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# Superconvergence for Optimal Control Problems Governed by Semi-linear Elliptic Equations

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**Abstract** In this paper, we will investigate the superconvergence of the finite element approximation for quadratic optimal control problem governed by semi-linear elliptic equations. The state and co-state variables are approximated by the piecewise linear functions and the control variable is approximated by the piecewise constant functions. We derive the superconvergence properties for both the control variable and the state variables. Finally, some numerical examples are given to demonstrate the theoretical results.

**Keywords** Finite element approximation · Superconvergence · Semi-linear elliptic equation · Optimal control problem · Interpolate operator

## 1 Introduction

In this paper, we shall study the superconvergence of the finite element approximation for the following optimal control problem

$$\min_{u\in K}\left\{\frac{1}{2}\|y-y_d\|^2+\frac{1}{2}\|u\|^2\right\}, \tag{1.1}$$

$$-\mathrm{div}(A\nabla y)+\phi(y)=f+Bu, \qquad x\in\Omega, \tag{1.2}$$

$$y|_{\partial\Omega}=0, \tag{1.3}$$

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where $\Omega$ is a convex bounded domain in $\mathbb{R}^2$ with smooth boundary. The details will be specified later. Problem (1.1)–(1.3) appears, for example, in temperate control problems, see [13].

Finite element approximation of optimal control problems plays a very important role in numerical methods for these problems. There have been extensively studies on this aspect, see, for example, [1, 2, 6–11, 14, 16, 18, 23, 24, 29, 30]. A systematic introduction of finite element method for PDEs and optimal control problem can be found in, for example, [12, 15, 27], and [28].

For optimal control problem governed by semi-linear elliptic state equations, a priori error estimates of finite element approximation were studied in, for example, [3] and [23]. A posteriori error estimates for this problem were discussed by Huang et al. [17].

In [20, 26], superconvergence properties of the control variable for the linear elliptic control problem are presented. Yang and Chang in [31] showed the superconvergence properties for optimal control problem of bilinear type. The purpose of this paper is to extend the superconvergence property of [20] to the above semi-linear control problem and also to get the superconvergence properties for the state variables.

The paper is organized as follows: In Sect. 2, we shall give a brief review on the finite element method and then construct the approximation schemes for the model optimal control problem. In Sect. 3, we shall give some intermediate error estimates which is useful to derive the superconvergence. In Sect. 4, superconvergence results for both control and state variables were derived. In Sect. 5, we will give some applications of the results derived in Sect. 4. In Sect. 6, some numerical examples are given to demonstrate our theoretical results. In the last section we briefly discuss some possible future work.

In this paper we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on $\Omega$ with a norm $\|\cdot\|_{m,p}$ given by

$$\|\phi\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha\phi\|_{L^p(\Omega)}^p,$$

a semi-norm $|\cdot|_{m,p}$ given by

$$|\phi|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha\phi\|_{L^p(\Omega)}^p.$$

We set

$$W_0^{m,p}(\Omega) = \{\phi \in W^{m,p}(\Omega) : \phi|_{\partial\Omega} = 0\}.$$

For simplicity, we denote $H^m(\Omega) = W^{m,2}(\Omega)$ and $\|\cdot\|_m = \|\cdot\|_{m,2}$ when $p = 2$. In addition, $c$ and $C$ denotes a general positive constant independent of $h$.

## 2 Preliminaries

In this section we will study the finite element approximation of (1.1)–(1.3). In the rest of the paper, we shall take the state space $V = H_0^1(\Omega)$, the control space $U = L^2(\Omega)$. Let $B$ be a linear continuous operator from $U$ to $H$, Let $K$ denotes the admissible set of the control variable defined by $K = \{u \in U : u \ge 0\}$. Let

$$a(y, w) = \int_\Omega (A\nabla y)\cdot\nabla w, \quad \forall y, w \in V,$$

$$(u,v)=\int_\Omega uv,\quad \forall (u,v)\in U\times U,$$
$$A(\cdot)=(a_{i,j}(\cdot))_{n\times n}\in (W^{1,\infty}(\Omega))^{n\times n},$$

satisfying that there is a constant $c>0$ such that for any vector $\mathbf{X}\in\mathbb{R}^n$,

$$\mathbf{X}^t A\mathbf{X}\geq c\|\mathbf{X}\|^2_{\mathbb{R}^n}.$$

It follows from the assumptions on $A$ that there are constants $c, C>0$ such that $\forall y, w\in V$

$$a(y,y)\geq c\|y\|_V^2,\qquad |a(y,w)|\leq C\|y\|_V\|w\|_V.$$

Then the standard weak formula for the state equation reads: find $y(u)\in V$ such that

$$a(y(u),w)+(\phi(y(u)),w)=(f+Bu,w),\quad \forall w\in V,$$

where we assume that the function $\phi(\cdot)\in W^{2,\infty}(-R,R)$ for any $R>0$, $\phi'(y)\in L^2(\Omega)$ for any $y\in H^1(\Omega)$, and $\phi'\geq 0$. Thus the above equation has a unique solution.

