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Efficient MFS Algorithms for Inhomogeneous Polyharmonic Problems

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Abstract In this work we develop an efficient algorithm for the application of the method of fundamental solutions to inhomogeneous polyharmonic problems, that is problems governed by equations of the form $\Delta^{\ell} u = f$, $\ell \in \mathbb{N}$, in circular geometries. Following the ideas of Alves and Chen (Adv. Comput. Math. 23:125–142, 2005), the right hand side of the equation in question is approximated by a linear combination of fundamental solutions of the Helmholtz equation. A particular solution of the inhomogeneous equation is then easily obtained from this approximation and the resulting homogeneous problem in the method of particular solutions is subsequently solved using the method of fundamental solutions. The fact that both the problem of approximating the right hand side and the homogeneous boundary value problem are performed in a circular geometry, makes it possible to develop efficient matrix decomposition algorithms with fast Fourier transforms for their solution. The efficacy of the method is demonstrated on several test problems.

Keywords Method of fundamental solutions · Polyharmonic equations · Circulant matrices · Fast Fourier transforms

1 Introduction

In this work we apply the ideas of Alves and Chen [2] in order to solve inhomogeneous elliptic problems using the method of fundamental solutions (MFS) [11, 13, 14, 26]. In [2], the authors use the method of particular solutions (MPS) to solve inhomogeneous elliptic boundary value problems by first approximating the right hand side by a linear combination of fundamental solutions of the Helmholtz equation. This is in contrast to the conventional approach in which the right hand side is approximated by radial basis functions (RBFs), see e.g. [7, 12, 17], linear combinations of Chebyshev polynomials, see e.g. [15, 33, 35], or monomials, see e.g. [8, 15, 34]. Once the approximation in terms of fundamental solutions

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of the Helmholtz equation has been obtained, one may easily construct a particular solution of the inhomogeneous equation. In the MPS (see e.g. [15]), this particular solution is then subtracted from the solution of the inhomogeneous problem yielding a homogeneous problem which may be solved using the MFS. The technique proposed in [2] has been successfully applied for the solution of several inverse problems [4, 5, 29, 37]. In this study we apply this technique to problems in circular domains. This leads to systems in which the coefficient matrices possess circulant [10] or block circulant structures and can thus be solved using matrix decomposition algorithms (MDAs) [6] and fast Fourier transforms (FFTs). Such algorithms have been used extensively with the method of fundamental solutions for the solution of homogeneous [20, 23] and inhomogeneous problems (using RBFs) [24, 25].

The current work is closely related to a recent study on the efficient MDA-FFT application of the so-called MFS-K method to inhomogeneous problems in circular geometries [22]. In the MFS-K method which was introduced in [3], the so-called Kansa method [19] is applied to inhomogeneous elliptic boundary value problems using linear combinations of fundamental solutions of the Helmholtz equation. In contrast to the current approach, this linear combination is collocated simultaneously in the interior of the domain to satisfy the differential equation and on the boundary to satisfy the boundary conditions. The MFS-K has also been used for the solution of inhomogeneous second-order equations with variable coefficients [38] and to heat conduction problems [39]. Note that the method proposed in [2] has also been used in the context of the boundary knot method in [18].

The paper is organized as follows. In Sect. 2 we briefly describe the method of particular solutions for the solution of the Poisson equation. In Sect. 3 we describe a MDA for the approximation of a function in a circular domain. In Sect. 4 we present MDAs for the solution of the homogeneous boundary value problems resulting from the application of the method of particular solutions to various inhomogeneous polyharmonic boundary value problems in circular domains. The general algorithmic strategy to be followed is given in Sect. 5 while several numerical examples are presented in Sect. 6. Finally, some conclusions and ideas about future work are given in Sect. 7.

2 The Method of Particular Solutions

In order to describe the MPS, we consider, for example, the solution of the inhomogeneous boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2\colon\ |\boldsymbol{x}| < \varrho\}$, with boundary $\partial\Omega$ and f and g are given functions.

Let u^p be a particular solution of the Poisson equation satisfying

$$\Delta u^p = f \quad \text{in } \Omega. \tag{2.2}$$

Now if we let the solution of problem (2.1) be

$$u = u^p + u^h, \tag{2.3}$$

clearly u^h satisfies the homogeneous boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $h = g - u^p$.

The idea is therefore to first construct an approximation to the function f in (2.1) and then, from it, construct an approximate particular solution u^p of the Poisson equation in Ω. We then, using the approximate particular solution to generate the boundary condition, solve the homogeneous boundary problem (2.4) to obtain the homogeneous solution u^h. Finally, the approximation u of the solution of problem (2.1) is obtained from (2.3).

3 Approximation of Functions

From the description of the MPS in Sect. 2, our first objective is to approximate a given function f in the domain Ω. This is achieved using a MDA similar to, but simpler than, the one proposed in [22], in which both the differential equation and the boundary conditions are collocated simultaneously to yield the solution of elliptic boundary value problems. In the current approach the algorithm is simpler in the sense that only the right-hand side function f is collocated.

3.1 The Method

The function f is approximated by

$$f_{MN}(A, \boldsymbol{Q}; P) = \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}\Phi_{k_m}(P, Q_n), \qquad P \in \overline{\Omega} = \Omega \cup \partial\Omega, \tag{3.1}$$

where $A = (a_{mn})_{m,n=1}^{M,N}$ is the matrix of unknown coefficients, $\boldsymbol{Q}$ is a N-vector containing the coordinates of the singularities Q_n, $n = 1, \dots, N$, which lie outside $\overline{\Omega}$, and $\Phi_k(P, Q)$ is a fundamental solution of the Helmholtz operator $-(\Delta + k^2)$ given by

$$\Phi_k(P, Q) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|P - Q|), \tag{3.2}$$

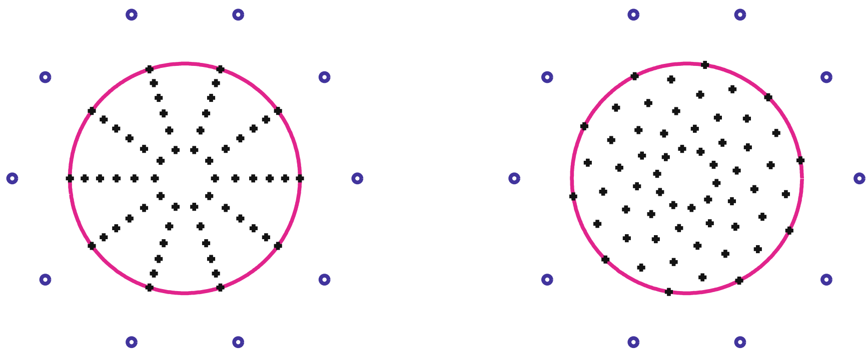
with $|P - Q|$ denoting the distance between the points P and Q. In (3.2), $H_0^{(1)}$ denotes the Hänkel function of the first kind and order zero. The test frequencies k_m, $m = 1, \dots, M$ are chosen such that $0 < k_1 < k_2 < \cdots < k_M$. The singularities $Q_\ell = (x_{Q_\ell}, y_{Q_\ell})$ are fixed, as usual, on a curve similar to $\partial\Omega$ [16], namely a circle Γ concentric to Ω and defined by $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^2\colon\ |\boldsymbol{x}| = R\}$, where $R > \varrho$. The collocation points $\{P_{ij}\}_{i,j=1}^{M,N} = \{(x_{P_{ij}}, y_{P_{ij}})\}_{i,j=1}^{M,N}$ are placed in $\overline{\Omega}$ in the following way.

$$\begin{aligned} &x_{P_{ij}} = \varrho_i \cos\frac{2(j-1+\alpha_i)\pi}{N}, \qquad y_{P_{ij}} = \varrho_i \sin\frac{2(j-1+\alpha_i)\pi}{N}, \\ &\quad i = 1, \dots, M;\ j = 1, \dots, N, \end{aligned} \tag{3.3}$$

where $0 < \varrho_1 < \varrho_2 < \cdots < \varrho_M = \varrho$. The singularities are distributed on the circle Γ as follows.

$$x_{Q_\ell} = R\cos\frac{2(\ell-1)\pi}{N}, \qquad y_{Q_\ell} = R\sin\frac{2(\ell-1)\pi}{N}, \quad \ell = 1, \dots, N. \tag{3.4}$$

In (3.3) the parameters $-1/2 \le \alpha_i \le 1/2$, $i = 1, \dots, M$, represent rotations of $\frac{2\pi\alpha_i}{N}$ of the collocation points with respect to the singularities. These rotations enable us to produce a



(a) $\alpha_i = 0,\ i = 1,\dots,M$ (b) $\alpha_i = (-1)^i/4,\ i = 1,\dots,M$

Fig. 1 Typical distribution of collocation (+) and singularities (o), for disk with $\varrho_i = (i/M)^{3/4}$, $i = 1,\dots,M$

more uniform distribution of the collocation points [20, 24]. Typical distributions of collocation points and singularities, as well as the effect of rotating the collocation points may be observed in Fig. 1.

