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An Equal-Order DG Method for the Incompressible Navier-Stokes Equations

Bernardo Cockburn · Guido Kanschat · Dominik Schötzau

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Abstract We introduce and analyze a discontinuous Galerkin method for the incompressible Navier-Stokes equations that is based on finite element spaces of the same polynomial order for the approximation of the velocity and the pressure. Stability of this equal-order approach is ensured by a pressure stabilization term. A simple element-by-element post-processing procedure is used to provide globally divergence-free velocity approximations. For small data, we prove the existence and uniqueness of discrete solutions and carry out an error analysis of the method. A series of numerical results are presented that validate our theoretical findings.

Keywords Discontinuous Galerkin methods · Equal-order methods · Incompressible Navier-Stokes equations

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B. Cockburn
School of Mathematics, University of Minnesota, Vincent Hall, Minneapolis, MN 55455, USA
e-mail: cockburn@math.umn.edu

G. Kanschat
Department of Mathematics, Texas A&M University, Mailstop 3368, College Station, TX 77843, USA
e-mail: kanschat@tamu.edu

D. Schötzau (✉)
Mathematics Department, University of British Columbia, 121-1984 Mathematics Road, Vancouver V6T 1Z2, Canada
e-mail: schoetzau@math.ubc.ca

1 Introduction

In this paper, we introduce and study, theoretically as well as computationally, a new discontinuous Galerkin (DG) method for the stationary incompressible Navier-Stokes equations, namely,

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \nabla p = \mathbf{f} &\quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 &\quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} &\quad \text{on } \Gamma = \partial\Omega. \end{aligned} \tag{1.1}$$

Here, ν is the kinematic viscosity, $\mathbf{u}$ the velocity field, p the pressure, and $\mathbf{f}$ an external body force. For simplicity, we take Ω to be a polygonal domain in $\mathbb{R}^2$.

This paper is the last in a series of papers devoted to the study of DG methods for incompressible flows. Indeed, in [5], a local discontinuous Galerkin (LDG) method for the Stokes system was introduced. The method was then extended to the Oseen problem in [6]; see also the review [7]. The extension to the stationary incompressible Navier-Stokes was carried out later in [8], where an unexpected difficulty, introduced by the presence of the nonlinearity of the equations, was uncovered. It can be stated, roughly speaking, as follows: it is not possible to devise *locally conservative* DG methods which are also *energy-stable*, unless the approximate velocity is *globally divergence-free*. In [8], it was shown how to overcome this difficulty by using a sequence of *suitably* defined DG solutions of Oseen problems. Let us sketch it.

A classic argument to find the solution of problem (1.1) consists in generating a sequence $\{(\mathbf{u}^k, p^k)\}_{k\geq 0}$ of solutions of Oseen problems

$$\begin{aligned} -\nu\Delta\mathbf{u}^{k+1} + (\mathbf{w}\cdot\nabla)\mathbf{u}^{k+1} + \nabla p^{k+1} = \mathbf{f} &\quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u}^{k+1} = 0 &\quad \text{in } \Omega, \\ \mathbf{u}^{k+1} = \mathbf{0} &\quad \text{on } \Gamma, \end{aligned}$$

where $\mathbf{w} = \mathbf{u}^k$. This sequence converges to the solution $(\mathbf{u}, p)$ of (1.1) provided the data satisfies a *small data* condition of the form

$$\nu^{-2}\|\mathbf{f}\|_0 < \rho_\Omega, \tag{1.2}$$

for a positive constant ρ_Ω only depending on Ω; see, for example, [14]. The method proposed in [8] is a discrete version of this idea. It consists in generating a sequence $\{(\mathbf{u}_h^k, p_h^k)\}_{k\geq 0}$ of approximations to the above solutions which then converge to the approximation sought. In this case, $(\mathbf{u}_h^{k+1}, p_h^{k+1})$ is given by a DG discretization of the above Oseen problem where the flow field $\mathbf{w} = \widetilde{P}(\mathbf{u}_h^k)$ is now taken to be a post-processed velocity, obtained from $\mathbf{u}_h^k$ in such a way that it turns out to be globally divergence-free. The post-processing operator $\widetilde{P}$ can be computed in a simple, element-by-element fashion. DG methods in which the post-processing $\widetilde{P}$ can be taken as the identity were briefly mentioned in [8] and explicitly further explored in [9, 11]; these methods provide exactly divergence-free velocities.

In [8], the case of approximate velocities which are piecewise polynomials of degree k and approximate pressures that are piecewise polynomials of degree $k-1$ was considered. In this paper, we treat the more complicated case of equal-order spaces where the approximate pressures are also piecewise polynomials of degree k. Although the elements are now no longer matched properly with respect to their approximation properties, equal-order approaches are often observed to be more computationally efficient than their mixed-order counterparts; see for example the numerical tests in [5] for Stokes flow problems.

For the non-linear Navier-Stokes equations, the main difficulty in the analysis of the equal-order case stems from the fact that, in order to compensate for using a bigger polynomial space for the approximate pressure, the numerical traces for the velocity and pressure cannot be taken as in [8], that is, as

$$\widehat{\mathbf{u}}_h^p = \{\{\mathbf{u}_h\}\} \quad \text{and} \quad \widehat{p}_h = \{\{p_h\}\},$$

but have to be modified as follows:

$$\widehat{\mathbf{u}}_h^p = \{\{\mathbf{u}_h\}\} + \gamma_E [\![p_h \mathbf{n}]\!] \quad \text{and} \quad \widehat{p}_h = \{\{p_h\}\},$$

where the *stabilization* function γ_E is strictly positive. This function then introduces a pressure stabilization term in the variational formulation of the method. As a direct consequence, the above-mentioned post-processing projection $\widetilde{P}$ depends not only on $\mathbf{u}_h$ but also on p_h, since $\widehat{\mathbf{u}}_h^p$ depends on $\mathbf{u}_h$ and p_h. If the problem were linear, this would not cause any major difficulty, as was shown in [5, 6] and [15]. However, the presence of the nonlinearity renders the analysis of this modification quite delicate. The main contribution of this paper is to show that the *optimal* order of convergence can be obtained for the DG norm error of the approximation in the velocity. On the other hand, the theoretical and observed convergence rates for the L^2-norm errors in the pressure are suboptimal, as is expected for equal-order methods.

The paper is organized as follows. In Sect. 2, we describe the equal-order DG method. In Sect. 3, we state and discuss our main results: the existence and uniqueness of DG approximations and a-priori error estimates. Sects. 4, 5, 6 and 7 are devoted to the proofs of these results. In Sect. 8, we display numerical results showing that the orders of convergence predicted by our theoretical estimates are sharp. We end in Sect. 9 with some concluding remarks.

2 Equal-Order DG Discretization

2.1 Notation

We consider subdivisions $\mathbb{T}_h$ of (global) mesh size h of the domain into triangles or rectangles K. We assume that the meshes do not have hanging nodes and are shape-regular. We denote by $\mathbb{E}_h$ the set of all edges of $\mathbb{T}_h$, by $\mathbb{E}_h^I$ the set of all interior edges, and by $\mathbb{E}_h^B$ the set of all boundary edges. As usual, we write h_E for the length of edge E. Finally, we denote by $\mathbb{E}(K)$ the set of all elemental edges of K.

Using the space $P_k(K)$ of polynomials of degree up to k on K, we define the discontinuous finite element space

$$P_h^k = \{v \in L^2(\Omega) : v|_K \in P_k(K),\ K \in \mathbb{T}_h\}.$$

For a given polynomial order $k \geq 1$, we introduce the discrete velocity and pressure spaces by

$$\mathbf{V}_h = (P_h^k)^2, \qquad Q_h = \left\{ q \in P_h^k : \int_\Omega q\, d\mathbf{x} = 0 \right\}. \tag{2.1}$$

In order to define derivative operators on these spaces, we introduce average and jump operators on edges. To do so, let K^+ and K^- be two adjacent elements of $\mathbb{T}_h$ that share an interior edge $E = \partial K^+ \cap \partial K^- \in \mathbb{E}_h^I$. Let φ be any piecewise smooth function (vector- or scalar-valued), and let us denote by $\varphi^\pm$ the traces of φ on E taken from within the interior of $K^\pm$. On the edge E, we then define the mean value $\{\!\{\cdot\}\!\}$ as

$$\{\!\{\varphi\}\!\} = \frac{1}{2}(\varphi^+ + \varphi^-).$$

Furthermore, for piecewise smooth vector-valued function, we define the jumps across E as

$$\begin{aligned} [\![\mathbf{v} \otimes \mathbf{n}]\!] &= \mathbf{v}^+ \otimes \mathbf{n}_{K^+} + \mathbf{v}^- \otimes \mathbf{n}_{K^-}, \\ [\![\mathbf{v} \cdot \mathbf{n}]\!] &= \mathbf{v}^+ \cdot \mathbf{n}_{K^+} + \mathbf{v}^- \cdot \mathbf{n}_{K^-}, \\ [\![q\mathbf{n}]\!] &= q^+\mathbf{n}_{K^+} + q^-\mathbf{n}_{K^-}, \end{aligned}$$

where $\mathbf{n}_{K^\pm}$ denotes the outward unit normal vector on the boundary $\partial K^\pm$. For a boundary edge $E = \partial K \cap \Gamma \in \mathbb{E}_h^B$, we analogously define $\{\!\{\varphi\}\!\} = \varphi^+$, $[\![\mathbf{v} \otimes \mathbf{n}]\!] = \mathbf{v}^+ \otimes \mathbf{n}$, $[\![\mathbf{v} \cdot \mathbf{n}]\!] = \mathbf{v}^+ \cdot \mathbf{n}$ and $[\![q\mathbf{n}]\!] = q^+\mathbf{n}$, where we now denote by φ^+, $\mathbf{v}^+$, q^+ the traces of φ, $\mathbf{v}$, q on ∂K, respectively, and by $\mathbf{n}$ the unit outward normal on Γ.

