Galerkin methods and cannot be easily amended to the standard finite element analysis framework; it can provide quantitative error comparisons among different schemes [28, 29]; and it can be used to prove superconvergence and time evolution of errors for the DG method [8, 9].

This rest of the paper is organized as follows. In Sect. 2, we describe the four DG methods for the model heat equation. In Sect. 3, we write out in detail the Fourier technique to analyze the errors of the four DG methods, and show the numerical errors match well with the analytical predictions. Finally, some concluding remarks are given in Sect. 4.

2 Direct Discontinuous Galerkin Method and Its Variations

Again, in this paper we focus on the model heat equation

$$U_t - U_{xx} = 0, \quad \text{for } x \in [0, 2\pi]$$

with initial condition $u(x,0) = \sin(\alpha x)$ and suitable given boundary condition.

Let us denote $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, $j = 1, \ldots, N$, as a mesh for $[0, 2\pi]$, where $x_{\frac{1}{2}} = 0$ and $x_{N+\frac{1}{2}} = 2\pi$. We denote the center of each cell by $x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$ and the size of each cell by $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. The cells do not need to be uniform for the numerical methods, but for simplicity of analysis we will consider only uniform meshes in this paper and will

denote the uniform mesh size by Δx. The numerical solution u and the test function v are piecewise polynomials of degree k. In a word, for any time $t \in [0, T]$, $u \in \mathbb{V}_{\Delta x}$, where

$$\mathbb{V}_{\Delta x} := \{v \in L^2(0, 2\pi) \ : \ v|_{I_j} \in P^k(I_j), \ j = 1, \ldots, N\},$$

and $P^k(I_j)$ denotes the space of polynomials in I_j with degree at most k. In this paper we use lower letters to represent numerical solutions and test functions. We are now ready to formulate the DG schemes.

2.1 Direct Discontinuous Galerkin Method

We multiply heat equation (1.1) by an arbitrary test function $v(x) \in \mathbb{V}_{\Delta x}$, integrate over the cell I_j, have the integration by parts, and formally we obtain,

$$\int_{I_j} u_t v dx - \widehat{u_x} v \Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} + \int_{I_j} u_x v_x dx = 0, \tag{2.1}$$

where

$$\widehat{u_x} v \Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} := (\widehat{u_x})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (\widehat{u_x})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}.$$

This is the starting point to design the direct discontinuous Galerkin method. Note here and below we adopt the following notations:

$$u^{\pm} = u(x \pm 0, t), \qquad [u] = u^+ - u^-, \qquad \overline{u} = \frac{u^+ + u^-}{2}.$$

Motivated by the solution derivative trace formula of the heat equation with discontinuous initial data, in [16] we introduce a numerical flux formula at the cell interface $x_{j\pm 1/2}$ by the following form,

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] + \beta_2 (\Delta x)^3 [u_{xxxx}] + \cdots. \tag{2.2}$$

The numerical flux $\widehat{u_x}$ which approximates u_x at the discontinuous cell interface $x_{j\pm 1/2}$ thus involves the average $\overline{u_x}$ and the jumps of even order derivatives of u. The coefficients $\beta_0, \beta_1, \ldots$ are chosen to ensure the stability and convergence of the method. The numerical flux $\widehat{u_x}$ thus defined is conservative and consistent to the solution derivative. We then introduce a concept of admissibility for the numerical fluxes. The admissibility condition serves as a criterion for selecting suitable numerical fluxes (β_i's) to guarantee nonlinear stability of the DDG method and the corresponding convergence of the method. Indeed in the linear case, for the error in a parabolic energy norm $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$, convergence rate of order $(\Delta x)^k$ is obtained for p^k polynomial approximations. Numerically we observe that the scheme is sensitive to the choice of β_i's to obtain optimal $(k+1)$th order of accuracy under L^2 norm, especially for the case $k = even$. For p^2 quadratic polynomial approximation, only the numerical flux

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \frac{\Delta x}{12} [u_{xx}]$$

with $\beta_1 = \frac{1}{12}$ gives optimal 3rd order of accuracy, all other admissible numerical fluxes with $\beta_1 \neq \frac{1}{12}$ degenerate the scheme to a 2nd order. On the other hand, the scheme accuracy is not

sensitive to the choice of β_0. In the following Sect. 3.1 the Fourier type error analysis confirms these numerical results that only $\beta_1 = \frac{1}{12}$ gives the optimal 3rd order of accuracy. As for higher order polynomial approximations with even k and $k \geq 4$, it remains an unsettled issue on how to find suitable β_i's to ensure optimal $(k+1)$th order of accuracy.

2.2 Direct Discontinuous Galerkin Method with Interface Corrections

In [17], we add interface correction terms in the DDG scheme formulation and obtain the following refined DDG method,

$$\int_{I_j} u_t v dx - \widehat{u_x} v\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} + \int_{I_j} u_x v_x dx + \underline{\frac{1}{2}[u](v_x)^-_{j+1/2} + \frac{1}{2}[u](v_x)^+_{j-1/2}} = 0. \tag{2.3}$$

The Numerical flux is taken as

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}], \tag{2.4}$$

where the second derivative jump term is still included and higher order ($k \geq 4$) derivatives jump terms are dropped. Extra interface terms are added due to the observation that the test function v is also discontinuous at the cell interfaces, the slope of the test function will contribute at interfaces whenever $[u]$ is non-zero. The refined DDG method with such numerical fluxes enjoys the optimal $(k+1)$th order of accuracy for all p^k polynomial approximations.

If we take $\beta_1 = 0$ in (2.4), the numerical flux reduces to

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x}, \tag{2.5}$$

in which case the scheme reduces to the classical symmetric Interior Penalty (IP) method [1]. It is well known that the penalty parameter (β_0 in (2.5)) depends on k the order of the approximate polynomial, and needs to be large enough to stabilize the scheme.

