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On the Suboptimality of the p-Version Interior Penalty Discontinuous Galerkin Method

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Abstract We address the question of the rates of convergence of the p-version interior penalty discontinuous Galerkin method (p-IPDG) for second order elliptic problems with non-homogeneous Dirichlet boundary conditions. It is known that the p-IPDG method admits slightly suboptimal a-priori bounds with respect to the polynomial degree (in the Hilbertian Sobolev space setting). An example for which the suboptimal rate of convergence with respect to the polynomial degree is both proven theoretically and validated in practice through numerical experiments is presented. Moreover, the performance of p-IPDG on the related problem of p-approximation of corner singularities is assessed both theoretically and numerically, witnessing an almost doubling of the convergence rate of the p-IPDG method.

Keywords Discontinuous Galerkin method · Interior penalty · A priori error estimation · p-version · Suboptimality

1 Introduction

Discontinuous Galerkin (DG) methods for elliptic problems have gained popularity in recent years. Their great flexibility in the design of finite element methods make them good contenders in the area of hp-adaptive algorithms. Moreover, DG methods have shown to

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be accurate and stable numerical methods for the numerical approximation of convection-dominated convection-diffusion problems (see, e.g., [5] and the references therein). Historically, DG methods incorporate ideas from the classical Nitsche's method for the treatment of non-homogeneous Dirichlet boundary conditions [10] and from the penalty method [1].

To the best of our knowledge, the sharpest known general error bounds (in the Hilbertian Sobolev space setting) for the hp-version interior penalty DG method for second-order elliptic PDEs are due to Rivière, Wheeler and Girault [11] and Houston, Schwab and Süli [9]; when the error is measured in the (natural) energy norm, the a-priori bounds are optimal with respect to the meshsize h but are suboptimal with respect to the polynomial degree p by half an order of p.

Optimal error bounds for the hp-version interior penalty DG method are known in the case where the underlying discontinuous Galerkin finite element space admits an H^1-conforming subspace of the same polynomial order up to the boundary; see, e.g., [6, Theorem 8.2] and the subsequent discussion therein. In the case where the discontinuous Galerkin finite element space does *not* admit such an H^1-conforming subspace (e.g., when the mesh is highly irregular or when the Dirichlet boundary conditions are not represented exactly as traces of finite element functions), hp-optimal error bounds in the energy norm have been derived in [7] for the case of quadrilateral elements, provided the analytical solution admits additional regularity in the framework of augmented Sobolev spaces. Recently, in [13], a variation of the interior penalty DG method (which includes an additional penalization term resembling the local discontinuous Galerkin method) is proposed and hp-optimal error bounds are proven for the case of homogeneous Dirichlet boundary conditions.

In this work, we focus on the p-version interior penalty discontinuous Galerkin finite element method (p-IPDG), addressing the question of the rates of convergence for second order elliptic problems with *non-homogeneous* Dirichlet boundary conditions, when the underlying analytical solution belongs to a (standard) Hilbertian Sobolev space H^k. More specifically, we present an example for which the suboptimal rate of convergence with respect to the polynomial degree is both proven theoretically and validated through numerical experiments; hence, the known a-priori bounds from the literature [9, 11] are sharp, i.e., the p-IPDG method is indeed suboptimal by half an order of p.

Furthermore, we investigate the question of convergence rates for the p-IPDG in the case where the exact solution of the Dirichlet problem admits corner singularities of type r^α, $\alpha > 0$, at a vertex of the computational domain Ω (where r is the distance from the vertex). For the standard conforming p-version finite element method applied to the Poisson problems with corner singularities, it is well-known that the convergence rate is twice that predicted by standard a-priori bounds based on the regularity of the solution in Hilbertian Sobolev spaces [3]. Here, we show that a nearly order-doubling is also witnessed for the p-IPDG method through an analytical and numerical study.

The rest of this work is organised as follows. Section 2 contains the model problem and definition of the interior penalty discontinuous Galerkin method. In Sect. 3, the question of suboptimal rate of convergence for the p-IPDG method is studied. In Sect. 4, some approximation bounds on the p-convergence of the L^2-projection operator are presented, which are subsequently utilized in Sect. 5, where the convergence behaviour of p-IPDG on corner singularities is discussed. Section 6 contains some final comments.

2 Preliminaries

Let Ω be a bounded open polygonal domain in $\mathbb{R}^2$. We consider the Poisson problem

$$-\Delta u = f \quad \text{in } \Omega, \tag{2.1}$$

with $f \in L^2(\Omega)$ together with Dirichlet boundary conditions

$$u = g_{\mathrm{D}} \quad \text{on } \partial\Omega. \tag{2.2}$$

The norm of $L^2(\omega)$, for $\omega \subset \Omega$ will be denoted by $\|\cdot\|_{L^2(\omega)}$. We shall also denote by $H^s(\omega)$ the standard Hilbertian Sobolev space of index $s \geq 0$ of real-valued functions defined on $\omega \subset \Omega$ and by $\|\cdot\|_{H^s(\omega)}$ it corresponding norm. We shall also refer in passing to the notions of an *augmented Sobolev space* (see [7] for the definition of augmented Sobolev spaces) and to function spaces constructed via the real method of interpolation (see, e.g., [14]).

