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Galerkin Methods for Stochastic Hyperbolic Problems Using Bi-Orthogonal Polynomials

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Abstract This work is concerned with scalar transport equations with random transport velocity. We first give some sufficient conditions that can guarantee the solution to be in appropriate random spaces. Then a Galerkin method using bi-orthogonal polynomials is proposed, which decouples the equation in the random spaces, yielding a sequence of uncoupled equations. Under the assumption that the random wave field has a structure of the truncated KL expansion, a principle on how to choose the orders of the approximated polynomial spaces is given based on the sensitivity analysis in the random spaces. By doing this, the total degree of freedom can be reduced significantly. Numerical experiments are carried out to illustrate the efficiency of the proposed method.

Keywords Hyperbolic equation · Regularity · Stochastic Galerkin methods · Bi-orthogonal

1 Introduction

In simulating complex physical or engineering systems, there are always some uncertain factors associated with the physical problems, such as constitutive laws, boundary and initial conditions, transport coefficients, source terms, geometric irregularities (e.g., roughness), etc. In order to obtain reliable numerical predictions, one has to include uncertainty quantification and to consider the randomness effects.

The methods for solving SPDEs include Monte Carlo and sampling based methods [3, 5], perturbation methods [11, 12], the generalized polynomial chaos (gPC) methods [17, 19], and so on. The gPC method, which was first introduced in [7, 16] as PC methods and was

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recently developed by Xiu et al. [17–19], has become one of the most widely used methods. With gPC, stochastic solutions are expressed as orthogonal polynomials of the input random parameters, and different types of orthogonal polynomials can be chosen to achieve better convergence. It is essentially a spectral representation in random space, and exhibits fast convergence when the solution depends smoothly on the random parameters. However, when the governing equations take complicated forms, numerical implementations of stochastic Galerkin methods can become nontrivial. Typically, the resulting systems are usually coupled together, and some decoupling techniques need to be used to remain efficiency [20].

When the governing equations take a hyperbolic type, additional difficulties are involved, as the hyperbolic equations have worst regularity properties among all the three types of equations. In [8], Gottlieb and Xiu applied the gPC-Galerkin methods to solve the random transport equation with a random wave speed. In [13], Tang and Zhou proposed some rigorous regularity analysis for the same problem and demonstrated the convergence of the stochastic collocation methods.

In this work, we consider the scalar transport equations whose transport velocity is a random field. We first gave the sufficient conditions that allow the underlying solution to be in BV and H^k in the random spaces. A special case when the wave speed is a random variable was considered by Tang and Zhou [13]. Then, a Galerkin method using bi-orthogonal polynomials is proposed for the problem, which decouples the equation in the random spaces, yielding a sequence of uncoupled equations. Under the assumption that the random wave field has a structure of the truncated KL expansion, a principle on how to choose the orders of the approximated polynomial spaces is given based on the sensitivity analysis in the random spaces, with which the total degree of freedom can be significantly reduced. Numerical experiments are carried out to illustrate the efficiency of the method.

The paper is composed of the following sections. Section 2 is devoted to the regularity analysis. The Galerkin methods based on the bi-orthogonal polynomials and the related sensitivity analysis are described in Sect. 3. Numerical examples are given in Sect. 4 to support the theoretical analysis. The final section gives some conclusion remarks.

2 Problem Definition and Regularity in Various Spaces

Random Transport Equation Let $x \in D \equiv [-1, 1]$ be the spatial coordinates, T denote the set $[0, T]$ for time variable t, and $(\Omega, \mathcal{A}, \mathcal{P})$ be a complete probability space, whose event space is Ω ($\omega \in \Omega$ is the event) and is equipped with σ-algebra $\mathcal{A}$, and $P : \mathcal{A} \to [0, 1]$ is a probability measure. By such definitions, $y(\omega)$ admits a random variable. Consider the stochastic linear wave equation: Find a random function, $u : T \times D \times \Omega \to \mathbb{R}$ such that P-almost everywhere (a.e.) in Ω, or in other words, almost surely (a.s.) the following equation holds:

$$\frac{\partial u(x, t, y(\omega))}{\partial t} = c(x, y(\omega)) \frac{\partial u(x, t, y(\omega))}{\partial x}, \tag{2.1a}$$

$$u(x, t = 0, y(\omega)) = u_0(x, y(\omega)). \tag{2.1b}$$

A well-posed set of boundary conditions is given by:

$$u(-1, t; y(\omega)) = u_L(t; y(\omega)), \qquad c(-1, y(\omega)) < 0, \tag{2.2a}$$

$$u(1, t; y(\omega)) = u_R(t; y(\omega)), \qquad c(1, y(\omega)) > 0. \tag{2.2b}$$

Equations (2.1)–(2.2) complete the set up of the problem.

For ease of notations, we will omit ω in the remaining of the paper. Then the random field becomes $c(x, y)$. One can also view y as a parameter in some parametric spaces. Below we will analyze the solution regularity in the H^1 and BV random spaces.

2.1 Regularity in H^1

Without lose of generality, we use D to indicate the physical space and Γ the parametric space. Since stochastic functions intrinsically have different structure with respect to y and with respect to x, the analysis of numerical approximations requires tensor spaces. The details for the definition can be found in [2]. Following [2], $u \in L^2 \otimes H^k(D)$ implies that $u(\cdot, y; t) \in H^k(D)$ a.e. on Γ and $u(x, t; \cdot) \in L^2(\Gamma)$ a.e. on D. Moreover, we have (for every fixed $t < T$) the isomorphism $L^2 \otimes H^k(D) \simeq L^2(\Gamma; H^k(D)) \simeq H^k(D; L^2(\Gamma))$ with the definitions

$$
\begin{aligned}
&L^2(\Gamma; H^k(D)) \\
&\quad = \left\{ v : \Gamma \times D \to \mathbb{R} \mid v \text{ is strongly measurable and } \int_\Gamma \|v(\cdot, y, t)\|^2_{H^k(D)} dy < +\infty \right\}; \\
&H^k(D; L^2(\Gamma)) \\
&\quad = \left\{ v : \Gamma \times D \to \mathbb{R} \mid v \text{ is strongly measurable and } \forall |\alpha| \le k, \exists\ \partial_\alpha v \in L^2(\Gamma) \otimes L^2(D), \right. \\
&\qquad \left. \int_\Gamma \int_D \partial_\alpha v \varphi(x, y) dx dy = (-1)^{|\alpha|} \int_\Gamma \int_D v(x, y, t) \partial_\alpha \varphi(x, y) dx dy \ \forall \varphi \in C_0^\infty(\Gamma \times D) \right\}.
\end{aligned}
$$

We also denote

$$
\Gamma^+ = \left\{ y \,\middle|\, y \in \Gamma,\ c(1, y) > 0 \right\}, \quad \text{and} \quad \Gamma^- = \left\{ y \,\middle|\, y \in \Gamma,\ c(-1, y) < 0 \right\}.
$$

With the above definitions, we now introduce the following lemma.