We recast (1.1)–(1.3) in the following weak form: find $(y,u)\in V\times U$ such that

$$\min_{u\in K\subset U}\left\{\frac{1}{2}\|y-y_d\|^2+\frac{1}{2}\|u\|^2\right\},\tag{2.1}$$
$$a(y,w)+(\phi(y),w)=(f+Bu,w),\quad \forall w\in V=H_0^1(\Omega).\tag{2.2}$$

It is well known (see, e.g., [22]) that the control problem (2.1)–(2.2) has a unique solution $(y,u)$ and that if a pair $(y,u)$ is the solution of (2.1)–(2.2), then there is a co-state $p\in V$ such that the triplet $(y,p,u)$ satisfies the following optimality conditions:

$$a(y,w)+(\phi(y),w)=(f+Bu,w),\quad \forall w\in V=H_0^1(\Omega),\tag{2.3}$$
$$a(q,p)+(\phi'(y)p,q)=(y-y_d,q),\quad \forall q\in V=H_0^1(\Omega),\tag{2.4}$$
$$(u+B^*p,v-u)\geq 0,\quad \forall v\in K\subset U=L^2(\Omega),\tag{2.5}$$

where $B^*$ is the adjoint operator of $B$.

In the following we construct the finite element approximation for the optimal control problem (2.1)–(2.2). For ease of exposition we will assume that $\Omega$ is a polygon. Let $T_h$ be a quasi-uniform partition of $\Omega$ into triangles. And let $h$ be the maximum diameter of $T$ in $T_h$.

Moreover, we set

$$U_h=\{\tilde{u}\in U:\ \tilde{u}|_T \text{ is constant on } T\in T_h\},$$
$$V_h=\{y_h\in C(\overline{\Omega}):\ y_h|_T\in\mathbb{P}_1,\ \forall T\in T_h\},$$

where $\mathbb{P}_1$ is the polynomials of degree less than or equal to 1 for triangulation. Let $K_h\subset K\cap U_h$ be a closed convex set. Now, the finite element approximation of the problem (2.1)–(2.2) is as follows:

$$\min_{u_h\in K_h}\left\{\frac{1}{2}\|y_h-y_d\|^2+\frac{1}{2}\|u_h\|^2\right\},\tag{2.6}$$
$$a(y_h,w_h)+(\phi(y_h),w_h)=(f+Bu_h,w_h),\quad \forall w_h\in V_h\subset H_0^1(\Omega).\tag{2.7}$$

The control problem (2.6)–(2.7) has a unique solution $(y_h, u_h)$, and a pair $(y_h, u_h)$ is the solution of (2.6)–(2.7) if and only if their is a co-state $p_h$ such that the triplet $(y_h, p_h, u_h)$ satisfying the following optimal conditions:

$$a(y_h, w_h) + (\phi(y_h), w_h) = (f + Bu_h, w_h), \quad \forall w_h \in V_h \subset H_0^1(\Omega), \tag{2.8}$$

$$a(q_h, p_h) + (\phi'(y_h)p_h, q_h) = (y_h - y_d, q_h), \quad \forall q_h \in V_h \subset H_0^1(\Omega), \tag{2.9}$$

$$(u_h + B^* p_h, v_h - u_h) \geq 0, \quad \forall v_h \in K_h \subset L^2(\Omega). \tag{2.10}$$

## 3 Intermediate Error Estimates

First, we shall use some intermediate variables. For any $\tilde{u} \in K$, let $(y(\tilde{u}), p(\tilde{u}))$ be the solution of the following equations:

$$a(y(\tilde{u}), w) + (\phi(y(\tilde{u})), w) = (f + B\tilde{u}, w), \quad \forall w \in V, \tag{3.1}$$

$$a(q, p(\tilde{u})) + (\phi'(y(\tilde{u}))p(\tilde{u}), q) = (y(\tilde{u}) - y_d, q), \quad \forall q \in V. \tag{3.2}$$

Then, for any $\tilde{u} \in K$, let $(y_h(\tilde{u}), p_h(\tilde{u}))$ be the solution of the following equations:

$$a(y_h(\tilde{u}), w_h) + (\phi(y_h(\tilde{u})), w_h) = (f + B\tilde{u}, w_h), \quad \forall w_h \in V_h, \tag{3.3}$$

$$a(q_h, p_h(\tilde{u})) + (\phi'(y_h(\tilde{u}))p_h(\tilde{u}), q_h) = (y_h(\tilde{u}) - y_d, q_h), \quad \forall q_h \in V_h. \tag{3.4}$$

Thus, we have $(y, p) = (y(u), p(u))$, $(y_h, p_h) = (y_h(u_h), p_h(u_h))$.

We introduce the standard $L^2(\Omega)$-orthogonal projection $Q_h : U \to U_h$, which satisfies: for any $\psi \in U$

$$(\psi - Q_h\psi, \mu_h) = 0, \quad \forall \mu_h \in U_h, \tag{3.5}$$

and the elliptic projection $R_h : V \to V_h$, which satisfies: for all $w \in V$

$$a(w - R_h w, w_h) = 0, \quad w_h \in V_h. \tag{3.6}$$

We have the following approximation properties:

$$\|\psi - Q_h\psi\|_{-s} \leq Ch^{1+s}|\psi|_1, \quad s = 0, 1, \tag{3.7}$$

$$\|w - R_h w\| \leq Ch^2\|w\|_2, \quad \text{for } w \in H^2(\Omega). \tag{3.8}$$

**Lemma 3.1** *Let $u$ be the solution of* (2.3)–(2.4), *for $h$ sufficiently small, there exists a positive constant $C$ such that*

$$\|y(Q_h u) - y(u)\|_1 \leq Ch^2, \tag{3.9}$$

$$\|p(Q_h u) - p(u)\|_1 \leq Ch^2. \tag{3.10}$$

*Proof* Choose $\tilde{u} = Q_h u$ and $\tilde{u} = u$ in (3.1)–(3.2), respectively, then we have the following error equations

$$a(y(Q_h u) - y(u), w) + (\phi(y(Q_h u)) - \phi(y(u)), w) = (B(Q_h u - u), w), \tag{3.11}$$

$$a(q, p(Q_h u) - p(u)) + (\phi'(y(Q_h u))p(Q_h u) - \phi'(y(u))p(u), q)$$
$$= (y(Q_h u) - y(u), q), \tag{3.12}$$

for any $w \in V$ and $q \in V$.

