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The Mortar-Discontinuous Galerkin Method for the 2D Maxwell Eigenproblem

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Abstract We consider discontinuous Galerkin (DG) approximations of the Maxwell eigenproblem on meshes with hanging nodes. It is known that while standard DG methods provide spurious-free and accurate approximations on the so-called k-irregular meshes, they may generate spurious solutions on general irregular meshes. In this paper we present a mortar-type method to cure this problem in the two-dimensional case. More precisely, we introduce a projection based penalization at non-conforming interfaces and prove that the obtained DG methods are spectrally correct. The theoretical results are validated in a series of numerical experiments on both convex and non convex problem domains, and with both regular and discontinuous material coefficients.

Keywords Discontinuous Galerkin methods · Maxwell's equations · Eigenvalue problems · Mortar methods

1 Introduction

The theory of discontinuous Galerkin (DG) approximations of the Maxwell eigenproblem with discontinuous material coefficients was developed in [7]. In that paper, under the assumption of regular meshes (i.e., with no hanging nodes), necessary and sufficient conditions under which a given DG method provides a spurious-free approximation for the Maxwell

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eigenproblem were identified. Numerical experiments reported in [6] showed that standard DG methods provide accurate and spurious-free approximations of the Maxwell eigenproblem also on some non-regular meshes, namely on k-regular meshes, where the hanging nodes are regularly spaced (the theory of [7] could be easily extended to this case). On the other hand, on general non-regular meshes they may generate spurious solutions, according to the results reported in [18] and in [6].

In this paper we introduce a cure of this problem in the two-dimensional case. We take inspiration from the literature about domain decomposition methods on non-matching grids, and the main ideas come from the mortar method which provides optimal treatment of non-conforming interfaces. We refer the reader to [4] for the definition of the method, and to [3] and [2] for its extensions to electromagnetic problems. More precisely, we introduce a projection-based corrections for the treatment of jumps across the non-conforming interfaces. The method we propose is related to the mortar-Nitsche method (see e.g., [13]) since a suitable mortar projection of jumps across the non-conforming interfaces is penalized. We adopt this approach instead of the classical mortar technique because it is more coherent with the discontinuous Galerkin paradigms.

As far as the spectral theory is concerned, the main difference with respect to the setting of [7] is that the inclusion of the discrete kernel into the continuous kernel of the Maxwell operator is no longer valid. Therefore, we need to modify the theory of [7] in order to prove that no pollution of the spectrum is generated by the mortar-DG method. On the other hand, the result on the non-pollution of the eigenspaces is weaker that the corresponding one of [7]: we can prove non-pollution of the eigenspaces for all the eigenvalues but the one associated with the kernel of the Maxwell operator. We point out that some of the tools used in our analysis are specific to the two-dimensional case and some preliminary numerical tests not reported in this paper suggest that, in the three-dimensional case, a straightforward extension of this method might present spurious solutions. Finally, our analysis applies also to the case when conforming Nédélec elements of the second family are used away from the non-conformity of the mesh. This means, in particular, that our analysis provides spectral correctness also for a mortar-type discretisation of Maxwell equations.

This paper is organized as follows: the continuous Maxwell eigenproblem and its mortar-DG discretisation are introduced in Sect. 2 and Sect. 3, respectively; the theoretical analysis of the mortar-DG method is developed in Sect. 4; finally, in Sect. 5, numerical experiments carried out on both convex and non convex problem domains, and with both regular and discontinuous material coefficients are presented, demonstrating the theoretical results.

2 2D Model Problem

Let Ω be a simply-connected bounded Lipschitz polygon in $\mathbb{R}^2$ with outward normal and counterclockwise tangent unit vectors $\mathbf{n}$ and $\mathbf{t}$, respectively, on $\partial\Omega$. Consider the eigenproblem: find $\mathbf{u} \neq 0$ and ω such that

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) = \omega^2 \varepsilon \mathbf{u},$$

with $\mathbf{u} \cdot \mathbf{t} = 0$, where μ and ε are the magnetic permeability and the electric permittivity, respectively. More precisely, $\varepsilon = \varepsilon(\mathbf{x})$ is a second order, real, symmetric tensor-valued function satisfying

$$\varepsilon_\star(\mathbf{x}) \leq \sum_{i,j=1}^{2} \varepsilon_{i,j} \xi_i \xi_j \leq \varepsilon^\star(\mathbf{x}) \quad \text{a.e. in } \Omega, \ \forall \xi \in \mathbb{R}^2, \ \|\xi\| = 1,$$

where $\varepsilon^\star$, $\varepsilon_\star \in L^\infty(\Omega)$, whereas $\mu = \mu(\mathbf{x})$ is a scalar function such that $\mu_\star(\mathbf{x}) = \mu(\mathbf{x}) = \mu^\star(\mathbf{x})$, with $\mu^\star$, $\mu_\star \in L^\infty(\Omega)$; we assume that both $\inf_{\mathbf{x}\in\Omega} \varepsilon_\star(\mathbf{x})$ and $\inf_{\mathbf{x}\in\Omega} \mu_\star(\mathbf{x})$ are strictly positive. Finally, we assume that there exists a partition of Ω into Lipschitz subdomains such that in each of them ε is smooth and μ is constant.

We define, as usual, the following spaces of complex functions:

$$H(\text{curl}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \nabla \times \mathbf{v} \in L^2(\Omega)\},$$

$$H_0(\text{curl}; \Omega) = \{\mathbf{v} \in H(\text{curl}; \Omega) : \mathbf{v} \cdot \mathbf{t} = \mathbf{0} \text{ on } \partial\Omega\},$$

$$H_0(\text{curl}^0; \Omega) = \{\mathbf{v} \in H_0(\text{curl}; \Omega) : \nabla \times \mathbf{v} = \mathbf{0}\},$$

$$H(\text{div}^0_\varepsilon; \Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \nabla \cdot (\varepsilon\mathbf{v}) = 0\}.$$

We set

$$\mathbf{V} = H_0(\text{curl}; \Omega), \qquad \mathbf{V}^0 = H_0(\text{curl}^0; \Omega), \qquad \mathbf{W} = \mathbf{V} \cap H(\text{div}^0_\varepsilon; \Omega).$$

We denote by $(\cdot, \cdot)$ the standard inner product in $L^2(\Omega)^2$ given by $(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \overline{\mathbf{v}}\, d\mathbf{x}$, and write $L^2_\varepsilon(\Omega)^2$ for the space $L^2(\Omega)^2$ endowed with the ε-weighted inner product $(\mathbf{u}, \mathbf{v})_\varepsilon = \int_\Omega \varepsilon\mathbf{u} \cdot \overline{\mathbf{v}}\, d\mathbf{x}$. The L^2-norm and the L^2_ε-norm are clearly equivalent, due to the assumptions on ε.

We endow $\mathbf{V}$ with the seminorm $|\mathbf{v}|_\mathbf{V} = \|\mu^{-1/2}\nabla \times \mathbf{v}\|_{0,\Omega}$, the inner product $(\mathbf{u}, \mathbf{v})_\mathbf{V} = (\mu^{-1}\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + (\mathbf{u}, \mathbf{v})_\varepsilon$ and the norm $\|\mathbf{v}\|^2_\mathbf{V} = \|\mu^{-1/2}\nabla \times \mathbf{v}\|^2_{0,\Omega} + \|\varepsilon^{1/2}\mathbf{v}\|^2_{0,\Omega}$.

Define the (hermitian) bilinear forms $a : \mathbf{V} \times \mathbf{V} \to \mathbb{C}$ and $b : \mathbf{V} \times \mathbf{V} \to \mathbb{C}$ as

$$a(\mathbf{u}, \mathbf{v}) = (\mu^{-1}\nabla \times \mathbf{u}, \nabla \times \mathbf{v}),$$

$$b(\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{v})_\varepsilon.$$

The variational formulation of the eigenproblem we are interested in is the following: find $(\mathbf{0} \neq \mathbf{u}, \omega) \in \mathbf{W} \times \mathbb{C}$ such that

$$a(\mathbf{u}, \mathbf{v}) = \omega^2(\mathbf{u}, \mathbf{v})_\varepsilon \quad \forall\mathbf{v} \in \mathbf{W}.$$

A standard way to discretise this problem consists in neglecting the constraint $\mathbf{u} \in \mathbf{W}$ and adding a zero frequency eigenspace corresponding to the *infinite-dimensional* space $\mathbf{V}^0$, leading to the following variational problem.

Problem 1 Find $(\mathbf{0} \neq \mathbf{u}, \omega) \in \mathbf{V} \times \mathbb{C}$:

$$a(\mathbf{u}, \mathbf{v}) = \omega^2(\mathbf{u}, \mathbf{v})_\varepsilon \quad \forall\mathbf{v} \in \mathbf{V}.$$

For the purpose of the analysis, following [8], we introduce the following auxiliary eigenproblem with positive definite operator.

Problem 2 Find $(\mathbf{0} \neq \mathbf{u}, \widetilde{\omega}) \in \mathbf{V} \times \mathbb{C}$:

$$b(\mathbf{u}, \mathbf{v}) = \widetilde{\omega}^2(\mathbf{u}, \mathbf{v})_\varepsilon \quad \forall\mathbf{v} \in \mathbf{V}.$$

The eigenvalues of Problem 1 and those of Problem 2 are such that $\widetilde{\omega}^2 = \omega^2 + 1$; thus, $\widetilde{\omega}^2 = 1$ is an eigenvalue of Problem 2 with infinite multiplicity and associated eigenspace $\mathbf{V}^0$.

Define the solution operator $A : L^2(\Omega)^2 \to \mathbf{V}$ as follows: given $\mathbf{f} \in L^2(\Omega)^2$, $A\mathbf{f}$ is the (unique) element of $\mathbf{V}$ which satisfies

$$b(A\mathbf{f}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_\varepsilon \quad \forall \mathbf{v} \in \mathbf{V}.$$

We have that $A \in \mathcal{L}(L^2(\Omega)^2, \mathbf{V})$. Notice that $(\mathbf{u}, \omega)$ is an eigenpair of Problem 1 if and only if $(\mathbf{u}, \lambda = \frac{1}{\omega^2+1})$ is an eigenpair of A.

Denote by $\sigma(A)$ and $\rho(A)$ the spectrum and the resolvent set (in the complex plane), respectively, of the solution operator A. Finally, for any $z \in \rho(A)$, we define the resolvent operator $R_z(A) = (z - A)^{-1}$ from $\mathbf{V}$ to $\mathbf{V}$.

3 The Mortar-DG Method

Let $\Omega = \Omega_1 \cup \Omega_2$ be a non-overlapping subdomain partition with simply-connected Ω_1 and Ω_2, with interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$; let $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ be simplicial meshes of Ω_1 and Ω_2, respectively, non-matching at Γ, and set $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2$. The mesh parameter h is defined as $h = \max_{K\in\mathcal{T}_h}\{h_K\}$, with $h_K = \mathrm{diam}(K)$.

We assume μ to be piecewise constant on $\mathcal{T}_h$ and ε piecewise smooth on $\mathcal{T}_h$. For the minimal assumptions on the regularity of the components of ε, we refer to [17, Sect. 4.2].

We denote by $\mathcal{F}_h^1$ and $\mathcal{F}_h^2$ the sets of all faces (edges) of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$, respectively, which do not lie on Γ, and by $\mathcal{F}_h^\Gamma$ the set of all faces of $\mathcal{T}_h^2$ which lie on Γ. On $\mathcal{F}_h := \mathcal{F}_h^1 \cup \mathcal{F}_h^2 \cup \mathcal{F}_h^\Gamma$, we define the *mesh size function* $\mathtt{h} \in L^\infty(\mathcal{F}_h)$ by

$$\mathtt{h}(\mathbf{x}) = h_f\, \mathtt{m}(\mathbf{x}) \quad \mathbf{x} \in f,\ f \in \mathcal{F}_h,$$

where h_f is the diameter of f, and $\mathtt{m} \in L^\infty(\mathcal{F}_h)$ is defined as follows: if μ_K is the restriction of μ to K, then $\mathtt{m}(\mathbf{x}) = \min\{\mu_{K^+}, \mu_{K^-}\}$ if $\mathbf{x}$ is in the interior of $\partial K^+ \cap \partial K^-$, and $\mathtt{m}(\mathbf{x}) = \mu_K$ if $\mathbf{x}$ is in the interior of $\partial K \cap \partial\Omega$.

Introduce the DG finite element spaces:

$$\mathbf{V}_h = \{\mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}_{|_K} \in \mathcal{P}^\ell(K)^2 \ \ \forall K \in \mathcal{T}_h\}, \quad \ell \ge 1$$

where $\mathcal{P}^\ell(K)$ is the space of complex polynomials of total degree at most ℓ on K. We set $\mathbf{V}_h^i := \mathbf{V}_{h|_{\Omega_i}}$, and $\mathbf{w}^i := \mathbf{w}_{|_{\Omega_i}}$, $i = 1, 2$.

Define the DG trace spaces

$$\begin{aligned}\Phi_h &= \{\psi \in L^2(\Gamma) : \psi_{|_f} = (\mathbf{v}\cdot\mathbf{t}_f)_{|_f} \text{ for some } \mathbf{v} \in \mathbf{V}_h^2 \text{ and } \forall f \in \mathcal{F}_h^\Gamma\} \\ &= \{\psi \in L^2(\Gamma) : \psi_{|_f} \in \mathcal{P}^\ell(f)\ \forall f \in \mathcal{F}_h^\Gamma\}\end{aligned}$$

and denote by $\Pi_{h,\Gamma}$ the $L^2(\Gamma)$-projection onto Φ_h, i.e., for any $\varphi \in \Phi_h$,

$$\int_{\mathcal{F}_h^\Gamma} (\varphi - \Pi_{h,\Gamma}\varphi)\,\overline{\psi}\, ds = 0 \quad \forall \psi \in \Phi_h.$$

If K^- and K^+ are two adjacent elements, on $\partial K^- \cap \partial K^+$ we define the weighted averages and jumps by $\{\{\mathbf{w}\}\}_\delta = \delta\mathbf{w}_{|_{K^-}} + (1-\delta)\mathbf{w}_{|_{K^+}}$ and $[[\mathbf{w}]]_T = \mathbf{w}_{|_{K^-}} \cdot \mathbf{t}^- + \mathbf{w}_{|_{K^+}} \cdot \mathbf{t}^+$, where $\mathbf{t}^\pm$ are the counterclockwise oriented tangential unit vectors to $\partial K^\pm$. In particular, on Γ, we understand Ω_1 as 'minus' side and Ω_2 as 'plus' side; therefore, $\{\{\mathbf{w}\}\}_\delta = \delta\mathbf{w}_{|_{\Omega_1}} + (1-\delta)\mathbf{w}_{|_{\Omega_2}}$.