Lemma 2.1 *Consider problem* (2.1)–(2.2). *If* $c_x(x, y) < C$ *is bounded and the following conditions are satisfied*:

$$
\int_\Gamma \int_D \rho(y) \Big(\partial_x u_0(x; y) \Big)^2 dx dy < \infty, \tag{2.3a}
$$

$$
\int_0^T \int_{\Gamma^+} \frac{\rho(y)}{c(1, y)} \Big(\partial_t u_R(t; y) \Big)^2 dy dt < \infty, \tag{2.3b}
$$

$$
\int_0^T \int_{\Gamma^-} \frac{\rho(y)}{|c(-1, y)|} \Big(\partial_t u_L(t; y) \Big)^2 dy dt < \infty, \tag{2.3c}
$$

then

$$
\int_\Gamma \int_D \rho(y) u_x^2 dx dy < C(T), \quad 0 < t \le T, \tag{2.4}
$$

where $\rho(y) > 0$ *is the probability distribution function and* $C(T)$ *is a positive constant depending on* T.

Proof Using (2.1) and (2.2), we have, for $c(-1, y) < 0$,

$$u_x(-1, t; y) = \frac{1}{c(-1, y)} u_t(-1, t; y) = \frac{1}{c(-1, y)} \partial_t u_L(t; y). \tag{2.5}$$

Similarly, for $c(1, y) > 0$,

$$u_x(1, t; y) = \frac{1}{c(1, y)} \partial_t u_R(t; y). \tag{2.6}$$

It follows from the governing equation (2.1) that

$$\begin{aligned} \partial_t (u_x^2) &= c(x, y) \partial_x (u_x^2) + 2 c_x(x, y) u_x^2 \\ &= \left(c(x, y) u_x^2\right)_x + c_x(x, y) u_x^2, \quad x \in D,\ t > 0, \end{aligned}$$

which leads to

$$\begin{aligned} &\partial_t \int_D \rho(y) u_x^2 dx - \rho(y) \int_D c_x(x, y) u_x^2 dx \\ &\quad = \rho(y) \left(c(1, y) u_x^2(1, t; y) - c(-1, y) u_x^2(-1, t; y)\right) \\ &\quad \le \begin{cases} 0 & \text{if } c(1, y) < 0,\ c(-1, y) > 0, \\ \rho(y) c(1, y) u_x^2(1, t; y) & \text{if } c(1, y) > 0,\ c(-1, y) > 0, \\ -\rho(y) c(-1, y) u_x^2(-1, t; y) & \text{if } c(1, y) < 0,\ c(-1, y) < 0, \\ \rho(y) c(1, y) u_x^2(1, t; y) - \rho(y) c(-1, y) u_x^2(-1, t; y) & \text{if } c(1, y) > 0,\ c(-1, y) < 0. \end{cases} \end{aligned}$$

The above result, together with (2.5) and (2.6), yields

$$\begin{aligned} &\frac{d}{dt} \int_\Gamma \int_D \rho(y) u_x^2 dx dy \\ &\quad \le \int_{\Gamma^+} \frac{\rho(y)}{c(1, y)} \left(\partial_t u_R(t; y)\right)^2 dy + \int_{\Gamma^-} \frac{\rho(y)}{|c(-1, y)|} \left(\partial_t u_L(t; y)\right)^2 dy \\ &\qquad + C \int_\Gamma \int_D \rho(y) u_x^2 dx dy. \end{aligned} \tag{2.7}$$

The desired estimate is obtained by using the assumption (2.3) and the Gronwall inequality. □

Theorem 2.1 *Consider problem* (2.1)–(2.2). *Assume that there exists a constant* C *such that*

$$|c_y(x, y)|,\ |c_x(x, y)| \le C, \quad \textit{almost everywhere in } D \otimes \Gamma. \tag{2.8}$$

If the assumption (2.3) *holds and furthermore if*

$$\int_\Gamma \int_D \rho(y) (\partial_y u_0(x; y))^2 dx dy < \infty, \tag{2.9a}$$

$$\int_0^T \int_{\Gamma^+} \frac{\rho(y)}{c(1, y)} (\partial_y u_R(t; y))^2 dy dt < \infty, \tag{2.9b}$$

$$\int_0^T \int_{\Gamma^-} \frac{\rho(y)}{|c(-1,y)|}(\partial_y u_L(t;y))^2 dydt < \infty, \tag{2.9c}$$

then

$$\int_\Gamma \int_D \rho(y) u_y^2 dxdy < C(T), \quad 0 < t \le T, \tag{2.10}$$

where $C(T)$ is a non-negative number depending on T.

Proof Differentiating both sides of (2.1) with respect to y gives

$$(u_y)_t = c_y(x,y)u_x + c(x,y)(u_y)_x,$$

which yields

$$(u_y^2)_t = 2c_y(x,y)u_x u_y + c(x,y)(u_y^2)_x.$$

Integrating the above equation with respect to x leads to

$$\begin{aligned}
&\partial_t \int_D \rho(y) u_y^2 dx \\
&\quad = 2\rho(y)\int_D c_y(x,y)u_x u_y dx + \rho(y)\int_D c_x(x,y)u_y^2 dx \\
&\qquad + \rho(y)\Big(c(1,y)u_y^2(1,t;y) - c(-1,y)u_y^2(-1,t;y)\Big).
\end{aligned}$$

Note

$$\begin{aligned}
&\rho(y)\left(c(1,y)u_y^2(1,t;y) - c(-1,y)u_y^2(-1,t;y)\right) \\
&\le \begin{cases}
0 & \text{if } c(1,y) < 0,\ c(-1,y) > 0, \\
\frac{\rho(y)}{c(1,y)}(\partial_y u_R)^2(t;y) & \text{if } c(1,y) > 0,\ c(-1,y) > 0, \\
-\frac{\rho(y)}{c(-1,y)}(\partial_y u_L)^2(t;y) & \text{if } c(1,y) < 0,\ c(-1,y) < 0, \\
\frac{\rho(y)}{c(1,y)}(\partial_y u_R)^2(t;y) - \frac{\rho(y)}{c(-1,y)}(\partial_y u_L)^2(t;y) & \text{if } c(1,y) > 0,\ c(-1,y) < 0,
\end{cases}
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Gamma \int_D \rho(y) u_y^2 dxdy \\
&\quad \le C\int_\Gamma \int_D \rho(y) u_x^2 dxdy + C\int_\Gamma \int_D \rho(y) u_y^2 dxdy \\
&\qquad + \int_{\Gamma^+} \frac{\rho(y)}{c(1,y)}(\partial_y u_R(t;y))^2 dy + \int_{\Gamma^-} \frac{\rho(y)}{|c(-1,y)|}(\partial_y u_L(t;y))^2 dy,
\end{aligned} \tag{2.11}$$

where the boundedness assumption of ∇c is used. The desired estimate (2.10) follows from Lemma 2.1, the Gronwall inequality and the assumption (2.9). □

Remark 2.1 It is clear that if the boundary data and the initial data satisfy some further assumptions, then the solution of problem (2.1)–(2.2) can have higher regularity. A more detailed set of conditions can be found following the above procedures, which will be omitted here.

2.2 Regularity in *BV* Space

Using a similar trick used in the proof of Lemma 2.1, we can obtain the following result.