The unknown coefficients $(a_{mn})_{m,n=1}^{M,N}$ are determined by collocating the approximation f_{MN} at the points $\{P_{ij}\}_{i,j=1}^{M,N}$. More precisely, for the set of points on the circle with radius ϱ_i for each $i = 1,\dots,M$, we have

$$f_{MN}(A, \boldsymbol{Q}; P_{ij}) = f(P_{ij}), \quad j = 1,\dots,N. \tag{3.5}$$

Substitution of expression (3.1) into (3.5) yields

$$\sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}\Phi_{k_m}(P_{ij}, Q_n) = f(P_{ij}), \quad j = 1,\dots,N, \tag{3.6}$$

or

$$\sum_{n=1}^{N} a_{1n}\Phi_{k_1}(P_{ij}, Q_n) + \sum_{n=1}^{N} a_{2n}\Phi_{k_2}(P_{ij}, Q_n) + \cdots + \sum_{n=1}^{N} a_{Mn}\Phi_{k_M}(P_{ij}, Q_n) = f(P_{ij}),$$
$$j = 1,\dots,N. \tag{3.7}$$

Equations (3.7) can be written as

$$\sum_{m=1}^{M} F_{i,m}\boldsymbol{a}_m = \boldsymbol{f}_i, \quad m = 1,\dots,M, \tag{3.8}$$

where

$$F_{i,m} = \left(\Phi_{k_m}(P_{ij}, Q_n)\right)_{j,n=1}^{N}, \qquad \boldsymbol{a}_m = [a_{m1}, a_{m2},\dots,a_{mN}]^T, \quad m = 1,\dots,M, \tag{3.9}$$

and

$$\boldsymbol{f}_i = \left[f(P_{i1}), f(P_{i2}),\dots,f(P_{iN})\right]^T. \tag{3.10}$$

Equivalently, (3.8) can be written as the $MN \times MN$ system

$$F\boldsymbol{a} = \begin{pmatrix} F_{1,1} & F_{1,2} & \dots & F_{1,M} \\ F_{2,1} & F_{2,2} & \dots & F_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ F_{M,1} & F_{M,2} & \dots & F_{M,M} \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_M \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \vdots \\ \boldsymbol{f}_M \end{pmatrix} = \boldsymbol{f}. \tag{3.11}$$

It can be easily observed that each of the $N \times N$ submatrices $F_{i,m}$, $i, m = 1, \dots, M$ in the coefficient matrix in (3.11) is circulant [10].

3.2 The Algorithm

3.2.1 Preliminaries

A basic tool in the algorithm for the efficient solution of system (3.11) is the unitary $N \times N$ Fourier matrix

$$U_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \bar{\omega} & \bar{\omega}^2 & \cdots & \bar{\omega}^{N-1} \\ 1 & \bar{\omega}^2 & \bar{\omega}^4 & \cdots & \bar{\omega}^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \bar{\omega}^{N-1} & \bar{\omega}^{2(N-1)} & \cdots & \bar{\omega}^{(N-1)(N-1)} \end{pmatrix}, \quad \text{where } \omega = \mathrm{e}^{2\pi \mathrm{i}/N}. \tag{3.12}$$

In the sequel, we shall also use extensively some properties of circulant matrices [10]. In particular, if the $N \times N$ matrix $C = (c_{ij})_{i,j=1}^N$ is circulant, then it can be fully described by the elements of its first row and we write

$$C = \operatorname{circ}(c_{11}, c_{12}, \dots, c_{1N}).$$

Further, we have that

$$U_N C U_N^* = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_N), \tag{3.13}$$

where

$$\lambda_j = \sum_{k=1}^{N} c_{1k} \omega^{(k-1)(j-1)}, \quad j = 1, \dots, N. \tag{3.14}$$

3.2.2 Matrix Decomposition Algorithm

If I_M is the $M \times M$ identity matrix, pre-multiplication of (3.11) by $I_M \otimes U_N$ yields

$$(I_M \otimes U_N) F \big(I_M \otimes U_N^*\big)(I_M \otimes U_N)\boldsymbol{a} = (I_M \otimes U_N)\boldsymbol{f} \quad \text{or} \quad \tilde{F}\tilde{\boldsymbol{a}} = \tilde{\boldsymbol{f}}, \tag{3.15}$$

where

$$\begin{aligned}
\tilde{F} &= (I_M \otimes U_N) F \big(I_M \otimes U_N^*\big) \\
&= \begin{pmatrix} U_N F_{1,1} U_N^* & U_N F_{1,2} U_N^* & \cdots & U_N F_{1,M} U_N^* \\ U_N F_{2,1} U_N^* & U_N F_{2,2} U_N^* & \cdots & U_N F_{2,M} U_N^* \\ \vdots & \vdots & & \vdots \\ U_N F_{M,1} U_N^* & U_N F_{M,2} U_N^* & \cdots & U_N F_{M,M} U_N^* \end{pmatrix} \\
&= \begin{pmatrix} D_{1,1} & D_{1,2} & \cdots & D_{1,M} \\ D_{2,1} & D_{2,2} & \cdots & D_{2,M} \\ \vdots & \vdots & & \vdots \\ D_{M,1} & D_{M,2} & \cdots & D_{M,M} \end{pmatrix}
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
\tilde{\boldsymbol{a}} &= (I_M \otimes U_N)\boldsymbol{a} = \begin{pmatrix} U_N \boldsymbol{a}_1 \\ U_N \boldsymbol{a}_2 \\ \vdots \\ U_N \boldsymbol{a}_M \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{a}}_1 \\ \tilde{\boldsymbol{a}}_2 \\ \vdots \\ \tilde{\boldsymbol{a}}_M \end{pmatrix}, \\
\tilde{\boldsymbol{f}} &= (I_M \otimes U_N)\boldsymbol{f} = \begin{pmatrix} U_N \boldsymbol{f}_1 \\ U_N \boldsymbol{f}_2 \\ \vdots \\ U_N \boldsymbol{f}_M \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{f}}_1 \\ \tilde{\boldsymbol{f}}_2 \\ \vdots \\ \tilde{\boldsymbol{f}}_M \end{pmatrix}.
\end{aligned} \tag{3.17}$$

From property (3.13), in (3.16), each of the $N \times N$ matrices $D_{i,m}$, $i, m = 1, \ldots, M$ is diagonal. If, in particular

$$D_{i,m} = \operatorname{diag}\big(D_{i,m}^1, D_{i,m}^2, \ldots, D_{i,m}^N\big) \quad \text{and} \quad F_{i,m} = \operatorname{circ}\big(F_{i,m}^1, F_{i,m}^2, \ldots, F_{i,m}^N\big),$$

we have, from property (3.14), for $i, m = 1, \ldots, M$,

$$D_{i,m}^j = \sum_{k=1}^{N} F_{i,m}^k \omega^{(k-1)(j-1)}, \quad j = 1, \ldots, N. \tag{3.18}$$

Since the matrix $\tilde{F}$ consists of M^2 blocks of order N each of which is diagonal, the solution of system (3.15) can be decomposed into solving the N systems of order M

$$E_j \boldsymbol{x}_j = \boldsymbol{y}_j, \quad j = 1, \ldots, N, \tag{3.19}$$

where

$$(E_j)_{i,m} = D_{i,m}^j \quad i, m = 1, \ldots, M$$

and

$$(\boldsymbol{x}_j)_i = (\tilde{\boldsymbol{a}}_i)_j, \qquad (\boldsymbol{y}_j)_i = (\tilde{\boldsymbol{f}}_i)_j, \quad i = 1, \ldots, M. \tag{3.20}$$

Having obtained the vectors $\boldsymbol{x}_j$, $j = 1, \ldots, N$, we can recover the vectors $\tilde{\boldsymbol{a}}_m$, $m = 1, \ldots, M$ and, subsequently, the vector $\boldsymbol{a}$ from (3.17).