For simplicity, we restrict ourselves to the case of an interior penalty (IPDG) discretisation in each subdomain, although other DG methods could be considered instead. More precisely, for $i = 1, 2$, let $a_h^i(\cdot,\cdot) : \mathbf{V}_h^i \times \mathbf{V}_h^i \to \mathbb{C}$ defined by

$$a_h^i(\mathbf{u}^i, \mathbf{v}^i) = \int_{\Omega_i} \mu^{-1} \nabla_h \times \mathbf{u}^i \cdot \nabla_h \times \overline{\mathbf{v}}^i \, d\mathbf{x} - \int_{\mathcal{F}_h^i} [\![\overline{\mathbf{v}}^i]\!]_T \cdot \{\!\{ \mu^{-1} \nabla_h \times \mathbf{u}^i \}\!\}_\delta \, ds \\ - \int_{\mathcal{F}_h^i} [\![\mathbf{u}^i]\!]_T \cdot \{\!\{ \mu^{-1} \nabla_h \times \overline{\mathbf{v}}^i \}\!\}_\delta \, ds + \int_{\mathcal{F}_h^i} \mathtt{a} [\![\mathbf{u}^i]\!]_T \cdot [\![\overline{\mathbf{v}}^i]\!]_T \, ds,$$

with $0 \le \delta \le 1$ and with stability parameter $\mathtt{a}$ chosen proportional to $\mathtt{h}^{-1}$.

We complete the definition of the mortar-DG method by including a projection based penalization at the non-conforming interface $\mathcal{F}_h^\Gamma$, together with consistency terms. Therefore, we consider the following complex-valued mortar-DG bilinear form on $\mathbf{V}_h \times \mathbf{V}_h$:

$$a_h(\mathbf{u}, \mathbf{v}) = a_h^1(\mathbf{u}^1, \mathbf{v}^1) + a_h^2(\mathbf{u}^2, \mathbf{v}^2) \\ - \int_{\mathcal{F}_h^\Gamma} [\![\overline{\mathbf{v}}]\!]_T \cdot \{\!\{ \mu^{-1} \nabla_h \times \mathbf{u} \}\!\}_\delta \, ds - \int_{\mathcal{F}_h^\Gamma} [\![\mathbf{u}]\!]_T \cdot \{\!\{ \mu^{-1} \nabla_h \times \overline{\mathbf{v}} \}\!\}_\delta \, ds \\ + \int_{\mathcal{F}_h^\Gamma} \mathtt{a} \, \Pi_{h,\Gamma} [\![\mathbf{u}]\!]_T \cdot \Pi_{h,\Gamma} [\![\overline{\mathbf{v}}]\!]_T \, ds,$$

with $\delta = 0$ in the integrals over $\mathcal{F}_h^\Gamma$. Again, the stability parameter $\mathtt{a}$ is chosen proportional to $\mathtt{h}^{-1}$. Notice that $a_h(\cdot,\cdot)$ is hermitian. We also define

$$b_h(\mathbf{u}, \mathbf{v}) = a_h(\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{v})_\varepsilon.$$

The mortar-DG approximation of Problem 1 reads as follows.

Problem 3 Find $(\mathbf{0} \neq \mathbf{u}_h, \omega_h) \in \mathbf{V}_h \times \mathbb{C}$:

$$a_h(\mathbf{u}_h, \mathbf{v}) = \omega_h^2 (\mathbf{u}_h, \mathbf{v})_\varepsilon \quad \forall \mathbf{v} \in \mathbf{V}_h.$$

Problem 3 is clearly consistent.

Remark 3.1 The case of conforming discretisations with the Nédélec elements of the second family within each subdomain is nothing but a particular case of the presented method.

Introduce the following seminorm and norm on $\mathbf{V}(h) := \mathbf{V} + \mathbf{V}_h$:

$$|\mathbf{v}|^2_{\mathbf{V}(h)} = \|\mu^{-1/2} \nabla_h \times \mathbf{v}\|^2_{0,\Omega_1 \cup \Omega_2} + \|\mathtt{h}^{-1/2} [\![\mathbf{v}]\!]_T\|^2_{0,\mathcal{F}_h^1 \cup \mathcal{F}_h^2} \\ + \|\mathtt{h}^{-1/2} \Pi_{h,\Gamma} [\![\mathbf{v}]\!]_T\|^2_{0,\mathcal{F}_h^\Gamma},$$

$$\|\mathbf{v}\|^2_{\mathbf{V}(h)} = |\mathbf{v}|^2_{\mathbf{V}(h)} + \|\varepsilon^{1/2} \mathbf{v}\|^2_{0,\Omega}.$$

The inner product in $\mathbf{V}_h$ is defined by

$$(\mathbf{v}, \mathbf{w})_{\mathbf{V}(h)} = (\mathbf{v}, \mathbf{w})_\varepsilon + \int_\Omega \mu^{-1} \nabla_h \times \mathbf{v} \cdot \nabla_h \times \overline{\mathbf{w}} \, d\mathbf{x} \\ + \int_{\mathcal{F}_h^1 \cup \mathcal{F}_h^2} \mathtt{h}^{-1} [\![\mathbf{v}]\!]_T \cdot [\![\overline{\mathbf{w}}]\!]_T \, ds + \int_{\mathcal{F}_h^\Gamma} \mathtt{h}^{-1} \Pi_{h,\Gamma} [\![\mathbf{v}]\!]_T \cdot \Pi_{h,\Gamma} [\![\overline{\mathbf{w}}]\!]_T \, ds.$$

Define the kernel of $a_h(\cdot,\cdot)$ and its $\mathbf{V}(h)$-orthogonal complement as follows:

$$K_h = \{\mathbf{v} \in \mathbf{V}_h : a_h(\mathbf{v},\mathbf{w}) = 0 \ \forall \mathbf{w} \in \mathbf{V}_h\},$$
$$K_h^\perp = \{\mathbf{v} \in \mathbf{V}_h : (\mathbf{v},\mathbf{w})_{\mathbf{V}(h)} = 0 \ \forall \mathbf{w} \in K_h\}.$$

We define the discrete solution operator $A_h : L^2(\Omega)^2 \to \mathbf{V}_h$ as follows: given $\mathbf{f} \in L^2(\Omega)^2$, $A_h\mathbf{f}$ is the (unique) element of $\mathbf{V}_h$ which satisfies

$$b_h(A_h\mathbf{f},\mathbf{v}) = (\mathbf{f},\mathbf{v})_\varepsilon \quad \forall \mathbf{v} \in \mathbf{V}_h.$$

The operator A_h is well defined and $A_h \in \mathcal{L}(L^2(\Omega)^2, \mathbf{V}_h)$ (see Remark 4.3).

As in the continuous case, we denote by $\sigma(A_h)$ and $\rho(A_h)$ the spectrum and the resolvent set, respectively, of the discrete solution operator A_h. Finally, for any $z \in \mathbb{C}$, we formally define the resolvent operator $R_z(A_h) = (z - A_h)^{-1}$ from $\mathbf{V}_h$ to $\mathbf{V}_h$.

4 Theoretical Analysis

The analysis of Problem 3 can be carried out by slightly modifying the theoretical setting of [7].

4.1 Preliminary Properties

In this section we state (as Properties) and prove the validity of the assumptions of the theory developed in [7], or their modifications when it is needed. Indeed, the theory developed in [7] needs to be modified due to a major difference: it is no longer true that $K_h \subset \mathbf{V}^0$ and this changes the spectral convergence theory.

It is straightforward to see that the following property holds true (compare with the second part of Assumption 1 of [7]).

Property 1 (Norm compatibility) *If* $\mathbf{v} \in \mathbf{V}$, *then* $|\mathbf{v}|_{\mathbf{V}(h)} = |\mathbf{v}|_{\mathbf{V}}$.

Notice that the first part of Assumption 1 of [7], namely, if $\mathbf{v} \in \mathbf{V}(h)$ and $|\mathbf{v}|_{\mathbf{V}(h)} = 0$, then $\mathbf{v} \in \mathbf{V}^0$, is not satisfied; as a consequence, the discrete kernel K_h is not contained in the continuous kernel $\mathbf{V}^0$; this is the reason why we need to slightly modify the theory of [7].

We can prove that also the following two properties are satisfied (compare with [7, Assumptions 2 and 3]).

Property 2 (Approximation property of $\mathbf{V}_h$) *There holds*

$$\lim_{h\to 0} \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{V}(h)} = 0 \quad \forall \mathbf{v} \in \mathbf{V}.$$

Property 3 (Coercivity in seminorm and continuity) *Provided that the stability parameter* $\mathtt{a}$ *is chosen as* $a_{IP}\mathtt{h}^{-1}$, *with* a_{IP} *sufficiently large, there exist positive constants* α, γ *independent of the mesh size such that*

$$\begin{aligned} a_h(\mathbf{v},\mathbf{v}) &\geq \alpha\, |\mathbf{v}|^2_{\mathbf{V}(h)} \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ |a_h(\mathbf{u},\mathbf{v})| &\leq \gamma\, \|\mathbf{u}\|_{\mathbf{V}(h)} \|\mathbf{v}\|_{\mathbf{V}(h)} \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{V}_h. \end{aligned}$$

Proposition 4.1 *Property 2 holds true.*

Proof Owing to the density of $C^\infty(\Omega)^2$ in $\mathbf{V}$, it is enough to prove the result for all $\mathbf{v} \in C^\infty(\Omega)^2$. To this end, given $\mathbf{v} \in C^\infty(\Omega)^2$, let $\mathbf{v}_h$ be the elementwise second family Nédélec's interpolation of $\mathbf{v}$. Then $\|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{V}(h)}$ tends to zero as $h \to 0$ owing to the usual approximation properties of the Nédélec elements and to L^2-stability of $\Pi_{h,\Gamma}$. □

Proposition 4.2 *Property 3 holds true.*

Proof The proof of the continuity exploits the fact that, for $\delta = 0$ in the integrals over $\mathcal{F}_h^\Gamma$, the assumption μ piecewise constant on $\mathcal{T}_h$ implies

$$\int_{\mathcal{F}_h^\Gamma} [\![\mathbf{v}]\!]_T \cdot \{\{\mu^{-1}\nabla \times \overline{\mathbf{w}}\}\}_\delta \, ds = \int_{\mathcal{F}_h^\Gamma} \Pi_{h,\Gamma}[\![\mathbf{v}]\!]_T \cdot \{\{\mu^{-1}\nabla \times \overline{\mathbf{w}}\}\}_\delta \, ds$$

for all $\mathbf{v}, \mathbf{w} \in \mathbf{V}_h$. The proof of the coercivity is standard. □

From here on, we assume $\mathtt{a} = a_{IP}\mathtt{h}^{-1}$, with a_{IP} such that the coercivity bound of Property 3 is satisfied.

Remark 4.3 From Property 3 it follows that

$$b_h(\mathbf{v}, \mathbf{v}) \geq \min\{\alpha, 1\}\|\mathbf{v}\|_{\mathbf{V}(h)}^2 \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{1}$$

and that

$$|\mathbf{v}|_{\mathbf{V}(h)} = 0 \quad \forall \mathbf{v} \in K_h. \tag{2}$$

The coercivity property (1) guarantees that, for any given $\mathbf{f} \in L^2(\Omega)^2$, there exists a unique $\mathbf{u}_h \in \mathbf{V}_h$ such that $b_h(\mathbf{u}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_\varepsilon$ for all $\mathbf{v} \in \mathbf{V}_h$, and $\|\mathbf{u}_h\|_{\mathbf{V}(h)} \leq C\|\mathbf{f}\|_{0,\Omega}$, with $C > 0$ independent of the mesh size and of the right-hand side $\mathbf{f}$.

Moreover, introducing the subspace

$$\mathbf{V}_h^c = \{\mathbf{v} \in \mathbf{V}_h : \mathbf{v}_{|\Omega_i} \in H(\text{curl}, \Omega_i),\ i = 1, 2,\ (\mathbf{v} \cdot \mathbf{t})_{|\partial\Omega} = 0,\ \Pi_{h,\Gamma}[\![\mathbf{v}]\!]_T = 0 \text{ on } \Gamma\}, \tag{3}$$

the identity (2) gives that

$$K_h = \{\mathbf{v} \in \mathbf{V}_h^c : (\nabla \times \mathbf{v})_{|\Omega_i} = 0,\ i = 1, 2\}. \tag{4}$$

For the following property (compare with [7, Assumption 4]), we introduce the broken spaces:

$$H^s(\mathcal{T}_h)^2 = \{\mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in H^s(K)^2\ \forall K \in \mathcal{T}_h\} \quad \text{for } s \geq 0,$$

$$H^r(\text{curl}; \mathcal{T}_h) = \{\mathbf{v} \in L^2(\Omega)^2 : \varepsilon\mathbf{v}|_K \in H^r(K)^2,$$
$$\mu^{-1}\nabla \times \mathbf{v}|_K \in H^r(K)\ \forall K \in \mathcal{T}_h\} \quad \text{for } r > 0,$$

and the norms:

$$\|\mathbf{v}\|_{H^s(\mathcal{T}_h)^2}^2 = \sum_{K \in \mathcal{T}_h} \|\mathbf{v}\|_{s,K}^2,$$

$$\|\mathbf{v}\|_{H^r(\text{curl};\mathcal{T}_h)}^2 = \sum_{K \in \mathcal{T}_h} \left(\|\varepsilon^{1/2}\mathbf{v}\|_{r,K}^2 + \|\mu^{-1/2}\nabla \times \mathbf{v}\|_{r,K}^2\right).$$

Property 4 (Convergence) *Let* $\mathbf{f}$ *be in* $H(\mathrm{div}^0_\varepsilon;\Omega)$; *denote by* $\mathbf{u}_s \in \mathbf{V}$ *the solution to the coercive source problem* $b(\mathbf{u}_s,\mathbf{v}) = (\mathbf{f},\mathbf{v})_\varepsilon$ *for all* $\mathbf{v}\in\mathbf{V}$, *and by* $\mathbf{u}_h \in \mathbf{V}_h$ *its Galerkin projection which satisfies* $b_h(\mathbf{u}_h,\mathbf{v}) = (\mathbf{f},\mathbf{v})_\varepsilon$ *for all* $\mathbf{v}\in\mathbf{V}_h$. *Whenever* $\mathbf{u}_s \in H^r(\mathrm{curl};\mathcal{T}_h)$, *with* $r>0$, *there exists a sequence* ξ_h, $\xi_h \to 0$ *as* $h\to 0$, *such that*

$$\|\mathbf{u}_s - \mathbf{u}_h\|_{\mathbf{V}(h)} \le \xi_h \left(\|\mathbf{u}_s\|_{H^r(\mathrm{curl};\mathcal{T}_h)} + \|\mathbf{f}\|_{0,\Omega}\right). \tag{5}$$

The bound (5), *together with the regularity results in* [9], *implies that*

$$\|\mathbf{u}_s - \mathbf{u}_h\|_{\mathbf{V}(h)} \le \xi_h \|\mathbf{f}\|_{0,\Omega} \quad \forall \mathbf{f}\in H(\mathrm{div}^0_\varepsilon;\Omega),$$

with $\xi_h \to 0$ *as* $h \to 0$.