Lemma 2.2 *Consider the problem* (2.1)–(2.2). *If* c_x *is bounded and the following conditions are satisfied*:

$$\int_\Gamma \int_D \rho(y)|\partial_x u_0(x;y)|dxdy < +\infty, \tag{2.12a}$$

$$\int_0^T \int_{\Gamma^+} \frac{\rho(y)}{c(1,y)}|\partial_t u_R(t;y)|dydt < +\infty, \tag{2.12b}$$

$$\int_0^T \int_{\Gamma^-} \frac{\rho(y)}{|c(-1,y)|}|\partial_t u_L(t;y)|dydt < +\infty, \tag{2.12c}$$

then we have

$$\int_\Gamma \int_D \rho(y)|\partial_x u(x,t;y)|dxdy < C(T), \quad 0 < t \le T, \tag{2.13}$$

where $C(T)$ *is a constant dependent on* T.

Theorem 2.2 *Consider problem* (2.1)–(2.2). *Assume that the assumption* (2.8) *holds*, *i.e.*, c_x, c_y *are bounded in the distribution sense. If the assumption* (2.12) *holds and furthermore if*

$$\int_\Gamma \int_D \rho(y)|\partial_y u_0(x;y)|dxdy < +\infty, \tag{2.14a}$$

$$\int_0^T \int_{\Gamma^+} \frac{\rho(y)}{|c(1,y)|}|\partial_y u_R(t;y)|dydt < +\infty, \tag{2.14b}$$

$$\int_0^T \int_{\Gamma^-} \frac{\rho(y)}{|c(-1,y)|}|\partial_y u_L(t;y)|dydt < +\infty, \tag{2.14c}$$

then we have

$$\int_\Gamma \int_D \rho(y)|u_y(x,t;y)|dxdy < C(T), \quad 0 < t \le T, \tag{2.15}$$

where $C(T)$ *is a finite number depending on* T.

Proof Differentiating both sides of (2.1) with respect to y and multiplying the resulting equation by $\operatorname{sgn}(u_y)$ yield

$$|u_y|_t = c_y u_x \operatorname{sgn}(u_y) + c(x,y)|u_y|_x, \tag{2.16}$$

where $\operatorname{sgn}(u_y)$ gives the sign of u_y. Integrating the above equation with respect to x leads to

$$\begin{aligned}&\partial_t \int_D \rho(y)|u_y|dx \\ &\quad = \rho(y)\int_D c_y(x,y)u_x \operatorname{sgn}(u_y)dx + \rho(y)\int_D c_x(x,y)|u_y|dx\end{aligned}$$

$$+\rho(y)\Big[c(1,y)|u_y(1,t;y)|-c(-1,y)|u_y(-1,t;y)|\Big].$$

Using

$$\rho(y)\left(c(1,y)|u_y(1,t;y)|-c(-1,y)|u_y(-1,t;y)|\right)$$
$$\leq\begin{cases}0 & \text{if } c(1,y)<0, c(-1,y)>0,\\ \frac{\rho(y)}{|c(1,y)|}|\partial_y u_L(t;y)| & \text{if } c(1,y)>0, c(-1,y)>0,\\ \frac{\rho(y)}{|c(-1,y)|}|\partial_y u_R(t;y)| & \text{if } c(1,y)<0, c(-1,y)<0,\\ \frac{\rho(y)}{|c(1,y)|}|\partial_y u_L(t;y)|+\frac{\rho(y)}{|c(-1,y)|}|\partial_y u_R(t;y)| & \text{if } c(1,y)>0, c(-1,y)<0\end{cases}$$

yields

$$\begin{aligned}\frac{d}{dt}\int_\Gamma\int_D \rho(y)|u_y|dxdy \leq C\int_\Gamma\int_D \rho(y)|u_x|dxdy + C\int_\Gamma\int_D \rho(y)|u_y|dxdy\\ +\int_{\Gamma^+}\frac{\rho(y)}{|c(1,y)|}|\partial_y u_R(t;y)|dy\\ +\int_{\Gamma^-}\frac{\rho(y)}{|c(-1,y)|}|\partial_y u_L(t;y)|dy,\end{aligned}\tag{2.17}$$

where the boundedness assumption of ∇c is used. The desired estimate (2.15) follows from Lemma 2.2, the Gronwall inequality and the assumption (2.14). □

3 Numerical Methods

In many problems the source of randomness can be approximated using just a small number of uncorrelated, sometimes independent, random variables; take, for example, the case of a truncated Karhunen-Lòeve expansion [7]. This motivates us to make the following assumption which is frequently used by various authors (see, e.g., [10, 15]).

Assumption 3.1 The random wave field has a form:

$$\kappa(x,\omega)=\kappa(x,y_1,\ldots,y_N)\quad \text{on } \Omega\times D,\tag{3.1}$$

where $\{y_i\}_{i=1}^N$ are real valued random variables with mean value zero and unit variance.

Example (Karhunen–Lòeve Expansion) Denote the mean and the covariance of $\kappa(x,y)$ as

$$\kappa_0(x)=\int_\Omega \kappa(x,y(\omega))dP(\omega)$$

and

$$\mathrm{Cov}_\kappa(x,x')=\int_\Omega(\kappa(x,y(\omega))-\kappa_0)(\kappa(x',y(\omega))-\kappa_0)dP(\omega)$$

respectively. The KL expansion represents $\kappa(x,y(\omega))$ in the series form as

$$\kappa(x,\omega)=\kappa_0(x)+\sum_{i=1}^{\infty}\sqrt{\lambda_i}\kappa_i(x)y_i(\omega),\tag{3.2}$$

where $\{\lambda_i\}_{i=1}^{\infty}$ and $\{\kappa_i\}_{i=1}^{\infty}$ are the eigenvalues and orthogonal eigenfunctions of $\mathrm{Cov}_\kappa(x, x')$, i.e.,

$$\int_\Omega \mathrm{Cov}_\kappa(x, z)\kappa_i(z)dz = \lambda_i\kappa_i(x).$$

This series converges in the mean-square sense. By definition, $\mathrm{Cov}_\kappa(x, x')$ is symmetric and positive semidefinite. Moreover, there exists a countable sequence of eigenpairs $\{\lambda_i, \kappa_i\}_{i=1}^{\infty}$ satisfying

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots \to 0,$$

and the eigenfunctions $\{\kappa_i(x)\}$ are orthogonal in $L^2(D)$. Furthermore, $\{y_i(\omega)\}$ is a set of uncorrelated random variables with mean value zero. If the eigenfunctions are normalized, $\{y_i(\omega)\}$ all have unit variance. Then the N-term truncation of KL expansion reads

$$\kappa^N(x, \omega) = \kappa_0(x) + \sum_{i=1}^{N} \sqrt{\lambda_i}\kappa_i(x)y_i(\omega). \tag{3.3}$$

In the remaining of this work, we assume that the random wave field has a form of (3.3) and $y_i(\omega)$ has some identical distribution with probability density function $\rho_i(y_i)$. In what follows, we will denote with $\Gamma_i \equiv y_i(\Omega)$ the image of y_i, $\Gamma = \prod_{i=1}^{N} \Gamma_i$. We also use $\rho(y)$ to stand for the accumulation density function. Consequently, the random wave problem becomes

$$\begin{cases} \partial_t u(x, t, y) = c(x, y)\partial_x u(x, t, y), & \\ u(1, t; y) = u_R(t, y), & c(1, y) > 0, \\ u(-1, t; y) = u_L(t, y), & c(-1, y) < 0, \\ u(x, 0; y) = u_0(x, y), & \end{cases} \tag{3.4}$$

where $y = (y_i, \ldots, y_N)$ is a N-variable random vector and $\kappa(x, y)$ has the form of (3.3). Many standard numerical methods can be used to solve the problem.

