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Third Order Explicit Runge-Kutta Discontinuous Galerkin Method for Linear Conservation Law with Inflow Boundary Condition

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Abstract In this paper we will present the stability in L^2-norm and the optimal a priori error estimate for the Runge-Kutta discontinuous Galerkin method to solve linear conservation law with inflow boundary condition. Semi-discrete version and fully-discrete version of this method are considered respectively, where time is advanced by the explicit third order total variation diminishing Runge-Kutta algorithm. To avoid the reduction of accuracy, two correction techniques are given for the intermediate boundary condition. Numerical experiments are also given to verify the above results.

Keywords Discontinuous Galerkin finite element · Runge-Kutta · Inflow boundary condition · Stability · Error estimate

1 Introduction

The first Discontinuous Galerkin (DG) method was introduced in 1973 by Reed and Hill [8], in the framework of linear neutron transport. Then it was developed into the Runge-Kutta discontinuous Galerkin (RKDG) scheme by Cockburn and his collaborators for nonlinear hyperbolic systems. Recently this method has been used widely for the simulation of conservation laws, even for the other problems with high order derivatives. For a fairly complete set of references on RKDG methods as well as their implementation and applications, see the review paper by Cockburn and Shu [4, 5].

The DG method uses a completely discontinuous piecewise polynomial space for the numerical solution and the test functions, so it possesses several properties to make it very attractive for practical computations, such as parallelization, adaptivity, and simple treatment of boundary conditions. The most important properties of DG method is its strong stability and high-order accuracy; as a result, it is very good at capturing discontinuous jumps.

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Many error analysis for the semidiscrete version of the DG method have been carried out for linear conservation law, for example, [5, 6]. Recently, Zhang and Shu have analyzed the fully discrete version coupled with the explicit total variation diminishing Runge-Kutta (TVDRK) algorithm, given by Shu [9]. The explicit second order TVDRK time-marching is studied for scalar nonlinear conservation law in [10], and for symmetrizable system of nonlinear conservation laws in [11]. The explicit third order TVDRK (TVDRK3) time-marching is also considered in [12].

However, most of the above analysis are given for the periodic boundary conditions. In this paper we will continue our work and study the RKDG method for conservation law with inflow boundary condition. As a model, we would like to consider the following one dimensional linear problem

$$\begin{cases} u_t + \beta u_x = 0, & x \in I = (a,b),\ t \in (0,T]; \\ u(x,0) = f(x), & x \in I; \\ u(a,t) = g(t), & t \in (0,T], \end{cases} \tag{1}$$

where, for simplicity, we assume the convection speed β is a positive constant, and consequently $x = a$ is the inflow boundary. In contract with the periodic boundary condition, we will pay more attention to the treatment of inflow boundary condition, especially, for the fully discrete version of this method with the explicit TVDRK3 time-marching. By virtue of theoretical analysis, two correction techniques for avoiding the reduction of accuracy are presented for this type of higher order algorithm.

The content of this paper is organized as follows. In Sect. 2 we give the discontinuous finite element space as well as its properties. In Sect. 3 we present the semi-discrete version of DG method, and then the stability result and an optimal a priori error estimate. Section 4 is the main body of this paper, where the fully discrete version of DG method coupled with the explicit TVDRK3 time-marching is discussed in detail. To do that, we first abstract each step of the considered algorithm into a black box, and then set up an elemental estimate. As a direct application, we then obtain the stability result and an optimal error estimate for this algorithm. During this process, some treatments on the inflow boundary are given to avoid the reduction of accuracy. Finally, numerical experiments and concluding remarks are given respectively in Sects. 5 and 6.

2 Preliminaries

2.1 Finite Element Space and Projections

Let $\{x_j\}_{j=0}^N$ be a partition of the interval $I = (a,b)$, where $x_0 = a$ and $x_N = b$. Denote each cell by $I_j = (x_{j-1}, x_j)$ with the length $h_j = x_j - x_{j-1}$, for $j = 1, 2, \ldots, N$. Then we define the mesh $\mathcal{T}_h = \{I_j : j = 1, \ldots, N\}$, with the mesh parameter $h = \max_{1\le j\le N} h_j$. For simplicity of analysis, the mesh is assumed to be regular, namely, the ratio of h over h_j, for $j = 1, 2, \ldots, N$, is upper bounded by a fixed positive constant.

Associated the mesh $\mathcal{T}_h$, we define the so-called broken Sobolev space

$$H^{1,h}(\mathcal{T}_h) = \{v \in L^2(I) \colon v|_{I_j} \in H^1(I_j), j = 1, 2, \ldots, N\}, \tag{2}$$

since the functions in $H^{1,h}(\mathcal{T}_h)$ are allowed to have discontinuities across element interfaces. For any $p \in H^{1,h}(\mathcal{T}_h)$, at each element boundary point there are two limits from different directions, namely, the left-side value p^- and the right-side value p^+. The jump and the mean at the element boundary point, respectively, are denoted by $[\![p]\!] = p^+ - p^-$, and $\{\!\{p\}\!\} = \frac{1}{2}(p^+ + p^-)$.

By $\mathbb{P}^k(\Omega)$ we denote the space of polynomials in Ω of degree at most k. The discontinuous finite element space is defined as

$$V_h = \{v \in L^2(0,1) : v|_{I_j} \in \mathbb{P}^k(I_j),\, j = 1, \ldots, N\}. \tag{3}$$

Note that $V_h \subset H^{1,h}(\mathcal{T}_h)$. As a standard trick in DG analysis, two types of projections are used in this paper. Due to the discontinuity property of the finite element space, these projections are locally defined on each element I_j.

The first one is the standard L^2-projection, denoted by $\mathbb{P}_h$. For any function $p \in L^2(0,1)$, the projection $\mathbb{P}_h p$ is defined as the unique function in V_h such that

$$\int_{I_j} (\mathbb{P}_h p(x) - p(x)) v_h(x)\,\mathrm{d}x = 0, \quad \forall v_h \in \mathbb{P}^k(I_j),\ 1 \le j \le N. \tag{4}$$

This projection is often used to approximate the initial solution, and also used to obtain the quasi-optimal error estimate.

The other is the so-called Castillo's projection [2], denoted by $\mathbb{R}_h$, in order to obtain the optimal error estimate. For any function $p \in L^2(0,1)$, the projection $\mathbb{R}_h p$ is defined as the unique function in V_h such that

$$\int_{I_j} (\mathbb{R}_h p(x) - p(x)) v_h(x)\,\mathrm{d}x = 0, \quad \forall v_h(x) \in \mathbb{P}^{k-1}(I_j),\ 1 \le j \le N, \tag{5a}$$

with the exact collocation at the downwind endpoint x_j, namely

$$(\mathbb{R}_h p)_j^- = p_j^-, \tag{5b}$$

since the convection speed is assumed to be $\beta > 0$. If $\beta < 0$, the definition for this projection is similar. Note that the projection $\mathbb{R}_h$ is defined well only for $k \ge 1$.

If $k = 0$, the considered DG scheme is equivalent to the finite volume method. For the error estimate for finite volume method, please see [7]. Throughout this paper we assume $k \ge 1$.

2.2 The Properties of Finite Element Space

To present the properties of the finite element space, we will use some traditional notations of Sobolev space. For any integer $s \ge 0$, let $H^s(\Omega)$ represent the well-known Sobolev space equipped with the norm $\|\cdot\|_{s,\Omega}$, which consists of functions with (distributional) derivatives of order not greater than s in $L^2(\Omega)$. Next, let $L^\infty(0,T;H^s(\Omega))$ represent the space-time space with the norm $\|\cdot\|_{L^\infty(H^s(\Omega))}$, which consists of functions with $\|u(x,t)\|_{s,\Omega}$ bounded uniformly for any time $t \in [0,T]$. Further, let the scalar inner product on $L^2(\Omega)$ be denoted by $(\cdot,\cdot)_\Omega$, and the associated norm be denoted by $\|\cdot\|_\Omega$. If the subscript $\Omega = I$, we omit it.

In what follows we will use C to denote a general positive constant independent of n, h and τ, which may have a different value in each occurrence.

For any function $p(x) \in H^{k+1}(I)$, there hold the following approximation properties. Denote by $\eta = p(x) - \mathbb{Q}_h p(x)$ the approximation error, where $\mathbb{Q}_h$ is one of projections mentioned above. By a standard scaling argument, it is easy to obtain that

$$\|\eta\| + h\|\eta_x\| + h^{1/2}\|\eta\|_{\Gamma_h} \le Ch^{k+1}, \tag{6}$$

where $C > 0$ is a constant independent of h and solely depends on $\|p\|_{k+1}$; see [2, 3]. Here Γ_h is the union of all element interface points, and the L^2-norm on Γ_h is defined as

$$\|v\|_{\Gamma_h} = \left[\sum_{1\le j\le N} (v_{j-1}^+)^2 + (v_j^-)^2 \right]^{1/2}, \quad v \in H^{1,h}(\mathcal{T}_h). \tag{7}$$

It is worthy to point out that the exact collocation at element boundary point, say, $\eta_j^- = 0$ for $j = 1, 2, \ldots, N$, holds only for the projection $\mathbb{R}_h$ but not for the projection $\mathbb{P}_h$.