In conclusion, the MDA can be summarized as follows:

Algorithm 1

Step 1: `Compute` $\tilde{\boldsymbol{f}}_m = U_N \boldsymbol{f}_m$, $m = 1, \ldots, M$.
Step 2: `Construct the diagonal matrices` $D_{i,m}$ `from` (3.18).
Step 3: `Solve the` N, $M \times M$ `systems` (3.19) `to obtain the` $\{\boldsymbol{x}_j\}_{j=1}^N$, `and subsequently the` $\{\tilde{\boldsymbol{a}}_m\}_{m=1}^M$ `from` (3.20).
Step 4: `Recover the vector of coefficients` $\boldsymbol{a}$ `from` $\boldsymbol{a}_m = U_N^* \tilde{\boldsymbol{a}}_m$, $m = 1, \ldots, M$.

As well documented, in Steps 1, 2 and 4 FFTs are used while the most expensive part of the algorithm is the solution of N linear systems, each of order M. The FFTs are carried out using the `Matlab` [30] commands `fft` and `ifft` while the Hänkel function $H_0^{(1)}(z)$ is calculated using the `Matlab` function `besselh(0,1,`z`)`. The cost of Steps 1 and 4 is thus $\mathcal{O}(MN \log N)$, the cost of Step 2 is $\mathcal{O}(M^2 N \log N)$ and of the cost of Step is 3 $\mathcal{O}(M^3 N)$.

4 Polyharmonic Boundary Value Problems

In this section we consider the method of particular solutions for the solution of certain inhomogeneous polyharmonic boundary value problems. In particular, we consider the polyharmonic equation (see [27])

$$\Delta^\ell u = f \quad \text{in } \Omega, \tag{4.1}$$

where $\ell \in \mathbb{N}$.

In the case ℓ is odd, that is $\ell = 2p - 1$ for some $p \in \mathbb{N}$, (4.1) is associated with the boundary conditions

$$u, \frac{\partial u}{\partial n}, \Delta u, \frac{\partial \Delta u}{\partial n}, \ldots, \frac{\partial \Delta^{p-2} u}{\partial n} \text{ and } \Delta^{p-1} u \quad \text{specified on } \partial\Omega. \tag{4.2}$$

In the case ℓ is even, that is $\ell = 2p$ for some $p \in \mathbb{N}$, (4.1) is associated with the boundary conditions

$$u, \frac{\partial u}{\partial n}, \Delta u, \frac{\partial \Delta u}{\partial n}, \ldots, \Delta^{p-1} u \text{ and } \frac{\partial \Delta^{p-1} u}{\partial n}, \quad \text{specified on } \partial\Omega. \tag{4.3}$$

In (4.2)–(4.3), $\frac{\partial}{\partial n}$ denotes differentiation in the outward normal direction $\boldsymbol{n} = (n_x, n_y)$.

Note that in the cases examined in the sequel we shall use the fact that for the functions Φ_{k_m} defined in (3.2),

$$\Delta^\ell \Phi_{k_m} = (-1)^\ell k_m^{2\ell} \Phi_{k_m} \quad \text{in } \Omega,\ \ell \in \mathbb{N}. \tag{4.4}$$

4.1 Poisson Problems

We first consider the solution of the inhomogeneous boundary value (2.1). First, we seek a particular solution of the Poisson equation u_{MN}^p satisfying

$$\Delta u_{MN}^p = f_{MN} \quad \text{in } \Omega. \tag{4.5}$$

From (4.4) each function Φ_{k_m} satisfies the equation

$$\big(\Delta + k_m^2\big)\Phi_{k_m} = 0 \quad \text{in } \Omega, \tag{4.6}$$

we may readily obtain a particular solution of the Poisson equation from the expression

$$u_{MN}^p(A, \boldsymbol{Q}; P) = -\sum_{m=1}^{M}\sum_{n=1}^{N} \frac{a_{mn}}{k_m^2}\Phi_{k_m}(P, Q_n), \quad P \in \overline{\Omega} = \Omega \cup \partial\Omega. \tag{4.7}$$

In accordance to (2.3), we let the solution of problem (2.1) be

$$u_{MN} = u_{MN}^p + u_{MN}^h, \tag{4.8}$$

where clearly u_{MN}^h satisfies the homogeneous boundary value problem (2.4) with $h = g - u_{MN}^p$.

This homogeneous problem can be solved efficiently using the MFS as follows (see e.g. [31]). We approximate the solution of problem (2.4) by

$$u_{MN}^h(\boldsymbol{c}, \boldsymbol{Q}; P) = \sum_{j=1}^{N} c_j K_1(P, Q_j), \quad P \in \overline{\Omega}, \tag{4.9}$$

where $K_1(P, Q) = -\frac{1}{2\pi}\log|P - Q|$ is a fundamental solution of the Laplace equation.

A set of collocation points $\{P_i\}_{i=1}^N$ is placed on $\partial\Omega$ as follows. If $P_i = (x_{P_i}, y_{P_i})$, then we take for $i = 1, \ldots, N$

$$x_{P_i} = \varrho \cos\frac{2(i-1)\pi}{N}, \qquad y_{P_i} = \varrho \sin\frac{2(i-1)\pi}{N}. \tag{4.10}$$

We choose the sources $\{Q_j\}_{j=1}^N$ as in (3.4).

The coefficients $\boldsymbol{c}$ are determined so that the boundary condition is satisfied at the boundary points

$$u_{MN}^h(\boldsymbol{c}, \boldsymbol{Q}; P_i) = h(P_i), \quad i = 1, \ldots, N. \tag{4.11}$$

This yields a $N \times N$ linear system of the form

$$B\boldsymbol{c} = \boldsymbol{h} \tag{4.12}$$

where $\boldsymbol{h} = (h(P_1), h(P_2), \ldots, h(P_N))^T$ and the elements of matrix B are given by

$$B_{i,j} = -\frac{1}{2\pi}\log|P_i - Q_j|.$$

The matrix B is clearly circulant with $B = \text{circ}\{b_1, b_2, \ldots, b_N\}$ where $b_j = B_{1,j}$, $j = 1, \ldots, N$.

4.1.1 Matrix Decomposition Algorithm

Upon premultiplication by the matrix U_N defined in (3.12), using property (3.13), system (4.12) can now be written as

$$U_N B U_N^* U_N \boldsymbol{c} = U_N \boldsymbol{h} \tag{4.13}$$

or

$$D\hat{\boldsymbol{c}} = \hat{\boldsymbol{h}},$$

where

$$\hat{\boldsymbol{c}} = U_N \boldsymbol{c} \quad \text{and} \quad \hat{\boldsymbol{h}} = U_N \boldsymbol{h}.$$

Moreover, from property (3.14)

$$D = \mathrm{diag}(d_1, d_2, \ldots, d_N), \tag{4.14}$$

where

$$d_j = \sum_{k=1}^{N} b_k \omega^{(k-1)(j-1)}, \quad j = 1, \ldots, N. \tag{4.15}$$

If $\hat{\boldsymbol{h}} = (\hat{h}_1, \ldots, \hat{h}_N)^T$, the solution is thus clearly,

$$\hat{c}_i = \frac{\hat{h}_i}{d_i}, \quad i = 1, \ldots, N. \tag{4.16}$$

Having obtained $\hat{\boldsymbol{c}}$, we can find $\boldsymbol{c}$ from

$$\boldsymbol{c} = U_N^* \hat{\boldsymbol{c}}.$$

The algorithm for calculating u_{MN}^h can thus be summarized as follows.

Algorithm 2

Step 1: `Compute` $\hat{\boldsymbol{h}} = U_N \boldsymbol{h}$.
Step 2: `Construct the diagonal matrix` D `from` (4.15).
Step 3: `Evaluate` $\hat{\boldsymbol{c}}$ `from` (4.16).
Step 4: `Compute` $\boldsymbol{c} = U_N^* \hat{\boldsymbol{c}}$.

In Steps 1 and 4, the operations can be carried out via FFTs at a cost of order $\mathcal{O}(N \log N)$ operations. FFTs can also be used for the evaluation of the diagonal entries in Step 2 at a cost of $\mathcal{O}(N \log N)$. Clearly, the cost of Step 3 is $\mathcal{O}(N)$.

4.2 Biharmonic Problems

We next consider the solution of the inhomogeneous biharmonic boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = g_1 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial n} = g_2 & \text{on } \partial\Omega, \end{cases} \tag{4.17}$$

where f, g_1 and g_2 are given functions.