For $y \in \Gamma$, we use the double orthogonal polynomial function space introduced in [15] to approximate $L^2(\Gamma, \rho)$, by which we can decouple the equation in the random spaces, and get a sequence of uncoupled equations. This kind of polynomial space has also been used in [2, 10] for different problems. In the following, we construct the double orthogonal basis. For any $r \in \mathbb{N}$, the space of single-variable polynomials of degree at most r is denoted by P_r. For $\mathbf{r} = (r_1, r_2, \ldots, r_M) \in \mathbb{N}^M$, we construct the multi-variable polynomial space

$$P_{\mathbf{r}} = P_{r_1} \otimes P_{r_2} \otimes \cdots \otimes P_{r_M} \in L^2(\Gamma, \rho).$$

For the spaces $P_{r_i}, i = 1, \ldots, M$, we use the double orthogonal functions, denoted by $\{\phi_{j,i}\}_{j=0}^{r_i}$ as basis functions. More precisely, we require

$$\begin{cases} \int_{\Gamma_i} \rho(y_i)\phi_{j,i}\phi_{k,i}dy_i = \delta_{jk}, & j, k = 1, \ldots, r_i, \\ \int_{\Gamma_i} y_i\rho(y_i)\phi_{j,i}\phi_{k,i}dy_i = C_{k,i}\delta_{jk}, & j, k = 1, \ldots, r_i, \end{cases} \tag{3.5}$$

where $\{C_{k,i}\}_{k=0}^{r_i}$ are nonzero constants. Then we can construct a basis function of $P_{\mathbf{r}}$ by selecting one polynomial basis function from each $P_{r_i}, i = 1, \ldots, M$ and multiplying these selected M basis functions together. It is clear that if given $\mathbf{r} = (r_1, r_2, \ldots, r_M)$, we have totally $N_y = \prod_{i=1}^{M}(r_i + 1)$ basis for $P_{\mathbf{r}}$.

Let $\mathbf{i} = (i_1, i_2, \ldots, i_M)$. If $0 \le i_l \le r_l$, $\forall 1 \le l \le M$, then we say that $\mathbf{i} \le \mathbf{r}$. There are total N_y multi-index $\mathbf{i}$ less than or equal to $\mathbf{r}$. Each $\mathbf{i}$ corresponds to one basis function for $P_{\mathbf{r}}$. Then the basis functions for $P_{\mathbf{r}}$ form the set

$$\left\{ \Phi_{\mathbf{i}}(y) \middle| \Phi_{\mathbf{i}}(y) = \prod_{k=1}^{M} \phi_{i_k,k}(y_k),\ i_k \in \{0, 1, \ldots, r_k\} \right\}_{\mathbf{i} \le \mathbf{r}}. \tag{3.6}$$

In Sect. 8.7.2 in [9], it was shown that finding the above basis for $\{P_{r_j}\}_{j=1}^{N}$ results in eigenproblems, and the computational work for the eigenproblems is negligible comparing to the cost required to solve the coupled equations if r_j is not large.

The basis functions defined by (3.5) satisfy the following equations:

$$\int_{\Gamma} y_k \rho(y) \Phi_{\mathbf{i}}(y) \Phi_{\mathbf{j}}(y) dy = C_{i_k,k} \delta_{\mathbf{ij}}, \tag{3.7}$$

where $y_k \in \{y_k\}_{k=1}^{M}$. To see why (3.7) holds, we first note that

$$\Phi_{\mathbf{i}}(y) = \phi_{i_1}(y_1) \cdots \phi_{i_M}(y_M), \qquad \Phi_{\mathbf{j}}(y) = \phi_{j_1}(y_1) \cdots \phi_{j_M}(y_M),$$

which yields

$$\begin{aligned} &\int_{\Gamma} y_k \rho_k(y) \Phi_{\mathbf{i}}(y) \Phi_{\mathbf{j}}(y) dy \\ &= \Big(\int y_k \phi_{i_k}(y_k) \phi_{j_k}(y_k) dy_k \Big) \Big(\prod_{l=1, l \ne k}^{M} \int \rho_l \phi_{i_l}(y_l) \phi_{j_l}(y_l) dy_k \Big) = C_{i_k,k} \delta_{\mathbf{ij}}. \end{aligned}$$

3.1 Decoupling Procedure

By the discussions above, we construct the approximated solution by

$$u = \sum_{\mathbf{i} \le \mathbf{r}} u_{\mathbf{i}} \Phi_{\mathbf{i}}(y), \qquad u_0 = \sum_{\mathbf{i} \le \mathbf{r}} u_{\mathbf{0i}} \Phi_{\mathbf{i}}(y).$$

Substituting the above equations into the wave equation (2.1) and make a Galerkin projection, we will arrive at N_y independent deterministic equations as following

$$\frac{\partial u_{\mathbf{i}}}{\partial t} = \kappa_{\mathbf{i}}(x) \frac{\partial u_{\mathbf{i}}}{\partial x}, \quad \mathbf{i} \le \mathbf{r}, \tag{3.8}$$

$$u_{\mathbf{i}}(x, 0) = u_{0\mathbf{i}}(x), \tag{3.9}$$

where

$$\kappa_{\mathbf{i}}(x) = \kappa_0(x) + \sum_{k=1}^{M} \kappa_k(x) C_{i_k}.$$

For boundary conditions, we claim that $\{C_{k,i}\}_{k=1}^{r_i}$ are all located in the support of the variable y_i. To see this, note

$$\bar{y}_i \le C_{k,i} = \int_{\Gamma_i} y_i \rho(y_i) \phi_{k,i}^2 dy_i \le \bar{\bar{y}}_i, \quad k = 1, \ldots, r_i,$$

where $\bar{y}_i$ and $\bar{\bar{y}}_i$ are the lower bound and upper bound of the variable y_i respectively. So it is natural for us to impose the boundary conditions by

$$u_{\mathbf{i}}(1,t) = u_{R\mathbf{i}}(t), \qquad \kappa_i(1) > 0,$$
$$u_{\mathbf{i}}(-1,t) = u_{L\mathbf{i}}(t), \qquad \kappa_i(-1) < 0,$$

where $\{u_{R\mathbf{i}}(t), u_{L\mathbf{i}}(t)\}_{\mathbf{i}\le\mathbf{r}}$ are the expansion coefficients of the initial condition $u_R(y,t)$ and $u_L(y,t)$ based on the basis $\{\Phi_{\mathbf{i}}\}_{\mathbf{i}<\mathbf{r}}$. The underlying problem now becomes

$$\begin{cases} \frac{\partial u_{\mathbf{i}}}{\partial t} = \kappa_{\mathbf{i}}(x)\frac{\partial u_{\mathbf{i}}}{\partial x}, & \mathbf{i} \le \mathbf{r}. \\ u_{\mathbf{i}}(1,t) = u_{R\mathbf{i}}(t), & \kappa_{\mathbf{i}}(1) > 0, \\ u_{\mathbf{i}}(-1,t) = u_{L\mathbf{i}}(t), & \kappa_{\mathbf{i}}(-1) < 0, \\ u_{\mathbf{i}}(x,0) = u_{0\mathbf{i}}(x). \end{cases} \tag{3.10}$$