Let $\mathcal{T}$ be a subdivision of the polygonal domain Ω into disjoint open elements κ constructed via affine mappings $F_\kappa : \hat{\kappa} \to \kappa$ from some reference simplex or rectangle $\hat{\kappa}$, which are assumed to be constructed so as to ensure that the union of the closures of the elements $\kappa \in \mathcal{T}$ forms a covering of the closure of Ω, i.e., $\bar{\Omega} = \bigcup_{\kappa \in \mathcal{T}} \bar{\kappa}$.

Definition 2.1 Let $\mathbf{p} := (p_\kappa : \kappa \in \mathcal{T})$ be the vector containing the polynomial degrees of the elements in a given subdivision $\mathcal{T}$ as described above. We define the *finite element space* $S^{\mathbf{p}}$ with respect to $\mathcal{T}$ and $\mathbf{p}$ by

$$S^{\mathbf{p}} := \{u \in L^2(\Omega) : u|_\kappa \circ F_\kappa \in \mathcal{P}_{p_\kappa}(\hat{\kappa})\},$$

where $\mathcal{P}_p(\hat{\kappa})$ is the space of polynomials of degree at most p when $\hat{\kappa}$ is the reference simplex and of degree p in each variable, when $\hat{\kappa}$ is the reference square.

We shall assume throughout that the mesh is fixed with a meshsize vector $\mathbf{h} := (h_\kappa : \kappa \in \mathcal{T})$ and the local polynomial degree vector $\mathbf{p}$, with $p_\kappa \geq 1$ for each $\kappa \in \mathcal{T}$, varies. For $\mathbf{p}$ we assume *bounded local variation* as $p_\kappa \to \infty$ for convergence, i.e., there exists a constant $\rho \geq 1$, independent of $\mathbf{p}$, such that, for any pair of elements κ and κ' in $\mathcal{T}$ which share a side, we have $\rho^{-1} \leq p_\kappa / p_{\kappa'} \leq \rho$.

We denote by Γ the union of all open one-dimensional element faces associated with the subdivision $\mathcal{T}$. We also assume that Γ can be decomposed into two disjoint subsets $\partial\Omega$ and $\Gamma_{\mathrm{int}} := \Gamma \backslash \partial\Omega$.

Further, we introduce some trace operators. Let κ, κ' be two (generic) elements sharing an interface $e \subset \Gamma_{\mathrm{int}}$. Define the outward normal unit vectors n^+ and n^- on e corresponding to $\partial\kappa$ and $\partial\kappa'$, respectively. Let $q \in S^{\mathbf{p}}$ and $\phi \in [S^{\mathbf{p}}]^2$. Then, with $q^+ := q|_{e \cap \partial\kappa}$, $q^- := q|_{e \cap \partial\kappa'}$ and $\phi^+ := \phi|_{e \cap \partial\kappa}$, $\phi^- := \phi|_{e \cap \partial\kappa'}$, we set

$$\{q\}|_e := \frac{1}{2}(q^+ + q^-), \qquad \{\phi\}|_e := \frac{1}{2}(\phi^+ + \phi^-),$$
$$[\![q]\!]|_e := q^+ n^+ + q^- n^-.$$

If $e \subset \partial\Omega$, we define

$$\{q\}|_e := q^+, \qquad \{\phi\}|_e := \phi^+, \qquad [\![q]\!]|_e := q^+ n,$$

where n denotes the outward normal unit vector to $\partial\Omega$.

The *discontinuous Galerkin finite element method* for the problem (2.1), (2.2) reads:

$$\text{Find} \quad u_{\mathrm{DG}} \in S^{\mathbf{p}} \quad \text{such that } B(u_{\mathrm{DG}}, v) = l(v) \quad \forall v \in S^{\mathbf{p}}, \tag{2.3}$$

where

$$B(u,v) := \sum_{\kappa \in \mathcal{T}} \int_\kappa \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Gamma (\theta [\![u]\!] \cdot \{\nabla v\} - \{\nabla u\} \cdot [\![v]\!] + \sigma [\![u]\!] \cdot [\![v]\!]) \,\mathrm{d}s, \tag{2.4}$$

and

$$l(v) := \int_\Omega f v \,\mathrm{d}x + \int_{\partial\Omega} g_{\mathrm{D}} (\theta \nabla v \cdot n + \sigma v) \,\mathrm{d}s,$$

with $\theta \in \{-1, 1\}$, and a function σ to be defined later. If $\theta = -1$ we shall refer to the method as the *symmetric version*, and if $\theta = 1$ we shall speak about (2.3) as the *non-symmetric* version of the IPDG method. In the simpler case when $\Gamma_{\mathrm{int}} = \emptyset$ (i.e., we have a one-element mesh), we recover the p-version of the classical *Nitsche's method* for the imposition of non-homogeneous Dirichlet boundary conditions.

Related to the bilinear form, we define the (natural) DG-energy norm:

$$|\!|\!| w |\!|\!| := \left(\sum_{\kappa \in \mathcal{T}} \| \nabla w \|^2_{L^2(\kappa)} + \| \sqrt{\sigma} [\![w]\!] \|^2_{L^2(\Gamma)} \right)^{1/2},$$

for any function $w \in L^2(\Omega)$ such that $w|_\kappa \in H^1(\kappa)$ for all $\kappa \in \mathcal{T}$.