2.2 The Oseen Problem

We begin the description of the equal-order method by defining the DG discretization of the Oseen problem

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{w} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} &\quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 &\quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} &\quad \text{on } \Gamma. \end{aligned}$$

Here, $\mathbf{w}$ is a generic divergence-free flow field in the space

$$\mathbf{J}(\Omega) = \left\{\mathbf{w} \in H^{\text{div}}(\Omega) : \nabla \cdot \mathbf{w} \equiv 0 \text{ in } \Omega\right\},$$

where $H^{\text{div}}(\Omega)$ is the space of vector functions in $L^2(\Omega)^2$ whose divergence is in $L^2(\Omega)$.

It reads as follows: for $\mathbf{w} \in \mathbf{J}(\Omega)$, find $\mathbf{u} \in \mathbf{V}_h$ and $p \in Q_h$ such that

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}) + o_h(\mathbf{w}; \mathbf{u}, \mathbf{v}) + b_h(\mathbf{v}, p) &= \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ -b_h(\mathbf{u}, q) + c_h(p, q) &= 0, \quad \forall q \in Q_h. \end{aligned} \tag{2.2}$$

Here, the forms a_h, o_h and b_h correspond to ν times the Laplacian, the advection term and the divergence operator, respectively. The form c_h is associated with the pressure jumps and plays a stabilization role. These forms are specified below.

2.2.1 The Discrete Laplacian

Since DG formulations for the Laplacian have been intensely researched during the last years, see for example [2], we refrain from listing special discretizations and will simply make assumptions on the method used.

To state them we need to introduce the standard DG norm on the space $\mathbf{V}_h$:

$$\|\mathbf{u}\|_{1,h}^2 = \sum_{K\in\mathbb{T}_h} \|\nabla\mathbf{u}\|_{L^2(K)}^2 + \sum_{E\in\mathbb{E}_h} \int_E \frac{\kappa_0}{h_E} |[\![\mathbf{u}\otimes\mathbf{n}]\!]|^2 ds,$$

where κ_0 is a positive parameter depending on the polynomial degree k, but not on the mesh size h.

Assumption 2.1 With $\mathbf{u}$ the solution to the vector-valued equation

$$-\nu\Delta\mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma,$$

we make the following assumptions on the discretizing DG bilinear form a_h:

- *Boundedness:* there is a positive constant c_a independent of ν and the mesh size h such that for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}_h$

$$|a_h(\mathbf{u},\mathbf{v})| \le \nu c_a \|\mathbf{u}\|_{1,h}\|\mathbf{v}\|_{1,h}. \tag{2.3a}$$

- *Stability:* there is a positive constant α independent of ν and the mesh size h such that for any $\mathbf{u}\in\mathbf{V}_h$

$$a_h(\mathbf{u},\mathbf{u}) \ge \nu\alpha\|\mathbf{u}\|_{1,h}^2. \tag{2.3b}$$

- *Consistency:* for the solution $\mathbf{u}$ above and any $\mathbf{v}\in\mathbf{V}_h$ there holds

$$a_h(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}. \tag{2.3c}$$

- *Approximation property:* if the solution $\mathbf{u}$ above belongs to $H^{s+1}(\Omega)^2$, for some exponent $s \ge 1$, and $\mathbf{P}_{\mathbf{V}_h}\mathbf{u}$ is its L^2-projection onto $\mathbf{V}_h$, then there is a positive constant C independent of ν and the mesh size h such that for any $\mathbf{v}\in\mathbf{V}_h$

$$|a_h(\mathbf{u}-\mathbf{P}_{\mathbf{V}_h}\mathbf{u},\mathbf{v})| \le Ch^{\min\{s,k\}}\nu^{\frac{1}{2}}\|\mathbf{u}\|_{s+1}\nu^{\frac{1}{2}}\|\mathbf{v}\|_{1,h}. \tag{2.3d}$$

We remark, that the interior penalty methods and the LDG methods are among those DG schemes for which these assumptions hold. Note also that, while the energy norm error estimates from [5, 6, 8] were proven for the LDG discretization of the Laplacian, their proofs rely on these assumptions only; see also [2, 10]. For the optimal L^2-norm or negative-order norm error estimates derived in [5, 6], also *adjoint consistency* of the bilinear form a_h is needed: for the solution $\mathbf{u}$ above and any $\mathbf{v}\in\mathbf{V}_h$ there holds

$$a_h(\mathbf{v},\mathbf{u}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}.$$

In the following, we will not make use of any duality argument. Therefore, adjoint consistency is not necessary for the analysis presented in this article.

2.2.2 The Advection Form

For the convective term, we take the standard upwind flux introduced in [13], namely,

$$o_h(\mathbf{w};\mathbf{u},\mathbf{v}) = \sum_{K\in\mathbb{T}_h}\left(-\int_K \mathbf{u}\otimes\mathbf{w} : \nabla\mathbf{v}\,d\mathbf{x} - \int_{\partial K}(\mathbf{w}\cdot\mathbf{n})\mathbf{u}^{\uparrow}\cdot\mathbf{v}\,ds\right),$$

where $\mathbf{u}^{\uparrow}$ is the upwind value of $\mathbf{u}$ on ∂K with respect to $\mathbf{w}$, namely,

$$\mathbf{u}^{\uparrow}(\mathbf{x}) = \begin{cases} \lim_{\epsilon\downarrow 0}\mathbf{u}(\mathbf{x}-\epsilon\mathbf{w}(\mathbf{x})), & \mathbf{x}\in\partial K\setminus\Gamma_-,\\ \mathbf{0}, & \mathbf{x}\in\partial K\cap\Gamma_-, \end{cases}$$

with Γ_- denoting the inflow part of Γ:

$$\Gamma_- = \{\mathbf{x}\in\Gamma : \mathbf{w}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) < 0\}.$$

As observed in [13], the following result holds.

Proposition 2.2 *If* $\mathbf{w}\in\mathbf{J}(\Omega)$, *we have*

$$O_h(\mathbf{w};\mathbf{u},\mathbf{u}) \geq 0, \quad \forall\mathbf{u}\in\mathbf{V}_h.$$

Let us further recall the Lipschitz continuity of the form o_h with respect to the convective term; cf. [8, Proposition 4.2]. To that end, we define

$$\mathbf{V}(h) = H_0^1(\Omega)^2 + \mathbf{V}_h.$$

We now have the following result.

Proposition 2.3 *Let* $\mathbf{w}_1,\mathbf{w}_2\in\mathbf{J}(\Omega)$, $\mathbf{u}\in\mathbf{V}(h)$ *and* $\mathbf{v}\in\mathbf{V}_h$. *Then we have*

$$|o_h(\mathbf{w}_1;\mathbf{u},\mathbf{v}) - o_h(\mathbf{w}_2;\mathbf{u},\mathbf{v})| \leq c_o\,\|\mathbf{w}_1-\mathbf{w}_2\|_{1,h}\,\|\mathbf{u}\|_{1,h}\,\|\mathbf{v}\|_{1,h},$$

with a constant $c_o > 0$ *that is independent of* ν *and the mesh size* h.

2.2.3 The Discrete Divergence

We take

$$b_h(\mathbf{u},q) = \sum_{K\in\mathbb{T}_h}\int_K \mathbf{u}\cdot\nabla q\,d\mathbf{x} - \sum_{E\in\mathbb{E}_h}\int_E \{\!\{\mathbf{u}\}\!\}\cdot[\![q\mathbf{n}]\!]\,ds,$$

$$c_h(p,q) = \sum_{E\in\mathbb{E}_h^I}\nu^{-1}\gamma_0 h_E\int_E [\![p\mathbf{n}]\!]\cdot[\![q\mathbf{n}]\!]\,ds,$$

where γ_0 is a positive parameter. These forms correspond to a discretization of the incompressibility condition on the exact velocity similar to that used in [8], namely,

$$-\int_K \mathbf{u}_h\cdot\nabla q\,d\mathbf{x} + \int_{\partial K}\widehat{\mathbf{u}}_h^p\cdot\mathbf{n}_K q\,ds = 0. \qquad (2.4)$$

Here, the numerical fluxes associated with the incompressibility constraint, $\widehat{\mathbf{u}}_h^p$ and $\widehat{p}_h$, are defined as follows. If $E \in \mathbb{E}_h^I$ is an interior edge, we take

$$\widehat{\mathbf{u}}_h^p = \{\{\mathbf{u}_h\}\} + \gamma_E [\![p_h \mathbf{n}]\!], \qquad \widehat{p}_h = \{\{p_h\}\}, \tag{2.5a}$$

for a stabilization function which, in accordance with the discussion in [6, Appendix], we take as

$$\gamma_E = \nu^{-1} \gamma_0 h_E. \tag{2.5b}$$

Note that if we set $\gamma_0 = 0$, we obtain the numerical traces used in [8] for the mixed-order DG method. On boundary edges $E \in \mathbb{E}_h^B$, we set

$$\widehat{\mathbf{u}}_h^p = \mathbf{0}, \qquad \widehat{p}_h = p_h. \tag{2.5c}$$

After integration by parts, it can be seen that the form b_h is bounded.

Proposition 2.4 *For functions $\mathbf{v} \in \mathbf{V}_h$ and $q \in Q_h$, the form b_h is bounded by*

$$|b_h(\mathbf{v}, q)| \le c_b \|\mathbf{v}\|_{1,h} \|q\|_0,$$

with a constant $c_b > 0$ that is independent of ν and the mesh size h.