We will also use the following inverse properties [3]. For any $v_h \in V_h$, there exist two positive constants μ_1 and μ_2, independent of v_h and h, such that

$$\text{(i)} \quad \|v_{h,x}\| \le \mu_1 h^{-1} \|v_h\|; \qquad \text{(ii)} \quad \|v_h\|_{\Gamma_h} \le \mu_2 h^{-1/2} \|v_h\|. \tag{8}$$

Here $v_{h,x}$ denotes the spatial derivative of v_h. For simplicity, we will denote $\mu = \max\{\mu_1, (\mu_2)^2\}$ as the uniform inverse constant.

3 Semi-discrete DG Method

First we multiply an arbitrary test function v_h on both sides of the first equation in (1), and integrate by parts in each element I_j. By defining the suitable numerical flux at the element boundary point, we will get the semi-discrete version of DG method: fine the map $u_h(t) : [0, T] \to V_h$ such that

$$\int_{I_j} \frac{\mathrm{d}u_h(t)}{\mathrm{d}t} v_h \,\mathrm{d}x = \mathcal{H}_j(u_h(t), v_h), \quad \forall v_h \in V_h,\ \forall t \in (0, T], \tag{9}$$

where the initial value is taken as the approximation of $f(x)$, for example, $u_h(0) = \mathbb{P}_h f(x)$. Here the compact notation $\mathcal{H}_j(\cdot, \cdot)$ describes the DG spatial discretization in the element I_j. For any function w and $v \in H^{1,h}(\mathcal{T}_h)$, it reads

$$\mathcal{H}_j(w, v) = \int_{I_j} \beta w v_x \,\mathrm{d}x - \hat{h}_j v_j^- + \hat{h}_{j-1} v_{j-1}^+, \tag{10}$$

where $\hat{h}_j = \hat{h}(w_j^-, w_j^+)$ is the monotone numerical flux depending on two values at the element boundary point x_j. In practice, for the considered problem (1) the numerical flux is often taken as the classical upwind type, namely, $\hat{h}_j = \beta w_j^-$ for $j = 0, 1, \ldots, N$, since $\beta > 0$.

The general treatment in the DG framework is that we introduce the boundary condition into the numerical flux at the boundary points. For the periodic boundary condition the numerical flux $\hat{h}$ is defined clearly at the left and the right boundary. However, for the inflow boundary condition, there is a little trouble. We have to define the numerical flux according to the position of the element boundary point. Namely, we take the numerical flux $\hat{h}_j = \beta w_j^-$ for $j = 1, 2, \ldots, N$, and take

$$\hat{h}_0(t) = \beta g(t), \quad t \in [0, T], \tag{11}$$

for $j = 0$, according to the inflow boundary condition at $x = a$. Now the definition for the semi-discrete DG scheme is completed.

For notation's simplicity, we would like to write the above scheme into a compact form

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}u_h(t), v_h\right) = \mathcal{H}(g(t); u_h(t), v_h), \quad \forall v_h \in V_h, \tag{12}$$

where $\mathcal{H}(g(t); u_h(t), v)$ is the DG spatial discretization over all elements. We divide this term into two parts, such as

$$\mathcal{H}(g(t); u_h(t), v_h) = \mathcal{H}^{\mathrm{c}}(u_h(t), v_h) + \mathcal{H}^{\mathrm{b}}(g(t); v_h), \tag{13}$$

where the bilinear form $\mathcal{H}^{\mathrm{c}}$ and the linear form $\mathcal{H}^{\mathrm{b}}$ are respectively defined by

$$\mathcal{H}^{\mathrm{c}}(w, v) = (\beta w, v_x) + \sum_{1\le j\le N-1} \hat{h}(w)_j [\![v]\!]_j - \hat{h}(w)_N v_N^-, \quad \forall w, v \in H^{1,h}(\mathcal{T}_h); \tag{14a}$$

$$\mathcal{H}^{\mathrm{b}}(g; v) = \beta g(t) v_0^+, \quad \forall v \in H^{1,h}(\mathcal{T}_h). \tag{14b}$$

Here the abbreviations "c" and "b" stand for "core" and "boundary" of the elements respectively. Further, for any function w, we denote the jump at boundary of computation domain as $[\![w]\!]_0 = w^+$ and $[\![w]\!]_N = w_N^-$, and denote the square of all jumps at every interface points by

$$[\![\![w]\!]\!]^2 = \sum_{0\le j\le N} [\![w]\!]_j^2. \tag{15}$$

Similar as those properties [12] for periodic boundary condition, the bilinear form $\mathcal{H}^{\mathrm{c}}(\cdot,\cdot)$ are also approximating skew-symmetrical, negative semi-definite, and continuous in the finite element space. These properties are given one by one in the next lemma. Since the proof is trivial and almost same as that in [12], we omit it here.

Lemma 3.1 *The bilinear form $\mathcal{H}^{\mathrm{c}}(\cdot,\cdot)$ has the following properties*:

$$\mathcal{H}^{\mathrm{c}}(w, v) + \mathcal{H}^{\mathrm{c}}(v, w) = -\sum_{0\le j\le N} |\beta| [\![w]\!]_j [\![v]\!]_j, \quad \forall w, v \in H^{1,h}(\mathcal{T}_h); \tag{16a}$$

$$\mathcal{H}^{\mathrm{c}}(v, v) = -\frac{|\beta|}{2} [\![\![v]\!]\!]^2, \quad \forall v \in H^{1,h}(\mathcal{T}_h); \tag{16b}$$

$$|\mathcal{H}^{\mathrm{c}}(w, v)| \le \gamma_0 \mu |\beta| h^{-1} \|w\| \|v\|, \quad \forall w, v \in V_h, \tag{16c}$$

where γ_0 is a positive constant independent of h and μ.

Remark 3.1 In [12] we have defined this constant as $\gamma_0 = \sqrt{2} + 1$, by using the inverse properties (i) and (ii), and a simple application of Cauchy-Schwarz inequality.

Based on this lemma, we are easy to obtain the stability result and error estimate for the semi-discrete version of the DG method.

Theorem 3.1 *For the semi-discrete DG method, we have, for any $t \in [0, T]$, that*

$$\|u_h\|^2(t) \le \|u_h\|^2(0) + \int_0^t |\beta| g^2(s)\,\mathrm{d}s. \tag{17}$$

Proof Below we drop the time argument t for simplicity. Taking the test function $v_h = u_h$ in algorithm (12), then we can use identity (16b) in Lemma 3.1 to get

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}u_h, u_h\right) = \mathcal{H}^{\mathrm{c}}(u_h, u_h) + \mathcal{H}^{\mathrm{b}}(g; u_h) = -\frac{|\beta|}{2}[\![\![u_h]\!]\!]^2 + \beta g u_{h,0}^{+}. \tag{18}$$

Since $|u_{h,0}^{+}| \le [\![\![u_h]\!]\!]$, an application of Young's inequality to the last term yields that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2 + \frac{1}{2}|\beta|[\![\![u_h]\!]\!]^2 \le \frac{1}{2}|\beta|g^2 + \frac{1}{2}|\beta|[\![\![u_h]\!]\!]^2. \tag{19}$$

So we complete the proof of this theorem by an integration in time. □

Theorem 3.2 *Let u and u_h be the solution of model problem (1) and the semi-discrete version of DG method (12), respectively. Assume that u and u_t are both in $L^\infty(0, T; H^{k+1})$, then there exists a positive constant C independent of h such that*

$$\|u - u_h\|_{L^\infty(L^2)} \le Ch^{k+1}. \tag{20}$$

Proof As the usual treatment in the analysis for finite element method, we divide the error into two parts, namely $u - u_h = \xi - \eta$, where $\xi = \mathbb{Q}_h u - u_h$ and $\eta = \mathbb{Q}_h u - u$. Here and below we drop the time arguments x and t, for simplicity. To obtain the optimal error estimate, we take the projection as $\mathbb{Q}_h = \mathbb{R}_h$.