With $\beta_1 \neq 0$ in (2.4), numerically we observe that $(k+1)$th order of accuracy is obtained for p^k polynomials with a fixed β_0. For instance, optimal accuracy is obtained for all p^k up to $k = 9$ when taking $(\beta_0, \beta_1) = (2, \frac{1}{12})$ in (2.4). Thus the second derivative jump term $[u_{xx}]$ is important, it indeed provides leverage to compensate the β_0 term. Moreover, there exists a large class of β_0 and β_1 that lead to optimal order of accuracy with p^k polynomial approximations. In the following Sect. 3.2, the Fourier type error analysis results are consistent with these numerical tests.

2.3 Direct Discontinuous Galerkin Method with Symmetric/Nonsymmetric Structure

We see the DDG method (2.1)–(2.2) and the DDG method with interface corrections (2.3)–(2.4) are schemes lack of symmetric structures, and thus are not adjoint-consistent. To further study the DDG method, we introduce a numerical flux term $\widetilde{v_x}$ for the test function v and add it into the original DDG scheme (2.1). This test function numerical flux $\widetilde{v_x}$ shares similar format to the solution numerical flux $\widehat{u_x}$ given above. With more interface terms, now we obtain the following DG methods with symmetric ($\sigma = +1$) or nonsymmetric structures ($\sigma = -1$),

$$\int_{I_j} u_t v dx - \widehat{u_x} v\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}} + \int_{I_j} u_x v_x dx + \sigma(\widetilde{v_x}[u]_{j+1/2} + \widetilde{v_x}[u]_{j-1/2}) = 0, \tag{2.6}$$

where the numerical flux is defined as

$$\begin{aligned} \widehat{u_x} &= \beta_{0u}\frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x[u_{xx}], \\ \widetilde{v_x} &= \beta_{0v}\frac{[v]}{\Delta x} + \overline{v_x} + \beta_1 \Delta x[v_{xx}]. \end{aligned} \tag{2.7}$$

Notice that the test function v is taken to be none zero only inside the cell I_j, thus at the cell interface $x_{j\pm 1/2}$ only one side contributes to the computation of $\widetilde{v_x}$. Now, summing the scheme (2.6) over all computational cells I_j (here we assume the periodic boundary condition for simplicity of explanation), we obtain the primal weak formulation as the following,

$$\int_0^{2\pi} u_t v dx + \mathbb{B}(u, v) = 0, \tag{2.8}$$

with the bilinear form defined as,

$$\mathbb{B}(u, v) = \sum_{j=1}^{N} \int_{I_j} u_x v_x \, dx + \sum_{j=1}^{N} (\widehat{u_x}[v])_{j+1/2} + \sigma \sum_{j=1}^{N} ([u]\widetilde{v_x})_{j+1/2}. \tag{2.9}$$

With the numerical flux formula of $\widehat{u_x}$ and $\widetilde{v_x}$ as defined in (2.7), it can be seen that the bilinear form $\mathbb{B}(u, v)$ has an obvious symmetry for $\sigma = +1$ case. That is, $\mathbb{B}(u, v) = \mathbb{B}(v, u)$. In [26], this symmetric structure helps us prove the optimal $(k+1)$th order of accuracy in the $L^2(L^2)$ sense for any p^k polynomial approximations. The admissibility analysis in [26] shows us that any $\beta_0 = \beta_{0u} + \beta_{0v}$ and β_1 coefficients that satisfy the following quadratic form inequality lead to an admissible numerical flux (2.7), and guarantees the optimal convergence of the symmetric DDG scheme,

$$\beta_{0u} + \beta_{0v} \geq 1 + 8\left(\beta_1^2\left(\frac{k^2(k^2-1)^2}{3}\right) - \beta_1\left(\frac{k^2(k^2-1)}{2}\right) + \frac{k^2}{4}\right), \quad \text{for } k \geq 1.$$

For the purpose of this paper where we mainly consider piecewise quadratic p^2 approximations, here we choose $\beta_{0u} = \beta_{0v} = 1$ and $\beta_1 = 1/12$ in the following Sect. 3.3 to carry out the Fourier analysis of the symmetric DDG method.

Taking $\sigma = -1$ in (2.6), we end up having a new DG method with nonsymmetric structure in [30]. This nonsymmetric DG method is closely related to the Baumann-Oden method [5] and the NIPG method [20]. The essential difference is that the second derivative jump term $[u_{xx}]$ and $[v_{xx}]$ are included in the numerical flux formula (2.7). When taking $v = u$ in (2.8) to proceed the stability analysis, we obtain a stability result that is the same as the Baumann-Oden method (with $\beta_{0u} = \beta_{0v}$) and the NIPG method (with $\beta_{0u} > \beta_{0v}$) as below,

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u^2 dx + \sum_{j=1}^{N}\int_{I_j} (u_x)^2 \, dx + (\beta_{0u} - \beta_{0v}) \sum_{j=1}^{N} \frac{[u]_{j+1/2}^2}{\Delta x} = 0. \tag{2.10}$$

Even our nonsymmetric DG method shares the same stability with the Baumann-Oden method and the NIPG method, the numerical performance is quite different. Numerically we obtain $(k+1)$th order of accuracy with any p^k polynomial approximations for our nonsymmetric DG method [30]. As is well known that both the Baumann-Oden method and the NIPG method lose one order for p^k polynomial approximations when k is even. With the

second derivative jump term included in the numerical flux we see the optimal convergence is recovered. In the following Sect. 3.3, we take $\beta_{0u} = 1$ and $\beta_{0v} = 1$ coupled with $\beta_1 = 1/12$ in (2.7) to perform Fourier analysis of the nonsymmetric DG method, the Fourier type error analysis further confirms the optimal convergence of the method.

Up to now, we have taken the method of lines approach and have left time variable t continuous. For time discretization, the explicit third order TVD Runge-Kutta method [22, 23] was used in order to match the accuracy in space. CFL conditions are taken as the standard ones for DG methods, see [10].

3 Fourier Analysis for the L^2 Errors

In this section, we will rewrite the four discontinuous Galerkin methods as finite difference schemes under the assumption of periodic boundary condition and uniform mesh, then perform Fourier type error analysis on the four schemes and compare the error estimates with the numerical results. For the L^2 error we show that optimal order of convergence can be obtained for all the four DG methods with suitable numerical fluxes.