Moreover, the form b_h satisfies the following stability condition on $\mathbf{V}_h \times Q_h$. This condition has been shown in [15] and in a slightly different form in [5]. To state it, we need to introduce the semi norm $|\cdot|_h$ on Q_h given by

$$|p|_h^2 = \sum_{E \in \mathbb{E}_h^I} \gamma_0 h_E \|[\![p]\!]\|_{L^2(E)}^2.$$

Proposition 2.5 *There exist constants $\beta_1 > 0$ and $\beta_2 > 0$ independent of ν and the mesh size h such that*

$$\sup_{0 \ne \mathbf{v} \in \mathbf{V}_h} \frac{b_h(\mathbf{v}, q)}{\|\mathbf{v}\|_{1,h}} \ge \beta_1 \|q\|_0 - \beta_2 |q|_h, \quad \forall q \in Q_h.$$

2.3 Post-Processing

Next, we introduce the local post-processing. Let the function $(\mathbf{u}, p)$ be piecewise smooth, and let $\widehat{\mathbf{u}}^p$ be the numerical flux associated with $(\mathbf{u}, p)$ as in (2.5). We define the post-processed velocity $\mathbf{w} = \widetilde{P}(\mathbf{u}, p)$ as follows. The restriction of $\mathbf{w}$ to K is the function in $\mathbf{BDM}_k(K)$ defined via the following moments on K and its edges E:

$$\int_E \mathbf{w} \cdot \mathbf{n}_K \varphi \, ds = \int_E \widehat{\mathbf{u}}^p \cdot \mathbf{n}_K \varphi \, ds, \quad \forall \varphi \in P_k(E), \; E \in \mathbb{E}(K), \tag{2.6a}$$

$$\int_K \mathbf{w} \cdot \mathbf{v} d\mathbf{x} = \int_K \mathbf{u} \cdot \mathbf{v} d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{M}_k(K). \tag{2.6b}$$

Here, when K is a triangle, we have

$$\mathbf{BDM}_k(K) = \big(P_k(K)\big)^2,$$
$$\mathbf{M}_k(K) = \big\{\nabla\varphi + \nabla\times(b_K\psi) : (\varphi,\psi)\in P_{k-1}(K)\times P_{k-2}(K)\big\},$$

with b_K being the elemental polynomial bubble in $P_3(K)$ vanishing on ∂K. When K is a rectangle, we have

$$\mathbf{BDM}_k(K) = \big(P_k(K)\big)^2 \oplus \mathrm{span}\big\{\nabla\times(x^{k+1}\,y), \nabla\times(x\,y^{k+1})\big\},$$
$$\mathbf{M}_k(K) = \big(P_{k-2}(K)\big)^2.$$

Note that these moments are closely related to those of the standard Brezzi-Douglas-Marini (BDM) projection; see [4, Sect. III.3]. In fact, it can be easily seen that, if $(\mathbf{u}, p)$ are in $H^2(\Omega)^2 \times H^1(\Omega)$, then $\widetilde{P}(\mathbf{u}, p)$ is simply the BDM projection of $\mathbf{u}$ (cf. [4, Sect. III.3]). Evidently, the post-processed velocity $\mathbf{w}$ is well-defined and can be computed in an element-by-element fashion. Moreover, it belongs to the space

$$\widetilde{\mathbf{V}}_h = \{\mathbf{v}\in H^{\mathrm{div}}(\Omega) : \mathbf{v}|_K \in \mathbf{BDM}_k(K),\ K\in\mathbb{T}_h\}.$$

The main property of $\mathbf{w} = \widetilde{P}(\mathbf{u}_h, p_h)$ is that it is exactly divergence-free provided that $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ satisfies the weak incompressibility condition in (2.2).

Proposition 2.6 *Let* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *satisfy the second equation in* (2.2) *and let* $\mathbf{w} = \widetilde{P}(\mathbf{u}_h, p_h) \in \widetilde{\mathbf{V}}_h$ *be the post-processed velocity associated with* $(\mathbf{u}_h, p_h)$. *Then* $\mathbf{w}$ *is exactly divergence-free and belongs to* $\mathbf{J}(\Omega)$.

Proof For $q \in Q_h$, we obtain

$$\int_\Omega \nabla\cdot\mathbf{w}q\,d\mathbf{x} = \sum_{K\in\mathbb{T}_h}\left(-\int_K \mathbf{w}\cdot\nabla q\,d\mathbf{x} + \int_{\partial K}\mathbf{w}\cdot\mathbf{n}_K q\,ds\right)$$
$$= \sum_{K\in\mathbb{T}_h}\left(-\int_K \mathbf{u}_h\cdot\nabla q\,d\mathbf{x} + \int_{\partial K}\widehat{\mathbf{u}}_h^p\cdot\mathbf{n}_K q\,ds\right),$$

by the definition of $\mathbf{w}$ in (2.6). Finally, rewriting the weak incompressibility condition in (2.2) in the flux formulation (2.4), we obtain

$$\int_\Omega \nabla\cdot\mathbf{w}q\,d\mathbf{x} = 0.$$

We claim that $\nabla\cdot\mathbf{w}\in Q_h$. As a consequence, we conclude that $\nabla\cdot\mathbf{w}\equiv 0$ in Ω.

To prove the claim, we note that $\nabla\cdot\mathbf{w}|_K \in P_k(K)$ for all $K\in\mathbb{T}_h$ and

$$\int_\Omega \nabla\cdot\mathbf{w}\,d\mathbf{x} = \int_\Gamma \mathbf{w}\cdot\mathbf{n}\,ds = \int_\Gamma \widehat{\mathbf{u}}_h^p\cdot\mathbf{n}\,ds = 0,$$

which in turn is a consequence of the definition of $\mathbf{w}$ in (2.6) and the fact that we have $\widehat{\mathbf{u}}_h^p = \mathbf{0}$ on Γ. This completes the proof. □

The following stability result will be crucial in the analysis of the method. It is analogous to the stability result of Proposition 4.6 in [8].

Proposition 2.7 *For piecewise polynomial functions* $\mathbf{u}$ *and* p, *the post-processing operator* $\widetilde{P}$ *admits the stability estimate*

$$\|\widetilde{P}(\mathbf{u}, p)\|_{1,h} \leq c_s \left(\|\mathbf{u}\|_{1,h}^2 + \alpha^{-1}\nu^{-2}|p|_h^2\right)^{\frac{1}{2}},$$

with a constant $c_s > 0$ *that is independent of* ν *and the mesh size* h.

In the estimate of Proposition 2.7, the coercivity constant α of the bilinear form a_h is introduced for convenience only; this allows us to avoid certain maximum terms in the proof of existence and uniqueness of discrete solutions. The proof of this result will be presented in Sect. 4.

2.4 The Navier-Stokes Equations

We are finally ready to present the equal-order DG method for the incompressible Navier-Stokes equation (1.1). It is nothing else than the DG discretization of the Oseen problem (2.2) where the convective velocity field $\mathbf{w}_h$ is the post-processed velocity $\mathbf{w}_h = \widetilde{P}(\mathbf{u}_h, p_h)$: find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in Q_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}) + o_h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}) + b_h(\mathbf{v}, p_h) &= \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ -b_h(\mathbf{u}_h, q) + c_h(p_h, q) &= 0, \quad \forall q \in Q_h. \end{aligned} \tag{2.9}$$

As in [8], we compute the solution of this nonlinear system by the Picard iteration mentioned in Sect. 1. We rewrite it as follows. For $\mathbf{w} \in \widetilde{\mathbf{V}}_h \cap \mathbf{J}(\Omega)$, let $(T_u(\mathbf{w}), T_p(\mathbf{w})) \in \mathbf{V}_h \times Q_h$ be the solution of

$$\begin{aligned} a_h(T_u(\mathbf{w}), \mathbf{v}) + o_h(\mathbf{w}; T_u(\mathbf{w}), \mathbf{v}) + b_h(\mathbf{v}, T_p(\mathbf{w})) &= \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ -b_h(T_u(\mathbf{w}), q) + c_h(T_p(\mathbf{w}), q) &= 0, \quad \forall q \in Q_h. \end{aligned}$$

Furthermore, define $\widetilde{T}$ as the post-processing applied to the Oseen solution, namely

$$\widetilde{T} : \widetilde{\mathbf{V}}_h \cap \mathbf{J}(\Omega) \to \widetilde{\mathbf{V}}_h \cap \mathbf{J}(\Omega), \qquad \widetilde{T}(\mathbf{w}) = \widetilde{P}(T_u(\mathbf{w}), T_p(\mathbf{w})).$$

Obviously, the field $\mathbf{w}_h$ in (2.9) must be a fixed point of $\widetilde{T}$. To find it, we set $\mathbf{w}_h^0 = \mathbf{0}$, and generate the sequence $\{\mathbf{w}_h^k\}_{k\geq 0}$ by using the Picard iteration

$$\mathbf{w}_h^{k+1} = \widetilde{T}(\mathbf{w}_h^k). \tag{2.10}$$

3 Main Results

3.1 Existence and Uniqueness of Approximate Solutions

Our first main result is concerned with the convergence of the Picard iteration and with the existence and uniqueness of the approximate solutions provided by the equal-order DG method. To state it, we need to introduce the following discrete *small data* condition:

$$\nu^{-2}\|\mathbf{f}\|_0 < \rho_1 \quad \text{for } \rho_1 = \frac{\alpha^2}{c_o c_p c_s}, \tag{3.1}$$

where $c_p > 0$ is the constant in the Poincaré inequality

$$\|\mathbf{v}\|_{1,h} \geq c_p^{-1} \|\mathbf{v}\|_0, \quad \forall \mathbf{v} \in \mathbf{V}(h). \tag{3.2}$$

For a proof of this inequality, see, for example, [1, Lemma 2.1] or [3].

Theorem 3.1 *Consider the equal-order DG method given by (2.9) for which the assumptions (2.3a) and (2.3b) on the bilinear form a_h hold. Under the small data condition (3.1), the Picard iteration in (2.10) converges to a unique fixed point $\mathbf{w}_h \in \widetilde{\mathbf{V}}_h \cap \mathbf{J}(\Omega)$, and the method has a unique solution $\mathbf{w}_h \in \widetilde{\mathbf{V}}_h \cap \mathbf{J}(\Omega)$ and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$.*

Moreover, the solution satisfies the stability bound

$$\left(\|\mathbf{u}_h\|_{1,h}^2 + \alpha^{-1}\nu^{-2}|p_h|_h^2\right)^{\frac{1}{2}} \leq \frac{c_p}{\alpha\nu}\|\mathbf{f}\|_0 \quad \textit{and} \quad \|\mathbf{w}_h\|_{1,h} \leq \frac{c_p c_s}{\alpha\nu}\|\mathbf{f}\|_0.$$

The proof of Theorem 3.1 is presented in Sect. 5.