As in Sect. 4.1, we seek a particular solution of the biharmonic equation u_{MN}^p satisfying

$$\Delta^2 u_{MN}^p = f_{MN} \quad \text{in } \Omega. \tag{4.18}$$

From (4.4), Φ_{k_m} satisfies the equation

$$\left(\Delta^2 - k_m^4\right)\Phi_{k_m} = 0 \quad \text{in } \Omega. \tag{4.19}$$

Therefore, we may readily obtain a particular solution of the biharmonic equation from the expression

$$u_{MN}^p(A, \boldsymbol{Q}; P) = \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{a_{mn}}{k_m^4}\Phi_{k_m}(P, Q_n), \quad P \in \overline{\Omega} = \Omega \cup \partial\Omega. \tag{4.20}$$

We let the solution of problem (4.17) be

$$u_{MN} = u_{MN}^p + u_{MN}^h, \tag{4.21}$$

where clearly u_{MN}^h satisfies the homogeneous biharmonic boundary value problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = h_1 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial n} = h_2 & \text{on } \partial\Omega, \end{cases} \tag{4.22}$$

where $h_1 = g_1 - u_{MN}^p$ and $h_2 = g_2 - \frac{\partial u_{MN}^p}{\partial n}$. Note that in the calculation of h_2 we shall use the fact that

$$\frac{\partial H_0^{(1)}}{\partial n}(k|P-Q|) = -kH_1^{(1)}(k|P-Q|)\left(\frac{x_P - x_Q}{|P-Q|}n_x + \frac{y_P - y_Q}{|P-Q|}n_y\right). \tag{4.23}$$

This homogeneous problem can be solved efficiently using the MFS as follows (see e.g. [32]). We approximate the solution of problem (4.22) by

$$u_{MN}^h(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{Q}; P) = \sum_{j=1}^{N} c_j K_1(P, Q_j) + \sum_{j=1}^{N} d_j K_2(P, Q_j), \quad P \in \overline{\Omega}, \tag{4.24}$$

where $K_2(P, Q) = -\frac{1}{8\pi}|P-Q|^2 \log|P-Q|$ is a fundamental solution of the biharmonic equation.

The coefficients $\boldsymbol{c}$ and $\boldsymbol{d}$ are determined so that the boundary conditions are satisfied at the boundary points

$$u_{MN}^h(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{Q}; P_i) = h_1(P_i), \qquad \frac{\partial u_{MN}^h}{\partial n}(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{Q}; P_i) = h_2(P_i), \quad i = 1, \dots, N. \tag{4.25}$$

This yields a $2N \times 2N$ linear system of the form

$$\left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)\left(\begin{array}{c} \boldsymbol{c} \\ \hline \boldsymbol{d} \end{array}\right) = \left(\begin{array}{c} \boldsymbol{h}^1 \\ \hline \boldsymbol{h}^2 \end{array}\right), \tag{4.26}$$

where $\boldsymbol{h}^1 = (h_1(P_1), h_1(P_2), \dots, h_1(P_N))^T$, $\boldsymbol{h}^2 = (h_2(P_1), h_2(P_2), \dots, h_2(P_N))^T$, and the elements of matrices B_{mn}, $m, n = 1, 2$ are given by

$$(B_{11})_{i,j} = -\frac{1}{2\pi}\log|P_i - Q_j|, \qquad (B_{12})_{i,j} = -\frac{1}{8\pi}|P_i - Q_j|^2 \log|P_i - Q_j|,$$

$$(B_{21})_{i,j} = -\frac{1}{2\pi}\left(\frac{(x_{P_i}-x_{Q_j})n_x+(y_{P_i}-y_{Q_j})n_y}{|P_i-Q_j|^2}\right),$$

$$(B_{22})_{i,j} = -\frac{1}{8\pi}(1+2\log|P_i-Q_j|)\big((x_{P_i}-x_{Q_j})n_x+(y_{P_i}-y_{Q_j})n_y\big), \quad i,j=1,\ldots,N.$$

The matrices B_{mn}, $m,n=1,2$ are clearly circulant with $B_{mn}=\text{circ}\{b_1^{mn},b_2^{mn},\ldots,b_N^{mn}\}$ where $b_j^{mn}=(B_{mn})_{1,j}$, $j=1,\ldots,N$.

4.2.1 Matrix Decomposition Algorithm

Upon premultiplication by the matrix $I_2\otimes U_N$, where I_2 is the 2×2 identity matrix, using property (3.13), system (4.26) can now be written as

$$(I_2\otimes U_N)\left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22}\end{array}\right)(I_2\otimes U_N^*)(I_2\otimes U_N)\left(\begin{array}{c}\boldsymbol{c}\\ \hline \boldsymbol{d}\end{array}\right) = (I_2\otimes U_N)\left(\begin{array}{c}\boldsymbol{h}^1\\ \hline \boldsymbol{h}^2\end{array}\right) \tag{4.27}$$

or

$$\left(\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22}\end{array}\right)\left(\begin{array}{c}\hat{\boldsymbol{c}}\\ \hline \hat{\boldsymbol{d}}\end{array}\right) = \left(\begin{array}{c}\hat{\boldsymbol{h}}^1\\ \hline \hat{\boldsymbol{h}}^2\end{array}\right), \tag{4.28}$$

where

$$\hat{\boldsymbol{c}} = U_N\boldsymbol{c}, \qquad \hat{\boldsymbol{d}} = U_N\boldsymbol{d}, \qquad \hat{\boldsymbol{h}}^1 = U_N\boldsymbol{h}^1 \quad \text{and} \quad \hat{\boldsymbol{h}}^2 = U_N\boldsymbol{h}^2.$$

Also,

$$D_{mn} = \text{diag}\big(d_1^{mn}, d_2^{mn},\ldots,d_N^{mn}\big), \tag{4.29}$$

where

$$d_j^{mn} = \sum_{k=1}^{N} b_k^{mn}\omega^{(k-1)(j-1)}, \quad j=1,\ldots,N,\ m,n=1,2. \tag{4.30}$$

If $\hat{\boldsymbol{h}}^m = (\hat{h}_1^m,\ldots,\hat{h}_N^m)^T$, $m=1,2$, the solution can be obtained by solving the N independent 2×2 systems,

$$\begin{pmatrix} d_\ell^{11} & d_\ell^{12} \\ d_\ell^{21} & d_\ell^{22}\end{pmatrix}\begin{pmatrix}\hat{c}_\ell \\ \hat{d}_\ell\end{pmatrix} = \begin{pmatrix}\hat{h}_\ell^1 \\ \hat{h}_\ell^2\end{pmatrix}, \quad \ell=1,\ldots,N, \tag{4.31}$$

which yields

$$\begin{pmatrix}\hat{c}_\ell \\ \hat{d}_\ell\end{pmatrix} = \frac{1}{(d_\ell^{11}d_\ell^{22}-d_\ell^{21}d_\ell^{12})}\begin{pmatrix} d_\ell^{22}\hat{h}_\ell^1 - d_\ell^{12}\hat{h}_\ell^2 \\ -d_\ell^{21}\hat{h}_\ell^1 + d_\ell^{11}\hat{h}_\ell^2\end{pmatrix}, \quad \ell=1,\ldots,N. \tag{4.32}$$

Having obtained $\hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{d}}$, we can find $\boldsymbol{c}$ and $\boldsymbol{d}$ from

$$\boldsymbol{c} = U_N^*\hat{\boldsymbol{c}}, \qquad \boldsymbol{d} = U_N^*\hat{\boldsymbol{d}}.$$

The algorithm for calculating u_{MN}^h can thus be summarized as follows.

Algorithm 3

Step 1: `Compute` $\hat{\boldsymbol{h}}^m = U_N \boldsymbol{h}^m$, $m = 1, 2$.
Step 2: `Construct the diagonal matrices` D_{mn}, $m, m = 1, 2$ `from` (4.30).
Step 3: `Evaluate` $\hat{\boldsymbol{c}}$, $\hat{\boldsymbol{d}}$ `from` (4.32).
Step 4: `Compute` $\boldsymbol{c} = U_N^* \hat{\boldsymbol{c}}$, $\boldsymbol{d} = U_N^* \hat{\boldsymbol{d}}$.

As in Algorithm 2, FFTs can be used in Steps 1, 2 and 4 with similar cost. In (4.23), the Hänkel function $H_1^{(1)}(z)$ is calculated using the `Matlab` function `besselh(1,1,z)`. Note that, alternatively, in `Fortran` the calculation of the Hänkel functions could carried out using the code `MJY01A` from [40].