We multiply the test function on the first equation of model problem (1), and make an integration by parts to yield the variation form of model problem (1). Noticing the inflow boundary condition and the continuity of u, this variation form is as same as semi-discrete algorithm (12), which reads

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}, v\right) = \mathcal{H}(g; u, v), \quad \forall v \in H^{1,h}(\mathcal{T}_h). \tag{21}$$

Then we subtract semi-discrete algorithm (12) from variation formula (21), with the same test function $v = v_h = \xi$ in both equations. This gives the identity

$$(\xi_t, \xi) = \mathcal{H}^{\mathrm{c}}(\xi, \xi) - \mathcal{H}^{\mathrm{c}}(\eta, \xi) + (\eta_t, \xi). \tag{22}$$

By using again identity (16b) in Lemma 3.1, we get $\mathcal{H}^{\mathrm{c}}(\xi, \xi) = -\frac{1}{2}|\beta|[\![\![\xi]\!]\!]^2$. By the virtue of the definition of projection $\mathbb{R}_h$, (5a) and (5b), the expression of (14a) implies that

$$\mathcal{H}^{\mathrm{c}}(\eta, \xi) = \sum_{1\le j\le N}\left[\int_{I_j} \beta\eta\xi_x \,\mathrm{d}x + \beta\eta_j^{-}[\![\xi]\!]_j\right] = 0, \tag{23}$$

since $[\![\xi]\!]_N = \xi_N^{-}$. Using Cauchy-Schwarz inequality to get $|(\eta_t, \xi)| \le (\|\eta_t\|^2 + \|\xi\|^2)/2$, then from (22) we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2 + \frac{1}{2}|\beta|[\![\![\xi]\!]\!]^2 \le \frac{1}{2}\|\eta_t\|^2 + \frac{1}{2}\|\xi\|^2, \quad \forall t \in [0, T]. \tag{24}$$

Since (6) holds for both projections, it follows from the initial setting $u_h(0) = \mathbb{P}_h f(x)$ that $\|\xi(0)\| \le Ch^{k+1}$. Also we have $\|\eta\|_{L^\infty(L^2)} \le Ch^{k+1}$ form (6), since we assume $u \in$

$L^\infty(0, T; H^{k+1})$. An application of Gronwall's inequality to (24) yields that

$$\|\xi(t)\|^2 + \int_0^t |\beta| [\![\xi(s)]\!]^2 \,\mathrm{d}s \le C \int_0^t \|\eta_t(s)\|^2 \,\mathrm{d}s + C\|\xi(0)\|^2$$
$$\le Ch^{2k+2}, \quad \forall t \in [0, T]. \tag{25}$$

Finally, we use (6) again and obtain $\|u(t) - u_h(t)\|^2 \le Ch^{2k+2}$ for any $t \in [0, T]$. It completes the proof of this theorem. □

4 RKDG Scheme with TVDRK3 Time-Marching

In this section we would like to obtain the stability result and an optimal error estimate for the fully discrete version of RKDG method, coupled with the explicit TVDRK3 time marching. In this paper we refer to this algorithm under consideration as RKDG3.

4.1 RKDG3

Let τ be the time step. In general, it maybe changes for different n; however, for simplicity we take it as a constant in this paper.

The RKDG3 algorithm is defined as follows. First, we set the initial value $u_h^0 = \mathbb{P}_h f(x)$. Then for each $n \ge 0$, the approximate solution from the time $n\tau$ to the next time $(n+1)\tau$ is obtained by finding successively $u_h^{n,1}$, $u_h^{n,2}$ and u_h^{n+1} in the finite element space V_h, such that for any $v_h \equiv v_h(x) \in V_h$, there hold three variation forms

$$(u_h^{n,1}, v_h) = (u_h^n, v_h) + \tau\mathcal{H}(g^n; u_h^n, v_h), \tag{26a}$$

$$(u_h^{n,2}, v_h) = \frac{3}{4}(u_h^n, v_h) + \frac{1}{4}(u_h^{n,1}, v_h) + \frac{\tau}{4}\mathcal{H}(g^{n,1}; u_h^{n,1}, v_h), \tag{26b}$$

$$(u_h^{n+1}, v_h) = \frac{1}{3}(u_h^n, v_h) + \frac{2}{3}(u_h^{n,2}, v_h) + \frac{2\tau}{3}\mathcal{H}(g^{n,2}; u_h^{n,2}, v_h), \tag{26c}$$

where the last term in each equation is the DG spatial discretization $\mathcal{H}(\star; \cdot, \cdot)$, which has been defined in (14). The including values g^n, $g^{n,1}$ and $g^{n,2}$ are the approximations of the boundary condition $g(t)$ at different time stages. We postpone the detailed setting of these values here, and will discuss them in the error analysis.

In order to obtain the stability and error estimate for the above fully discrete version in a uniform framework, we would like to abstract each time-marching of the RKDG3 algorithm into the following black box: given an input function $w_h^n \in V_h$, find successively the solution $w_h^{n,1}$, $w_h^{n,2}$ and w_h^{n+1} in the finite element space V_h, such that the variation forms

$$(w_h^{n,1}, v_h) = (w_h^n, v_h) + \tau\mathcal{H}^{\mathrm{c}}(w_h^n, v_h) + \tau\mathcal{L}^n(v_h), \tag{27a}$$

$$(w_h^{n,2}, v_h) = \frac{3}{4}(w_h^n, v_h) + \frac{1}{4}(w_h^{n,1}, v_h) + \frac{\tau}{4}\mathcal{H}^{\mathrm{c}}(w_h^{n,1}, v_h) + \frac{\tau}{4}\mathcal{L}^{n,1}(v_h), \tag{27b}$$

$$(w_h^{n+1}, v_h) = \frac{1}{3}(w_h^n, v_h) + \frac{2}{3}(w_h^{n,2}, v_h) + \frac{2\tau}{3}\mathcal{H}^{\mathrm{c}}(w_h^{n,2}, v_h) + \frac{2\tau}{3}\mathcal{L}^{n,2}(v_h), \tag{27c}$$

hold for any test function $v_h \in V_h$. For this black box, w_h^{n+1} is the output function, $w_h^{n,1}$ and $w_h^{n,2}$ are the so-called intermediate solution. For convenience, we also denote $\mathcal{L}^{n,0}(\cdot) = \mathcal{L}^n(\cdot)$.

In (27), the bilinear form $\mathcal{H}^{\mathrm{c}}(\cdot,\cdot)$ is defined as same as (14a) throughout this paper, while the linear forms $\mathcal{L}^{n,\ell}(\cdot)$ maybe have different definition for different purposes. For example, in the implementation of RKDG3 scheme and its stability analysis these linear forms are given by $\mathcal{L}^{n,\ell}(v)=\mathcal{H}^{\mathrm{b}}(g^{n,\ell};v)=\beta g^{n,\ell}v_0^+$, due to (26) and (13). In the error estimate, these linear forms will be redefined again; see (53).

4.2 Abstract Analysis

For linear evolution equation, the explicit TVDRK3 time-marching is equal to the Taylor expansion up to the third order time derivatives, which have a strong relationship with some certain linear combinations of the intermediate solution at each time stage of the RKDG3 algorithm. Thus we follow [12] and define for black box (27) the quantities

$$\begin{aligned} &\mathbb{D}_1 w_h^n = w_h^{n,1} - w_h^n, \qquad \mathbb{D}_2 w_h^n = 2w_h^{n,2} - w_h^{n,1} - w_h^n, \\ &\mathbb{D}_3 w_h^n = w_h^{n+1} - 2w_h^{n,2} + w_h^n. \end{aligned} \tag{28}$$

By virtue of some linear combinations of each variation form in the black box, it is easy to see that the above quantities have a nice relation. We proclaim only these results in the following lemma. For more details about its proof, see [12].

Lemma 4.1 *For any $v_h\in V_h$, we have the following identities*

$$(\mathbb{D}_1 w_h^n, v_h) = \tau\mathcal{H}^{\mathrm{c}}(w_h^n, v_h) + \tau\mathcal{L}^n(v_h), \tag{29a}$$

$$(\mathbb{D}_2 w_h^n, v_h) = \frac{\tau}{2}\mathcal{H}^{\mathrm{c}}(\mathbb{D}_1 w_h^n, v_h) + \frac{\tau}{2}\mathbb{D}_1\mathcal{L}^n(v_h), \tag{29b}$$

$$(\mathbb{D}_3 w_h^n, v_h) = \frac{\tau}{3}\mathcal{H}^{\mathrm{c}}(\mathbb{D}_2 w_h^n, v_h) + \frac{\tau}{3}\mathbb{D}_2\mathcal{L}^n(v_h), \tag{29c}$$

where $\mathbb{D}_1\mathcal{L}^n(\cdot)$ and $\mathbb{D}_2\mathcal{L}^n(\cdot)$ are linear combinations of the linear forms $\mathcal{L}^n(\cdot)$, $\mathcal{L}^{n,1}(\cdot)$ and $\mathcal{L}^{n,2}(\cdot)$. In detail, they read

$$\mathbb{D}_1\mathcal{L}^n(v_h) = \mathcal{L}^{n,1}(v_h) - \mathcal{L}^n(v_h), \tag{29d}$$

$$\mathbb{D}_2\mathcal{L}^n(v_h) = 2\mathcal{L}^{n,2}(v_h) - \mathcal{L}^{n,1}(v_h) - \mathcal{L}^n(v_h). \tag{29e}$$

Although at this time the linear forms in black box (27) are not given explicitly, they are assumed to have the following continuity properties: there exist positive constants $\rho_{n,\ell}$, $\sigma_{n,\ell}$, $\pi_{n,\ell}$ and $\theta_{n,\ell}$ independent of v_h, such that

$$|\mathcal{L}^{n,\ell}(v_h)| \le \rho_{n,\ell}[\![v_h]\!] + \sigma_{n,\ell}\|v_h\|, \quad \ell=0,1,2; \tag{30a}$$

$$|\mathbb{D}_\ell\mathcal{L}^n(v_h)| \le \pi_{n,\ell}[\![v_h]\!] + \theta_{n,\ell}\|v_h\|, \quad \ell=1,2. \tag{30b}$$

These bounded constants will be determined later, which are not same for different purpose.

