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Vanishing Moment Method and Moment Solutions for Fully Nonlinear Second Order Partial Differential Equations

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Abstract This paper concerns with numerical approximations of solutions of fully nonlinear second order partial differential equations (PDEs). A new notion of weak solutions, called moment solutions, is introduced for fully nonlinear second order PDEs. Unlike viscosity solutions, moment solutions are defined by a constructive method, called the vanishing moment method, and hence, they can be readily computed by existing numerical methods such as finite difference, finite element, spectral Galerkin, and discontinuous Galerkin methods. The main idea of the proposed vanishing moment method is to approximate a fully nonlinear second order PDE by a higher order, in particular, a quasilinear fourth order PDE. We show by various numerical experiments the viability of the proposed vanishing moment method. All our numerical experiments show the convergence of the vanishing moment method, and they also show that moment solutions coincide with viscosity solutions whenever the latter exist.

Keywords Fully nonlinear PDEs · Monge-Ampère type equations · Moment solutions · Vanishing moment method · Viscosity solutions · Finite element method · Mixed finite element method · Spectral and discontinuous Galerkin methods

1 Introduction

Fully nonlinear PDEs are those PDEs which depend nonlinearly on the highest order derivatives of unknown functions. Fully nonlinear PDEs arise from many areas in science and engineering such as kinetic theory, materials science, differential geometry, general relativity, optimal control, mass transportation, image processing and computer vision, meteorology,

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and semigeostrophic fluid dynamics. They constitute the most difficult class of differential equations to analyze analytically and to approximate numerically (see [13, 15, 44, 45, 49, 58] and references therein).

The general fully nonlinear first order PDE has the form

$$F(\nabla u(x), u(x), x) = 0, \quad x \in \Omega \subset \mathbf{R}^n. \tag{1.1}$$

The best known examples include the Eikonal equation

$$|\nabla u(x)| = f(x), \quad x \in \Omega,$$

and the Hamilton-Jacobi equation [25, 45]

$$H(\nabla u(x)) = 0, \quad x \in \Omega.$$

The general fully nonlinear second order PDE, which will be the focus of this paper, takes the form

$$F(D^2u(x), \nabla u(x), u(x), x) = 0, \quad x \in \Omega, \tag{1.2}$$

where throughout this paper $D^2u(x)$ denotes the Hessian matrix of u at x. The best known examples are the Monge-Ampère type equations [45, 50, 56]

$$\det(D^2u(x)) = f(\nabla u(x), u(x), x), \quad x \in \Omega,$$

and the Bellman equations [44, 45]

$$\sup_{\theta\in\Theta} L_\theta(D^2u(x), \nabla u(x), u(x), x) = 0, \tag{1.3}$$

where $\det(D^2u(x))$ stands for the determinant of $D^2u(x)$, and L_θ is a given family of linear second order differential operators.

For the fully nonlinear first order PDEs, tremendous progress has been made in the past three decades. A revolutionary viscosity solution theory has been established (cf. [24–26, 44]) and wealthy amount of efficient and robust numerical methods and algorithms have been developed and implemented (cf. [10, 20, 27, 57, 66, 70, 71]). However, for fully nonlinear second order PDEs, the situation is strikingly different. On one hand, there have been enormous advances on PDE analysis in the past two decades after the introduction of the notion of viscosity solutions by M. Crandall and P.L. Lions in 1983 (cf. [11–13, 25, 45, 50]). On the other hand, in contrast to the success of the PDE analysis, numerical solutions for general fully nonlinear second order PDEs (except in the case of Bellman type PDEs, see below for details) is mostly an untouched area, and computing viscosity solutions of fully nonlinear second order PDEs has been impracticable. There are several reasons for this lack of progress. Firstly, the strong nonlinearity is an obvious one. Secondly, the conditional uniqueness (i.e., uniqueness holds only in certain class of functions) of solutions is difficult to handle numerically. Lastly and most importantly, the notion of viscosity solutions, which is not variational, has no equivalence at the discrete level.

To see the above points, let us consider the following model Dirichlet problem for the Monge-Ampère equation:

$$\det(D^2u) = f \quad \text{in } \Omega, \tag{1.4}$$

$$u = g \quad \text{on } \partial\Omega. \tag{1.5}$$

It is well-known that for non-strictly convex domain Ω, the above problem does not have classical solutions in general even if f, g and $\partial\Omega$ are smooth (see [45]). Classical result of A.D. Aleksandrov states that the Dirichlet problem with $f > 0$ has a unique generalized solution in the class of convex functions (cf. [1, 17]). Major progress on analysis of problem (1.4)–(1.5) has been made later by using the *viscosity solution* concept and machinery (cf. [13, 25, 50]). We recall that a *convex* function $u \in C^0(\overline{\Omega})$ satisfying $u = g$ on $\partial\Omega$ is called a *viscosity subsolution (resp. viscosity supersolution)* of (1.4) if for any $\varphi \in C^2$ there holds $\det(D^2\varphi(x_0)) \geq f(x_0)$ (resp. $\det(D^2\varphi(x_0)) \leq f(x_0)$) provided that $u - \varphi$ has a local maximum (resp. a local minimum) at $x_0 \in \Omega$. $u \in C^0(\overline{\Omega})$ is called a *viscosity solution* if it is both a *viscosity subsolution* and a *viscosity supersolution*.

First, the reason to restrict the admissible set to be the set of convex functions is that the Monge-Ampère equation is *elliptic* only in that set [50]. It should be noted that in general, the Dirichlet problem (1.4)–(1.5) may have other (nonconvex) solutions besides the unique convex solution. Multiple solutions are often expected for the Monge-Ampère type PDEs and for other fully nonlinear second order PDEs. It is easy to see that if one discretizes (1.4) straightforwardly using the finite difference method, one immediately loses control on which solution the numerical scheme approximates even assuming that the nonlinear discrete problem has solutions. *Second*, the situation is even worse if one tries to formulate a Galerkin type method (such as the finite element method and the spectral Galerkin method), because there is no variational or weak formulation to start with. In fact, this is clear from the definition of *viscosity solutions*. It is not defined by the traditional integration by parts approach. Instead, it is defined by a "*differentiation by parts*" (a terminology introduced in [24, 26]) approach. Although the "*differentiation by parts*" approach has worked remarkably well for establishing the viscosity solution theory for fully nonlinear second order PDEs in the past two decades, it is extremely difficult (if all possible) to mimic it at the discrete level. *Third*, regardless which method is used, one can easily envisage that the anticipated algebraic problem from the discretization of a fully nonlinear PDE such as the Monge-Ampère equation must be very difficult to solve due to the nonuniqueness of solutions and very strong nonlinearity.

Nevertheless, a few recent numerical attempts and results have been known in the literature. In [65] Oliker and Prussner proposed a finite difference scheme for computing Aleksandrov measure induced by D^2u (and obtained the solution u of (1.4) as a by-product) in 2-d. The scheme is extremely geometric and difficult to use and to generalize to other fully nonlinear second order PDEs. In [7] Barles and Souganidis showed that any monotone, stable and consistent finite difference scheme converges to the viscosity solution provided that there exists a comparison principle for the limiting equation. Their result provides a guideline for constructing convergent finite difference methods although it did not address how to construct such a scheme. Very recently, Oberman [64] was able to construct some wide stencil finite difference schemes which fulfill the criterion listed in [7] for the Monge-Ampère type equations. In [5] Baginski and Whitaker proposed a finite difference scheme for the Gauss curvature equation (see Sect. 4) in 2-d by mimicking the unique continuation method (used to prove existence of the PDE) at the discrete level. Finally, in a series of papers [28–31] Dean and Glowinski proposed an augmented Lagrange multiplier method and a least squares method for problem (1.4)–(1.5) and the Pucci's equation (cf. [13, 45]) in 2-d by treating the Monge-Ampère equation and Pucci's equation as a constraint and using a variational criterion to select a particular solution. Numerical experiments were reported in [5, 28–31, 64, 65], however, convergence analysis was not addressed except in [64].

In addition, we like to remark that there is a considerable amount of literature available on using finite difference methods to approximate viscosity solutions of fully nonlinear second order Bellman type PDE (1.3) arising from stochastic optimal control. See [6, 7, 53, 55]. Due to the special nonlinearity of the Bellman type PDEs, the approach used and the methods proposed in those papers unfortunately could not be extended to other types of second order fully nonlinear PDEs since the construction of those methods critically relies on the linearity of the operators L_θ.

The first goal of this paper is to introduce a new weak solution concept and a method to construct such a solution for second order fully nonlinear PDEs, in particular, for the Monge-Ampère type equations. These new weak solutions are called *moment solutions* and the method to construct such a moment solution is called *the vanishing moment method*. The crux of this new method is that we approximate a second order fully nonlinear PDE by a family of *quasilinear* higher order (in particular, fourth order) PDEs. The limit of the solutions of the higher order PDEs, if exists, is defined as a moment solution of the original fully nonlinear second order PDE. Hence, moment solutions are constructive by nature. The second goal of this paper is to present a number of numerical methods for computing moment solutions of fully nonlinear second order PDEs, and to present extensive numerical experiment results to demonstrate the convergence and effectiveness of the proposed vanishing moment methodology. Indeed, one of advantages of the vanishing moment method is that it allows one to use wealthy amount of existing numerical methods and algorithms as well as computer codes for linear and quasilinear fourth order PDEs to solve fully nonlinear second order PDEs. The third and last goal of this paper is to show using numerical studies that the notion of moment solutions generalizes the notion of viscosity solutions in the sense that the former coincides with the later whenever the later exists. These numerical studies indeed motivate us to analyze convergence of the vanishing moment method for the Monge-Ampère equation in the case of two spatial dimensions [35].