We use the direct discontinuous Galerkin scheme (2.1) to demonstrate the procedure of the Fourier analysis. After picking a local basis for the space $\mathbb{V}_{\Delta x}$ and inverting a local $(k+1) \times (k+1)$ mass matrix (which could be done by hand), the DDG scheme (2.1) can be written as

$$\frac{d}{dt}\vec{u}_j = \frac{1}{\Delta x^2}(A\vec{u}_{j-1} + B\vec{u}_j + C\vec{u}_{j+1}) \tag{3.1}$$

where $\vec{u}_j$ is a small vector of length $k+1$ containing the coefficients of the solution u in the local basis inside cell I_j, and A, B and C are $(k+1) \times (k+1)$ constant matrices. If we choose the degrees of freedom for the kth degree polynomial inside the cell I_j as the point values of the solution, denoted by

$$u_{j+\frac{2i-k}{2(k+1)}}, \quad i = 0, \ldots, k,$$

at the $k+1$ equally spaced points

$$\left(j - \frac{1}{2} + \frac{2i-k}{2(k+1)}\right)\Delta x, \quad i = 0, \ldots, k,$$

then the DDG scheme written in terms of these degrees of freedom becomes a finite difference scheme on a globally uniform mesh (with a mesh size $\Delta x/(k+1)$); however they are not standard finite difference schemes because each point in the group of $k+1$ points belonging to the cell I_j obeys a different form of finite difference scheme. Let us discuss the procedure in detail using the piecewise quadratic case $k = 2$. The degrees of freedom are now the point values at the $3N$ uniformly spaced points

$$u_{j-\frac{1}{3}}, u_j, u_{j+\frac{1}{3}}, \quad j = 1, \ldots, N.$$

The solution inside the cell I_j is then represented by

$$u(x) = u_{j-\frac{1}{3}}\phi_{j-\frac{1}{3}}(x) + u_j\phi_j(x) + u_{j+\frac{1}{3}}\phi_{j+\frac{1}{3}}(x),$$

where $\phi_{j-\frac{1}{3}}(x)$, $\phi_j(x)$, $\phi_{j+\frac{1}{3}}(x)$ are the Lagrange interpolation polynomials at points $x_{j-\frac{1}{3}}$, x_j, $x_{j+\frac{1}{3}}$. Now we obtain the finite difference representation of the DDG method (2.1)–(2.2) as in (3.1) with solution vector defined as

$$\vec{u}_j = \begin{pmatrix} u_{j-\frac{1}{3}} \\ u_j \\ u_{j+\frac{1}{3}} \end{pmatrix} \tag{3.2}$$

and scheme matrices defined as

$$\begin{aligned}
A &= \begin{pmatrix} \frac{23}{32}(-4+\beta_0+24\beta_1) & \frac{-24}{48}(-18+5\beta_0+72\beta_1) & \frac{23}{32}(-8+5\beta_0+24\beta_1) \\ \frac{-9}{32}(-4+\beta_0+24\beta_1) & \frac{3}{16}(-18+5\beta_0+72\beta_1) & \frac{-9}{32}(-8+5\beta_0+24\beta_1) \\ \frac{-1}{32}(-4+\beta_0+24\beta_1) & \frac{1}{48}(-18+5\beta_0+72\beta_1) & \frac{-1}{32}(-8+5\beta_0+24\beta_1) \end{pmatrix} \\
B &= \begin{pmatrix} \frac{-3}{16}(6+19\beta_0+88\beta_1) & \frac{-1}{24}(18-55\beta_0-814\beta_1) & \frac{3}{16}(10-3\beta_0-88\beta_1) \\ \frac{9}{16}(14+3\beta_0+24\beta_1) & \frac{-3}{8}(42+5\beta_0+72\beta_1) & \frac{9}{16}(14+3\beta_0+24\beta_1) \\ \frac{3}{16}(10-3\beta_0+88\beta_1) & \frac{-1}{24}(18-55\beta_0-814\beta_1) & \frac{-3}{16}(6+19\beta_0+88\beta_1) \end{pmatrix} \\
C &= \begin{pmatrix} \frac{1}{32}(8-5\beta_0-24\beta_1) & \frac{1}{48}(-18+5\beta_0+72\beta_1) & \frac{1}{32}(4-\beta_0-24\beta_1) \\ \frac{9}{32}(8-5\beta_0-24\beta_1) & \frac{3}{16}(-18+5\beta_0+72\beta_1) & \frac{9}{32}(4-\beta_0-24\beta_1) \\ \frac{-23}{32}(8-5\beta_0-24\beta_1) & \frac{-1}{48}(-18+5\beta_0+72\beta_1) & \frac{-23}{32}(4-\beta_0-24\beta_1) \end{pmatrix}.
\end{aligned} \tag{3.3}$$

We perform the following standard Fourier analysis as on a finite difference method. The Fourier analysis depends on the assumption of uniform mesh and periodic boundary condition. We make an ansatz of the solution with the form

$$\begin{pmatrix} u_{j-\frac{1}{3}}(t) \\ u_j(t) \\ u_{j+\frac{1}{3}}(t) \end{pmatrix} = \begin{pmatrix} \hat{u}_{j-\frac{1}{3}}(t) \\ \hat{u}_j(t) \\ \hat{u}_{j+\frac{1}{3}}(t) \end{pmatrix} e^{i\alpha x_j} \tag{3.4}$$

and substitute the ansatz (3.4) in the DDG scheme (3.1) to find the evolution equation for the coefficient vector as

$$\frac{d}{dt}\begin{pmatrix} \hat{u}_{j-\frac{1}{3}}(t) \\ \hat{u}_j(t) \\ \hat{u}_{j+\frac{1}{3}}(t) \end{pmatrix} = G(\alpha, \Delta x)\begin{pmatrix} \hat{u}_{j-\frac{1}{3}}(t) \\ \hat{u}_j(t) \\ \hat{u}_{j+\frac{1}{3}}(t) \end{pmatrix} \tag{3.5}$$