3 The Convergence of the p-IPDG Method

For the above DG method the following error bound holds, [7, 9, 11]:

Theorem 3.1 *Let Ω be a polygonal domain, $\mathcal{T}$ a regular subdivision of Ω into shape-regular elements. We define*

$$\sigma := C_\sigma \left\{ \frac{\mathbf{p}^2}{\mathbf{h}} \right\},$$

for some $C_\sigma > 0$ (large enough) independent of $\mathbf{h}$ and of $\mathbf{p}$, but dependent on ρ (which measures the local variation of the polynomial degree). If $u \in H^1(\Omega)$ is such that $u|_\kappa \in H^{k_\kappa+1}(\kappa)$ for all $\kappa \in \mathcal{T}$ then the solution $u_{\mathrm{DG}} \in S^{\mathbf{p}}$ satisfies:

$$|\!|\!| u - u_{\mathrm{DG}} |\!|\!|^2 \lesssim \sum_{\kappa \in \mathcal{T}} \frac{h_\kappa^{2s_\kappa}}{p_\kappa^{2k_\kappa - 1}} \| u \|^2_{H^{k_\kappa+1}(\kappa)}, \tag{3.1}$$

with $1 \le s_\kappa \le \min\{p_\kappa, k_\kappa\}$, $p_\kappa \ge 1$.

(Here and in the remainder of this work $A \lesssim B$ and $A \gtrsim B$ is used instead of $A \le CB$ and $A \ge CB$, respectively, for some positive generic constant C independent of $\mathbf{p}$.) Henceforth, we assume that σ is given by the formula in Theorem 3.1.

Hence, in particular, assuming a fixed mesh and a uniform polynomial degree $p_\kappa = p$, we conclude that for $u \in H^{k+1}(\Omega)$ and for $1 \le s \le \min\{p, k\}$, we have

$$|\!|\!| u - u_{\mathrm{DG}} |\!|\!| \lesssim p^{-k+1/2} \| u \|_{H^{k+1}(\Omega)}, \tag{3.2}$$

i.e., the p-IPDG method (and its special case when $\Gamma_{\text{int}} = \emptyset$, the Nitsche's method) converge at a suboptimal rate with respect to the polynomial degree p, by half an order of p.

In some cases it is possible to construct a non-trivial H^1-conforming subspace of the finite element space $S^{\mathbf{p}}$ up to the boundary with the same local polynomial degrees as $S^{\mathbf{p}}$, thereby facilitating the existence of an H^1-conforming interpolant [3, 12] of the analytical solution onto the finite element space, with hp-optimal convergence properties. Using this interpolant in the error analysis of the IPDG method, one can recover hp-optimal bounds (see, e.g., [6, Theorem 8.2] and the subsequent discussion therein). For instance, when the Dirichlet data g_{D} can be represented exactly as traces of finite element functions from $S^{\mathbf{p}}$ and the mesh contains simple hanging nodes (i.e., one hanging node per edge), a-priori error bounds of the form

$$|||u - u_{\text{DG}}||| \lesssim p^{-k} \|u\|_{H^{k+1}(\Omega)}, \tag{3.3}$$

for $u \in H^{k+1}(\Omega)$, have been shown [6]. This provides motivation to seek the cause of the potential p-suboptimality in the general error bound (3.2) to boundary effects.

Remark 3.2 The hp-optimal rate of convergence for general Dirichlet boundary conditions can be recovered (when the mesh consists of quadrilateral elements), if we make additional regularity assumptions on the analytical solutions, namely, by assuming that it belongs element-wise to an augmented Sobolev space; we refer to [7] for more details.

Example 3.3 We consider the boundary-value problem

$$-\Delta u = f \quad \text{in } \Omega := [-1, 1] \times [0, 2]$$

with Dirichlet boundary conditions and f such that

$$u(x, y) = (x^2 + y^2)^{\alpha/2},$$

for $\alpha \geq 1$ with $\alpha \notin 2\mathbb{N}_0$. We approximate u using the p-version IPDG method on a fixed regular mesh $\mathcal{T}$ (i.e., not containing any hanging nodes), which is constructed so that the origin $(0, 0)$ is situated at the midpoint of the face of an element; we denote by u_{DG} the approximation of u by the p-IPDG method.

The key point of the setup of Example 3.3 is that the singularity is *not* located at a vertex of mesh, and we have $u \in H^{\alpha+1-\epsilon}(\Omega)$, for all $\epsilon > 0$. Therefore, the bound (3.2) implies

$$|||u - u_{\text{DG}}||| \lesssim p^{-\alpha+1/2+\epsilon},$$

for all $\epsilon > 0$, as $p \to \infty$. The sharpness of this bound is settled by the following result, concluding that the p-IPDG is indeed suboptimal in p by half an order of p.

Proposition 3.4 *For the Dirichlet problem described in Example 3.3 we have*

$$|||u - u_{\text{DG}}||| \gtrsim p^{-\alpha+1/2},$$

as $p \to \infty$.

Proof We denote by

$$E_p(v) := \inf_{v_p \in \mathcal{P}_p([-1,1])} \|v - v_p\|_{L^2(-1,1)},$$

i.e., the best approximation in the L^2-norm of a function $v \in L^2([-1,1])$ by univariate polynomials of degree p on the interval $[-1,1]$. Theorem 9 of [8] implies that for functions of the form $v = |x|^\alpha$ for $\alpha > 0$ and $\alpha \notin 2\mathbb{N}_0$, we have

$$E_p(v) \gtrsim p^{-\alpha-1/2}, \tag{3.4}$$

for some constant $C > 0$. Moreover, it is a straightforward matter to see (cf. [8, Lemma 2]) that a rescaling of the domain from $[-1,1]$ to $[-h/2, h/2]$, for some $h > 0$ has only the effect of altering the constant in (3.4) (since we have assumed a fixed mesh).