First, choose $w = y(Q_h u) - y(u)$ in (3.11), we have

$$
\begin{aligned}
&a(y(Q_h u) - y(u), y(Q_h u) - y(u)) + (\phi(y(Q_h u)) - \phi(y(u)), y(Q_h u) - y(u)) \\
&\quad = (B(Q_h u - u), y(Q_h u) - y(u)). \qquad (3.13)
\end{aligned}
$$

Now, we estimate the right hand side of (3.13). Using the continuity of $B$ and (3.7), we have

$$
\begin{aligned}
&(B(Q_h u - u), y(Q_h u) - y(u)) = (Q_h u - u, B^*(y(Q_h u) - y(u))) \\
&\quad \le C\|B^*(y(Q_h(u)) - y(u))\|_1 \cdot \|Q_h u - u\|_{-1} \\
&\quad \le Ch^2\|u\|_1 \cdot \|y(Q_h u) - y(u)\|_1. \qquad (3.14)
\end{aligned}
$$

From (3.13), (3.14) and the assumption of $A$ and $\phi(\cdot)$, we have

$$
\begin{aligned}
&c\|y(Q_h u) - y(u)\|_1^2 \\
&\quad \le a(y(Q_h u) - y(u), y(Q_h u) - y(u)) \\
&\qquad + (\phi(y(Q_h u)) - \phi(y(u)), y(Q_h u) - y(u)) \\
&\quad = (B(Q_h u - u), y(Q_h u) - y(u)) \\
&\quad \le Ch^2\|y(Q_h u) - y(u)\|_1, \qquad (3.15)
\end{aligned}
$$

then, (3.9) can be obtained from (3.15).

Choose $q = p(Q_h u) - p(u)$ in (3.12), we have

$$
\begin{aligned}
&a(p(Q_h u) - p(u), p(Q_h u) - p(u)) \\
&\quad + (\phi'(y(Q_h u))p(Q_h u) - \phi' y(u)p(u), p(Q_h u) - p(u)) \\
&= (y(Q_h u) - y(u), p(Q_h u) - p(u)), \qquad (3.16)
\end{aligned}
$$

namely,

$$
\begin{aligned}
&a(p(Q_h u) - p(u), p(Q_h u) - p(u)) \\
&\quad + (\phi'(y(Q_h u))(p(Q_h u) - p(u)), p(Q_h u) - p(u)) \\
&= (y(Q_h u) - y(u), p(Q_h u) - p(u)) \\
&\quad + (p(u)(\phi'(y(u)) - \phi'(y(Q_h u))), p(Q_h u) - p(u)). \qquad (3.17)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
(y(Q_h u) - y(u), p(Q_h u) - p(u)) &\le C\|y(Q_h u) - y(u)\| \cdot \|p(Q_h u) - p(u)\| \\
&\le Ch^2\|p(Q_h u) - p(u)\|_1. \qquad (3.18)
\end{aligned}
$$

Using the assumption of $\phi(\cdot)$ and (3.9), we have

$$
\begin{aligned}
&(p(u)(\phi'(y(u)) - \phi'(y(Q_h u))), p(Q_h u) - p(u)) \\
&\quad \le C\|p(u)\|_{0,4}\|\phi'(y(u)) - \phi'(y(Q_h u))\| \cdot \|p(Q_h u) - p(u)\|_{0,4} \\
&\quad \le C\|p(u)\|_1\|\phi\|_{W^{2,\infty}}\|y(u) - y(Q_h u)\| \cdot \|p(Q_h u) - p(u)\|_1 \\
&\qquad \le Ch^2\|p(Q_h u) - p(u)\|_1, \qquad (3.19)
\end{aligned}
$$

where we used the embedding $\|v\|_{0,4} \le C\|v\|_1$. Then, using (3.17), (3.18), (3.19) and the assumption of $\phi(\cdot)$, we have

$$\begin{aligned}
&c\|p(Q_hu) - p(u)\|_1^2 \\
&\quad \le a(p(Q_hu) - p_h(u), p(Q_hu) - p(u)) \\
&\qquad + (\phi'(y(Q_hu)(p(Q_hu) - p(u)), p(Q_hu) - p(u)) \\
&\quad = (y(Q_hu) - y(u), p(Q_hu) - p(u)) \\
&\qquad + (p(u)(\phi'(y(u)) - \phi'(y(Q_hu))), p(Q_hu) - p(u)) \\
&\quad \le Ch^2\|p(Q_hu) - p(u)\|_1, \qquad (3.20)
\end{aligned}$$

which implies (3.12). □

**Lemma 3.2** *For any $\tilde{u} \in K$, if the intermediate solution satisfies*

$$y(\tilde{u}), p(\tilde{u}) \in H^2(\Omega),$$

*and $\Omega$ is convex, then, we have*

$$\|y_h(\tilde{u}) - R_hy(\tilde{u})\|_1 \le Ch^2, \qquad (3.21)$$

$$\|p_h(\tilde{u}) - R_hp(\tilde{u})\|_1 \le Ch^2. \qquad (3.22)$$

*Proof* According to Theorem 8.2.9 in [25], the following estimate hold

$$\|y(\tilde{u}) - y_h(\tilde{u})\| + \|p(\tilde{u}) - p_h(\tilde{u})\| \le Ch^2. \qquad (3.23)$$