Now we are going to build up a basic estimate for black box (27). To do that, we first take the test function v_h as w_h^n, $4w_h^{n,1}$ and $6w_h^{n,2}$ in (27a), (27b) and (27c), respectively, and then sum them up to obtain the following energy equality

$$3\|w_h^{n+1}\|^2 - 3\|w_h^n\|^2 = \Pi_1+\Pi_2+\Pi_3, \tag{31}$$

where

$$\Pi_1 = \tau\big[\mathcal{H}^c(w_h^n, w_h^n) + \mathcal{H}^c(w_h^{n,1}, w_h^{n,1}) + 4\mathcal{H}^c(w_h^{n,2}, w_h^{n,2})\big], \tag{32a}$$

$$\Pi_2 = \tau\big[\mathcal{L}^n(w_h^n) + \mathcal{L}^{n,1}(w_h^{n,1}) + 4\mathcal{L}^{n,2}(w_h^{n,2})\big], \tag{32b}$$

$$\Pi_3 = \|2w_h^{n,2} - w_h^{n,1} - w_h^n\|^2 + 3(w_h^{n+1} - w_h^n, w_h^{n+1} - 2w_h^{n,2} + w_h^n). \tag{32c}$$

Below we will estimate each above term separately.

It is easy to estimate the first two terms Π_1 and Π_2. By using identity (16b) in Lemma 3.1, it is easy to see that

$$\Pi_1 = -\frac{\tau|\beta|}{2}\Big[[\![w_h^n]\!]^2 + [\![w_h^{n,1}]\!]^2 + 4[\![w_h^{n,2}]\!]^2\Big]. \tag{33}$$

Noticing the assumption (30) for the linear forms $\mathcal{L}^{n,\ell}(\cdot)$, $\ell = 0, 1, 2$, after some simple applications of Young's inequality for those terms including the boundary jump, we have that

$$\begin{aligned}
\Pi_2 &\le \tau\big[\rho_{n,0}[\![w_h^n]\!] + \rho_{n,1}[\![w_h^{n,1}]\!] + 4\rho_{n,2}[\![w_h^{n,2}]\!] + \sigma_{n,0}\|w_h^n\| + \sigma_{n,1}\|w_h^{n,1}\| + 4\sigma_{n,2}\|w_h^{n,2}\|\big] \\
&\le \frac{|\beta|\tau}{4}\Big[[\![w_h^n]\!]^2 + [\![w_h^{n,1}]\!]^2 + 4[\![w_h^{n,2}]\!]^2\Big] + \tau\Theta_1^n,
\end{aligned} \tag{34}$$

where Θ_1^n depends on those constants in assumption (30), and is defined as

$$\Theta_1^n = \frac{1}{|\beta|}\big[\rho_{n,0}^2 + \rho_{n,1}^2 + 4\rho_{n,2}^2\big] + \sigma_{n,0}\|w_h^n\| + \sigma_{n,1}\|w_h^{n,1}\| + 4\sigma_{n,2}\|w_h^{n,2}\|. \tag{35}$$

For the third term Π_3, we would like to express it in term of $\mathbb{D}_1 w_h^n$, $\mathbb{D}_2 w_h^n$ and $\mathbb{D}_3 w_h^n$. Noticing the identity $w_h^{n+1} - w_h^n = \mathbb{D}_1 w_h^n + \mathbb{D}_2 w_h^n + \mathbb{D}_3 w_h^n$, we have the equivalent representation

$$\Pi_3 = (\mathbb{D}_2 w_h^n, \mathbb{D}_2 w_h^n) + 3(\mathbb{D}_3 w_h^n, \mathbb{D}_1 w_h^n) + 3(\mathbb{D}_3 w_h^n, \mathbb{D}_2 w_h^n) + 3(\mathbb{D}_3 w_h^n, \mathbb{D}_3 w_h^n). \tag{36}$$

Each above term on the right-hand side is denoted respectively by Λ_i, $(i = 1, 2, 3, 4)$, which will be estimated separately below.

First we estimate the sum of Λ_1 and Λ_2. To do that, we use identities (29b) and (29c) in Lemma 4.1, with the test functions $v_h = \mathbb{D}_2 w_h^n$ and $v_h = \mathbb{D}_1 w_h^n$, respectively. Then it follows from a direct application of (16a) in Lemma 3.1, that

$$\begin{aligned}
\Lambda_1 + \Lambda_2 &= -(\mathbb{D}_2 w_h^n, \mathbb{D}_2 w_h^n) + 2(\mathbb{D}_2 w_h^n, \mathbb{D}_2 w_h^n) + 3(\mathbb{D}_3 w_h^n, \mathbb{D}_1 w_h^n) \\
&= -\|\mathbb{D}_2 w_h^n\|^2 + \tau\mathcal{H}^c(\mathbb{D}_1 w_h^n, \mathbb{D}_2 w_h^n) + \tau\mathbb{D}_1\mathcal{L}^n(\mathbb{D}_2 w_h^n) + \tau\mathcal{H}^c(\mathbb{D}_2 w_h^n, \mathbb{D}_1 w_h^n) \\
&\quad + \tau\mathbb{D}_2\mathcal{L}^n(\mathbb{D}_1 w_h^n) \\
&= -\|\mathbb{D}_2 w_h^n\|^2 - \tau\sum_{0\le j\le N}|\beta|[\![\mathbb{D}_1 w_h^n]\!]_j[\![\mathbb{D}_2 w_h^n]\!]_j + \tau\mathbb{D}_1\mathcal{L}^n(\mathbb{D}_2 w_h^n) + \tau\mathbb{D}_2\mathcal{L}^n(\mathbb{D}_1 w_h^n).
\end{aligned}$$

We first use assumption (30) to bound the last two terms, and then use Cauchy inequality and Young's inequality to estimate those terms including the boundary jump. Finally it yields that

$$\Lambda_1 + \Lambda_2 \le -\|\mathbb{D}_2 w_h^n\|^2 + \tau\left[\frac{|\beta|}{16}[\![\mathbb{D}_1 w_h^n]\!]^2 + 4|\beta|[\![\mathbb{D}_2 w_h^n]\!]^2\right]$$
$$+ \tau\left[\frac{|\beta|}{16}[\![\mathbb{D}_1 w_h^n]\!]^2 + \frac{4}{|\beta|}\pi_{n,2}^2\right]$$
$$+ \tau\left[\frac{1}{4}|\beta|[\![\mathbb{D}_2 w_h^n]\!]^2 + \frac{1}{|\beta|}\pi_{n,1}^2\right] + \tau\big[\theta_{n,2}\|\mathbb{D}_1 w_h^n\| + \theta_{n,1}\|\mathbb{D}_2 w_h^n\|\big].$$

Next we turn to estimate the term Λ_3. To do that, we take the test function $v_h = \mathbb{D}_2 w_h^n$ in identity (29c) in Lemma 4.1, and then use identity (16b) in Lemma 3.1. It yields

$$\Lambda_3 = 3(\mathbb{D}_3 w_h^n, \mathbb{D}_2 w_h^n) = \tau\mathcal{H}^{c}(\mathbb{D}_2 w_h^n, \mathbb{D}_2 w_h^n) + \tau\mathbb{D}_2\mathcal{L}^n(\mathbb{D}_2 w_h^n)$$
$$= -\frac{|\beta|\tau}{2}[\![\mathbb{D}_2 w_h^n]\!]^2 + \tau\mathbb{D}_2\mathcal{L}^n(\mathbb{D}_2 w_h^n)$$
$$\le -\frac{|\beta|\tau}{4}[\![\mathbb{D}_2 w_h^n]\!]^2 + \frac{\tau}{|\beta|}\pi_{n,2}^2 + \tau\theta_{n,2}\|\mathbb{D}_2 w_h^n\|,$$

where in the last step we have used assumption (30) and used Young's inequality to control the jump term.

Now we estimate the term Λ_4, which need a more analysis than the formers. We first take the test function $v_h = \mathbb{D}_3 w_h^n$ in identity (29c) in Lemma 4.1. Denote by $\lambda = \mu|\beta|\tau h^{-1}$ the so-called CFL number. By virtue of assumption (30) for the linear forms, a simple manipulation yields that

$$\|\mathbb{D}_3 w_h^n\|^2 = (\mathbb{D}_3 w_h^n, \mathbb{D}_3 w_h^n) = \frac{\tau}{3}\mathcal{H}^{c}(\mathbb{D}_2 w_h^n, \mathbb{D}_3 w_h^n) + \frac{\tau}{3}\mathbb{D}_2\mathcal{L}^n(\mathbb{D}_3 w_h^n)$$
$$\le \frac{1}{3}\gamma_0\lambda\|\mathbb{D}_2 w_h^n\|\|\mathbb{D}_3 w_h^n\| + \frac{1}{3}\tau\big[\pi_{n,2}[\![\mathbb{D}_3 w_h^n]\!] + \theta_{n,2}\|\mathbb{D}_3 w_h^n\|\big]$$
$$\le \frac{1}{3}\big[\gamma_0\lambda\|\mathbb{D}_2 w_h^n\| + \tau[\pi_{n,2}\mu_2 h^{-1/2} + \theta_{n,2}]\big]\|\mathbb{D}_3 w_h^n\|,$$

where in the second step we have used the continuity property (16c) in Lemma 3.1, and in the last step we have used the inverse inequality (ii). Consequently, by the simple inequality $(a+b)^2 \le 2(a^2+b^2)$ we get

$$\Lambda_4 = 3\|\mathbb{D}_3 w_h^n\|^2 \le \frac{2}{3}\gamma_0^2\lambda^2\|\mathbb{D}_2 w_h^n\|^2 + \frac{2}{3}\tau^2[\pi_{n,2}\mu_2 h^{-1/2} + \theta_{n,2}]^2.$$