3.2 Inf-sup Stability

Next, we present the inf-sup condition that we use to derive our error estimates. To that end, we introduce the global form

$$\mathcal{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{v}, q) = a_h(\mathbf{u}, \mathbf{v}) + o_h(\mathbf{w}; \mathbf{u}, \mathbf{v}) + b_h(\mathbf{v}, p) - b_h(\mathbf{u}, q) + c_h(p, q),$$

and the norm

$$|||\,(\mathbf{u}, p)\,||| = \left(\nu\,\|\mathbf{u}\|_{1,h}^2 + \nu^{-1}\,\|p\|_0^2 + \nu^{-1}\,|p|_h^2\right)^{\frac{1}{2}}.$$

As a consequence of the generalized inf-sup condition for b_h in Proposition 2.5 and the introduction of the pressure stabilization form c_h, the following stability result holds for the form $\mathcal{A}_h$.

Proposition 3.2 *Assume the small data condition (3.1), and let $\mathbf{w} \in \mathbf{J}(\Omega)$ be bounded by*

$$\|\mathbf{w}\|_{1,h} \leq \frac{c_s c_p}{\alpha\nu}\|\mathbf{f}\|_0.$$

Then there is a constant $c_{\mathcal{A}} > 0$ independent of ν and the mesh size h such that

$$\inf_{(\mathbf{u},p)\in\mathbf{V}_h\times Q_h}\ \sup_{(\mathbf{v},q)\in\mathbf{V}_h\times Q_h} \frac{\mathcal{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{v}, q)}{|||\,(\mathbf{u}, p)\,|||\,|||\,(\mathbf{v}, q)\,|||} \geq c_{\mathcal{A}}^{-1}.$$

The proof of Proposition 3.2 is given in Sect. 6.

3.3 Error Estimates

Our second main result are a-priori error estimates for the equal-order DG method. To derive them, we need a small data condition which is slightly different than that in (3.1), namely:

$$\nu^{-2}\|\mathbf{f}\|_0 < \rho_2 \quad \text{for } \rho_2 = \left(2c_{\mathcal{A}}c_o c_p c_s \max\{1, \alpha^{-\frac{1}{2}}\}\right)^{-1}. \tag{3.3}$$

In the following, we assume for simplicity that the strictest of the small data conditions in (1.2), (3.1) and (3.3) holds, namely:

$$\nu^{-2}\|\mathbf{f}\|_0 < \min\{\rho_\Omega, \rho_1, \rho_2\}. \tag{3.4}$$

This also ensures the existence and uniqueness of Navier-Stokes solutions and their DG approximations. We then have the following error estimates.

Theorem 3.3 *Assume that the small data condition* (3.4) *holds. Let the solution* $(\mathbf{u}, p)$ *of the incompressible Navier-Stokes equations* (1.1) *belong to*

$$(\mathbf{u}, p) \in H^{s+1}(\Omega)^2 \times H^s(\Omega), \quad s \geq 1.$$

Let $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be the solution of the equal-order DG method given by* (2.9) *for which the assumptions* (2.3) *on the bilinear form* a_h *hold, and let* $\mathbf{w}_h = \widetilde{P}(\mathbf{u}_h, p_h)$ *be the post-processed exactly divergence-free velocity in* $\widetilde{\mathbf{V}}_h \times \mathbf{J}(\Omega)$. *Then we have the error estimates*

$$\begin{aligned}&\nu^{\frac{1}{2}}\|\mathbf{u}-\mathbf{w}_h\|_{1,h} + |||(\mathbf{u}-\mathbf{u}_h, p-p_h)|||\\ &\quad \leq Ch^{\min\{s,k\}}\left(\nu^{\frac{1}{2}}\|\mathbf{u}\|_{s+1} + \nu^{-\frac{1}{2}}\|p\|_s\right),\end{aligned}$$

with a constant $C > 0$ *that is independent of* ν *and the mesh size* h.

The proof of Theorem 3.3 is given in Sect. 7.

Remark 3.4 For smooth solutions, the result in Theorem 3.3 predicts convergence rates of the order $\mathcal{O}(h^k)$ for the error measured in the norm $|||\cdot|||$. This rate is optimal with respect to the DG norm error in the velocity. With respect to the L^2-norm error in the pressure, this result is suboptimal by one order since polynomials of degree k are used to approximate the pressure variable. The numerical results in Sect. 8 indicate that these suboptimal rates in the pressure are in fact observed in practice. For the DG norm error in the post-processed velocity, rates of the order $\mathcal{O}(h^k)$ are predicted, which is verified in our numerical experiments as well. In this sense, the error estimate of Theorem 3.3 is sharp.

4 Proof of Proposition 2.7

For each element $K \in \mathbb{T}_h$, we introduce the space

$$\mathcal{V}(K) = \left\{\mathbf{u} \in \mathbf{BDM}_k(K) : \int_K \mathbf{u}\cdot\mathbf{v}\,d\mathbf{x} = 0,\ \forall \mathbf{v} \in \mathbf{M}_k(K)\right\},$$

and define

$$\|\mathbf{u}\|_{\partial K}^2 = \sum_{E\in\mathbb{E}(K)} \int_E |\mathbf{u}\cdot\mathbf{n}_K|^2\,ds.$$

Obviously, $\|\cdot\|_{\partial K}$ is a norm on $\mathcal{V}(K)$. This follows from the fact that all the interior BDM moments are zero for functions in $\mathcal{V}(K)$; see [4].

Lemma 4.1 *There is a constant $C > 0$ independent of the mesh size h such that*

$$h_K^{-1}\|\mathbf{u}\|_{L^2(K)} + |\mathbf{u}|_{H^1(K)} \le C h_K^{-\frac{1}{2}} \|\mathbf{u}\|_{\partial K}$$

for any $\mathbf{u} \in \mathcal{V}(K)$.

Proof On the reference element $\widetilde{K}$, the equivalence of norms on finite dimensional spaces shows that there is $\widetilde{C} > 0$ such that

$$\|\widetilde{\mathbf{u}}\|_{L^2(\widetilde{K})} + |\widetilde{\mathbf{u}}|_{H^1(\widetilde{K})} \le \widetilde{C} \|\widetilde{\mathbf{u}}\|_{\partial \widetilde{K}}$$

for all $\widetilde{\mathbf{u}} \in \mathcal{V}(\widetilde{K})$.

Let us then consider an arbitrary triangle K and $\mathbf{u} \in \mathcal{V}(K)$. We define

$$\widetilde{\mathbf{u}}(\widetilde{\mathbf{x}}) = \frac{1}{\det(DF_K)} DF_K \mathbf{u}(F_K(\widetilde{\mathbf{x}})),$$

where $F_K : \widetilde{K} \to K$ is the affine elemental mapping and DF_K its Jacobian. This mapping has the property that $\widetilde{\mathbf{u}} \in \mathcal{V}(\widetilde{K})$; see [4, Sect. III.1]. Furthermore, scaling arguments show that

$$\|\mathbf{u}\|_{0,K} \le C \|\widetilde{\mathbf{u}}\|_{0,\widetilde{K}}, \qquad |\mathbf{u}|_{1,K} \le C h_K^{-1} |\widetilde{\mathbf{u}}|_{1,\widetilde{K}},$$

with a constant $C > 0$ only depending on the shape-regularity of the mesh; cf. [4, Sect. III.1].

Next $E \in \mathbb{E}(K)$ be an edge of ∂K. The corresponding edge on the reference element $\widetilde{K}$ is denoted by $\widetilde{E}$. Using the properties of the mapping $\mathbf{u} \mapsto \widetilde{\mathbf{u}}$ and the density argument from [14, Equation (3.4.2)], we conclude that

$$\begin{aligned}
\|\widetilde{\mathbf{u}} \cdot \mathbf{n}_{\widetilde{K}}\|_{L^2(\widetilde{E})} &= \sup_{\widetilde{\varphi} \in P_k(\widetilde{E})} \frac{\int_{\widetilde{E}} \widetilde{\mathbf{u}} \cdot \mathbf{n}_{\widetilde{K}} \widetilde{\varphi}\, d\widetilde{s}}{\|\widetilde{\varphi}\|_{L^2(\widetilde{E})}} \\
&\le C h_K^{\frac{1}{2}} \sup_{\varphi \in P_k(E)} \frac{\int_E \mathbf{u} \cdot \mathbf{n}_K \varphi\, ds}{\|\varphi\|_{L^2(E)}} \le C h_K^{\frac{1}{2}} \|\mathbf{u} \cdot \mathbf{n}_K\|_{L^2(E)}.
\end{aligned}$$

Here, we have also used that, if $\widetilde{\varphi}(\widetilde{\mathbf{x}}) = \widetilde{\varphi}(F_K(\widetilde{\mathbf{x}}))$, then $\|\varphi\|_{L^2(E)} \le C h_K^{\frac{1}{2}} \|\widetilde{\varphi}\|_{L^2(\widetilde{E})}$.