Proposition 4.4 *Property* 4 *holds true.*

Proof Following the same arguments as in [7], we observe that the result is a consequence of the coercivity in Property 3 and of the following continuity property: for all $\mathbf{w}\in H^r(\mathrm{curl};\mathcal{T}_h)$ with $\nabla_h \times (\mu^{-1}\nabla_h\times\mathbf{w}) \in L^2(\Omega)$, and $\mathbf{v}_h \in \mathbf{V}_h$, we have

$$|a_h(\mathbf{w},\mathbf{v}_h)| \le C\|\mathbf{w}\|_{\mathbf{V}_\sigma(h)}\|\mathbf{v}_h\|_{\mathbf{V}(h)}, \tag{6}$$

with a constant $C>0$ independent of the mesh size, where the $\mathbf{V}_\sigma(h)$-norm is defined as follows. If $0<\sigma<\min\{1/2,r\}$, then, for all $\mathbf{w}\in H^r(\mathrm{curl};\mathcal{T}_h)$ with $\nabla_h\times(\mu^{-1}\nabla_h\times\mathbf{w})\in L^2(\Omega)$, we set

$$\|\mathbf{w}\|^2_{\mathbf{V}_\sigma(h)} = \|\mathbf{w}\|^2_{\mathbf{V}(h)} + |\mathbf{w}|^2_{\mathbf{V}_\sigma(h)},$$

with

$$|\mathbf{w}|^2_{\mathbf{V}_\sigma(h)} = \sum_{K\in\mathcal{T}_h}(h_K^{2\sigma}\mu_K\|\mu^{-1}\nabla\times\mathbf{w}\|^2_{\sigma,K} + h_K^2\mu_K\|\nabla\times(\mu^{-1}\nabla\times\mathbf{w})\|^2_{0,K})$$

where μ_K stands for $\mu_{|K}$. Note that there exists a sequence η_h, $\eta_h\to 0$ when $h\to 0$ such that for all $\mathbf{w}\in H^r(\mathrm{curl},\mathcal{T}_h)$ such that $\nabla_h\times(\mu^{-1}\nabla_h\times\mathbf{w})\in L^2(\Omega)$, it holds

$$\inf_{\mathbf{v}_h\in\mathbf{V}_h}\|\mathbf{w}-\mathbf{v}_h\|_{\mathbf{V}_\sigma(h)} \le \eta_h\left(\|\mathbf{w}\|_{H^r(\mathrm{curl};\mathcal{T}_h)} + \left(\sum_{K\in\mathcal{T}_h)}\mu_K\|\nabla\times(\mu^{-1}\nabla\times\mathbf{w})\|^2_{0,K}\right)^{1/2}\right).$$

In order to prove (6), first of all, it is easy to see that

$$\begin{aligned}|a_h(\mathbf{w},\mathbf{v}_h)| \le C\,\|\mathbf{w}\|_{\mathbf{V}(h)}\|\mathbf{v}_h\|_{\mathbf{V}(h)} &+ C'\int_{\mathcal{F}_h^1\cap\mathcal{F}_h^2}[\![\overline{\mathbf{v}}_h]\!]_T\,\{\!\{\mu^{-1}\nabla_h\times\mathbf{w}\}\!\}_\delta\,ds\\ &+ C'\int_{\mathcal{F}_h^\Gamma}[\![\overline{\mathbf{v}}_h]\!]_T\,(\mu^{-1}\nabla_h\times\mathbf{w})_{|\Omega_2}\,ds.\end{aligned}$$

Consequently, in order to get (6), it is enough to show that, if $f\in\mathcal{F}_h$ and K is an element having f as a *complete* face, then

$$\int_f [\![\overline{\mathbf{v}}_h]\!]_T\,(\mu^{-1}\nabla_h\times\mathbf{w})_{|K}\,ds \le C|\mathbf{w}|_{\mathbf{V}_\sigma(h)}\|\mathbf{v}_h\|_{\mathbf{V}(h)}.$$

The proof of this estimate can be carried out by adapting to the 2D case the steps of [7, Proof of Lemma 8.2], where the 3D case was considered (the changes in the scalings actually compensate and the obtained result is the same). □

We have the following result, whose proof can be carried out exactly as in [7, Propositions 3.2 and 3.3] (notice that since $a_h(\cdot,\cdot)$ is hermitian, all the discrete eigenvalues are real).

Proposition 4.5 *If $\lambda_h \in \sigma(A_h)$, then $0 < \lambda_h \leq 1$; $1 \in \sigma(A_h)$ and its associated eigenspace is K_h. Moreover, for all eigenfunctions $\mathbf{v}_1$, $\mathbf{v}_2$ associated with different eigenvalues, it holds $(\mathbf{v}_1, \mathbf{v}_2)_\varepsilon = b_h(\mathbf{v}_1, \mathbf{v}_2) = 0$; in particular, if $\mathbf{v} \neq \mathbf{0}$ is an eigenfunction of A_h associated with an eigenvalue $\lambda_h \neq 1$, then $(\mathbf{v}, \mathbf{w})_\varepsilon = (\mathbf{v}, \mathbf{w})_{\mathbf{V}(h)} = b_h(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in K_h$.*

4.2 Characterization of the Kernel

Setting

$$Q_h = \{q \in L^2(\Omega) : q_{|K} \in \mathcal{P}^{\ell+1}(K) \;\; \forall K \in \mathcal{T}_h\}$$

(and $Q_h^i := Q_{h\,|\Omega_i}$, $i = 1, 2$), and defining the subspace

$$Q_h^c = \{q \in Q_h : q_{|\Omega_i} \in H^1(\Omega_i),\; i = 1, 2,\; q_{|\partial\Omega} = 0,\; \Pi_{h,\Gamma}[\![\nabla_h q]\!]_T = 0 \text{ on } \Gamma\},$$

from (4) we have the characterization

$$K_h = \nabla_h Q_h^c, \tag{7}$$

i.e., any $\mathbf{v} \in K_h$ can be expressed as $\nabla_h p$, with $p \in Q_h^c$. Notice that since $\partial\Gamma \subset \partial\Omega$, if $p \in Q_h^c$, setting $[\![p]\!] := (p_{|\Omega_1} - p_{|\Omega_2})_{|\Gamma}$, $[\![p]\!] \in H_0^1(\Gamma)$ and, by integration by parts, we obtain

$$\int_\Gamma [\![p]\!]\, \partial_t \overline{\varphi}\, ds = 0 \quad \forall \varphi \in \Phi_h \cap H^1(\Gamma), \tag{8}$$

where ∂_t denotes the tangential derivative along Γ. In particular, $\int_\Gamma [\![p]\!]\, ds = 0$.

In the following, we will also need to characterize $K_h^\perp \cap \mathbf{V}_h^c$, with V_h^c defined as in (3). To this aim, denote by K_h^c the kernel of $a_h(\cdot,\cdot)$ in $\mathbf{V}_h^c$, i.e.,

$$K_h^c = \{\mathbf{v} \in \mathbf{V}_h^c : a_h(\mathbf{v}, \mathbf{w}) = 0\; \forall \mathbf{w} \in \mathbf{V}_h^c\}.$$

It is immediate to see that $K_h = K_h^c$, and then

$$K_h^\perp \cap \mathbf{V}_h^c = (K_h^c)^\perp, \tag{9}$$

where $(K_h^c)^\perp$ denotes the $\mathbf{V}(h)$-orthogonal complement of K_h^c in $\mathbf{V}_h^c$.

4.3 Main Properties

We have the following property which is a stronger form of the Gap property defined in [7].

Property 5 (Strong gap property) *For all h small enough, for any $\mathbf{w}_h \in K_h^{\perp}$ there exists $\mathbf{w} = \mathbf{w}(h) \in \mathbf{W}$ such that*

$$\|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega} \le \eta_h \|\mathbf{w}_h\|_{\mathbf{V}(h)},$$

with $\eta_h \to 0$ as $h \to 0$.

In order to prove Property 5, we need an approximation result, which extends that of [15, Proposition 1] to our situation, and a stable orthogonal decomposition of $\mathbf{V}_h$.

Proposition 4.6 *Let $\mathbf{V}_h^c$ be defined as in (3). For any $\mathbf{v} \in \mathbf{V}_h$ there exists $\mathbf{v}^c \in \mathbf{V}_h^c$ such that*

$$\|\mathbf{v} - \mathbf{v}^c\|_{0,\Omega}^2 \le C\big[\|\mathrm{h}^{1/2}[\![\mathbf{v}]\!]_T\|_{0,\mathcal{F}_h^1\cup\mathcal{F}_h^2}^2 + \|\mathrm{h}^{1/2}\Pi_{h,\Gamma}[\![\mathbf{v}]\!]_T\|_{0,\mathcal{F}_h^\Gamma}^2\big],$$
$$\|\mathbf{v} - \mathbf{v}^c\|_{\mathbf{V}(h)}^2 \le C\big[\|\mathrm{h}^{-1/2}[\![\mathbf{v}]\!]_T\|_{0,\mathcal{F}_h^1\cup\mathcal{F}_h^2}^2 + \|\mathrm{h}^{-1/2}\Pi_{h,\Gamma}[\![\mathbf{v}]\!]_T\|_{0,\mathcal{F}_h^\Gamma}^2\big],$$

with a constant $C > 0$ independent of $\mathbf{v}$ and of the mesh size.

Proof Notice that $\mathbf{V}_h^c$ is the space of the Nédélec elements of the second family in Ω_1 and Ω_2, with zero tangential trace on $\partial\Omega$ and with the constraint $\Pi_{h,\Gamma}[\![\mathbf{v}]\!]_T = 0$ on the interface Γ.

Fix $\mathbf{v} \in \mathbf{V}_h$. We construct $\mathbf{v}^c \in \mathbf{V}_h^c$ by modifying the construction given in [15, Appendix] along Γ. More precisely, the volume moments of $\mathbf{v}^c$ coincide with those of $\mathbf{v}$; the face moments of $\mathbf{v}^c$ are zero on the faces on $\partial\Omega$ and are equal to the mean values of the face moments of $\mathbf{v}$ from the two sides on the faces in $\mathcal{F}_h^1$ and $\mathcal{F}_h^2$ which do not lie on $\partial\Omega$ (no hanging nodes); finally, the face moments of $\mathbf{v}^c$ coincide with those of $\mathbf{v}$ on the faces on Γ from the Ω_1 side and are equal to the face moments of $\Pi_{h,\Gamma}\mathbf{v}_{|\Omega_1}$ on the faces on Γ from the Ω_2 side (all the above moments are non averaged moments).

We can prove that, for any $\mathbf{w} \in \mathcal{P}^\ell(K)^2$ of volume moments $\{w_K^i\}_{i=1}^{N_v}$ and face moments $\{w_{K,f}^i\}_{i=1}^{N_f}$, with N_v and N_f being the number of volume and face moments, respectively, we have

$$h_K^{-2}\|\mathbf{w}\|_{0,K}^2 + \|\nabla\times\mathbf{w}\|_{0,K}^2 \le C h_K^{-2}\left(\sum_{f\subset\partial K}\sum_{i=1}^{N_f}|w_{K,f}^i|^2 + \sum_{i=1}^{N_v}|w_K^i|^2\right), \tag{10}$$

with a constant $C > 0$ independent of the element size.

In fact, on the reference element, the bound (10) follows from the representation of functions in discrete spaces in terms of linear combinations of Lagrange basis functions with respect to the given moments and the Cauchy-Schwarz inequality.

Consider now a general element $K \in \mathcal{T}_h$, image of the reference element $\widehat{K}$ under the affine mapping F_K, where $F_K(\widehat{\mathbf{x}}) = B_K\widehat{\mathbf{x}} + b_K$, with $B_K \in \mathbb{R}^{2\times 2}$ and $b_K \in \mathbb{R}^2$. Since the transformation $\mathbf{w}\circ F_K = B_K^{-T}\widehat{\mathbf{w}}$ preserves the moments, and

$$\|\mathbf{w}\|_{0,K}^2 \le C\|\widehat{\mathbf{w}}\|_{0,\widehat{K}}^2, \qquad \|\nabla\times\mathbf{w}\|_{0,K}^2 \le C h_K^{-2}\|\widehat{\nabla}\times\widehat{\mathbf{w}}\|_{0,\widehat{K}}^2,$$

with a constant $C > 0$ independent of the mesh size (see, e.g., [5, Proposition 3.1]), the bound (10) is obtained.

Denoting by $v^i_{K,f}$ and $\widetilde{v}^i_{K,f}$ the moments of the face $f \subset \partial K$ of $\mathbf{v}$ and $\mathbf{v}^c$, respectively, for $i = 1, \ldots, N_f$, since the volume moments of $\mathbf{v} - \mathbf{v}^c$ are zero, we have

$$h_K^{-2}\|\mathbf{v} - \mathbf{v}^c\|^2_{0,K} + \|\nabla \times (\mathbf{v} - \mathbf{v}^c)\|^2_{0,K} \leq C h_K^{-2} \left(\sum_{f \subset \partial K} \sum_{i=1}^{N_f} |v^i_{K,f} - \widetilde{v}^i_{K,f}|^2 \right).$$

If $f \not\subset \Gamma$, if K and K' are the two elements sharing f, we have

$$\sum_{i=1}^{N_f} |v^i_{K,f} - \widetilde{v}^i_{K,f}|^2 = \frac{1}{4} \sum_{i=1}^{N_f} |v^i_{K,f} - v^i_{K',f}|^2 \leq C|f| \int_f |[\![\mathbf{v}]\!]_T|^2;$$

if $f \subset \Gamma$ belongs to $K \in \mathcal{T}_h^1$ we have

$$\sum_{i=1}^{N_f} |v^i_{K,f} - \widetilde{v}^i_{K,f}|^2 = 0,$$

while if $f \subset \Gamma$ belongs to $K \in \mathcal{T}_h^2$ we have

$$\sum_{i=1}^{N_f} |v^i_{K,f} - \widetilde{v}^i_{K,f}|^2 \leq C|f| \int_f |\Pi_{h,\Gamma}[\![\mathbf{v}]\!]_T|^2.$$

Therefore, owing to the shape regularity assumption, we obtain

$$\begin{aligned} &h_K^{-2}\|\mathbf{v} - \mathbf{v}^c\|^2_{0,K} + \|\nabla \times (\mathbf{v} - \mathbf{v}^c)\|^2_{0,K} \\ &\quad \leq C h_K^{-1} \left(\sum_{f \subset \partial K \setminus \Gamma} \int_f |[\![\mathbf{v}]\!]_T|^2 + \sum_{f \subset \partial K \cap \Gamma} \int_f |\Pi_h^2 [\![\mathbf{v}]\!]_T|^2 \right), \end{aligned}$$

from which the result follows after summation over all elements. □

Corollary 4.7 *For any* $\mathbf{w} \in K_h^\perp$ *there exists* $\mathbf{w}^c \in K_h^\perp \cap \mathbf{V}_h^c$ *such that*

$$\begin{aligned} \|\mathbf{w} - \mathbf{w}^c\|^2_{0,\Omega} &\leq C\big[\|\mathrm{h}^{1/2}[\![\mathbf{w}]\!]_T\|^2_{0,\mathcal{F}_h^1 \cup \mathcal{F}_h^2} + \|\mathrm{h}^{1/2}\Pi_{h,\Gamma}[\![\mathbf{w}]\!]_T\|^2_{0,\mathcal{F}_h^\Gamma}\big], \\ \|\mathbf{w} - \mathbf{w}^c\|^2_{\mathbf{V}(h)} &\leq C\big[\|\mathrm{h}^{-1/2}[\![\mathbf{w}]\!]_T\|^2_{0,\mathcal{F}_h^1 \cup \mathcal{F}_h^2} + \|\mathrm{h}^{-1/2}\Pi_{h,\Gamma}[\![\mathbf{w}]\!]_T\|^2_{0,\mathcal{F}_h^\Gamma}\big], \end{aligned} \tag{11}$$

with a constant $C > 0$ *independent of* $\mathbf{w}$ *and of the mesh size.*

Proof For each $\mathbf{w} \in K_h^\perp$, let $\mathbf{v}^c$ be the element of $\mathbf{V}_h^c$ given by Proposition 4.6. Due to (9), we can write $\mathbf{v}^c = \mathbf{k} + \mathbf{w}^c$, with $\mathbf{k} \in K_h^c$ and $\mathbf{w}^c \in (K_h^c)^\perp = K_h^\perp \cap \mathbf{V}_h^c$. By definition, it holds

$$\|\mathbf{w} - \mathbf{v}^c\|^2_{0,\Omega} = \|\mathbf{w} - \mathbf{w}^c\|^2_{0,\Omega} + \|\mathbf{k}\|^2_{0,\Omega}$$

and

$$\|\mathbf{w} - \mathbf{v}^c\|^2_{\mathbf{V}(h)} = \|\mathbf{w} - \mathbf{w}^c\|^2_{\mathbf{V}(h)} + \|\mathbf{k}\|^2_{0,\Omega},$$

from which the result readily follows. □

Proposition 4.8 *Let $\mathbf{V}_h^c$ be defined as in Proposition 4.6. There exists a complement $\mathbf{V}_h^\perp$ of $\mathbf{V}_h^c$ in $\mathbf{V}_h$ such that the decomposition $\mathbf{V}_h = \mathbf{V}_h^c + \mathbf{V}_h^\perp$ is stable in the $\mathbf{V}(h)$-norm, i.e., there exists a constant $C > 0$ independent of the mesh size such that, for all $\mathbf{v}_h \in \mathbf{V}_h$, we have*

$$\mathbf{v}_h = \mathbf{v}_h^c + \mathbf{v}_h^\perp, \quad \|\mathbf{v}_h^c\|_{\mathbf{V}(h)} + \|\mathbf{v}_h^\perp\|_{\mathbf{V}(h)} \le C\|\mathbf{v}_h\|_{\mathbf{V}(h)}.$$

Proof See [7, Proposition 7.5]. □

We can now prove Property 5. Following the same approach as in [8] (see also [17, Theorem 7.17 and Theorem 7.18]), we first prove Property 5 when $\varepsilon = I$, then we derive the general case.