Now we substitute the above estimates about terms Λ_1, Λ_2, Λ_3 and Λ_4, into formula (36). Since $\lambda = \mu|\beta|\tau h^{-1}$ and $(\mu_2)^2 \le \mu$, it yields that

$$\Pi_3 \le \frac{|\beta|\tau}{8}[\![\mathbb{D}_1 w_h^n]\!]^2 + 4|\beta|\tau[\![\mathbb{D}_2 w_h^n]\!]^2 + \left(\frac{2}{3}\gamma_0^2\lambda^2 - 1\right)\|\mathbb{D}_2 w_h^n\|^2 + \tau\Theta_2^n$$
$$\le \frac{|\beta|\tau}{4}\big[[\![w_h^n]\!]^2 + [\![w_h^{n,1}]\!]^2\big] + 8|\beta|\tau\|\mathbb{D}_2 w_h^n\|_{\Gamma_h}^2 + \left(\frac{2}{3}\gamma_0^2\lambda^2 - 1\right)\|\mathbb{D}_2 w_h^n\|^2 + \tau\Theta_2^n$$
$$\le \frac{|\beta|\tau}{2}\big[[\![w_h^n]\!]^2 + [\![w_h^{n,1}]\!]^2\big] + \left(\frac{2}{3}\gamma_0^2\lambda^2 + 8\lambda - 1\right)\|\mathbb{D}_2 w_h^n\|^2 + \tau\Theta_2^n, \qquad (37)$$

where in the second step we have used the simple inequality $[\![v]\!]^2 \le 2\|v\|^2_{\Gamma_h}$, and in the last step we have used the inverse inequality (ii). Here $\Theta_2^n = \Theta_{21}^n + \Theta_{22}^n$ is resulted from assumption (30) for the linear forms $\mathbb{D}_\ell \mathcal{L}^n(\cdot)$, in which each term is given as

$$\Theta_{21}^n = \frac{1}{|\beta|}\left[\pi_{n,1}^2 + 5\pi_{n,2}^2\right] + \frac{2}{3}\tau[\pi_{n,2}\mu_2 h^{-1/2} + \theta_{n,2}]^2, \tag{38a}$$

$$\Theta_{22}^n = \left[\theta_{n,2}\|\mathbb{D}_1 w_h^n\| + \theta_{n,1}\|\mathbb{D}_2 w_h^n\| + \tau\theta_{n,2}\|\mathbb{D}_2 w_h^n\|\right]. \tag{38b}$$

Finally we insert the above estimates about Π_1, Π_2 and Π_3 into energy identity (31), and get the following estimate

$$3\|w_h^{n+1}\|^2 - 3\|w_h^n\|^2 + 2|\beta|[\![w_h^{n,2}]\!]^2\tau \le P(\lambda)\|\mathbb{D}_2 w_h^n\|^2 + \tau\Theta_1^n + \tau\Theta_2^n, \tag{39}$$

where

$$P(\lambda) = \frac{2}{3}\gamma_0^2\lambda^2 + 8\lambda - 1. \tag{40}$$

This implies the following lemma for black box (27).

Lemma 4.2 *Assume $w_h^{n+1} \in V_h$ is the output function by the black box (27), with the input function $w_h^n \in V_h$. If the CFL number λ is small enough such that $P(\lambda) \le 0$, then we have*

$$3\|w_h^{n+1}\|^2 - 3\|w_h^n\|^2 + 2|\beta|[\![w_h^{n,2}]\!]^2\tau \le \tau\Theta_1^n + \tau\Theta_2^n, \tag{41}$$

where Θ_1^n and Θ_2^n have been defined in (35) and (38), respectively.

In the next two subsections we will present the stability result and a priori error estimate by mean of this lemma. To do that, we only define the corresponding linear forms in black box (27) and the corresponding estimates for Θ_1^n and Θ_2^n.

4.3 Stability Analysis

To obtain the stability result, we have to give the explicit definition of linear forms $\mathcal{L}^{n,\ell}$, and the corresponding bounded constants in assumption (30). The nth time-marching in the RKDG3 algorithm is to get u_h^{n+1} from u_h^n by black box (27), with three linear forms

$$\mathcal{L}^{n,\ell}(v_h) = \beta g^{n,\ell} v_{h,0}^+, \quad \ell = 0, 1, 2. \tag{42}$$

It is easy to see that

$$|\mathcal{L}^{n,\ell}(v_h)| \le |\beta g^{n,\ell}||v_{h,0}^+| \le |\beta g^{n,\ell}|[\![v_h]\!], \quad \ell = 0, 1, 2, \tag{43a}$$

$$|\mathbb{D}_\ell \mathcal{L}^n(v_h)| \le |\beta \mathbb{D}_\ell g^n||v_{h,0}^+| \le |\beta \mathbb{D}_\ell g^n|[\![v_h]\!], \quad \ell = 1, 2. \tag{43b}$$

So the bounded constants in assumption (30) are determined as

$$\rho_{n,\ell} = |\beta g^{n,\ell}|, \qquad \sigma_{n,\ell} = 0, \quad \ell = 0, 1, 2; \tag{44a}$$

$$\pi_{n,\ell} = |\mathbb{D}_\ell g^n|, \qquad \theta_{n,\ell} = 0, \quad \ell = 1, 2. \tag{44b}$$

Consequently, from (35) and (38) we have

$$\Theta_1^n + \Theta_2^n = |\beta|\big[(g^n)^2 + (g^{n,1})^2 + 4(g^{n,2})^2 + (\mathbb{D}_1 g^n)^2 + (\mathbb{D}_2 g^n)^2\big] \\ \leq C|\beta|\big[(g^n)^2 + (g^{n,1})^2 + (g^{n,2})^2\big]. \tag{45}$$

Now we can obtain the L^2-norm stability for the RKDG3 algorithm by directly using the elemental lemma 4.2 and inequality (45).

Theorem 4.1 *Let u_h^n be the solution of the fully discrete version (26) of the DG method. For any time level n satisfying $(n+1)\tau \leq T$, we have the following stability result*

$$\|u_h^{n+1}\|^2 + \frac{2}{3}\sum_{m=0}^{n} |\beta| [\![u_h^{m,2}]\!]^2 \tau \leq \|u_h^0\|^2 + C\sum_{m=0}^{n}\big[(g^m)^2 + (g^{m,1})^2 + (g^{m,2})^2\big]\tau, \tag{46}$$

if the CFL number λ is small enough such that $P(\lambda) < 0$, where $\lambda = \mu|\beta|\tau h^{-1}$. In the above estimate the bounded constant $C > 0$ is independent of h, τ and u_h.

4.4 A Priori Error Estimate

In this subsection we would like to use Lemma 4.2 again and obtain the optimal a priori error estimate for the RKDG3 algorithm, under the careful treatment for the inflow boundary condition.

4.4.1 Error Equations

First we define the errors corresponding to different stages of the RKDG3 algorithm. To do that, we introduce the reference functions paralleled to the Runge-Kutta time discrete, following [12]. In detail, let $u^{(0)}(x,t) = u(x,t)$ be the exact solution of conservation law (1), and

$$u^{(1)}(x,t) = u^{(0)}(x,t) + \tau u_t^{(0)}(x,t), \tag{47a}$$

$$u^{(2)}(x,t) = \frac{3}{4}u^{(0)}(x,t) + \frac{1}{4}u^{(1)}(x,t) + \frac{1}{4}\tau u_t^{(1)}(x,t). \tag{47b}$$

Associated the construction of the explicit TVDRK3 time-marching, the local truncation error in time direction can be bounded by the following lemma.

Lemma 4.3 *If $u_{tttt} \in L^\infty(0,T;L^2)$, there exists the local truncation error in time direction $\mathcal{E}(x,t) \in L^\infty(0,T;L^2)$ such that*

$$u(x,t+\tau) = \frac{1}{3}u^{(0)}(x,t) + \frac{2}{3}u^{(2)}(x,t) + \frac{2}{3}\tau u_t^{(2)}(x,t) + \mathcal{E}(x,t), \quad x \in I, t+\tau \in (0,T], \tag{48}$$

where $\|\mathcal{E}(x,t)\|_{L^\infty(L^2)} \leq C\tau^4$ and the bounded constant $C > 0$ is independent of τ.