From (3.1)–(3.2) and (3.3)–(3.4), we have the following error equations:

$$a(y_h(\tilde{u}) - y(\tilde{u}), w_h) + (\phi(y_h(\tilde{u})) - \phi(y(\tilde{u})), w_h) = 0, \qquad (3.24)$$

$$\begin{aligned}
&a(q_h, p_h(\tilde{u}) - p(\tilde{u})) + (\phi'(y_h(\tilde{u}))p_h(\tilde{u}) - \phi'(y(\tilde{u}))p(\tilde{u}), q_h) \\
&\quad = (y_h(\tilde{u}) - y(\tilde{u}), q_h), \qquad (3.25)
\end{aligned}$$

for any $w_h \in V_h$ and $q_h \in V_h$. Using the definition of $R_h$, the above equation can be restated as

$$a(y_h(\tilde{u}) - R_hy(\tilde{u}), w_h) = (\phi(y(\tilde{u})) - \phi(y_h(\tilde{u})), w_h), \qquad (3.26)$$

$$\begin{aligned}
&a(q_h, p_h(\tilde{u}) - R_hp(\tilde{u})) + (\phi'(y_h(\tilde{u}))(p_h(\tilde{u}) - R_hp(\tilde{u})), q_h) \\
&\quad = (y_h(\tilde{u}) - y(\tilde{u}), q_h) \\
&\qquad + (p(\tilde{u})(\phi'(y(\tilde{u})) - \phi'(y_h(\tilde{u}))), q_h) + (\phi'(y_h(\tilde{u}))(p(\tilde{u}) - R_hp(\tilde{u})), q_h). \qquad (3.27)
\end{aligned}$$

First, let $w_h = y_h(\tilde{u}) - R_hy(\tilde{u})$ in (3.26), we have

$$\begin{aligned}
&c\|y_h(\tilde{u}) - R_hy(\tilde{u})\|_1^2 \\
&\quad \le a(y_h(\tilde{u}) - R_hy(\tilde{u}), y_h(\tilde{u}) - R_hy(\tilde{u})) \\
&\quad = (\phi(y(\tilde{u})) - \phi(y_h(\tilde{u})), y_h(\tilde{u}) - R_hy(\tilde{u}))
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\phi\|_{W^{1,\infty}}\|y(\tilde{u}) - y_h(\tilde{u})\| \cdot \|y_h(\tilde{u}) - R_h y(\tilde{u})\| \\
&\le Ch^2\|\phi\|_{W^{1,\infty}}\|y(\tilde{u})\|_2 \cdot \|y_h(\tilde{u}) - R_h y(\tilde{u})\| \\
&\le Ch^2\|y_h(\tilde{u}) - R_h y(\tilde{u})\|_1, \qquad (3.28)
\end{aligned}
$$

which implies (3.21).

Then, let $q_h = p_h(\tilde{u}) - R_h p(\tilde{u})$ in (3.27). Notice that

$$
\begin{aligned}
(y_h(\tilde{u}) - y(\tilde{u}), p_h(\tilde{u}) - R_h p(\tilde{u})) &\le \|y_h(\tilde{u}) - y(\tilde{u})\| \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\| \\
&\le Ch^2\|y(\tilde{u})\|_2 \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\| \\
&\le Ch^2\|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1. \qquad (3.29)
\end{aligned}
$$

Using the assumption of $\phi(\cdot)$, we have

$$
\begin{aligned}
&(p(\tilde{u})(\phi'(y(\tilde{u})) - \phi'(y_h(\tilde{u}))), p_h(\tilde{u}) - R_h p(\tilde{u})) \\
&\quad\le C\|p(\tilde{u})\|_{0,4}\|\phi'(y(\tilde{u})) - \phi'(y_h(\tilde{u}))\| \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\|_{0,4} \\
&\quad\le Ch^2\|p(\tilde{u})\|_1 \cdot \|\phi\|_{W^{2,\infty}}\|y(\tilde{u})\|_2 \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1 \\
&\quad\le Ch^2\|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1, \qquad (3.30)
\end{aligned}
$$

where we used the embedding $\|v\|_{0,4} \le C\|v\|_1$. Then, using the definition of $R_h$ and the assumption of $\phi(\cdot)$, we have

$$
\begin{aligned}
&(\phi'(y_h(\tilde{u}))(p(\tilde{u}) - R_h p(\tilde{u})), p_h(\tilde{u}) - R_h p(\tilde{u})) \\
&\quad\le C\|\phi\|_{W^{1,\infty}}\|p(\tilde{u}) - R_h p(\tilde{u})\| \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\| \\
&\quad\le Ch^2\|\phi\|_{W^{1,\infty}}\|p(\tilde{u})\|_2 \cdot \|p_h(\tilde{u}) - R_h p(\tilde{u})\| \\
&\quad\le Ch^2\|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1. \qquad (3.31)
\end{aligned}
$$

From (3.27) and (3.29)–(3.31), we have

$$
\begin{aligned}
&c\|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1^2 \\
&\quad\le a(p_h(\tilde{u}) - R_h p(\tilde{u}), p_h(\tilde{u}) - R_h p(\tilde{u})) \\
&\qquad + (\phi'(y_h(\tilde{u}))(p_h(\tilde{u}) - R_h p(\tilde{u})), p_h(\tilde{u}) - R_h p(\tilde{u})) \\
&\quad= (y_h(\tilde{u}) - y(\tilde{u}), p_h(\tilde{u}) - R_h p(\tilde{u})) + (p(\tilde{u})(\phi'(y(\tilde{u})) - \phi'(y_h(\tilde{u}))), p_h(\tilde{u}) - R_h p(\tilde{u})) \\
&\qquad + (\phi'(y_h(\tilde{u}))(p(\tilde{u}) - R_h p(\tilde{u})), p_h(\tilde{u}) - R_h p(\tilde{u})) \\
&\quad\le Ch^2\|p_h(\tilde{u}) - R_h p(\tilde{u})\|_1, \qquad (3.32)
\end{aligned}
$$

which implies (3.22). □

Let $y(u)$ and $y_h(u_h)$ be the solution of (2.2) and (2.7), respectively. Set

$$
J(u) = \left\{\frac{1}{2}\|y - y_d\|^2 + \frac{1}{2}\|u\|^2\right\},
$$

$$
J_h(u_h) = \left\{\frac{1}{2}\|y_h(u_h) - y_d\|^2 + \frac{1}{2}\|u_h\|^2\right\}.
$$