Let κ be the element whose boundary contains the origin $(0,0)$. We consider the part of the boundary $\partial\kappa_1 := [-h_\kappa/2, h_\kappa/2] \times \{0\}$ on which we have $u(x,0) = |x|^\alpha$. The bound (3.4) then implies that in this case we have

$$\|u - u_{\mathrm{DG}}\|_{L^2(\partial\kappa_1)} \gtrsim p^{-\alpha-1/2},$$

for some constant $C > 0$ independent of p. Therefore, recalling the definition of σ, we conclude that

$$\begin{aligned} |||u - u_{\mathrm{DG}}|||^2 &= \sum_{\kappa\in\mathcal{T}} \|\nabla(u - u_{\mathrm{DG}})\|^2_{L^2(\kappa)} + \|\sqrt{\sigma}(u - u_{\mathrm{DG}})\|^2_{L^2(\Gamma\setminus\partial\kappa_1)} \\ &\quad + \|\sqrt{\sigma}(u - u_{\mathrm{DG}})\|^2_{L^2(\partial\kappa_1)} \gtrsim p^{-2\alpha+1}. \end{aligned}$$ □

To investigate the setting of Example 3.3 numerically, we consider p-IPDG method with one element (i.e., $\Gamma_{\mathrm{int}} = \emptyset$); the p-convergence history is shown in Fig. 1. The error appears to oscillate in magnitude for even and for odd p. One possible reason behind this is the symmetry of the analytical solution (which is an even function); for low polynomial degrees p we observe that the even degree approximations are more accurate than the approximations with odd polynomial degree basis functions. We point out that quadruple precision arithmetic has been used in the numerical experiments of this work, along with a geometrically graded composite quadrature rules, graded towards the point $(0,0)$.

In Table 1, the error in the DG-norm is presented, grouped in even and odd polynomial degree approximations, along with the p-convergence rates $r(p)$ calculated as follows:

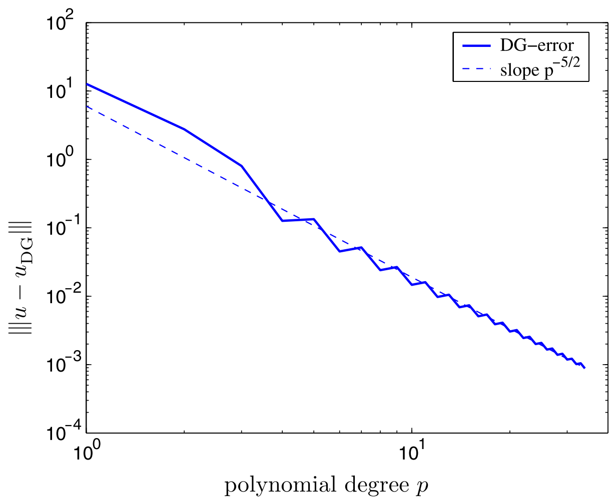

Fig. 1 Example 3.3: convergence history for the p-IPDG method for $\alpha = 3$

$$r(p) = -\frac{\log \mathrm{error}(p) - \log \mathrm{error}(p-2)}{\log p - \log(p-2)},$$

Table 1 Example 3.3: DG-norm errors and convergence rates for the p-IPDG method for $\alpha = 3$

p	$\|\|\|u - u_{\mathrm{DG}}\|\|\|$	$r(p)$	p	$\|\|\|u - u_{\mathrm{DG}}\|\|\|$	$r(p)$
2	2.7634+00	–	1	1.2728+01	–
4	1.2643−01	4.45	3	8.0330−01	2.51
6	4.5238−02	2.53	5	1.3381−01	3.51
8	2.4020−02	2.20	7	5.1617−02	2.83
10	1.4674−02	2.21	9	2.6717−02	2.62
12	9.7691−03	2.23	11	1.6029−02	2.55
14	6.9029−03	2.25	13	1.0535−02	2.51
16	5.0964−03	2.27	15	7.3725−03	2.50
18	3.8919−03	2.29	17	5.4024−03	2.48
20	3.0529−03	2.30	19	4.1008−03	2.48
22	2.4478−03	2.32	21	3.2011−03	2.47
24	1.9987−03	2.33	23	2.5562−03	2.47
26	1.6573−03	2.34	25	2.0801−03	2.47
28	1.3925−03	2.35	27	1.7198−03	2.47
30	1.1835−03	2.36	29	1.4414−03	2.47
32	1.0160−03	2.36	31	1.2223−03	2.47
34	8.7996−04	2.37	33	1.0473−03	2.47