The statistics of the solution can be found from the approximate solutions $u_{\mathbf{i}}(x)$. For example, the mean of $u(x,y)$ can be approximated by

$$\mathbb{E}(u_M) = \int_\Gamma \rho(y)u(x,y,t)dy = \sum_{\mathbf{i}\le\mathbf{r}} u_{\mathbf{i}}(x,t)\int_\Gamma \rho(y)\Phi_{\mathbf{i}}dy; \tag{3.11}$$

and similarly, the second-order moment of the solution can be calculated by

$$\int_\Gamma \rho(y)\Big(\sum_{\mathbf{i}\le\mathbf{r}} u_{\mathbf{i}}\Phi_{\mathbf{i}}\Big)^2 dy = \sum_{\mathbf{i}\le\mathbf{r}} (u_{\mathbf{i}})^2 \int_\Gamma \rho(y)(\Phi_{\mathbf{i}})^2 dy = \sum_{\mathbf{i}\le\mathbf{r}} \big(u_{\mathbf{i}}(x,t)\big)^2. \tag{3.12}$$

Remark 3.1 We remark that the methods for constructing the bi-orthogonal polynomials can be easily extended to problems of high dimension such as the following two-dimensional problem:

$$\frac{\partial u(x_1,x_2,t,y(\omega))}{\partial t} = \kappa_1(x_1,x_2,y(\omega))\frac{\partial u}{\partial x_1} + \kappa_2(x_1,x_2,y(\omega))\frac{\partial u}{\partial x_2}, \tag{3.13}$$

where κ_1, κ_2 are two independent random wave fields. In fact, we only need to construct the polynomials in each direction and then multiply them together. The extension to hyperbolic system is also straight forward.

Remark 3.2 It can be seen from (3.5) that the Galerkin methods with bi-orthogonal polynomials are equivalent to certain types of collocation methods. On the implementation side, there seems no difficulty to extend the numerical methods described in this section to the collocation approaches. However, one advantage of the Galerkin method is that its mathematical framework is more suitable for theoretical analysis.

3.2 Sensitivity Analysis: A Principle for Choosing the Space Dimension P_{r_j}

It is known from the above discussion that given index vector $\mathbf{r} = (r_1, r_2, \ldots, r_M)$ a total of $N_y = \prod_{i=1}^M (r_i + 1)$ basis exists for $P_{\mathbf{r}}$. If we use the same order (say, r) of polynomial spaces in each direction, we will arrive at $N_y = r^M$. This means N_y will become very large when r becomes large, which may become very expensive in practice. To this end, we claim that it is possible to use smaller r_j for larger j. To see this, we give some simple analysis for

the random wave equation in the following. Recall that the random wave field takes a form as truncated KL expansion, namely,

$$\kappa(x, y) = \kappa_0(x) + \sum_{i=1}^{N} \sqrt{\lambda_i}\kappa_i(x) y_i,$$

where $\{\lambda_i\}_{i=1}^{N}$ have a typical decay property as [7]

$$\lambda_i \sim \frac{1}{i^{2m}}, \quad m \geq 1.$$

Here the decay index m depends on the property of the given covariance function. We further assume $\{\kappa_i\}_{i=0}^{M}$ are constants for simplicity. If the problem is also compatible, then the exact solution for the proposed problem yields $u(x, y, t) = u_0(x + \kappa(y)t)$. It is easy to get

$$\left|\frac{\partial u(x, y, t)}{\partial y_i}\right| \sim \frac{1}{i^m}, \qquad \left|\frac{\partial^2 u(x, y, t)}{\partial y_i \partial y_j}\right| \sim \frac{1}{i^m j^m}. \tag{3.14}$$

Below, we give a detailed analysis for the problems whose random wave fields are associate with the space variable x. More precisely, we have the following lemma:

Lemma 3.1 *Consider the problem* (2.1)–(2.3) *with*

$$\kappa(x, y) = \kappa_0(x) + \sum_{i=1}^{N} \sqrt{\lambda_i}\kappa_i(x) y_i,$$

$$0 < \kappa_{\min} \leq \kappa(x, y) < \kappa_{\max}, \qquad |\partial_x \kappa(x, y)| < \bar{\kappa}_{\max}.$$

If the following conditions are satisfied:

$$\int_\Gamma \int_D \rho(y)\Big(\partial_x u_0(x; y)\Big)^2 dxdy < \infty, \tag{3.15a}$$

$$\int_0^T \int_\Gamma \rho(y)\Big(\partial_t u_R(t; y)\Big)^2 dydt < \infty, \tag{3.15b}$$

$$\int_0^T \int_\Gamma \rho(y)\Big(\partial_t u_L(t; y)\Big)^2 dydt < \infty, \tag{3.15c}$$

then

$$\int_\Gamma \int_D \rho(y) u_x^2 dxdy \leq C(T) < \infty, \quad 0 < t \leq T, \tag{3.16}$$

where $\rho(y) > 0$ *is the probability distribution function and* $C(T)$ *is a positive constant depending on* T.

By using the lemma above, we have the following theorem:

Theorem 3.1 *Consider the problem* (2.1)–(2.3). *Assume that the initial condition* u_0 *is deterministic and the boundary condition have such KL expansion expression as*:

$$u_R(y, t) = u_{R_0}(t) + \sum_{i=1}^{N} \sqrt{\mu_i} u_{R_i}(t) y_i, \qquad \int_0^T u_{R_i}^2(t) dt < \infty,$$

$$u_L(y,t) = u_{L_0}(t) + \sum_{i=1}^{N} \sqrt{\nu_i} u_{L_i}(t) y_i, \qquad \int_0^T u_{L_i}^2(t)dt < \infty.$$

If the assumption (3.15) *holds, then*

$$\int_\Gamma \int_D \rho(y) u_{y_i}^2 dxdy \le C(T)\left(\sqrt{\lambda_i} + \sqrt{\mu_i} + \sqrt{\nu_i}\right), \quad 0 < t \le T, \tag{3.17}$$

where $C(T)$ *is a finite number depending on* T.

The proof of the above Theorem is more or less the same as Theorems 2.1 and 2.2, and will be omitted here. Moreover, the high order derivatives of the solution can be bounded by using the same methods; e.g., we can obtain

$$\int_\Gamma \int_D \rho(y) \left(\frac{\partial^2 u}{\partial y_i \partial y_j}\right)^2 dxdy \lesssim C(T)\left(\lambda_i + \mu_i + \sqrt{\lambda_i \mu_i} + \sqrt{\lambda_i \nu_i}\right), \quad 0 < t \le T, \tag{3.18}$$

under some further assumptions on the initial-boundary conditions and the random wave field.