The estimates above allow us the conclude that

$$\begin{aligned}
h_K^{-1}\|\mathbf{u}\|_{L^2(K)} + |\mathbf{u}|_{H^1(K)} &\le C h_K^{-1} \left(\|\widetilde{\mathbf{u}}\|_{L^2(\widetilde{K})} + |\widetilde{\mathbf{u}}|_{H^1(\widetilde{K})} \right) \\
&\le C h_K^{-1} \|\widetilde{\mathbf{u}}\|_{\partial \widetilde{K}} \\
&\le C h_K^{-\frac{1}{2}} \|\mathbf{u}\|_{\partial K}.
\end{aligned}$$

This completes the proof. □

For $\lambda \in L^2(\mathbb{E}_h)^2$, we now let $\mathbf{U}_\lambda$ be defined elementwise as the unique polynomial in $\mathbf{BDM}_k(K)$ satisfying:

$$\int_E \mathbf{U}_\lambda \cdot \mathbf{n}_K \varphi\, ds = \int_E \lambda \cdot \mathbf{n}_K \varphi\, ds, \quad \forall \varphi \in P_k(E), E \in \mathbb{E}(K),$$

$$\int_K \mathbf{U}_\lambda \cdot \mathbf{v}\, d\mathbf{x} = 0, \quad \forall \mathbf{v} \in \mathbf{M}_k(K).$$

Clearly, we have that $\mathbf{U}_\lambda \in \widetilde{\mathbf{V}}_h$.

Proposition 4.2 *There is a constant $C > 0$ independent of the mesh size h such that*

$$\|\mathbf{U}_\lambda\|_{1,h}^2 \le C \sum_{E \in \mathbb{E}_h} h_E^{-1} \|\lambda\|_{L^2(E)}^2.$$

Proof For any element K, we have $\mathbf{U}_\lambda|_K \in \mathcal{V}(K)$. Hence, Lemma 4.1 and the shape-regularity of the mesh yield

$$\sum_{K \in \mathbb{T}_h} |\mathbf{U}_\lambda|_{H^1(K)}^2 \le \sum_{K \in \mathbb{T}_h} h_K^{-1} \|\mathbf{U}_\lambda\|_{\partial K}^2.$$

Let now $E \in \mathbb{E}(K)$ be an edge of ∂K. The inverse estimate

$$\|\mathbf{U}_\lambda\|_{L^2(E)}^2 \le C h_K^{-1} \|\mathbf{U}_\lambda\|_{L^2(K)}^2,$$

the shape-regularity of the mesh and the result in Lemma 4.1 for the L^2-norm show that

$$h_E^{-1} \|\mathbf{U}_\lambda\|_{L^2(E)} \le C h_K^{-1} \|\mathbf{U}_\lambda\|_{\partial K}.$$

It then follows readily that

$$\|\mathbf{U}_\lambda\|_{1,h}^2 \le \sum_{K \in \mathbb{T}_h} h_K^{-1} \|\mathbf{U}_\lambda\|_{\partial K}^2.$$

Testing the defining moments of $\mathbf{U}_\lambda$ with $\varphi = \mathbf{U}_\lambda \cdot \mathbf{n}_K$ and applying the Cauchy-Schwarz inequality, we obtain that, on each edge E,

$$\int_E |\mathbf{U}_\lambda \cdot \mathbf{n}_K|^2\, ds \le \int_E |\lambda \cdot \mathbf{n}_K|^2\, ds.$$

Taking into account the shape-regularity of the mesh, the assertion follows from the above estimates. □

We are now ready to complete the proof of Proposition 2.7. Let $(\mathbf{u}, p)$ be a piecewise polynomial function and let $\mathbf{w} = \widetilde{P}(\mathbf{u}, p)$ be the post-processed field obtained from $(\mathbf{u}, p)$. Taking into account the form of the flux (2.5) associated with the incompressibility, we have

$$\mathbf{w} = \mathbf{U}_1 + \mathbf{U}_2,$$

where

$$\int_E \mathbf{U}_1 \cdot \mathbf{n}_K \varphi\, ds = \int_E \{\{\mathbf{u}\}\} \cdot \mathbf{n}_K \varphi\, ds, \quad \forall \varphi \in P_k(E), E \in \mathbb{E}(K),$$

$$\int_K \mathbf{U}_1 \cdot \mathbf{v}\, d\mathbf{x} = \int_K \mathbf{u} \cdot \mathbf{v}\, d\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{M}_k(K),$$

and

$$\int_E \mathbf{U}_2 \cdot \mathbf{n}_K \varphi \, ds = \int_E \gamma_E [\![p]\!] \cdot \mathbf{n}_K \varphi \, ds, \quad \forall \varphi \in P_k(E), E \in \mathbb{E}(K) \cap \mathbb{E}_h^I,$$
$$\int_E \mathbf{U}_2 \cdot \mathbf{n}_K \varphi \, ds = 0, \quad \forall \varphi \in P_k(E), E \in \mathbb{E}(K) \cap \mathbb{E}_h^B,$$
$$\int_K \mathbf{U}_2 \cdot \mathbf{v} \, d\mathbf{x} = 0, \quad \forall \mathbf{v} \in \mathbf{M}_k(K).$$

The function $\mathbf{U}_1$ is the post-processed velocity used in the mixed-order approach in [8]. Hence, from [8, Proposition 4.6], there is a constant $C > 0$ independent of the mesh size h such that

$$\|\mathbf{U}_1\|_{1,h} \leq C \|\mathbf{u}\|_{1,h}.$$

Moreover, Proposition 4.2 implies that there is constant $C > 0$ independent of the mesh size h such that

$$\|\mathbf{U}_2\|_{1,h}^2 \leq C \sum_{E \in \mathbb{E}_h^I} h_E^{-1} \gamma_E^2 \|[\![p]\!]\|_{L^2(E)}^2.$$

Since $\gamma_E = \nu^{-1}\gamma_0 h_E$, see (2.5b), we have

$$\|\mathbf{w}\|_{1,h}^2 \leq C \left(\|\mathbf{u}\|_{1,h}^2 + \nu^{-2}|p|_h^2 \right) \leq c_s \left(\|\mathbf{u}\|_{1,h}^2 + \alpha^{-1}\nu^{-2}|p|_h^2 \right).$$

This shows the result of Proposition 2.7.

5 Proof of Theorem 3.1

To prove the existence and uniqueness of the approximate solutions of the equal-order DG method, we proceed as in [8] and mimic the proof of existence and uniqueness of the exact solution to problem (1.1); see also [14, Theorem 10.1.1]. Thus, we show that, under the small data condition (3.1), the Picard iteration (2.10) is a contraction on a suitably defined bounded set.

We begin by obtaining key estimates on the solution of the Oseen problem (2.2).

Lemma 5.1 *If* $\mathbf{w} \in \mathbf{J}(\Omega)$, *then the Oseen problem* (2.2) *admits a unique solution* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *with the stability estimate*

$$\left(\|\mathbf{u}_h\|_{1,h}^2 + \alpha^{-1}\nu^{-2}|p_h|_h^2 \right)^{\frac{1}{2}} \leq \frac{c_p}{\alpha \nu} \|\mathbf{f}\|_0.$$

Proof The existence and uniqueness of the solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ of the Oseen problem (2.2) can be obtained as in [6]; see also [15] for the Stokes equations. To obtain the estimates, we proceed as follows. Taking $\mathbf{v} = \mathbf{u}_h$ in the first equation of the formulation (2.2), $q = p_h$ in the second, and adding the resulting equations, we get

$$a_h(\mathbf{u}_h, \mathbf{u}_h) + o_h(\mathbf{w}; \mathbf{u}_h, \mathbf{u}_h) + c_h(p_h, p_h) = \int_\Omega \mathbf{f} \cdot \mathbf{u}_h \, dx.$$

Using the ellipticity assumption (2.3b) on the form a_h and the stability of the form $o_h(\mathbf{w};\mathbf{u}_h,\mathbf{u}_h)$ guaranteed by Proposition 2.2, we get that

$$\nu\alpha\|\mathbf{u}_h\|_{1,h}^2+\nu^{-1}|p_h|_h^2\le\int_\Omega \mathbf{f}\cdot\mathbf{u}_h\,dx,$$

and, after applying the Cauchy-Schwarz inequality, that

$$\nu\alpha\|\mathbf{u}_h\|_{1,h}^2+\nu^{-1}|p_h|_h^2\le\|\mathbf{f}\|_0\|\mathbf{u}_h\|_0\le c_p\|\mathbf{f}\|_0\|\mathbf{u}_h\|_{1,h},$$

by Poincaré's inequality (3.2). The estimate now follows. □

We are now ready to state and prove the main result of this subsection.

Lemma 5.2 *Under the small data condition* (3.1), *the mapping* $\widetilde{T}$ *defining the Picard iteration* (2.10) *is a contraction on the set*

$$\mathcal{K}_h=\left\{\mathbf{w}\in\widetilde{\mathbf{V}}_h\cap\mathbf{J}(\Omega):\|\mathbf{w}\|_{1,h}\le\frac{c_sc_p}{\alpha\nu}\|\mathbf{f}\|_0\right\}.$$

Proof First, we note that by the stability estimate of Proposition 2.7, we have that, for any $\mathbf{w}\in\mathcal{K}_h$,

$$\begin{aligned}\|\widetilde{T}(\mathbf{w})\|_{1,h}&=\|\widetilde{P}(T_u(\mathbf{w}),T_p(\mathbf{w}))\|_{1,h}\\&\le c_s\left(\|T_u(\mathbf{w})\|_{1,h}^2+\alpha^{-1}\nu^{-2}|T_p(\mathbf{w})|_h^2\right)^{\frac12}\\&\le\frac{c_sc_p}{\alpha\nu}\|\mathbf{f}\|_0,\end{aligned}$$

by Lemma 5.1. Hence, $\widetilde{T}$ maps $\mathcal{K}_h$ into itself.