The remainder of the paper is organized as follows. In Sect. 2, we introduce the abstract framework of moment solutions and the vanishing moment method for general fully nonlinear second order PDEs. In Sect. 3, we propose two classes of numerical discretization methods and briefly discuss solution algorithms. In Sect. 4, we apply the abstract framework to several classes of fully nonlinear second order PDEs which include the Monge-Ampère type equations, Pucci's extremal equations, the infinite Laplace equation, and parabolic fully nonlinear second order PDEs. In Sect. 5, we present many 2-d and 3-d numerical experiment results to demonstrate the convergence and effectiveness of the vanishing moment methodology, and provide numerical evidences of the agreement of moment solutions and viscosity solutions whenever the latter exists. The paper is concluded by a summary and some remarks in Sect. 6.

2 Vanishing Moment Method and the Notion of Moment Solutions

2.1 Preliminaries

Standard space notation will be adopted throughout this paper, we refer to [45, 56] for their exact definitions. Ω denotes a generic bounded domain in $\mathbf{R}^n$. $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are used to denote the L^2-inner products on Ω and on $\partial\Omega$, respectively. We assume $n \geq 2$, except in Sects. 3 and 5, where we restrict $n = 2, 3$ when we develop numerical methods and perform numerical experiments.

Since the notion of viscosity solutions has been and will continue to be referred many times (also, it is closely related to the notion of moment solutions to be described later in this paper), for readers' convenience, we briefly recall its definition and history and refer to [13, 25, 26, 33] for detailed discussions.

Definition 2.1 Suppose $F : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is continuous (nonlinear) function.

(i) A function $u \in C^0(\Omega)$ is called a *viscosity subsolution* of (1.1) if, for every C^1 function $\varphi = \varphi(x)$ such that $u - \varphi$ has a local maximum at $x^0 \in \Omega$, there holds

$$F(\nabla\varphi(x^0), \varphi(x^0), x^0) \le 0.$$

(ii) A function $u \in C^0(\Omega)$ is called a *viscosity supersolution* of (1.1) if, for every C^1 function $\varphi = \varphi(x)$ such that $u - \varphi$ has a local minimum at $x^0 \in \Omega$, there holds

$$F(\nabla\varphi(x^0), \varphi(x^0), x^0) \ge 0.$$

(iii) A function $u \in C^0(\Omega)$ is called a *viscosity solution* of (1.1) if it is both a viscosity subsolution and a viscosity supersolution.

It should be pointed out that the above definition is a modern definition of viscosity solutions for (1.1). It can be regarded as a "differentiation by parts" definition (cf. [26]). However, viscosity solutions were first introduced differently by a vanishing viscosity procedure (cf. [26]), that is, (1.1) is approximated by the *quasilinear* second order PDEs

$$-\varepsilon\Delta u^\varepsilon + F(\nabla u^\varepsilon, u^\varepsilon, x) = 0,$$

and the limit $\lim_{\varepsilon\to 0^+} u^\varepsilon$, if it exists, is called a viscosity solution of (1.1). It was later proved that the two definitions are equivalent for equation (1.1) (cf. [24]).

Another important reason to favor the modern "differentiation by parts" definition is that the definition and the notion of viscosity solutions can be readily extended to fully nonlinear second order PDEs.

Definition 2.2 Suppose $F : \mathbf{R}^{n\times n} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is continuous (nonlinear) function.

(i) A function $u \in C^0(\Omega)$ is called a *viscosity subsolution* of (1.2) if, for every C^2 function $\varphi = \varphi(x)$ such that $u - \varphi$ has a local maximum at $x^0 \in \Omega$, there holds

$$F(D^2\varphi(x^0), \nabla\varphi(x^0), \varphi(x^0), x^0) \ge 0.$$

(ii) A function $u \in C^0(\Omega)$ is called a *viscosity supersolution* of (1.2) if, for every C^2 function $\varphi = \varphi(x)$ such that $u - \varphi$ has a local minimum at $x^0 \in \Omega$, there holds

$$F(D^2\varphi(x^0), \nabla\varphi(x^0), \varphi(x^0), x^0) \le 0.$$

(iii) A function $u \in C^0(\Omega)$ is called a *viscosity solution* of (1.2) if it is both a viscosity subsolution and a viscosity supersolution.

As it is now known, a successful theory of viscosity solutions has been established for fully nonlinear second order PDEs in the past two decades (cf. [13, 25, 50]). On the other hand, it should be noted that the phrase "viscosity solution" loses its original meaning in this theory since it has nothing to do with the vanishing viscosity method in the case of fully nonlinear second order PDEs. We recall that to establish the existence of viscosity solutions, the technique used to substitute for the vanishing viscosity method in the theory is the classical Perron's method (cf. [13, 25]). To the best of our knowledge, viscosity solutions of fully nonlinear second order PDEs were never defined by a limiting process like one described above for the Hamilton-Jacobi equation.

2.2 General Framework of the Vanishing Moment Method

For the reasons and difficulties explained in Sect. 1, as far as we can see, it is unlikely (at least very difficult if all possible) that one can *directly* approximate viscosity solutions of general fully nonlinear second order PDEs such as Monge-Ampère type equations using any available numerical methodology (finite difference method, finite element method, spectral method, meshless method etc.). From a computational point of view, the notion of viscosity solutions is a "bad" notion for fully nonlinear second order PDEs because it is not constructive nor variational, so one has no handle on how to compute such a solution.

In searching for a "better" notion of weak solutions for fully nonlinear second order PDEs, we are inspired by the following simple but crucial observation: *the essence of the vanishing viscosity method for the Hamilton-Jacobi equation and the original notion of viscosity solutions is to approximate a lower order fully nonlinear PDE by a family of quasilinear higher order PDEs*. This observation motivates us to apply the above principle to fully nonlinear second order PDE (1.2), which is exactly what we are going to do in this paper. That is, we approximate equation (1.2) by the following quasilinear higher order PDEs:

$$G_\varepsilon(D^r u^\varepsilon) + F(D^2 u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x) = 0 \quad (r \geq 3, \varepsilon > 0), \tag{2.1}$$

where $\{G_\varepsilon\}$ is a family of suitably chosen linear or quasilinear differential operators of order r. The above approximation then naturally leads to the next definition.

Definition 2.3 Suppose that u^ε solves (2.1) for each $\varepsilon > 0$, we call $\lim_{\varepsilon\to 0^+} u^\varepsilon$ a *moment solution* of (1.2) provided that the limit exists. We also call this limiting process *the vanishing moment method*.

Clearly, the above definition is a loose definition since the operator G_ε is not specified, nor is the meaning of the limit, but they will become clear later in this section. We note that the reason to use the terminology "moment solution" will also be explained later in this section, and the notion of moment solutions and the vanishing moment method are clearly in the spirit of the (original) notion of viscosity solutions and the vanishing viscosity method [26].

To establish a complete theory of moment solutions and vanishing moment method for fully nonlinear second order PDEs, there are many issues we must address. For instance,

- How to choose the operator G_ε?
- What additional boundary condition(s) should u^ε satisfy?
- Does the limit $\lim_{\varepsilon\to 0^+} u^\varepsilon$ always exist? If it does, what is the rate of convergence?
- How do moment solutions relate to viscosity solutions?
- How to solve (2.1) numerically?
- Error estimates, nonlinear solvers, computer implementations.

As expected, we do not have answers for all the questions now, nor do we intend to address all of them in this paper. Instead, the focus of this paper is to develop the framework for moment solutions and the vanishing viscosity method, and to present numerical evidences to show effectiveness of the method and to justify the proposed approach. On the other hand, the above issues will be addressed elsewhere [35, 62].

Although the choices for G_ε are abundant and flexible, the following are some guidelines for choosing a good operator G_ε.

(a) G_ε must be a linear or quasilinear operator.
(b) $G_\varepsilon \to 0$ in some reasonable sense as $\varepsilon \to 0^+$.
(c) $G_\varepsilon(D^r u)$ is better to be elliptic, in particular, when PDE (1.2) is elliptic.
(d) Equation (2.1) should be relatively easy to solve numerically.

Since an elliptic operator is necessarily of even order, guideline (c) above implies that r must be an even number in (2.1). Hence, the lowest order of (2.1) is $r = 4$. When talking about fourth order elliptic operators, the biharmonic operator stands out immediately. So we let

$$G_\varepsilon(D^4 v) := -\varepsilon \Delta^2 v,$$

and (2.1) becomes

$$-\varepsilon \Delta^2 u^\varepsilon + F(D^2 u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x) = 0. \tag{2.2}$$

After the differential operator G_ε is chosen, next we need to take care of the boundary conditions. Here, we only consider the Dirichlet problem for (1.2). Suppose that

$$u = g \quad \text{on } \partial\Omega. \tag{2.3}$$

It is obvious that we need to impose

$$u^\varepsilon = g \quad \text{or} \quad u^\varepsilon \approx g \quad \text{on } \partial\Omega. \tag{2.4}$$

Moreover, since (2.2) is a fourth order PDE, in order to uniquely determine u^ε we need to impose an additional boundary condition for u^ε. Mathematically, many boundary conditions can be used for this purpose. Physically, any additional boundary condition will introduce a "boundary layer", so a better choice would be one which minimizes the boundary layer. Here, we propose to use one of following three boundary conditions.

$$\Delta u^\varepsilon = c_\varepsilon \ (> 0) \quad \text{or} \quad \frac{\partial \Delta u^\varepsilon}{\partial \mathbf{n}} = c_\varepsilon \quad \text{or} \quad D^2 u^\varepsilon \mathbf{n} \cdot \mathbf{n} = c_\varepsilon \quad \text{on } \partial\Omega. \tag{2.5}$$

In particular, the first two boundary conditions, which are natural boundary conditions, have an advantage in PDE convergence analysis [35, 62]. Another valid boundary condition is the Neumann boundary condition $\frac{\partial u^\varepsilon}{\partial \mathbf{n}} = c_\varepsilon$ on $\partial\Omega$. But since this is an essential boundary condition, it produces a larger boundary layer than the above three boundary conditions. The rationale for picking the above boundary conditions is that we implicitly impose an extra boundary condition $\varepsilon^m \Delta u^\varepsilon + u^\varepsilon = g + \varepsilon^m c_\varepsilon$ on $\partial\Omega$ with $c_\varepsilon > 0$, which is a higher order perturbation of the original Dirichlet boundary condition $u^\varepsilon = g$ on $\partial\Omega$. Intuitively, we expect and hope that the extra boundary condition converges to the original Dirichlet boundary condition as ε tends to zero for sufficiently large positive integer m.