Proof Noticing definition (47), a simple manipulation yields that

$$\begin{aligned}\frac{1}{3}u^{(0)}+\frac{2}{3}u^{(2)}+\frac{2}{3}\tau u_t^{(2)} &= \frac{5}{6}u^{(0)}+\frac{1}{6}u^{(1)}+\frac{1}{6}\tau u_t^{(1)}+\frac{2}{3}\tau\frac{\partial}{\partial t}\left[\frac{3}{4}u^{(0)}+\frac{1}{4}u^{(1)}+\frac{1}{4}\tau u_t^{(1)}\right]\\ &= \frac{5}{6}u^{(0)}+\frac{1}{6}u^{(1)}+\frac{1}{2}\tau u_t^{(0)}+\frac{1}{3}\tau u_t^{(1)}+\frac{1}{6}\tau^2 u_{tt}^{(1)}\\ &= u^{(0)}+\tau u_t^{(0)}+\frac{1}{2}\tau^2 u_{tt}^{(0)}+\frac{1}{6}\tau^3 u_{ttt}^{(0)},\end{aligned}$$

where we have dropped the arguments (x,t) for notation's simplicity. Consequently, from Taylor's expansion in time direction we have

$$u(x,t+\tau)=\frac{1}{3}u^{(0)}(x,t)+\frac{2}{3}u^{(2)}(x,t)+\frac{2}{3}\tau u_t^{(2)}(x,t)+\int_t^{t+\tau}\frac{u_{tttt}(s)}{6}(t+\tau-s)^3\,\mathrm{d}s, \quad (49)$$

where the last integration in (49) is the local truncation error, denoted by $\mathcal{E}(x,t)$. Hence, under the assumption of this lemma we have $\|\mathcal{E}(x,t)\|_{L^\infty(L^2)}\le C\tau^4$, where the bounded constant $C>0$ is independent of τ. It completes the proof of this lemma. □

Denote, respectively, the error of the fully discrete algorithm at each time stage by

$$e^{n,0}=u^{(0)}(x,t^n)-u_h^n,\qquad e^{n,1}=u^{(1)}(x,t^n)-u_h^{n,1},\qquad e^{n,2}=u^{(2)}(x,t^n)-u_h^{n,2}. \quad (50)$$

Similar as the analysis for semidiscrete version, we also take the projection $\mathbb{Q}_h=\mathbb{R}_h$, and divide each above stage error into the form $e^{n,\ell}=\eta^{n,\ell}-\xi^{n,\ell}$, where

$$\xi^{n,\ell}=u_h^{n,\ell}-\mathbb{Q}_h u^{(\ell)}(x,t^n),\qquad \eta^{n,\ell}=u^{(\ell)}(x,t^n)-\mathbb{Q}_h u^{(\ell)}(x,t^n),\quad \ell=0,1,2. \quad (51)$$

For notation's simplicity, below we will omit the argument x and denote $u^{(\ell)}(x,t)=u^{(\ell)}(t)$. We also omit the index of time stage for the above notations if $\ell=0$.

In what follows we are going to estimate $\xi^{n,\ell}$ and $\eta^{n,\ell}$ separately. The approximation error $\eta^{n,\ell}$ are resulted from the projection, which can be estimated easily from the approximation property of finite element space. However, we have to make great efforts to estimate $\xi^{n,\ell}\in V_h$, starting from its error equations at any time stages.

The error equations are obtained in two steps. In the first step, we give the variation forms for the reference values. To this end, we multiply the test function $v\in H^{1,h}(\mathcal{T}_h)$ on the both side of identities (47a), (47b) and (48). Let the time be $t=t^n$. We transfer the time derivatives to spatial derivatives, by virtue of the first equation in (1). Then an integration by parts gives the variation formulas for $u^{(0)}(t^n)$, $u^{(1)}(t^n)$ and $u^{(2)}(t^n)$. Noticing the continuity of above variables and the inflow boundary condition, one can get the variation formulas almost same as the RKDG3 algorithm. To be more specific, they are given as follows

$$\begin{aligned}(u^{(1)}(t^n),v)&=(u^{(0)}(t^n),v)+\tau\mathcal{H}(u^{(0)}(t^n);u^{(0)}(t^n),v),\\ (u^{(2)}(t^n),v)&=\frac{3}{4}(u^{(0)}(t^n),v)+\frac{1}{4}(u^{(1)}(t^n),v)+\frac{\tau}{4}\mathcal{H}(u^{(1)}(t^n);u^{(1)}(t^n),v),\\ (u^{(0)}(t^{n+1}),v)&=\frac{1}{3}(u^{(0)}(t^n),v)+\frac{2}{3}(u^{(2)}(t^n),v)+(\mathcal{E}^n,v)+\frac{2\tau}{3}\mathcal{H}(u^{(2)}(t^n);u^{(2)}(t^n),v),\end{aligned}$$

where v is any function in $H^{1,h}(\mathcal{T}_h)$, and $\mathcal{E}^n=\mathcal{E}(x,t^n)$.

In the second step, since $V_h \subset H^{1,h}(\mathcal{T}_h)$, we subtract above variation formulas from the RKDG3 algorithm, and obtain the following error equations under the framework of black box (27): for any test function $v_h \in V_h$, there hold

$$(\xi^{n,1}, v_h) = (\xi^n, v_h) + \tau\mathcal{H}^c(\xi^n, v_h) + \tau\mathcal{L}^n(v_h), \tag{52a}$$

$$(\xi^{n,2}, v_h) = \frac{3}{4}(\xi^n, v_h) + \frac{1}{4}(\xi^{n,1}, v_h) + \frac{\tau}{4}\mathcal{H}^c(\xi^{n,1}, v_h) + \frac{\tau}{4}\mathcal{L}^{n,1}(v_h), \tag{52b}$$

$$(\xi^{n+1}, v_h) = \frac{1}{3}(\xi^n, v_h) + \frac{2}{3}(\xi^{n,2}, v_h) + \frac{2\tau}{3}\mathcal{H}^c(\xi^{n,2}, v_h) + \frac{2\tau}{3}\mathcal{L}^{n,2}(v_h). \tag{52c}$$

Here the bilinear form $\mathcal{H}^c(\cdot,\cdot)$ is still defined same as (14a). However, the linear forms $\mathcal{L}^{n,\ell}(\cdot)$ are different with those in the stability analysis. In the process of error estimate, $\mathcal{L}^{n,\ell}(\cdot)$ are made up of the inflow boundary condition, the local truncation error in time direction, and the linear combinations of approximation errors at different time stages.

For notation's simplicity, we would like to write the linear form in a uniform construction

$$\mathcal{L}^{n,\ell}(v_h) = \frac{1}{\tau}(\zeta^{n,\ell}, v_h) - \mathcal{H}^c(\eta^{n,\ell}, v_h) - \beta\vartheta^{n,\ell}v_{h,0}^+, \quad \ell = 0, 1, 2. \tag{53}$$

Each term on the right-hand side of (53) is defined with different meaning. The first term represents the time discretization, since $\zeta^{n,\ell}$ is composed of the time-marching of approximation error and the truncation error in time, say,

$$\begin{aligned}&\zeta^{n,0} = \eta^{n,1} - \eta^n, \qquad \zeta^{n,1} = 4\eta^{n,2} - 3\eta^n - \eta^{n,1},\\ &\zeta^{n,2} = (3\eta^{n+1} - \eta^n - 2\eta^{n,2} + 3\mathcal{E}^n)/2.\end{aligned} \tag{54}$$

The second one represents the DG spatial error without the inflow boundary condition, and the last one represents the inflow boundary error, where

$$\vartheta^{n,\ell} = u^{(\ell)}(a, t^n) - g^{n,\ell}, \quad \ell = 0, 1, 2. \tag{55}$$

Here, $u^{(\ell)}$ is the reference functions defined by (47), and $g^{n,\ell}$ is the setting of the inflow boundary at each time stage.

4.4.2 Estimates the Linear Forms (53)

We would like to estimate $\xi^{n,\ell}$ by using Lemma 4.2, since error equation (52) is given in the form of black box (27). To this end, we need to determine those bounded constants in assumption (30) for the linear form (53). This depends on the estimates for the stage errors and their combinations.

Below we would like to assume the exact solution of model problem (1) is smooth enough such that u, u_t and $u_{tt} \in L^\infty(0, T; H^{k+1})$. It follows from (6) that

$$\|\eta^n\| + \|\eta^{n,1}\| + \|\eta^{n,2}\| \le Ch^{k+1}, \quad \forall n : n\tau \le T; \tag{56}$$

where the constant $C > 0$ is independent of n, h and τ. Next we estimate the time-marching of stage error, by the fact that the projection $\mathbb{Q}_h$ is linear and independent of time.

For any given constants $\{c_\ell\}_{\ell=0}^2$ satisfying $\sum_{\ell=0}^2 c_\ell = 0$, denote $L_\star^n(u) \equiv \sum_{\ell=0}^2 c_\ell u^{(\ell)}(t^n)$. From definition (47) it follows that $L_\star^n(u)$ is equal to a certain linear combination of $u^{(0)}(t^n)$, $\tau u_t^{(0)}(t^n)$ and $\tau u_t^{(1)}(t^n)$. Consequently $L_\star^n(u) \in H^{k+1}$. The corresponding projection error is

$\sum_{\ell=0}^{2} c_\ell \eta^{n,\ell}$. Then it follows from (6) that $\|\sum_{\ell=0}^{2} c_\ell \eta^{n,\ell}\| \le Ch^{k+1}\tau$, where the constant $C>0$ solely depends on $\|u_t\|_{L^\infty(H^{k+1})}$ and $\|u_{tt}\|_{L^\infty(H^{k+1})}$.

Further denote $\tilde{L}^n_\star(u) \equiv u(x,t^{n+1}) - u(x,t^n)$, which satisfies $\|\tilde{L}^n_\star(u)\|_{k+1} \le \tau\|u_t\|_{L^\infty(H^{k+1})}$. Noticing the projection error is $\eta^{n+1}-\eta^n$, it follows from (6) that $\|\eta^{n+1}-\eta^n\| \le Ch^{k+1}\tau$, where the constant $C>0$ solely depends on $\|u_t\|_{L^\infty(H^{k+1})}$.