4.3 Triharmonic Problems

We next consider the solution of the inhomogeneous triharmonic boundary value problem [27, 28]

$$\begin{cases} \Delta^3 u = f & \text{in } \Omega, \\ u = g_1 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial n} = g_2 & \text{on } \partial\Omega, \\ \Delta u = g_3 & \text{on } \partial\Omega, \end{cases} \tag{4.33}$$

where f, g_1, g_2 and g_3 are given functions.

We seek a particular solution of the triharmonic equation u_{MN}^p satisfying

$$\Delta^3 u_{MN}^p = f_{MN} \quad \text{in } \Omega. \tag{4.34}$$

From (4.4), each Φ_{k_m} satisfies the equation

$$\left(\Delta^3 + k_m^6\right)\Phi_{k_m} = 0 \quad \text{in } \Omega. \tag{4.35}$$

Therefore, we may readily obtain a particular solution of the triharmonic equation from the expression

$$u_{MN}^p(A, \boldsymbol{Q}; P) = -\sum_{m=1}^{M}\sum_{n=1}^{N} \frac{a_{mn}}{k_m^6} \Phi_{k_m}(P, Q_n), \qquad P \in \overline{\Omega} = \Omega \cup \partial\Omega. \tag{4.36}$$

We let the solution of problem (4.33) be

$$u_{MN} = u_{MN}^p + u_{MN}^h, \tag{4.37}$$

where clearly u_{MN}^h satisfies the homogeneous triharmonic boundary value problem

$$\begin{cases} \Delta^3 u = 0 & \text{in } \Omega, \\ u = h_1 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial n} = h_2 & \text{on } \partial\Omega, \\ \Delta u = h_3 & \text{on } \partial\Omega, \end{cases} \tag{4.38}$$

where $h_1 = g_1 - u_{MN}^p$, $h_2 = g_2 - \frac{\partial u_{MN}^p}{\partial n}$ and $h_3 = g_3 - \Delta u_{MN}^p$.

This homogeneous problem can be solved efficiently using the MFS as follows. We approximate the solution of problem (4.38) by

$$u^h_{MN}(\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q};P)=\sum_{j=1}^{N}c_jK_1(P,Q_j)+\sum_{j=1}^{N}d_jK_2(P,Q_j)+\sum_{j=1}^{N}e_jK_3(P,Q_j),\quad P\in\overline{\Omega}, \tag{4.39}$$

where $K_3(P,Q)=-\frac{1}{128\pi}|P-Q|^4\log|P-Q|$ is a fundamental solution of the triharmonic equation [7, 27].

The coefficients $\boldsymbol{c},\boldsymbol{d}$ and $\boldsymbol{e}$ are determined so that the boundary conditions are satisfied at the boundary points

$$\begin{aligned}&u^h_{MN}(\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q};P_i)=h_1(P_i),\qquad \frac{\partial u^h_{MN}}{\partial n}(\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q};P_i)=h_2(P_i),\\&\Delta u^h_{MN}(\boldsymbol{c},\boldsymbol{d},\boldsymbol{Q};P_i)=h_3(P_i),\quad i=1,\dots,N.\end{aligned} \tag{4.40}$$

This yields a $3N\times 3N$ linear system of the form

$$\left(\begin{array}{c|c|c}B_{11}&B_{12}&B_{13}\\\hline B_{21}&B_{22}&B_{23}\\\hline B_{31}&B_{32}&B_{33}\end{array}\right)\left(\begin{array}{c}\boldsymbol{c}\\\hline\boldsymbol{d}\\\hline\boldsymbol{e}\end{array}\right)=\left(\begin{array}{c}\boldsymbol{h}^1\\\hline\boldsymbol{h}^2\\\hline\boldsymbol{h}^3\end{array}\right), \tag{4.41}$$

where $\boldsymbol{h}^m=(h_m(P_1),h_m(P_2),\dots,h_m(P_N))^T$, $m=1,2,3$, and the elements of matrices B_{mn}, $m,n=1,3$ are given by

$$\begin{aligned}
(B_{11})_{i,j}&=-\frac{1}{2\pi}\log|P_i-Q_j|,\qquad (B_{12})_{i,j}=-\frac{1}{8\pi}|P_i-Q_j|^2\log|P_i-Q_j|,\\
(B_{13})_{i,j}&=-\frac{1}{128\pi}|P_i-Q_j|^4\log|P_i-Q_j|,\\
(B_{21})_{i,j}&=-\frac{1}{2\pi}\left(\frac{(x_{P_i}-x_{Q_j})n_x+(y_{P_i}-y_{Q_j})n_y}{|P_i-Q_j|^2}\right),\\
(B_{22})_{i,j}&=-\frac{1}{8\pi}(1+2\log|P_i-Q_j|)\big((x_{P_i}-x_{Q_j})n_x+(y_{P_i}-y_{Q_j})n_y\big),\\
(B_{23})_{i,j}&=-\frac{1}{128\pi}|P_i-Q_j|^2(1+4\log|P_i-Q_j|)\big((x_{P_i}-x_{Q_j})n_x+(y_{P_i}-y_{Q_j})n_y\big),\\
(B_{31})_{i,j}&=0,\qquad (B_{32})_{i,j}=-\frac{1}{8\pi}(4\log|P_i-Q_j|+4),\\
(B_{33})_{i,j}&=-\frac{1}{128\pi}8|P_i-Q_j|^2(2\log|P_i-Q_j|+1),
\end{aligned}$$

for $i,j=1,\dots,N$.

The matrices B_{mn}, $m,n=1,2,3$ are clearly circulant with $B_{mn}=\mathrm{circ}\{b_1^{mn},b_2^{mn},\dots,b_N^{mn}\}$ where $b_j^{mn}=(B_{mn})_{1,j}$, $j=1,\dots,N$. Note that $B_{31}=0$ is trivial.

4.3.1 Matrix Decomposition Algorithm

Upon premultiplication by the matrix $I_3 \otimes U_N$, where I_3 is the 3×3 identity matrix, system (4.41) can now be written as

$$(I_3 \otimes U_N) \left(\begin{array}{c|c|c} B_{11} & B_{12} & B_{13} \\ \hline B_{21} & B_{22} & B_{23} \\ \hline B_{31} & B_{32} & B_{33} \end{array}\right) \big(I_3 \otimes U_N^*\big)(I_3 \otimes U_N) \left(\begin{array}{c} \boldsymbol{c} \\ \hline \boldsymbol{d} \\ \hline \boldsymbol{e} \end{array}\right) = (I_3 \otimes U_N) \left(\begin{array}{c} \boldsymbol{h}^1 \\ \hline \boldsymbol{h}^2 \\ \hline \boldsymbol{h}^3 \end{array}\right) \quad (4.42)$$

or

$$\left(\begin{array}{c|c|c} D_{11} & D_{12} & D_{13} \\ \hline D_{21} & D_{22} & D_{23} \\ \hline 0 & D_{32} & D_{33} \end{array}\right) \left(\begin{array}{c} \hat{\boldsymbol{c}} \\ \hline \hat{\boldsymbol{d}} \\ \hline \hat{\boldsymbol{e}} \end{array}\right) = \left(\begin{array}{c} \hat{\boldsymbol{h}}^1 \\ \hline \hat{\boldsymbol{h}}^2 \\ \hline \hat{\boldsymbol{h}}^3 \end{array}\right), \quad (4.43)$$

where

$$\hat{\boldsymbol{c}} = U_N \boldsymbol{c}, \qquad \hat{\boldsymbol{d}} = U_N \boldsymbol{d}, \qquad \hat{\boldsymbol{e}} = U_N \boldsymbol{e}, \quad \text{and} \quad \hat{\boldsymbol{h}}^m = U_N \boldsymbol{h}^m, \; m = 1, 2, 3.$$

Also (excluding the zero matrix D_{31}),

$$D_{mn} = \operatorname{diag}\big(d_1^{mn}, d_2^{mn}, \ldots, d_N^{mn}\big), \quad (4.44)$$

where

$$d_j^{mn} = \sum_{k=1}^{N} b_k^{mn} \omega^{(k-1)(j-1)}, \qquad j = 1, \ldots, N, \; m, n = 1, 2, 3. \quad (4.45)$$

If $\hat{\boldsymbol{h}}^m = (\hat{h}_1^m, \ldots, \hat{h}_N^m)^T$, $m = 1, 2, 3$, the solution can be obtained by solving the N independent 3×3 systems,

$$\begin{pmatrix} d_\ell^{11} & d_\ell^{12} & d_\ell^{13} \\ d_\ell^{21} & d_\ell^{22} & d_\ell^{23} \\ 0 & d_\ell^{32} & d_\ell^{33} \end{pmatrix} \begin{pmatrix} \hat{c}_\ell \\ \hat{d}_\ell \\ \hat{e}_\ell \end{pmatrix} = \begin{pmatrix} \hat{h}_\ell^1 \\ \hat{h}_\ell^2 \\ \hat{h}_\ell^3 \end{pmatrix}, \quad \ell = 1, \ldots, N, \quad (4.46)$$

which the solution of which yields $\hat{\boldsymbol{c}}, \hat{\boldsymbol{d}}$ and $\hat{\boldsymbol{e}}$.