We now remark that when $n = 2$ in mechanical applications, u^ε often stands for the vertical displacement of a plate and $D^2 u^\varepsilon$ is the *moment tensor*. In the weak formulation, the biharmonic term becomes $-\varepsilon(D^2 u^\varepsilon, D^2 v)$ which should vanish as $\varepsilon \to 0^+$. This is the very reason why we call $\lim_{\varepsilon\to 0^+} u^\varepsilon$, if it exists, a moment solution and call the limiting process the vanishing moment method.

In summary, we propose to approximate the fully nonlinear second order Dirichlet problem (1.2), (2.3) by the quasilinear fourth order boundary value problem (2.2), (2.4), (2.5). Since we expect $u^\varepsilon \in W^{m,p}(\Omega)$ for $m \geq 2$, $p \geq 2$, the convergence $\lim_{\varepsilon\to 0^+} u^\varepsilon$ in Definition 2.3 can be understood in H^2-topology or in H^1-topology. To distinguish these different limits, we introduce the following refined definition of Definition 2.3.

Definition 2.4 Suppose that $u^\varepsilon \in H^2(\Omega) \cap C^0(\overline{\Omega})$ solves problem (2.2), (2.4), (2.5). $\lim_{\varepsilon\to 0^+} u^\varepsilon$ is called a *weak* (resp. *strong moment solution*) to problem (1.2), (2.3) if the convergence holds in H^1-weak (resp. H^2-weak) topology.

Remark 2.1 Since weak moment solutions do not have second order weak derivatives, they are very hard (if all possible) to identify. On the other hand, since strong moment solutions do have second order weak derivatives, naturally they are expected to satisfy the PDE (1.2) almost everywhere in Ω and to fulfill the boundary condition (2.3) pointwise on $\partial\Omega$. In the remainder of this paper moment solutions will always mean weak moment solutions.

3 Discretization and Solution Methods

The vanishing moment method reduces the problem of solving (1.2), (2.3) to a problem of solving (2.2), $(2.4)_1$, $(2.5)_1$ for each fixed $\varepsilon > 0$. Since (2.2) is a nonlinear biharmonic equation, one can use any of the wealthy amount of existing numerical methods for biharmonic problems to discretize the equation. Although other types of numerical methods are applicable, in this paper we focus on Galerkin type methods such as finite element methods, mixed finite element methods, discontinuous and spectral Galerkin methods [8, 9, 18, 22]. Throughout this section, we assume $n = 2, 3$.

3.1 Finite Element Methods in 2-d

In the two-dimensional case many finite element methods, such as conforming Argyris, Bell, Bogner-Fox-Schmit and Hsieh-Clough-Tocher elements and nonconforming Adini, Morley, and Zienkiewicz elements, were extensively developed in 60's and 70's for the biharmonic problems. A beautiful theory of plate finite element methods was also established (cf. [18]). Naturally, one would want to solve problem (2.2), $(2.4)_1$, $(2.5)_1$ by using and adapting these well-known plate finite element methods. That is exactly what we are going to do next. For the sake of presentation clarity, here we only discuss the conforming finite element methods, and refer to [62] for a detailed development of nonconforming finite element methods for problem (2.2), $(2.4)_1$, $(2.5)_1$.

The variational formulation for (2.2), $(2.4)_1$, $(2.5)_1$ is defined as: Find $u^\varepsilon \in H^2(\Omega)$ with $u^\varepsilon = g$ a.e. on $\partial\Omega$ such that for any $v \in H^2(\Omega) \cap H^1_0(\Omega)$ there holds

$$-\varepsilon\big(\Delta u^\varepsilon, \Delta v\big) + \big(F(D^2 u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x), v\big) = -\Big\langle \varepsilon c_\varepsilon, \frac{\partial v}{\partial \mathbf{n}}\Big\rangle, \tag{3.1}$$

where throughout the paper $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote respectively the L^2-inner products on $L^2(\Omega)$ and on $L^2(\partial\Omega)$.

Let $\mathcal{T}_h$ be a quasiuniform triangular or rectangular mesh with mesh size $h \in (0, 1)$ for the domain $\Omega \subset \mathbf{R}^2$. Let $U^h_g \subset H^2(\Omega)$ denote one of the conforming finite element spaces (as mentioned above) whose functions take the boundary value g at all nodes on $\partial\Omega$. Then our finite element method is defined as: Find $u^\varepsilon_h \in U^h_g$ such that

$$-\varepsilon\big(\Delta u^\varepsilon_h, \Delta v_h\big) + \big(F(D^2 u^\varepsilon_h, \nabla u^\varepsilon_h, u^\varepsilon_h, x), v_h\big) = -\Big\langle \varepsilon c_\varepsilon, \frac{\partial v_h}{\partial \mathbf{n}}\Big\rangle, \quad \forall v_h \in U^h_0. \tag{3.2}$$

In Sect. 5, we shall present several numerical experiment results for the Monge-Ampère type equations to show the excellent performance of the Argyris finite element method. Convergence and error analysis of the above scheme and other finite element schemes will be presented in forthcoming papers (also see [62]).

3.2 Mixed Finite Element Methods in 2-d and 3-d

Along with the theory of plate finite element methods, another beautiful theory of mixed finite element methods was also extensively developed in 70's and 80's for the biharmonic problems in 2-d (cf. [9, 18, 34]). It is interesting to point out that all these 2-d mixed finite element methods can be easily generalized to solving 3-d biharmonic problems and general quasilinear fourth order PDEs (cf. [32, 40, 41]).

Because the Hessian matrix D^2u^ε appears in (2.2) in a nonlinear fashion, to design a mixed method we are "forced" to introduce $\sigma^\varepsilon := D^2u^\varepsilon$ (not $v^\varepsilon := \Delta u^\varepsilon$ alone) as an additional variable, so the mixed method simultaneously seeks u^ε and σ^ε. This observation then excludes the usage of the popular family of Ciarlet-Raviart mixed finite element methods (originally designed for the biharmonic problems) [18, 19]. On the other hand, the observation suggests to try Hermann-Miyoshi mixed elements [34, 52, 59, 60, 67] and Hermann-Johnson mixed elements [34, 52, 54] which both use σ^ε as an additional variable.

To define Hermann-Miyoshi type mixed finite element methods, we first derive the following mixed variational formulation for problem (2.2), $(2.4)_1$, $(2.5)_3$: Find $(u^\varepsilon, \sigma^\varepsilon) \in V_g \times W_\varepsilon$ such that

$$(\sigma^\varepsilon, \mu) + (\nabla u^\varepsilon, \operatorname{div}\mu) = \sum_{i=1}^{n-1} \left\langle \frac{\partial g}{\partial \tau_i}, \mu\mathbf{n}\cdot\tau_i \right\rangle \quad \forall \mu \in W_0, \tag{3.3}$$

$$\varepsilon(\operatorname{div}\sigma^\varepsilon, \nabla v) + \big(F(\sigma^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x), v\big) = (f, v) \quad \forall v \in V_0, \tag{3.4}$$

where $\tau_i, i = 1, 2, \dots, (n-1)$ denote a set of $(n-1)$ linearly independent tangential directions at each point on $\partial\Omega$, $\frac{\partial g}{\partial \tau_i}$ denotes the tangential derivative of g along τ_i, and

$$\begin{aligned} V_g &:= \big\{v \in H^1(\Omega);\ v|_{\partial\Omega} = g\big\}, \qquad V_0 := \big\{v \in H^1(\Omega);\ v|_{\partial\Omega} = 0\big\}, \\ W_\varepsilon &:= \big\{\mu \in [H^1(\Omega)]^{n\times n};\ \mu_{ij} = \mu_{ji}, \mu\mathbf{n}\cdot\mathbf{n}|_{\partial\Omega} = c_\varepsilon\big\}, \\ W_0 &:= \big\{\mu \in [H^1(\Omega)]^{n\times n};\ \mu_{ij} = \mu_{ji}, \mu\mathbf{n}\cdot\mathbf{n}|_{\partial\Omega} = 0\big\}. \end{aligned}$$

Let $\mathcal{T}_h$ be a quasiuniform triangular or rectangular mesh if $n = 2$ and be a quasiuniform tetrahedral or 3-d rectangular mesh if $n = 3$ for the domain Ω. Let $V^h \subset H^1(\Omega)$ be the Lagrange finite element space consisting of continuous piecewise polynomials of degree $k(\geq 2)$ associated with the mesh $\mathcal{T}_h$. Let

$$\begin{aligned} V_g^h &:= V^h \cap V_g, \qquad V_0^h := V^h \cap V_0, \\ W_\varepsilon^h &:= [V^h]^{n\times n} \cap W_\varepsilon, \qquad W_0^h := [V^h]^{n\times n} \cap W_0. \end{aligned}$$