Then, the reduced problems of (2.1) and (2.6) read as

$$\min_{u\in K}\{J(u)\}, \tag{3.33}$$

and

$$\min_{u_h\in K_h}\{J_h(u_h)\}, \tag{3.34}$$

respectively. It can be shown that

$$\begin{aligned}(J'(u), v) &= (u + B^* p, v),\\ (J'(u_h), v) &= (u_h + B^* p(u_h), v),\\ (J'(Q_h u), v) &= (Q_h u + B^* p(Q_h u), v),\\ (J'_h(u_h), v) &= (u_h + B^* p_h, v),\end{aligned}$$

where $p(u_h)$ and $p(Q_h u)$ are solutions of (3.1)–(3.2) with $\tilde{u} = u_h$ and $\tilde{u} = Q_h u$, respectively.

In many application, $J(\cdot)$ is uniform convex near the solution $u$. The convexity of $J(\cdot)$ is closely related to the second order sufficient conditions of the control problem, which are assumed in many studies on numerical methods of the problem. Then, there is a constant $c > 0$, independent of h, such that

$$(J'(Q_h u) - J'(u_h), Q_h u - u_h) \geq c\|Q_h u - u_h\|^2, \tag{3.35}$$

where $u$ and $u_h$ are solutions of (3.33) and (3.34) respectively, $Q_h u$ is the orthogonal projection of $u$ which is defined in (3.5). We shall assume the above inequality throughout this paper. More discussion of this can be found in, for example, [4] and [5].

## 4 Superconvergence Properties

In this section, we will discuss the superconvergence for both the control variable and the state variables. Let $\pi^c$ be the average operator defined in [20]. Let

$$\begin{aligned}\Omega^+ &= \{\cup\tau : \tau \subset \Omega, u|_\tau > 0\},\\ \Omega^0 &= \{\cup\tau : \tau \subset \Omega, u|_\tau = 0\},\\ \Omega^b &= \Omega\backslash(\Omega^+ \cup \Omega^0).\end{aligned}$$

In this paper, we assume that $u$ and $T_h$ are regular such that $\mathrm{meas}(\Omega^b) \leq Ch$.

**Theorem 4.1** *Let $u$ be the solution of* (2.3)–(2.5) *and $u_h$ be the solution of* (2.8)–(2.10). *We assume that the exact control and state solution satisfy*

$$u,\ u + B^* p \in W^{1,\infty}(\Omega),$$

*and*

$$y(u),\ p(u) \in H^2(\Omega).$$

*Then, we have*

$$\|Q_h u - u_h\| \le Ch^{\frac{3}{2}}. \tag{4.1}$$

*Proof* Let $v = u_h$ in (2.5) and $v_h = Q_h u$ in (2.10), then add the two inequalities we have

$$(u_h + B^* p_h - u - B^* p, Q_h u - u_h) + (u + B^* p, Q_h u - u) \ge 0. \tag{4.2}$$

Hence

$$\begin{aligned}
&(Q_h u - u_h, Q_h u - u_h)\\
&\quad = (u - u_h, Q_h u - u_h)\\
&\quad \le (B^* p_h - B^* p, Q_h u - u_h) + (u + B^* p, Q_h u - u).
\end{aligned} \tag{4.3}$$

For the first term of (4.3), we divide it into three parts,

$$\begin{aligned}
&(B^* p_h - B^* p, Q_h u - u_h)\\
&\quad = (B^* p_h - B^* p(u_h), Q_h u - u_h) + (B^* p(u_h) - B^* p(Q_h u), Q_h u - u_h)\\
&\qquad + (B^* p(Q_h u) - B^* p(u), Q_h u - u_h),
\end{aligned} \tag{4.4}$$

then, from (4.3)–(4.4), we have

$$\begin{aligned}
&(Q_h u - u_h, Q_h u - u_h) - (B^* p(u_h) - B^* p(Q_h u), Q_h u - u_h)\\
&\quad \le (B^* p_h - B^* p(u_h), Q_h u - u_h) + (B^* p(Q_h u) - B^* p(u), Q_h u - u_h)\\
&\qquad + (u + B^* p, Q_h u - u).
\end{aligned} \tag{4.5}$$

According to (3.23),

$$\begin{aligned}
(B^* p(u_h) - B^* p_h, Q_h u - u_h) &\le C\|B^*(p(u_h) - p_h)\| \cdot \|Q_h u - u_h\|\\
&\le Ch^2 \|Q_h u - u_h\|.
\end{aligned} \tag{4.6}$$

For the second term of (4.3)

$$(u + B^* p, Q_h u - u) = \int_{\Omega^+} + \int_{\Omega^0} + \int_{\Omega^b} (u + B^* p, Q_h u - u)dx.$$