Now, let $\mathbf{w}_1$ and $\mathbf{w}_2$ be in $\mathcal{K}_h$, and set $(\mathbf{u}_1,p_1)=(T_u(\mathbf{w}_1),T_p(\mathbf{w}_1))$ and $(\mathbf{u}_2,p_2)=(T_u(\mathbf{w}_2),T_p(\mathbf{w}_2))$. It can be readily seen that the difference $(\mathbf{u},p)=(\mathbf{u}_1-\mathbf{u}_2,p_1-p_2)$ solves

$$a_h(\mathbf{u},\mathbf{v})+o_h(\mathbf{w}_1;\mathbf{u},\mathbf{v})+b_h(\mathbf{v},p)-b_h(\mathbf{u},q)+c_h(p,q)=L_h(\mathbf{v}),$$

for any $(\mathbf{v},q)\in\mathbf{V}_h\times Q_h$. Here, $L_h(\mathbf{v})$ is the functional

$$L_h(\mathbf{v})=o_h(\mathbf{w}_1;\mathbf{u}_2,\mathbf{v})-o_h(\mathbf{w}_2;\mathbf{u}_2,\mathbf{v}).$$

Thus, if we take $\mathbf{v}=\mathbf{u}$ and $q=p$, we obtain

$$a_h(\mathbf{u},\mathbf{u})+o_h(\mathbf{w}_1;\mathbf{u},\mathbf{u})+c_h(p,p)=L_h(\mathbf{u}),$$

and hence, proceeding as in the proof of the previous result, that

$$\nu\alpha\|\mathbf{u}\|_{1,h}^2+\nu^{-1}|p|_h^2\le L_h(\mathbf{u}).$$

Let us now estimate the functional $L_h(\mathbf{u})$. In view of the Lipschitz continuity of the Oseen form o_h in Proposition 2.3, we have

$$|L_h(\mathbf{u})|\le c_o\|\mathbf{w}_1-\mathbf{w}_2\|_{1,h}\|\mathbf{u}_2\|_{1,h}\|\mathbf{u}\|_{1,h},$$

and by the estimate of solutions of the discrete Oseen problem (2.2), Lemma 5.1, applied to $\mathbf{u}_2$, we obtain

$$|L_h(\mathbf{u})| \leq \frac{c_o c_p}{\alpha\nu}\|\mathbf{f}\|_0\|\mathbf{w}_1 - \mathbf{w}_2\|_{1,h}\|\mathbf{u}\|_{1,h}.$$

This implies that

$$\nu\alpha\|\mathbf{u}\|_{1,h}^2 + \nu^{-1}|p|_h^2 \leq \frac{c_o c_p}{\alpha\nu}\|\mathbf{f}\|_0\|\mathbf{w}_1 - \mathbf{w}_2\|_{1,h}\|\mathbf{u}\|_{1,h},$$

and hence that

$$\left(\|\mathbf{u}\|_{1,h}^2 + \alpha^{-1}\nu^{-2}|p|_h^2\right)^{\frac{1}{2}} \leq \frac{c_o c_p}{\alpha^2\nu^2}\|\mathbf{f}\|_0\|\mathbf{w}_1 - \mathbf{w}_2\|_{1,h}.$$

Finally, by the stability property of the post-processing $\widetilde{P}$ of Proposition 2.7, we obtain

$$\begin{aligned}\|\widetilde{T}(\mathbf{w}_1) - \widetilde{T}(\mathbf{w}_2)\|_{1,h} &\leq c_s\left(\|\mathbf{u}\|_{1,h}^2 + \alpha^{-1}\nu^{-2}|p|_h^2\right)^{\frac{1}{2}} \\ &\leq \frac{c_o c_p c_s}{\alpha^2\nu^2}\|\mathbf{f}\|_0\|\mathbf{w}_1 - \mathbf{w}_2\|_{1,h}.\end{aligned}$$

By the small data condition (3.1), we conclude that $\widetilde{T}$ is a contraction on $\mathcal{K}_h$. This completes the proof. □

Banach's fixed point theorem now implies that the Picard iteration in (2.10) converges to a unique fixed point $\mathbf{w}_h$ in $\mathcal{K}_h$: $\mathbf{w}_h = \widetilde{T}(\mathbf{w}_h)$. Thus, the unique solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ of the Oseen problem (2.2) associated with the convective field $\mathbf{w}_h$ is the unique solution of (2.9). This completes the proof of Theorem 3.1.

6 Proof of Proposition 3.2

To prove Proposition 3.2, fix $(\mathbf{u}, p) \in \mathbf{V}_h \times Q_h$. By the coercivity of a_h in (2.3b), the non-negativity of o_h in Proposition 2.2, and the definition of $|\cdot|_h$, we see that

$$\begin{aligned}\mathcal{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{u}, p) &= a_h(\mathbf{u}, \mathbf{u}) + o_h(\mathbf{w}; \mathbf{u}, \mathbf{u}) + c_h(p, p) \\ &\geq \nu\alpha\|\mathbf{u}\|_{1,h}^2 + \nu^{-1}|p|_h^2.\end{aligned} \tag{6.1}$$

Further, due to the generalized inf-sup condition for b_h in Proposition 2.5, there is a velocity $\bar{\mathbf{v}} \in \mathbf{V}_h$ such that

$$b_h(\bar{\mathbf{v}}, p) \geq \beta_1\|p\|_0^2 - \beta_2\|p\|_0|p|_h, \quad \|\bar{\mathbf{v}}\|_{1,h} \leq \|p\|_0.$$

From the boundedness of a_h and b_h, (2.3a) and Proposition 2.4, the continuity property of o_h in Proposition 2.3, and the bound for $\bar{\mathbf{v}}$, we obtain

$$\begin{aligned}\mathcal{A}_h(\mathbf{w})(\mathbf{u}, p; \bar{\mathbf{v}}, 0) &= a_h(\mathbf{u}, \bar{\mathbf{v}}) + o_h(\mathbf{w}; \mathbf{u}, \bar{\mathbf{v}}) + b_h(\bar{\mathbf{v}}, p) \\ &\geq \beta_1\|p\|_0^2 - \beta_2\|p\|_0|p|_h - |a_h(\mathbf{u}, \bar{\mathbf{v}})| - |o_h(\mathbf{w}; \mathbf{u}, \bar{\mathbf{v}})| \\ &\geq \beta_1\|p\|_0^2 - \beta_2\|p\|_0|p|_h - (\nu c_a + c_o\|\mathbf{w}\|_{1,h})\|\mathbf{u}\|_{1,h}\|p\|_0.\end{aligned}$$

The small data condition (3.1) and the bound on $\mathbf{w}$ in Proposition 3.2 imply that

$$c_o \|\mathbf{w}\|_{1,h} \le \frac{c_o c_s c_p}{\alpha \nu} \|\mathbf{f}\|_0 \le \alpha \nu.$$

Therefore,

$$\mathcal{A}_h(\mathbf{w})(\mathbf{u}, p; \bar{\mathbf{v}}, 0) \ge \beta_1 \|p\|_0^2 - \beta_2 \|p\|_0 \, |p|_h - c_1 \nu \|\mathbf{u}\|_{1,h} \|p\|_0, \tag{6.2}$$

with $c_1 = c_a + \alpha$. Next, we set $(\mathbf{v}, q) = (\mathbf{u}, p) + \delta \nu^{-1}(\bar{\mathbf{v}}, 0)$ with $\delta > 0$ still to be chosen. From the bounds in (6.1), (6.2), and the geometric-arithmetic inequality, we readily conclude that

$$\begin{aligned}
\mathcal{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{v}, q) &= \mathcal{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{u}, p) + \delta \nu^{-1} \mathcal{A}_h(\mathbf{w})(\mathbf{u}, p; \bar{\mathbf{v}}, 0) \\
&\ge \left(\alpha - \delta \frac{\varepsilon_1 c_1}{2}\right) \nu \|\mathbf{u}\|_{1,h}^2 + \left(1 - \delta \frac{\beta_2 \varepsilon_2}{2}\right) \nu^{-1} \, |p|_h^2 \\
&\quad + \delta \left(\beta_1 - \frac{c_1}{2\varepsilon_1} - \frac{\beta_2}{2\varepsilon_2}\right) \nu^{-1} \|p\|_0^2,
\end{aligned}$$

with parameters $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ still at our disposal. Choosing ε_1 and ε_2 sufficiently large shows that

$$\begin{aligned}
&\mathcal{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{v}, q) \\
&\quad \ge (\alpha - \delta c_2) \nu \|\mathbf{u}\|_{1,h}^2 + \frac{\delta \beta_1}{3} \nu^{-1} \|p\|_0^2 + (1 - \delta c_3) \nu^{-1} \, |p|_h^2,
\end{aligned}$$

with constants $c_2 > 0$ and $c_3 > 0$ that are independent of ν and the mesh size h. Next, by choosing δ sufficiently small, we deduce the existence of a constant $c_4 > 0$ independent of ν and the mesh size h such that

$$\mathcal{A}_h(\mathbf{w})(\mathbf{u}, p; \mathbf{v}, q) \ge c_4 ||| \, (\mathbf{u}, p) \, |||^2. \tag{6.3}$$

On the other hand, due the bound for $\bar{\mathbf{v}}$, we have

$$\begin{aligned}
||| \, (\mathbf{v}, q) \, ||| &\le ||| \, (\mathbf{u}, p) \, ||| + \delta \nu^{-1} ||| \, (\bar{\mathbf{v}}, 0) \, ||| \\
&\le ||| \, (\mathbf{u}, p) \, ||| + \delta \nu^{-\frac{1}{2}} \|\bar{\mathbf{v}}\|_{1,h} \\
&\le ||| \, (\mathbf{u}, p) \, ||| + \delta \nu^{-\frac{1}{2}} \|p\|_0 \le c_5 ||| \, (\mathbf{u}, p) \, |||.
\end{aligned} \tag{6.4}$$

The bounds in (6.3) and (6.4) imply the result of Proposition 3.2.

7 Proof of Theorem 3.3

7.1 Preliminaries

To prove Theorem 3.3, let $(\mathbf{u}, p)$ denote the solution of the Navier-Stokes equations. We recall from [8, Equation (1.5)] that the velocity field $\mathbf{u}$ satisfies the stability bound

$$\|\mathbf{u}\|_{1,h} \le \frac{c_p}{\nu} \|\mathbf{f}\|_0. \tag{7.1}$$

Further, let $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ be the DG approximation from (2.9) and $\mathbf{w}_h = \widetilde{P}(\mathbf{u}_h, p_h) \in \widetilde{\mathbf{V}}_h \cap \mathbf{J}(\Omega)$ the post-processed velocity field.