Theorem 4.9 *Assume that $\varepsilon = I$. Then, Property 5 holds true. Moreover, with the notation of Property 5, it holds:*

$$\|\mathbf{w}\|_{\mathbf{V}(h)} \le C\|\mathbf{w}_h\|_{\mathbf{V}(h)}, \tag{12}$$

with a constant $C > 0$ independent of $\mathbf{w}_h$ and of the mesh size.

Proof We are given with a $\mathbf{w}_h \in K_h^\perp$, by Corollary 4.7 and Proposition 4.8, we know that there exists a $\boldsymbol{\psi}_h^c \in K_h^\perp \cap \mathbf{V}_h^c$ such that

$$\|\mathbf{w}_h - \boldsymbol{\psi}_h^c\|_{0,\Omega} \le Ch\|\mathbf{w}_h\|_{\mathbf{V}(h)} \quad \text{and} \quad \|\boldsymbol{\psi}_h^c\|_{\mathbf{V}(h)} \le C\|\mathbf{w}_h\|_{\mathbf{V}(h)}. \tag{13}$$

Let $\mathbf{w} \in \mathbf{W}$ be the unique solution of the problem

$$\nabla \times \mathbf{w} = \nabla_h \times \boldsymbol{\psi}_h^c. \tag{14}$$

Recall that $\mathbf{W} \hookrightarrow H^{1/2+\sigma}(\Omega)^2$, with $0 < \sigma \le 1/2$, with σ depending only on Ω (see, e.g., [1]). The well-posedness of the problem (14), the use of the Poincaré-Friedrichs inequality on vector fields in $\mathbf{W}$ and the second bound in (13) imply then

$$\|\mathbf{w}\|_{1/2+\sigma,\Omega} \le C\|\mathbf{w}\|_{\mathbf{V}(h)} \le C\|\nabla_h \times \boldsymbol{\psi}_h^c\|_{0,\Omega} \le C\|\mathbf{w}_h\|_{\mathbf{V}(h)},$$

i.e., (12).

We estimate now the quantity $\|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega}^2$. Let Π_h^M be a projector from $\mathbf{W}$ to $\mathbf{V}_h^c$. We split the quantity to be estimated into three addends:

$$\begin{aligned}\|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega}^2 &= (\mathbf{w} - \Pi_h^M\mathbf{w}, \mathbf{w} - \mathbf{w}_h) + (\Pi_h^M\mathbf{w} - \boldsymbol{\psi}_h^c, \mathbf{w} - \mathbf{w}_h)\\ &\quad + (\boldsymbol{\psi}_h^c - \mathbf{w}_h, \mathbf{w} - \mathbf{w}_h)\\ &=: T_1 + T_2 + T_3,\end{aligned}$$

and we estimate them separately.

1st term. We first need to choose a suitable projection Π_h^M. Let Q_h^Γ be the restriction to Γ of Q_h^2 (see Sect. 4.2), i.e.,

$$Q_h^\Gamma = \{q \in H_0^1(\Gamma) : q_{|I} \in \mathcal{P}^{\ell+1}(I) \ \ \forall I \in \mathcal{F}_h^\Gamma\}$$

(recall that $\mathcal{F}_h^\Gamma = \mathcal{T}^2_{h\,|\Gamma}$). We remark that

$$\varphi \in \Phi_h \ : \int_\Gamma \varphi\, ds = 0 \quad \Longleftrightarrow \quad \exists \chi_h \in Q_h^\Gamma \ : \ \varphi = \partial_t \chi_h.$$

Define $I_h\mathbf{w}$ by $(I_h\mathbf{w})_{|_{\Omega^i}} = I_h^i\mathbf{w}_{|_{\Omega^i}}$, $i = 1, 2$, where I_h^i is the edge element interpolation operator in Ω^i, $i = 1, 2$. Since $\int_\Gamma (I_h^1\mathbf{w} - I_h^2\mathbf{w}) \cdot \boldsymbol{\tau}\, ds = 0$, we have $\int_\Gamma \Pi_{h,\Gamma}(I_h^1\mathbf{w} - I_h^2\mathbf{w}) \times \boldsymbol{\tau}\, ds = 0$, i.e. there exists a $\chi_h \in Q_h^\Gamma$ such that

$$\Pi_{h,\Gamma}(I_h^1\mathbf{w} - I_h^2\mathbf{w}) \cdot \boldsymbol{\tau} = \partial_t \chi_h.$$

Now, we denote by $\mathcal{H}_h : Q_h^\Gamma \to Q_h^2$ the discrete harmonic extension from Γ to Ω_2. It is easy to see that, if we define

$$\Pi_h^M\mathbf{w} := I_h\mathbf{w} + \nabla\mathcal{H}_h(\chi_h), \tag{15}$$

then $\Pi_h^M\mathbf{w} \in \mathbf{V}_h^c$. The projection $\Pi_h^M\mathbf{w}$ is called in the literature *mortar* interpolation operator; see [3] and [2]. We can estimate the quantity $\|\mathbf{w} - \Pi_h^M\mathbf{w}\|_{0,\Omega}$ as follows:

$$\begin{aligned}\|\mathbf{w} - \Pi_h^M\mathbf{w}\|_{0,\Omega} &\le \|\mathbf{w} - I_h\mathbf{w}\|_{0,\Omega} + \|\nabla\mathcal{H}_h(\chi_h)\|_{0,\Omega} \\ &\le C_1 h^{1/2+\sigma}\|\mathbf{w}\|_{1/2+\sigma,\Omega} + C_2\|\chi_h\|_{1,\Gamma},\end{aligned} \tag{16}$$

where we have used the regularity of $\mathbf{w} \in H^{1/2+\sigma}(\Omega)^2$ and the fact that I_h is well defined on vector fields in $H^{1/2+\sigma}(\Omega)^2$. Since $\chi_h \in H_0^1(\Gamma)$, we have a Poincaré inequality

$$\|\chi_h\|_{1,\Gamma} \le C\|\partial_t\chi_h\|_{0,\Gamma}.$$

Finally, using the $L^2(\Gamma)$ stability of $\Pi_{h,\Gamma}$, we have

$$\begin{aligned}\|\partial_t\chi_h\|_{0,\Gamma} &\le C\|(I_h^1\mathbf{w} - I_h^2\mathbf{w}) \cdot \boldsymbol{\tau}\|_{0,\Gamma} \\ &\le C\|(I_h^1\mathbf{w} - \mathbf{w}) \cdot \boldsymbol{\tau}\|_{0,\Gamma} + C\|(\mathbf{w} - I_h^2\mathbf{w}) \cdot \boldsymbol{\tau}\|_{0,\Gamma} \\ &\le Ch^\sigma|\mathbf{w}|_{\sigma,\Gamma} \le Ch^\sigma\|\mathbf{w}\|_{1/2+\sigma,\Omega},\end{aligned}$$

where we have used again the regularity of $\mathbf{w} \in H^{1/2+\sigma}(\Omega)$. Therefore, using the Cauchy-Schwarz inequality and the estimate (12), we obtain

$$T_1 \le Ch^\sigma\|\mathbf{w}\|_{1/2+\sigma,\Omega}\|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega} \le C_1 h^\sigma\|\mathbf{w}_h\|_{\mathbf{V}(h)}\|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega}.$$

2nd term. It is easy to see that $\Pi_h^M\mathbf{w} - \boldsymbol{\psi}_h^c \in \mathbf{V}_h^c$. Moreover, thanks to the structure of (15), the identity (14), and the commuting diagram properties for the edge element interpolation operator I_h, we have:

$$\nabla_h \times (\Pi_h^M\mathbf{w} - \boldsymbol{\psi}_h^c) = \nabla_h \times (I_h\mathbf{w} - \boldsymbol{\psi}_h^c) = 0.$$

Thus, we have $\Pi_h^M\mathbf{w} - \boldsymbol{\psi}_h^c \in K_h$. Thus, owing to (7), there exists a $\xi_h \in Q_h^c$ such that $\Pi_h^M\mathbf{w} - \boldsymbol{\psi}_h^c = \nabla_h\xi_h$. Since $\mathbf{w}_h \in K_h^\perp$, by integrating by parts, we obtain

$$T_2 = (\nabla_h\xi_h, \mathbf{w} - \mathbf{w}_h) = \int_\Gamma [\![\xi_h]\!]\, \overline{\mathbf{w}} \cdot \mathbf{n}\, ds.$$

On Γ, by surjectivity of the operator ∂_t, since $\mathbf{w} \cdot \mathbf{n} \in H^\sigma(\Gamma)$, there exists a $\varphi \in H^{1+\sigma}(\Gamma)$ such that $\partial_t\varphi = \mathbf{w} \cdot \mathbf{n}$. Using (9) and recalling that $[\![\xi_h]\!] \in H_0^1(\Gamma)$ we obtain that, for any $\varphi_h \in \Phi_h \cap H^1(\Gamma)$,

$$\begin{aligned}\int_\Gamma [\![\xi_h]\!]\,\overline{\mathbf{w}}\cdot\mathbf{n}\,ds &= \int_\Gamma [\![\xi_h]\!]\partial_t(\overline{\varphi}-\overline{\varphi}_h)\,ds = \int_\Gamma \partial_t[\![\xi_h]\!](\overline{\varphi}-\overline{\varphi}_h)\,ds\\ &\le C\,h^\sigma\,\|\partial_t[\![\xi_h]\!]\|_{-1,\Gamma}\|\varphi\|_{1+\sigma,\Gamma}\\ &\le C\,h^\sigma\,\|\mathbf{w}_h\|^2_{\mathbf{V}(h)},\end{aligned}\tag{17}$$

where we have used that φ_h is at least a piecewise polynomial of degree 1, and the following estimate:

$$\begin{aligned}\|\partial_t[\![\xi_h]\!]\|_{-1,\Gamma} \le \|\nabla_h\xi_h\|_{0,\Omega} &\le \|\Pi_h^M\mathbf{w}\|_{0,\Omega} + \|\boldsymbol{\psi}_h^c\|_{0,\Omega}\\ &\le C_1\|\mathbf{w}\|_{1/2+\sigma,\Omega} + C_2\|\mathbf{w}_h\|_{\mathbf{V}(h)}\\ &\le C\|\mathbf{w}_h\|_{\mathbf{V}(h)}.\end{aligned}$$

Therefore, we have the bound

$$T_2 \le C_2\,h^\sigma\,\|\mathbf{w}_h\|^2_{\mathbf{V}(h)}.$$

3rd term. From the estimate (13), we have

$$T_3 \le C_3h\|\mathbf{w}_h\|_{\mathbf{V}(h)}\|\mathbf{w}-\mathbf{w}_h\|_{0,\Omega}.$$

Collecting the estimates from the previous steps, we obtain

$$\begin{aligned}\|\mathbf{w}-\mathbf{w}_h\|^2_{0,\Omega} \le\ & C_1h^\sigma\|\mathbf{w}_h\|_{\mathbf{V}(h)}\|\mathbf{w}-\mathbf{w}_h\|_{0,\Omega}\\ &+C_2h^\sigma\,\|\mathbf{w}_h\|^2_{\mathbf{V}(h)}\\ &+C_3h\|\mathbf{w}_h\|_{\mathbf{V}(h)}\,\|\mathbf{w}-\mathbf{w}_h\|_{0,\Omega},\end{aligned}\tag{18}$$

which implies

$$\|\mathbf{w}-\mathbf{w}_h\|_{0,\Omega} \le Ch^{\sigma/2}\|\mathbf{w}_h\|_{\mathbf{V}(h)},$$

hence the result. □

We now turn to the proof of Property 5 for general piecewise smooth tensors ε. To this aim, we need some preliminary results.

As a consequence of Theorem 4.9, we have the following proposition.

Proposition 4.10 $\mathbf{K}_h$ *is approximating in* $\mathbf{V}^0$ *and* $K_h^\perp$ *is approximating in* $\mathbf{W}$, *i.e.*,

$$\lim_{h\to 0}\inf_{\mathbf{k}_h\in K_h}\|\mathbf{k}-\mathbf{k}_h\|_{\mathbf{V}(h)} = 0\quad \forall\,\mathbf{k}\in\mathbf{V}^0,\tag{19}$$

$$\lim_{h\to 0}\inf_{\mathbf{w}_h\in K_h^\perp}\|\mathbf{w}-\mathbf{w}_h\|_{\mathbf{V}(h)} = 0\quad \forall\,\mathbf{w}\in\mathbf{W}.\tag{20}$$

Proof (i) *Proof of* (19). Since both the continuous and discrete kernel do not depend upon ε, we set $\varepsilon = I$ in this part of the proof. Let Π_h be the $\mathbf{V}(h)$-orthogonal projection from $\mathbf{V}$ onto $\mathbf{V}_h$; $(I-\Pi_h)\in\mathcal{L}(\mathbf{V},\mathbf{V}(h))$ and, owing to Property 2, $(I-\Pi_h)\to 0$ as $h\to 0$ pointwise. Denote by P_h and Q_h the $\mathbf{V}(h)$-orthogonal projections of $\mathbf{V}_h$ onto $K_h^\perp$ and K_h, respectively (recall that we have set, for the moment, $\varepsilon = I$). Set $\mathbf{k}_h = Q_h\Pi_h\mathbf{k}$ and $\mathbf{w}_h := P_h\Pi_h\mathbf{k}$. Notice that, owing to (7), $\mathbf{k}_h = \nabla_h p_h^c$ for some $p_h^c \in Q_h^c$.