Obviously, $(Q_h u - u)|_{\Omega^0} = 0$. From (2.5), we have pointwise a.e. $(u + B^* p) \ge 0$, we choose $\tilde{u}|_{\Omega^+} = 0$ and $\tilde{u}|_{\Omega\backslash\Omega^+} = u$, so that $(u + B^* p, u)|_{\Omega^+} \le 0$. Hence, $(u + B^* p)|_{\Omega^+} = 0$. Then

$$\begin{aligned}
(u + B^* p, Q_h u - u) &= (u + B^* p, Q_h u - u)_{\Omega^b}\\
&\le (u + B^* p - \pi^c(u + B^* p), Q_h u - u)_{\Omega^b}\\
&\le Ch^2 \|u + B^* p\|_{1,\Omega^b} \|u\|_{1,\Omega^b}\\
&\le Ch^2 \|u + B^* p\|_{1,\infty} \|u\|_{1,\infty} \cdot \text{meas}(\Omega^b)\\
&\le Ch^3.
\end{aligned} \tag{4.7}$$

According to (3.35), the left hand of (4.5) can be restated as:

$$\begin{aligned}(Q_hu-u_h, Q_hu-u_h) &- (B^*p(u_h)-B^*p(Q_hu), Q_hu-u_h)\\ &= (Q_hu+B^*p(Q_hu), Q_hu-u_h)-(u_h+B^*p(u_h), Q_hu-u_h)\\ &= (J'(Q_hu)-J'(u_h), Q_hu-u_h)\\ &\ge c\|Q_hu-u_h\|^2. \end{aligned} \tag{4.8}$$

Then, combining Lemma 3.1 and (4.5)–(4.8), we have

$$\|Q_hu-u_h\|\le Ch^{\frac{3}{2}},$$

which complete the proof of Theorem 4.1. □

**Theorem 4.2** *Let $u$ be the solution of* (2.3)–(2.5), *$u_h$ be the solution of* (2.8)–(2.10) *and $\Omega$ is convex. We assume that the exact control and state solution satisfy*

$$u,\ u+B^*p\in W^{1,\infty}(\Omega),$$

*and*

$$y(u),\ p(u)\in H^2(\Omega).$$

*Then, we have*

$$\|y_h-R_hy\|_1\le Ch^{\frac{3}{2}}, \tag{4.9}$$

$$\|p_h-R_hp\|_1\le Ch^{\frac{3}{2}}. \tag{4.10}$$

*Proof* From (2.3)–(2.4) and (2.8)–(2.9). We have the following error equations

$$a(y_h-y, w_h)+(\phi(y_h)-\phi(y), w_h)=(B(u_h-u), w_h),\quad \forall w_h\in V_h, \tag{4.11}$$

$$a(q_h, p_h-p)+(\phi'(y_h)p_h-\phi'(y)p, q_h)=(y_h-y, q_h),\quad \forall q_h\in V_h. \tag{4.12}$$

Using the definition of $R_h$, we have

$$\begin{aligned}a(y_h-R_hy, w_h)&+(\phi(y_h)-\phi(R_hy), w_h)\\ &=(B(u_h-u), w_h)+(\phi(y)-\phi(R_hy), w_h),\end{aligned} \tag{4.13}$$

$$\begin{aligned}a(q_h, p_h-R_hp)&+(\phi'(y_h)(p_h-R_hp), q_h)\\ &=(y_h-y, q_h)+(\phi'(y_h)(p-R_hp), q_h)+(p(\phi'(y)-\phi'(y_h)), q_h),\end{aligned} \tag{4.14}$$

for any $w_h\in V_h$ and $q_h\in V_h$.

First, take $w_h=y_h-R_hy$ in (4.13) and using the assumption of $\phi(\cdot)$, we have

$$\begin{aligned}c\|y_h-R_hy\|_1^2 &\le a(y_h-R_hy, y_h-R_hy)+(\phi(y_h)-\phi(R_hy), y_h-R_hy)\\ &=(B(u_h-Q_hu), y_h-R_hy)+(B(Q_hu-u), y_h-R_hy)+(\phi(y)-\phi(R_hy), y_h-R_hy)\\ &\le C(\|u_h-Q_hu\|\cdot\|y_h-R_hy\|+\|Q_hu-u\|_{-1}\|B^*(y_h-R_hy)\|_1\end{aligned}$$

$$
\begin{aligned}
&+\|\phi\|_{1,\infty}\|y-R_hy\|\cdot\|y_h-R_hy\|)\\
&\le C(h^{\frac{3}{2}}\|y_h-R_hy\|_1+h^2\|u\|_1\|y_h-R_hy\|_1+h^2\|\phi\|_{1,\infty}\|y\|_2\|y_h-R_hy\|_1)\\
&\le Ch^{\frac{3}{2}}\|y_h-R_hy\|_1, \qquad (4.15)
\end{aligned}
$$

which implies (4.9).

Then, we take $q_h = p_h - R_h p$ in (4.14). Notice that

$$
\begin{aligned}
(y_h-y, p_h-R_hp) &= (y_h-R_hy, p_h-R_hp)+(R_hy-y, p_h-R_hp)\\
&\le Ch^{\frac{3}{2}}\|p_h-R_hp\|. \qquad (4.16)
\end{aligned}
$$

Using the definition of $R_h$ and the assumption of $\phi(\cdot)$, we have

$$
\begin{aligned}
(\phi'(y_h)(p-R_hp), p_h-R_hp) &\le Ch^2\|\phi\|_{1,\infty}\|p\|_2\|p_h-R_hp\|\\
&\le Ch^2\|p_h-R_hp\|, \qquad (4.17)
\end{aligned}
$$

and

$$
\begin{aligned}
&(p(\phi'(y)-\phi'(y_h)), p_h-R_hp)\\
&\quad\le C\|\phi\|_{2,\infty}(p(y-y_h), p_h-R_hp)\\
&\quad\le C\|\phi\|_{2,\infty}\|y-y_h\|\cdot\|p\|_{0,4}\|p_h-R_hp\|_{0,4}\\
&\quad\le C\|\phi\|_{2,\infty}(\|y-R_hy\|+\|R_hy-y_h\|)\|p\|_1\|p_h-R_hp\|_1\\
&\quad\le Ch^{\frac{3}{2}}\|p_h-R_hp\|_1. \qquad (4.18)
\end{aligned}
$$