We note that, under the regularity assumptions of Theorem 3.3, we have the error equation

$$\mathcal{A}_h(\mathbf{u})(\mathbf{u}, p; v, q) - \mathcal{A}_h(\mathbf{w}_h)(\mathbf{u}_h, p_h; \mathbf{v}, q) = 0 \tag{7.2}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$. Here, we have used the consistency of all the DG forms involved; cf. assumption (2.3c) on the bilinear form a_h.

7.2 Proof of the Error Estimates

Let us now bound the errors $\mathbf{e}_u = \mathbf{u} - \mathbf{u}_h$, $e_p = p - p_h$ and $\mathbf{e}_w = \mathbf{u} - \mathbf{w}_h$. To that end, we proceed in several steps.

Step 1 (decomposition of the errors): As usual, we decompose the errors into two parts:

$$\mathbf{e}_u = \eta_u + \xi_u = \left(\mathbf{u} - \mathbf{\Pi}_{\mathbf{V}_h}\mathbf{u}\right) + \left(\mathbf{\Pi}_{\mathbf{V}_h}\mathbf{u} - \mathbf{u}_h\right),$$
$$e_p = \eta_p + \xi_p = \left(p - \Pi_{Q_h} p\right) + \left(\Pi_{Q_h} p - p_h\right),$$

where $\mathbf{\Pi}_{\mathbf{V}_h}$ and Π_{Q_h} denote the L^2-projections onto $\mathbf{V}_h$ and Q_h, respectively. Using the triangle inequality and the approximation properties of the L^2-projections, we obtain that

$$|||\,(\mathbf{e}_u, e_p)\,||| \le |||\,(\eta_u, \eta_p)\,||| + |||\,(\xi_u, \xi_p)\,||| \le C\mathcal{E}_s(\mathbf{u}, p) + |||\,(\xi_u, \xi_p)\,|||, \tag{7.3}$$

where we set, for notational convenience,

$$\mathcal{E}_s(\mathbf{u}, p) = h^{\min\{s,k\}}\left(\nu^{\frac{1}{2}}\|\mathbf{u}\|_{s+1} + \nu^{-\frac{1}{2}}\|p\|_s\right).$$

Hence, to bound the error in $\mathbf{u}$ and p, we need to further estimate the term $|||\,(\xi_u, \xi_p)\,|||$ on the right-hand side of (7.3).

Step 2 (a preliminary bound for (ξ_u, ξ_p)): We claim that

$$|||\,(\xi_u, \xi_p)\,||| \le \left(2c_s \max\{1, \alpha^{-\frac{1}{2}}\}\right)^{-1}\nu^{\frac{1}{2}}\|\mathbf{e}_w\|_{1,h} + C\mathcal{E}_s(\mathbf{u}, p), \tag{7.4}$$

with a constant $C > 0$ that is independent of ν and the mesh size h.

Proof To prove this bound, we start from the inf-sup condition in Proposition 3.2: there is a test function $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$ such that $|||\,(\mathbf{v}, q)\,||| = 1$ and

$$c_{\mathcal{A}}^{-1}|||\,(\xi_u, \xi_p)\,||| \le \mathcal{A}_h(\mathbf{w}_h)(\xi_u, \xi_p; \mathbf{v}, q).$$

Note that, by Lemma 5.2 and the small data condition (3.4), Proposition 3.2 is applicable. We then rewrite the right-hand side of the above inequality by using the error equation (7.2) and obtain that

$$\begin{aligned}\mathcal{A}_h(\mathbf{w}_h)(\xi_u, \xi_p; \mathbf{v}, q) &= o_h(\mathbf{w}_h; \xi_u, \mathbf{v}) + o_h(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}) - o_h(\mathbf{u}; \mathbf{u}, \mathbf{v}) \\ &\quad - a_h(\eta_u, \mathbf{v}) - b_h(\mathbf{v}, \eta_p) + b_h(\eta_u, q) - c_h(\eta_p, q).\end{aligned}$$

We continue to express the difference between two of the above trilinear forms as follows:

$$\begin{aligned} &o_h(\mathbf{w}_h;\mathbf{u}_h,\mathbf{v}) - o_h(\mathbf{u};\mathbf{u},\mathbf{v}) \\ &\quad = -o_h(\mathbf{w}_h;\xi_u,\mathbf{v}) - o_h(\mathbf{w}_h;\eta_u,\mathbf{v}) + o_h(\mathbf{w}_h;\mathbf{u},\mathbf{v}) - o_h(\mathbf{u};\mathbf{u},\mathbf{v}). \end{aligned}$$

Combining these identities yields

$$c_{\mathcal{A}}^{-1} ||| (\xi_u, \xi_p) ||| \le T_1 + T_2 + T_3, \tag{7.5}$$

where

$$\begin{aligned} T_1 &= |a_h(\eta_u,\mathbf{v})| + |b_h(\mathbf{v},\eta_p)| + |b_h(\eta_u,q)| + |c_h(\eta_p,q)|, \\ T_2 &= |o_h(\mathbf{w}_h;\eta_u,\mathbf{v})|, \\ T_3 &= |o_h(\mathbf{w}_h;\mathbf{u},\mathbf{v}) - o_h(\mathbf{u};\mathbf{u},\mathbf{v})|. \end{aligned}$$

Let us now bound these terms.

For T_1, we use the approximation property (2.3d) for a_h, the analogous approximation properties for b_h and c_h derived in [5, Corollary 3.8] and the fact that $||| (\mathbf{v}, q) ||| = 1$. We obtain that

$$T_1 \le C\mathcal{E}_s(\mathbf{u}, p) ||| (\mathbf{v}, q) ||| \le C\mathcal{E}_s(\mathbf{u}, p),$$

with a constant $C > 0$ independent of ν and the mesh size h.

To estimate T_2, we use the continuity bound of o_h in Proposition 2.3, the stability result for $\widetilde{P}$ in Proposition 2.7 and the a-priori bound for $(\mathbf{u}_h, p_h)$ in Theorem 3.1:

$$\begin{aligned} T_2 &\le c_o c_s \left(\|\mathbf{u}_h\|_{1,h}^2 + \alpha^{-1}\nu^{-2}|p_h|_h^2\right)^{\frac{1}{2}} \|\eta_u\|_{1,h}\|\mathbf{v}\|_{1,h} \\ &\le \frac{c_o c_s c_p}{\alpha\nu^2}\|\mathbf{f}\|_0 \nu^{\frac{1}{2}}\|\eta_u\|_{1,h}\nu^{\frac{1}{2}}\|\mathbf{v}\|_{1,h}. \end{aligned}$$

Hence, the small data condition (3.4), the fact that $||| (\mathbf{v}, q) ||| = 1$ and the approximation properties of the L^2-projection $\mathbf{\Pi}_{\mathbf{V}_h}$ yield

$$T_2 \le C\mathcal{E}_s(\mathbf{u}, p),$$

with a constant $C > 0$ independent of ν and the mesh size h.

To estimate T_3, we proceed similarly and apply the continuity of o_h in Proposition 2.3, the stability bound for $\mathbf{u}$ in (7.1) and the fact that $||| (\mathbf{v}, q) ||| = 1$. This results in

$$T_3 \le c_o\|\mathbf{e}_w\|_{1,h}\|\mathbf{u}\|_{1,h}\|\mathbf{v}\|_{1,h} \le c_o c_p \nu^{-2}\|\mathbf{f}\|_0\nu^{\frac{1}{2}}\|\mathbf{e}_w\|_{1,h}.$$

From (7.5) and the bounds for the terms T_1, T_2 and T_3, we now deduce that

$$||| (\xi_u, \xi_p) ||| \le c_{\mathcal{A}} c_o c_p \nu^{-2}\|\mathbf{f}\|_0 \nu^{\frac{1}{2}}\|\mathbf{e}_w\|_{1,h} + C\mathcal{E}_s(\mathbf{u}, p),$$

for a constant $C > 0$ independent of ν and the mesh size h. Using the small data condition (3.4), we see that

$$c_{\mathcal{A}} c_o c_p \nu^{-2}\|\mathbf{f}\|_0 \le \left(2c_s \max\{1, \alpha^{-\frac{1}{2}}\}\right)^{-1},$$

which concludes the proof of (7.4). □

Step 3 (a preliminary bound for $\mathbf{e}_w$*):* Next, we show that there holds

$$\nu^{\frac{1}{2}} \|\mathbf{e}_w\|_{1,h} \le c_s \max\{1, \alpha^{-\frac{1}{2}}\} ||| (\xi_u, \xi_p) ||| + C\mathcal{E}_s(\mathbf{u}, p), \tag{7.6}$$

with a constant $C > 0$ independent of ν and the mesh size h.