We know that the approximation error by orthogonal polynomials usually depends on the derivatives of the exact solution. Equation (3.18) and Theorem 3.1 suggest that the leading error terms come from the first derivative with respect to y_i when i is small. This tells us that we can use relatively large r_i when i, is small, and use relatively small r_i when i, is large. Consequently, the resulting system is of reasonably small size. Such "dimension-adaptivity" approach has been used by many researchers for different application problems, see, e.g., [4, 6]. The main strategy can be summarized as follows: assume that the Total Degree (TD) principle is used to deal with the high-dimension cases, that is, we chose the orders by

$$\sum_{i=1}^{M} r_i \le P,$$

where P is the "total order" which is given beforehand. The modified TD is to chose the order in an anisotropic way, i.e., to use

$$\sum_{i=1}^{M} (r_i + \alpha_i) \le P,$$

where α_i are some non-negative integers. In general, α_i are chosen relatively large when the ith direction is less important.

4 Numerical Example

It is pointed out in Sect. 3 that different orthogonal polynomials can be constructed according to different weights (i.e., different distributions). This implies that order reductions may be affected by difference choice of distributions. On the other hands, the results of (3.17)–(3.18) are independent of the distribution types, which means that for different distributions similar "anisotropic" approaches can be used and similar sensitivity results can be obtained. Therefore we will employ the uniform distribution in this numerical experiment section.

Example 4.1 To illustrate efficiency of the proposed method in random spaces, we consider the following simple example:

$$\begin{cases} \partial_t u(x,t,y) = \kappa(y)\partial_x u(x,t,y), \\ u(1,t,y) = \sin(1+\kappa(y)t), \quad \kappa(y) > 0, \\ u(-1,t,y) = \sin(-1+\kappa(y)t), \quad \kappa(y) < 0, \\ u(x,t=0,y) = \sin(x), \end{cases} \tag{4.1}$$

where

$$\kappa(y) = 0.5 * \sum_{n=1}^{4} \frac{1}{n^2} y_n, \tag{4.2}$$

and each y_n has a uniform distribution in $(-1, 1)$. Note in this example, κ depends only on the variable y, and this will enable us to check the convergence results in random spaces easily. In the computation, we need to check the sign of each $\kappa_{\mathbf{i}}$ and impose the numerical boundary conditions correspondingly. The errors between the numerical solutions u_{num} and the exact solutions u_{exa} are defined as follows, the error in mean:

$$e_{\text{mean}} = \max_x |\mathbb{E}(u_{num}) - \mathbb{E}(u_{exa})|,$$

the error in standard deviation (STD):

$$e_{\text{std}} = \max_x |\sigma_{u_{num}} - \sigma_{u_{exa}}|,$$

and the mean-square error:

$$e_2 = \max_x \left(\mathbb{E}(u_{num} - u_{exa})^2\right)^{1/2}.$$

In Fig. 1, we plot the exact and numerical solutions at different times, where we can see that the numerical solutions match well with the exact solutions when very lower order $(r_1, r_2, r_3, r_4) = (3, 2, 1, 1)$ of approximate spaces are used. In Fig. 2 (Top), we plot the uniform convergence results, where "uniform" means that the same order is used in every direction. We can see that the methods converge fast. In Fig. 2 (bottom), we plot the convergence results versus time t. The influence of time to the convergence rate is clear, and it is typically a linear dependence. The same phenomenon was observed in [8] for gPC methods and [13] for collocation methods. This "long-time integral" phenomenon is notorious, see, e.g., Wan and Karniadakis [14].

In Table 1, we give some convergence results for nonuniform approximation for $t = 2$. It is observed from the table that enhancing the order for lower index i gives much better convergence than doing the same for the larger i. For example, by comparing the results for $(r_1, r_2, r_3, r_4) = (1, 3, 3, 2), (2, 3, 3, 2), (3, 3, 3, 2)$ we can see an order of 2 error reduction each time; but there is no significant change in errors by using $(r_1, r_2, r_3, r_4) = (2, 3, 3, 1), (2, 3, 3, 2), (2, 3, 3, 3$.

To see the nonuniform convergence, we consider the case that

$$\kappa(y) = 1 + 0.5 * \left(\sum_{n=1}^{3} \frac{1}{n^2} y_n + 10^{-8} y_4\right), \tag{4.3}$$

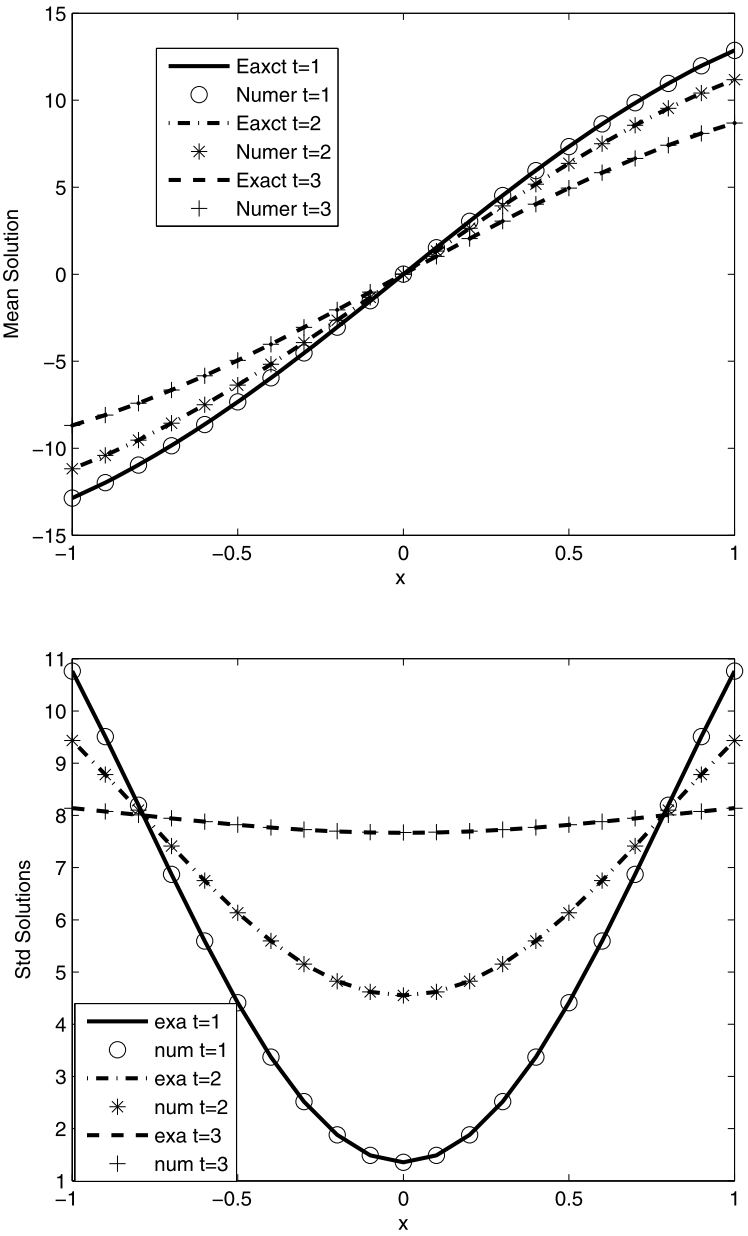

Fig. 1 Example 4.1: Mean solutions of different times (*top*), and Std solutions of different times (*bottom*). Orders of (3, 2, 1, 1) are used

namely, λ_4 is much more smaller than other eigenvalues. In Fig. 3, we plot the nonuniform convergence result for the mean error. We first fix the orders of the last three direction as $(r_1, r_2, r_3, r_4) = (\bullet, 4, 4, 2)$, and let the order of the first direction change from 1 to 7. In this case, the error goes to zero exponentially (solid line). We then fix the first three orders