From (4.14) and (4.16)–(4.18), we have

$$
\begin{aligned}
c\|p_h-R_hp\|_1^2 &\le a(p_h-R_hp, p_h-R_hp)+(\phi'(y_h)(p_h-R_hp), p_h-R_hp)\\
&= (y_h-y, p_h-R_hp)+(\phi'(y_h)(p-R_hp), p_h-R_hp)\\
&\quad+(p(\phi'(y)-\phi'(y_h)), p_h-R_hp)\\
&\le Ch^{\frac{3}{2}}\|p_h-R_hp\|_1, \qquad (4.19)
\end{aligned}
$$

which implies (4.10). □

## 5 Application

In this section, some applications of the results derived in Sect. 4 will be presented.

First, we will introduce a higher order interpolation operator $I_{2h}^2$ which is presented by Lin and Zhu [21] and $I_h$ be the Lagrangian interpolate operator. They satisfy the following properties:

$$\|v-I_{2h}^2v\|_1 \le Ch^2\|v\|_3, \quad \forall v\in H^3(\Omega), \qquad (5.1)$$

$$I_{2h}^2 I_h = I_{2h}^2, \qquad (5.2)$$

$$\|I_{2h}^2 v\|_1 \le C\|v\|_1, \quad \forall v\in V_h. \qquad (5.3)$$

Using these properties we have

**Theorem 5.1** *Suppose that all the conditions of Theorem 4.2 are valid. Moreover, we will assume that $y, p \in H^3$. Then, we have*

$$\|y - I_{2h}^2 y_h\|_1 \le Ch^{\frac{3}{2}}, \tag{5.4}$$

$$\|p - I_{2h}^2 p_h\|_1 \le Ch^{\frac{3}{2}}. \tag{5.5}$$

*Proof* Obviously, by (5.2) and (5.3), we have

$$y - I_{2h}^2 y_h = y - I_{2h}^2 y + I_{2h}^2(I_h y - R_h y) + I_{2h}^2(R_h y - y_h), \tag{5.6}$$

$$\|y - I_{2h}^2 y_h\|_1 \le \|y - I_{2h}^2 y\|_1 + C\|I_h y - R_h y\|_1 + C\|R_h y - y_h\|_1. \tag{5.7}$$

According to Theorem 2.1.1 in [21], we have

$$\|I_h y - R_h y\|_1 \le Ch^2\|y\|_3. \tag{5.8}$$

Combining with (4.9) and (5.1), we complete the proof of (5.4). Similarly, we have (5.5). □

Then, let us construct the recovery operator $G_h$. Let $G_h v$ be a continuous piecewise linear function (without zero boundary constraint). The value of $G_h v$ on the nodes are defined by lease-squares argument on an element patches surrounding the nodes, the details can be refer to the definition of $R_h$ in [20].

**Theorem 5.2** *Let $u$ and $u_h$ be the solutions of* (2.3)–(2.5) *and* (2.8)–(2.10), *respectively. Assume that $u \in W^{1,\infty}(\Omega)$ and $\Omega$ is convex. Then,*

$$\|u - G_h u_h\| \le Ch^{\frac{3}{2}}. \tag{5.9}$$

*Proof* Let $Q_h u$ be defined in (3.5). Then,

$$\|u - G_h u_h\| \le \|u - G_h u\| + \|G_h u - G_h Q_h u\| + \|G_h Q_h u - G_h u_h\|. \tag{5.10}$$

According to Lemma 4.2 in [20], we have

$$\|u - G_h u\| \le Ch^{\frac{3}{2}}. \tag{5.11}$$

Using the definition of $G_h$, we have

$$G_h u = G_h Q_h u, \tag{5.12}$$

and

$$\|G_h Q_h u - G_h u_h\| \le C\|Q_h u - u_h\|. \tag{5.13}$$

Combining (4.1) and (5.10)–(5.13), we complete the proof of Theorem 5.2. □

**Table 1** The error of Example 1 on a sequential uniform refined meshes

| Resolution | $\|u-u_h\|$ | $\|Q_hu-u_h\|$ | $\|u-G_hu_h\|$ |
|---|---|---|---|
| $16\times 16$ | 2.551E-2 | 3.598E-3 | 2.362E-2 |
| $32\times 32$ | 1.295E-2 | 1.340E-3 | 9.459E-3 |
| $64\times 64$ | 6.516E-3 | 4.677E-4 | 3.525E-3 |
| $128\times 128$ | 3.267E-3 | 1.699E-4 | 1.268E-3 |

## 6 Numerical Examples

In this section, we carry out some numerical examples to demonstrate our theoretical results. The optimal problem was solved numerically by a precondition projection algorithm, see for instance [17], with codes developed based on AFEPack [19]. The discretization was already described in Sect. 2: the state and co-state equations were approximated by piecewise linear functions, whereas the control function $u$ is discretized by piecewise constant functions. In our examples, we choose the domain $\Omega=[0,1]\times[0,1]$ and $B=I$.