It follows from Lemma 4.3 that $\|\mathcal{E}^n\| = \mathcal{O}(\tau^4)$ holds uniformly. By (54), the definition of $\zeta^{n,\ell}$, we have

$$\|\zeta^{n,\ell}\| \le C(h^{k+1}\tau + \delta_{2\ell}\tau^4), \quad \ell = 0,1,2, \tag{57}$$

where the constant $C>0$ is independent of n,h, and τ. Here $\delta_{2\ell}$ is the usual Kronecker symbol, i.e., $\delta_{2\ell}=1$ if $\ell=2$; otherwise, $\delta_{2\ell}=0$.

Lemma 4.4 *There exist positive constants C independent of n,h,τ and v_h, such that*

$$|\mathcal{L}^{n,\ell}(v_h)| \le |\beta\vartheta^{n,\ell}| [\![v_h]\!] + C\big[h^{k+1} + \delta_{2\ell}\tau^3\big]\|v_h\|, \quad \forall v_h \in V_h,\ \ell = 0,1,2, \tag{58a}$$

$$|\mathbb{D}_\ell\mathcal{L}^{n}(v_h)| \le |\beta\mathbb{D}_\ell\vartheta^{n}| [\![v_h]\!] + C\big[h^{k+1} + \delta_{2\ell}\tau^3\big]\|v_h\|, \quad \forall v_h \in V_h,\ \ell = 1,2, \tag{58b}$$

where $\vartheta^{n,\ell} = u^{(\ell)}(a,t^n) - g^{n,\ell}$.

Proof The estimates are almost same, so we only prove estimate (58a) for $\ell=2$. Since (57), we get $|\frac{1}{\tau}(\zeta^{n,2},v_h)| \le C(h^{k+1}+\tau^3)\|v_h\|$. Same as (23), the projection $\mathbb{R}_h$ implies $\mathcal{H}^c(\eta^{n,2},v_h)=0$. Further, we also have $|\beta\vartheta^{n,2}v^+_{h,0}| \le |\beta\vartheta^{n,2}| [\![v_h]\!]$. Now we complete the proof of this lemma. □

As a direct conclusion of Lemma 4.4, the bounded constants in assumption (30) for the linear form (53) are determined as

$$\rho_{n,\ell} = |\beta\vartheta^{n,\ell}|, \qquad \sigma_{n,\ell} = C\big[h^{k+1}+\delta_{2\ell}\tau^3\big], \quad \ell=0,1,2; \tag{59a}$$

$$\pi_{n,\ell} = |\beta\mathbb{D}_\ell\vartheta^{n}|, \qquad \theta_{n,\ell} = C\big[h^{k+1}+\delta_{2\ell}\tau^3\big], \quad \ell=1,2. \tag{59b}$$

By noticing the definitions (35) and (38), we use triangle inequality to both $\|\mathbb{D}_\ell\xi^n\|$ and $|\mathbb{D}_\ell\vartheta^n|$ for $\ell=1,2$, and then have the following error estimate for the linear form (53) such that

$$\Theta_1^n + \Theta_2^n \le C(h^{2k+2}+\tau^6) + \mathcal{E}_b^n + \mathcal{E}_{\mathrm{FE}}^n. \tag{60a}$$

Here, $\mathcal{E}_b^n$ and $\mathcal{E}_{\mathrm{FE}}^n$ are the boundary error and the global L^2-error, respectively. They are given as

$$\mathcal{E}_b^n = C\big[(\vartheta^n)^2 + (\vartheta^{n,1})^2 + (\vartheta^{n,2})^2\big], \qquad \mathcal{E}_{\mathrm{FE}}^n = C\big[\|\xi^n\|^2 + \|\xi^{n,1}\|^2 + \|\xi^{n,2}\|^2\big]. \tag{60b}$$

Before we use Lemma 4.2 to obtain the optimal a priori error estimate, we need to estimate the above two quantities in the next two subsections.

4.4.3 Estimate the Boundary Errors

The natural expectation is that the boundary error does not destroy the accurate of the RKDG3 algorithm for periodic boundary condition, namely, the boundary error should be bounded by the form $\vartheta^{n,\ell} = \mathcal{O}(h^{k+1}+\tau^3)$ for $\ell=0,1,2$. Below we will discuss $\vartheta^{n,\ell}$ under the different setting of boundary condition.

Recall that, in the explicit TVDRK3 time-marching, the intermediate solutions are given at the certain time stages. Therefore, an immediate setting is to take the intermediate boundary condition $g^{n,\ell} = g^{n,\ell}_{\text{Exact}}$ as the exact boundary condition at these time stages, such as

$$g^{n}_{\text{Exact}} = g(t^n), \qquad g^{n,1}_{\text{Exact}} = g(t^n+\tau), \qquad g^{n,2}_{\text{Exact}} = g\left(t^n + \frac{1}{2}\tau\right). \tag{61}$$

To verify whether there holds $\vartheta^{n,\ell} = u^{(\ell)}(a,t^n) - g^{n,\ell}_{\text{Exact}} = \mathcal{O}(h^{k+1}+\tau^3)$, we express the reference value at the inflow boundary in term of the exact boundary information $g(t)$. Since $u(a,t) = g(t)$ along the inflow boundary, it is easy to see that $u^{(0)}(a,t^n) = g(t^n)$, and

$$u^{(1)}(a,t^n) = g(t^n) + \tau g'(t^n), \tag{62a}$$

$$u^{(2)}(a,t^n) = g(t^n) + \frac{1}{2}\tau g'(t^n) + \frac{1}{4}\tau^2 g''(t^n). \tag{62b}$$

Obviously $\vartheta^n = 0$, and the simple Taylor expansions for $g^{n,1}_{\text{Exact}}$ and $g^{n,2}_{\text{Exact}}$ show that

$$\vartheta^{n,1} = u^{(1)}(a,t^n) - g^{n,1}_{\text{Exact}} = \frac{1}{2}\tau^2 g''(t^n) + \mathcal{O}(\tau^3), \tag{63a}$$

$$\vartheta^{n,2} = u^{(2)}(a,t^n) - g^{n,2}_{\text{Exact}} = \frac{1}{8}\tau^2 g''(t^n) + \mathcal{O}(\tau^3). \tag{63b}$$

If $g''(t^n) \neq 0$, this so-called exact treatment for the inflow boundary condition leads to a reduction of accuracy that the third order temporal error decrease to two order.

To avoid the reduction of accuracy, in this paper we will consider two correction techniques, which are referred to as strategies (I) and (II).

Strategy (I) is to take the local solution of each Runge-Kutta time-marching, namely

$$g^{n,\ell} = g^{n,\ell}_{\text{I}} = u^{(\ell)}(a,t^n), \quad \ell = 0,1,2, \tag{64}$$

where $u^{(\ell)}(a,t^n)$, $\ell = 0,1,2$, are given by (62). From (70), we can see that this strategy (I) does not bring any error at the inflow boundary. However, this treatment is not easy to implement in a uniform coding.

On contract with strategy (I), strategy (II) is to take the global solution of Runge-Kutta time marching. That is to say, we define the following ordinary differential equation

$$\begin{cases} w_t(t) = g'(t), & t > 0 \\ w(0) = g(0), \end{cases} \tag{65}$$

in which the exact solution is just the given boundary condition $g(t)$. Next we apply the same explicit TVDRK3 algorithm to get the approximation $w^{n,\ell}$ at every time stages. Then we take the inflow boundary condition as $g^{n,\ell} = g^{n,\ell}_{\text{II}} = w^{n,\ell}$. This correction technique has been proposed by Carpenter and Gottlieb [1].

Now we check the order of the boundary error $\vartheta^{n,\ell}$ for $\ell = 0,1,2$. It is well known that the explicit TVDRK3 algorithm has error estimate $g(t^n) - w^n = \mathcal{O}(\tau^3)$. Consequently, $\vartheta^n = u^{(0)}(a,t^n) - g^n_{\text{II}} = g(t^n) - w^n = \mathcal{O}(\tau^3)$. Hence it follows from (62a) and (62b), respectively, that

$$\begin{aligned} \vartheta^{n,1} &= u^{(1)}(a,t^n) - g^{n,1}_{\text{II}} = u^{(1)}(a,t^n) - w^{n,1} \\ &= [g(t^n) + \tau g'(t^n)] - [w^n + \tau g'(t^n)] = g(t^n) - w^n = \mathcal{O}(\tau^3), \end{aligned} \tag{66a}$$

$$\begin{aligned}\vartheta^{n,2} &= u^{(2)}(a,t^n) - g_{\mathrm{II}}^{n,2} = u^{(2)}(a,t^n) - w^{n,2}\\ &= \left[g(t^n) + \frac{1}{2}\tau g'(t^n) + \frac{1}{4}\tau^2 g''(t^n)\right] - \left[\frac{3}{4}w^n + \frac{1}{4}w^{n,1} + \frac{1}{4}\tau g'(t^{n+1})\right]\\ &= \left[g(t^n) + \frac{1}{2}\tau g'(t^n) + \frac{1}{4}\tau^2 g''(t^n)\right] - \left[\frac{3}{4}w^n + \frac{1}{4}(w^n + \tau g'(t^n)) + \frac{1}{4}\tau g'(t^{n+1})\right]\\ &= g(t^n) - w^n + \frac{1}{4}\tau^2 g''(t^n) + \frac{1}{4}\tau\big[g'(t^n) - g'(t^{n+1})\big] = \mathcal{O}(\tau^3),\end{aligned} \tag{66b}$$

where in the last step in (66b) we have used Taylor's expansion for $g'(t^{n+1})$ at the point $t = t^n$.