Table 2 Example 3.3: $\|\nabla(u - u_{\mathrm{DG}})\|_{L^2(\Omega)}$-error and convergence rates for $\alpha = 3$

p	$\|\nabla(u - u_{\mathrm{DG}})\|_{L^2(\Omega)}$	$r(p)$	p	$\|\nabla(u - u_{\mathrm{DG}})\|_{L^2(\Omega)}$	$r(p)$
2	1.7896+00	–	1	8.7949+00	–
4	5.3593−02	5.06	3	4.4791−01	2.71
6	1.5150−02	3.12	5	4.8958−02	4.33
8	6.5414−03	2.92	7	1.4964−02	3.52
10	3.4303−03	2.89	9	6.5336−03	3.30
12	2.0212−03	2.90	11	3.4306−03	3.21
14	1.2898−03	2.91	13	2.0217−03	3.17
16	8.7268−04	2.93	15	1.2901−03	3.14
18	6.1753−04	2.94	17	8.7284−04	3.12
20	4.5281−04	2.94	19	6.1763−04	3.11
22	3.4175−04	2.95	21	4.5287−04	3.10
24	2.6419−04	2.96	23	3.4179−04	3.09
26	2.0839−04	2.96	25	2.6422−04	3.09
28	1.6724−04	2.97	27	2.0841−04	3.09
30	1.3623−04	2.97	29	1.6726−04	3.08
32	1.1243−04	2.98	31	1.3624−04	3.08
34	9.3853−05	2.98	33	1.1244−04	3.07

Table 3 Example 3.3: $\|\sqrt{\sigma}(u-u_{\rm DG})\|_{L^2(\partial\Omega)}$-error and convergence rates for $\alpha=3$

p	$\|\sqrt{\sigma}(u-u_{\rm DG})\|_{L^2(\partial\Omega)}$	$r(p)$	p	$\|\sqrt{\sigma}(u-u_{\rm DG})\|_{L^2(\partial\Omega)}$	$r(p)$
2	2.1056+00	–	1	9.2003+00	–
4	1.1451−01	4.20	3	6.6684−01	2.39
6	4.2626−02	2.44	5	1.2454−01	3.28
8	2.3113−02	2.13	7	4.9401−02	2.75
10	1.4267−02	2.16	9	2.5906−02	2.57
12	9.5577−03	2.20	11	1.5657−02	2.50
14	6.7814−03	2.23	13	1.0339−02	2.48
16	5.0211−03	2.25	15	7.2588−03	2.47
18	3.8426−03	2.27	17	5.3314−03	2.47
20	3.0191−03	2.29	19	4.0541−03	2.46
22	2.4238−03	2.30	21	3.1689−03	2.46
24	1.9812−03	2.32	23	2.5333−03	2.46
26	1.6442−03	2.33	25	2.0633−03	2.46
28	1.3825−03	2.34	27	1.7071−03	2.46
30	1.1757−03	2.35	29	1.4316−03	2.46
32	1.0098−03	2.36	31	1.2147−03	2.46
34	8.7494−03	2.36	33	1.0412−03	2.46

with $p=4,6,8,\dots$ or $p=3,5,7,\dots$, respectively, where error(p) denotes the approximation error in the respective (semi)norm when polynomial degree p basis functions are used.

Next, we investigate the p-convergence of the individual components of the DG-norm error. To this end, recalling that we work on a single-element mesh (and, therefore, $\Omega=\kappa$ and $\Gamma=\partial\kappa$), we study the p-convergence of the errors

$$\|\nabla(u-u_{\rm DG})\|_{L^2(\Omega)} \quad \text{and} \quad \|\sqrt{\sigma}(u-u_{\rm DG})\|_{L^2(\partial\Omega)},$$

as $p\to\infty$. The errors and the corresponding p-convergence rates are given in Tables 2 and 3, where the dominance of the term $\|\sqrt{\sigma}(u-u_{\rm DG})\|_{L^2(\partial\Omega)}$ over the term $\|\nabla(u-u_{\rm DG})\|_{L^2(\Omega)}$ is manifested; indeed the p-convergence behaviour of $\|\sqrt{\sigma}(u-u_{\rm DG})\|_{L^2(\partial\Omega)}$ essentially determines the p-convergence of the DG-norm error (cf. Table 1). We note that for odd p the convergence rate appears to be increasing towards the value 5/2.

4 Properties of the L^2-Projection Operator

In the present section, we refine the analysis of [9] of the properties of the L^2-projection operator. We start with an improvement of the one-dimensional result [9, Lemma 3.5].

Lemma 4.1 *Let $I=(-1,1)$ and denote $\Pi_p: L^2(I)\to\mathcal{P}_p$ the L^2-projection operator. Then*

$$|(\Pi_p u)(\pm 1)| \lesssim \|u\|_{L^2(I)}^{1/2}\|u\|_{H^1(I)}^{1/2} \quad \forall u\in H^1(I). \tag{4.1}$$

In particular, therefore,

$$|(\Pi_p u)(\pm 1)| \lesssim \|u\|_{B^{1/2}_{2,1}(I)} \quad \forall u \in B^{1/2}_{2,1}(I). \tag{4.2}$$

Here, the space $B^{1/2}_{2,1}(I) = (L^2(I), H^1(I))_{1/2,1}$ *is the interpolation space obtained by the real method* (*see, e.g.,* [14]).