*Example 1* The example is to solve the following two dimension semi-linear elliptic control problem

$$\begin{aligned} &\min_{u\in K}\left\{\frac{1}{2}\|y-y_d\|^2+\frac{1}{2}\|u-u_0\|^2\right\},\\ &\text{s.t.}\quad -\Delta y+y^3=f+u,\quad u\ge 0, \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} y&=\sin(\pi x_1)\sin(\pi x_2),\\ p&=\sin(\pi x_1)\sin(\pi x_2),\\ u_0&=1.0-\sin\frac{\pi x_1}{2}-\sin\frac{\pi x_2}{2},\\ u&=\max(u_0-p,0),\\ f&=2\pi^2y+y^3-u,\\ y_d&=y-2\pi^2p-3y^2p. \end{aligned} \tag{6.2}$$

The dual equation of the state equation is

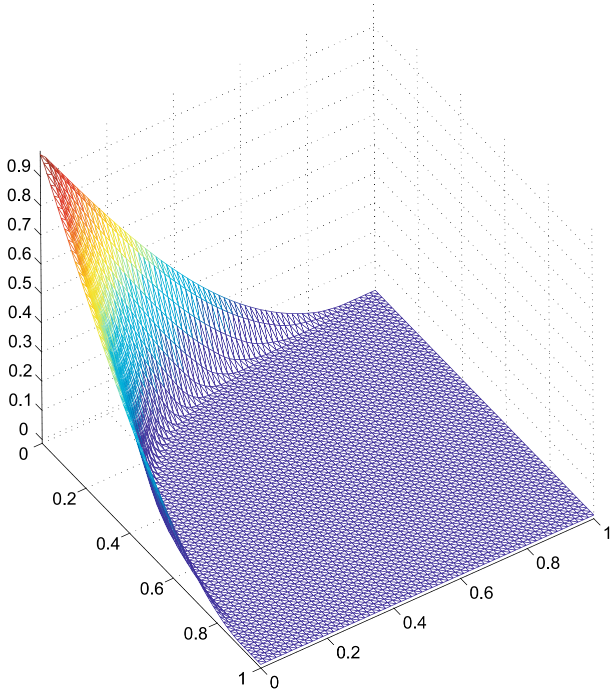
$$-\Delta p+3y^2p=y-y_d. \tag{6.3}$$

In Table 1, the error $\|u-u_h\|$, $\|Q_hu-u_h\|$ and $\|u-G_hu_h\|$ obtained on a sequence of uniformly refined meshes are shown. In Fig. 1, the profile of the numerical solution of $u$ on the $64\times 64$ mesh grid is plotted.

*Example 2* We consider the following semi-linear elliptic optimal control problem

$$\begin{aligned} &\min_{u\in K}\left\{\frac{1}{2}\|y-y_d\|^2+\frac{1}{2}\|u-u_0\|^2\right\},\\ &\text{s.t.}\quad -\Delta y+y^3=f+u,\quad u\ge 0, \end{aligned} \tag{6.4}$$

**Fig. 1** The profile of the numerical solution of Example 1 on 64 × 64 triangle mesh




where

$$
\begin{aligned}
y_d &= \sin(2\pi x_1) + \sin(2\pi x_2), \\
y &= y_d, \\
u_0 &= \max(4\pi^2 y_d, 0), \\
u &= u_0, \qquad (6.5)\\
f &= 4\pi^2 y + y^3 - u, \\
p &= 0.
\end{aligned}
$$

The dual equation of the state equation is

$$
-\Delta p + 3y^2 p = y - y_d. \qquad (6.6)
$$

Table 2 shows the numerical results of both $\|u - u_h\|$, $\|Q_h u - u_h\|$ and $\|u - G_h u_h\|$ on a sequential uniformly refined meshes. Figure 2 shows the numerical solution on the 64 × 64 mesh grid.

## 7 Conclusion

In this paper, we present the superconvergence analysis of the finite element approximation for optimal control problems governed by semi-linear elliptic equations. Our superconvergence analysis for the semi-linear elliptic equations by standard finite element method seems

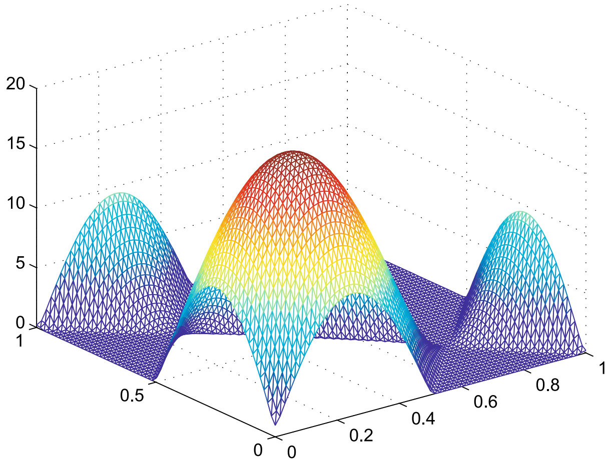



**Fig. 2** The profile of the numerical solution of Example 2 on 64 × 64 triangle mesh

**Table 2** The error of Example 2 on a sequential uniform refined meshes

| Resolution | $\|u-u_h\|$ | $\|Q_hu-u_h\|$ | $\|u-G_hu_h\|$ |
|---|---|---|---|
| 16 × 16 | 6.390E-1 | 4.559E-2 | 4.701E-1 |
| 32 × 32 | 3.214E-1 | 1.608E-2 | 1.689E-1 |
| 64 × 64 | 1.611E-1 | 5.681E-3 | 5.973E-2 |
| 128 × 128 | 8.065E-2 | 2.008E-3 | 2.106E-2 |

to be new, and these results can be extended to general convex problems. We shall study the superconvergence for optimal control problems governed by semi-linear parabolic equations. Furthermore we shall study the superconvergence of these problems by mixed finite element method.

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