Based on the variational formulation (3.3)–(3.4) we define our (Hermann-Miyoshi type) mixed finite element methods as follows: Find $(u_h^\varepsilon, \sigma_h^\varepsilon) \in V_g^h \times W_\varepsilon^h$ such that

$$(\sigma_h^\varepsilon, \mu_h) + (\nabla u_h^\varepsilon, \operatorname{div}\mu_h) = \sum_{i=1}^{n-1} \left\langle \frac{\partial g}{\partial \tau_i}, \mu_h\mathbf{n}\cdot\tau_i \right\rangle \quad \forall \mu_h \in W_0^h, \tag{3.5}$$

$$\varepsilon(\operatorname{div}\sigma_h^\varepsilon, \nabla v_h) + \big(F(\sigma_h^\varepsilon, \nabla u_h^\varepsilon, u_h^\varepsilon, x), v_h\big) = (f, v_h) \quad \forall v_h \in V_0^h. \tag{3.6}$$

Similarly, we can define variants of the above scheme as those proposed in [67] as well as Hermann-Johnson type mixed methods. In Sect. 5, we shall present several numerical experiment results for the above scheme applying to the Monge-Ampère type equations. Convergence and error analysis of the above scheme and other mixed finite element schemes will be presented in a forthcoming paper (also see [62]).

Remark 3.1 Besides the finite element and mixed finite element discretization methods, one can also approximate problem (2.2), $(2.4)_1$, $(2.5)_3$ by discontinuous Galerkin methods [2, 21, 22, 36, 37, 46, 61] and spectral Galerkin methods [8, 14, 68]. It should be pointed out that these methods are dimension-independent, hence, can be used in both 2-d and 3-d cases. We refer to [62] for a detailed exposition.

3.3 Remarks on Fully Nonlinear Second Order Parabolic Equations

By adopting the method of lines approach, generalizations of the numerical methods discussed in previous subsections to the corresponding parabolic equations (4.11) and (4.12) are standard (cf. [32, 37] and references therein). Assuming that an implicit time stepping method such as the backward Euler and the Crank-Nicolson schemes are used for time discretization, then at each time step we only need to solve a fully nonlinear elliptic equation of the form (2.2). As a result, all numerical methods discussed in Sects. 3.1–3.2 immediately apply. On the other hand, it should be pointed out that the convergence and error analysis of all fully discrete schemes are expected to be harder, in particular, establishing error estimates which depend on ε^{-1} *polynomially* instead of *exponentially* will be very challenging (cf. [38–43]).

3.4 Remarks on Nonlinear Solvers and Preconditioning

After equations (2.2) and (4.11) are discretized by any of the above discretization methods, we get a strong nonlinear algebraic system to solve. To this end, one has to use one or another iterative solution method to do the job. In all numerical experiments to be given in Sect. 5, we use preconditioned Newton iterative methods as our nonlinear solvers. A few fixed point iterations might be needed to generate initial guess for Newton type iterative methods. Another strategy which we are currently investigating is the following "multi-resolution" strategy: first compute a numerical solution using a relatively large ε, then use the computed solution as an initial guess for the Newton method at a finer resolution ε. Regarding to preconditioning, we use the simple ILU preconditioner in all simulations of Sect. 5. We plan to use more sophisticated multigrid and Schwarz (or domain decomposition) preconditioners when the amount of computations becomes intensive and large in 3-d.

4 Applications

In this section, we shall apply the vanishing moment methodology outlined in the previous section to several classes of specific second order fully nonlinear PDEs.

4.1 Monge-Ampère Type Equations

Monge-Ampère type equations refer to a class of fully nonlinear second order PDEs of the form (cf. [12, 13, 17, 45, 50])

$$F(D^2u^0, Du^0, u^0, x) := \det(D^2u^0) - f(\nabla u^0, u^0, x) = 0. \tag{4.1}$$

Note that from now on we shall always use u^0 to denote a solution of a fully nonlinear second order PDE we intend to solve. Equation (4.1) reduces to the classical Monge-Ampère equation

$$\det(D^2u^0) = f(x)$$

if $f(\nabla u^0, u^0, x) = f(x) > 0$, and to the Gauss curvature equation

$$\det(D^2u^0) = K(1 + |\nabla u^0|^2)^{\frac{n+2}{2}}$$

if $f(\nabla u^0, u^0, x) = K(1+|\nabla u^0|^2)^{\frac{n+2}{2}}$, where the constant K is a prescribed Gauss curvature. Monge-Ampère type equations are the best known fully nonlinear second order PDEs. They arise in differential geometry and applications such as mass transportation and meteorology. It is well-known that Monge-Ampère type equations are elliptic only in the set of convex functions (cf. [45, 50]), so their viscosity solutions are defined as convex functions in the sense of Definition 2.2.

The vanishing moment approximation (2.2) to (4.1) reads as:

$$-\varepsilon\Delta^2 u^\varepsilon + \det(D^2u^\varepsilon) = f(\nabla u^\varepsilon, u^\varepsilon, x) \quad (\varepsilon > 0). \tag{4.2}$$

For each fixed $\varepsilon > 0$, this is a quasilinear fourth order PDE with Hessian type nonlinearity. It is complemented by boundary conditions $(2.4)_1$ and $(2.5)_1$ (or $(2.5)_2$).

For the classical Monge-Ampère equation, when $n = 2$ it can be shown that [35, 62]

- For each fixed small $\varepsilon > 0$, problem (4.2), $(2.4)_1$, $(2.5)_1$ (or $(2.5)_2$) has a unique weak solution in $W^{3,2}(\Omega)$.
- $\det(D^2u^\varepsilon) > 0$ and $\Delta u^\varepsilon > 0$ in Ω for sufficiently small $\varepsilon > 0$.
- u^ε is convex in Ω for sufficiently small $\varepsilon > 0$.
- The Dirichlet problem for the classical Monge-Ampère equation has a unique convex moment solution, which coincides with the unique convex viscosity solution of the same problem.

The above results immediately imply that in two dimensions, the vanishing moment method indeed works for the classical Monge-Ampère equation, and the notion of moment solutions and the notion of viscosity solutions are equivalent.

Remark 4.1 Recall that when $n = 2$, the Dirichlet problem (1.4)–(1.5) has *exactly* two solutions (see [23]). An amazing numerical discovery to be given in Sect. 5 is that if we restrict ε in (4.2) to $\varepsilon < 0$, then $\lim_{\varepsilon\to 0^-} u^\varepsilon$ also exists and the limit is nothing but the other solution of problem (1.4)–(1.5) which is *concave* (see Figs. 1–6)!

4.2 Pucci's Equations

Pucci's extremal equations are referred to the following two families of fully nonlinear PDEs (cf. [13, 45])

$$M_\alpha[u] := \alpha\Delta u + (1 - n\alpha)\lambda_n(D^2u^0) = f(x), \tag{4.3}$$

$$m_\alpha[u] := \alpha\Delta u + (1 - n\alpha)\lambda_1(D^2u^0) = f(x) \tag{4.4}$$

for $0 < \alpha \le \frac{1}{n}$. Where $\lambda_n(D^2u^0)$ and $\lambda_1(D^2u^0)$ denote the maximum and minimum eigenvalues of the Hessian matrix D^2u^0. In the 2-d case, the above equations can be rewritten in terms of Δu^0 and $\det(D^2u^0)$ (cf. [31]).

The vanishing moment approximations to (4.3) and (4.4) are defined as

$$-\varepsilon\Delta^2 u^\varepsilon + \alpha\Delta u^\varepsilon + (1-n\alpha)\lambda_n(D^2u^\varepsilon) = f(x), \tag{4.5}$$

$$-\varepsilon\Delta^2 u^\varepsilon + \alpha\Delta u^\varepsilon + (1-n\alpha)\lambda_1(D^2u^\varepsilon) = f(x), \tag{4.6}$$

which should be complemented by boundary conditions $(2.4)_1$ and $(2.5)_1$ (or $(2.5)_2$).

4.3 Infinite Laplace Equation

T he infinite Laplace equation refers to the following *degenerate quasilinear* PDE:

$$F(D^2u^0, \nabla u^0, u^0, x) := \Delta_\infty u^0 = 0, \tag{4.7}$$

where

$$\Delta_\infty u^0 := D^2u^0\nabla u^0 \cdot \nabla u^0.$$

$\Delta_\infty u^0$ can be regarded as the limit of the p-Laplacian $\Delta_p u^0 := \mathrm{div}(|\nabla u^0|^{p-2}\nabla u^0)$ as $p \to \infty$. It also can be derived as the Euler-Lagrange equation of the L^∞ functional

$$I(v) := \text{ess sup}_{x\in\Omega}|\nabla v(x)|,$$

whose minimizers are often called "absolute minimizers" [3]. Besides its mathematical appeals, the infinite Laplace equation also arises from image processing, geography, and geology applications [4, 15]. Although the infinite Laplace equation is only a degenerate quasilinear PDE (not a fully nonlinear PDE), it is very difficult to solve numerically. This is because the infinite Laplace equation does not have classical solutions in general [3], and since it is not in divergence form, its weak solutions are defined and understood in the viscosity sense. We refer to [63] for recent developments on finite difference approximations of the infinite Laplace equation.