where the amplification matrix $G(\alpha, \Delta x)$ is given by

$$G(\alpha, \Delta x) = \frac{1}{\Delta x^2}\left(Ae^{-i\alpha\Delta x} + B + Ce^{i\alpha\Delta x}\right), \tag{3.6}$$

and matrices A, B, C are from (3.3), which is the reformulation of the DDG scheme (2.1) with Lagrange interpolation polynomials as basis functions. Now the error estimates can be performed based on the matrix $G(\alpha, \Delta x)$. The general solution of the ODE (3.5) is given by

$$\begin{pmatrix} \hat{u}_{j-\frac{1}{3}}(t) \\ \hat{u}_j(t) \\ \hat{u}_{j+\frac{1}{3}}(t) \end{pmatrix} = a_1 e^{\lambda_1 t} V_1 + a_2 e^{\lambda_2 t} V_2, + a_3 e^{\lambda_3 t} V_3 \tag{3.7}$$

where λ_1, λ_2, λ_3, and V_1, V_2, V_3 are the eigenvalues and the corresponding eigenvectors of the amplification matrix $G(\alpha, \Delta x)$ respectively.

To fit the given initial condition $u(x,0) = \sin(\alpha x)$, we have

$$u_{j-\frac{1}{3}}(0) = e^{i\alpha x_{j-\frac{1}{3}}}, \qquad u_j(0) = e^{i\alpha x_j}, \qquad u_{j+\frac{1}{3}}(0) = e^{i\alpha x_{j+\frac{1}{3}}} \tag{3.8}$$

whose imaginary part is our initial condition for the heat equation. Thus we require, at $t = 0$,

$$\begin{pmatrix} \hat{u}_{j-\frac{1}{3}}(0) \\ \hat{u}_j(0) \\ \hat{u}_{j+\frac{1}{3}}(0) \end{pmatrix} = \begin{pmatrix} e^{i\frac{-\alpha\Delta x}{3}} \\ 1 \\ e^{i\frac{\alpha\Delta x}{3}} \end{pmatrix} e^{i\alpha x_j}.$$

This initial condition fitting determines the coefficients a_1, a_2, and a_3 in the general solution formula (3.7). Thus we can explicitly write out the solution of the DDG scheme (2.1). Through a simple Taylor expansion, we can check the imaginary part of $u_{j\pm\frac{1}{3}}(t)$, $u_j(t)$ and find out the leading order in the error. In the following sections, we will perform the Fourier analysis to the four discontinuous Galerkin methods for the model heat equation with p^2 quadratic polynomial approximations. The predicted errors through the above Fourier analysis match well with the numerical solution errors.

3.1 DDG Method

In this section, we perform Fourier analysis on the DDG method (2.1) with numerical flux taken in the form of

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}].$$

We carry out three cases studies with I: $(\beta_0, \beta_1) = (\frac{7}{6}, \frac{1}{12})$; II: $(\beta_0, \beta_1) = (3, \frac{1}{12})$; and III: $(\beta_0, \beta_1) = (3, \frac{1}{4})$. We pick three groups of (β_0, β_1) in the numerical flux formula to show that the DDG scheme accuracy is sensitive to the β_1 coefficient, but not to the β_0 coefficient. It turns out that 3rd order of accuracy is obtained only with $\beta_1 = \frac{1}{12}$ and the DDG scheme degenerates to 2nd order with $\beta_1 \neq \frac{1}{12}$.

Case I: $\beta_0 = \frac{7}{6}$ and $\beta_1 = \frac{1}{12}$

In this case, the numerical flux is given as

$$\widehat{u_x} = \frac{7}{6}\frac{[u]}{\Delta x} + \bar{u}_x + \frac{\Delta x}{12}[u_{xx}].$$

Through the Fourier analysis, we obtain the DDG solution in cell I_j as

$$\mathrm{Im}\{u_{j-1/3}(t)\} = e^{-\alpha^2 t}\sin(\alpha x_{j-1/3}) + \frac{5}{648}\Delta x^3 e^{-\alpha^2 t}\cos(\alpha x_{j-1/3}) + O(\Delta x^4), \tag{3.9}$$

and

$$\mathrm{Im}\{u_j(t)\} = e^{-\alpha^2 t}\sin(\alpha x_j) + \frac{-23 + 132t}{60480}\Delta x^4 e^{-\alpha^2 t}\sin(\alpha x_j) + O(\Delta x^6). \tag{3.10}$$

The result for $\mathrm{Im}\{u_{j+\frac{1}{3}}(t)\}$ is similar to $\mathrm{Im}\{u_{j-\frac{1}{3}}(t)\}$. With the initial condition $u(x,0) = \sin(\alpha x)$, we know the exact solution of the heat equation is $u(x,t) = e^{-\alpha^2 t}\sin(\alpha x)$. We see

Table 1 DDG scheme (2.1) with Case I numerical flux. L^2 and L^∞ errors for u, measured at the point $x_{j-\frac{1}{3}}$ of the cells

Δx	Numerical solutions				Predicted by analysis			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
$2\pi/40$	1.08E-05	–	1.53E-05	–	2.11E-05	–	2.98E-05	–
$2\pi/80$	2.49E-06	2.12	3.52E-06	2.12	2.64E-06	3.00	3.73E-06	3.00
$2\pi/160$	3.30E-07	2.92	4.67E-07	2.92	3.30E-07	3.00	4.67E-07	3.00
$2\pi/320$	4.13E-08	3.00	5.83E-08	3.00	4.13E-08	3.00	5.83E-08	3.00
$2\pi/640$	5.16E-09	3.00	7.29E-09	3.00	5.16E-09	3.00	7.29E-09	3.00

Table 2 DDG scheme (2.1) with Case I numerical flux. L^2 and L^∞ errors for u, measured at the center x_j of the cells