Proof We only show the multiplicative inequality (4.1) since the bound (4.2) follows from [14, Lemma 25.3]. Also, we will only consider the case $p \geq 2$ and restrict out attention to evaluation at the right endpoint $+1$. Since $\|u\|^2_{L^\infty(I)} \lesssim \|u\|_{L^2(I)}\|u\|_{H^1(I)}$, it suffices to establish the inequality for $u - \Pi_p u$. Following [9, Lemma 3.5], we expand u and u' in Legendre series:

$$u = \sum_{i=0}^{\infty} u_i L_i, \quad u_i = \frac{2i+1}{2}\int_I u(x)\,\mathrm{d}x,$$
$$u' = \sum_{i=0}^{\infty} b_i L_i, \quad b_i = \frac{2i+1}{2}\int_I u'(x)\,\mathrm{d}x.$$

Orthogonality properties of the Legendre polynomials L_i imply (see the proof of [9, Lemma 3.5] for details)

$$u_i = \frac{b_{i-1}}{2i-1} - \frac{b_{i+1}}{2i+3}, \quad i \geq 2. \tag{4.3}$$

Since $L_i(1) = 1$ for all $i \in \mathbb{N}_0$, we get

$$(u - \Pi_p u)(1) = \sum_{i=p+1}^{\infty} u_i = \frac{b_p}{2p+1} + \frac{b_{p+1}}{2p+3},$$

and therefore

$$|(u - \Pi_p u)(1)|^2 = \left|\sum_{i=p+1}^{\infty} u_i\right|^2 \leq 2\left(\frac{b_p}{2p+1}\right)^2 + 2\left(\frac{b_{p+1}}{2p+3}\right)^2.$$

The terms in the last expression are now estimated using a telescoping sum:

$$\begin{aligned}
\left(\frac{b_p}{2p+1}\right)^2 &= \sum_{r=p}^{\infty}\left(\frac{b_r}{2r+1}\right)^2 - \left(\frac{b_{r+2}}{2(r+2)+1}\right)^2 \\
&= \sum_{r=p}^{\infty}\left(\frac{b_r}{2r+1} - \frac{b_{r+2}}{2(r+2)+1}\right)\left(\frac{b_r}{2r+1} + \frac{b_{r+2}}{2(r+2)+1}\right) \\
&= \sum_{r=p}^{\infty} u_{r+1}\left(\frac{b_r}{2r+1} + \frac{b_{r+2}}{2(r+2)+1}\right) \\
&= \sum_{r=p}^{\infty} u_{r+1}\frac{1}{\sqrt{2(r+1)+1}}\sqrt{2(r+1)+1}\left(\frac{b_r}{2r+1} + \frac{b_{r+2}}{2(r+2)+1}\right)
\end{aligned}$$

$$\lesssim \left(\sum_{r=p}^{\infty} \frac{1}{2(r+1)+1}|u_r|^2\right)^{1/2} \left(\sum_{r=p}^{\infty}(2r+1)\left(\frac{b_r}{2r+1}\right)^2\right)^{1/2}$$

$$\lesssim \|u\|_{L^2(-1,1)}\|u'\|_{L^2(-1,1)},$$

where we have used $\|u\|^2_{L^2(I)} = \sum_{i=0}^{\infty} |u_i|^2 \frac{2}{2i+1}$ and $\|u'\|^2_{L^2(I)} = \sum_{i=0}^{\infty} |b_i|^2 \frac{2}{2i+1}$. We therefore conclude $|(u - \Pi_p u)(1)|^2 \lesssim \|u\|_{L^2(I)}\|u'\|_{L^2(I)}$ as desired. □

Tensorization of Lemma 4.1 gives a result for the square.

Lemma 4.2 *Let $I = (-1, 1)$ and $S = I^2$. Denote by $\Pi_p : L^2(S) \to \mathcal{P}_p$ the $L^2(S)$-projection. Then,*

$$\|\Pi_p u\|^2_{L^2(\partial S)} \lesssim \|u\|_{L^2(S)}\|u\|_{H^1(S)} \quad \forall u \in H^1(S).$$

This implies

$$\|\Pi_p u\|_{L^2(\partial S)} \lesssim \|u\|_{B^{1/2}_{2,1}(S)} \quad \forall u \in B^{1/2}_{2,1}(S).$$

Proof The two-dimensional L^2-projection is the tensor product of one-dimensional projection operators: $\Pi_p = \Pi^x_p \circ \Pi^y_p$. Then, letting $\Gamma = I \times \{1\}$ be one edge of S:

$$\|\Pi_p u\|^2_{L^2(\Gamma)} = \int_{x\in I} \left|\Pi^y_p(\Pi^x_p u)(x, 1)\right|^2 \mathrm{d}x$$

$$\lesssim \int_{x\in I} \|\Pi^x_p u(x, \cdot)\|_{L^2(I)}(\|\Pi^x_p u(x, \cdot)\|_{L^2(I)} + \|\partial_y \Pi^x_p u(x, \cdot)\|_{L^2(I)})\,\mathrm{d}x$$

$$\lesssim \|u\|^2_{L^2(S)} + \int_{x\in I} \|\Pi^x_p u(x, \cdot)\|_{L^2(I)}\|\Pi^x_p \partial_y u(x, \cdot)\|_{L^2(I)}\,\mathrm{d}x.$$

For any $t > 0$, we can estimate further

$$\int_{x\in I} \|\Pi^x_p u(x, \cdot)\|_{L^2(I)}\|\Pi^x_p \partial_y u(x, \cdot)\|_{L^2(I)}\,\mathrm{d}x$$

$$\lesssim \int_{x\in I} t^{-1}\|\Pi^x_p u(x, \cdot)\|^2_{L^2(I)} + t\|\Pi^x_p \partial_y u(x, \cdot)\|^2_{L^2(I)}\,\mathrm{d}x$$

$$\lesssim t^{-1}\|u\|^2_{L^2(S)} + t\|\nabla u\|^2_{L^2(S)}.$$

Optimizing t gives

$$\int_{x\in I} \|\Pi^x_p u(x, \cdot)\|_{L^2(I)}\|\Pi^x_p \partial_y u(x, \cdot)\|_{L^2(I)} \lesssim \|u\|_{L^2(S)}\|u\|_{H^1(S)}.$$ □

Remark 4.3 1. Lemma 4.1 together with standard polynomial approximation properties implies for every $k > 1/2$:

$$|(u - \Pi_p u)(\pm 1)| \le C_k p^{-(k-1/2)}\|u\|_{H^k(-1,1)} \quad \forall u \in H^k(-1, 1),$$

for some constant $C_k > 0$, depending on the Sobolev index k.