Here, we propose the following vanishing moment approximation for (4.7):

$$-\varepsilon\Delta^2 u^\varepsilon + \Delta_\infty u^\varepsilon = 0, \tag{4.8}$$

which is complemented by boundary conditions $(2.4)_1$ and $(2.5)_1$ (or $(2.5)_2$). In Sect. 5, we shall present numerical results which show that the vanishing moment approximation converges to the unique viscosity solution of the Dirichlet problem for (4.7). This is another example which shows that the notion of moment solutions and the notion of viscosity solutions coincide.

Remark 4.2 It is easy to see that the above vanishing moment method also applies to the p-Laplacian equation $-\Delta_p u^0 = f$ for $1 \le p < \infty$.

4.4 Second Order Fully Nonlinear Parabolic PDEs

We first like to note that there are several different versions of legitimate parabolic generalizations to elliptic PDE (1.2) (cf. [56, 69]). In this paper, we shall only consider the following widely studied (and it turns out to be the "easiest") class of fully nonlinear second order parabolic PDEs:

$$F(D^2u^0, \nabla u^0, u^0, x, t) - u^0_t = 0, \tag{4.9}$$

assuming that $F(D^2u^0, \nabla u^0, u^0, x, t)$ is elliptic. Clearly, this is the most natural parabolic generalization to (1.2). For example, the corresponding parabolic Monge-Amperé type equation reads as

$$\det(D^2u^0) - u^0_t = f(\nabla u^0, u^0, x, t) \geq 0. \tag{4.10}$$

In the past two decades the viscosity solution theory has been well developed for equations (4.9) and (4.10), see [51, 56, 69]. On the other hand, numerical approximation to these fully nonlinear parabolic PDEs is a completely untouched area. To the best our of knowledge, no numerical result (in fact, no attempt) is known in the literature.

Similarly, we can define the vanishing moment method and the notion of moment solutions for initial and initial-boundary value problems for (4.9), and then ask the same questions as we did in Sect. 2.2. We leave this as an exercise to interested readers and refer to [62] for a detailed exposition.

Following the derivation of Sect. 2.2, we propose the following vanishing moment approximations to (4.9) and (4.10), respectively,

$$F(D^2u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x, t) - \varepsilon\Delta^2 u^\varepsilon - u^\varepsilon_t = 0, \tag{4.11}$$

$$\det(D^2u^\varepsilon) - \varepsilon\Delta^2 u^\varepsilon - u^\varepsilon_t = f(\nabla u^\varepsilon, u^\varepsilon, x, t), \tag{4.12}$$

each of the above equations is a quasilinear fourth order parabolic PDEs.

5 Numerical Experiments

In this section, we shall present a number of numerical experiment results obtained by using the vanishing moment method together with the numerical methods proposed in Sect. 3. Both 2-d and 3-d tests will be presented. All the 3-d tests are obtained by a Hermann-Miyoshi type mixed finite element method, while the 2-d tests are computed by using both the Argyris (plate) finite element method and the Hermann-Miyoshi mixed finite element method. In all our numerical tests, we use $c_\varepsilon = \varepsilon$.

5.1 Two-Dimensional Numerical Experiments

The numerical solutions of the first seven tests are computed using the Argyris finite element method.

Test 1: In this test we solve the Monge-Ampère problem (1.4)–(1.5) on the unit square $\Omega = (0, 1)^2$ with the following data:

$$f(x, y) \equiv 1, \qquad g(x, y) \equiv 0.$$

We remark that problem (1.4)–(1.5) has a unique convex viscosity solution but does not have a classical solution (cf. [31, 50]).

Recall that the vanishing moment approximation of (1.4)–(1.5) is problem (4.2), $(2.4)_1$, $(2.5)_1$ with the above f and g. We discretize problem (4.2), $(2.4)_1$, $(2.5)_1$ using the Argyris plate element as described in Sect. 3.1. Figure 1 displays the computed (moment) solutions using $\varepsilon = 10^{-3}$ (left graph) and $\varepsilon = -10^{-3}$ (right graph). Clearly, the vanishing moment approximations correctly capture the convex viscosity solution (left graph) and the concave viscosity solution (right graph) (see [35] for a rigorous proof). Hence, the moment solutions coincide with the viscosity solutions.

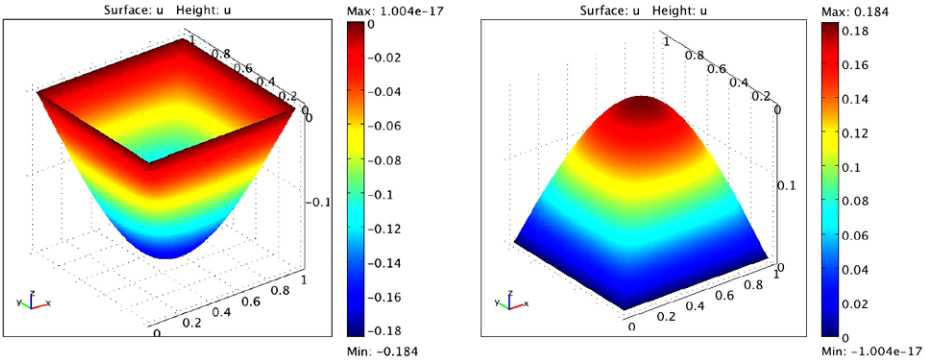

Fig. 1 Computed (moment) solutions of Test 1: Graph on *left* corresponds to $\varepsilon = 10^{-3}$ and graph on *right* corresponds to $\varepsilon = -10^{-3}$. $h = 0.013$ in both cases

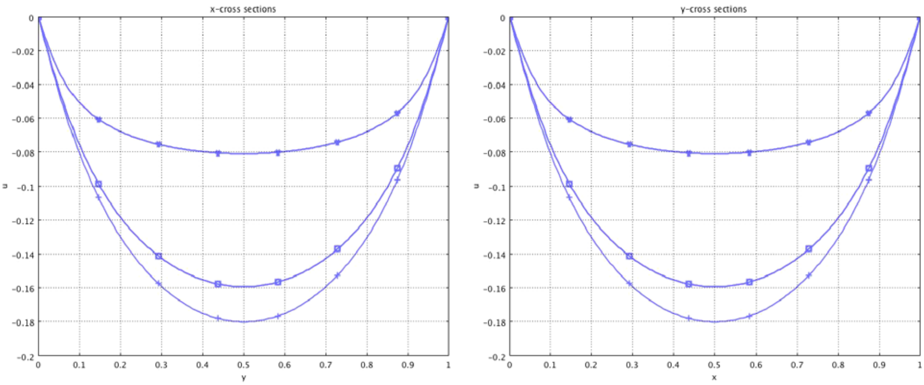

Fig. 2 x-cross sections (*left figure*) of the left graph in Fig. 1 at $x = 0.1, 0.3, 0.5, 0.7, 0.9$ (indicated respectively by *asterisk*, *circle*, *plus sign*, *square*, and *triangle*); y-cross sections (*right figure*) of the left graph in Fig. 1 at $y = 0.1, 0.3, 0.5, 0.7, 0.9$ (indicated respectively by *asterisk*, *circle*, *plus sign*, *square*, and *triangle*)

To have a better view of the convexity of the computed solution, we also plot selected cross sections of the left figure in Fig. 2. The cross sections clearly show that the computed solution is a convex function. In particular, there is no visible smear at the boundary.

Test 2: The only difference between this test and Test 1 is that the datum functions are now chosen as

$$f(x,y) = (1 + x^2 + y^2)e^{(x^2+y^2)}, \qquad g(x,y) = \begin{cases} e^{y^2/2} & \text{if } x = 0, \\ e^{x^2/2} & \text{if } y = 0, \\ e^{(1+x^2)/2} & \text{if } y = 1, \\ e^{(1+y^2)/2} & \text{if } x = 1, \end{cases}$$

so that $u^0(x,y) = e^{(x^2+y^2)/2}$ is the exact solution of problem (1.4)–(1.5). Clearly, u^0 is a convex function, hence u^0 must be the unique convex viscosity solution of problem (1.4)–(1.5) (cf. [50]).

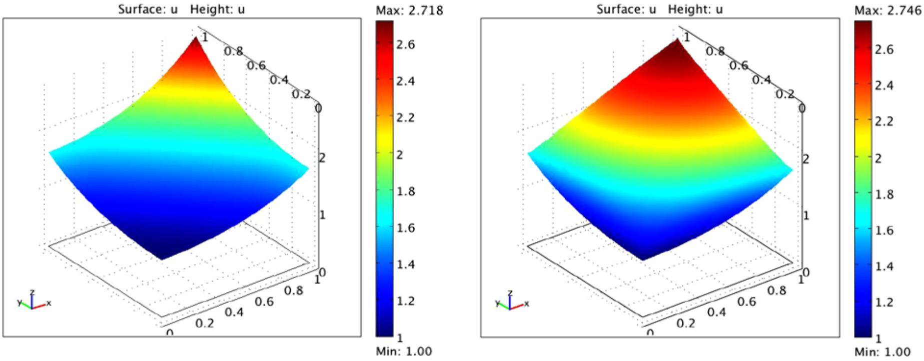

Fig. 3 Computed (moment) solutions of Test 2: Graph on *left* corresponds to $\varepsilon = 10^{-3}$ and graph on *right* corresponds to $\varepsilon = -10^{-3}$. $h = 0.016$ in both cases

Figure 3 shows the computed (moment) solutions using $\varepsilon = 10^{-3}$ (left graph) and $\varepsilon = -10^{-3}$ (right graph). Again, the vanishing moment approximations correctly capture the convex viscosity solution u^0 (left graph) and the concave viscosity solution (right graph). Hence, the moment solutions coincide with the viscosity solutions.