Let $\mathbf{w}$ be the element of $\mathbf{w}$ 'close' to $\mathbf{w}_h$ as from Theorem 4.9. Due to the definition of $\mathbf{w}_h$ and $(\cdot,\cdot)_{\mathbf{V}(h)}$, the orthogonality of $\mathbf{k}$ and $\mathbf{w}$, the Cauchy-Schwartz inequality and Theorem 4.9, we have

$$\|\mathbf{w}_h\|^2_{\mathbf{V}(h)} = (P_h\Pi_h\mathbf{k}, \mathbf{w}_h)_{\mathbf{V}(h)} = (\mathbf{k}, \mathbf{w}_h)_1 = (\mathbf{k}, \mathbf{w}_h - \mathbf{w})_1 \le \eta_h\|\mathbf{k}\|_{0,\Omega}\|\mathbf{w}_h\|_{\mathbf{V}(h)},$$

from which

$$\|\mathbf{w}_h\|_{\mathbf{V}(h)} \le \eta_h\|\mathbf{k}\|_{0,\Omega}. \tag{21}$$

The triangle inequality, (21) gives

$$\|\mathbf{k} - \mathbf{k}_h\|_{\mathbf{V}(h)} \le \|\mathbf{k} - \Pi_h\mathbf{k}_h\|_{\mathbf{V}(h)} + \|\mathbf{w}_h\|_{\mathbf{V}(h)}$$

hence (19), due to Property 2 and (21).

(ii) *Proof of* (20). Let Π_h be again the $\mathbf{V}(h)$-orthogonal projection from $\mathbf{V}$ onto $\mathbf{V}_h$, P_h and Q_h the $\mathbf{V}(h)$-orthogonal projections of $\mathbf{V}_h$ onto $K_h^\perp$ and K_h, respectively (note that now ε is any). Given a $\mathbf{w} \in \mathbf{W}$, the following chain of equalities holds for all $\mathbf{k}_h = \nabla_h p_h^c$, $\mathbf{k}_h \in K_h$ and $p_h^c \in Q_h^c$:

$$(Q_h\Pi_h\mathbf{w}, \mathbf{k}_h)_{\mathbf{V}(h)} = (\mathbf{w}, \mathbf{k}_h)_{\mathbf{V}(h)} = (\mathbf{w}, \mathbf{k}_h)_\varepsilon = \int_\Gamma (\varepsilon\,\mathbf{w}\cdot\mathbf{n})[\![\,\overline{p}_h^c]\!]. \tag{22}$$

For any $\varphi_h \in \Phi_h \cap H^1(\Gamma)$, owing to (8) we have

$$\begin{aligned}\int_\Gamma (\varepsilon\,\mathbf{w}\cdot\mathbf{n})[\![\,\overline{p}_h^c]\!]\,ds &= \int_\Gamma (\varepsilon\,\mathbf{w}\cdot\mathbf{n} - \partial_t\varphi_h)\,[\![\,\overline{p}_h^c]\!]\,ds \\ &\le |[\![p_h^c]\!]|_{1/2,\Gamma}\|\varepsilon\,\mathbf{w}\cdot\mathbf{n} - \partial_t\varphi_h\|_{-1/2,\Gamma}.\end{aligned}$$

The continuity of trace, the Poincaré inequality, the equivalence between the $L^2(\Omega)$-norm and the $L^2_\varepsilon(\Omega)$-norm and the definition of p_h^c give

$$|[\![p_h^c]\!]|_{1/2,\Gamma} \le C\|\varepsilon^{1/2}\mathbf{k}_h\|_{0,\Omega}.$$

Due to the smoothness of $\mathbf{w}$ (indeed $\mathbf{w} \in H^\sigma(\Omega)^2$, for some $0 < \sigma < 1/2$, implies that $\varepsilon\,\mathbf{w}\cdot\mathbf{n} \in H^{-1/2+\sigma}(\Gamma)$), and the fact that the space $\{\partial_t\varphi_h\,,\ \varphi_h \in \Phi_h \cap H^1(\Gamma)\}$ contains all piecewise constant functions,

$$\inf_{\varphi_h\in\Phi_h\cap H^1(\Gamma)} \|\varepsilon\,\mathbf{w}\cdot\mathbf{n} - \partial_t\varphi_h\|_{-1/2,\Gamma} \le Ch^\sigma\|\mathbf{w}\|_{\mathbf{V}(h)}.$$

Therefore, by choosing $\mathbf{k}_h = Q_h\Pi_h\mathbf{w}$ in (22), we have:

$$\|Q_h\Pi_h\mathbf{w}\|_{\mathbf{V}(h)} = \|\varepsilon^{1/2}Q_h\Pi_h\mathbf{w}\|_{0,\Omega} \le Ch^\sigma\|\mathbf{w}\|_{\mathbf{V}(h)}.$$

Hence, (20) follows by triangle inequality and Property 2. □

Corollary 4.11 *Let* $0 < \sigma < 1/2$; *there exists a sequence* ξ_h, $\xi_h \to 0$ *when* $h \to 0$, *such that*

$$\inf_{\mathbf{k}_h\in K_h} \|\mathbf{k} - \mathbf{k}_h\|_{\mathbf{V}(h)} \le \xi_h\|\mathbf{k}\|_{\sigma,\Omega} \quad \forall\mathbf{k} \in \nabla H_0^1(\Omega) \cap H^\sigma(\Omega)^2.$$

Proof It is a direct consequence of Proposition 4.10 and the compactness of the injection of $\nabla H_0^1(\Omega) \cap H^\sigma(\Omega)^2$ in $\mathbf{V}^0$. □

Theorem 4.12 *Property 5 holds true.*

Proof Along this proof, we explicitly indicate with a subscript the dependence of $\mathbf{V}(h)$ on ε by writing $\mathbf{W}_1$, $\mathbf{V}_1(h)$ and $\mathbf{W}_\varepsilon$, $\mathbf{V}_\varepsilon(h)$ for the cases $\varepsilon = I$ and general ε, respectively.

Fix $\mathbf{w}_h \in K_h^\perp$. Due to (7) and the definition of the $\mathbf{V}_\varepsilon(h)$-norm, we have

$$(\mathbf{w}_h, \nabla_h q_h^c)_{\mathbf{V}_\varepsilon(h)} = (\mathbf{w}_h, \nabla_h q_h^c)_\varepsilon = 0 \quad \forall q_h^c \in Q_h^c. \tag{23}$$

Now, decompose $\mathbf{w}_h$ as $\mathbf{w}_h = \widetilde{\mathbf{w}}_h + \nabla_h p_h^c$, with $p_h^c \in Q_h^c$ and $\widetilde{\mathbf{w}}_h$ in the $\mathbf{V}_1(h)$-orthogonal complement of $\nabla_h Q_h^c$ in $\mathbf{V}_h$. We have

$$(\widetilde{\mathbf{w}}_h, \nabla_h q_h^c)_{\mathbf{V}_1(h)} = (\widetilde{\mathbf{w}}_h, \nabla_h q_h^c)_1 = 0 \quad \forall q_h^c \in Q_h^c. \tag{24}$$

From Theorem 4.9, provided that h is small enough, given $\widetilde{\mathbf{w}}_h$, there exists $\widetilde{\mathbf{w}} \in \mathbf{W}_1$ such that

$$\|\widetilde{\mathbf{w}} - \widetilde{\mathbf{w}}_h\|_{0,\Omega} \le \widetilde{\eta}_h \|\widetilde{\mathbf{w}}_h\|_{\mathbf{V}_1(h)}, \tag{25}$$

with $\widetilde{\eta}_h \to 0$ as $h \to 0$. Moreover, using (12), we also know that there is a $\sigma > 0$ such that $\widetilde{\mathbf{w}} \in H^{1/2+\sigma}(\Omega)^2$ and,

$$\|\widetilde{\mathbf{w}}\|_{1/2+\sigma,\Omega} \le C\|\widetilde{\mathbf{w}}\|_{\mathbf{V}_1(h)} \le C\|\widetilde{\mathbf{w}}_h\|_{\mathbf{V}_1(h)}. \tag{26}$$

Notice that, owing to (24) and the Cauchy-Schwarz inequality, we have $\|\widetilde{\mathbf{w}}_h\|^2_{\mathbf{V}_1(h)} = (\widetilde{\mathbf{w}}_h, \widetilde{\mathbf{w}}_h + \nabla_h p_h^c)_{\mathbf{V}_1(h)} = (\widetilde{\mathbf{w}}_h, \mathbf{w}_h)_{\mathbf{V}_1(h)} \le \|\widetilde{\mathbf{w}}_h\|_{\mathbf{V}_1(h)}\|\mathbf{w}_h\|_{\mathbf{V}_1(h)}$, and then, due to equivalence between the $\mathbf{V}_1(h)$-norm and the $\mathbf{V}_\varepsilon(h)$-norm, we have

$$\|\widetilde{\mathbf{w}}_h\|_{\mathbf{V}_1(h)} \le C\|\mathbf{w}_h\|_{\mathbf{V}_\varepsilon(h)}, \tag{27}$$

with C only depending on ε.

Now, decompose $\widetilde{\mathbf{w}} \in \mathbf{W}_1$ as $\widetilde{\mathbf{w}} = \mathbf{w} + \nabla p$, with $\mathbf{w} \in \mathbf{W}_\varepsilon$ and $p \in H_0^1(\Omega)$.

Note that p verifies the following variational equality:

$$\int_\Omega \varepsilon \nabla p \cdot \nabla \overline{q} = \int_\Omega \varepsilon \widetilde{\mathbf{w}} \cdot \nabla \overline{q} \quad \forall q \in H_0^1(\Omega).$$

The regularity theory for the Poisson problem implies that there exists a $\lambda > 0$ depending upon ε and σ, such that $p \in H^{1+\lambda}(\Omega)$. Using also (26) and (27), it holds:

$$\|p\|_{1+\lambda,\Omega} \le C\|\widetilde{\mathbf{w}}\|_{\mathbf{V}_1(h)} \le C\|\widetilde{\mathbf{w}}_h\|_{\mathbf{V}_1(h)} \le C\|\mathbf{w}_h\|_{\mathbf{V}_\varepsilon(h)}. \tag{28}$$

We prove that

$$\|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega} \le \eta_h \|\mathbf{w}_h\|_{\mathbf{V}_\varepsilon(h)},$$

with $\eta_h \to 0$ as $h \to 0$.

In order to do that, write

$$\begin{aligned}\|\varepsilon^{1/2}(\mathbf{w} - \mathbf{w}_h)\|^2_{0,\Omega} &= (\mathbf{w} - \mathbf{w}_h, \mathbf{w} - \widetilde{\mathbf{w}})_\varepsilon + (\mathbf{w} - \mathbf{w}_h, \widetilde{\mathbf{w}} - \widetilde{\mathbf{w}}_h)_\varepsilon \\ &\quad + (\mathbf{w} - \mathbf{w}_h, \widetilde{\mathbf{w}}_h - \mathbf{w}_h)_\varepsilon \\ &=: T_1 + T_2 + T_3,\end{aligned}$$

and estimate T_1, T_2 and T_3 separately.

1st term. From $\widetilde{\mathbf{w}} - \mathbf{w} = \nabla p$, $(\mathbf{w}, \nabla p)_\varepsilon = 0$ and (23), we have

$$T_1 = (\mathbf{w} - \mathbf{w}_h, -\nabla p)_\varepsilon = (\mathbf{w}_h, \nabla p - \nabla_h q_h^c)_\varepsilon \leq \|\varepsilon \mathbf{w}_h\|_{0,\Omega} \|\nabla p - \nabla_h q_h^c\|_{0,\Omega}$$

for all $q_h^c \in Q_h^c$. From Proposition 4.11 and the stability estimate (28), we have

$$T_1 \leq \xi_h \|\varepsilon \mathbf{w}_h\|_{0,\Omega} \|\mathbf{w}_h\|_{\mathbf{V}_\varepsilon(h)}.$$

2nd term. For the second term, by using the Cauchy-Schwarz inequality, the equivalence between the $L^2(\Omega)$-norm and the $L^2_\varepsilon(\Omega)$-norm, and the bounds (25) and (27), we have

$$\begin{aligned} T_2 &\leq \|\varepsilon^{1/2}(\mathbf{w} - \mathbf{w}_h)\|_{0,\Omega} \|\varepsilon^{1/2}(\widetilde{\mathbf{w}} - \widetilde{\mathbf{w}}_h)\|_{0,\Omega} \\ &\leq C\widetilde{\eta}_h \|\mathbf{w}_h\|_{\mathbf{V}_\varepsilon(h)} \|\varepsilon^{1/2}(\mathbf{w} - \mathbf{w}_h)\|_{0,\Omega}. \end{aligned}$$

3rd term. Since $\widetilde{\mathbf{w}}_h - \mathbf{w}_h = \nabla_h p_h^c$, by using (23), integration by parts and the fact that $\nabla \times (\varepsilon \mathbf{w}) = 0$, we have

$$T_3 = (\mathbf{w} - \mathbf{w}_h, \nabla_h p_h^c)_\varepsilon = (\mathbf{w}, \nabla_h p_h^c)_\varepsilon = \int_\Gamma (\varepsilon \mathbf{w} \cdot \mathbf{n}) [\![\overline{p}_h^c]\!] \, ds.$$

Using the same techniques as in the proof of (20) in Proposition 4.10, we deduce that

$$T_3 \leq C h^\sigma \|\varepsilon^{1/2} \nabla_h p_h^c\|_{0,\Omega} \|\mathbf{w}\|_{\mathbf{V}_\varepsilon(h)}.$$

Thus, using the stability $\|\mathbf{w}\|_{\mathbf{V}_\varepsilon(h)} \leq C\|\widetilde{\mathbf{w}}\|_{\mathbf{V}_\varepsilon(h)} \leq C\|\mathbf{w}_h\|_{\mathbf{V}_\varepsilon(h)}$ (see also (28)), the stability $\|\nabla_h p_h^c\|_{0,\Omega} \leq C\|\mathbf{w}_h\|_{\mathbf{V}_1(h)}$ and the equivalence between the $\mathbf{V}_1(h)$-norm and the $\mathbf{V}_\varepsilon(h)$-norm, we obtain

$$\begin{aligned} T_3 &\leq C h^\sigma \|\mathbf{w}\|_{\mathbf{V}_\varepsilon(h)} \|\nabla_h p_h^c\|_{0,\Omega} \\ &\leq C\, h^\sigma \|\mathbf{w}_h\|_{\mathbf{V}_\varepsilon(h)} \|\mathbf{w}_h\|_{\mathbf{V}_1(\Omega)} \leq C h^\sigma \|\mathbf{w}_h\|^2_{\mathbf{V}_\varepsilon(h)}, \end{aligned}$$

hence the result. □

We turn now the attention to another important consequence of Property 5.