2. Lemma 4.2 can be generalized to hypercubes in $\mathbb{R}^d$, $d \ge 2$.

We conclude this section with a statement about the approximation properties of the L^2-projector on squares.

Lemma 4.4 *Let $S = (-1, 1)^2$ and denote by $\Pi_p : L^2(S) \to \mathcal{P}_p$ the $L^2(S)$-projection. Then for all $u \in H^1(S)$:*

$$\|u - \Pi_p u\|_{H^1(S)} \lesssim \sqrt{p} \inf_{q \in \mathcal{P}_p} \|u - q\|_{H^1(S)},$$

$$\|u - \Pi_p u\|_{L^2(\partial S)} \lesssim \inf_{q \in \mathcal{P}_p} \|u - q\|_{B^{1/2}_{2,1}(S)} \lesssim \inf_{q \in \mathcal{P}_p} \|u - q\|^{1/2}_{L^2(S)} \|u - q\|^{1/2}_{H^1(S)}.$$

Proof The first estimate follows immediately from [4, Theorem 2.4], which states $\|\Pi_p u\|_{H^1(S)} \lesssim \sqrt{p}\|u\|_{H^1(S)}$ for all $u \in H^1(S)$. The second bound follows from Lemma 4.2. □

5 Convergence in the Presence of Corner Singularities

When the exact solution u admits a corner singularity of type r^α (here (r, θ) denote polar coordinates and the origin is assumed to be a vertex of Ω), it is known [3] that the *conforming* p-version finite element method applied to the problem (2.1), (2.2) obeys a bound of the form

$$\|\nabla(u - u_{p,\mathrm{conf}})\| \lesssim p^{-2\alpha},$$

where $u_{p,\mathrm{conf}}$ denotes the conforming p-version finite element approximation. This is the well-known *order-doubling phenomenon* of the p-version finite element method in the presence of corner singularities. (Note that $u = r^\alpha \in H^{\alpha+1-\epsilon}(\Omega)$ for all $\epsilon > 0$.)

In order to investigate the convergence behaviour of the p-IPDG method for problems with corner singularities, we consider the following example.

Example 5.1 We consider the boundary-value problem

$$-\Delta u = f \quad \text{in } \Omega := [0, 1] \times [0, 1]$$

with Dirichlet boundary conditions and f such that

$$u(x, y) = (x^2 + y^2)^{\alpha/2}.$$

We approximate u using the p-IPDG method on a fixed Cartesian mesh $\mathcal{T}$, denoting the p-IPDG solution by u_{DG}.

Before formulating the convergence result for the p-IPDG applied to Example 5.1, we state an approximation result:

Lemma 5.2 *Let $S = (0, 1)^2$ and a function $u \in H^1(S)$ be given of the form $u(r, \theta) = r^\alpha \phi(\theta)$, where (r, θ) are polar coordinates with respect to the origin. The function ϕ is assumed to be smooth. Let $\Pi_p^{H^1}$ and $\Pi_p^{L^2}$ denote the $H^1(S)$ and $L^2(S)$-projection operators onto $\mathcal{P}_p$. Then:*

$$\|u - \Pi_p^{H^1} u\|_{H^1(S)} + p\|u - \Pi_p^{H^1} u\|_{L^2(S)} + p^{1/2}\|u - \Pi_p^{H^1} u\|_{L^2(\partial S)} \lesssim p^{-2\alpha}, \quad (5.1)$$

$$p^{-1/2}\|u - \Pi_p^{L^2} u\|_{H^1(S)} + p^2\|u - \Pi_p^{L^2} u\|_{L^2(S)} + p^{3/4}\|u - \Pi_p^{L^2} u\|_{L^2(\partial S)} \lesssim p^{-2\alpha}. \quad (5.2)$$

Proof By [2, Theorem 2.6], we have $\|u - \Pi_p^{H^1} u\|_{H^1(S)} \lesssim p^{-2\alpha}$. A standard duality argument for the convex domain $S = (0,1)^2$ then gives additionally $\|u - \Pi_p^{H^1} u\|_{L^2(S)} \lesssim p^{-2\alpha-1}$. The estimate for $\|u - \Pi_p^{H^1} u\|_{L^2(\partial S)}$ finally follows from the multiplicative trace inequality.

The H^1-estimate in (5.2) follows from (5.1) and Lemma 4.4. The L^2-bound follows from [2, Theorems 3.10, 3.12]. These two estimates together with Lemma 4.4 imply the $L^2(\partial S)$-bound. □

Proposition 5.3 *Let $\alpha > 2$. Then the p-IPDG method for the problem described in Example 5.1, obeys the bound*

$$|||u - u_{\mathrm{DG}}||| \lesssim p^{-2\alpha+1/2}, \quad (5.3)$$

as $p \to \infty$.