Test 3: Similar to Test 2, the only difference between this test and Test 1 is that the datum functions are now chosen as

$$f(x,y) = \frac{1}{x^2+y^2}, \qquad g(x,y) = \begin{cases} \frac{2\sqrt{2}}{3} y^{\frac{3}{2}} & \text{if } x = 0, \\ \frac{2\sqrt{2}}{3} x^{\frac{3}{2}} & \text{if } y = 0, \\ \frac{2\sqrt{2}}{3} (1+x^2)^{\frac{3}{4}} & \text{if } y = 1, \\ \frac{2\sqrt{2}}{3} (1+y^2)^{\frac{3}{4}} & \text{if } x = 1, \end{cases}$$

so that $u^0(x,y) = \frac{2\sqrt{2}}{3}(x^2+y^2)^{\frac{3}{4}}$ is the unique convex viscosity solution of problem (1.4)–(1.5).

Figure 4 displays the computed (moment) solutions using $\varepsilon = 10^{-3}$ (left graph) and $\varepsilon = -10^{-3}$ (right graph). As expected, the vanishing moment approximations correctly capture the convex viscosity solution u^0 (left graph) and the concave viscosity solution (right graph). Hence, the moment solutions coincide with the viscosity solutions.

Test 4: Again, the only difference between this test and Test 1 is that the datum functions are now chosen as

$$f(x,y) = (1-x-y)^2, \qquad g \equiv 0.$$

On the other hand, mathematically there is a significant difference between these two test problems. Note that $f(x,y) = 0$ on the line $x + y = 1$ in the domain $\Omega = (0,1)^2$. Hence, problem (1.4)–(1.5) is known as a *degenerate* Monge-Ampère problem (cf. [50]).

Figure 5 displays the computed (moment) solutions using $\varepsilon = 10^{-3}$ (left graph) and $\varepsilon = -10^{-3}$ (right graph). Once again, the vanishing moment approximations correctly capture the convex viscosity solution (left graph) and the concave viscosity solution (right graph). Hence, the moment solutions coincide with the viscosity solutions. In addition, our numerical result shows that the vanishing moment method is robust with respect to the degeneracy of the underlying PDE.

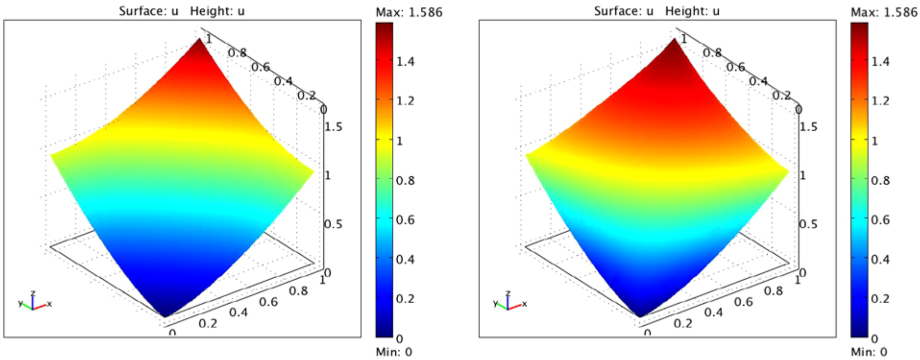

Fig. 4 Computed (moment) solutions of Test 3: Graph on *left* corresponds to $\varepsilon = 10^{-3}$ and graph on *right* corresponds to $\varepsilon = -10^{-3}$. $h = 0.016$ in both cases

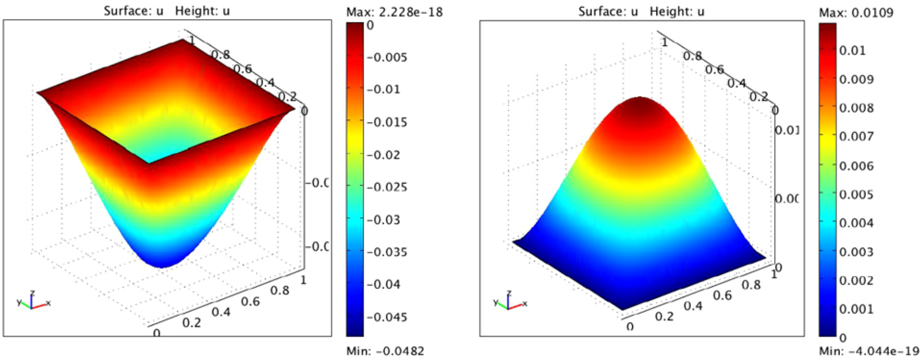

Fig. 5 Computed (moment) solutions of Test 4: Graph on *left* corresponds to $\varepsilon = 10^{-3}$ and graph on *right* corresponds to $\varepsilon = -10^{-3}$. $h = 0.013$ in both cases

Test 5: Once again, the only difference between this test and Test 1 is that the datum functions are now chosen as

$$f(x, y) = x^2 - y^2, \qquad g \equiv 0.$$

Mathematically, the difference between this test problem and Test 1 is even more dramatic because not only $f(x, y) = 0$ on the line $x - y = 0$, *but also f changes sign* (hence the PDE changes type) in Ω. To the best of our knowledge, there is no viscosity solution theory for this type Monge-Ampère problems in the literature. However, the vanishing moment method seems to work well for this problem. Our numerical results indicate existence of both convex and concave moment solutions.

Figure 6 displays the computed convex (moment) solution using $\varepsilon = 10^{-3}$ (left graph) and the computed concave (moment) solution using $\varepsilon = -10^{-3}$ (right graph).

Again, to have a better view of the convexity of the computed solution, we also plot selected cross sections of the left figure in Fig. 7. The cross sections clearly show that the computed solution is a convex function. In particular, there is no visible smear at the boundary.

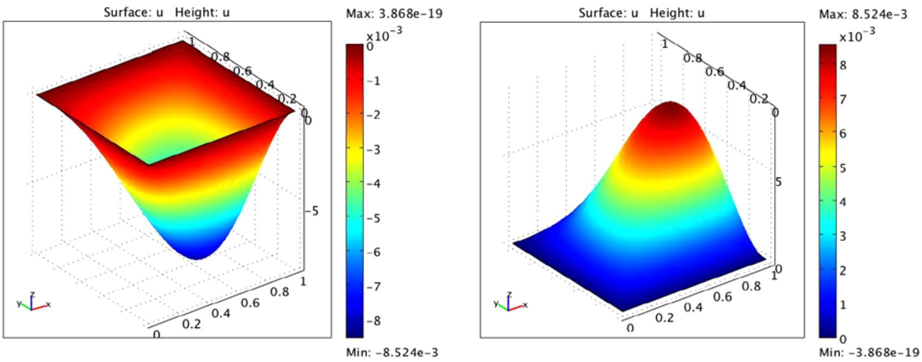

Fig. 6 Computed (moment) solutions of Test 5: Graph on *left* corresponds to $\varepsilon = 10^{-3}$ and graph on *right* corresponds to $\varepsilon = -10^{-3}$. $h = 0.013$ in both cases

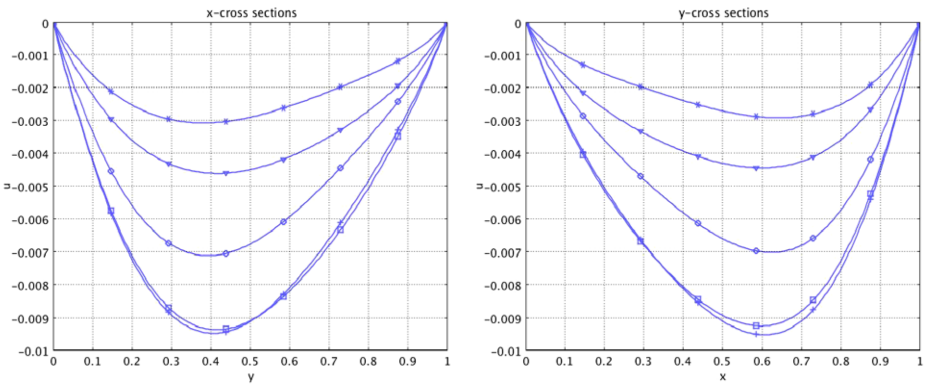

Fig. 7 x-cross sections (*left figure*) of the left graph in Fig. 6 at $x = 0.1, 0.3, 0.5, 0.7, 0.9$ (indicated respectively by *asterisk*, *circle*, *plus sign*, *square*, and *triangle*); y-cross sections (*right figure*) of the left graph in Fig. 6 at $y = 0.1, 0.3, 0.5, 0.7, 0.9$ (indicated respectively by *asterisk*, *circle*, *plus sign*, *square*, and *triangle*)

Test 6: In this test we solve the following Gauss curvature (or $\mathcal{K}$-surface) equation (cf. [47–49])

$$\det(D^2u^0) = K(1 + |\nabla u^0|^2)^2 \quad \text{in } \Omega := (-0.57, 0.57)^2, \tag{5.1}$$

$$u^0 = x^2 + y^2 - 1 \quad \text{on } \partial\Omega, \tag{5.2}$$

where $K > 0$ is a given constant Gauss curvature. Note that the above problem is a special case of problem (4.1), (2.3) with $f(\nabla u^0, u^0, x, y) = K(1 + |\nabla u^0|^2)^{\frac{n+2}{2}}$, $n = 2$, and $g(x, y) = x^2 + y^2 - 1$.