Property 6 (Discrete Friedrichs inequality) *Let $\mathbf{w}_h \in K_h^\perp$, then there exists a $\beta > 0$ independent of the mesh size, such that*

$$\|\mathbf{w}_h\|^2_{0,\Omega} \leq \beta\, a_h(\mathbf{w}_h, \mathbf{w}_h).$$

Theorem 4.13 *Property 5 implies Property 6.*

Proof This proof follows the lines of [8, Proposition 2.19] (see also [17, Lemma 7.20]). First, if $a_h(\mathbf{w}_h, \mathbf{w}_h) = 0$, then it is easy to see that $\mathbf{w}_h \in K_h$. Thus $\mathbf{w}_h = 0$. This implies that there exists a $\beta = \beta(h)$, i.e., possibly depending upon h. We prove that β is independent of h by contradiction.

First, we prove the statement for functions $\mathbf{w}_h^c$ in $K_h^\perp \cap \mathbf{V}_h^c$. Note that

$$a_h(\mathbf{w}_h^c, \mathbf{w}_h^c) = \|\nabla_h \times \mathbf{w}_h^c\|^2_{0,\Omega}.$$

If β was depending on h, there would be a sequence a mesh sizes $h(n)$, $h(n) \to 0$ when $n \to \infty$, and vector fields $\mathbf{w}^c_{h(n)}$ such that

$$\|\mathbf{w}^c_{h(n)}\|_{0,\Omega} = 1, \quad \|\nabla_h \times \mathbf{w}^c_{h(n)}\|_{0,\Omega} = 1/n.$$

Now, using Property 5 and its proof, we know that, for each $\mathbf{w}^c_{h(n)}$, there exists a $\mathbf{w}(n) \in \mathbf{W}$, such that

$$\nabla \times \mathbf{w}(n) = \nabla_h \times \mathbf{w}^c_{h(n)}, \quad \|\mathbf{w}^c_{h(n)} - \mathbf{w}(n)\|_{0,\Omega} \le \eta_{h(n)} \|\mathbf{w}^c_{h(n)}\|_{\mathbf{V}(h(n))}. \tag{29}$$

Thus, we know $\|\mathbf{w}(n)\|_{0,\Omega} \le 1 + \eta_{h(n)}(1 + 1/n)$ and, the compactness of the $L^2(\Omega) \hookrightarrow \mathbf{W}$ implies that there exists a $\mathbf{w} \in L^2(\Omega)$ and a subsequence of $\{\mathbf{w}(n)\}$, still denoted by $\{\mathbf{w}(n)\}$, such that:

$$\mathbf{w}(n) \to \mathbf{w} \quad \text{in } L^2(\Omega) \quad \text{and} \quad \nabla \cdot (\varepsilon \mathbf{w}) = 0.$$

Since $\|\nabla \times \mathbf{w}(n)\|_{0,\Omega} = \|\nabla_h \times \mathbf{w}^c_{h(n)}\|_{0,\Omega} = 1/n$, by uniqueness of the limit, $\nabla \times \mathbf{w} = 0$. Thus, $\mathbf{w} = 0$ and $\|\mathbf{w}(n)\|_{0,\Omega} \to 0$ as $n \to \infty$. Using again (29), we have then

$$\|\mathbf{w}^c_{h(n)}\|_{0,\Omega} \le \|\mathbf{w}(n)\|_{0,\Omega} + \eta_{h(n)}(1 + 1/n),$$

which implies $\|\mathbf{w}^c_{h(n)}\|_{0,\Omega} \to 0$, which contradicts the assumption.

Given now $\mathbf{w}_h \in K_h^\perp$, by Corollary 4.7 there exists a $\mathbf{w}^c_h \in K_h^\perp \cap \mathbf{V}^c_h$ such that (11) holds true. Then, we can conclude that

$$\begin{aligned}
\|\mathbf{w}_h\|^2_{0,\Omega} &\le C\Big[\|\mathbf{w}^c_h\|^2_{0,\Omega} + \|\mathrm{h}^{1/2}[\![\mathbf{w}_h]\!]_T\|^2_{0,\mathcal{F}^1_h\cup\mathcal{F}^2_h} + \|\mathrm{h}^{1/2}\Pi_{h,\Gamma}[\![\mathbf{w}_h]\!]_T\|^2_{0,\mathcal{F}^\Gamma_h}\Big] \\
&\le C\Big[\|\nabla_h \times \mathbf{w}^c_h\|^2_{0,\Omega} + \|\mathrm{h}^{1/2}[\![\mathbf{w}_h]\!]_T\|^2_{0,\mathcal{F}^1_h\cup\mathcal{F}^2_h} + \|\mathrm{h}^{1/2}\Pi_{h,\Gamma}[\![\mathbf{w}_h]\!]_T\|^2_{0,\mathcal{F}^\Gamma_h}\Big] \\
&\le C\Big[\|\nabla_h \times \mathbf{w}_h\|^2_{0,\Omega} + \|\mathrm{h}^{-1/2}[\![\mathbf{w}_h]\!]_T\|^2_{0,\mathcal{F}^1_h\cup\mathcal{F}^2_h} + \|\mathrm{h}^{-1/2}\Pi_{h,\Gamma}[\![\mathbf{w}_h]\!]_T\|^2_{0,\mathcal{F}^\Gamma_h}\Big] \\
&\le C a_h(\mathbf{w}_h, \mathbf{w}_h).
\end{aligned}$$

□

4.4 Non-Pollution of the Spectrum and Isolation of the Discrete Kernel

The present section is devoted to the proof that Problem 3 is a spurious free approximation of Problem 1 in the sense of [8] and is intimately related to [7, Sect. 4]. We recall that for an approximation to be spurious free, two main properties must hold: (i) there are no spurious eigenvalues, (ii) the discrete kernel is isolated. Following [11], the main theoretical tool for (i) is the proof of convergence for $(A - A_h)$ when applied to discrete fields. The main difference between the present theory and the one developed in [7] is that we can prove such a convergence (see Theorem 4.14 below) only for discrete fields in $K_h^\perp$ and not for fields in $\mathbf{V}_h$. This fact will not have consequences on the main result of this section, i.e., Theorem 4.19, but it will weaken the theory of the non-pollution of eigenspaces. Namely, we will prove non-pollution of eigenspaces for all eigenvalues but the value 1 associated to the kernel (see Theorem 4.24 below).

In order to prove non-pollution of the spectrum, we need some intermediate steps.

Theorem 4.14 (Convergence in mesh-dependent norm) *For all h small enough*,

$$\|(A - A_h)\mathbf{f}_h\|_{\mathbf{V}(h)} \le \eta_h \|\mathbf{f}_h\|_{\mathbf{V}(h)} \quad \forall \mathbf{f}_h \in K_h^\perp,$$

with $\eta_h \to 0$ *as* $h \to 0$.

Proof The proof is a slight modification of that of [7, Proposition 4.4]. We report here the complete proof, for convenience.

Fix $\mathbf{f}_h \in K_h^{\perp}$ and let $\mathbf{f} \in \mathbf{W}$ be as in Property 5; thus,

$$\|(A - A_h)\mathbf{f}_h\|_{\mathbf{V}(h)} \le \|(A - A_h)(\mathbf{f} - \mathbf{f}_h)\|_{\mathbf{V}(h)} + \|(A - A_h)\mathbf{f}\|_{\mathbf{V}(h)}. \tag{30}$$

Owing to the continuity of A_h (see Remark 4.3) and Property 5, for the first term at the right-hand side of (30) we have

$$\begin{aligned}\|(A - A_h)(\mathbf{f} - \mathbf{f}_h)\|_{\mathbf{V}(h)} &\le (\|A\|_{\mathcal{L}(L^2(\Omega)^2,\mathbf{V})} + \|A_h\|_{\mathcal{L}(L^2(\Omega)^2,\mathbf{V}_h)})\|\mathbf{f} - \mathbf{f}_h\|_{0,\Omega} \\ &\le C\eta_h\|\mathbf{f}_h\|_{\mathbf{V}(h)}.\end{aligned}$$

For the second term at the right-hand side of (30), from Property 4, Property 5 and the definition of the $\mathbf{V}(h)$-norm, we have

$$\|(A - A_h)\mathbf{f}\|_{\mathbf{V}(h)} \le \xi_h\|\mathbf{f}\|_{0,\Omega} \le \xi_h(\|\mathbf{f} - \mathbf{f}_h\|_{0,\Omega} + \|\mathbf{f}_h\|_{0,\Omega}) \le \xi_h(\eta_h + 1)\|\mathbf{f}_h\|_{\mathbf{V}(h)},$$

and the proof is complete. □

Theorem 4.14 corresponds to the validity of [7, Property 2], i.e., the uniform convergence to 0 of $A - A_h : \mathbf{V}_h \to \mathbf{V}(h)$. As we have already pointed out, Theorem 4.14 is weaker than [7, Property 2] since it proves only the convergence to 0 for $A - A_h : K_h^{\perp} \to \mathbf{V}(h)$. Fortunately, the following lemmas still hold.

Lemma 4.15 *Fix* $0 \ne z \in \rho(A)$. *There exists a positive constant* C *only depending upon* Ω *and* $|z|$ *such that, for all* $\mathbf{f} \in \mathbf{V}(h)$,

$$\|(z - A)\mathbf{f}\|_{\mathbf{V}(h)} \ge C\|\mathbf{f}\|_{\mathbf{V}(h)}.$$

Proof See proof of [7, Lemma 4.6]. □

The result of [7, Theorem 4.7] holds true, although its proof needs to be modified. We first state the following lemma.

Lemma 4.16 *If* $0 \ne z \in \rho(A)$, *we have that* $(z - A_h)\mathbf{f} \in K_h^{\perp}$ *if and only if* $\mathbf{f} \in K_h^{\perp}$.

Proof For any $\mathbf{f}^0 \in K_h$, since $(A_h\mathbf{f}, \mathbf{f}^0)_{\mathbf{V}(h)} = (A_h\mathbf{f}, \mathbf{f}^0)_\varepsilon = (\mathbf{f}, \mathbf{f}^0)_\varepsilon = (\mathbf{f}, \mathbf{f}^0)_{\mathbf{V}(h)}$, we have

$$\left((z - A_h)\mathbf{f}, \mathbf{f}^0\right)_{\mathbf{V}(h)} = z\left(\mathbf{f}, \mathbf{f}^0\right)_{\mathbf{V}(h)},$$

and, since $z \ne 1$, the result immediately follows. □

Theorem 4.17 *Fix* $0 \ne z \in \rho(A)$. *For* h *small enough, there exists a positive constant* C *only depending upon* Ω *and* $|z|$ *such that, for all* $\mathbf{f} \in \mathbf{V}_h$,

$$\|(z - A_h)\mathbf{f}\|_{\mathbf{V}(h)} \ge C\|\mathbf{f}\|_{\mathbf{V}(h)}.$$

Proof If $\mathbf{f} \in K_h^{\perp}$, then the result follows exactly as in the proof of [7, Theorem 4.7], due to Theorem 4.14. In the general case, decompose $\mathbf{f} = \mathbf{f}^0 + \mathbf{f}^{\perp}$, with $\mathbf{f}^0 \in K_h$ and $\mathbf{f}^{\perp} \in K_h^{\perp}$. Due to orthogonality, we have $\|\mathbf{f}\|^2_{\mathbf{V}(h)} = \|\mathbf{f}^0\|^2_{\mathbf{V}(h)} + \|\mathbf{f}^{\perp}\|^2_{\mathbf{V}(h)}$.

Since $A_h\mathbf{f}^0 = \mathbf{f^0}$, we have

$$\begin{aligned}\|(z - A_h)\mathbf{f}\|^2_{\mathbf{V}(h)} &= \|(z-1)\mathbf{f}^0 + (z - A_h)\mathbf{f}^{\perp}\|^2_{\mathbf{V}(h)} \\ &= |z-1|^2\|\mathbf{f}^0\|^2_{\mathbf{V}(h)} + \|(z - A_h)\mathbf{f}^{\perp}\|^2_{\mathbf{V}(h)} + 2\left((z-1)\mathbf{f}^0, (z - A_h)\mathbf{f}^{\perp}\right)_{\mathbf{V}(h)}.\end{aligned}$$

The inner product on the right-hand side is zero, due to Lemma 4.16; then, the first part of this proof allows us to conclude that

$$\begin{aligned}\|(z - A_h)\mathbf{f}\|^2_{\mathbf{V}(h)} &= |z-1|^2\|\mathbf{f}^0\|^2_{\mathbf{V}(h)} + \|(z - A_h)\mathbf{f}^{\perp}\|^2_{\mathbf{V}(h)} \\ &\geq C(\|\mathbf{f^0}\|^2_{\mathbf{V}(h)} + \|\mathbf{f}^{\perp}\|^2_{\mathbf{V}(h)}) = C\|\mathbf{f}\|^2_{\mathbf{V}(h)},\end{aligned}$$

since $z \neq 1$. □

Theorem 4.17 implies that, for any $0 \neq z \in \rho(A)$ and h small enough, $(z - A_h)$ is an invertible operator and the following result holds true.

Corollary 4.18 *Let $F \subset \rho(A)$ be closed. Then, there exists a positive constant C independent of the mesh size such that, for h small enough, we have*

$$\|R_z(A_h)\|_{\mathcal{L}(\mathbf{V}_h, \mathbf{V}_h)} \leq C$$

for all $z \in F$, with $C > 0$ independent of the mesh size.

Proof See proof of [7, Corollary 4.8]. □

The non-pollution of the spectrum is now a direct consequence of Corollary 4.18 and we state it here in the form of a theorem:

Theorem 4.19 (Non-pollution of the spectrum) *Let $G \subset \mathbb{C}$ be an open set containing $\sigma(A)$. Then, for h small enough, $\sigma(A_h) \subset G$.*

Remark 4.20 For fixed $z \in \rho(A)$ and $\mathbf{f} \in \mathbf{V}(h)$, we can write

$$\|(z - A)\mathbf{f}\|_{\mathbf{V}(h)} \leq |z|\|\mathbf{f}\|_{\mathbf{V}(h)} + \|A\mathbf{f}\|_{\mathbf{V}} \leq |z|\|\mathbf{f}\|_{\mathbf{V}(h)} + C\|\mathbf{f}\|_{0,\Omega} \leq C(|z|)\|\mathbf{f}\|_{\mathbf{V}(h)},$$

owing to the stability estimate of the continuous problem and the definition of the $\mathbf{V}(h)$-norm. This, together with the result of Lemma 4.15, implies that, for all fixed $0 \neq z \in \rho(A)$, $(z - A) : \mathbf{V}(h) \to \mathbf{V}(h)$ is a continuous invertible operator with continuous inverse. An immediate consequence of this fact is the analogue of Corollary 4.18: Let $F \subset \rho(A)$ be closed. Then, there exists a positive constant C independent of the mesh size such that, for all $z \in F$,

$$\|R_z(A)\|_{\mathcal{L}(\mathbf{V}(h), \mathbf{V}(h))} \leq C.$$

Let us know turn to the *isolation of the discrete kernel*. The following result holds.