Having obtained $\hat{\boldsymbol{c}}, \hat{\boldsymbol{d}}$ and $\hat{\boldsymbol{e}}$, we can find $\boldsymbol{c}, \boldsymbol{d}$ and $\boldsymbol{e}$ from

$$\boldsymbol{c} = U_N^* \hat{\boldsymbol{c}}, \qquad \boldsymbol{d} = U_N^* \hat{\boldsymbol{d}}, \qquad \boldsymbol{e} = U_N^* \hat{\boldsymbol{e}}.$$

The algorithm for calculating u_{MN}^h can thus be summarized as follows.

Algorithm 4

Step 1: `Compute` $\hat{\boldsymbol{h}}^m = U_N \boldsymbol{h}^m$, $m = 1, 2, 3$.
Step 2: `Construct the diagonal matrices` D_{mn}, $m = 1, 2, 3$ `from` (4.45).
Step 3: `Evaluate` $\hat{\boldsymbol{c}}, \hat{\boldsymbol{d}}, \hat{\boldsymbol{e}}$ `from` (4.46).
Step 4: `Compute` $\boldsymbol{c} = U_N^* \hat{\boldsymbol{c}}$, $\boldsymbol{d} = U_N^* \hat{\boldsymbol{d}}$, $\boldsymbol{e} = U_N^* \hat{\boldsymbol{e}}$.

As in Algorithms 2 and 3, FFTs can be used in Steps 1, 2 and 4 with similar costs.

5 Solution Strategy

By combining Algorithm 1 of Sect. 3 and Algorithms 2, 3 and 4 described in Sect. 4.3 we can obtain the solution of the general inhomogeneous polyharmonic problem (4.1)–(4.2)/(4.3) from the following steps.

Step 1: `Use Algorithm 1 to obtain approximation` (3.1) `to the right hand side` f `in problem` (2.1).
Step 2: `Construct an approximation` u_{MN}^p `of the particular solution from` (3.1) `using` (4.4).
Step 3: `Solve the resulting homogeneous boundary value problem using the appropriate MFS algorithm to obtain` u_{MN}^h. `(Algorithm 2 for` (2.4), `Algorithm 3 for` (4.22) `and Algorithm 4 for` (4.38)).
Step 4: `Obtain the approximation` u_{MN} `to boundary value problem` (4.1)–(4.2)/(4.3) `from` (4.8).

The most expensive part in the above process is Step 1 with a dominant cost of $O(NM^3)$, due to the solution of N linear systems of order M.

The above strategy can be easily extended to more general polyharmonic problems (4.1)–(4.2)/(4.3) with the particular solution taken equal to

$$u_{MN}^p(A, \boldsymbol{Q}; P) = (-1)^\ell \sum_{m=1}^{M} \sum_{n=1}^{N} \frac{a_{mn}}{k_m^{2\ell}} \Phi_{k_m}(P, Q_n), \qquad P \in \overline{\Omega} = \Omega \cup \partial\Omega. \tag{5.1}$$

The fundamental solutions of the corresponding operators can be derived from the formulæ provided in e.g. [7, 9].

6 Numerical Examples

In all numerical examples considered in the unit disk ($\varrho = 1$) we took collocation points described by $\varrho_i = (i/M)^{3/4}$ and $\alpha_i = (-1)^i/4$, $i = 1, \ldots, M$ in (3.3) (see also, Fig. 1). Following the recommendations of [2, 3], we chose the frequencies $k_m = 2m, m = 1, \ldots, M$. A somehow related discussion on the choice of the frequencies and singularities to avoid singular coefficient matrices may be found in [36]. We calculated the maximum relative error in both the approximation of the right hand side f and the approximation of the solution u in boundary value problem (2.1) on a grid of 25×50 uniformly distributed points in $\overline{\Omega}$, and will be denoting them by E_f and E_u, respectively. These values were obtained for a range of values of $R > 1$. It should be noted that the evaluation of the exact right hand sides f from the solutions u for some of the examples considered is very tedious. As a result, in some cases we have used repeatedly the differentiation command `diff` from the Symbolic Math Toolbox of `Matlab`.

In the following examples we shall consider the following exact solutions ([2]):

- $u_1(x, y) = \cos(x + y)$,
- $u_2(x, y) = \sin(y - x^2)$,
- $u_3(x, y) = \dfrac{1}{1 + x^4 + y^2}$.

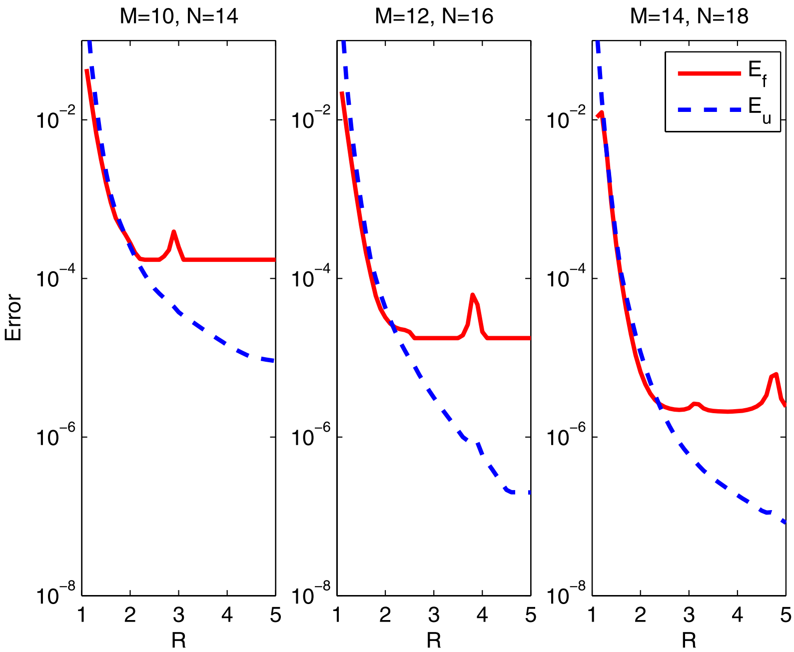


Fig. 2 Results for Example 1

6.1 Poisson Problems

We first consider the Dirichlet problem (2.1) on the unit disk with f and g corresponding to the exact solutions u_1, u_2 and u_3. In all cases, it can be observed that the error decreases when we increase M and N.

Example 1 (Exact solution $u = u_1$) In Fig. 2 we present the maximum relative errors E_f and E_u versus R for the three cases $M = 10$, $N = 14$, $M = 12$, $N = 16$, $M = 14$, $N = 18$.

Example 2 (Exact solution $u = u_2$) In Fig. 3 we present the maximum relative errors E_f and E_u versus R for the three cases $M = 8$, $N = 16$, $M = 10$, $N = 20$ and $M = 12$, $N = 24$.

Example 3 (Exact solution $u = u_3$) In Fig. 4 we present the maximum relative errors E_f and E_u versus R for the three cases $M = 12$, $N = 36$, $M = 14$, $N = 42$ and $M = 16$, $N = 48$.

6.2 Biharmonic Problems

We next consider the biharmonic problem (4.17) on the unit disk with f, g_1 and g_2 corresponding to the exact solutions u_1, u_2 and u_3. As in the Poisson problems, it can be observed that the error decreases when we increase M and N.

Example 4 (Exact solution $u = u_1$) In Fig. 5 we present the maximum relative errors E_f and E_u versus R for the three cases $M = 10$, $N = 20$, $M = 14$, $N = 28$, $M = 18$, $N = 36$.