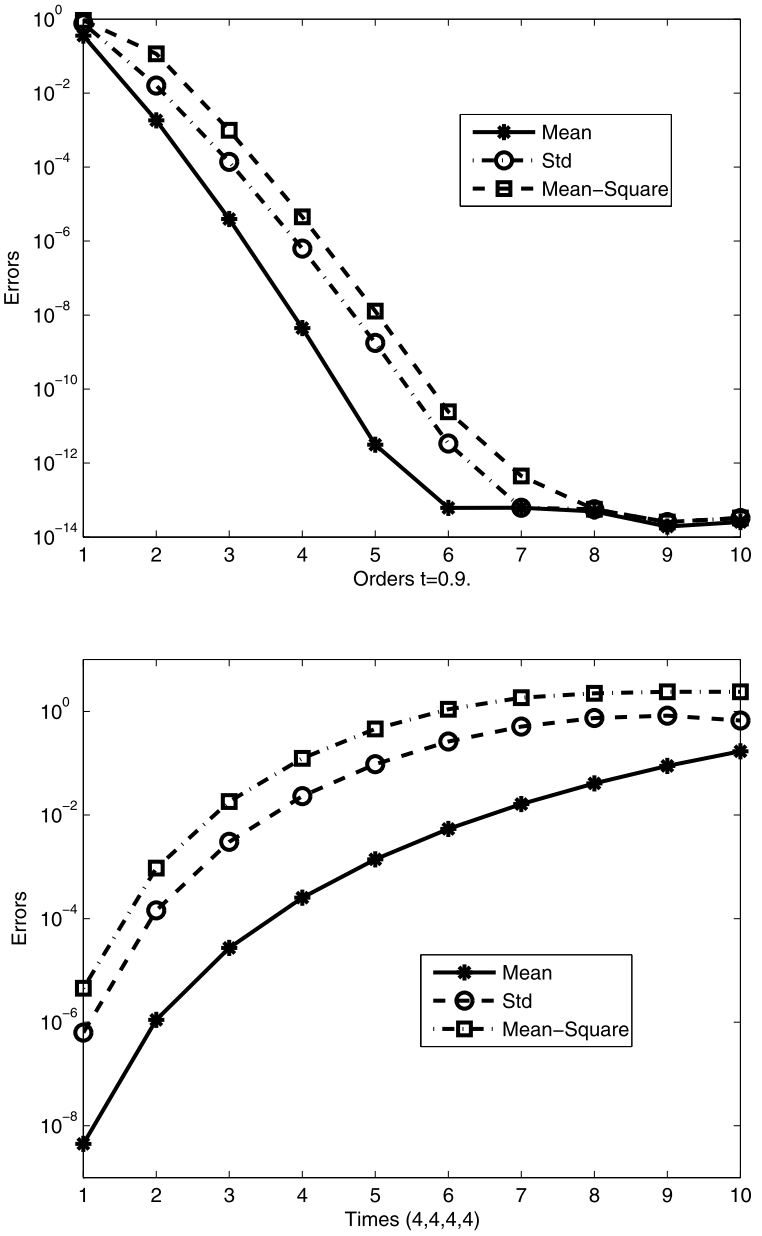

Fig. 2 Example 4.1: Uniform convergence results for $t = 0.9$ (*top*), and Convergence results for different times (*bottom*), with a fixed approximate order of (4, 4, 4, 4)

as $(r_1, r_2, r_3, r_4) = (2, 4, 4, \bullet)$, and let the last order change from 1 to 7. In this case, the errors with differ N do note have much difference (dashed line). We also plot the errors for different order in the second direction (dashdot line), where we fix the other three orders as $(r_1, r_2, r_3, r_4) = (5, \bullet, 4, 2)$. This is a good support for the nonuniform convergence for

Table 1 Example 4.1: Nonuniform convergence results for different directions ($t = 2$)

Order	e_{mean}	e_{std}	e_2	Order	e_{mean}	e_{std}	e_2
(1,3,3,1)	0.5544	1.2568	2.5614	(2,3,3,2)	3.1792e-3	2.8139e-2	2.7634e-1
(1,3,3,2)	0.5522	1.2517	2.5446	(2,3,3,3)	3.1260e-3	2.8109e-2	2.8549e-1
(2,3,3,1)	6.8164e-3	3.3813e-2	3.3347e-1	(3,3,3,2)	5.5919e-5	2.4277e-4	8.0764e-3

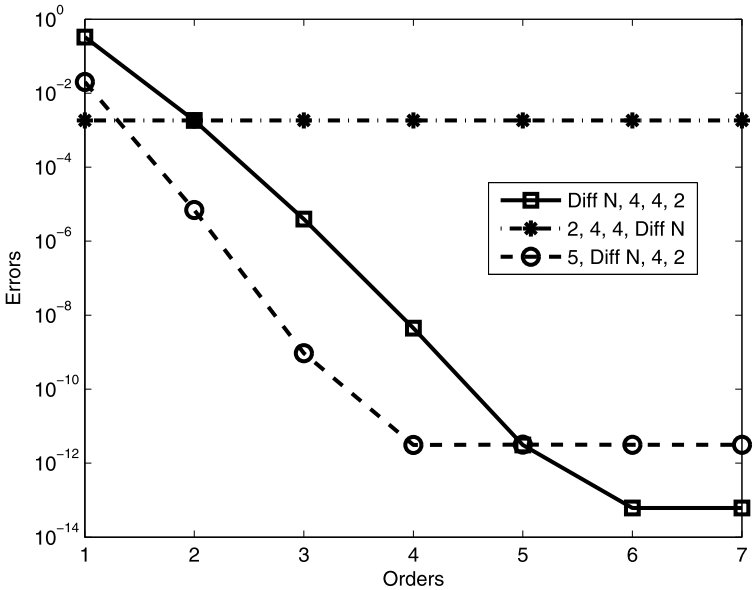

Fig. 3 Example 4.1: Nonuniform convergence results for $t = 1$. The *solid line* is for $(\bullet, 4, 4, 2)$, the first order changes from 1 to 7. The *dashdot line* is for $(2, 4, 4, \bullet)$, the fourth order changes from 1 to 7. The *dashed line* is for $(5, \bullet, 4, 2)$, the second order changes from 1 to 7

different directions. Note we use the framework of the Galerkin methods which for smooth solutions, usually 3–5 order of expansion is enough to get a satisfactory convergence results. Using the principle to choose relatively small order for large i, we then end up with only few uncoupled equations.