Theorem 4.21 (Isolation of discrete kernel) *There exists* $0 < \beta < 1$ *independent of the mesh size such that*, *if* $1 \neq \lambda_h \in \sigma(A_h)$, *then*

$$\mathrm{Re}[\lambda_h] \leq \beta.$$

Proof It is a direct consequence of Property 6 which is proved to hold thanks to Theorems 4.12 and 4.13. This result corresponds to [7, Proposition 4.1]. □

Remark 4.22 Consider the indefinite Maxwell source problem: given $\mathbf{f} \in L^2(\Omega)^d$ and $\omega \in \mathbb{R}$ such that ω^2 is not an eigenvalue of Problem 1, find $\mathbf{u} \in \mathbf{V}$ such that

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \omega^2 \varepsilon \mathbf{u} = \mathbf{f}.$$

With the same abstract arguments as in [7, Sect. 6], based on Theorem 4.17 and Corollary 4.18, one can prove that, provided that h is sufficiently small, the mortar-DG method

$$a_h(\mathbf{u}_h, \mathbf{v}) - \omega^2 (\mathbf{u}_h, \mathbf{v})_\varepsilon = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h,$$

is well-posed (existence and uniqueness of discrete solutions and continuous dependence on $\mathbf{f}$) and the following inf-sup condition holds true: there exists a constant $\kappa > 0$ independent of h such that

$$\inf_{\mathbf{0} \neq \mathbf{u}_h \in \mathbf{V}_h} \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{\mathrm{Re}[a_h(\mathbf{u}_h, \mathbf{v}_h) - \omega^2 (\mathbf{u}_h, \mathbf{v}_h)_\varepsilon]}{\|\mathbf{u}_h\|_{\mathbf{V}(h)} \|\mathbf{v}_h\|_{\mathbf{V}(h)}} \geq \kappa.$$

This inf-sup condition, together with the consistency, the coercivity in Property 3, and the continuity property (6) established within the proof of Proposition 4.4, are the key ingredients in the proof of the convergence of the mortar-DG method for the indefinite source problem.

4.5 Non-Pollution and Completeness of the Eigenspaces, and Completeness of the Spectrum

Let λ be an eigenvalue of A with algebraic multiplicity m, and let Γ be a circle in the complex plane centered at λ which lies in $\rho(A)$ and does not enclose any other point of $\sigma(A)$. According to [16, p. 178], we define the *spectral projections* E and, for h small enough, E_h from $\mathbf{V}_h$ into $\mathbf{V}(h)$ by

$$E = E_\lambda = \frac{1}{2\pi i} \int_\Gamma R_z(A)\, dz, \qquad E_h = E_{h,\lambda} = \frac{1}{2\pi i} \int_\Gamma R_z(A_h)\, dz, \tag{31}$$

respectively. Theorem 4.19 guarantees that, for h small enough, E_h is well defined.

We need to slightly modify the theory of [7], and we obtain a result on the non-pollution of the eigenspaces which is weaker than that of [7], namely, we have non-pollution only for the eigenspaces associated with discrete eigenvalues different from 1.

Theorem 4.23 *For h small enough*,

$$\|(E - E_h)\mathbf{f}\|_{\mathbf{V}(h)} \leq \xi_h \|\mathbf{f}\|_{\mathbf{V}(h)} \quad \forall \mathbf{f} \in K_h^\perp,$$

with $\xi_h \to 0$ *as* $h \to 0$.

Proof Since $R_z(A) - R_z(A_h) = R_z(A)(A - A_h)R_z(A_h)$, due to the fact that, if $\mathbf{f} \in K_h^\perp$ then $R_z(A_h)\mathbf{f} \in K_h^\perp$ (see Lemma 4.16), from Remark 4.20, Theorem 4.14 and Corollary 4.18, we have

$$\|R_z(A)(A - A_h)R_z(A_h)\mathbf{f}\|_{\mathbf{V}(h)} \le \|R_z(A)\|_{\mathcal{L}(\mathbf{V}(h),\mathbf{V}(h))}\|(A - A_h)R_z(A_h)\mathbf{f}\|_{\mathbf{V}(h)} \le C\eta_h\|\mathbf{f}\|_{\mathbf{V}(h)}.$$

The result immediately follows. □

If Y and Z are closed subspaces of $\mathbf{V}(h)$, we define

$$\delta_h(\mathbf{x}, Y) := \inf_{\mathbf{y}\in Y}\|\mathbf{x} - \mathbf{y}\|_{\mathbf{V}(h)}, \qquad \delta_h(Y, Z) := \sup_{\substack{\mathbf{y}\in Y\\ \|\mathbf{y}\|_{\mathbf{V}(h)}=1}} \delta_h(\mathbf{y}, Z),$$

$$\widehat{\delta}_h(Y, Z) := \max\{\delta_h(Y, Z), \delta_h(Z, Y)\}.$$

The following result is the analogue of [7, Theorem 4.11] for eigenspaces associated with discrete eigenvalues different from 1.

Theorem 4.24 (Non-pollution of the eigenspaces) *If* $\lambda \neq 1$, *we have*

$$\lim_{h\to 0}\delta_h(E_h(\mathbf{V}_h), E(\mathbf{V})) = 0.$$

Proof The proof is basically the same as that of [7, Theorem 4.11] (the only change is at the very end; we report here all the step for convenience).

We start by observing that $E(\mathbf{V}) = E(L^2(\Omega)^d)$. Indeed, $E(\mathbf{V})$ is the projection onto the eigenspace associated with the eigenvalue λ of the operator $A : \mathbf{V} \to \mathbf{V}$, and $E(L^2(\Omega)^d)$ is the projection onto the eigenspace associated with the eigenvalue λ of the operator $A : L^2(\Omega)^d \to L^2(\Omega)^d$ (see, e.g., [12, Theorem 5, p. 579]). Since all eigenfunctions of $A : L^2(\Omega)^d \to L^2(\Omega)^d$ are in $\mathbf{V}$, the two eigenspaces coincide, i.e., $E(\mathbf{V}) = E(L^2(\Omega)^d)$. Therefore,

$$\sup_{\substack{\mathbf{y}_h\in E_h(\mathbf{V}_h)\\ \|\mathbf{y}_h\|_{\mathbf{V}(h)}=1}} \inf_{\mathbf{x}\in E(\mathbf{V})}\|\mathbf{y}_h - \mathbf{x}\|_{\mathbf{V}(h)} = \sup_{\substack{\mathbf{y}_h\in E_h(\mathbf{V}_h)\\ \|\mathbf{y}_h\|_{\mathbf{V}(h)}=1}} \inf_{\mathbf{x}\in E(L^2(\Omega)^d)}\|\mathbf{y}_h - \mathbf{x}\|_{\mathbf{V}(h)}$$

$$= \sup_{\substack{\mathbf{y}_h\in E_h(\mathbf{V}_h)\\ \|y_h\|_{\mathbf{V}(h)}=1}} \inf_{\mathbf{x}\in L^2(\Omega)^d}\|E_h\mathbf{y}_h - E\mathbf{x}\|_{\mathbf{V}(h)},$$

where in the last step we have used that $E_h\mathbf{y}_h = \mathbf{y}_h$ for all $\mathbf{y}_h \in E_h(\mathbf{V}_h)$. Taking $\mathbf{x} = \mathbf{y}_h$, since $\mathbf{y}_h \in K_h^\perp$, Theorem 4.23 allows us to conclude. □

Theorem 4.25 (Completeness of the eigenspaces) *If* $E = E_\lambda$ *is associated with an eigenvalue* $\lambda \neq 1$, *then*

$$\lim_{h\to 0}\delta_h(E(\mathbf{V}), E_h(\mathbf{V}_h)) = 0.$$

Proof The proof follows the same lines as that of [7, Theorem 4.12]. Here, since the bound of Theorem 4.23 holds true for function in $K_h^\perp$ only, we use Proposition 4.10 instead of Property 2. We report here the complete proof, for convenience.

Since $EE\mathbf{y} = E\mathbf{y}$ for all $\mathbf{y} \in \mathbf{V}$, we can write

$$\delta_h(E(\mathbf{V}), E_h(\mathbf{V}_h)) = \sup_{\substack{\mathbf{x}\in E(\mathbf{V}) \\ \|\mathbf{x}\|_{\mathbf{V}(h)}=1}} \inf_{\mathbf{x}_h\in\mathbf{V}_h} \|E\mathbf{x} - E_h\mathbf{x}_h\|_{\mathbf{V}(h)}.$$

Fix $\mathbf{x} \in E(\mathbf{V})$. Then, $E\mathbf{x} = \mathbf{x}$ and $\mathbf{x} \in \mathbf{W}$. By Proposition 4.10, there exists $\widetilde{\mathbf{x}}_h \in K_h^{\perp}$ such that

$$\lim_{h\to 0} \|\mathbf{x} - \widetilde{\mathbf{x}}_h\|_{\mathbf{V}(h)} = 0. \tag{32}$$

Therefore,

$$\begin{aligned} \inf_{\mathbf{x}_h\in\mathbf{V}_h} \|E\mathbf{x} - E_h\mathbf{x}_h\|_{\mathbf{V}(h)} &\le \|E\mathbf{x} - E_h\widetilde{\mathbf{x}}_h\|_{\mathbf{V}(h)} \\ &\le \|E(\mathbf{x} - \widetilde{\mathbf{x}}_h)\|_{\mathbf{V}(h)} + \|(E - E_h)\widetilde{\mathbf{x}}_h\|_{\mathbf{V}(h)} \\ &\le \|E\|_{\mathcal{L}(\mathbf{V}(h),\mathbf{V}(h))}\|\mathbf{x} - \widetilde{\mathbf{x}}_h\|_{\mathbf{V}(h)} + \eta_h\|\widetilde{\mathbf{x}}_h\|_{\mathbf{V}(h)}, \end{aligned}$$

with $\eta_h \to 0$ as $h \to 0$ (see Theorem 4.23); the first term at right-hand side also tends to zero, as $h \to 0$, due to (32). Since $E(\mathbf{V})$ is the eigenspace associated with $\lambda \neq 1$, it is finite dimensional; therefore, pointwise convergence implies uniform convergence in $E(\mathbf{V})$, and the result readily follows. □

Finally, as far as the completeness of the spectrum is concerned, the analogue of [7, Theorem 4.13] holds true with the same proof.

Theorem 4.26 (Completeness of the spectrum) *For all* $\lambda \in \sigma(A)$,

$$\lim_{h\to 0} \delta_h(\lambda, \sigma(A_h)) = 0.$$

5 Numerical Results

We implemented the discrete Problem 3 following [14] and we applied it in four example meshes and geometries that follow. The mortar mesh for each case was formed by choosing the edges from one of the sides of the non-conforming interface. In all of the following experiments we chose a penalty parameter of the form $\mathtt{a} = 10\frac{(\ell+1)^2}{h}$.

5.1 Example 1: Square Cavity + Single Mortar

In this example we let $\Omega = (0,\pi)^2$ with $\mu = 1$ and $\varepsilon = I$ with corresponding eigenvalues $\lambda = (n^2 + m^2)$ for non-negative integers n, m with at least one of n, m being non-zero. The base mesh chosen for our initial experiments is shown in Fig. 1. It consists of two conforming blocks with a non-conforming interface at $x = \pi/2$.

In our first experiment we applied a standard IPDG formulation on a sequence of meshes obtained by uniform refinement of the base mesh. For each mesh, we computed all the eigenvalues in a range bracketing 0 to 8. We show in Table 1 the computed eigenvalues, and we see that there is pollution of the eigenvalues near zero and also extra eigenvalues interleave the expected eigenvalues.

Furthermore, in Table 2, we show that similar pollution happens for a IPDG discretisation when we keep the mesh fixed but increase the polynomial order of approximation, ℓ.

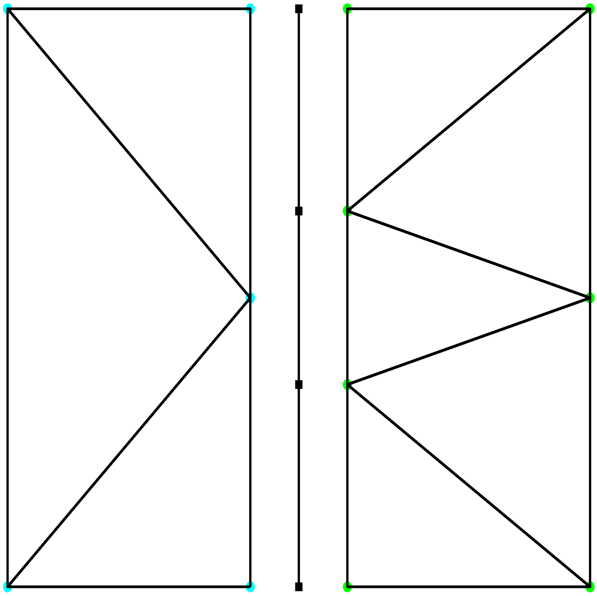

Fig. 1 Exploded view of the base non-conforming mesh blocks and mortar meshes used for Examples 1

Table 1 Eigenvalues obtained with IPDG compared with expected eigenvalues for the $\mu = 1$ and $\varepsilon = I$ material square domain shown in Fig. 1. Here $\ell = 1$, and $\mathtt{a} = \frac{10(\ell+1)^2}{h}$

Expected	h	$h/2$	$h/4$	$h/8$
1.0000	0.0000	1.0318	0.0468	0.0097
1.0000	0.5293	1.0468	0.0569	0.0130
2.0000	1.1267	2.1841	1.0079	0.0856
4.0000	1.1772	2.9721	1.0119	0.0924
4.0000	3.1432	3.2261	2.0472	1.0020
5.0000	4.9785	4.3796	2.8705	1.0030
5.0000	7.4444	4.6601	3.1185	1.2375
8.0000	8.1280	5.7319	4.1109	1.3953
–	–	6.2757	4.1741	2.0117
–	–	–	5.2056	3.5326
–	–	–	5.3193	3.7611
–	–	–	8.5813	4.0286
–	–	–	–	4.0448
–	–	–	–	5.0521
–	–	–	–	5.0779
–	–	–	–	8.1464

In contrast, in Table 3 we show that when we use the mortar-DG method, which is based on a projection based penalization at the non-conforming interface (IPPDG from here on) there is no pollution for the low eigenvalues on h-refinement.

And similarly in Table 4 we also see no pollution for a IPPDG discretisation when the mesh size is kept fixed and the order is increased, furthermore we see exponential convergence for each eigenvalue.