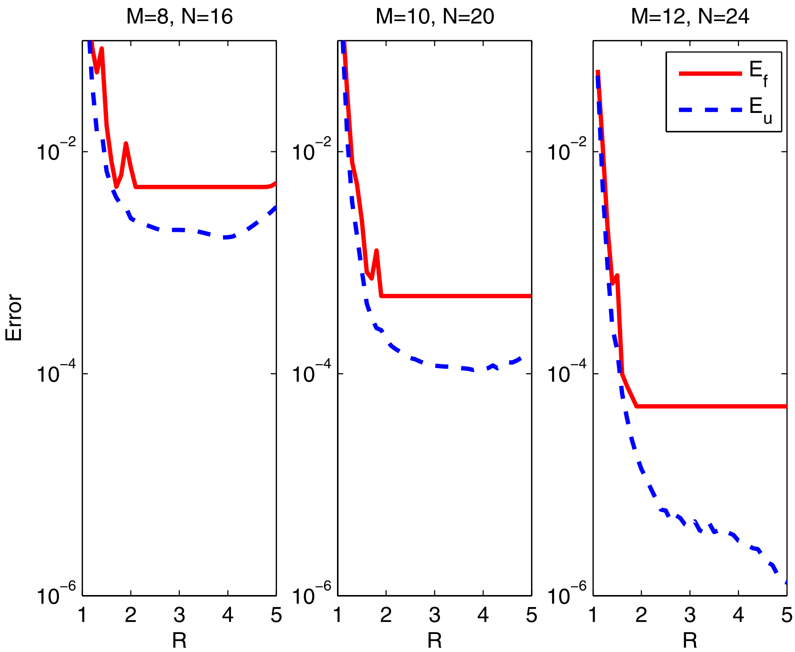


Fig. 3 Results for Example 2

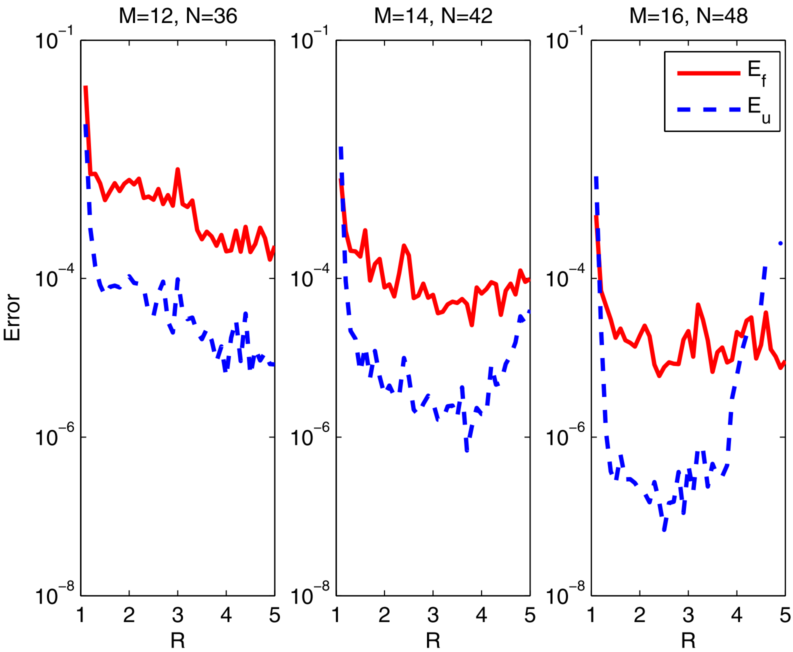


Fig. 4 Results for Example 3

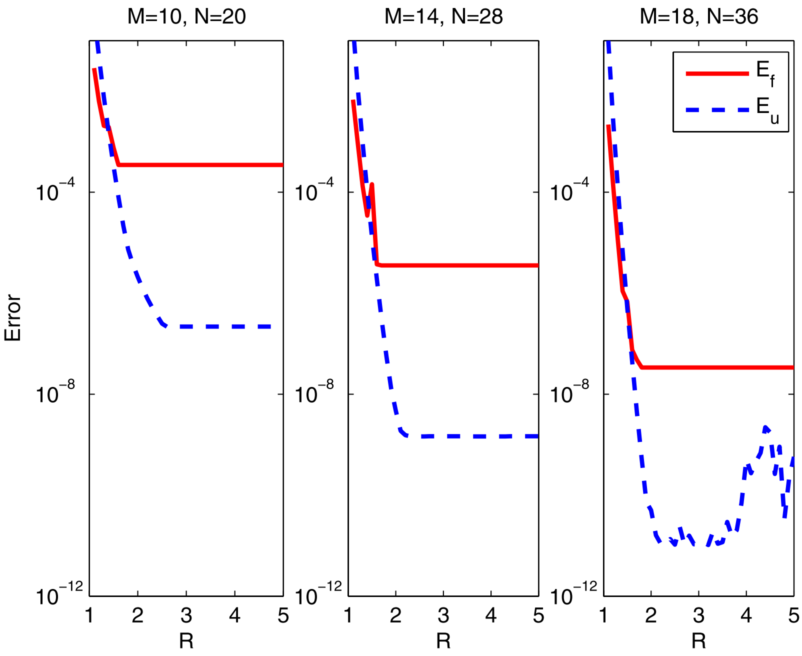


Fig. 5 Results for Example 4

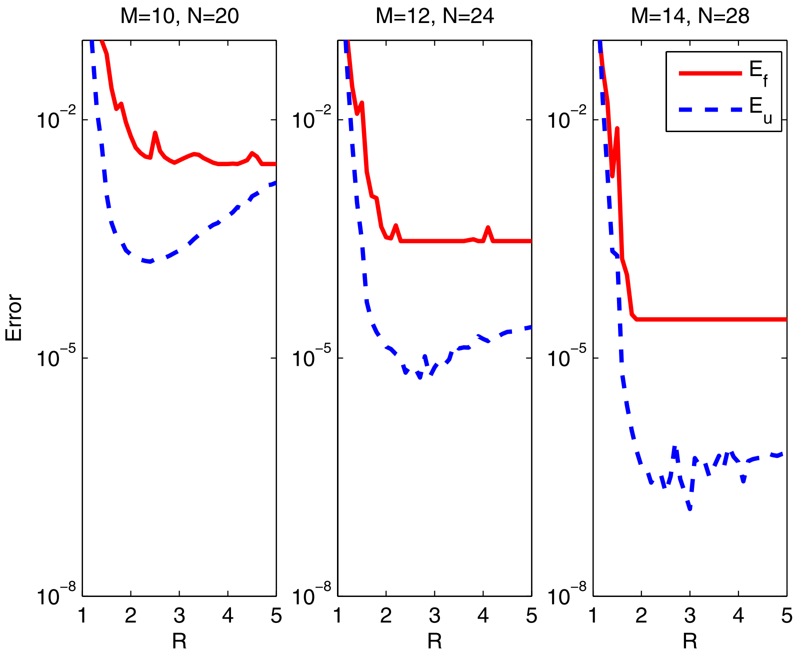


Fig. 6 Results for Example 5

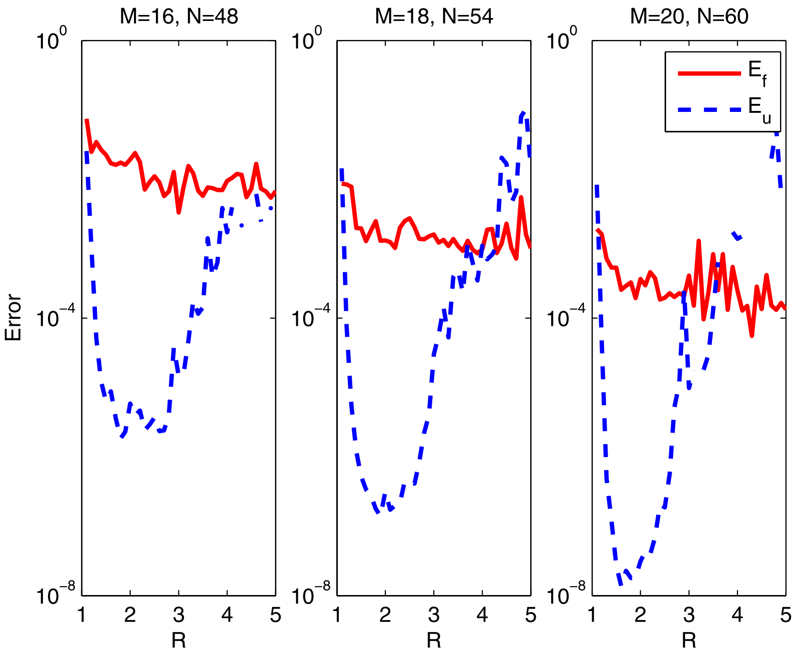


Fig. 7 Results for Example 6

Example 5 (Exact solution $u = u_2$) In Fig. 6 we present the maximum relative errors E_f and E_u versus R for the three cases $M = 10$, $N = 20$, $M = 12$, $N = 24$ and $M = 24$, $N = 28$.