Δx	Numerical solutions				Predicted by analysis			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
$2\pi/40$	2.48E-07	–	3.51E-07	–	1.62E-07	–	2.30E-07	–
$2\pi/80$	1.09E-08	4.51	1.54E-08	4.51	1.02E-08	4.00	1.44E-08	4.00
$2\pi/160$	6.34E-10	4.11	8.97E-10	4.11	6.34E-10	4.00	8.97E-10	4.00
$2\pi/320$	3.96E-11	4.00	5.61E-11	4.00	3.96E-11	4.00	5.61E-11	4.00
$2\pi/640$	2.48E-12	4.00	3.50E-12	4.00	2.48E-12	4.00	3.50E-12	4.00

Table 3 DDG scheme (2.1) with numerical flux of Case I and Case II

Δx	Case I: $\beta_0 = \frac{7}{6}, \beta_1 = \frac{1}{12}$				Case II: $\beta_0 = 3, \beta_1 = \frac{1}{12}$			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
$2\pi/40$	5.45E-05	–	2.19E-05	–	7.55E-05	–	2.70E-05	–
$2\pi/80$	1.06E-05	2.36	3.69E-06	2.57	1.12E-05	2.75	3.87E-06	2.80
$2\pi/160$	1.41E-06	2.92	4.84E-07	2.93	1.41E-06	3.00	4.84E-07	3.00
$2\pi/320$	1.76E-07	3.00	6.05E-08	3.00	1.76E-07	3.00	6.05E-08	3.00
$2\pi/640$	2.20E-08	3.00	7.56E-09	3.00	2.20E-08	3.00	7.56E-09	3.00

the leading errors for points $x_{j-\frac{1}{3}}$ and $x_{j+\frac{1}{3}}$ are 3rd order and is 4th order for point x_j. We compare the numerical solution errors and the analytical errors (the leading terms in (3.10) and (3.9)) with $\alpha = 0.01$ and final time $t = 4\pi$, and list them in Table 1 for point $x_{j-1/3}$ and in Table 2 for point x_j. We see the two errors match very well with each other.

Note that when measuring as finite element solutions (using 40 uniformly spaced sampling points per cell), the scheme accuracy in both L^2 and L^∞ is 3rd order, see the left column of Table 3.

Case II: $\beta_0 = 3\ \beta_1 = \frac{1}{12}$

The numerical flux now becomes

$$\widehat{u}_x = 3\frac{[u]}{\Delta x} + \bar{u}_x + \frac{\Delta x}{12}[u_{xx}].$$

The Fourier analysis gives the DDG solution with such numerical flux as

$$\mathrm{Im}\{u_{j-1/3}(t)\} = e^{-\alpha^2 t}\sin(\alpha x_{j-1/3}) + \frac{5}{648}\Delta x^3 e^{-\alpha^2 t}\cos(\alpha x_{j-1/3}) + O(\Delta x^4), \tag{3.11}$$

and

$$\mathrm{Im}\{u_j(t)\} = e^{-\alpha^2 t}\sin(\alpha x_j) + \frac{49}{77760}\Delta x^4 e^{-\alpha^2 t}\sin(\alpha x_j) + O(\Delta x^6). \tag{3.12}$$

Again, the result for $\mathrm{Im}\{u_{j+\frac{1}{3}}(t)\}$ is similar to $\mathrm{Im}\{u_{j-\frac{1}{3}}(t)\}$. Similar to Case I, 3rd order of accuracy (the leading term in (3.11)) is obtained for points $x_{j\pm\frac{1}{3}}$ and 4th order for point x_j. We also compare the numerical solution errors and the analytical errors with $\alpha = 0.01$ and $t = 4\pi$, and the two errors match well with each other. We use this example to show that the scheme accuracy is not sensitive to the choice of β_0 for the DDG method (2.1) with quadratic approximations. In Table 3 we list the errors measured as a finite element method for this case and case I together, and we see the numerical solution errors and orders are similar to each other with different β_0's.

Case III: $\beta_0 = 3$ and $\beta_1 = \frac{1}{4}$

For this case, the numerical flux is

$$\widehat{u}_x = 3\frac{[u]}{\Delta x} + \bar{u}_x + \frac{\Delta x}{4}[u_{xx}], \tag{3.13}$$

and the Fourier analysis of the DDG method gives the solutions as

$$\mathrm{Im}\{u_{j-\frac{1}{3}}(t)\} = e^{-\alpha^2 t}\sin(\alpha x_{j-\frac{1}{3}}) + \frac{t}{18}\Delta x^2 e^{-\alpha^2 t}\sin(\alpha x_{j-\frac{1}{3}}) + O(\Delta x^3), \tag{3.14}$$

and

$$\mathrm{Im}\{u_j(t)\} = e^{-\alpha^2 t}\sin(\alpha x_j) + \frac{t}{18}\Delta x^2 e^{-\alpha^2 t}\sin(\alpha x_j) + O(\Delta x^4). \tag{3.15}$$

The result for $\mathrm{Im}\{u_{j+\frac{1}{3}}(t)\}$ is similar to $\mathrm{Im}\{u_{j-\frac{1}{3}}(t)\}$. Notice the leading errors degenerate to 2nd order for all the three points $x_{j-\frac{1}{3}}, x_j, x_{j+\frac{1}{3}}$. We list the numerical errors and the analytical errors for point $x_{j-\frac{1}{3}}$ in Table 4. Again the numerical solution errors and the analytical errors match well on the three points. In Table 5 at the left column we list the numerical errors for point x_j and at the right column we list the errors and orders of the

Table 4 DDG scheme (2.1) with Case III numerical flux. L^2 and L^∞ errors for u, measured at the point $x_{j-\frac{1}{3}}$ of the cells

Δx	Numerical solutions				Predicted by analysis			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
$2\pi/40$	1.23E-04	–	1.75E-04	–	1.07E-04	–	1.52E-04	–
$2\pi/80$	2.79E-05	2.14	3.95E-05	2.15	2.69E-05	2.00	3.80E-06	2.00
$2\pi/160$	6.78E-06	2.04	9.59E-06	2.04	6.71E-06	2.00	9.49E-06	2.00
$2\pi/320$	1.68E-06	2.01	2.38E-06	2.01	1.68E-06	2.00	2.37E-06	2.00
$2\pi/640$	4.20E-07	2.00	5.94E-07	2.00	4.20E-07	2.00	5.94E-07	2.00