Table 2 Eigenvalues obtained with IPDG compared with expected eigenvalues for the $\mu = 1$ and $\varepsilon = I$ material square domain shown in Fig. 1. Here $\mathsf{a} = \frac{10(\ell+1)^2}{h}$

Expected	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
1.0000	0.5293	0.0889	0.0079	0.0005	0.0000
1.0000	1.1267	1.0036	0.2352	0.0166	0.0009
2.0000	1.1772	1.0043	1.0001	1.0000	0.5898
4.0000	3.1432	2.0157	1.0009	1.0000	1.0000
4.0000	4.9785	2.0826	2.0059	2.0000	1.0000
5.0000	7.4444	4.0489	4.0272	4.0001	2.0000
5.0000	8.1280	4.2297	4.0366	4.0142	4.0000
8.0000	–	5.2783	5.0576	5.0029	4.0002
–	–	6.1367	5.0708	5.0379	5.0002
–	–	–	8.2023	5.4277	5.0004
–	–	–	–	8.0945	8.0016

Table 3 First eight discrete eigenvalues, obtained with IPPDG, compared with expected eigenvalues for the $\mu = 1$ and $\varepsilon = I$ material square domain shown in Fig. 1. Here $\ell = 1$, and $\mathsf{a} = \frac{10(\ell+1)^2}{h}$

Expected	h	$h/2$	$h/4$	$h/8$	Est. order
1.0000	1.1255	1.0318	1.0079	1.0020	1.995
1.0000	1.1769	1.0467	1.0119	1.0030	1.966
2.0000	3.2260	2.1978	2.0473	2.0117	2.219
4.0000	4.9419	4.3794	4.1109	4.0286	1.690
4.0000	7.9118	4.6567	4.1738	4.0448	2.127
5.0000	–	5.7296	5.2055	5.0521	2.158
5.0000	–	6.2847	5.3171	5.0777	2.004
8.0000	–	–	8.5808	8.1463	2.110

Table 4 Eigenvalues obtained with IPPDG compared with expected eigenvalues for the $\mu = 1$ and $\varepsilon = I$ material square domain shown in Fig. 1. Here $\mathsf{a} = \frac{10(\ell+1)^2}{h}$

Expected	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
1.0000	1.1255	1.0036	1.0001	1.0000	1.0000
1.0000	1.1769	1.0043	1.0009	1.0000	1.0000
2.0000	3.2260	2.0153	2.0059	2.0000	2.0000
4.0000	4.9419	4.0468	4.0269	4.0001	4.0000
4.0000	7.9118	4.2261	4.0366	4.0142	4.0002
5.0000	–	5.2737	5.0571	5.0029	5.0002
5.0000	–	6.1114	5.0707	5.0379	5.0004
8.0000	–	–	8.2024	8.0944	8.0016

5.2 Example 2: Square Cavity + Multiple Mortars

To examine the impact of including the projection operator in the stabilization term we created a base mesh for Ω by combining four unrelated meshes for each of four quadrants as shown in Fig. 2.

We performed an h and p uniform refinement study for IPPDG discretisations using three mesh levels obtained from the base mesh using uniform refinement and $\ell = 1, 2, 3$. In Table 5 we show that the first eight eigenvalues are computed correctly, without any spurious eigenvalues, using IPPDG with $\ell = 1$. In Table 6 we further show that for each

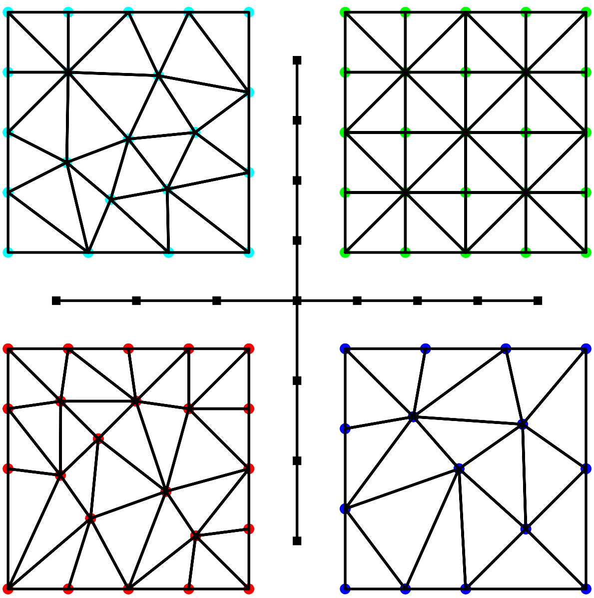

Fig. 2 Exploded view of the base non-conforming mesh blocks and mortar meshes used for Example 2

Table 5 First eight discrete eigenvalues compared with expected eigenvalues for the $\mu = 1$ and $\varepsilon = I$ material square domain shown in Fig. 2. Here $\ell = 1$, and $\mathtt{a} = \frac{10(\ell+1)^2}{h}$

Expected	h	$h/2$	$h/4$	$h/8$	Est. order
2.4674	2.4924	2.4737	2.4690	2.4678	2.000
2.4674	2.4937	2.4740	2.4691	2.4678	1.999
4.9348	5.0368	4.9607	4.9413	4.9364	1.991
9.8696	10.2721	9.9709	9.8949	9.8759	1.997
9.8696	10.3381	9.9881	9.8993	9.8770	1.994
12.3370	12.9674	12.4971	12.3772	12.3471	1.990
12.3370	13.0874	12.5283	12.3850	12.3490	1.989
19.7392	21.5189	20.1950	19.8539	19.7679	1.986

Table 6 Convergence rates for the first eight eigenvalues for Example 2 using $\ell = 1, 2, 3$

Expected	$\ell = 1$	$\ell = 2$	$\ell = 3$
1.0000	2.000	3.990	5.959
1.0000	1.998	4.007	6.010
2.0000	1.986	4.005	5.948
4.0000	1.995	3.967	5.982
4.0000	1.990	3.969	5.957
5.0000	1.985	3.946	5.986
5.0000	1.983	3.983	5.911
8.0000	1.978	3.915	5.963

of the first eight eigenvalues the IPPDG method obtains optimal order of convergence with rates approximately equal to 2ℓ for $\ell = 1, 2, 3$.

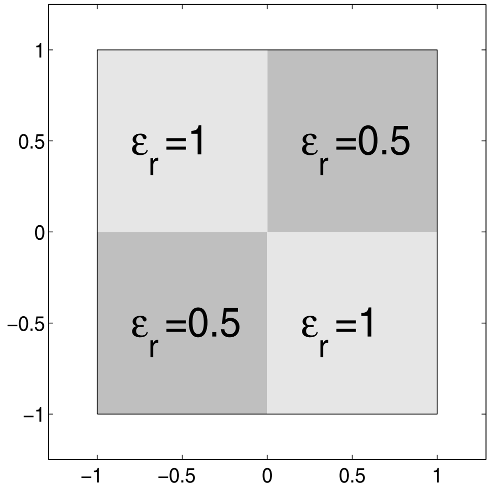

Fig. 3 Checkered material geometry

Table 7 First ten discrete eigenvalues compared with reference results from BENCHMAX for the $\varepsilon_r = 1$ & $\varepsilon_r = 0.5$ checker material square domain shown in Fig. 2. Here $\ell = 1$, and $\mathsf{a} = \frac{10(\ell+1)^2}{h}$

Expected	h	$h/2$	$h/4$	$h/8$	Est. order
3.3175	3.3514	3.3262	3.3197	3.3181	1.986
3.3663	3.3942	3.3729	3.3678	3.3667	2.137
6.1864	6.3133	6.2188	6.1945	6.1884	1.987
13.9263	14.5407	14.0817	13.9653	13.9361	1.993
15.0830	15.7391	15.2512	15.1253	15.0936	1.984
15.7789	16.6867	16.0127	15.8377	15.7936	1.982
18.6433	19.6901	18.9100	18.7102	18.6600	1.989
25.7975	28.2979	26.4491	25.9620	25.8387	1.975
29.8524	32.6070	30.5827	30.0373	29.8988	1.966
30.5379	–	–	30.7069	30.5798	3.602

5.3 Example 3: Checkered Material Cavity

In this example we let $\Omega = (-1, 1)^2$ with $\mu = 1$ and $\varepsilon = \varepsilon_r I$, with ε_r given by

$$\varepsilon_r(x, y) = \begin{cases} 1 & \text{for } (x, y) \in (-1, 0) \times (0, 1) \cup (0, 1) \times (-1, 0) \\ 1/2 & \text{for } (x, y) \in (-1, 0) \times (-1, 0) \cup (0, 1) \times (0, 1) \end{cases} \quad (33)$$

as represented in Fig. 3. The first ten eigenvalues for this geometry are shown in the first column of Table 7. In the second to fifth columns of this table we show the computed eigenvalues for a sequence of four meshes obtained via uniform refinement and the IPPDG discretisation. The final column provides an estimate for the rate of convergence of the discrete eigenvalues to the reference values provided by BENCHMAX [10]. We note that no special adaptivity strategy was used to create the refined meshes, as adaptive algorithms for Maxwell's eigenvalue problem will be the focus of a future paper.

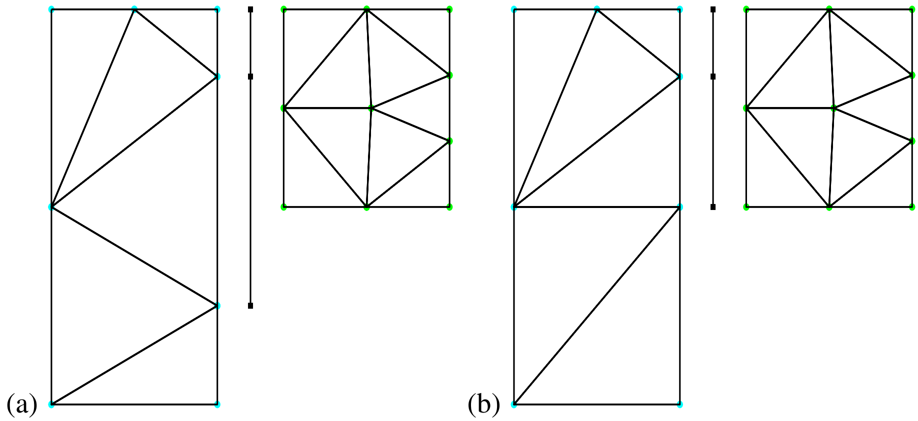

Fig. 4 Exploded view of the base non-conforming mesh blocks and mortar meshes used for L-shape geometry in Example 3. (**a**) Mesh is non-conforming at the reentrant corner. (**b**) Mesh is conforming at the reentrant corner

Table 8 Convergence of the first five discrete eigenvalues to reference results from BENCHMAX for the L-shape domain with base mesh and mortar as shown in Fig. 4a. Here $l = 1$, and $a = \frac{10(l+1)^2}{h}$

Expected	h	$h/2$	$h/4$	$h/8$	Est. order
1.4756	1.6067	1.4990	1.4801	1.4798	1.727
3.5340	3.8678	3.6258	3.5576	3.5398	1.949
9.8696	11.8062	10.4752	10.0351	9.9120	1.841
9.8696	–	10.6906	10.0882	9.9250	2.434
11.3895	–	12.2837	11.6206	11.4477	2.479

5.4 Example 4: L-shape Domain

In this example we let:

$$\Omega = (-1, 1)^2 \setminus \{[0, 1) \times [-1, 0)\} \tag{34}$$

with $\mu = 1$ and $\varepsilon = I$. The first five eigenvalues are 1.47562182408, 3.53403136678, π^2, π^2 and 11.3894793979 (see the BENCHMAX test suite at [10]). We created two different base meshes for a numerical refinement study of this problem, see Fig. 4. The first base mesh (Fig. 4a) consists of conforming left and right mesh blocks, with a non-conforming interface and mortar grid at $x = 0$. In this case there is an element in the left block, and corresponding mortar element that straddles the reentrant corner. In the second base mesh (Fig. 4b) no element straddles the reentrant corner.

For a sequence of meshes obtained by uniform refinement from the base mesh in Fig. 4a we see optimal order convergence rates, using IPPDG, for the first five eigenvalues as shown in Table 8. The theory discussed earlier in this paper does not treat this case, yet it still gives comparable convergence rates to those shown for the theoretically tractable second case in Table 9.

Table 9 Convergence of the first five discrete eigenvalues to reference results from BENCHMAX for the L-shape domain with base mesh and mortar as shown in Fig. 4b. Here $l = 1$, and $a = \frac{10(l+1)^2}{h}$

Expected	h	$h/2$	$h/4$	$h/8$	Est. order
1.4756	1.5033	1.4782	1.4738	1.4741	1.316
3.5340	3.7965	3.6095	3.5538	3.5390	1.906
9.8696	12.0068	10.5669	10.0588	9.9178	1.829
9.8696	–	10.6416	10.0768	9.9225	2.453
11.3895	–	12.0822	11.5825	11.4393	2.536

6 Conclusions

We have introduced and analyzed a mortar-DG method which provide a spurious-free approximation of the 2D Maxwell eigenproblem on general meshes with hanging nodes. Numerical tests have confirmed that the mortar-type correction at the non-conforming interfaces actually remove the spurious eigenmodes generated by standard DG methods. Appropriate definitions of the non-conforming interface meshes and projections in order to extend our mortar-DG method to the 3D case are still under study.

References

1. Amrouche, C., Bernardi, C., Dauge, M., Girault, V.: Vector potentials in three-dimensional non-smooth domains. Math. Methods Appl. Sci. **21**, 823–864 (1998)
2. Ben Abdallah, A., Ben Belgacem, F., Maday, Y., Rapetti, F.: Mortaring the two-dimensional edge finite elements for the discretization of some electromagnetic models. Math. Models Methods Appl. Sci. **14**(11), 1635–1656 (2004)
3. Ben Belgacem, F., Buffa, A., Maday, Y.: The mortar finite element method for 3D Maxwell equations: first results. SIAM J. Numer. Anal. **39**(3), 880–901 (2001). (Electronic)
4. Bernardi, C., Maday, Y., Patera, A.T.: A new nonconforming approach to domain decomposition: The mortar elements method. In: Brezis, H., Lions, J.L. (eds.) Nonlinear Partial Differential Equations and Their Applications, pp. 13–51. Pitman, London (1994)
5. Buffa, A., Christiansen, S.H.: A dual finite element complex on the barycentric refinement. Math. Comput. **76**(260), 1743–1769 (2007). (Electronic)
6. Buffa, A., Houston, P., Perugia, I.: Discontinuous Galerkin computation of the maxwell eigenvalues on simplicial meshes. J. Comput. Appl. Math. **204**(2), 317–333 (2007)
7. Buffa, A., Perugia, I.: Discontinuous Galerkin approximation of the Maxwell eigenproblem. SIAM J. Numer. Anal. **44**, 2198–2226 (2006)
8. Caorsi, S., Fernandes, P., Raffetto, M.: On the convergence of Galerkin finite element approximations of electromagnetic eigenproblems. SIAM J. Numer. Anal. **38**, 580–607 (2000)
9. Costabel, M., Dauge, M., Nicaise, S.: Singularities of Maxwell interface problems. Modél. Math. Anal. Numér. **33**, 627–649 (1999)
10. Dauge, M.: Benchmark computations for Maxwell equations for the approximation of highly singular solutions. http://perso.univ-rennes1.fr/monique.dauge/benchmax.html
11. Descloux, J., Nassif, N., Rappaz, J.: On spectral approximation Part 1. The problem of convergence. RAIRO. Modél. Math. Anal. Numér. **12**, 97–112 (1978)
12. Dunford, N., Schwartz, J.T.: Spectral Theory: Self Adjoint Operators in Hilbert Space. Interscience, New York (1963)
13. Heinrich, B., Pietsch, K.: Nitsche type mortaring of some elliptic problem with corner singularities. Computing **68**, 217–238 (2002)
14. Hesthaven, J.S., Warburton, T.: Nodal Discontinuous Galerkin Methods: Algorithms, Analysis, and Applications. Texts in Applied Math., vol. 54. Springer, New York (2008)
15. Houston, P., Perugia, I., Schneebeli, A., Schötzau, D.: Interior penalty method for the indefinite time-harmonic Maxwell equations. Numer. Math. **100**, 485–518 (2005)
16. Kato, T.: Perturbation Theory of Linear Operators. Springer, Berlin (1966)
17. Monk, P.: Finite Element Methods for Maxwell's Equations. Oxford Univ. Press, London (2003)
18. Warburton, T., Embree, M.: The role of the penalty in the local discontinuous Galerkin method for maxwell's eigenvalue problem. Comput. Methods Appl. Mech. Eng. **195**, 3205–3223 (2006)