Example 6 (Exact solution $u = u_3$) In Fig. 7 we present the maximum relative errors E_f and E_u versus R for the three cases $M = 16$, $N = 48$, $M = 18$, $N = 54$ and $M = 20$, $N = 60$. In this case, due to the complicated nature of the function f more degrees of freedom are necessary to approximate it satisfactorily.

6.3 Triharmonic Problems

We finally consider the triharmonic problem (4.33) on the unit disk with f, g_1, g_2 and g_3 corresponding to the exact solutions u_1, u_2 and u_3. As in the Poisson and biharmonic problems, it can be observed that the error decreases when we increase M and N.

Example 7 (Exact solution $u = u_1$) In Fig. 8 we present the maximum relative errors E_f and E_u versus R for the three cases $M = 10$, $N = 20$, $M = 12$, $N = 24$, $M = 14$, $N = 28$.

Example 8 (Exact solution $u = u_2$) In Fig. 9 we present the maximum relative errors E_f and E_u versus R for the three cases $M = 12$, $N = 24$, $M = 14$, $N = 28$ and $M = 16$, $N = 32$.

Example 9 (Exact solution $u = u_3$) In Fig. 10 we present the maximum relative errors E_f and E_u versus R for the three cases $M = 20$, $N = 60$, $M = 24$, $N = 72$ and $M = 30$, $N = 90$.

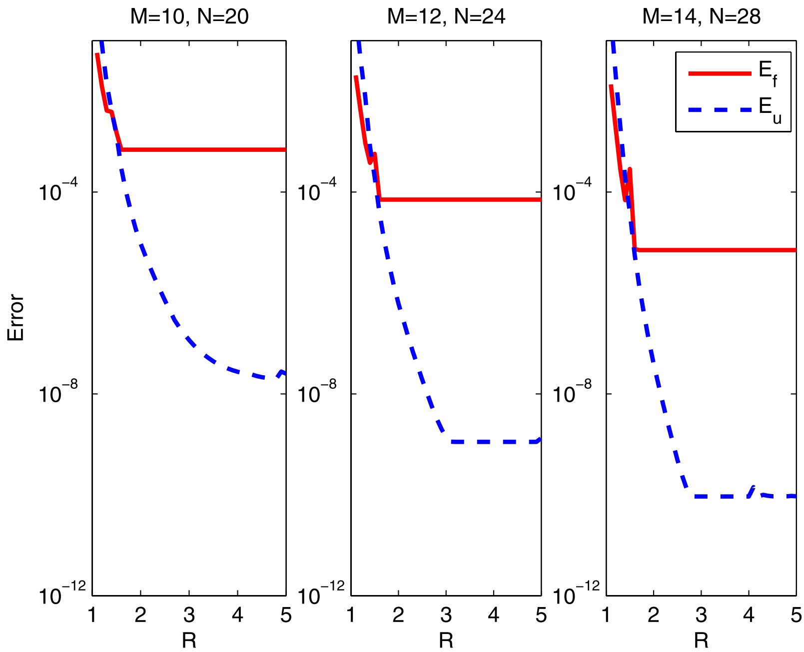


Fig. 8 Results for Example 7

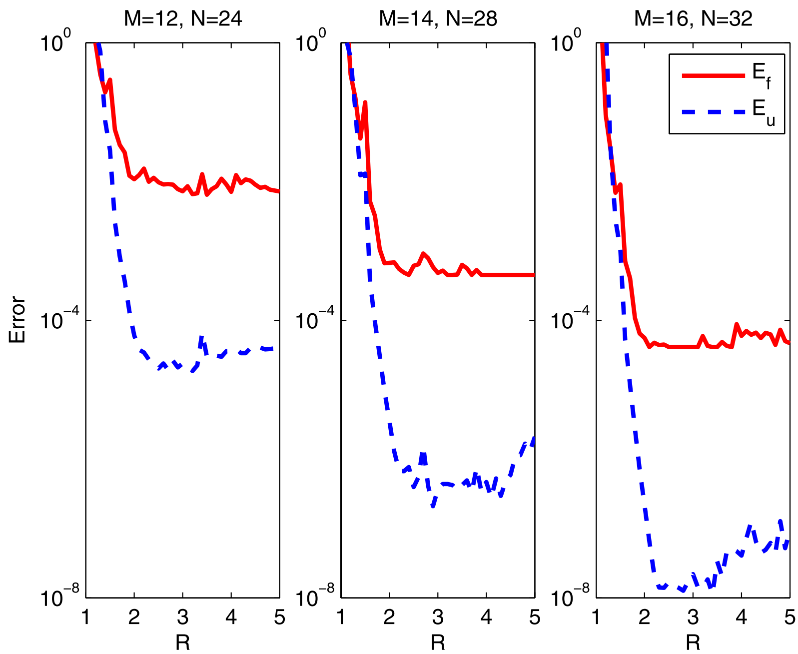


Fig. 9 Results for Example 8

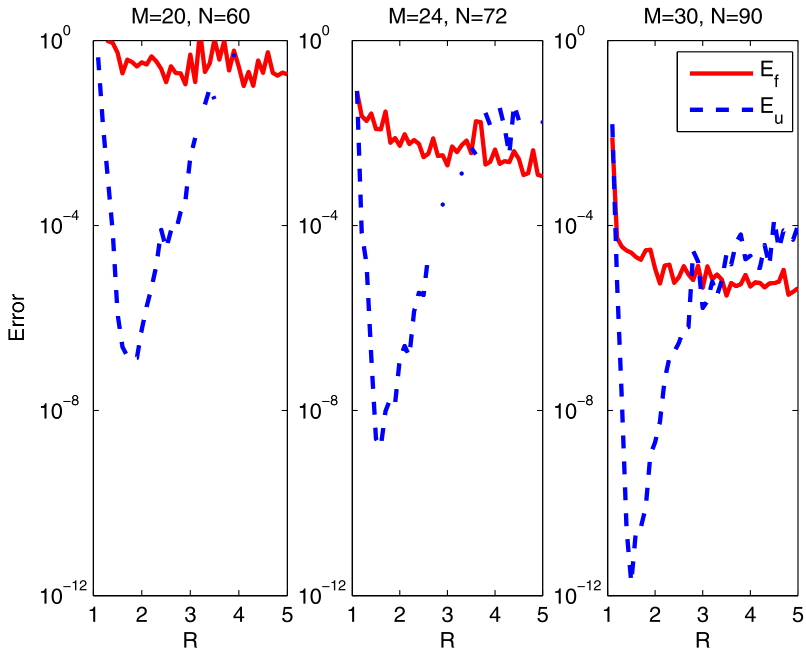


Fig. 10 Results for Example 9

In this case, a large number of degrees of freedom is necessary to satisfactorily approximate the complicated function $f = \Delta^3 u$.

7 Conclusions

In this work we propose efficient FFT-based matrix decomposition algorithms for the solution of inhomogeneous polyharmonic problems in circular domains. The inhomogeneous part of the governing polyharmonic equations is approximated using linear combinations of fundamental solutions of the Helmholtz equation as proposed in [2]. The evaluation of this approximation is carried out using a matrix decomposition algorithm which takes advantage of the block circulant structure of the matrices resulting from the collocation equations. Once the approximation is calculated, a particular solution of the inhomogeneous polyharmonic equation is easily constructed using the properties of the fundamental solutions of the Helmholtz operator. The particular solution is then subtracted from the problem yielding a homogeneous polyharmonic problem which can be easily solved using standard matrix decomposition algorithms for the MFS in circular domains. Several numerical examples have been considered with very satisfactory results.

The choice of the optimal positioning of the pseudoboundary in both stages of the solution problem remains a challenging problem as the behaviour of the error appears to depend differently on the distance of the pseudoboundary from the boundary in each of the two stages. Research in this direction could be carried out using some of the methods recently proposed in the literature [1, 21]. Also, the choice of the frequencies k_m in (3.1) remains a delicate issue and could be the subject of a future study. The extension of the current approach to polyharmonic problems in axisymmetric three-dimensional domains is currently under investigation.

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