Table 5 DDG scheme (2.1) with Case III numerical flux

h	Numerical solutions at the point x_j				Measuring as finite element solutions			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
$2\pi/40$	1.08E-04	–	1.52E-04	–	7.07E-04	–	1.78E-04	–
$2\pi/80$	2.69E-05	2.00	3.80E-05	2.00	1.71E-04	2.05	3.97E-05	2.16
$2\pi/160$	6.72E-06	2.00	9.50E-06	2.00	4.25E-05	2.01	9.60E-06	2.05
$2\pi/320$	1.68E-06	2.00	2.37E-06	2.00	1.06E-05	2.00	2.38E-06	2.01
$2\pi/640$	4.20E-07	2.00	5.94E-07	2.00	2.65E-06	2.00	5.94E-07	2.00

method measured as finite element solutions (using 40 uniformly spaced sampling points per cell). Note that only 2nd order is obtained in L^2 and L^∞. Here we take $\alpha = 0.1$ and $t = 4\pi$. This example shows that the scheme accuracy is sensitive to the choice of β_1 for the DDG method with quadratic approximations. With $\beta_1 \neq 1/12$, the scheme degenerates to a 2nd order method. The numerical results match well with the theoretical predictions through the Fourier analysis.

3.2 DDG Method with Interface Corrections

In this section, we perform Fourier analysis on the refined DDG method, namely the DDG with interface corrections (2.3) with quadratic p^2 polynomial approximations. We study two cases with I: $(\beta_0, \beta_1) = (3, \frac{1}{4})$ and II: $(\beta_0, \beta_1) = (3, \frac{1}{12})$, to illustrate that there exists a large class of admissible numerical fluxes leading to optimal convergence for the refined DDG method with interface corrections. The following Fourier analysis confirms that 3rd order of accuracy is obtained with such admissible numerical fluxes.

Case I: $\beta_0 = 3$ and $\beta_1 = \frac{1}{4}$

The numerical flux is

$$\widehat{u}_x = 3\frac{[u]}{\Delta x} + \bar{u}_x + \frac{\Delta x}{4}[u_{xx}].$$

With Fourier analysis the solution at point $x_{j-1/3}$ is given as

$$\mathrm{Im}\{u_{j-1/3}(t)\} = e^{-\alpha^2 t}\sin(\alpha x_{j-1/3}) + \frac{23}{648}\Delta x^3 e^{-\alpha^2 t}\cos(\alpha x_{j-1/3}) + O(\Delta x^4), \tag{3.16}$$

and the solution at point x_j is given as

$$\mathrm{Im}\{u_j(t)\} = e^{-\alpha^2 t}\sin(\alpha x_j) - \frac{19+72t}{8640}\Delta x^4 e^{-\alpha^2 t}\sin(\alpha x_j) + O(\Delta x^6). \tag{3.17}$$

Result for $\mathrm{Im}\{u_{j+\frac{1}{3}}(t)\}$ is similar to $\mathrm{Im}\{u_{j-\frac{1}{3}}(t)\}$. We see the leading error for points $x_{j\pm\frac{1}{3}}$ is 3rd order, and is 4th order for point x_j. These analytical results are consistent with the numerical results at the points, see Table 6 for point $x_{j-1/3}$ and Table 7 for point x_j with $\alpha = 0.01$ and $t = 4\pi$.

When measuring as finite element solutions, the scheme accuracy (in L^2 and L^∞) is 3rd order, see the left column of Table 8.

Case II: $\beta_0 = 3$ and $\beta_1 = \frac{1}{12}$

Table 6 Refined DDG scheme (2.3) with Case I numerical flux. L^2 and L^∞ errors for u, measured at the point $x_{j-\frac{1}{3}}$ of the cells

Δx	Numerical solutions				Predicted by analysis			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
$2\pi/40$	6.83E-05	–	9.64E-05	–	9.72E-05	–	1.37E-04	–
$2\pi/80$	1.20E-05	2.50	1.70E-05	2.50	1.21E-05	3.00	1.72E-05	3.00
$2\pi/160$	1.52E-06	2.99	2.15E-06	2.99	1.52E-06	3.00	2.15E-06	3.00
$2\pi/320$	1.90E-07	3.00	2.68E-07	3.00	1.90E-07	3.00	2.68E-07	3.00
$2\pi/640$	2.37E-08	3.00	3.35E-08	3.00	2.37E-08	3.00	3.35E-08	3.00

Table 7 Refined DDG scheme (2.3) with Case I numerical flux. L^2 and L^∞ errors for u, measured at the center x_j of the cells

Δx	Numerical solutions				Predicted by analysis			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
$2\pi/40$	1.06E-06	–	1.50E-06	–	9.50E-07	–	1.34E-06	–
$2\pi/80$	5.96E-08	4.15	8.43E-08	4.15	5.94E-08	4.00	8.40E-08	4.00
$2\pi/160$	3.71E-09	4.01	5.25E-09	4.01	3.71E-09	4.00	5.25E-09	4.00
$2\pi/320$	2.32E-10	4.00	3.30E-10	4.00	2.32E-10	4.00	3.30E-10	4.00
$2\pi/640$	1.45E-11	4.00	2.05E-11	4.00	1.45E-11	4.00	2.05E-11	4.00

Table 8 Refined DDG scheme (2.3) solution errors and orders with numerical flux of Case I and Case II

Δx	Case I				Case II			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
$2\pi/40$	3.13E-04	–	1.04E-04	–	6.56E-05	–	2.37E-05	–
$2\pi/80$	5.58E-05	2.49	2.00E-05	2.38	1.12E-05	2.56	3.85E-06	2.62
$2\pi/160$	7.04E-06	2.99	2.52E-06	2.99	1.41E-06	2.99	4.84E-07	2.99
$2\pi/320$	8.80E-07	3.00	3.15E-07	3.00	1.76E-07	3.00	6.05E-08	3.00
$2\pi/640$	1.10E-07	3.00	3.94E-08	3.00	2.20E-08	3.00	7.56E-09	3.00