Proof For simplicity of presentation, we consider the case of a one-element mesh, i.e., $\mathcal{T} = \{\Omega\}$; the general case follows analogously. We consider the extension $\tilde{B}$ of the bilinear form (2.4) into $(H^1(\Omega) + S^{\mathbf{p}}) \times (H^1(\Omega) + S^{\mathbf{p}})$ defined by

$$\tilde{B}(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} (\theta u \Pi_p(\nabla v) \cdot n - \Pi_p(\nabla u) \cdot n v + \sigma u v) \, ds, \quad (5.4)$$

where $\Pi_p : L^2(\Omega) \to [S^{\mathbf{p}}]^2$ here denotes the (component-wise) L^2-projection operator onto $[S^{\mathbf{p}}]^2$. Note that $\tilde{B} = B$ on $S^{\mathbf{p}} \times S^{\mathbf{p}}$. Also $\tilde{B}$ is coercive and continuous with respect to the energy norm (these properties can be verified using standard arguments). Therefore, Strang's Second Lemma implies

$$|||u - u_{\mathrm{DG}}||| \lesssim \inf_{v \in S^{\mathbf{p}}} |||u - v||| + \sup_{\chi \in S^{\mathbf{p}} \setminus \{0\}} \frac{|R(u,\chi)|}{|||\chi|||}, \quad (5.5)$$

where the residual $R(u,\chi) := B(u,\chi) - \tilde{B}(u,\chi)$ equals

$$R(u,\chi) = \int_{\partial\Omega} (\nabla u \cdot n - \Pi_p(\nabla u) \cdot n) \chi \, ds.$$

Since the components of ∇u of have the form $r^{\alpha-1}\widetilde{\phi}(\theta)$ for a smooth $\widetilde{\phi}$, we get from Lemma 5.2

$$\begin{aligned} \sup_{\chi \in S^{\mathbf{p}} \setminus \{0\}} \frac{|R(u,\chi)|}{|||\chi|||} &\le \|\sigma^{-1/2}(\nabla u \cdot n - \Pi_p(\nabla u) \cdot n)\|_{L^2(\partial\Omega)} \lesssim p^{-1}\|\nabla u - \Pi_p(\nabla u)\|_{L^2(\partial\Omega)} \\ &\lesssim p^{-1} p^{-2(\alpha-1)-3/4} = p^{-2\alpha+1/4}. \quad (5.6) \end{aligned}$$

Additionally, Lemma 5.2 implies

$$\inf_{q \in S^{\mathbf{p}}} |||u - q||| \lesssim p^{-2\alpha+1/2},$$

which allows us to conclude the proof. □

Table 4 Example 5.1: p-convergence of $|||u - u_{DG}|||$ for $\alpha = 3$

p	$\|\|\|u - u_{DG}\|\|\|$	$r(p)$	p	$\|\|\|u - u_{DG}\|\|\|$	$r(p)$
1	1.5850+01	–	18	4.4933−06	5.84
2	3.4096+00	2.21	19	3.2758−06	5.84
3	2.2218−01	6.73	20	2.4270−06	5.84
4	3.9386−02	6.01	21	1.8243−06	5.85
5	8.7734−03	6.72	22	1.3895−06	5.85
6	2.8793−03	6.11	23	1.0710−06	5.85
7	1.1399−03	6.01	24	8.3472−07	5.85
8	5.1657−04	5.92	25	6.5718−07	5.85
9	2.5833−04	5.88	26	5.2211−07	5.86
10	1.3929−04	5.86	27	4.1855−07	5.85
11	7.9756−05	5.85	28	3.3817−07	5.86
12	4.7967−05	5.84	29	2.7512−07	5.87
13	3.0056−05	5.84	30	2.2542−07	5.87
14	1.9499−05	5.83	31	1.8602−07	5.85
15	1.3033−05	5.83	32	1.5429−07	5.89
16	8.9413−06	5.83	33	1.2914−07	5.78
17	6.2750−06	5.84	34	1.0853−07	5.82

To investigate the convergence history numerically, we consider the p-IPDG with one element (i.e., $\Gamma_{\text{int}} = \emptyset$); the errors in the DG-norm along with the corresponding convergence rates calculated in this case by the formula

$$r(p) = -\frac{\log \text{error}(p) - \log \text{error}(p-1)}{\log p - \log(p-1)},$$

are given in Table 4. The convergence rate appears to be higher than what the bound (5.3) suggests for this case (i.e., rate $11/2$), but lower than 6, which is the expected convergence rate for the conforming p-version finite element method for the same problem. It is not known at this point, if the bound (5.3) is sharp, or if it can be further improved.

6 Conclusions

From the discussion above, we conclude that error bounds of the form (3.1) are sharp and that the p-suboptimality is a result of suboptimal approximation of Dirichlet boundary conditions (assuming that the underlying mesh is reasonable, e.g., containing simple hanging nodes [12]). Hence, in situations where the Dirichlet boundary conditions are represented exactly (or approximated super-optimally) by traces of finite element functions, the p-IPDG method converges optimally. We also investigated the question of convergence rates for the p-IPDG method for problems with corner singularities, finding that the convergence rate of the p-IPDG method for such problems appears to be slightly suboptimal.

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