Proof Take an arbitrary function $\widetilde{\mathbf{v}} \in \widetilde{\mathbf{V}}_h$. By the triangle inequality,

$$\|\mathbf{e}_w\|_{1,h} \le \|\mathbf{u} - \widetilde{\mathbf{v}}\|_{1,h} + \|\widetilde{\mathbf{v}} - \mathbf{w}_h\|_{1,h}.$$

The second term on the right-hand side of this inequality can be estimated by employing the property that $\widetilde{P}(\widetilde{\mathbf{v}}, 0) = \widetilde{\mathbf{v}}$, the stability bound for $\widetilde{P}$ in Proposition 2.7 and the fact that $|p|_h = 0$ for $p \in H^1(\Omega)$. This yields

$$\begin{aligned}
\|\widetilde{\mathbf{v}} - \mathbf{w}_h\|_{1,h} &= \|\widetilde{P}(\widetilde{\mathbf{v}} - \mathbf{u}_h, p_h)\|_{1,h} \\
&\le c_s \left(\|\widetilde{\mathbf{v}} - \mathbf{u}_h\|_{1,h}^2 + \alpha^{-1}\nu^{-2}|p_h|_h^2\right)^{\frac{1}{2}} \\
&\le c_s \nu^{-\frac{1}{2}} \max\{1, \alpha^{-\frac{1}{2}}\} \left(\nu\|\widetilde{\mathbf{v}} - \mathbf{u}_h\|_{1,h}^2 + \nu^{-1}|p - p_h|_h^2\right)^{\frac{1}{2}} \\
&= c_s \nu^{-\frac{1}{2}} \max\{1, \alpha^{-\frac{1}{2}}\} ||| (\widetilde{\mathbf{v}} - \mathbf{u}_h, e_p) |||.
\end{aligned}$$

Since $(\widetilde{\mathbf{v}} - \mathbf{u}_h, e_p) = (\widetilde{\mathbf{v}} - \mathbf{u}, 0) + (\mathbf{e}_u, e_p)$, a repeated application of the triangle inequality results in

$$\begin{aligned}
\|\widetilde{\mathbf{v}} - \mathbf{w}_h\|_{1,h} &\le c_s\, \nu^{-\frac{1}{2}} \max\{1, \alpha^{-\frac{1}{2}}\} \left(||| (\mathbf{u} - \widetilde{\mathbf{v}}, 0) ||| + ||| (\mathbf{e}_u, e_p) |||\right) \\
&\le c_s \nu^{-\frac{1}{2}} \max\{1, \alpha^{-\frac{1}{2}}\} ||| (\mathbf{e}_u, e_p) ||| + C\|\mathbf{u} - \widetilde{\mathbf{v}}\|_{1,h} \\
&\le c_s \nu^{-\frac{1}{2}} \max\{1, \alpha^{-\frac{1}{2}}\} ||| (\xi_u, \xi_p) ||| \\
&\quad + C ||| (\eta_u, \eta_p) ||| + C\, \|\mathbf{u} - \widetilde{\mathbf{v}}\|_{1,h},
\end{aligned}$$

with a constant $C > 0$ independent of ν and the mesh size h.

Therefore, we conclude that

$$\begin{aligned}
\nu^{\frac{1}{2}} \|\mathbf{e}_w\|_{1,h} &\le c_s \max\{1, \alpha^{-\frac{1}{2}}\} ||| (\xi_u, \xi_p) ||| + C ||| (\eta_u, \eta_p) ||| \\
&\quad + C \inf_{\widetilde{\mathbf{v}} \in \widetilde{\mathbf{V}}_h} \nu^{\frac{1}{2}} \|\mathbf{u} - \widetilde{\mathbf{v}}\|_{1,h}.
\end{aligned}$$

The bound in (7.6) now follows from the approximation properties of the L^2-projections and the ones of the space $\widetilde{\mathbf{V}}_h$. □

Step 4 (conclusion of the result in Theorem 3.3): We are now ready to prove our error estimates. Indeed, by combining the results (7.4) and (7.6) of Steps 2 and 3, respectively, we obtain

$$\begin{aligned}
||| (\xi_u, \xi_p) ||| &\le \left(2c_s \max\{1, \alpha^{-\frac{1}{2}}\}\right)^{-1} \nu^{\frac{1}{2}} \|\mathbf{e}_w\|_{1,h} + C\mathcal{E}_s(\mathbf{u}, p) \\
&\le \frac{1}{2} ||| (\xi_u, \xi_p) ||| + C\mathcal{E}_s(\mathbf{u}, p).
\end{aligned}$$

Absorbing the first term on the right-hand side of this inequality in the left-hand side shows that

$$\|\!| (\xi_u, \xi_p) \|\!| \le C\mathcal{E}_s(\mathbf{u}, p), \tag{7.7}$$

with a constant $C > 0$ independent of ν and the mesh size h. The error estimate for $\|\!| (\mathbf{e}_u, e_p) \|\!|$ now follows immediately from (7.3) and (7.7).

To prove the estimate for the error $\mathbf{e}_w = \mathbf{u} - \mathbf{w}_h$ in the post-processed velocity, we now only need to combine the bounds in (7.6) and (7.7):

$$\|\mathbf{e}_w\|_{1,h} \le C\|\!| (\xi_u, \xi_p) \|\!| + C\mathcal{E}_s(\mathbf{u}, p) \le C\mathcal{E}_s(\mathbf{u}, p),$$

with a constant $C > 0$ independent of ν and the mesh size h. This completes the proof of Theorem 3.3.

8 Numerical Experiments

In this section, we carry out numerical experiments that confirm our theoretical findings. The test problem we use was introduced by Kovasznay [12]. On the square $\Omega = [-\frac{1}{2}, \frac{3}{2}] \times [0, 2]$, this solution is

$$\begin{aligned} u_1(x, y) &= 1 - e^{\lambda x}\cos(2\pi y), \\ u_2(x, y) &= \frac{\lambda}{2\pi} e^{\lambda x}\sin(2\pi y), \\ p(x, y) &= -\frac{1}{2} e^{2\lambda x} + \bar{p}, \end{aligned}$$

where

$$\lambda = \frac{-8\pi^2}{\nu^{-1} + \sqrt{\nu^{-2} + 16\pi^2}}, \qquad \bar{p} = 2\int_{-\frac{1}{2}}^{\frac{3}{2}} e^{2\lambda x}\,dx.$$

We impose this function as Dirichlet boundary condition on the boundary of Ω and choose the viscosity $\nu = 1/10$.

In our implementation, we use the spaces $\mathbf{V}_h$ and Q_h in (2.1) for different values of k on sequences of uniform meshes made of squares of size $h = 2^{-\ell}$ for some integer ℓ. As a DG discretization for the Laplacian, we choose the symmetric interior penalty form a_h with penalty parameter equal to $\nu k(k+1)/h$. It satisfies the properties listed in Assumption 2.1 with $\kappa_0 = k(k+1)$. The pressure stabilization parameter γ_E in (2.5b) is chosen as $\gamma_E = \nu^{-1}h/10$ independent of k. We use standard DG terms to incorporate the inhomogeneous Dirichlet boundary conditions.

In Table 1, we report the history of convergence of the method. There, we denote by $\mathbf{e}_u$ and $\mathbf{e}_w$ the errors $\mathbf{e}_u = \mathbf{u} - \mathbf{u}_h$ and $\mathbf{e}_w = \mathbf{u} - \mathbf{w}_h$, respectively. We see the expected convergence of order $\mathcal{O}(h^k)$ for the energy norm of the error, as well as for the standard DG norm $\|\cdot\|_{1,h}$ of the errors in $\mathbf{u}_h$ and $\mathbf{w}_h$, in agreement with Theorem 3.3. Additionally, we see that the post-processing improves the error. The norm of the pressure converges with the order $\mathcal{O}(h^k)$ for $k = 1$ and $k = 3$; a slightly higher order is observed for $k = 2$.

In Table 2, we report the maximum norm of the (piecewise) divergence of the approximate velocities. Note that $\nabla \cdot \mathbf{u}_h$ does not converge faster than $\|\mathbf{e}_u\|_{1,h}$, and that the post-processed divergence is of the order of the computational accuracy.

Table 1 History of convergence of the errors for the velocities $\mathbf{u}_h$, $\mathbf{w}_h$ and the pressure p_h for the Kovasznay flow ($\nu = 1/10$). Polynomials of degree k and meshes made of squares of size h are used

h	$\vert\vert\vert (\mathbf{e}_u, e_p) \vert\vert\vert$	Order	$\Vert \mathbf{e}_u \Vert_{1,h}$	$\Vert \mathbf{e}_w \Vert_{1,h}$	$\Vert e_p \Vert_0$	Order
$k = 1$ (linear)						
2^{-5}	7.81e−1	1.04	1.85e−0	1.36e−0	5.72e−2	1.17
2^{-6}	3.81e−1	1.03	9.13e−1	6.72e−1	2.87e−2	1.00
2^{-7}	1.88e−1	1.02	4.53e−1	3.34e−1	1.45e−2	0.98
2^{-8}	9.37e−2	1.01	2.26e−1	1.66e−1	7.30e−3	0.99
$k = 2$ (quadratic)						
2^{-5}	5.67e−2	2.01	1.27e−1	9.61e−2	1.45e−3	2.92
2^{-6}	1.42e−2	2.00	3.17e−2	2.32e−2	2.07e−4	2.81
2^{-7}	3.54e−3	2.00	7.93e−3	5.73e−3	3.19e−5	2.70
2^{-8}	8.86e−4	2.00	1.98e−3	1.43e−3	5.19e−6	2.62
$k = 3$ (cubic)						
2^{-5}	3.00e−3	2.98	6.05e−3	6.18e−3	1.92e−4	3.18
2^{-6}	3.76e−4	3.00	7.52e−4	7.48e−4	2.32e−5	3.04
2^{-7}	4.71e−5	3.00	9.37e−5	9.19e−5	2.89e−6	3.01
2^{-8}	5.88e−6	3.00	1.17e−5	1.14e−5	3.61e−7	3.00

Table 2 History of convergence of the divergence of each of the two approximate velocities inside the elements for the Kovasznay flow ($\nu = 1/10$). Polynomials of degree k and meshes made of squares of size h are used.

h	$\sup \vert\nabla \cdot \mathbf{u}_h\vert$			$\sup \vert\nabla \cdot \mathbf{w}_h\vert$		
	$k = 1$	$k = 2$	$k = 3$	$k = 1$	$k = 2$	$k = 3$
2^{-5}	1.08e−0	1.01e−1	8.45e−3	1.30e−07	2.80e−09	6.93e−11
2^{-6}	3.87e−1	2.16e−2	1.02e−3	4.20e−09	8.44e−11	6.54e−11
2^{-7}	1.64e−1	4.90e−3	1.27e−4	2.36e−10	9.78e−11	9.30e−11
2^{-8}	8.35e−2	1.16e−3	1.59e−5	2.43e−10	3.19e−10	1.88e−10

9 Concluding Remarks

Let us end by pointing out that what has been done here for the two dimensional case can also be carried out for the three-dimensional one. Tetrahedra and cubic prisms could be easily handled.

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