The numerical flux becomes

$$\widehat{u}_x = 3\frac{[u]}{\Delta x} + \bar{u}_x + \frac{\Delta x}{12}[u_{xx}],$$

and the Fourier analysis gives the DDG interface correction solutions as

$$\mathrm{Im}\{u_{j-1/3}(t)\} = e^{-\alpha^2 t}\sin(\alpha x_{j-1/3}) + \frac{5}{648}\Delta x^3 e^{-\alpha^2 t}\cos(\alpha x_{j-1/3}) + O(\Delta x^4), \tag{3.18}$$

for point $x_{j-1/3}$, and

$$\mathrm{Im}\{u_j(t)\} = e^{-\alpha^2 t}\sin(\alpha x_j) + \frac{11-12t}{8640}\Delta x^4 e^{-\alpha^2 t}\sin(\alpha x_j) + O(\Delta x^6) \tag{3.19}$$

for point x_j. The result for $\mathrm{Im}\{u_{j+\frac{1}{3}}(t)\}$ is similar to $\mathrm{Im}\{u_{j-\frac{1}{3}}(t)\}$. Again, we obtain 3rd order for points $x_{j\pm\frac{1}{3}}$, and 4th order for point x_j. They are consistent with the numerical results on the three points.

When measuring as finite element solutions, the scheme accuracy (in L^2 and L^∞) is 3rd order. In Table 8 we list the solution errors and orders with the left column corresponding to the Case I numerical flux, and the right column corresponding to the Case II numerical flux. We see 3rd order of accuracy is obtained with these numerical fluxes. The above Fourier analysis confirms that there exists a large class of admissible numerical fluxes that lead to optimal convergence for the DDG method with interface corrections.

3.3 Variations of DDG Method with Symmetric/Nonsymmetric Structure

In this section we perform Fourier analysis to the two new DG methods (2.6) with numerical flux (2.7). The first case is about the symmetric version of the DDG method with $\sigma = +1$ in (2.6), and the second case is about the nonsymmetric version with $\sigma = -1$ in (2.6). The Fourier analysis gives 3rd order of convergence for both two schemes with p^2 approximations, and the analytical results show some interesting connections between the two DG methods.

Case I: *A variation of the DDG method with symmetric structure*

The scheme is defined as

$$\int_{I_j} u_t v dx + \int_{I_j} u_x v_x dx - (\widehat{u}_x)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + (\widehat{u}_x)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} + ([u]\widetilde{v}_x)_{j+\frac{1}{2}} + ([u]\widetilde{v}_x)_{j-\frac{1}{2}} = 0 \tag{3.20}$$

with the numerical flux given as

$$\begin{cases} \widehat{u_x} = \frac{[u]}{\Delta x} + \overline{u_x} + \frac{\Delta x}{12}[u_{xx}], \\ \widetilde{v_x} = \frac{[v]}{\Delta x} + \overline{v_x} + \frac{\Delta x}{12}[v_{xx}]. \end{cases} \tag{3.21}$$

The Fourier analysis gives the DG solutions at points $x_{j-1/3}, x_j, x_{j+1/3}$ as

$$\begin{cases} \mathrm{Im}\{u_{j-1/3}(t)\} = e^{-\alpha^2 t}\sin(\alpha x_{j-1/3}) + \frac{25-18t}{2160}\Delta x^3 e^{-\alpha^2 t}\cos(\alpha x_{j-1/3}) + O(\Delta x^4), \\ \mathrm{Im}\{u_j(t)\} = e^{-\alpha^2 t}\sin(\alpha x_j) + \frac{25-54t}{6480}\Delta x^3 e^{-\alpha^2 t}\cos(\alpha x_j) + O(\Delta x^4), \\ \mathrm{Im}\{u_{j+1/3}(t)\} = e^{-\alpha^2 t}\sin(\alpha x_{j+1/3}) - \frac{25+54t}{6480}\Delta x^3 e^{-\alpha^2 t}\cos(\alpha x_{j+1/3}) + O(\Delta x^4). \end{cases} \tag{3.22}$$

The leading errors in the solutions (3.22) are all 3rd order. Again they match well with the numerical results at the points, see Table 9 for point $x_{j-1/3}$, Table 10 for point x_j and Table 11 for point $x_{j+1/3}$ with $\alpha = 0.1$ and $t = 4\pi$.

When measuring as finite element solutions, we obtain 3rd order of accuracy in L^2 and L^∞, see the left column of Table 12. Even in [26] we prove the optimal convergence of the symmetric DDG method, here the Fourier analysis further confirms the optimal 3rd order of accuracy for quadratic approximations.

Case II: *A DG method with nonsymmetric structure*

With $\sigma = -1$ in (2.6), the scheme is defined as

$$\int_{I_j} u_t v dx + \int_{I_j} u_x v_x dx - (\widehat{u}_x)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + (\widehat{u}_x)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} - ([u]\widetilde{v}_x)_{j+\frac{1}{2}} - ([u]\widetilde{v}_x)_{j-\frac{1}{2}} = 0. \tag{3.23}$$

Table 9 DDG method with symmetric structure (3.20)–(3.21). L^2 and L^∞ errors for u, measured at the point $x_{j-\frac{1}{3}}$ of the cells

Δx	Numerical solutions				Predicted by analysis			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
$2\pi/40$	2.55E-05	–	3.62E-05	–	2.54E-05	–	3.59E-05	–
$2\pi/80$	3.18E-06	3.00	4.50E-06	3.00	3.18E-06	3.00	4.50E-06	3.00
$2\pi/160$	3.98E-07	3.00	5.62E-07	3.00	3.98E-07	3.00	5.62E-07	3.00
$2\pi/320$	4.97E-08	3.00	7.03E-08	3.00	4.97E-08	3.00	7.03E-08	3.00
$2\pi/640$	6.21E-09	3.00	8.79E-09	3.00	6.21E-09	3.00	8.79E-09	3.00