Example 4.2 Our second example is a simple hyperbolic system:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & \mu \\ \xi & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x, \qquad \begin{pmatrix} u \\ u \end{pmatrix}_{t=0} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

where $\mu = \mu(\bar{y}) > 0$ and $\xi = \xi(\tilde{y}) > 0$, and $\bar{y}, \tilde{y}$ are random vectors. This simple system shares many important properties with some other equations, such us the second order wave equation and the Maxwell equation. In the computation, we assume

$$\mu(\bar{y}) = 1 + 0.5 * \sum_{i=1}^{3} \frac{1}{i^2} \bar{y}_i, \qquad \xi(\tilde{y}) = 1 + 0.5 * \sum_{j=1}^{3} \frac{1}{j^2} \tilde{y}_j,$$

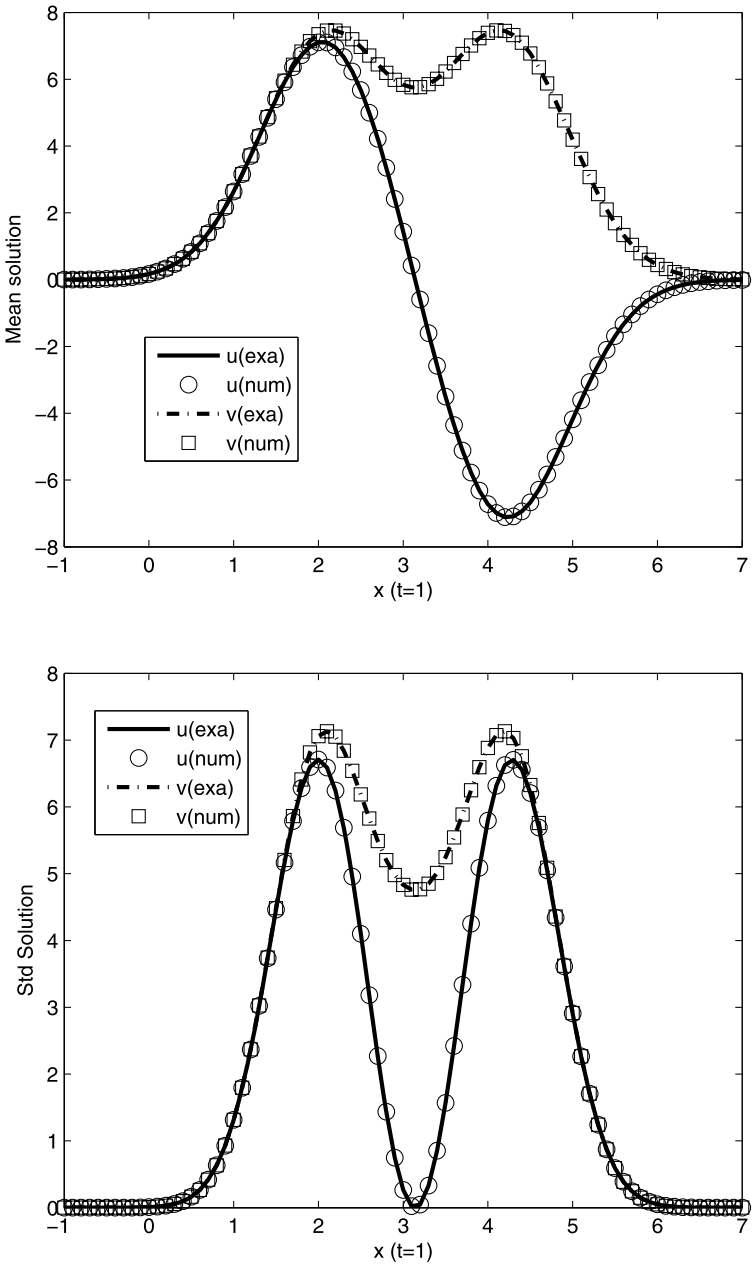

Fig. 4 Example 4.2: Mean solutions for $t = 1$ (*top*), and Std solutions at $t = 1$ (*bottom*). Each random vector has an approximate order of (3, 2, 2)

where $\{\bar{y}_i\}_{i=1}^3$, and $\{\tilde{y}_j\}_{j=1}^3$ are random variables having uniform distribution in $(-1, 1)$, namely, there are totally six dimension of random spaces. The initial condition is $u_0 = 0$,

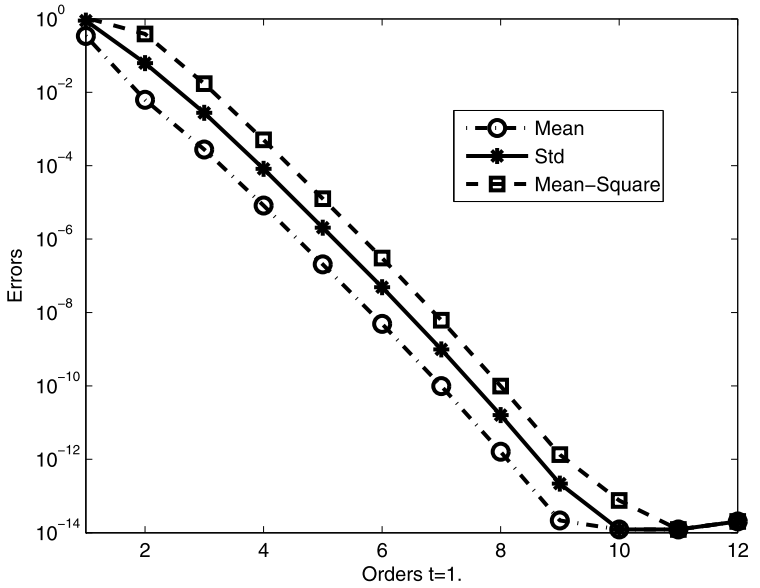

Fig. 5 Example 4.2: Uniform convergence results at $t = 0.8$

$v_0 = 2f(x)$, where $f(x) = \exp(-(x-\pi)^2)$. The exact solution is

$$u_{exa} = \sqrt{\mu}\big(f(x+ct) - f(x-ct)\big), \qquad v_{exa} = \sqrt{\xi}\big(f(x+ct) + f(x-ct)\big),$$

where $c = \sqrt{\mu\xi}$. We assume again the random variables have uniform distribution in $(-1, 1)$. In Fig. 4, we plot the exact and numerical solutions (both mean and std) at $t = 1$, where very low orders $(r_1, \ldots, r_6) = (3, 2, 2; 3, 2, 2)$ are used. It is again observed that the numerical solutions match well with the exact solutions. In Fig. 5, we also give the uniform convergence results for the underlying hyperbolic system problem.

5 Conclusion

In this work, we first give some regularity results to the scalar transport equations whose transport velocity is a random field. A Galerkin method using bi-orthogonal polynomials is proposed for the first order wave equation with a random wave field, and hyperbolic systems with a random coefficient matrix. This decouples the equations in the random space, yielding just a number of uncoupled systems. The double orthogonal polynomials are able to perform the decoupling whenever the random variables in the KL expansion are independent. Otherwise, an iterative procedure should be applied, and at each step we may use the product of the marginal densities $\prod_{i=1}^{N} \rho_i$, as an approximation of the density ρ, to accelerate the convergence [1]. The method proposed here uses a Galerkin framework but results in a uncoupled system like those given by the collocation methods and other simple based methods.

In Sect. 3.1, we give a principle of how to choose the orders of the approximate polynomial spaces, with which the total freedom will be significant reduced. The method in some sense overcomes "*curse of dimensionality*." Numerical examples are given in Sect. 4, which

give a good support to the analysis in Sect. 3.1. The principle can also be used for other problems, e.g., stochastic elliptic problem [1].

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