It is known [47] that there exists $K^* > 0$ such that for each $K \in [0, K^*)$ problem (5.1)–(5.2) (with more general Dirichlet data) has a unique convex viscosity solution. Theoretically, it is very difficult to give an accurate estimate for the curvature upper bound K^*. However, this offers an ideal opportunity for numerical analysts to help and contribute. It turns out that the vanishing moment method proposed in this paper works very well for such a problem. Hence, it might provide a useful tool and answer to the challenge.

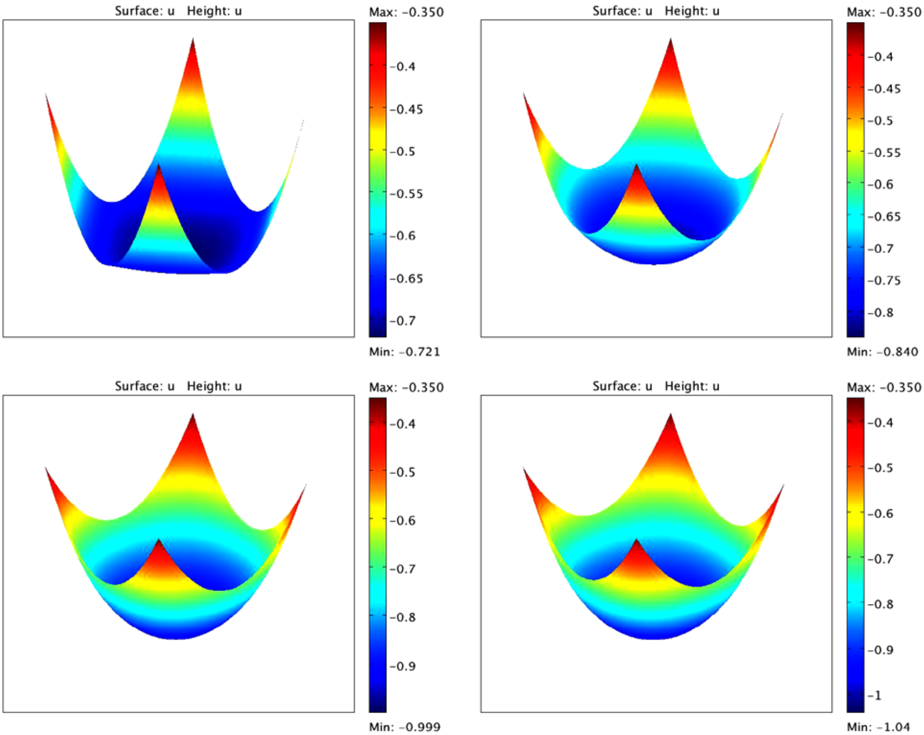

Fig. 8 Computed (moment) solutions of Test 6: $\varepsilon = 10^{-3}$, $h = 0.016$, and $K = 0.1, 1, 2, 2.1$. Graphs are arranged row-wise

Since we are only interested in convex solutions of the Gauss curvature equation, we restrict $\varepsilon > 0$ in (4.2). Figure 8 displays the computed convex (moment) solution using $\varepsilon = 10^{-3}$ and $K = 0.1, 1, 2, 2.1$, respectively. We note that our computer code stops producing a convergent numerical solution for $K = 2.2$. Hence we conjecture that $K^* \approx 2.1$ for the above test problem.

Test 7: In this test, we solve problem (4.7), (2.3) over the domain $\Omega = (-\frac{1}{2}, \frac{1}{2})^2$ with the following boundary datum function

$$g(x, y) = \begin{cases} (\frac{1}{2})^{\frac{4}{3}} - y^{\frac{4}{3}} & \text{if } x = -\frac{1}{2}, \\ x^{\frac{4}{3}} - (\frac{1}{2})^{\frac{4}{3}} & \text{if } y = -\frac{1}{2}, \\ (\frac{1}{2})^{\frac{4}{3}} - y^{\frac{4}{3}} & \text{if } x = \frac{1}{2}, \\ x^{\frac{4}{3}} - (\frac{1}{2})^{\frac{4}{3}} & \text{if } y = \frac{1}{2}, \end{cases}$$

so that $u^0(x, y) = x^{\frac{4}{3}} - y^{\frac{4}{3}}$ is the unique viscosity solution (cf. [3]).

We remark that this is an important example in the theory of absolutely minimizing functions since u^0 is the least regular absolutely minimizing function known in the case of the Euclidean norm (cf. [3] and the references therein). It is easy to check that u^0 is a Hölder continuous function with exponent $\frac{1}{3}$. However, it is not twice differentiable on the axes. We

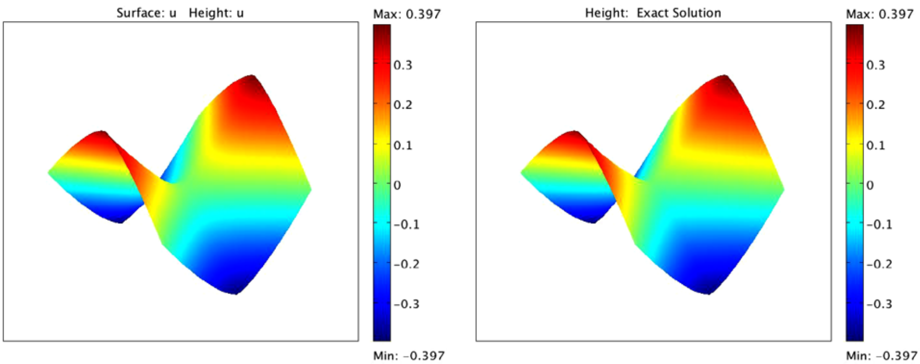

Fig. 9 Computed moment solution (*left graph*) and the exact viscosity solution (*right graph*) of Test 7. $\varepsilon = 10^{-3}$, $h = 0.016$

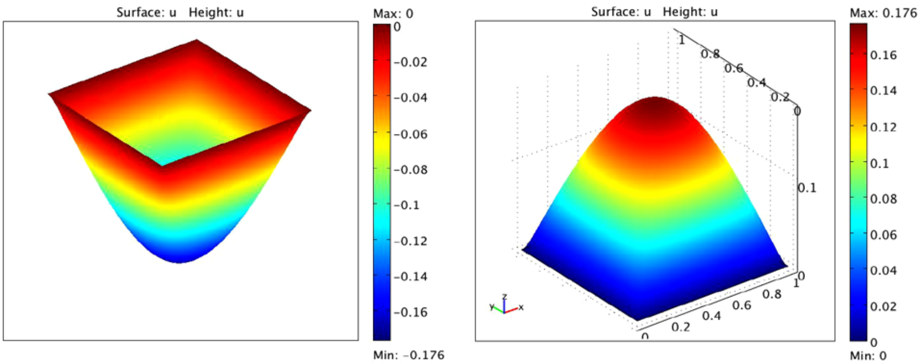

Fig. 10 Computed (moment) solutions of Test 8: Graph on *left* is the computed u_h^ε with $\varepsilon = 10^{-3}$ and graph on *right* is the computed u_h^ε with $\varepsilon = -10^{-3}$. $h = 0.012$ in both cases

also note that (see Sect. 4.3) the infinite Laplace (4.7) is only a (degenerate) quasilinear (instead of a fully nonlinear) second order PDE. However, the complicated and nondivergence structure makes the infinite Laplace equation very difficult to analyze theoretically and to compute numerically.

Figure 9 displays the computed (moment) solution using $\varepsilon = 10^{-3}$ (left graph) and the exact solution u^0 (right graph). Once again, the vanishing moment approximation correctly captures the viscosity solution u^0. Hence, the moment solution coincides with the viscosity solution.

The numerical solutions of the next two tests are obtained by using the Hermann-Miyoshi mixed finite element method with piecewise quadratic shape functions.

Test 8: This test is a re-run of Test 1 but using the quadratic Hermann-Miyoshi mixed finite element method. Figure 10 is the counterpart of Fig. 1. We remark that the mixed method also produces an approximation to the Hessian matrix D^2u^ε, which is not shown here. Since the difference (not shown here) between the computed solutions of Test 1 and Test 8 is very small, we conclude that the numerical results of the two tests have the same accuracy. However, it should be noted that the mixed method runs about 20 times faster than

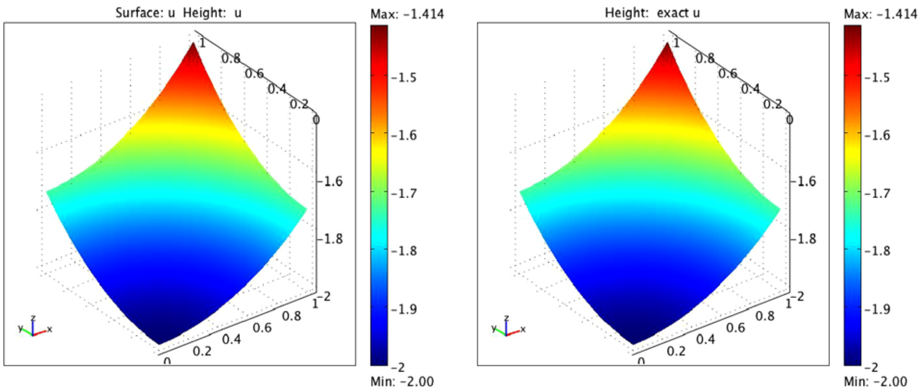

Fig. 11 Computed (moment) solution (*left graph*) and the exact solution (*right graph*) of Test 9 with $\varepsilon = 10^{-3}$, $h = 0.012$

the Argyris method on this test problem even though the former uses a finer mesh than the latter does.