The above analysis show that although the strategy (II) introduces an error at each time stage, but this error is more or less the same as the temporal error in the interior domain. Therefore, DG approximations near the boundary will be better than with the exact boundary values.

4.4.4 Optimal Error Estimate

Now we would like to obtain a priori optimal error estimate if we use the strategy (I) and (II) to cope with the inflow boundary condition to ensure

$$\mathcal{E}_b^n \le C(h^{2k+2} + \tau^6), \tag{67}$$

where the bounded constant $C > 0$ is dependent of n, h and τ.

Noticing Lemma 4.2 and (60a), we have to only estimate the second term $\mathcal{E}_{\mathrm{FE}}^n$ in (60b). For this purpose, we give the following lemma.

Lemma 4.5 *There exists constant $C > 0$ independent of h and τ, such that*

$$\|\xi^{n,1}\|^2 \le C\big[\|\xi^n\|^2 + \tau|\beta|(\vartheta^{n,0})^2 + \tau^2 h^{2k+2}\big], \tag{68a}$$

$$\|\xi^{n,2}\|^2 \le C\big[\|\xi^n\|^2 + \|\xi^{n,1}\|^2 + \tau|\beta|(\vartheta^{n,1})^2 + \tau^2 h^{2k+2}\big]. \tag{68b}$$

Proof The proofs for the above conclusions are same, so we only prove the first one now. To do that, we take the test function $v_h = \xi^n$ in (26a). Then we use the continuity inequality (16c) in Lemma 3.1, and the inverse property (ii), to get that

$$\begin{aligned}\|\xi^{n,1}\|^2 &\le \|\xi^n\|\|\xi^{n,1}\| + \gamma_0\lambda\|\xi^n\|\|\xi^{n,1}\| + \tau\big[\rho_{n,0}[\![\xi^{n,1}]\!] + \sigma_{n,0}\|\xi^{n,1}\|\big]\\ &\le \big[(1+\gamma_0\lambda)\|\xi^n\| + \tau\mu_2 h^{-\frac{1}{2}}\rho_{n,0} + \tau\sigma_{n,0}\big]\|\xi^{n,1}\|.\end{aligned}$$

By inserting (59), and dropping the term $\|\xi^{n,1}\|$ on both side, we obtain conclusion (68a), where the bounded constant $C > 0$ depends on λ. It completes the proof of this lemma. □

As a direct corollary of Lemma 4.5, we can bound $\mathcal{E}_{\mathrm{FE}}^n$ in form

$$\mathcal{E}_{\mathrm{FE}}^n \le C\big[\|\xi^n\|^2 + \tau|\beta|(\vartheta^{n,0})^2 + \tau|\beta|(\vartheta^{n,1})^2 + \tau^2 h^{2k+2}\big] \le C\big[\|\xi^n\|^2 + \tau\mathcal{E}_b^n + \tau^2 h^{2k+2}\big], \tag{69}$$

which also depends on the boundary error $\mathcal{E}_b^n$. Applying Lemma 4.2, as well as (60a), (67) and (69), we have for any $n : n\tau \le T$ that

$$3\|\xi^{n+1}\|^2 - 3\|\xi^n\|^2 + 2|\beta|[\![\xi^{n,2}]\!]^2\tau \le C\big[\|\xi^n\|^2 + h^{2k+2} + \tau^6\big]\tau, \tag{70}$$

since $\tau < 1$, where the bounded constant $C > 0$ is independent of n, h and τ.

As we have shown before, there holds $\|\xi^0\| \le Ch^{k+1}$. Then an application of discrete Gronwall's inequality for (70), as well as the approximation property (56), yields that

$$\|e^n\|^2 \le C(h^{2k+2} + \tau^6), \quad \forall n : n\tau \le T, \tag{71}$$

where the bounded constant $C > 0$ is independent of n, h and τ. Now we obtain the optimal error estimate for the RKDG3 algorithm with both strategies (I) and (II) for the inflow boundary condition. We conclude these results in the following theorem.

Theorem 4.2 *Let u_h be the numerical solution of the fully discrete version* (27) *of the DG method coupled with the explicit TVDRK*3 *time marching, where the inflow boundary condition is approximated by either strategy* (I) *or strategy* (II), *and the finite element space V_h is made up of piecewise polynomials with degree $k \ge 1$, defined on the regular triangulations of $I = (a, b)$.*

Let u is the exact solution of problem (1), *which is sufficiently smooth such that $u, u_t, u_{tt} \in L^\infty(0, T; H^{k+1})$ and $u_{tttt} \in L^\infty(0, T; L^2)$. Then there exists a positive constant C independent of h and τ, such that*

$$\max_{n\tau \le T} \|u(t^n) - u_h^n\| \le C(h^{k+1} + \tau^3), \tag{72}$$

if the CFL number λ is small enough such that $P(\lambda) < 0$, where $\lambda = \mu|\beta|\tau h^{-1}$.

Remark 4.1 This error estimate as well as the above stability result can be extended easily to the multidimensional problem. There are no essential difficulties since our analysis is based on the general energy analysis, which can be applied to arbitrary mesh with varying sharp and size.

Remark 4.2 If we take the standard L^2-projection, we will lost a half order and get the quasi-optimal error estimate, because of the approximation property (6) and the loss of exact collocation at the downwind endpoint of each element.

5 Numerical Experiments

To illustrate the above analysis numerically, we use the fully discrete version (27) of the DG method coupled with the explicit TVDRK3 time-marching to solve $u_t + u_x = 0$ with the exact solution $u(x, t) = \sin(x - t)$. The spatial domain is $I = (0, 4\pi)$, and the initial value is given by the exact solution at $t = 0$. Two types of boundary conditions will be considered in our test. One is the periodic condition $u(0, t) = u(4\pi, t)$, and the other is the inflow boundary condition $u(0, t) = \sin(-t)$.

In our test, we always take the finite element space as the piecewise quadratic polynomials, and set the final time $T = 20$. The time step is always taken as $\tau = 0.16h$. In Table 1 we list the error and the approximating error order in L^2-norm and L^∞-norm, respectively, for the periodic boundary condition and for three kinds of treatments of the inflow boundary.

One can see the optimal error order for periodic boundary condition, and the reduction of accuracy if we take the exact approximation (64) on the inflow boundary. For the strategies (I) and (II) the error order is restored to be optimal in L^2-norm and L^∞-norm, respectively, However, there are no obvious difference in numerical results for the both correction techniques.

Table 1 Errors and orders for different boundary conditions and different treatments for inflow boundary condition. Here N is the number in the partition of $(0, 4\pi)$

	N	L^∞-error	L^∞-order	L^2-error	L^2-order
Periodic b.c.	40	5.10e−4		4.81e−4	
	80	6.50e−5	2.97	1.64e−5	3.03
	160	8.18e−6	2.99	7.35e−6	3.01
	320	1.03e−6	2.99	9.17e−7	3.00
	640	1.29e−7	3.00	1.15e−7	3.00
	1280	1.61e−8	3.00	1.43e−8	3.00
Exact for inflow b.c.	40	6.14e−4		4.13e−4	
	80	1.04e−4	2.56	5.17e−5	3.00
	160	2.10e−5	2.31	6.55e−6	2.98
	320	6.18e−6	1.77	8.88e−7	2.88
	640	1.39e−6	2.15	1.19e−7	2.91
	1280	3.00e−7	2.22	1.59e−8	2.89
Local for inflow b.c.	40	5.14e−4		4.02e−4	
	80	6.45e−5	3.00	5.01e−5	3.01
	160	8.08e−6	3.00	6.26e−6	3.00
	320	1.01e−6	3.00	7.82e−7	3.00
	640	1.27e−7	3.00	9.77e−8	3.00
	1280	1.59e−8	3.00	1.22e−8	3.00
Global for inflow b.c.	40	5.04e−4		4.03e−4	
	80	6.31e−5	3.00	5.02e−5	3.01
	160	7.92e−6	3.00	6.26e−6	3.00
	320	9.93e−7	3.00	7.83e−7	3.00
	640	1.24e−7	3.00	9.77e−8	3.00
	1280	1.55e−8	3.00	1.22e−8	3.00

6 Concluding Remarks

In this paper we discuss the detailed treatment for inflow boundary condition, under the general RKDG framework for linear conservation law. When the explicit TVDRK3 (one type of high order) time-marching is used, the boundary condition must been defined in a reasonable form to avoid the reduction of accuracy. Both theoretical analysis and numerical experiment show strategy (I) and (II) are good candidate techniques to restore the algorithm to be optimal order error estimate. In the further work we will extend our analysis on these correction techniques to nonlinear conservation laws, in which the approximating skew-symmetric property (16a) does not hold again. Also we will consider the other types of equations with high order derivatives.

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