Table 10 DDG method with symmetric structure (3.20)–(3.21). L^2 and L^∞ errors for u, measured at the point x_j of the cells

Δx	Numerical solutions				Predicted by analysis			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
$2\pi/40$	6.78E-06	–	9.58E-06	–	6.79E-06	–	9.61E-06	–
$2\pi/80$	8.49E-07	3.00	1.20E-06	3.00	8.49E-07	3.00	1.20E-06	3.00
$2\pi/160$	1.06E-07	3.00	1.50E-07	3.00	1.06E-07	3.00	1.57E-07	3.00
$2\pi/320$	1.33E-08	3.00	1.88E-08	3.00	1.33E-08	3.00	1.88E-08	3.00
$2\pi/640$	1.66E-09	3.00	2.35E-09	3.00	1.66E-09	3.00	2.35E-09	3.00

Table 11 DDG method with symmetric structure (3.20)–(3.21). L^2 and L^∞ errors for u, measured at the point $x_{j+\frac{1}{3}}$ of the cells

Δx	Numerical solutions				Predicted by analysis			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
$2\pi/40$	5.34E-06	–	7.53E-06	–	5.06E-06	–	7.15E-06	–
$2\pi/80$	6.42E-07	3.06	9.07E-07	3.05	6.33E-07	3.00	8.95E-07	3.00
$2\pi/160$	7.94E-08	3.02	1.12E-07	3.01	7.91E-08	3.00	1.12E-07	3.00
$2\pi/320$	9.91E-09	3.00	1.40E-08	3.00	9.89E-09	3.00	1.40E-08	3.00
$2\pi/640$	1.24E-09	3.00	1.75E-09	3.00	1.24E-09	3.00	1.75E-09	3.00

Take $\beta_{0u} = \beta_{0v} = 1$ in (2.7), the numerical flux becomes

$$\begin{cases} \widehat{u_x} = \frac{[u]}{\Delta x} + \overline{u_x} + \frac{\Delta x}{12}[u_{xx}], \\ \widetilde{v_x} = \frac{[v]}{\Delta x} + \overline{v_x} + \frac{\Delta x}{12}[v_{xx}]. \end{cases} \tag{3.24}$$

Plug in this numerical flux (3.24) into (3.23), we end up having a DG scheme with no penalty term (the solution jump term), which is similar to the Baumann-Oden method [5] except now we have extra second derivative jump terms included in the scheme definition. We choose such a numerical flux to show that the optimal 3rd order convergence is recovered with our nonsymmetric DG method.

Table 12 L^2 and L^∞ errors and orders for the symmetric DDG method (3.20)–(3.21) and the nonsymmetric DG method (3.23)–(3.24)

Δx	Symmetric case				Nonsymmetric case			
	L^2 error	Order	L^∞ error	Order	L^2 error	Order	L^∞ error	Order
$2\pi/40$	8.45E-05	–	3.71E-05	–	8.45E-05	–	3.71E-05	–
$2\pi/80$	1.05E-05	3.01	4.62E-06	3.00	1.05E-05	3.00	4.62E-06	3.00
$2\pi/160$	1.32E-06	3.00	5.78E-07	3.00	1.32E-06	3.00	5.78E-07	3.00
$2\pi/320$	1.64E-07	3.00	7.22E-08	3.00	1.64E-07	3.00	7.22E-08	3.00
$2\pi/640$	2.06E-08	3.00	9.02E-09	3.00	2.06E-08	3.00	9.02E-09	3.00

The Fourier analysis gives the DG solutions (3.23)–(3.24) at the points as

$$\begin{cases} \mathrm{Im}\{u_{j-1/3}(t)\} = e^{-\alpha^2 t}\sin(\alpha x_{j-1/3}) + \frac{25+54t}{6480}\Delta x^3 e^{-\alpha^2 t}\cos(\alpha x_{j-1/3}) + O(\Delta x^4), \\ \mathrm{Im}\{u_j(t)\} = e^{-\alpha^2 t}\sin(\alpha x_j) + \frac{-25+54t}{2160}\Delta x^3 e^{-\alpha^2 t}\cos(\alpha x_j) + O(\Delta x^4), \\ \mathrm{Im}\{u_{j+1/3}(t)\} = e^{-\alpha^2 t}\sin(\alpha x_{j+1/3}) + \frac{-25+18t}{2160}\Delta x^3 e^{-\alpha^2\alpha^2 t}\cos(\alpha x_{j+1/3}) + O(\Delta x^4). \end{cases} \tag{3.25}$$

The leading errors in (3.25) are all 3rd order. Again the analytical errors match well with the numerical errors at the three points.

When measuring as finite element solutions, 3rd order of accuracy (in L^2 and L^∞) is obtained and the errors and orders are listed in Table 12 at the right column. We see the numerical solution errors of the nonsymmetric DG method are similar to the solution errors of the symmetric DDG method. When comparing at point values, it shows that the two errors are the same at x_j in the absolute value sense, and the error of the Case I symmetric DDG method at point $x_{j-\frac{1}{3}}$ is identical to the error of the Case II nonsymmetric DG method at point $x_{j+\frac{1}{3}}$ in the absolute value sense, and similarly the error at point $x_{j+\frac{1}{3}}$ of Case I is same as the error at point $x_{j-\frac{1}{3}}$ of Case II.

4 Conclusion

In this paper, we carry out the Fourier type error analysis of the recent four discontinuous Galerkin methods with p^2 quadratic approximations, namely the DDG method [16]; the DDG method with interface corrections [17]; the DDG method with symmetric structure [26]; and a DG method with nonsymmetric structure [30]. We choose Lagrange polynomials as basis functions, thus the degree of freedom can be taken as the solution values on the uniformly distributed mesh points. The discontinuous Galerkin method is then rewritten in a finite difference format. Standard Fourier type technique is used to analyze the error behavior of the four discontinuous Galerkin methods. It is verified that optimal 3rd order of accuracy can be obtained for all the four discontinuous Galerkin methods with suitable numerical fluxes. The theoretical predicted errors match well with the numerical results.

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