Test 9: This test solves, using the quadratic Hermann-Miyoshi mixed finite method, the Monge-Ampère problem (1.4)–(1.5) on the unit square $\Omega = (0,1)^2$ with the following data:

$$f(x,y) = \frac{4}{(4 - x^2 - y^2)^2}, \qquad g(x,y) = -\sqrt{4 - x^2 - y^2}$$

so that $u^0 = -\sqrt{4 - x^2 - y^2}$ is the exact convex solution. We note that problem (1.4)–(1.5) has exactly two solutions, one is convex and the other is concave (cf. [23]).

Figure 11 displays the computed (moment) solution u_h^ε using $\varepsilon = 10^{-3}$ (left graph) and the exact solution (right graph). As expected, the vanishing moment approximation correctly captures the convex viscosity solution u^0. Hence the moment solution coincides with the viscosity solution. Again, we remark that the mixed method also gives an approximation to the Hessian matrix $D^2 u^\varepsilon$, which is not shown here, and the mixed method runs about 20 times faster than the Argyris method for solving this test problem.

5.2 Three-Dimensional Numerical Experiments

In this subsection we present two numerical tests on computing moment (and viscosity) solutions of the Monge-Ampère problem (1.4)–(1.5) in the unit cube $\Omega = (0,1)^3$. Numerical approximations of fully nonlinear PDEs in 3-d is known to be very difficult. To the best of our knowledge, no 3-d numerical results are given in the literature for the Monge-Ampère type fully nonlinear PDEs.

Test 10: Consider the Monge-Ampère problem (1.4)–(1.5) on the unit cube $\Omega = (0,1)^3$ with the following data:

$$f(x,y,z) = (1 + x^2 + y^2 + z^2)\exp\left(\frac{3(x^2 + y^2 + z^2)}{2}\right),$$

$$g(x,y,z) = \exp\left(\frac{x^2 + y^2 + z^2}{2}\right).$$

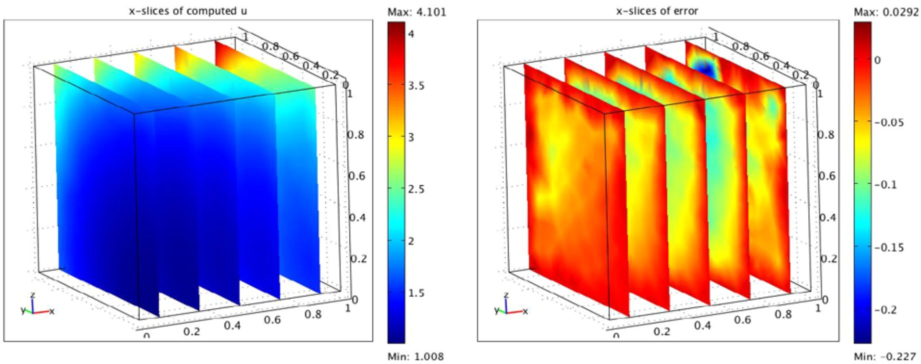

Fig. 12 x-slices of computed (moment) solution and its error of Test 10 with $\varepsilon = 10^{-3}$, $h = 0.058$

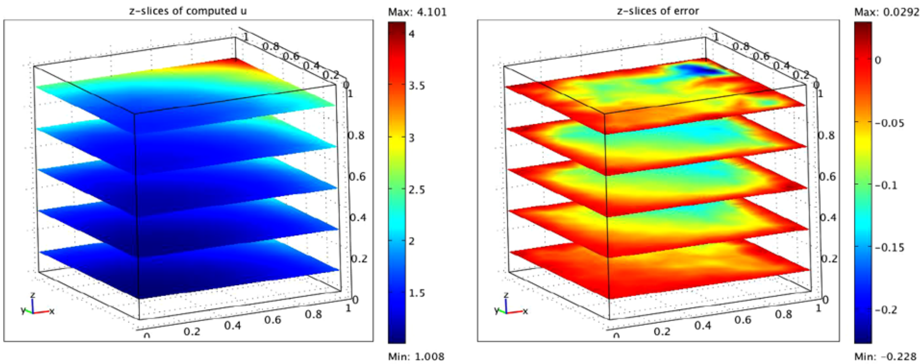

Fig. 13 z-slices of computed (moment) solution and its error of Test 10 with $\varepsilon = 10^{-3}$, $h = 0.058$

It is easy to verify that $u^0 = \exp(\frac{x^2+y^2+z^2}{2})$ is the unique exact (convex) solution. We compute this solution using the vanishing moment method combined with a generalized Hermann-Miyoshi type mixed finite element method using linear shape functions.

Figure 12 displays color plots of five x-slices of the computed (moment) solution u_h^ε (left graph) and its corresponding error (right graph). Figure 13 displays color plots of five z-slices of the computed (moment) solution u_h^ε (left graph) its corresponding error (right graph). As expected, the vanishing moment approximation correctly captures the convex viscosity solution u^0. Hence the moment solution coincides with the viscosity solution.

Test 11: Our last numerical test solves the 3-d generalization of the test problem in Test 1. That is, we assume u^0 satisfies the Monge-Ampère problem (1.4)–(1.5) in $\Omega = (0,1)^3$ with the data

$$f(x,y,z) \equiv 1, \qquad g(x,y,z) \equiv 0.$$

We remark that the above problem has a unique convex viscosity solution but does not have a classical solution (cf. [31, 50]). There is no explicit solution formula for the boundary value problem.

Figure 14 displays color plots of x-slices (left graph) and z-slices (right graph) of the computed (moment) solution u_h^ε using a generalized linear Hermann-Miyoshi mixed

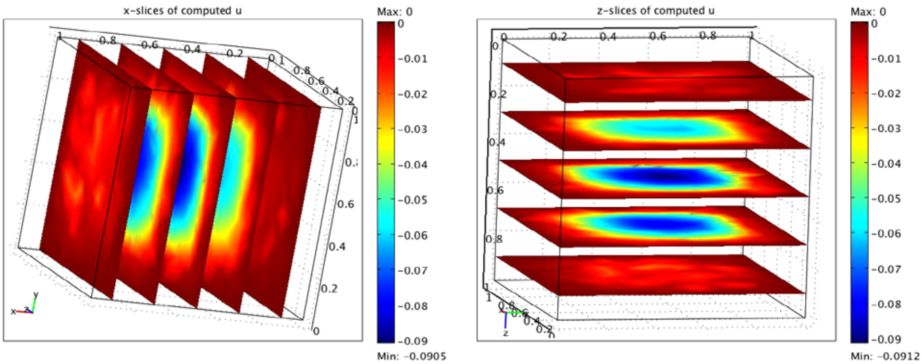

Fig. 14 x-slices (*left*) and z-slices (*right*) of computed (moment) solution of Test 11 with $\varepsilon = 10^{-3}$, $h = 0.058$

method. Once again, the vanishing moment approximation correctly captures the convex viscosity solution u^0. Hence the moment solution coincides with the viscosity solution.

We conclude this section by noting that one difficult and important issue is the rate of convergence for each proposed numerical method. While it is hard to determine a rate of convergence for $u^0 - u^\varepsilon$ analytically, we are able to derive a rate of convergence (in powers of h) for $u^\varepsilon - u_h^\varepsilon$ for a fixed ε in the case of the Monge-Ampère equation (see [62]). In addition, we have done a detailed computational study for determining the rates of convergence in terms of powers of ε for the error $u^0 - u_h^\varepsilon$ (and $\sigma^\varepsilon - \sigma_h^\varepsilon$ in the case of mixed methods), and numerically examine what is the "best" mesh size h in relation to ε in order to achieve these rates. Our numerical study suggests the following rates of convergence (see [62]):

$$\|u^0 - u^\varepsilon\|_{H^2} = O(\varepsilon^{\frac{1}{4}}), \qquad \|u^0 - u^\varepsilon\|_{H^1} = O(\varepsilon^{\frac{1}{2}}), \qquad \|u^0 - u^\varepsilon\|_{L^2} = O(\varepsilon),$$

which should serve as good guidelines for a possible rigorous justification.

6 Conclusions

In this paper we introduce a new notion of weak solutions, called *moment solutions*, through a constructive limiting process, called *the vanishing moment method*, for fully nonlinear second order PDEs. The notion of moment solutions and the vanishing moment method are exactly in the same spirit as the original notion of viscosity solutions and the vanishing viscosity method proposed by M. Crandall and P.L. Lions in [26] for the Hamilton-Jacobi equations, which is based on the idea of approximating a fully nonlinear PDE by a quasilinear higher order PDE. We first present a general framework of the vanishing moment method and the notion of moment solutions in Sect. 2. We then apply the general framework to several classes of PDEs including the Monge-Ampère type equations, Pucci's extremal equations, the infinite Laplace equation, and fully nonlinear second order parabolic PDEs. We then propose two classes of numerical methods to discretize the fourth order "regularized/perturbed" vanishing moment approximation equations. Finally, we present a number of numerical experiments using the vanishing moment methodology together with the proposed numerical methods to demonstrate convergence and effectiveness of the method. The numerical experiments also show the relationship between the notion of moment solutions and the notion of viscosity solution for fully nonlinear second order PDEs.

This paper provides a practical and systematic methodology/approach for approximating fully nonlinear second order PDEs. As a by-product, the moment solution concept gives some insights for the understanding of viscosity solutions, and might also provide a logical and natural generalization/extension for the notion of viscosity solution, especially, in the cases where there is no theory or the existing viscosity solution theory fails (e.g. the Monge-Ampère equations of hyperbolic type [16] and systems of fully nonlinear second order PDEs).

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